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Michael Hitrik
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Algebraic and Analytic Microlocal Analysis

AAMA, Evanston, Illinois, USA, 2012 and 2013

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Editors
Michael Hitrik
Department of Mathematics
UCLA
Los Angeles, CA, USA

Boris Tsygan
Department of Mathematics
Northwestern University
Evanston, IL, USA

Dmitry Tamarkin
Department of Mathematics
Northwestern University
Evanston, IL, USA

Steve Zelditch
Department of Mathematics
Northwestern University
Evanston, IL, USA

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Preface

On May 14–26, 2012, and May 20–24, 2013, two workshops took place at the Northwestern University Mathematics Department, as part of the emphasis year in microlocal analysis. The main subjects of the workshops were algebraic and analytic microlocal analysis, respectively. The organizers of the algebraic workshop in 2012 were Dmitry Tamarkin and Boris Tsygan, while the analytic workshop in 2013 was organized by Michael Hitrik and Steve Zelditch. This volume consists of articles expanding on some mini-courses and talks presented at the workshops. Altogether, they span over a large variety of topics, ranging from foundational material discussed in the mini-courses to advanced research level papers.

We shall now proceed to give a description of the chapters of the volume. When doing so, we shall discuss separately contributions of the authors corresponding to each of the two workshops.

Algebraic Microlocal Analysis

The contributions of Losev, Schapira, Tamarkin, and Tsygan are devoted to algebraic microlocal analysis. The discipline started in 1959 when Mikio Sato introduced hyperfunctions. This is the starting point of Schapira's article. A hyperfunction on $\mathbb{R}$ is a boundary value of an analytic function, that is, a complex analytic function on $\mathbb{C}/\mathbb{R}$ up to an analytic function on $\mathbb{C}$. More generally, for a real analytic manifold M which is the real part of a complex analytic manifold M, the sheaf of hyperfunctions on M is the cohomology of $\mathcal{O}_X$ with supports in M, twisted by the orientation sheaf. Distributions are examples of hyperfunctions. Sato defined the microlocal version of hyperfunctions, namely the sheaf of microfunctions that lives not on M but on the cotangent bundle T^*M. This sheaf is obtained from $\mathcal{O}_X$ by another fundamental sheaf-theoretic construction, namely by microlocalization (which is the Fourier-Sato transform of specialization). A hyperfunction defines a microfunction whose support is a closed conical subset of T^*M. This support is called the microsupport of the original hyperfunction.

Next, we turn to explaining how to interpret solutions of linear PDE in terms of homological algebra of sheaves. Given a differential operator P and a solution u of $Pu = 0$, the formula $Q \mapsto Qu$ defines a homomorphism of $\mathcal{D}$-modules $\mathcal{D}/\mathcal{D}P \to \mathcal{O}$ where $\mathcal{D}$ is the ring of all differential operators and $\mathcal{O}$ could be any space of (generalized) functions containing u. Set $\mathcal{M} = \mathcal{D}/\mathcal{D}P$. The short complex $\mathcal{O} \xrightarrow{P} \mathcal{O}$ computes $\mathrm{Ext}^\bullet(\mathcal{M}, \mathcal{O})$ because it can be interpreted as

$$\mathrm{Hom}_{\mathcal{D}}(\mathcal{D} \xrightarrow{\cdot P} \mathcal{D}, \mathcal{O}).$$

This suggests the definition of the complex of sheaves, or rather its image in the derived category of sheaves,

$$\mathrm{Sol}(\mathcal{M}) = \mathbb{R}\mathrm{Hom}(\mathcal{M}, \mathcal{O}_X)$$

for any sheaf $\mathcal{M}$ of $\mathcal{D}_X$-modules where $\mathcal{D}_X$ is the sheaf of algebras of holomorphic differential operators and $\mathcal{O}_X$ is the sheaf of holomorphic functions. As above, $\mathcal{O}_X$ can be replaced by any reasonable sheaf of (generalized) functions.

Remark 1 Note that a $\mathcal{D}_X$-module is a generalization of a bundle with a flat connection. For the latter, one can define the De Rham complex which is a complex of sheaves on X. This easily generalizes to any $\mathcal{D}_X$-module. The De Rham complex $\mathrm{DR}^\bullet(\mathcal{M})$ is very close to the sheaf of solutions, in fact it is the same up to some duality. (Of course one of the functors is contravariant and the other is covariant).

A complex $\mathcal{M}$ of $\mathcal{D}_X$-modules (with an additional condition, namely the existence of a good filtration) naturally gives rise to a microlocal object, namely a sheaf of $\mathcal{O}_{T^*X}$-modules $\mathrm{gr}(\mathcal{M})$. This is due to the fact that the sheaf of algebras of differential operators $\mathcal{D}_X$ is filtered by order, and its associated graded is $\mathcal{O}_{T^*X}$ The support of the cohomology of $\mathrm{gr}(\mathcal{M})$ is a conical closed subset $\mathrm{SS}(\mathcal{M})$ of T^*X which is called the singular support of $\mathcal{M}$. When $\mathcal{M}$ is of the form $\mathcal{D} \xrightarrow{\cdot P} \mathcal{D}$ then $\mathrm{gr}(\mathcal{M})$ is given by $\mathcal{O}_{T^*X} \xrightarrow{\sigma(P)} \mathcal{O}_{T^*X}$ where $\sigma(P)$ is the principal symbol of P. The singular support is therefore the characteristic variety of P, i.e., the subset of T^*X where $\sigma(P)$ is degenerate.

When one studies a real analytic differential operator P on a real analytic M as above, one can interpret the complex $\mathcal{O}_M \xrightarrow{P} \mathcal{O}_M$ as $\mathrm{Sol}(\mathcal{M}) \otimes_{\mathbb{C}_X} \mathbb{C}_M$ where $\mathcal{M} = \mathcal{D}_X \xrightarrow{\cdot P} \mathcal{D}_X$.

So far, we have seen two prominent applications of homological algebra of sheaves in microlocal analysis. One is due to the fact that, when M is real analytic and X is its complexification, hyperfunctions and microfunctions on M can be obtained by standard sheaf-theoretical constructions from the sheaf $\mathcal{O}_X$ of holomorphic functions. The other is due to the fact that, for a complex analytic manifold X, a sheaf $\mathcal{M}$ of $\mathcal{D}_X$-modules defines a sheaf of $\mathbb{C}_X$-modules, via one of the two related constructions Sol or $\mathrm{DR}^\bullet$. The latter suggests that, for a sheaf $\mathcal{F}$ on a manifold M, one can define its microsupport in T^*M. This was carried out by

Kashiwara and Schapira for any C^∞ manifold M (in which case the relation between $\mathcal{D}$-modules and sheaves is far from direct). The microsupport $\mu\mathrm{Supp}(\mathcal{F})$ is a conical closed subset of T^*M. A fundamental theorem says that when $\mathcal{F} = \mathrm{Sol}(\mathcal{M})$ on a complex manifold X, then $\mu\mathrm{Supp}(\mathcal{F}) = \mathrm{SS}(\mathcal{M})$. If N is a submanifold of M, $\mu\mathrm{Supp}(\mathbb{C}_N)$ is the conormal bundle of N.

Given the link between $\mathcal{D}$-modules and sheaves, one could expect that a sheaf $\mathcal{F}$ on M defines a sheaf on T^*M, similarly to a $\mathcal{D}_X$-module $\mathcal{M}$ defining an $\mathcal{O}_{T^*X}$-module $\mathrm{gr}(\mathcal{M})$. What actually happens that two sheaves $\mathcal{F}$ and $\mathcal{G}$ on M define the sheaf $\mu\mathrm{hom}(\mathcal{F},\mathcal{G})$ on T^*M. This sheaf is supported on $\mu\mathrm{Supp}(\mathcal{F}) \cap \mu\mathrm{Supp}(\mathcal{G})$. Its (derived) direct image to M is the usual (derived) sheaf of morphisms $\mathbb{R}\mathcal{H}om(\mathcal{F},\mathcal{G})$.

In light of the above, we see that a real analytic differential operator P is elliptic if and only if

$$\mathrm{SS}(\mathcal{M}) \cap \mu\mathrm{Supp}(\mathbb{C}_M) \subset T^*_M M$$

This gives the motivation for the definition due to Schapira and Schneiders: An elliptic pair on a complex manifold X is an $\mathbb{R}$-constructible sheaf $\mathcal{F}$ together with a $\mathcal{D}_X$-module $\mathcal{M}$ such that

$$\mathrm{SS}(\mathcal{M}) \cap \mu\mathrm{Supp}(\mathcal{F}) \subset T^*_M M. \tag{1}$$

The complex $\mathcal{O}_M \xrightarrow{P} \mathcal{O}_M$ generalizes to $\mathrm{DR}^\bullet(\mathcal{M}) \otimes_{\mathbb{C}_X} \mathcal{F}$. Schapira and Schneiders proved that, when the intersection in (1) is compact, then $\mathbb{R}\Gamma(M, \mathrm{DR}^\bullet(\mathcal{M}) \otimes_{\mathbb{C}_X} \mathcal{F})$ has finite-dimensional total cohomology (Theorem 1.5.1 and Corollary 1.5.2; a key result is the generalization of the elliptic regularity theorem). Therefore, the Euler characteristic of this complex is well defined, an invariant that generalizes the index of a real analytic elliptic differential operator P. Schapira and Schneiders proved that this invariant can be computed as the integral of some cohomology class ovet T^*X. They conjectured a formula for this cohomology class in terms of the Chern character of $\mathrm{gr}(\mathcal{M})$, the Todd class of T_X, and the microlocal Euler class of $\mathcal{F}$ (see below). This conjecture was proved by Bressler, Nest, and Tsygan.

The above is the topic of Schapira's lecture 1. Lecture 2 discussed trace-like invariants of objects such as $\mathcal{D}$-modules and $\mathbb{R}$-constructible sheaves. A recipient of such invariants is *microlocal cohomology* of X, i.e., cohomology with given support of T^*X with coefficients in the sheaf $p^{-1}(\omega_X)$ where ω_X is the dualizing sheaf on X. In other words, when X is oriented, the microlocal cohomology is the middle cohomology of X with supports in a given (conical closed) subset Λ. For an $\mathbb{R}$-constructible sheaf $\mathcal{F}$ with microsupport contained in Λ, one defines its microlocal Euler class $\mu eu(\mathcal{F})$ in $H^0_\Lambda(T^*X, p^{-1}(\omega_X))$; for a coherent $\mathcal{D}_X$-module $\mathcal{M}$ with singular support contained in Λ', one defines its characteristic class $\mathrm{hh}_{\mathcal{E}}(\mathcal{M})$ in $H^0_{\Lambda'}(T^*X, p^{-1}(\omega_X))$. Note that, if $\Lambda \cap \Lambda'$ is compact, then $\mu eu(\mathcal{F}) \smile \mathrm{hh}_{\mathcal{E}}(\mathcal{M})$ is in

$H_c^{\mathrm{top}}(T^*X)$; the theorem of Schapira and Schneiders says that the Euler characteristic of the elliptic pair $(\mathcal{M},\mathcal{F})$ is the integral of this class over T^*X.

Lecture 2 starts with the definition of Hochschild homology. For an associate algebra A, its Hochschild homology $\mathrm{HH}(A)$ can be interpreted as a universal trace-like invariant of A. In particular, $\mathrm{HH}_0(A)=A/[A,A]$ and for any finitely generated projective module M over A, its Hattori-Stallings trace is a well-defined element of $\mathrm{HH}_0(A)$. In fact, HH_0 does not change when one passes to a matrix algebra, and, if M is the image of an idempotent e, the class of e in the quotient HH_0 is well defined.

Hochschild homology is naturally defined in greater generality than for associative algebras, namely for differential graded categories. In this generality, the Hochschild homology class of an object is even easier to define; it is in fact almost tautological. The question now becomes to compute this homology in the following examples: (a) the dg category $\mathrm{Sh}_\Lambda(X)$ of ($\mathbb{R}$-constructible) complexes of sheaves on X with microsupport in Λ and (b) the dg category $\mathcal{D}_X-\mathrm{mod}'_\Lambda$ of perfect complexes of $\mathcal{D}_X$-modules with singular support in Λ'. The construction of $\mathrm{hh}_{\mathcal{E}}(\mathcal{M})$ does in fact come from a morphism

$$\mathrm{HH}(\mathcal{D}_X-\mathrm{mod}_{\Lambda'})\to H^\bullet_{\Lambda'}(T^*X,p^{-1}\omega_X)$$

This map is very plausibly an isomorphism. In contrast to this, as far as we know, there is nothing of the sort that is known in case a). The question how to describe $\mathrm{HH}(\mathrm{Sh}_\Lambda(X))$ is rather central not only in microlocal analysis but in symplectic topology and also in Langlands duality theory.

Remark 2 The above is very interesting when one replaces the Hochschild homology by other invariants of dg categories, such as the (negative, periodic) cyclic homology, or algebraic K theory, or perhaps some sort of a universal, or motivic, invariant. If $\mathcal{H}$ is such an invariant then the dg functor

$$\mathrm{Ell\ pairs}(X)\to\mathrm{Perf}(\mathbb{C});(\mathcal{M},\mathcal{F})\mapsto\mathbb{R}\Gamma(\mathrm{DR}^\bullet(\mathcal{M})\otimes_{\mathbb{C}_X}\mathcal{F}) \tag{2}$$

induces a map

$$\mathcal{H}(\mathrm{Ell\ pairs}(X))\to\mathcal{H}(\mathrm{Perf}(\mathbb{C}))\xrightarrow{\sim}\mathcal{H}(\mathbb{C}) \tag{3}$$

A generalized Schapira–Schneiders theorem would provide a formula for this map. The works of Beilinson [1] and Patel [3], and a very recent preprint of Groechenig [2], are closely related to the where cases $\mathcal{M}=\mathcal{O}_X$ and $\mathcal{F}$ any constructible sheaf on M, resp. $\mathcal{F}=\mathbb{C}_X$ and $\mathcal{M}$ a coherent algebraic $\mathcal{D}_X$-module.

Lecture 3 is devoted to a sheaf-theoretic interpretation of new classes of functions and distributions, as well as to its applications. These new classes are tempered C^∞ functions, tempered distributions, and Whitney C^∞ functions on a real analytic manifold X. On a complex analytic manifold X one also defines tempered and Whitney analytic functions. That they can be interpreted in terms of sheaves is quite nontrivial because growth properties of functions are not local. But it had been shown that tempered functions and distributions form sheaves on the *subanalytic site* of X. An alternative and closely related sheaf-theoretical description of these classes of (generalized) functions is in terms of *indsheaves* of Kashiwara and Schapira.

To explain the applications of these techniques to $\mathcal{D}$-modules, recall the functor $\mathrm{DR}^\bullet$ from $\mathcal{D}_X$-modules to sheaves. Kashiwara proved that this is an equivalence of derived categories between *regular holonomic* $\mathcal{D}_X$-modules and $\mathbb{C}$-constructible sheaves. This is called the Riemann–Hilbert correspondence because it assigns to a bundle with a flat connection the sheaf of its De Rham complexes whose degree zero cohomology is the sheaf of flat sections. When one considers all (possibly irregular) holonomic $\mathcal{D}_X$-modules, the De Rham functor has no chance of being an equivalence because, for example, on $X = \mathbb{C}^*$ the flat connection $\left(\frac{d}{dz} + \frac{1}{z^2}\right)dz$ has a global flat section $\exp\left(\frac{1}{z}\right)$ and cannot be distinguished from $\frac{d}{dz}dz$.

To establish a Riemann–Hilbert correspondence for all holonomic $\mathcal{D}_X$-modules, one introduces two new ideas. First, one considers solutions with values in tempered holomorphic functions. Second, one introduces the new variable t. In a series of recent works a Riemann–Hilbert correspondence was established between all holonomic $\mathcal{D}_X$-modules and *enhanced sheaves* on X (see references in 3.4 and 3.5).

To make a link between Schapira's lectures and other algebraic contributions in this volume, let us start by returning to microlocalization of differential operators. We already mentioned that $\mathrm{gr}(\mathcal{D}_X)$ lives naturally on the cotangent bundle. In what sense does $\mathcal{D}_X$ itself live on the cotangent bundle? In fact, one can define the sheaf of algebras of microdifferential operators $\mathcal{E}_{T^*X}$ whose direct image to X is $\mathcal{D}_X$ This sheaf plays a crucial role in constructing the microlocal class $\mathrm{hh}_{\mathcal{E}}$ in lecture 2. There is another, more algebraic way to pass to a sheaf on T^*X. If one replaces differential operators $\mathcal{D}_X$ by $\hbar$-differential operators $\mathcal{D}_X^\hbar$ (also called the Rees ring of $\mathcal{D}_X$), then one can replace $\mathcal{E}_{T^*X}$ by the sheaf $\mathbb{A}_{T^*X}$, the canonical deformation quantization of T^*X. Locally, this sheaf is isomorphic to $\mathcal{O}_{T^*X}[[\hbar]]$ with noncommutative multiplication which coincides with the standard one modulo $\hbar$. Deformation quantization is beyond the scope of Schapira's lectures in this volume (it is the subject of a large series of his joint works with Kashiwara).

Note also that the idea of introducing an extra variable to define enhanced sheaves (see above) was inspired by Tamarkin's contribution in this volume, which itself was inspired by the work of D'Agnolo, Dito, Polesello, and Schapira on deformation quantization.

Now note that deformation quantization can be defined for arbitrary symplectic (and, more generally, Poisson) manifolds. When one looks at constructions of symplectic topology such as Floer cohomology and Fukaya theory, one observes that many of them seem to be microlocal in nature, in the sense that they are based in a significant part on such microlocal notions as Lagrangian submanifolds, Maslov index, etc. In the early eighties, Feigin asked a question whether these constructions, or their analogues, can be carried out microlocally, for example in terms of some (enhanced) version of the category of modules over the canonical deformation quantization algebra. First steps in this direction were carried out by Bressler and Soibelman and by Kapustin and Witten. An answer for T^*X (in terms of constructible sheaves on X) was given by Nadler and Zaslow.

Tamarkin's paper in this volume provides another version of an answer for T^*X. He defines a category of sheaves on $X \times \mathbb{R}$ (note the extra variable) satisfying certain condition on their microsupport. A direct link to symplectic topology is not at all clear. What is established is that his category satisfies the same key properties as the Fukaya category.

More precisely, for two objects $\mathcal{F}$ and $\mathcal{G}$, $\mathrm{HOM}(\mathcal{F}, \mathcal{G})$ is a module over the Novikov ring Λ. For any object $\mathcal{F}$, its microsupport $\mu\mathrm{Supp}(\mathcal{F})$ is defined. Under some compactness assumptions, $\mathrm{HOM}(\mathcal{F}, \mathcal{G}) = 0$ if microsupports of $\mathcal{F}$ and $\mathcal{G}$ do not intersect. For any Hamiltonian isotopy Φ which is the identity outside a compact, a functor T_Φ is defined. One has $\mu\mathrm{Supp}(T_\Phi \mathcal{F}) = \Phi(\mu\mathrm{Supp}(\mathcal{F}))$. Finally, $\mathrm{HOM}(T_\Phi(\mathcal{F}), \mathcal{G}) \xrightarrow{\sim} \mathrm{HOM}(\mathcal{F}, \mathcal{G})$ modulo Λ-torsion. In particular, if $\mathrm{HOM}(\mathcal{F}, \mathcal{G})$ is not a torsion Λ-module then $\mu\mathrm{Supp}(\mathcal{F})$ cannot be displaced from $\mu\mathrm{Supp}(\mathcal{G})$ by a Hamiltonian isotopy (under a compactness condition).

Note also that Tamarkin's construction for a two-dimensional torus gives the answer similar to the Fukaya category as computed by Polishchuk and Zaslow. This is not part of Tamarkin's paper, but it is reviewed in Tsygan's contribution.

Tsygan's contribution to this volume should be viewed as related by a conjectural Riemann–Hilbert functor to Tamarkin's (or rather to the sequel [4] dealing with an arbitrary symplectic manifold). Instead of enhanced sheaves, it is based on enhanced modules over the canonical deformation quantization. For a symplectic manifold M, enhanced modules are dg modules over the sheaf of dg algebras $\mathcal{A}^\bullet_M$ with an extra structure. To describe this structure, note that the fundamental groupoid $\pi_1(M)$ acts on $\mathcal{A}^\bullet$ up to inner automorphisms (as defined in Section 5); the modules are required to have a compatible action of $\pi_1(M)$. For two such modules $\mathcal{V}^\bullet$ and $\mathcal{W}^\bullet$, the standard complex computing $\mathrm{Ext}_{\mathcal{A}^\bullet}(\mathcal{V}^\bullet, \mathcal{W}^\bullet)$ acquires an A_∞ action of $\pi_1(M)$ and therefore becomes an ∞-local system on M. (This is an expression of the standard fact that inner automorphisms act on Ext trivially).

As in the Fukaya category, one associates an object to a Lagrangian submanifold L of M, subject to some topological conditions. It may be worthwhile to describe it here in more detail, in order to clarify a connection with other contributions to this volume (algebraic and otherwise).

For this, let us come back to deformation quantization. As a version of what we did above, one can define it for $M = T^*X$ in terms of asymptotics of the product of (pseudo)differential operators depending on $\hbar$. Given a Lagrangian submanifold L of T^*X, one can extend this definition to asymptotics of these operators acting on ($\hbar$-dependent) Lagrangian distributions with wave front L. One gets a sheaf of modules over the canonical deformation quantization algebra $\mathbb{A}_M$ supported on L (this construction is reviewed in section 15). Actually, it gives rise to a dg module over a dg algebra, call it $\mathcal{B}^\bullet_L$, which is intermediate between $\mathbb{A}_M$ and $\mathcal{A}^\bullet_M$. It also carries an action of $\pi_1(L)$, of the type that we described above. To construct a module over $\mathcal{A}_M^\bullet$ with an action of $\pi_1(M)$ from a module over $\mathcal{B}_L^\bullet$, with the action of $\pi_1(M)$, one uses the standard procedure of induction.

The bigger algebra $\mathcal{A}_M^\bullet$ could be understood as describing asymptotics of the product of all $\hbar$-dependent Fourier integral operators. Its construction is based on Fedosov's technique of deformation quantization. Including asymptotics of FIO and Lagrangian distributions, or Guillemin and Sternberg's geometric asymptotics, into Fedosov's theory was suggested by Karabegov in the late nineties.

Losev's contribution to this volume is a survey on deformation quantization of certain *algebraic* varieties, namely symplectic resolutions of singularities V/Γ where V is a complex symplectic space and Γ is a finite group of symplectomorphisms. These resolutions are not so easy to describe; the first example is $T^*\mathbb{P}^1$ which is a resolution of $\mathbb{C}^2/(\mathbb{Z}/2)$. The latter can be identified with the nilpotent cone in $\mathfrak{sl}_2$, and the map to it from $T^*\mathbb{P}^1$ is the moment map for the standard $\mathfrak{sl}_2$ action. One way to approach these resolutions is to observe that they are Morita equivalent to an easily understood noncommutative algebra, namely the smash (or cross) product $\mathbb{C}[V]\#\Gamma$ of Γ and $\mathbb{C}[V]$. In fact, this algebra is the algebra of endomorphisms of a bundle, called the Procesi bundle, on the resolution. Now one can consider deformations of the smash product. The first tool of studying deformations of algebras is Hochschild *cohomology* (not homology discussed above). This cohomology is not hard to compute in our case, and all deformations can be classified. The easiest one is the smash product of Γ by the Weyl algebra of V, but there are more interesting deformations, namely *symplectic reflection algebras* of Etingof and Ginzburg. Now, one can do two things: using Morita equivalence, define a deformation of the symplectic resolution, or, observing that the idempotent $e = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\gamma$ deforms to an idempotent, and that

$$e(\mathbb{C}[V]\#\Gamma)e \xrightarrow{\sim} \mathbb{C}[\mathbb{V}]^\Gamma,$$

obtain a deformation of $\mathbb{C}[\mathbb{V}]^\Gamma$. The latter deformation is called a spherical symplectic reflection algebra.

Let us try to describe what we have in a little more analytic terms: Consider all the differential operators, plus a finite group of invertible FIOs. We can of course just generate an algebra of operators by them. But it turns out that there is a new, perhaps more interesting, product on this algebra. We get a new operator algebra $\mathcal{A}$ containing an idempotent e (the average of elements of the finite group). Elements of the subalgebra $e\mathcal{A}e$ have symbols that are functions not on the cotangent bundle but on a more nontrivial symplectic manifold.

Remark 3 Let us also note that the big algebra $\mathcal{A}^\bullet$ from Tsygan's article is an attempt to construct an asymptotic version of the algebra generated by differential operators and all FIO.

As explained in the article, symplectic resolutions can be obtained by Hamiltonian reduction. A particular example, the Hilbert scheme $\mathrm{Hilb}_n(\mathbb{C}^2)$ of ideals of codimension n of $\mathbb{C}[x, y]$, is obtained by Hamiltonian reduction from the cotangent bundle of the variety of framed representations of the one-loop quiver. The spherical symplectic reflection algebra can be obtained by quantum Hamiltonian reduction (Theorem 3.14). This is used to study its representation theory. Knowing representation theory of the spherical algebra on the one hand, and interpreting it as deformation quantization of the symplectic resolution on the other, is akin to saying that these representation operators have symbols that are functions on the resolutions, as mentioned above. To what extent one can advance this analogy, and how it could be used either in analysis or in this area of algebra, is not clear at the moment.

The article in this volume by no means cover all interesting developments in algebraic microlocal analysis. For example, a very extensive area of microlocal analysis in positive characteristic is virtually not discussed (except briefly in Losev's contribution).

Analytic Microlocal Analysis

The chapter by Robert Berman concerns a determinantal point process on a projective complex manifold whose underlying kernel is the Bergman kernel associated to a high power of a complex line bundle equipped with a Hermitian metric, which need not be positively curved. It is shown, in particular, that in the large particle number limit, the points concentrate in a droplet given by the support of an equilibrium measure and that the fluctuations around mean of smooth linear statistics are asymptotically normal and governed by a Gaussian free field.

The chapter by Bo Berndtsson studies geodesics in the space of positively curved metrics on a complex line bundle over a Kähler manifold and geodesics in the finite-dimensional space of Hermitian forms on the space of holomorphic sections of high powers of the line bundle and establishes certain finite-dimensional approximation results, in terms of associated spectral measures.

In their chapter, Yaiza Canzani, and John Toth study the nodal sets of eigenfunctions of the Laplacian on a compact real analytic two-dimensional manifold, in the semiclassical limit. An accurate upper bound is established on the number of intersections of the nodal sets with certain curves.

The chapter by Michael Christ proves optimal off-diagonal decay bounds for the Bergman kernels associated to high powers of a complex line bundle over a compact complex manifold, equipped with a positively curved C^∞ Hermitian metric. The following chapter, also by Christ, addresses the related question by Steve Zelditch of whether the exponentially fast decay of the Bergman kernel away from the diagonal, associated to a high power of a positively curved complex line bundle, implies that the corresponding curvature form is real analytic. The question is answered in the affirmative in a special case when the underlying manifold is the complex n-dimensional space.

The chapter by Michael Hitrik and Johannes Sjöstrand consists of two separate parts devoted to a package of results that form the core of Analytic Microlocal

Analysis: analytic pseudo-differential operators, FBI (or Bargmann) transforms to the complex domain, associated exponentially weighted spaces of analytic functions and Bergman projections, I-Lagrangian submanifolds and canonical relations, analytic Fourier integral operators in the complex domain, and conjugation of Toeplitz operators to analytic pseudo-differential operators.

There exist only partial expositions of this foundational material in the literature at present. The foundational text is J. Sjöstrand's 1982 Astérisque book, *Singularités analytiques microlocales*. This classic text presents some of the theory in a general context but much of the general theory can only be found in various articles of Sjöstrand, and much remains to be written in a systematic way.

The first part of the chapter gives a systematic exposition of the theory in the case of quadratic phase functions, i.e., the Bargmann–Fock metaplectic representation. The second part of the chapter is at a more advanced level and in some sense is a revision and extension of the Astérisque book cited above. The ideal would be to have a full exposition of the FBI package of results as in the Bargmann–Fock case but at the same level of generality as this chapter, but that would be a very arduous and lengthy project which remains for the future.

The chapter by Gilles Lebeau gives a detailed exposition of a theorem of L. Boutet de Monvel on the convergence in the complex domain of a series of eigenfunctions of the Laplacian on a real analytic compact manifold.

The purpose of the chapter by André Martinez, Shu Nakamura, and Vania Sordoni is to provide an overview of the work done by the authors, devoted to the propagation of singularities for second-order perturbations of the Laplacian in the real analytic category.

The chapter by Steve Zelditch and Peng Zhou studies spectral asymptotics for Toeplitz operators on high powers of a positively curved complex line bundle over a Kähler manifold and proves a two-term pointwise Weyl law for the kernels of spectral projections of the operator onto sums of eigenspaces of spectral width inversely proportional to the high power of the line bundle.

In his chapter, Maciej Zworski provides a novel definition of scattering resonances for Schrödinger operators. Namely, it is shown that the resonances can be defined as viscosity limits of eigenvalues of the operator obtained by perturbing the Schrödinger operator by a quadratic complex absorbing potential.

We would like to express our sincere gratitude to all the authors for their inspired lectures at the workshops and their contributions to this volume.

Los Angeles, USA
Evanston, USA
July 2018

Michael Hitrik
Dmitry Tamarkin
Boris Tsygan
Steve Zelditch

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Contents

Part I
Algebraic Microlocal Analysis

Procesi Bundles and Symplectic Reflection Algebras

Ivan Losev

Abstract In this survey we describe an interplay between Procesi bundles on symplectic resolutions of quotient singularities and Symplectic reflection algebras. Procesi bundles were constructed by Haiman and, in a greater generality, by Bezrukavnikov and Kaledin. Symplectic reflection algebras are deformations of skew-group algebras defined in complete generality by Etingof and Ginzburg. We construct and classify Procesi bundles, prove an isomorphism between spherical Symplectic reflection algebras, give a proof of wreath Macdonald positivity and of localization theorems for cyclotomic Rational Cherednik algebras.

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1 Introduction

1.1 Procesi Bundles: Hilbert Scheme Case

A Procesi bundle is a vector bundle of rank $n!$ on the Hilbert scheme $\mathrm{Hilb}_n(\mathbb{C}^2)$ whose existence was predicted by Procesi and proved by Haiman, [34]. This bundle was used by Haiman to prove a famous $n!$ conjecture in Combinatorics that, in turn, settles another famous conjecture: Schur positivity of Macdonald polynomials.

I. Losev (✉)
Department of Mathematics, University of Toronto, Toronto, ON M5S 2E4, Canada
e-mail: ivan.loseu@utoronto.ca

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1.1.1 n! Theorem

Consider the Vandermond determinant $\Delta(\underline{x})$, where we write $\underline{x}$ for $(x_1, \dots, x_n)$, it is given by $\Delta(\underline{x}) = \det(x_i^{j-1})_{i,j=1}^n$. Consider the space $\partial\Delta$ spanned by all partial derivatives of Δ. This space is graded and carries an action of the symmetric group $\mathfrak{S}_n$ (by permuting the variables $x_1, \dots, x_n$). A deeper fact is that $\dim \partial\Delta = n!$ (and $\partial\Delta \cong \mathbb{C}\mathfrak{S}_n$ as an $\mathfrak{S}_n$-module), in fact, $\partial\Delta$ coincides with the space of the $\mathfrak{S}_n$-harmonic polynomials, i.e., all polynomials annihilated by all elements of $\mathbb{C}[\underline{\partial}]^{\mathfrak{S}_n}$ without constant term.

One can ask if there is a two-variable generalization of that fact. We have several two-variable versions of Δ, one for each Young diagram λ with n boxes. Namely, let $(a_1, b_1), \dots, (a_n, b_n)$ be the coordinates of the boxes in λ, e.g., $\lambda = (3, 2)$ gives pairs $(0, 0), (1, 0), (2, 0), (0, 1), (1, 1)$.

$(0,1)$	$(1,1)$	
$(0,0)$	$(1,0)$	$(2,0)$

Then set $\Delta_\lambda(\underline{x}, \underline{y}) := \det(x_i^{a_j} y_i^{b_j})_{i,j=1}^n$ so that, for $\lambda = (n)$, we get $\Delta_\lambda(\underline{x}, \underline{y}) = \Delta(\underline{x})$, for $\lambda = (1^n)$, we get $\Delta_\lambda(\underline{x}, \underline{y}) = \Delta(\underline{y})$, while, for $\lambda = (2, 1)$, we get $\Delta_\lambda(\underline{x}, \underline{y}) = x_1 y_2 + x_2 y_3 + x_3 y_1 - x_2 y_1 - x_3 y_2 - x_1 y_3$.

Theorem 1.1 (Haiman's n! theorem). *The space $\partial\Delta_\lambda$ spanned by the partial derivatives of Δ_λ is isomorphic to $\mathbb{C}\mathfrak{S}_n$ as an $\mathfrak{S}_n$-module (where $\mathfrak{S}_n$ acts by permuting the pairs $(x_1, y_1), \dots, (x_n, y_n)$) and, in particular, has dimension $n!$.*

This is a beautiful result with an elementary statement and an extremely involved proof, [34].

1.1.2 Macdonald Positivity

Before describing some ideas of the proof that are relevant to the present survey, let us describe an application to *Macdonald polynomials*, particularly important and interesting symmetric polynomials with coefficients in $\mathbb{Q}(q, t)$. It will be more convenient for us to speak about representations of $\mathfrak{S}_n$ rather than about symmetric polynomials (they are related via taking the Frobenius character) and to deal with Haiman's modified Macdonald polynomials.

Definition 1.2 The modified Macdonald polynomial $\tilde{H}_\lambda$ is the Frobenius character of a bigraded $\mathfrak{S}_n$-module $P_\lambda := \bigoplus_{i,j\in\mathbb{Z}} P_\lambda[i, j]$ subject to the following three conditions

(a) The class of $P_\lambda \otimes \sum_{i=0}^n (-1)^i \bigwedge^i \mathbb{C}^n[1, 0]$ is expressed via the irreducibles V_μ with $\mu \geqslant \lambda$ (in the K_0 of the bigraded $\mathfrak{S}_n$-modules).
(b) $P_\lambda \otimes \sum_{i=0}^n (-1)^i \bigwedge^i \mathbb{C}^n[0, 1]$ is expressed via the irreducibles V_μ with $\mu \geqslant \lambda^t$.
(c) The trivial module $V_{(n)}$ occurs in P_λ once and in degree $(0, 0)$.

Here $\mu \geqslant \lambda$ is the usual dominance order on the set of Young diagrams (meaning that $\sum_{i=1}^k \mu_i \geqslant \sum_{i=1}^k \lambda_i$) and λ^t denotes the transpose of λ.

It is not clear from this definition that the representations P_λ exist (the statement on the level of virtual representations is easier but also non-trivial, this was known before Haiman's work).

Theorem 1.3 (Haiman's Macdonald positivity theorem). *A bigraded $\mathfrak{S}_n$-module P_λ exists (and is unique) for any λ. Moreover, P_λ coincides with $\partial\Delta_\lambda$, where $\partial\Delta_\lambda$ is is given the structure of a bigraded $\mathfrak{S}_n$-module as the quotient of $\mathbb{C}[\underline{\partial}_x, \underline{\partial}_y]$ (via $f \mapsto f\Delta_\lambda$).*

1.1.3 Hilbert Schemes and Procesi Bundles

The proofs of the two theorems above given in [34] are based on the geometry of the Hilbert schemes $\text{Hilb}_n(\mathbb{C}^2)$ of points in $\mathbb{C}^2$. A basic reference for Hilbert schemes of points on smooth surfaces is [53]. As a set, $\text{Hilb}_n(\mathbb{C}^2)$ consists of the ideals $J \subset \mathbb{C}[x, y]$ of codimension n. It turns out that $\text{Hilb}_n(\mathbb{C}^2)$ is a smooth algebraic variety of dimension $2n$. It admits a morphism (called the Hilbert-Chow map) to the variety $\text{Sym}_n(\mathbb{C}^2)$ of the unordered n-tuples of points in $\mathbb{C}^2$: to an ideal J one assigns its support, where points are counted with multiplicities. Of course, $\text{Sym}_n(\mathbb{C}^2)$ is nothing else but the quotient space $(\mathbb{C}^2)^{\oplus n}/\mathfrak{S}_n$, the affine algebraic variety whose algebra of functions is the invariant algebra $\mathbb{C}[\underline{x}, \underline{y}]^{\mathfrak{S}_n}$. The Hilbert-Chow map is a resolution of singularities.

Note that the two-dimensional torus $(\mathbb{C}^\times)^2$ acts on $\text{Hilb}_n(\mathbb{C}^2)$ and on $\text{Sym}_n(\mathbb{C}^2)$, the action is induced from the following action on $\mathbb{C}^2$: $(t, s).(a, b) := (t^{-1}a, s^{-1}b)$. The fixed points of this action on $\text{Hilb}_n(\mathbb{C}^2)$ correspond to the monomial ideals (=ideals generated by monomials) in $\mathbb{C}[x, y]$, they are in a natural one-to-one correspondence with Young diagrams (as before we fill a Young diagram with monomials and take the ideal spanned by all monomials that do not appear in the diagram). Let z_λ denote the fixed point corresponding to a Young diagram λ.

Following Haiman, consider the isospectral Hilbert scheme I_n, the reduced Cartesian product $\mathbb{C}^{2n} \times_{\text{Sym}_n(\mathbb{C}^2)} \text{Hilb}_n(\mathbb{C}^2)$, let $\eta : I_n \to \text{Hilb}_n(\mathbb{C}^2)$ be the natural morphism. It is finite of generic degree $n!$. The main technical result of Haiman, [34], is that I_n is Cohen-Macaulay and Gorenstein. So $\mathcal{P} := \eta_* \mathcal{O}_{I_n}$ is a rank $n!$ vector bundle on $\text{Hilb}_n(\mathbb{C}^2)$ (the Procesi bundle). By the construction, each fiber of this bundle carries an algebra structure that is a quotient of $\mathbb{C}[\underline{x}, \underline{y}]$. Let us write $\mathcal{P}_\lambda$ for the fiber of $\mathcal{P}$ in z_λ, this is an algebra that carries a natural bi-grading because the bundle $\mathcal{P}$ is $(\mathbb{C}^\times)^2$-equivariant by the construction. On the other hand, $\partial\Delta_\lambda$ is a quotient of $\mathbb{C}[\underline{\partial}_x, \underline{\partial}_y]$ by an ideal and so is also an algebra. The latter algebra is bigraded. Haiman has shown that $\mathcal{P}_\lambda \cong \partial\Delta_\lambda$, an isomorphism of bigraded algebras. This finishes the proof of Theorem 1.1.

Let us proceed to Theorem 1.3. The class in (a) of Definition 1.2 is that of the fiber at z_λ of the Koszul complex

$$\mathcal{P} \leftarrow \mathfrak{h}_{\underline{x}} \otimes \mathcal{P} \leftarrow \Lambda^2\mathfrak{h}_{\underline{x}} \otimes \mathcal{P} \leftarrow \dots, \quad (1)$$

where $\mathfrak{h}_{\underline{x}}$ is the span of $x_1, \dots, x_n$ viewed as endomorphisms of $\mathcal{P}$. Haiman has shown that I_n is flat over $\mathrm{Spec}(\mathbb{C}[\underline{x}])$ (with morphism $I_n \to \mathrm{Spec}(\mathbb{C}[\underline{x}, \underline{y}]) \to \mathrm{Spec}(\mathbb{C}[\underline{x}])$). It follows that (1) is a resolution of $\mathcal{P}/\mathfrak{h}_{\underline{x}}\mathcal{P}$. Now (a) follows from the claim that, for any Young diagram λ, the support of the isotypic V_λ-component in $\mathcal{P}/\mathfrak{h}_{\underline{x}}\mathcal{P}$ contains only points z_μ with $\mu \leqslant \lambda$. This was checked by Haiman. Part (b) is analogous, while (c) follows directly from the construction.

There are several other proofs of Theorem 1.3 available. Two of them use the geometry of Hilbert schemes and Procesi bundle, [8, 29]. We will discuss (a somewhat modified) approach from [8] in detail in Sect. 5.

1.2 Quotient Singularities and Symplectic Resolutions

1.2.1 Setting

Let V be a finite dimensional vector space over $\mathbb{C}$ equipped with a symplectic form $\Omega \in \bigwedge^2 V^*$. Let Γ be a finite subgroup of $\mathrm{Sp}(V)$. The invariant algebra $\mathbb{C}[V]^\Gamma$ is a graded Poisson algebra (as a subalgebra of $\mathbb{C}[V]$) and the corresponding quotient $V/\Gamma = \mathrm{Spec}(\mathbb{C}[V]^\Gamma)$ is a singular Poisson affine variety that comes with a $\mathbb{C}^\times$-action induced from the action on V by dilations: $t.v := t^{-1}v$.

1.2.2 Symplectic Resolutions

One can ask if there is a resolution of singularities of V/Γ that is nicely compatible with the Poisson structure (and with the $\mathbb{C}^\times$-action). This compatibility is formalized in the notion of a (conical) symplectic resolution.

Definition 1.4 Let X_0 be a singular normal affine Poisson variety such that the regular locus X_0^{reg} is symplectic. We say that a variety X equipped with a morphism $\rho : X \to X_0$ is a *symplectic resolution* of X_0 if X is symplectic (with form ω), ρ is a resolution of singularities and $\rho : \rho^{-1}(X_0^{reg}) \to X_0^{reg}$ is a symplectomorphism.

Definition 1.5 Further, suppose that X_0 is equipped with a $\mathbb{C}^\times$-action such that

- the corresponding grading $\mathbb{C}[X_0] = \bigoplus_{i \in \mathbb{Z}} \mathbb{C}[X_0]_i$ is positive, meaning that $\mathbb{C}[X_0]_i = \{0\}$ for $i < 0$ and $\mathbb{C}[X]_0 = \mathbb{C}$,
- and the Poisson bracket on $\mathbb{C}[X_0]$ has degree $-d$ for some fixed $d \in \mathbb{Z}_{>0}$: $\{\mathbb{C}[X_0]_i, \mathbb{C}[X_0]_j\} \subset \mathbb{C}[X_0]_{i+j-d}$ for all i, j.

We say that a symplectic resolution X is *conical* if it is equipped with a $\mathbb{C}^\times$-action making ρ equivariant.

The variety $X_0 = V/\Gamma$ is normal and carries a natural $\mathbb{C}^\times$-action (by dilations) as in Definition 1.5 with $d = 2$. Also note that the $\mathbb{C}^\times$-action on X automatically satisfies $t.\omega = t^{-d}\omega$. Finally, note that, under assumptions of Definition 1.4, we have $\mathbb{C}[X] = \mathbb{C}[X_0]$.

1.2.3 Symplectic Resolutions for Quotient Singularities

In the previous subsection, we have already seen an example of (V, Γ) such that V/Γ admits a conical symplectic resolution: $V = (\mathbb{C}^2)^{\oplus n}$, $\Gamma = \mathfrak{S}_n$, in this case one can take $X = \mathrm{Hilb}_n(\mathbb{C}^2)$ together with the Hilbert-Chow morphism, see [53, Section 1].

There are other examples as well. Let Γ_1 be a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$, such subgroups are classified (up to conjugacy) by Dynkin diagrams of ADE types. Say, the cyclic subgroup $\mathbb{Z}/(\ell+1)\mathbb{Z}$ (embedded into $\mathrm{SL}_2(\mathbb{C})$ via $n \mapsto \mathrm{diag}(\eta^n, \eta^{-n})$ with $\eta := \exp(2\pi\sqrt{-1}/(\ell+1)))$ corresponds to the diagram A_ℓ. The quotient singularity $\mathbb{C}^2/\Gamma_1$ admits a distinguished *minimal* resolution to be denoted by $\widetilde{\mathbb{C}^2/\Gamma_1}$. This resolution is conical symplectic, see, e.g., [53, Section 4.1].

The examples of $\mathfrak{S}_n$ and Γ_1 can be "joined" together. Consider the group $\Gamma_n := \mathfrak{S}_n \ltimes \Gamma_1^n$. It acts on $V_n := (\mathbb{C}^2)^{\oplus n}$: the symmetric group permutes the summands, while each copy of Γ_1 acts on its own summand. The quotient singularities V_n/Γ_n admit symplectic resolutions. For example, one can take $X := \mathrm{Hilb}_n(\widetilde{\mathbb{C}^2/\Gamma_1})$. But there are other conical symplectic resolutions of V_n/Γ_n, conjecturally, they are all constructed as Nakajima quiver varieties, we will recall the construction of these varieties in Sect. 3.1.4.

To finish this section, let us point out that, presently, two more pairs (V, Γ) such that V/Γ admits a symplectic resolutions are known, see [4, 5]. In this paper, we are not interested in these cases.

1.3 Procesi Bundles: General Case

1.3.1 Smash-Product Algebra

One nice feature of quotient singularities V/Γ is that they always have a nice resolution of singularities which is, however, noncommutative algebraic rather than algebro-geometric: the smash-product algebra $\mathbb{C}[V]\#\Gamma$ (a general notion of noncommutative resolutions of singularities is due to Bondal-Orlov, [13], and van den Bergh, [60]).

As a vector space, $\mathbb{C}[V]\#\Gamma$ is the tensor product $\mathbb{C}[V] \otimes \mathbb{C}\Gamma$, and the product on $\mathbb{C}[V]\#\Gamma$ is given by

$$(f_1 \otimes \gamma_1) \cdot (f_2 \otimes \gamma_2) = f_1\gamma_1(f_2) \otimes \gamma_1\gamma_2,\ f_1, f_2 \in \mathbb{C}[V], \gamma_1, \gamma_2 \in \Gamma,$$

where $\gamma_1(f_2)$ denotes the image of f_2 under the action of γ_1. The definition is arranged in such a way that a $\mathbb{C}[V]\#\Gamma$-module is the same thing as a Γ-equivariant $\mathbb{C}[V]$-module. Note that the algebra $\mathbb{C}[V]\#\Gamma$ is graded, for a homogeneous element f of degree n, the degree of $f \otimes \gamma$ is n.

Let us explain what we mean when we say that $\mathbb{C}[V]\#\Gamma$ is a resolution of singularities of V/Γ. Note that $\mathbb{C}[V]^\Gamma$ can be recovered from $\mathbb{C}[V]\#\Gamma$ in two different but related ways. First, we have an embedding $\mathbb{C}[V]^\Gamma \hookrightarrow \mathbb{C}[V]\#\Gamma$ given by $f \mapsto f \otimes 1$. The image lies in the center (this is easy) and actually coincides with the center (a bit harder). Second, consider the element $e \in \mathbb{C}\Gamma$, $e := |\Gamma|^{-1}\sum_{\gamma\in\Gamma}\gamma$, the averaging idempotent. Consider the subspace $e(\mathbb{C}[V]\#\Gamma)e \subset \mathbb{C}[V]\#\Gamma$. It is obviously closed under multiplication, and e is a unit with respect to the multiplication there. So $e(\mathbb{C}[V]\#\Gamma)e$ is an algebra, to be called the spherical subalgebra of $\mathbb{C}[V]\#\Gamma$. It is isomorphic to $\mathbb{C}[V]^\Gamma$, an isomorphism is given by $f \mapsto ef$.

Thanks to the realization of $\mathbb{C}[V]^\Gamma$ as a spherical subalgebra, we can consider the functor $M \mapsto eM : \mathbb{C}[V]\#\Gamma\text{-mod} \to \mathbb{C}[V]^\Gamma\text{-mod}$ (an analog of the morphism ρ). Note that the algebra $\mathbb{C}[V]\#\Gamma$ has finite homological dimension (because $\mathbb{C}[V]$ does) and so is "smooth". The algebra $\mathbb{C}[V]\#\Gamma$ is finite over $\mathbb{C}[V]^\Gamma$ which can be thought as an analog of ρ being proper. Also, after replacing $\mathbb{C}[V]\#\Gamma, \mathbb{C}[V]^\Gamma$ with sheaves $\mathcal{O}_{V^{reg}}\#\Gamma, \mathcal{O}_{V^{reg}/\Gamma}$ on V^{reg}/Γ, where

$$V^{reg} := \{v \in V | \Gamma_v = \{1\}\}, \tag{2}$$

the functor $M \mapsto eM$ becomes a category equivalence. This is an analog of ρ being birational.

1.3.2 Procesi Bundle: An Axiomatic Description

Now let X be a conical symplectic resolution of V/Γ. We want to relate X to $\mathbb{C}[V]\#\Gamma$.

Definition 1.6 A Procesi bundle $\mathcal{P}$ on X is a $\mathbb{C}^\times$-equivariant vector bundle on X together with an isomorphism $\mathrm{End}_{\mathcal{O}_X}(\mathcal{P}) \xrightarrow{\sim} \mathbb{C}[V]\#\Gamma$ of graded algebras over $\mathbb{C}[X] = \mathbb{C}[V]^\Gamma$ such that $\mathrm{Ext}^i(\mathcal{P}, \mathcal{P}) = 0$ for $i > 0$.

Note that the isomorphism $\mathrm{End}_{\mathcal{O}_X}(\mathcal{P}) \xrightarrow{\sim} \mathbb{C}[V]\#\Gamma$ gives a fiberwise Γ-action on $\mathcal{P}$. The invariant sheaf $e\mathcal{P}$ is a vector bundle of rank 1. We say that $\mathcal{P}$ is *normalized* if $e\mathcal{P} = \mathcal{O}_X$ (as a $\mathbb{C}^\times$-equivariant vector bundle). We can normalize an arbitrary Procesi bundle by tensoring it with $(e\mathcal{P})^*$. Below we only consider normalized Procesi bundles.

In particular, Haiman's Procesi bundle on $X = \mathrm{Hilb}_n(\mathbb{C}^2)$ fits the definition, this is essentially a part of [33, Theorem 5.3.2] (and is normalized). The existence of a Procesi bundle on a general X was proved by Bezrukavnikov and Kaledin in [11]. We will see that the number of different Procesi bundles on a symplectic resolution of $\mathbb{C}^{2n}/\Gamma_n$ equals $2|W|$ if $n > 1$, where W is the Weyl group of the Dynkin diagram corresponding to Γ_1. For example, when $\Gamma_1 = \mathbb{Z}/\ell\mathbb{Z}$, we get $W = \mathfrak{S}_\ell$ and so the number of different Procesi bundles is $2\ell!$.

1.4 Symplectic Reflection Algebras

1.4.1 Definition

Symplectic reflection algebras were introduced by Etingof and Ginzburg in [21]. Those are filtered deformations of $\mathbb{C}[V]\#\Gamma$.

By a symplectic reflection in Γ one means an element γ with $\mathrm{rk}(\gamma - 1_V) = 2$. Note that the rank has to be even: the image of $\gamma - 1_V$ is a symplectic subspace of V. By S we denote the set of all symplectic reflections in Γ, it is a union of conjugacy classes, $S = \sqcup_{i=1}^r S_i$. Now pick $t \in \mathbb{C}$ and $c = (c_1, \ldots, c_r) \in \mathbb{C}^r$. We define the algebra $H_{t,c}$ as the quotient of $T(V)\#\Gamma$ by the relations

$$u \otimes v - v \otimes u = t\Omega(u, v) + \sum_{i=1}^{r} c_i \sum_{s \in S_i} \Omega(\pi_s u, \pi_s v)s, \; u, v \in V. \tag{3}$$

Here we write π_s for the projection $V \twoheadrightarrow \mathrm{im}(s - 1_V)$ corresponding to the decomposition $V = \mathrm{im}(s - 1_V) \oplus \ker(s - 1_V)$.

As Etingof and Ginzburg checked in [21], the algebra $H_{t,c}$ satisfies the PBW property: if we filter $H_{t,c}$ by setting $\deg \Gamma = 0$, $\deg V = 1$, then $\mathrm{gr}\, H_{t,c} = \mathbb{C}[V]\#\Gamma$ (here we identify V with V^* by means of Ω so that $\mathbb{C}[V] \cong S(V)$). Moreover, we will see that $H_{t,c}$ satisfies a certain universality property so this deformation of $\mathbb{C}[V]\#\Gamma$ is forced on us, in a way.

1.4.2 Connection to Procesi Bundles

It may seem that Symplectic reflection algebras and Procesi bundles are not related. This is not so. It turns out that the algebra $H_{t,c}$ is the endomorphism algebra of a suitably understood deformation of a Procesi bundle $\mathcal{P}$. This connection is beneficial for studying both. On the Procesi side, it allows to classify Procesi bundles, [46], and prove the Macdonald positivity in the case of groups Γ_n with $\Gamma_1 = \mathbb{Z}/\ell\mathbb{Z}$, [8]. On the symplectic reflection side, it allows to relate the algebras $H_{t,c}$ to quantized Nakajima quiver varieties, see [20, 45] and references therein, which then allows to study the representation theory of $H_{t,c}$ ([12]) and to prove versions of Beilinson-Bernstein localization theorems, [25, 40]. Connections between Procesi bundles and Symplectic reflection algebras is the subject of this survey.

1.5 Notation and Conventions

Let us list some notation used in the paper.
Quantizations and deformations. We use the following conventions for quantizations. For a Poisson algebra A, we write $\mathcal{A}_\hbar$ for its formal quantization. When A is graded,

we write $\mathcal{A}$ for its filtered quantization. The notation $\mathcal{D}_\hbar$ is usually used for a formal quantization of a variety, while $\mathcal{D}$ usually denotes a filtered quantization.

When X is a conical symplectic resolution of singularities, we write $\tilde{X}$ for its universal conical deformation (over $H^2_{DR}(X)$) and $\tilde{\mathcal{D}}_\hbar$ stands for the canonical quantization of $\tilde{X}$.

Symplectic reflection groups and algebras. We write Γ_1 for a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$ and Γ_n for the semidirect product $\mathfrak{S}_n \ltimes \Gamma_1^n$. This semi-direct product acts on $V_n := \mathbb{C}^{2n}$. In the case when $\Gamma_1 = \{1\}$, we usually write V_n for $T^*\mathbb{C}^{n-1}$, where $\mathbb{C}^{n-1}$ is the reflection representation of $\mathfrak{S}_n$.

For a group Γ acting on a space V by linear symplectomorphisms, by S we denote the set of symplectic reflections in Γ. By e we denote the averaging idempotent of Γ. By $\mathbf{H}$ we denote the universal symplectic reflection algebra of (V, Γ). Its specializations are denoted by $H_{t,c}$.

Quotients and reductions. Let G be a group acting on a variety X. If G is finite and X is quasi-projective, then the quotient is denoted by X/G (note that this quotient may fail to exist when X is not quasi-projective). If G is reductive and X is affine, then $X/\!/G$ stands for the categorical quotient. A GIT quotient of X under the G-action with stability condition θ is denoted by $X/\!/^\theta G$.

When X is Poisson, and the G-action is Hamiltonian, we write $X/\!/\!/_\lambda G$ for $\mu^{-1}(\lambda)/\!/G$ and $X/\!/\!/^\theta_\lambda G$ for $\mu^{-1}(\lambda)/\!/^\theta G$.

Miscellaneous notation.

$\widehat{\otimes}$	the completed tensor product of complete topological vector spaces/ modules.
$(a_1, \dots, a_k)$	the two-sided ideal in an associative algebra generated by elements $a_1, \dots, a_k$.
$A^{\wedge_\chi}$	the completion of a commutative (or "almost commutative") algebra A with respect to the maximal ideal of a point $\chi \in \mathrm{Spec}(A)$.
$\mathbf{A}(V)$	the Weyl algebra of a symplectic vector space V.
$D(X)$	the algebra of differential operators on a smooth variety X.
$\mathbb{F}_q$	the finite field with q elements.
$\mathrm{gr}\,\mathcal{A}$	the associated graded vector space of a filtered vector space $\mathcal{A}$.
$H^i_{DR}(X)$	the ith De Rham cohomology of X with coefficients in $\mathbb{C}$.
$\mathcal{O}_X$	the structure sheaf of a scheme X.
$R_\hbar(\mathcal{A})$	$:= \bigoplus_{i\in\mathbb{Z}} \hbar^i \mathcal{A}_{\leqslant i}$:the Rees $\mathbb{C}[\hbar]$-module of a filtered vector space $\mathcal{A}$.
$\mathfrak{S}_n$	the symmetric group in n letters.
$S(V)$	the symmetric algebra of a vector space V.
$\mathrm{Sp}(V)$	the symplectic linear group of a symplectic vector space V.
$\Gamma(\mathcal{S})$	global sections of a sheaf $\mathcal{S}$.

2 Quantizations

In this section we review the quantization formalism. In Sect. 2.1 we discuss quantizations of Poisson algebras. There are two formalisms here: filtered quantizations and formal quantizations. We introduce both of them, discuss a relation between them and then give examples.

Then, in Sect. 2.2, we proceed to quantizations of non-necessarily affine Poisson algebraic varieties. Here we quantize the structure sheaf. We explain that to quantize an affine variety is the same thing as to quantize its algebra of functions. Then we mention a theorem of Bezrukavnikov and Kaledin classifying quantizations of symplectic varieties under certain cohomology vanishing conditions.

After that we proceed to modules over quantizations. We define coherent and quasi-coherent sheaves of modules and outline their basic properties. For a coherent sheaf of modules, we define its support. Then we discuss global section and localization functors and their derived analogs.

We finish this section by discussing Frobenius constant quantizations in positive characteristic.

2.1 Algebra Level

Here we will review formalisms of quantizations of Poisson algebras. Let A be a Poisson algebra (commutative, associative and with a unit).

2.1.1 Formal Quantizations

First, let us discuss formal quantizations. By a formal quantization of A we mean an associative $\mathbb{C}[[\hbar]]$-algebra $\mathcal{A}_\hbar$ equipped with an algebra isomorphism $\pi : \mathcal{A}_\hbar/(\hbar) \xrightarrow{\sim} A$ such that

(i) $\mathcal{A}_\hbar \cong A[[\hbar]]$ as a $\mathbb{C}[[\hbar]]$-module and this isomorphism intertwines π and the natural projection $A[[\hbar]] \to A$.
(ii) We have $\pi(\frac{1}{\hbar}[a,b]) \equiv \{\pi(a), \pi(b)\}$ (note that $\pi([a,b]) = [\pi(a), \pi(b)] = 0$ and so $\frac{1}{\hbar}[a,b]$ makes sense).

Condition (i) can be stated equivalently as follows: $\mathcal{A}_\hbar$ is flat over $\mathbb{C}[[\hbar]]$ and is complete and separated in the $\hbar$-adic topology.

2.1.2 Filtered Quantizations

Second, we will need the formalism of filtered quantizations. Suppose that A is equipped with an algebra grading, $A = \bigoplus_{i\in\mathbb{Z}} A_i$, that is compatible with $\{\cdot,\cdot\}$ in the following way: $\{A_i, A_j\} \subset A_{i+j-1}$.

First, we consider the case when the grading on A is non-negative: $A_i = \{0\}$ for $i < 0$. Then, by a filtered quantization of A one means a $\mathbb{Z}_{\geqslant 0}$-filtered algebra $\mathcal{A} = \bigcup_{i\geqslant 0} \mathcal{A}_{\leqslant i}$ together with a graded algebra isomorphism $\pi : \operatorname{gr}\mathcal{A} \xrightarrow{\sim} A$ such that, for $a \in \mathcal{A}_{\leqslant i}, b \in \mathcal{A}_{\leqslant j}$, one has $\{\pi(a + \mathcal{A}_{\leqslant i-1}), \pi(b + \mathcal{A}_{\leqslant j-1})\} = \pi([a,b] + \mathcal{A}_{\leqslant i+j-2})$ (note that $[a,b] \in \mathcal{A}_{\leqslant i+j-1}$ because $\operatorname{gr}\mathcal{A}$ is commutative).

2.1.3 Relation Between the Two Formalisms

Let us explain a connection between the two formalisms (that will also motivate the definition of a filtered quantization in the case when the grading on A has negative components). Take a filtered quantization $\mathcal{A}$ of A. Form the *Rees algebra* $R_\hbar(\mathcal{A}) := \bigoplus_{i\geqslant 0} \mathcal{A}_{\leqslant i}\hbar^i$ that is equipped with a graded algebra structure as a subalgebra in $\mathcal{A}[\hbar]$. We have natural identifications $R_\hbar(\mathcal{A})/(\hbar) \cong A$, $R_\hbar(\mathcal{A})/(\hbar-1) \cong \mathcal{A}$. The $\hbar$-adic completion $R_\hbar(\mathcal{A})^{\wedge_\hbar} := \varprojlim_{n\to+\infty} R_\hbar(\mathcal{A})/(\hbar^n)$ satisfies (i) and (ii) and so is a formal quantization of A. Moreover, it comes with a $\mathbb{C}^\times$-action by algebra automorphisms such that $t.\hbar = t\hbar$, $t \in \mathbb{C}^\times$: the action is given by $t.\sum_{i=0}^{+\infty} a_i\hbar^i := \sum_{i=0}^{+\infty} t^i a_i \hbar^i$. Clearly, the induced action on A coincides with the action coming from the grading. Conversely, suppose we have a formal quantization $\mathcal{A}_\hbar$ of A equipped with a $\mathbb{C}^\times$-action by algebra automorphisms such that $t.\hbar = t\hbar$ and the epimorphism π is $\mathbb{C}^\times$-equivariant. Assume, further, that the action is pro-rational meaning that it is rational on all quotients $\mathcal{A}_\hbar/(\hbar^n)$. Consider the subspace $\mathcal{A}_{\hbar,fin} \subset \mathcal{A}_\hbar$ consisting of all $\mathbb{C}^\times$-*finite* elements, i.e., those elements that are contained in some finite dimensional $\mathbb{C}^\times$-stable subspace. This is a $\mathbb{C}^\times$-stable $\mathbb{C}[\hbar]$-subalgebra of $\mathcal{A}_\hbar$. It is easy to see that π induces an isomorphism $\mathcal{A}_{\hbar,fin}/(\hbar) \cong A$. Then $\mathcal{A} := \mathcal{A}_{\hbar,fin}/(\hbar-1)$ is a filtered quantization.

2.1.4 Filtered Quantizations, General Case

Let us proceed to the case when the grading on A is not necessarily non-negative. We can still consider a formal quantization $\mathcal{A}_\hbar$ with a $\mathbb{C}^\times$-action as above, the subalgebra $\mathcal{A}_{\hbar,fin} \subset \mathcal{A}_\hbar$ and the quotient $\mathcal{A} := \mathcal{A}_{\hbar,fin}/(\hbar-1)$. It is still a filtered quantization in the sense explained above (with the difference that now we have a $\mathbb{Z}$-filtration rather than a $\mathbb{Z}_{\geqslant 0}$-filtration) but, moreover, the filtration on $\mathcal{A}$ has a special property: it is complete and separated meaning that a natural homomorphism $\mathcal{A} \to \varprojlim_{n\to-\infty} \mathcal{A}/\mathcal{A}_{\leqslant n}$ is an isomorphism. By a filtered quantization of A we now mean a $\mathbb{Z}$-filtered algebra $\mathcal{A}$, where the filtration is complete and separated, together with an isomorphism $\pi : \operatorname{gr}\mathcal{A} \xrightarrow{\sim} A$ of graded algebras such that $\{\pi(a+\mathcal{A}_{\leqslant i-1}), \pi(b+\mathcal{A}_{\leqslant j-1})\} = \pi([a,b]+\mathcal{A}_{\leqslant i+j-2})$.

Our conclusion is that the following two formalisms are equivalent: filtered quantizations and formal quantizations with a pro-rational $\mathbb{C}^\times$-action. To get from a filtered quantization $\mathcal{A}$ to a formal one, one takes $R_\hbar(\mathcal{A})^{\wedge_\hbar}$. To get from a formal quantization $\mathcal{A}_\hbar$ to a filtered one, one takes $\mathcal{A}_{\hbar,fin}/(\hbar-1)$.

2.1.5 Examples

Let us proceed to examples. In examples, one usually gets $\mathbb{Z}_{\geqslant 0}$-filtered quantizations, more general $\mathbb{Z}$-filtered or formal quantizations arise in various constructions (such as (micro)localization or completion).

Example 2.1 Let $\mathfrak{g}$ be a Lie algebra. Then, by the PBW theorem, the universal enveloping algebra $U(\mathfrak{g})$ is a filtered quantization of $S(\mathfrak{g})$.

Example 2.2 Let Y be an affine algebraic variety. The algebra $D(Y)$ of linear differential operators on Y (together with the filtration by the order of differential operators) is a filtered quantization of $\mathbb{C}[T^*Y]$.

Remark 2.3 Often one needs to deal with a more general compatibility condition between the grading and the bracket: $\{A_i, A_j\} \subset A_{i+j-d}$ for some fixed $d > 0$. In this case, one can modify the definitions of formal and filtered quantizations. Namely, in the definition of a formal quantization one can require that $[\mathcal{A}_\hbar, \mathcal{A}_\hbar] \subset \hbar^d \mathcal{A}_\hbar$ and $\pi(\frac{1}{\hbar^d}[a, b]) = \{\pi(a), \pi(b)\}$. The definition of a filtered quantization can be modified similarly.

Example 2.4 Let V be a symplectic vector space and $\Gamma \in \mathrm{Sp}(V)$ be a finite group. Consider $A = S(V)^\Gamma$ with Poisson bracket $\{\cdot, \cdot\}$ restricted from $S(V)$. In the notation of Remark 2.3, $d = 2$. As was essentially checked in [21], the spherical subalgebra $eH_{1,c}e$ (with a filtration restricted from $H_{1,c}$) is a quantization of $S(V)^\Gamma$ for any parameter c When $\Gamma = \{1_V\}$, we recover the usual Weyl algebra, $\mathbf{A}(V)$, of V.

To check that $eH_{1,c}e$ is a quantization carefully we note that the proof of Theorem 1.6 in *loc.cit.* shows that the bracket on $S(V)^\Gamma$ coming from the filtered deformation $eH_{1,c}e$ coincides with $a\{\cdot, \cdot\}$, where a is a nonzero number independent of c. Then we notice that for $c = 0$ we get $eH_{1,c}e = \mathbf{A}(V)^\Gamma$ and so $a = 1$.

In fact, in the previous example we often can also achieve $d = 1$. Namely, if $-1_V \in \Gamma$, then all degrees in $S(V)^\Gamma$ are even and so we can consider the grading $A = \bigoplus_{i \geqslant 0} A_i$ with A_i consisting of all homogeneous elements with usual degree $2i$. We introduce a filtration on $eH_{1,c}e$ in a similar way (this filtration is not restricted from $H_{1,c}$). Then we get a filtered quantization according to our original definition. When $\Gamma = \Gamma_n$, we only have $-1_V \notin \Gamma$ if $\Gamma = \mathbb{Z}/\ell\mathbb{Z}$ for odd ℓ. For $\Gamma = \mathbb{Z}/\ell\mathbb{Z}$ (and any ℓ), V splits as $\mathfrak{h} \oplus \mathfrak{h}^*$, where $\mathfrak{h} = \mathbb{C}^n$. We can grade $S(V)$ by setting $\deg \mathfrak{h}^* = 0$, $\deg \mathfrak{h} = 1$ and take the induced grading on $S(V)^\Gamma$ and the induced filtration on $H_{1,c}$.

2.2 Sheaf Level

Above, we were dealing with Poisson algebras or, basically equivalently, with affine Poisson algebraic varieties. Now we are going to consider general Poisson varieties (or schemes). Recall that by a Poisson variety one means a variety X such that the structure sheaf $\mathcal{O}_X$ is equipped with a Poisson bracket (meaning that all algebras of sections are Poisson and the restriction homomorphisms respect the Poisson brackets). In this case a quantization of X will be a (formal or filtered) quantization of $\mathcal{O}_X$ in the sense explained below in this section.

2.2.1 Formal Quantizations

We start with a formal setting. A quantization $\mathcal{D}_\hbar$ of X is a sheaf of $\mathbb{C}[[\hbar]]$-algebras on X (in the Zariski topology) together with an isomorphism $\pi : \mathcal{D}_\hbar/(\hbar) \xrightarrow{\sim} \mathcal{O}_X$ such that

(a) $\mathcal{D}_\hbar$ is flat over $\mathbb{C}[[\hbar]]$ (equivalently, there are no nonzero local sections annihilated by $\hbar$) and complete and separated in the $\hbar$-adic topology (meaning that $\mathcal{D}_\hbar \xrightarrow{\sim} \varprojlim_{n\to+\infty} \mathcal{D}_\hbar/(\hbar^n)$, where the inverse limit is taken in the category of sheaves).
(b) $\pi(\frac{1}{\hbar}[a,b]) = \{\pi(a), \pi(b)\}$ for any local sections a, b of $\mathcal{D}_\hbar$.

2.2.2 Motivation: Star-Products

The origins of this definition are in the deformation quantization introduced in [2]. Let us adopt this definition to our situation. Let A be a Poisson algebra. By a star-product on A one means a bilinear map $* : A \otimes A \to A[[\hbar]]$ subject to the following conditions:

(1) The $\mathbb{C}[[\hbar]]$-bilinear extension of $*$ to $A[[\hbar]]$ is associative and $1 \in A$ is a unit.
(2) $a * b \equiv ab \mod \hbar A[[\hbar]]$, $a * b - b * a \equiv \hbar\{a, b\} \mod \hbar^2 A[[\hbar]]$.

Of course, $A[[\hbar]]$ together with $*$ is a formal quantization of A in the sense of the previous section. Conversely, any formal quantization $\mathcal{A}_\hbar$ is isomorphic to $(A[[\hbar]], *)$.

Traditionally, one imposes an additional restriction on $*$: the locality axiom that requires that the coefficients D_i in the $\hbar$-adic expansion of $*$ ($a * b = \sum_{i=0}^\infty D_i(a, b)\hbar^i$) are bidifferential operators. If $*$ is local, then it naturally extends to any localization $A[a^{-1}]$. So, if $A = \mathbb{C}[X]$ for X affine, then a local star-product defines a quantization of $\mathcal{O}_X$.

Let us provide an example of a local star-product. Consider $A = \mathbb{C}[\underline{x}, \underline{y}]$ with standard Poisson bracket: $\{x_i, x_j\} = \{y_i, y_j\}$, $\{y_i, x_j\} = \delta_{ij}$. Then set

$$f * g = m \circ \exp(\hbar \sum_{i=1}^n \partial_{y_i} \otimes \partial_{x_i}) f \otimes g, \tag{4}$$

where $\mu : A \otimes A \to A$ is the usual commutative product. For example, we have $x_i * x_j = x_i x_j$, $y_i * y_j = y_i y_j$, $x_i * y_j = x_i y_j$, $y_j * x_i = x_i y_j + \hbar\delta_{ij}$. In this case, $A[\hbar]$ is closed with respect to $*$ and is identified with $R_\hbar(D(\mathbb{C}^n))$.

2.2.3 Algebra versus Sheaf Setting in the Affine Case

It turns out that any formal quantization $\mathcal{A}_\hbar$ of $\mathbb{C}[X]$ for an affine variety X defines a quantization of X. The reason is that we can localize elements of $\mathbb{C}[X]$ in $\mathcal{A}_\hbar$. The construction is as follows. Pick $f \in \mathbb{C}[X]$ and lift it to $\hat{f} \in \mathcal{A}_\hbar$. The operator ad $\hat{f}$

is nilpotent in $\mathcal{A}_\hbar/(\hbar^n)$ for any n and so the set $\{\hat{f}^n\} \subset \mathcal{A}_\hbar/(\hbar^n)$ satisfies the Ore conditions, hence the localization $\mathcal{A}_\hbar/(\hbar^n)[\hat{f}^{-1}]$ makes sense. It is easy to see that these localizations do not depend on the choice of the lift $\hat{f}$ and form an inverse system. We set $\mathcal{A}_\hbar[f^{-1}] := \varprojlim_{n\to+\infty} \mathcal{A}_\hbar/(\hbar^n)[\hat{f}^{-1}]$.

Exercise 2.5 Check that there is a unique sheaf $\mathcal{D}_\hbar$ in the Zariski topology on X such that $\mathcal{D}_\hbar(X_f) = \mathcal{A}_\hbar[f^{-1}]$ for any $f \in \mathbb{C}[X]$ and that this sheaf is a quantization of X.

So we see that there is a natural bijection between the quantizations of X and of $\mathbb{C}[X]$ (to get from a quantization of X to that of $\mathbb{C}[X]$ we just take the global sections). Thanks to this, we can view a quantization of a general variety X as glued from affine pieces.

2.2.4 Filtered Quantizations

Let us proceed to the filtered setting. Suppose that X is equipped with a $\mathbb{C}^\times$-action such that the Poisson bracket has degree -1. Obviously, for an arbitrary open $U \subset X$, the algebra $\mathbb{C}[U]$ does not need to be graded. However, it is graded when U is $\mathbb{C}^\times$-stable. By a *conical topology* on X we mean the topology, where "open" means Zariski open and $\mathbb{C}^\times$-stable. One can ask whether this topology is sufficiently rich, for example, whether any point has an open affine neighborhood.

Theorem 2.6 ([59], Section 3). *Suppose X is normal. Then any point in X has an open affine neighborhood in the conical topology.*

Below we always assume that X is normal. Note that $\mathcal{O}_X$ is a sheaf of graded algebras in the conical topology. By a filtered quantization of X we mean a sheaf $\mathcal{D}$ of filtered algebras (in the conical topology on X) equipped with an isomorphism $\pi : \operatorname{gr}\mathcal{D} \xrightarrow{\sim} \mathcal{O}_X$ of graded algebras such that the filtration on $\mathcal{D}$ is complete and separated and π is compatible with the Poisson brackets as in Sect. 2.1.2.

We still have a one-to-one correspondence between filtered quantizations and formal quantizations with $\mathbb{C}^\times$-actions. This works just as in Sect. 2.1.3 (note that $\mathcal{D}_{\hbar,fin}$ makes sense as a sheaf in conical topology).

2.2.5 Quantization in Families

Let X be a smooth scheme over a scheme $\mathcal{S}$. It still makes sense to speak about closed and non-degenerate forms in $\Omega^2(X/\mathcal{S})$. By a symplectic $\mathcal{S}$-scheme we mean a smooth $\mathcal{S}$-scheme X together with a closed non-degenerate form $\omega_\mathcal{S} \in \Omega^2(X/\mathcal{S})$. Note that from ω one can recover an $\mathcal{O}_\mathcal{S}$-linear Poisson bracket on X.

By a formal quantization $\mathcal{D}_\hbar$ of X we mean a sheaf of $\mathcal{O}_\mathcal{S}$-algebras on X satisfying conditions (a),(b) in Sect. 2.2.1.

Note that the definition above still makes sense when $\mathcal{S}$ is a formal scheme and X is a formal $\mathcal{S}$-scheme.

2.2.6 Classification Theorem

Let us finish this section with a classification theorem due to Bezrukavnikov and Kaledin, [10] (with a ramification given in [45]).

Theorem 2.7 *Let X be a smooth symplectic variety. Suppose $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$ (this holds when X is affine, for example). Then the formal quantizations of X are parameterized by $H^2_{DR}(X, \mathbb{C})[[\hbar]]$. If X has a $\mathbb{C}^\times$-action compatible with the bracket (where we have $d = 1$), then the filtered quantizations are in one-to-one correspondence with $H^2_{DR}(X, \mathbb{C})$.*

Even without the cohomology vanishing assumption, there is a so called *period map* Per from the set $\mathsf{Quant}(X)$ of formal quantizations of X (considered up to an isomorphism) to $H^2_{DR}(X)[[\hbar]]$. When the vanishing condition holds, this map is a bijection. The classification of filtered quantizations follows from the observation that once a quantization admits a $\mathbb{C}^\times$-action by automorphisms, its period lies in $H^2_{DR}(X) \subset H^2_{DR}(X)[[\hbar]]$ (and if the vanishing holds, the converse is also true), see [45, Section 2.3].

Assume until the end of the section that the vanishing condition holds.

A formal quantization $\mathcal{D}_\hbar$ having a $\mathbb{C}^\times$-action by automorphisms and satisfying $\mathrm{Per}(\mathcal{D}_\hbar) = 0$ has a nice property: it is *even*. When X is affine this means that the quantization can be realized by a star-product $f * g = \sum_{i=0}^\infty D_i(f, g)\hbar^i$ with $\deg D_i = -i$ and $D_i(f, g) = (-1)^i D_i(g, f)$. For general X, being even means that there is an antiautomorphism ϱ of $\mathcal{D}_\hbar$ that commutes with the $\mathbb{C}^\times$-action, is the identity modulo $\hbar$, and maps $\hbar$ to $-\hbar$. A classical example of an even quantization is as follows. Let Y be a smooth algebraic variety and $X = T^*Y$. Then we consider the differential operators twisted by half the canonical bundle, $D_Y^{K_Y/2}$. The corresponding formal quantization of T^*Y is even.

Let us finish this subsection with the discussion of the universal quantization. The variety X has a universal symplectic deformation $\widehat{X}$ over the formal disc $\mathcal{S}$ that is the formal neighborhood of 0 in $H^2_{DR}(X)$ (provided $H^i(X, \mathcal{O}_X) = 0$ for $i = 1, 2$), see [36]. The universality means that any other formal symplectic deformation of X is obtained from $\widehat{X}$ by pull-back. Further, there is a canonical quantization $\widehat{\mathcal{D}}_\hbar$ of $\widehat{X}/\mathcal{S}$. All quantizations of X are obtained by pulling back $\widehat{\mathcal{D}}_\hbar$. More precisely, we can view $\widehat{\mathcal{D}}_\hbar$ as a sheaf of $\mathbb{C}[[H^2_{DR}(X), \hbar]]$-algebras on X (via the sheaf-theoretic pull-back) and then we can obtain quantizations of X by base change to $\mathbb{C}[[\hbar]]$.

In the case when X, in addition, has a $\mathbb{C}^\times$-action rescaling the symplectic form, we can consider the universal $\mathbb{C}^\times$-equivariant deformation $\tilde{X}$ over $H^2_{DR}(X)$ as well as its canonical quantization $\tilde{\mathcal{D}}_\hbar$.

2.3 Modules Over Quantizations

Let X be a Poisson variety (or scheme). We are going to define coherent and quasi-coherent modules over filtered and formal quantizations of X (to be denoted by $\mathcal{D}$ and $\mathcal{D}_\hbar$, respectively).

2.3.1 Coherent Modules Over Formal Quantizations

By definition, a sheaf $\mathcal{M}_\hbar$ of $\mathcal{D}_\hbar$-modules on X is called *coherent* if $\mathcal{M}_\hbar/\hbar\mathcal{M}_\hbar$ is a coherent $\mathcal{O}_X$-module and $\mathcal{M}_\hbar$ is complete and separated in $\hbar$-adic topology.

Let X be affine and let $\mathcal{A}_\hbar := \Gamma(\mathcal{D}_\hbar)$. Let $\mathcal{N}_\hbar$ be a finitely generated $\mathcal{A}_\hbar$-module. Then it is easy to see that $\mathcal{N}_\hbar$ is complete and separated in the $\hbar$-adic topology. It follows that $\mathcal{D}_\hbar \otimes_{\mathcal{A}_\hbar} \mathcal{N}_\hbar$ is a coherent $\mathcal{D}_\hbar$-module. Conversely, for a coherent $\mathcal{D}_\hbar$-module $\mathcal{M}_\hbar$, the global sections $\Gamma(\mathcal{M}_\hbar)$ is a finitely generated $\mathcal{A}_\hbar$-module.

Lemma 2.8 *Let X be affine. Then the functors $\mathcal{D}_\hbar \otimes_{\mathcal{A}_\hbar} \bullet$ and $\Gamma(\bullet)$ are mutually quasi-inverse equivalences between the categories of coherent $\mathcal{D}_\hbar$-modules and finitely generated $\mathcal{A}_\hbar$-modules.*

Proof Note that these functors define compatible equivalences between the categories of coherent $\mathcal{D}_\hbar/(\hbar^n)$-modules and of finitely generated $\mathcal{A}_\hbar/(\hbar^n)$-modules for any n (which is proved in the same way as the classical statement for $n = 1$). Then we use that all objects we consider are complete and separated in the $\hbar$-adic topology. □

Clearly, if $U \subset X$ is a Zariski open subset and $\mathcal{M}_\hbar$ is a coherent $\mathcal{D}_\hbar$-module, then $\mathcal{M}_\hbar|_U$ is a coherent $\mathcal{D}_\hbar$-module. Cover X with open affine subvarieties, $X = \bigcup X^i$. Set $\mathcal{A}^i_\hbar := \Gamma(\mathcal{D}_\hbar|_{X^i})$, $\mathcal{N}^i_\hbar := \Gamma(\mathcal{D}_\hbar|_{X^i})$. Then $\mathcal{M}_\hbar$ gives rise to gluing isomorphisms between the localizations of $\mathcal{N}^i_\hbar, \mathcal{N}^j_\hbar$ to $X^i \cap X^j$ subject to the usual cocycle condition. Conversely, a collection of finitely generated $\mathcal{A}^i_\hbar$-modules $\mathcal{N}^i_\hbar$ with gluing isomorphisms subject to the cocycle condition gives rise to a coherent $\mathcal{D}_\hbar$-module. In particular, as in Algebraic geometry, being coherent is a local condition.

Also from Lemma 2.8 we easily see that a subsheaf and a quotient sheaf of a coherent $\mathcal{D}_\hbar$-module are coherent themselves. So the category $\mathrm{Coh}(\mathcal{D}_\hbar)$ of coherent $\mathcal{D}_\hbar$-modules is an abelian category.

2.3.2 Quasi-Coherent Modules Over Formal Quantizations

By a quasi-coherent $\mathcal{D}_\hbar$-module we mean a direct limit of coherent $\mathcal{D}_\hbar$-modules. Lemma 2.8 implies that, when X is affine, the category of quasi-coherent $\mathcal{D}_\hbar$-modules is equivalent to the category of $\Gamma(\mathcal{D}_\hbar)$-modules.

Analogously to the classical algebro-geometric result, the category $\mathrm{QCoh}(\mathcal{D}_\hbar)$ of quasi-coherent $\mathcal{D}_\hbar$-modules has enough injective objects. Note that the natural functor from $D^b(\mathrm{Coh}(\mathcal{D}_\hbar))$ to the full subcategory in $D^b(\mathrm{QCoh}(\mathcal{D}_\hbar))$ of all complexes with coherent homology is a category equivalence. This is because any quasi-coherent complex is a union of coherent subcomplexes, as in the usual Algebro-geometric situation.

2.3.3 Modules Over Filtered Quantizations

Let us proceed to modules over filtered quantizations. Let $\mathcal{M}$ be a sheaf of $\mathcal{D}$-modules. We say that $\mathcal{M}$ is *coherent* if it can be equipped with a global complete and separated filtration compatible with that on $\mathcal{D}$ and such that $\operatorname{gr}\mathcal{M}$ is a coherent sheaf on X (such a filtration is usually called *good*). The $\hbar$-adic completion of the Rees sheaf $R_\hbar(\mathcal{M})$ is then a $\mathbb{C}^\times$-equivariant coherent $\mathcal{D}_\hbar$-module. Conversely, if we take a $\mathbb{C}^\times$-equivariant coherent $\mathcal{D}_\hbar$-module $\mathcal{M}_\hbar$, take the $\mathbb{C}^\times$-finite part $\mathcal{M}_{\hbar,fin}$, then $\mathcal{M}_{\hbar,fin}/(\hbar-1)$ is a coherent $\mathcal{D}$-modules.

Lemma 2.9 *Consider the full subcategory* $\operatorname{Coh}^{\mathbb{C}^\times}(\mathcal{D}_\hbar)_{tor}$ *consisting of all modules that are torsion over* $\mathbb{C}[[\hbar]]$. *Then taking quotient by* $\hbar-1$ *gives rise to an equivalence* $\operatorname{Coh}^{\mathbb{C}^\times}(\mathcal{D}_\hbar)/\operatorname{Coh}^{\mathbb{C}^\times}(\mathcal{D}_\hbar)_{tor}\xrightarrow{\sim}\operatorname{Coh}(\mathcal{D})$.

Proof Let us produce a quasi-inverse functor. Of course, the $R_\hbar(\mathcal{D})$-module $R_\hbar(\mathcal{M})$ depends on the choice of a good filtration. Let F, F' be two good filtrations. Then one can find positive integers d_1, d_2 such that $\mathrm{F}_{i-d_1}\mathcal{M}\subset \mathrm{F}'_i\mathcal{M}\subset \mathrm{F}_{i+d_2}\mathcal{M}$ the inclusion of subsheaves (of vector spaces) in $\mathcal{M}$ (it is enough to check this claim for local sections over open subsets from an affine cover, where it is easy). It follows that modulo $\hbar$-torsion the sheaf $R_\hbar(\mathcal{M})$ is independent of the choice of a good filtration. Our quasi-inverse functor sends $\mathcal{M}$ to the $\hbar$-adic completion of $R_\hbar(\mathcal{M})$. To check that this is indeed a quasi-inverse functor is standard. □

2.3.4 Supports

For a coherent $\mathcal{D}_\hbar$-module $\mathcal{M}_\hbar$ we have the notion of support. By definition, $\operatorname{Supp}(\mathcal{M}_\hbar):=\operatorname{Supp}(\mathcal{M}_\hbar/\hbar\mathcal{M}_\hbar)$, this is a closed subvariety in X.

Now let $\mathcal{M}\in\operatorname{Coh}(\mathcal{D})$. Then we can take a good filtration on $\mathcal{M}$ and set $\operatorname{Supp}(\mathcal{M}):=\operatorname{Supp}(\operatorname{gr}\mathcal{M})$. By the argument in the proof of Lemma 2.9, the support of $\mathcal{M}$ is well-defined, i.e., it does not depend on the choice of a good filtration.

2.3.5 Global Sections and Localization

Let $\mathcal{D}$ be a filtered quantization of X. We have natural functors $\operatorname{Coh}(\mathcal{D})\to\Gamma(\mathcal{D})\text{-mod}$ of taking global sections (to be denoted by Γ) as well as a functor in the opposite direction $\operatorname{Loc}:\Gamma(\mathcal{D})\text{-mod}\to\operatorname{Coh}(\mathcal{D})$, $M\mapsto\mathcal{D}\otimes_{\Gamma(\mathcal{D})}M$.

Let us discuss a situation when these functors behave particularly nicely. Namely, let X be a conical symplectic resolution of singularities of an affine variety X_0. Note that, by the Grauert-Riemenschneider theorem, the higher cohomology of $\mathcal{O}_X$ vanish. This has the following corollary (the proof is left to the reader).

Lemma 2.10 *We have $H^i(\mathcal{D}) = 0$ for $i > 0$. Moreover, $\Gamma(\mathcal{D})$ is a quantization of X_0.*

Thanks to this lemma, it makes sense to consider derived functors $R\Gamma : D(\mathrm{Coh}(\mathcal{D})) \to D(\Gamma(\mathcal{D})\text{-mod})$ and $L\,\mathrm{Loc} : D(\Gamma(\mathcal{D})\text{-mod}) \to D(\mathrm{Coh}(\mathcal{D}))$. In fact, $R\Gamma$ is given by the Čech complex and so restricts to bounded (to the left and to the right) derived categories. The functor $L\,\mathrm{Loc}$ restricts to D^-'s. Lemma 2.10 implies that $R\Gamma \circ L\,\mathrm{Loc}$ is the identity on $D^-(\Gamma(\mathcal{D})\text{-mod})$. Furthermore, if $\Gamma(\mathcal{D})$ has finite homological dimension, then $L\,\mathrm{Loc}$ maps $D^b(\Gamma(\mathcal{D})\text{-mod})$ to $D^b(\mathrm{Coh}(\mathcal{D}))$ and is left inverse to $R\Gamma$. It is likely (and is proved in many cases, see, e.g., [49]) that $R\Gamma$ and $L\,\mathrm{Loc}$ are mutually quasi-inverse equivalences in this case.

2.4 Frobenius Constant Quantizations

Above, we were dealing with the case when the ground field is $\mathbb{C}$. Everything works the same for any algebraically closed field of characteristic 0. In this section we are going to work over an algebraically closed field $\mathbb{F}$ of positive characteristic.

The notions of filtered and formal quantizations still make sense, both for algebras and for varieties. But in positive characteristic we have an important special class of quantizations: Frobenius constant ones.

2.4.1 Basic Example

Let us start our discussion with an example of a quantization: the Weyl algebra $\mathbf{A}(V)$, where V is a symplectic $\mathbb{F}$-vector space. A new feature is that this algebra is finite over its center. Namely, for $v \in V \subset \mathbf{A}(V)$, the element $v^p \in \mathbf{A}(V)$ lies in the center. We have a semi-linear map $\iota : V \to \mathbf{A}(V)$ given by $v \mapsto v^p$ on $v \in V$ with central image that extends to a ring homomorphism $S(V) \to \mathbf{A}(V)$. The semi-linearity condition is $\iota(av) = \mathrm{Fr}(a)\iota(v)$, where $\mathrm{Fr} : \mathbb{F} \to \mathbb{F}$ is the Frobenius automorphism. Let $V^{(1)}$ denote the $\mathbb{F}$-vector space identified with V as an abelian group but with new multiplication by scalars: $a.v = \mathrm{Fr}^{-1}(a)v$. So ι becomes an algebra homomorphism when viewed as a map $S(V^{(1)}) \to \mathbf{A}(V)$, its image is usually called the p-center, in our case it coincides with the whole center. Another important feature of this example is that $\mathbf{A}(V)$ is an Azumaya algebra over $V^{(1)}$, i.e., $\mathbf{A}(V)$ is a vector bundle over $\mathrm{Spec}(S(V^{(1)}))$ and all (geometric) fibers are matrix algebras (of rank $p^{\dim V/2}$).

2.4.2 Definition

The notion of a Frobenius constant quantization that appeared in [11] generalizes the example in Sect. 2.4.1. We will give the definition in the filtered setting and only for symplectic varieties—we will only need it in this case. Let X be a smooth symplectic $\mathbb{F}$-variety equipped with an $\mathbb{F}^\times$-action rescaling the symplectic form (by the character $t \mapsto t^d$). Let $X^{(1)}$ be the $\mathbb{F}$-variety that is identified with X as a scheme over $\operatorname{Spec}(\mathbb{Z})$ but with twisted multiplication by scalars in the structure sheaf just as in Sect. 2.4.1. We have a natural morphism $\operatorname{Fr}: X \to X^{(1)}$ of $\mathbb{F}$-varieties and hence we have a sheaf $\operatorname{Fr}_*(\mathcal{O}_X)$ on $X^{(1)}$. This is a coherent sheaf of algebras and a vector bundle of rank $p^{\dim X}$.

Definition 2.11 A Frobenius constant quantization is a filtered sheaf $\mathcal{D}$ of Azumaya algebras on $X^{(1)}$ together with an isomorphism $\operatorname{gr}\mathcal{D} \xrightarrow{\sim} \operatorname{Fr}_* \mathcal{O}_X$ of graded algebras (in conical topology) that satisfies our usual compatibility condition on Poisson brackets.

It is not difficult to show that a Frobenius constant quantization gives rise to a filtered quantization of X. But, as we will see Sect. 3.3.3, not every filtered quantization arises this way.

2.4.3 Differential Operators

Let us give another example that should be thought as a global analog of Sect. 2.4.1. Let Y be a smooth $\mathbb{F}$-variety. Consider the sheaf D_Y of differential operators on Y. Let ξ be a vector field on an open subset $Y' \subset Y$. Define a vector field $\xi^{[p]}$ as follows. For every open affine subvariety $Y^0 \subset Y'$, we can regard ξ as a derivation of $\mathbb{F}[Y^0]$. The map $\xi^p : \mathbb{F}[Y^0] \to \mathbb{F}[Y^0]$ is again a derivation. The corresponding vector field on Y' (that is easily seen to be well-defined) is what we denote by $\xi^{[p]}$. It is easy to see that f^p, for a function f on Y, and $\xi^p - \xi^{[p]}$, for a vector field ξ (here ξ^p is taken with respect to the product on D_Y), are central. The maps $f \mapsto f^p, \xi \mapsto \xi^p - \xi^{[p]}$ give rise to a sheaf of algebras homomorphism $\pi_*\mathcal{O}_{(T^*Y)^{(1)}} \to \operatorname{Fr}_* D_Y$, where we write π for the projection $(T^*Y)^{(1)} = T^*(Y^{(1)}) \to Y^{(1)}$. The sheaf D_Y then becomes a Frobenius constant quantization of T^*Y.

To finish this section, let us mention that, under some restrictions on X, there is a classification of Frobenius constant quantizations, see [9].

3 Hamiltonian Reductions

In this section we recall the notions of the classical and quantum Hamiltonian reduction. The classical Hamiltonian reduction produces a new Poisson variety from an existing Poisson variety with suitable symmetries. The quantum Hamiltonian reduction does the same on the level of quantizations.

We start by discussing classical Hamiltonian reductions, Sect. 3.1. First, we recall Hamiltonian actions and moment maps. Then we define classical Hamiltonian reductions in the settings of categorical quotients and of GIT quotients. We then proceed to the construction and basic properties of Nakajima quiver varieties that are our main examples of Hamiltonian reductions. Next, we explain how quotient singularities V_n/Γ_n are realized as quiver varieties. Finally, we construct symplectic resolutions of singularities for V_n/Γ_n and establish, following Namikawa, some isomorphisms between some of these resolutions.

In Sect. 3.2 we proceed to quantum Hamiltonian reductions. We define them on the level of algebras and on the level of sheaves and compare the two levels. After that we state one of the main results of this survey: an isomorphism between spherical SRA for wreath-product groups and quantum Hamiltonian reductions. We finish this section by discussing a quantum version of Namikawa's Weyl group action.

Section 3.3 deals with Hamiltonian reductions for Frobenius constant quantization. We first recall some basic results on GIT in positive characteristic. Then we discuss Nakajima quiver varieties in sufficiently large positive characteristic. Finally, we prove, following Bezrukavnikov, Finkelberg and Ginzburg, that the quantum Hamiltonian reduction of a Frobenius constant quantization at an integral value of the quantum comoment map is Frobenius constant.

3.1 Classical Hamiltonian Reduction

3.1.1 Hamiltonian Group Actions

Let X be a Poisson variety (over an algebraically closed field) and let G be an algebraic group acting on X. The action induces a Lie algebra homomorphism $\mathfrak{g} \to \operatorname{Vect}(X)$, the image of $\xi \in \mathfrak{g}$ under this homomorphism will be denoted by ξ_X. We say that the G-action on X is *Hamiltonian*, if there is a G-equivariant linear map $\mathfrak{g} \to \mathbb{C}[X]$, $\xi \mapsto H_\xi$, such that $\{H_\xi, \cdot\} = \xi_X$. Note that this map is automatically a Lie algebra homomorphism. This map is called the *comoment map*, the dual map $\mu : X \to \mathfrak{g}^*$ is the *moment map*.

Let us provide two examples of Hamiltonian actions.

Example 3.1 Let Y be a smooth variety, G act on Y. Then $X := T^*Y$ carries a natural G-action. This action is Hamiltonian with $H_\xi = \xi_Y$ (viewed as a function on X).

Example 3.2 Let V be a vector space (with symplectic form Ω) and let G act on V by linear symplectomorphisms. The action is Hamiltonian with $H_\xi(v) = \frac{1}{2}\Omega(\xi v, v)$.

Below we will need a standard property of Hamiltonian actions.

Lemma 3.3 *Let $x \in X$. Then* $\operatorname{im} d_x\mu \subset \mathfrak{g}^*$ *coincides with the annihilator of* $\mathfrak{g}_x :=$ Lie(G_x). *In particular, μ is a submersion at x if and only if G_x is finite.*

3.1.2 Hamiltonian Reduction

Let A be a Poisson algebra and $\mathfrak{g}$ be a Lie algebra equipped with a Lie algebra homomorphism $\mathfrak{g} \to A, \xi \mapsto H_\xi$. Consider the ideal $I := A\{H_\xi, \xi \in \mathfrak{g}\}$. The adjoint action of $\mathfrak{g}$ on A preserves this ideal so we can take the invariants $A///_0\mathfrak{g} := (A/I)^{\mathfrak{g}}$. This algebra comes with a natural Poisson bracket: $\{a + I, b + I\} := \{a, b\} + I$ (but A/I has no Poisson bracket!).

This construction has several ramifications. First, let $\lambda : \mathfrak{g} \to \mathbb{C}$ be a character (i.e., a function vanishing on $[\mathfrak{g}, \mathfrak{g}]$). Then we can set $A///_\lambda\mathfrak{g} := (A/A\{H_\xi - \langle\lambda, \xi\rangle\})^{\mathfrak{g}}$. Also we can set $A///\mathfrak{g} := (A/A\{H_\xi, \xi \in [\mathfrak{g}, \mathfrak{g}]\})^{\mathfrak{g}}$. The latter is a Poisson $S(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])$-algebra whose specialization at $\lambda \in (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^*$ coincides with $A///_\lambda\mathfrak{g}$ provided that the $\mathfrak{g}$-action on $A/A\{H_\xi, \xi \in [\mathfrak{g}, \mathfrak{g}]\}$ is completely reducible.

Let us proceed to a geometric incarnation of this construction. Suppose the base field is of characteristic 0. To ensure a good behavior of quotients assume that G is a reductive group. Let X be an affine Poisson variety equipped with a Hamiltonian G-action. Then we can take $A := \mathbb{C}[X]$ together with the comoment map $\xi \mapsto H_\xi$. We set $A///_0G := (A/I)^G$, this algebra coincides with $\mathcal{A}///_0\mathfrak{g}$ when G is connected. It is finitely generated by the Hilbert theorem, here we use that G is reductive. The variety (or scheme) $\mathrm{Spec}(A///_0G)$ is nothing else but the categorical quotient $X///_0G := \mu^{-1}(0)//G$.

Here is a corollary of Lemma 3.3.

Corollary 3.4 *Suppose that X is smooth and symplectic and that the G-action on $\mu^{-1}(0)$ is free. Then $X///_0G$ is smooth and symplectic of dimension* $\dim X - 2\dim G$.

Proof The variety $\mu^{-1}(0)$ is smooth by Lemma 3.3. That the quotient is smooth of required dimension is a straightforward corollary of the Luna slice theorem, see, e.g., [57, Section 6.3].

The form on $X///_0G$ can be recovered as follows. Let Ω denote the form on X, $\iota : \mu^{-1}(0) \hookrightarrow X$ denote the inclusion map and $\pi : \mu^{-1}(0) \to X///_0G$ be the projection. Then there is a unique 2-form Ω_{red} on $X///_0G$ such that $\pi^*\Omega_{red} = \iota^*\Omega$ and this is the form we need. □

3.1.3 GIT Hamiltonian Reduction

We will be mostly interested in Hamiltonian reductions for linear actions $G \curvearrowright V$. The assumptions of Corollary 3.4 are not satisfied in this case. However, if one uses GIT quotients instead of the usual categorical quotients, one can often get a smooth symplectic variety that will be a resolution of the usual reduction $V///_0G$.

Let us recall the construction of a GIT quotient. Let G be a reductive algebraic group acting on an affine algebraic variety X. Fix a character $\theta : G \to \mathbb{C}^\times$. We use the additive notation for the multiplication of characters. Then consider the space $\mathbb{C}[X]^{G,n\theta}$ of $n\theta$-semiinvariants: $\mathbb{C}[X]^{G,n\theta} := \{f \in \mathbb{C}[X] | g.f := \theta(g)^n f\}$ (recall that $g.f(x) := f(g^{-1}x)$). Consider the graded algebra $\bigoplus_{n\geqslant 0} \mathbb{C}[X]^{G,n\theta}$, where $\deg \mathbb{C}[X]^{G,n\theta} := n$. Then we set $X//^{\theta}G := \mathrm{Proj}(\bigoplus_{n\geqslant 0}\mathbb{C}[X]^{G,n\theta})$, this is a projective variety over $X//G$. Note that we no longer have a morphism $X \to X//^{\theta}G$. Instead, consider the open subset of θ-*semistable* points $X^{\theta-ss}$, a point $x \in X$ is called semistable if there is $f \in \mathbb{C}[X]^{G,n\theta}$ for $n > 0$ with $f(x) \neq 0$. We clearly have a natural morphism $X^{\theta-ss} \to X//^{\theta}G$ that makes the following diagram commutative

$$\begin{array}{ccc} X^{\theta-ss} & \longrightarrow & X//^{\theta}G \\ \downarrow{\scriptstyle \subseteq} & & \downarrow \\ X & \longrightarrow & X//G \end{array}$$

The variety $X//^{\theta}G$ is glued from the varieties of the form $X_f//G$, where $f \in \mathbb{C}[X]^{G,n\theta}$ with some $n > 0$. The intersection of $X_f//G, X_g//G$ inside $X//^{\theta}G$ is identified with $X_{fg}//G$, where the inclusions $X_{fg}//G \hookrightarrow X_f//G, X_g//G$ are induced from the inclusions $X_{fg} \hookrightarrow X_f, X_g$ by passing to the quotients.

In the setting of Sect. 3.1.2, we set $X///_0^{\theta}G := \mu^{-1}(0)^{\theta-ss}//G$. This is a Poisson variety (the bracket comes from gluing together the brackets on the open subvarieties $X_f///_0G$) equipped with a projective morphism $X///_0^{\theta}G \to X///_0G$ of Poisson varieties. If X is smooth and symplectic, and the G-action on $\mu^{-1}(0)^{\theta-ss}$ is free, then $X///_0^{\theta}G$ is smooth and symplectic of dimension $\dim X - 2\dim G$. The symplectic form on $X///_0^{\theta}G$ is recovered similarly to the case of $X///_0G$ considered above.

3.1.4 Nakajima Quiver Varieties:Construction

Now we are going to introduce an important special class of varieties constructed by means of Hamiltonian reduction: Nakajima quiver varieties, introduced in [51], see also [54].

By a quiver, we mean an oriented graph. Formally, it can be presented as a quadruple $Q = (Q_0, Q_1, t, h)$, where Q_0, Q_1 are finite sets of vertices and arrows, respectively, and $t, h : Q_1 \to Q_0$ are maps that to an arrow a assign its tale and head.

Let us proceed to (framed) representations of Q. Fix two elements $v, w \in \mathbb{Z}_{\geqslant 0}^{Q_0}$ and set $V_i := \mathbb{C}^{v_i}$, $W_i := \mathbb{C}^{w_i}$, $i \in Q_0$. Consider the space

$$R(= R(Q, v, w)) := \bigoplus_{a\in Q_1}\mathrm{Hom}_{\mathbb{C}}(V_{t(a)}, V_{h(a)}) \oplus \bigoplus_{i\in Q_0}\mathrm{Hom}_{\mathbb{C}}(W_i, V_i).$$

An element of R can be thought as a collection of linear maps, one for each arrow, between the corresponding vector spaces, together with collections of vectors in each V_i. This description suggests a group of symmetry of R: we set $G := \prod_{i\in Q_0}\mathrm{GL}(V_i)$, this group acts by changing bases in the spaces V_i.

A character of G is of the form $g = (g_i)_{i\in Q_0} \mapsto \prod_{i\in Q_0}\det(g_i)^{\theta_i}$, where $\theta = (\theta_i)_{i\in Q_0} \in \mathbb{Z}^{Q_0}$. We will identify the character group of G with $\mathbb{Z}^{Q_0}$.

A Nakajima quiver variety $\mathcal{M}_{\lambda}^{\theta}(v, w)$ is, by definition, the reduction $T^*R///_{\lambda}^{\theta}G$. Here λ is a character of $\mathfrak{g}$, it can be thought as an element of $\mathbb{C}^{Q_0}$ via $\lambda(x) :=$

$\sum_{i\in Q_0}\lambda_i \operatorname{tr}(x_i)$. The moment map $\mu : T^*R \to \bigoplus_{i\in Q_0}\operatorname{End}(V_i) = \mathfrak{g}(\cong \mathfrak{g}^*)$ is explicitly given as follows:

$$(x_a, x_{a^*}, i_k, j_k)_{a\in Q_1, k\in Q_0} \mapsto \sum_{a\in Q_1}(x_a x_{a^*} - x_{a^*}x_a) - \sum_{k\in Q_0} j_k i_k,$$

where $x_a \in \operatorname{Hom}(V_{t(a)}, V_{h(a)}), x_{a^*} \in \operatorname{Hom}(V_{h(a)}, V_{t(a)}), i_k \in \operatorname{Hom}(V_k, W_k), j_k \in \operatorname{Hom}(W_k, V_k)$.

We also would like to remark that the quiver variety is independent of the choice of an orientation of Q. Indeed, let Q' be a quiver obtained from Q by changing the orientation of a single arrow a and let R' be the corresponding representation space. Then we have an isomorphism $T^*R \cong T^*R'$ that sends x_a to x'_{a^*}, x_{a^*} to $-x'_a$ and does not change the other components. This is a G-equivariant symplectomorphism that intertwines the moment maps and hence inducing a symplectomorphism of the corresponding Nakajima quiver varieties.

When $\lambda = 0$, we have a $\mathbb{C}^\times$-action on $\mathcal{M}_0^\theta(v, w)$ that rescales the Poisson structure. For example, one can take the action induced by the dilation action on T^*R, that is, $t.v := t^{-1}v, t \in \mathbb{C}^\times, v \in T^*R$ to be called the dilation action as well. Then the Poisson bracket on $\mathcal{M}_0^\theta(v, w)$ has degree -2. We can also have an action such that the Poisson bracket has degree -1 coming from $t.(r, r^*) := (r, t^{-1}r^*), r \in R, r^* \in R$.

3.1.5 Nakajima Quiver Varieties: Structural Results

Let us explain some structural results regarding the quiver varieties and the corresponding moment maps. We will need algebro-geometric properties of $\mu^{-1}(\lambda)$ and of $\mathcal{M}_\lambda^0(v, w)$ due to Crawley-Boevey and also a criterion for the freeness of the G-action on $\mu^{-1}(\lambda)^{\theta-ss}$ due to Nakajima.

Theorem 3.5 (Crawley-Boevey, [19]). *The scheme $\mathcal{M}_\lambda^0(v, w)$ is reduced and normal.*

We now want to provide a criterium for $\mu : T^*R \to \mathfrak{g}^*$ to be flat proved in [18]. Define the symmetrized Tits form $\mathbb{C}^{Q_0} \times \mathbb{C}^{Q_0} \to \mathbb{C}$:

$$(v^1, v^2) := \sum_{a\in Q_1}(v^1_{t(a)}v^2_{h(a)} + v^1_{h(a)}v^2_{t(a)}) - 2\sum_{i\in Q_0} v^1_i v^2_i$$

and quadratic maps $p, p_w : \mathbb{C}^{Q_0} \to \mathbb{C}$ by

$$p(v) := 1 - \frac{1}{2}(v, v), \quad p_w(v) := w \cdot v - \frac{1}{2}(v, v).$$

Theorem 3.6 (Crawley-Boevey, [18]). *The following two conditions are equivalent:*

(i) μ *is flat.*
(ii) $p_w(v) \geqslant p_w(v^0) + \sum_{i=1}^k p(v^i)$ *for any decomposition* $v = v^0 + v^1 + \ldots + v^k$ *with* $v^i \in \mathbb{Z}_{\geqslant 0}^{Q_0}$ *for* $i = 1, \ldots, k$.

Theorem 3.7 (Crawley-Boevey, [18]). *Suppose that, for a proper decomposition* $v = v^0 + v^1 + \cdots + v^k$, *we have* $p_w(v) > p_w(v^0) + \sum_{i=1}^k p(v^i)$. *Then* $\mu^{-1}(0)$ *is irreducible and a generic* G*-orbit there is closed and free.*

Let us proceed to a criterion for the action of G on $\mu^{-1}(\lambda)^{\theta-ss}$ to be free. We can view Q as a Dynkin diagram and form the corresponding Kac-Moody algebra $\mathfrak{g}(Q)$. Then $\mathbb{C}^{Q_0}$ gets identified with the dual of the Cartan of $\mathfrak{g}(Q)$ in such a way that the coordinate vector $\epsilon_i, i \in Q_0$, becomes a simple root. Then, [51], the action of G on $\mu^{-1}(\lambda)^{\theta-ss}$ is free if and only if there are no roots v' of $\mathfrak{g}(Q)$ such that $v' \leqslant v$ (component-wise) and $v' \cdot \theta = v' \cdot \lambda = 0$.

The equations $v' \cdot \theta = 0$, where v' is a root satisfying $v' \leqslant v, v' \cdot \lambda = 0$ split the character lattice into the union of cones. It is a classical fact from GIT, that when θ, θ' are generic and inside one cone, we have $\mu^{-1}(\lambda)^{\theta-ss} = \mu^{-1}(\lambda)^{\theta'-ss}$. So $\mathcal{M}_\lambda^\theta(v, w) = \mathcal{M}_\lambda^{\theta'}(v, w)$.

3.1.6 $\mathrm{Hilb}_n(\mathbb{C}^2)$ and $\mathbb{C}^{2n}/\mathfrak{S}_n$ as Quiver Varieties

Let Q be a quiver with a single vertex and a single loop (a.k.a. the Jordan quiver). We are going to show that $\mathrm{Hilb}_n(\mathbb{C}^2)$ is identified with $\mathcal{M}_0^{-1}(n, 1)$ and $\mathbb{C}^{2n}/\mathfrak{S}_n$ is identified with $\mathcal{M}_0^0(n, 1)$ (and the Hilbert-Chow map from Sect. 1.1.3 becomes the natural morphism $\mathcal{M}_0^{-1}(n, 1) \to \mathcal{M}_0^0(n, 1)$ from Sect. 3.1.3).

An identification $\mathcal{M}_0^{-1}(n, 1) \cong \mathrm{Hilb}_n(\mathbb{C}^2)$ is an easier part. We have $R = \mathrm{End}(\mathbb{C}^n) \oplus \mathbb{C}^n$. Using the trace pairing, we identify R^* with $\mathrm{End}(\mathbb{C}^n) \oplus \mathbb{C}^{n*}$ so that $T^*R = \mathrm{End}(\mathbb{C}^n)^{\oplus 2} \oplus \mathbb{C}^n \oplus \mathbb{C}^{n*}$. We write (A, B, i, j) for a typical point of T^*R. Identifying $\mathfrak{g}$ with $\mathfrak{g}^*$ again using the trace pairing, we can write the moment map $\mu : T^*R \to \mathfrak{g}$ as $\mu(A, B, i, j) = [A, B] + ij$.

Using the Hilbert-Mumford theorem from Invariant theory, see, e.g., [57, Section 5.3], one shows that $(T^*R)^{\theta-ss} = \{(A, B, i, j) | \mathbb{C}\langle A, B\rangle i = \mathbb{C}^n\}$. Then it is a nice Linear Algebra exercise to show that if $[A, B] + ij = 0$ and $\mathbb{C}\langle A, B\rangle i = \mathbb{C}^n$, then $j = 0$. This is based on an even nicer linear algebra fact: $A, B \in \mathrm{End}(\mathbb{C}^n)$ with $\mathrm{rk}[A, B] \leqslant 1$ are upper-triangular in some basis. So $\mu^{-1}(0)^{\theta-ss}//G = \{(A, B, i) | [A, B] = 0, \mathbb{C}[A, B]i = \mathbb{C}^n\}/G$ that recovers the classical description of $\mathrm{Hilb}_n(\mathbb{C}^2)$, see [53, Theorem 1.14].

An identification $\mathcal{M}_0^0(n, 1) \cong \mathbb{C}^{2n}/\mathfrak{S}_n$ is more subtle. An easy part is to construct a morphism $\iota : \mathbb{C}^{2n}/\mathfrak{S}_n \to \mathcal{M}_0^0(n, 1)$: we send $(\underline{x}, \underline{y}) \in \mathbb{C}^{2n}$ to $(\mathrm{diag}(\underline{x}), \mathrm{diag}(\underline{y}), 0, 0) \in \mu^{-1}(0)$ and this induces a morphism of quotients. Then one checks that ι is a closed embedding. For this, one uses a classical result of Weyl to see that polynomials of the form $\iota^* F_{m,n}$, where $F_{m,n}(A, B, i, j) := \mathrm{tr}(A^n B^m)$ generate the algebra $\mathbb{C}[\underline{x}, \underline{y}]^{\mathfrak{S}_n}$. It remains to prove that ι is surjective. This follows from the second linear algebra fact mentioned in the previous paragraph.

Lemma 3.8 *The isomorphism* $\mathcal{M}_0^0(n, 1) \cong \mathbb{C}^{2n}/\mathfrak{S}_n$ *intertwines the Poisson brackets.*

Proof Consider the principal open subsets

$$R^{reg} = \{(A, i) | A \text{ has distinct e-values}\}, \mathbb{C}^{n,reg} := \{(x_1, \dots, x_n) | x_i \neq x_j, \text{ for } i \neq j\}.$$

Note that under the above embedding $\mathbb{C}^{2n} \hookrightarrow T^*R$, we have $T^*\mathbb{C}^{n,reg} \hookrightarrow T^*R^{reg}$. Moreover, the pull-back of the symplectic form from T^*R^{reg} to $T^*\mathbb{C}^{n,reg}$ coincides with the natural symplectic form on the latter. Using the description of the symplectic form on the reduction, we conclude that the induced morphism of quotients $T^*\mathbb{C}^{n,reg}/\mathfrak{S}_n \to T^*R^{reg}///_0^0 G$ is a symplectomorphism. But $T^*R^{reg}///_0^G$ embeds as an open subset into $\mathcal{M}_0^0(n, 1)$ and the symplectomorphism above is the restriction of the isomorphism $\mathbb{C}^{2n}/\mathfrak{S}_n \xrightarrow{\sim} \mathcal{M}_0^0(n, 1)$ to $T^*\mathbb{C}^{n,reg}/\mathfrak{S}_n$. The claim of the lemma follows. □

3.1.7 McKay Correspondence

Let Γ_1 be a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$. It turns out that the singular Poisson variety V_n/Γ_n (where recall $V_n = \mathbb{C}^{2n}$) and its symplectic resolutions also can be realized as Nakajima quiver varieties.

The first step in this isomorphism is the McKay correspondence: a way to label the finite subgroups of $\mathrm{SL}_2(\mathbb{C})$ by Dynkin diagrams. Let Γ_1 be a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$ and let $N_0, \dots, N_r$ be the irreducible representations of Γ_1, where N_0 is the trivial representation. Let us define the McKay graph of Γ_1: its vertices are $0, 1, \dots, r$ and the number of edges (we consider a non-oriented graph) between i and j is $\dim \mathrm{Hom}_\Gamma(\mathbb{C}^2 \otimes N_i, N_j)$, note that this is well-defined because $\mathbb{C}^2$ is a self-dual representation of Γ and so the number of edges between i and j is the same as between j and i. McKay proved the following facts:

(i) The resulting graph is an extended Dynkin graph of types A, D, E and 0 is the extending vertex.
(ii) The vector $(\dim N_i)_{i=0}^r$ is the indecomposable imaginary root δ of the corresponding affine Kac-Moody algebra.

3.1.8 $\mathbb{C}^2/\Gamma_1$ as a Quiver Variety

Let Q be the McKay graph of Γ_1 with an arbitrary orientation. Then there is an isomorphism $\mathcal{M}_0^0(\delta, 0) \cong \mathbb{C}^2/\Gamma_1$.

Let us explain how this is established following [17, Section 8]. For this, we will need the representation varieties.

Let A be a finitely generated associative algebra and V be a vector space. Then the set $X := \mathrm{Hom}(A, \mathrm{End}(V))$ of algebra homomorphisms is an algebraic variety. More precisely, if A is the quotient of $\mathbb{C}\langle x_1, \dots, x_n\rangle$ by relations $F_\alpha(x_1, \dots, x_n)$, where α runs over an indexing set $\mathfrak{I}$, then $X = \{(A_1, \dots, A_n) \in \mathrm{End}(V) | F_\alpha(A_1, \dots, A_n) = 0, \alpha \in \mathfrak{I}\}$. The group $G := \mathrm{GL}(V)$ naturally acts on X and so we can form the quotient $X//G$ (called the representation variety). Recall that, in general, the points of $X//G$ correspond to the closed G-orbits on X, in our case an orbit is closed if its element is a semisimple representation.

This construction has various ramifications. For example, we can consider a semisimple finite dimensional subalgebra $A_0 \subset A$ and an A_0-module V. This leads to the variety X of A_0-linear homomomorphisms $A \to \mathrm{End}(V)$ acted on by the group G of A_0-linear automorphisms of V. In this situation we still can speak about representation varieties. We will realize $\mathcal{M}_0^0(\delta, 0)$, $\mathbb{C}^2/\Gamma_1$ as the representation varieties of this kind and then show that the algebras involved are Morita equivalent, this will yield an isomorphism of interest.

Let us start with $\mathbb{C}^2/\Gamma_1$. Set $A := \mathbb{C}[x, y]\#\Gamma_1$, $A_0 := \mathbb{C}\Gamma_1 \subset A$ and $V := \mathbb{C}\Gamma_1$, a regular representation. Then one can show that $\mathbb{C}^2/\Gamma_1$ is the representation variety for this triple.

Let us proceed to $\mathcal{M}_0^0(\delta, 0)$. Let $\bar{Q}$ be the *double quiver* of Q. It is obtained from Q by adding the inverse arrow to each arrow in Q. Formally, $\bar{Q}_0 = Q_0$, $\bar{Q}_1 = Q_1 \sqcup Q_1^*$, where Q_1^* is in bijection with Q_1, $a \mapsto a^*$, in such a way that $t(a^*) = h(a)$, $h(a^*) = t(a)$. Then form the *path algebra* $\mathbb{C}\bar{Q}$ of $\bar{Q}$, it has a basis consisting of the paths in $\bar{Q}$, the multiplication is given by concatenation (if two paths cannot be concatenated, the product is zero). This algebra is graded by the length of a path, where, by convention, the degree 0 paths are just vertices so the corresponding graded component $\mathbb{C}\bar{Q}_0$ is $\mathbb{C}^{Q_0}$.

Let us consider the quotient $\Pi^0(Q)$ of $\mathbb{C}\bar{Q}$ called the preprojective algebra. It is given by the following relation:

$$\sum_{a \in Q_1} [a, a^*] = 0.$$

Note that $\mathbb{C}^{Q_0}$ naturally embeds into $\Pi^0(Q)$. It is easy to see that $\mathcal{M}_0^0(\delta, 0)$ is the representation variety for the triple $(\Pi^0(Q), \mathbb{C}^{Q_0}, \bigoplus_{i \in Q_0} \mathbb{C}^{\delta_i})$.

It turns out that there is an idempotent $f \in \mathbb{C}\Gamma_1$ such that $f(\mathbb{C}[x, y]\#\Gamma_1)f \cong \Pi^0(Q)$. Namely, take primitive idempotents $f_i, i = 0, \dots, r$, in the matrix summands of $\mathbb{C}\Gamma_1$. Set $f := \sum_{i \in Q_0} f_i$. Obviously, $f(\mathbb{C}\Gamma_1)f \cong \mathbb{C}^{Q_0}$. Further, the construction of Q implies that $f(\mathrm{Span}(x, y) \otimes \mathbb{C}\Gamma_1)f \cong \mathbb{C}\bar{Q}_1$. These identifications induce an isomorphism $f(\mathbb{C}\langle x, y\rangle\#\Gamma_1)f \cong \mathbb{C}\bar{Q}$. Under this isomorphism, the ideal $f(xy - yx)f$ becomes $(\sum_{a \in Q_1}[a, a^*])$, see [17, Section 2]. Also note that the $\mathbb{C}^{Q_0}$-module $\bigoplus_{i \in Q_0} \mathbb{C}^{\delta_i}$ is nothing else but $f\mathbb{C}\Gamma_1$. Finally, note that f defines a Morita equivalence between $\mathbb{C}[x, y]\#\Gamma_1$, $\Pi^0(Q)$. An isomorphism $\mathbb{C}^2/\Gamma_1 \cong \mathcal{M}_0^0(\delta, 0)$ now follows from the next lemma, whose proof is left to the reader.

Lemma 3.9 *Let $A_0 \subset A$ and V be as above and let $f \in A_0$ be an idempotent giving a Morita equivalence. Then the representation varieties for (A, A_0, V) and (fAf, fA_0f, fV) are naturally isomorphic.*

Note that the algebras $\mathbb{C}[x, y]\#\Gamma_1$ and $\Pi^0(Q)$ are graded and an isomorphism $\Pi^0(Q) \cong \mathbb{C}[x, y]\#\Gamma_1$ preserves the grading. From here one easily deduces that the isomorphism $\mathbb{C}^2/\Gamma_1 \cong \mathcal{M}_0^0(\delta, 0)$ is equivariant with respect to the dilation $\mathbb{C}^\times$-actions.

3.1.9 V_n/Γ_n as a Quiver Variety

Let us proceed now to the case of an arbitrary n. Let $\epsilon_0 \in \mathbb{C}^{Q_0}$ be the coordinate vector at the extending vertex.

Proposition 3.10 *We have a $\mathbb{C}^\times$-equivariant isomorphism $\mathcal{M}_0^0(n\delta, \epsilon_0) \cong V_n/\Gamma_n$ (of Poisson schemes).*

Proof We have a diagonal embedding $T^*R(Q, \delta, 0)^{\oplus n} \to T^*R(Q, n\delta, \epsilon_0)$, compare to Sect. 3.1.6, that restricts to $\mu_1^{-1}(0)^n \hookrightarrow \mu^{-1}(0)$, where μ_1 stands for the moment map $T^*R(Q, \delta, 0) \to \mathfrak{gl}(\delta)^*$. This gives rise to a $\mathfrak{S}_n$-invariant morphism $\mathcal{M}_0^0(\delta, 0)^n \to \mathcal{M}_0^0(n\delta, \epsilon_0)$ and hence to a morphism $\iota : \mathbb{C}^{2n}/\Gamma_n = (\mathbb{C}^2/\Gamma_1)^n/\mathfrak{S}_n \to \mathcal{M}_0^0(n\delta, \epsilon_0)$. One can show that this morphism is bijective. Also it is $\mathbb{C}^\times$-equivariant, where the $\mathbb{C}^\times$-actions on $\mathbb{C}^{2n}/\Gamma_n, \mathcal{M}_0^0(n\delta, \epsilon_0)$ are induced from the dilation actions on $\mathbb{C}^{2n}, T^*R(Q, n\delta, \epsilon_0)$. It follows that ι is finite. By Theorem 3.5, $\mathcal{M}_0^0(n\delta, \epsilon_0)$ is normal and this implies that ι is an isomorphism.

We can make the isomorphism ι Poisson if we rescale it using the $\mathbb{C}^\times$-actions. This is a consequence of the following lemma. □

Lemma 3.11 ([21, Lemma 2.23]) *Let V be a symplectic vector space and $\Gamma \subset$ $\mathrm{Sp}(V)$ be a finite subgroup such that V is symplectically irreducible, i.e., there are no proper symplectic Γ-stable subspace in V. Then there are no nonzero brackets (=skew-symmetric bi-derivations) of degree < -2 on $\mathbb{C}[V]^\Gamma$. Further, the space of brackets of degree -2 is one-dimensional.*

One can ask why we use $\mathcal{M}_0^0(n\delta, \epsilon_0)$ instead of $\mathcal{M}_0^0(n\delta, 0)$ in the proposition. The reason is that the moment map for $T^*R(n\delta, \epsilon_0)$ is flat, this can be checked using Theorem 3.6.

3.1.10 Symplectic Resolutions of V_n/Γ_n

Here we will study symplectic resolutions of V_n/Γ_n constructed as non-affine Nakajima quiver varieties for generic stability conditions θ.

Let us consider the case $n = 1$ first. Let $\bar{G}$ denote the quotient of $G = \mathrm{GL}(\delta)$ modulo the one-dimensional torus $T_{const} := \{(x\,\mathrm{id}_{\mathbb{C}^{\delta_i}})_{i=0}^r, x \in \mathbb{C}^\times\}$. Note that the

G-action on $R := R(Q, \delta, 0)$ factors through $\bar{G}$. Analogously to Nakajima's result explained in Sect. 3.1.5, the group $\bar{G}$ acts freely on $\mu^{-1}(0)^{\theta-ss}$ if and only if $\theta \cdot \alpha \neq 0$ for every Dynkin root of Q (these are the roots $\alpha \in \mathbb{C}^{Q_0}$ with $\alpha_0 = 0$). For such θ, we get a conical symplectic resolution $\mathcal{M}_0^\theta(\delta, 0) \to \mathcal{M}_0^0(\delta, 0)$, this can be deduced, for example, from Theorem 3.7. Of course, all these resolutions are isomorphic to the minimal resolution $\widetilde{\mathbb{C}^2/\Gamma_1}$: there are just no other symplectic resolutions.

Let us proceed to the case $n > 1$. We get a projective morphism $\mathcal{M}_0^\theta(n\delta, \epsilon_0) \to \mathcal{M}_0^0(n\delta, \epsilon_0)$. Theorem 3.7 no longer applies, in fact, $\mu^{-1}(0)$ has $n+1$ irreducible components by [23, Section 3.2]. Still, $\mathcal{M}_0^\theta(n\delta, \epsilon_0) \to \mathcal{M}_0^0(n\delta, \epsilon_0)$ is a resolution of singularities. One just needs to check that the fiber over a generic point in $\mathcal{M}_0^0(n\delta, \epsilon_0)$ consists of a single point. A generic closed G-orbit in $\mu^{-1}(0)$ has a point of the form $r^1 \oplus \ldots \oplus r^n$, where $r^1, \ldots, r^n$ are pair-wise non-isomorphic simple representations of $\Pi^0(Q)$ of dimension δ. Then one can analyze the structure of the G-action near that orbit using a symplectic slice theorem, see, for example, [19, Section 4] or Sect. 4.3.3 below. This analysis shows that there is a unique semistable G-orbit containing Gr in its closure. So we see that $\mathcal{M}^\theta(n\delta, \epsilon_0) \to \mathcal{M}_0^0(n\delta, \epsilon_0)$ is a conical symplectic resolution.

3.1.11 Isomorphic Resolutions

Now let us discuss how many resolutions we get. The stability condition θ is generic if $\theta \cdot \delta \neq 0$ and $\theta \cdot v \neq 0$ for v of the form $v = \alpha + m\delta$, where α is a Dynkin root and $|m| < n$. So we get resolutions labeled by the open cones in the complement to these hyperplanes in $\mathbb{R}^n$. However, some of these resolutions are isomorphic: there is an action of $W \times \mathbb{Z}/2\mathbb{Z}$ on $\mathbb{Z}^{Q_0}$ such that, for θ, θ' lying in one orbit, the resolutions $\mathcal{M}_0^\theta(n\delta, \epsilon_0) \to \mathcal{M}_0^0(n\delta, \epsilon_0)$, $\mathcal{M}_0^{\theta'}(n\delta, \epsilon_0) \to \mathcal{M}_0^0(n\delta, \epsilon_0)$ are isomorphic (here W denotes the Weyl group of the Dynkin diagram obtained from Q by removing the vertex 0). This is a special case of a construction due to Namikawa, [55], that we are going to explain now.

Let $X \to X_0$ be an arbitrary conical symplectic resolution. The variety X_0 has finitely many symplectic leaves, [37]. Let $\mathcal{L}_1, \ldots, \mathcal{L}_k$ be the leaves of codimension 2. Take formal slices $\mathsf{S}_1^\wedge, \ldots, \mathsf{S}_k^\wedge$ through $\mathcal{L}_1, \ldots, \mathcal{L}_k$. The slices are formal neighborhoods of 0 in Kleinian singularities $\mathsf{S}_1, \ldots, \mathsf{S}_k$. From these Kleinian singularities one produces Weyl groups $\tilde{W}_1, \ldots, \tilde{W}_k$ (of the same types as the singularities) acting on the spaces $H^2(\mathsf{S}_k, \mathbb{C})$ identified with their reflection representations $\tilde{\mathfrak{h}}_i$. The fundamental group $\pi_1(\mathcal{L}_i)$ acts on the irreducible components of the exceptional divisor in S_i. Hence it also acts on $\tilde{W}_i$ (by diagram automorphisms) and on $\tilde{\mathfrak{h}}_i$. Set $W_i := \tilde{W}_i^{\pi_1(\mathcal{L}_i)}$, $\mathfrak{h}_i := \tilde{\mathfrak{h}}_i^{\mathcal{L}_i}$ so that W_i is a crystallographic reflection group and $\mathfrak{h}_i$ is its reflection representation. There is a natural restriction map $H^2(X) \to \mathfrak{h} := \bigoplus_i \mathfrak{h}_i$. Namikawa proved that this map is surjective. Furthermore, he has constructed a $W := \prod_i W_i$-action on $H^2_{DR}(X)$ that makes the map equivariant and is trivial on the kernel.

Let us return to our situation. The symplectic leaves in V/Γ are in one-to-one correspondence with conjugacy classes of stabilizers of points in V. The leaf corresponding to $\Gamma' \subset \Gamma$ is the image of $V^{\Gamma',reg} := \{v \in V | \Gamma_v = \Gamma'\}$ under the quotient morphism $\pi : V \to V/\Gamma$. The leaf is identified with $V^{\Gamma',reg}/N_\Gamma(\Gamma')$. So, in the case when $V = V_n$ and $\Gamma = \Gamma_n$, we get two leaves of codimension 2 (provided $\Gamma_1 \neq \{1\}$, in that case we get just one leaf of codimension 2). One of them, say $\mathcal{L}_1$, corresponds to $\Gamma_1 \subset \Gamma_n$ (the stabilizer of a point of the form $(0, p_1, \ldots, p_{n-1})$, where $p_1, \ldots, p_{n-1}$ are pairwise different points of $\mathbb{C}^2$). The other, say $\mathcal{L}_2$, corresponds to $\mathfrak{S}_2$ (the stabilizer of $(p_1, p_1, p_2, \ldots, p_{n-1})$). The fundamental group actions from the previous paragraph are easily seen to be trivial. So we get $W_1 = W$, $W_2 = \mathbb{Z}/2\mathbb{Z}$. Further, $H^2(X) = \mathbb{C}^{Q_0}$ and $\mathfrak{h}_1 = \{(x_i)_{i \in Q_0} | x \cdot \delta = 0\}$, $\mathfrak{h}_2 = \mathbb{C}\delta$. The group W_2 acts on $\mathbb{C}\delta$ by ± 1, while $\mathfrak{h}_1$ is identified with the Cartan space for W_1 via $(x_i)_{i \in Q_0} \mapsto \sum_{i=1}^r x_i \omega_i^\vee$, where we write $\omega_i^\vee$ for the fundamental coweights.

Let us remark that the W-action can be recovered by using the quiver variety setting as well, see [45, 48] for more detail.

3.2 *Quantum Hamiltonian Reduction*

Here we will explain a quantum counterpart of the constructions of the previous section.

3.2.1 Quantum Hamiltonian Reduction: Algebra Level

Let $\mathcal{A}$ be an associative algebra, $\mathfrak{g}$ a Lie algebra and $\Phi : \mathfrak{g} \to \mathcal{A}$ be a Lie algebra homomorphism. Then, for a character λ of $\mathfrak{g}$, set $\mathcal{I}_\lambda := \mathcal{A}\{x - \langle \lambda, x \rangle, x \in \mathfrak{g}\}$, this is a left ideal in $\mathcal{A}$ that is stable under the adjoint action of $\mathfrak{g}$. We set $\mathcal{A}///_\lambda \mathfrak{g} := (\mathcal{A}/\mathcal{I}_\lambda)^{\mathfrak{g}}$. This space has a natural associative product given by $(a + \mathcal{I}_\lambda)(b + \mathcal{I}_\lambda) := ab + \mathcal{I}_\lambda$. With this product, $\mathcal{A}///_\lambda \mathfrak{g}$ becomes naturally isomorphic to $\operatorname{End}_{\mathcal{A}}(\mathcal{A}/\mathcal{I}_\lambda)^{opp}$, an element $a + \mathcal{I}_\lambda$ gets mapped to the unique endomorphism sending $1 + \mathcal{I}_\lambda$ to $a + \mathcal{I}_\lambda$. We also have a universal variant of quantum Hamiltonian reduction: $\mathcal{A}///\mathfrak{g} := (\mathcal{A}/\mathcal{A}\Phi([\mathfrak{g}, \mathfrak{g}]))^{\mathfrak{g}}$.

Now suppose $\mathcal{A}$ is a filtered quantization of $\mathbb{C}[X]$, where X is an an affine Poisson variety (we assume that the bracket on $\mathbb{C}[X]$ has degree -1). Suppose that G acts on X in a Hamiltonian way and the functions $\mu^*(\xi)$ have degree 1 for all $\xi \in \mathfrak{g}$. By a quantization of the Hamiltonian G-action on $\mathbb{C}[X]$ we mean a rational G-action on $\mathcal{A}$ together with a G-equivariant map $\Phi : \mathfrak{g} \to \mathcal{A}$ such that

(i) the filtration on $\mathcal{A}$ is G-stable and the isomorphism $\operatorname{gr} \mathcal{A} \cong \mathbb{C}[X]$ is G-equivariant,
(ii) $\Phi(\xi)$ lies in $\mathcal{A}_{\leqslant 1}$ and coincides with $\mu^*(\xi)$ modulo $\mathcal{A}_{\leqslant 0}$,
(iii) and $[\Phi(\xi), \cdot] = \xi_{\mathcal{A}}$, where $\xi_{\mathcal{A}}$ is the derivation of $\mathcal{A}$ coming from the G-action.

Note that $\operatorname{gr}\mathcal{I}_\lambda \supset I := \mathbb{C}[X]\mu^*(\mathfrak{g})$ and so we have a surjective homomorphism $\mathbb{C}[X///_0G] \twoheadrightarrow \operatorname{gr}\mathcal{A}///_\lambda G$. We want to get a sufficient condition for $\operatorname{gr}\mathcal{I}_\lambda = I$ for all λ.

Lemma 3.12 *Let $\xi_1, \dots, \xi_n$ be a basis in $\mathfrak{g}$. Suppose $\mu^*(\xi_1), \dots, \mu^*(\xi_n)$ form a regular sequence. Then $\operatorname{gr}\mathcal{I}_\lambda = I$ for any λ.*

Proof The proof is based on the observation that the 1st homology in the Koszul complex associated to $\mu^*(\xi_1), \dots, \mu^*(\xi_n)$ is zero. In other words, if $f_1, \dots, f_n \in \mathbb{C}[X]$ are such that $\sum_{i=1}^n f_i\mu^*(\xi_i) = 0$, then there are $f_{ij} \in \mathbb{C}[X]$ with $f_{ij} = -f_{ji}$ and $f_i = \sum_{j=1}^n f_{ij}\mu^*(\xi_j)$. Details of the proof are left to the reader. □

So if G is reductive and the assumptions of Lemma 3.12 hold, then $\mathcal{A}///_\lambda\mathfrak{g}$ is a filtered quantization of $\mathbb{C}[X///_0G]$.

We can also give the definition of a quantization of a Hamiltonian action in the setting of formal quantizations. One should modify (i)-(iii) as follows. In (i) one requires the G-action to be $\mathbb{C}[[\hbar]]$-linear and the isomorphism $\mathcal{A}_\hbar/\hbar\mathcal{A}_\hbar \cong \mathbb{C}[X]$ has to be G-equivariant. In (ii), one requires that $\Phi(\xi)$ coincides with $\mu^*(\xi)$ modulo $\hbar$. In (iii) one requires $\frac{1}{\hbar}[\Phi(\xi), \cdot] = \xi_{\mathcal{A}_\hbar}$. We then can consider reductions of the form $\mathcal{A}_\hbar///_{\lambda(\hbar)}G$, where $\lambda(\hbar)$ is an element in $(\mathfrak{g}^{*G})[[\hbar]]$. If G is reductive, and the elements $\mu^*(\xi_i) - \langle\lambda(0), \xi_i\rangle$, $i = 1, \dots, n$, form a regular sequence in $\mathbb{C}[X]$, then $\mathcal{A}_\hbar///_{\lambda(\hbar)}G$ is a formal quantization of $\mathbb{C}[X///_{\lambda(0)}G]$.

3.2.2 Quantum Hamiltonian Reduction: Sheaf Level

Let X be a smooth affine symplectic algebraic variety equipped with a Hamiltonian action of G and let θ be a character of G. Assume that, for a basis $\xi_1, \dots, \xi_n$ of $\mathfrak{g}$, the elements $\mu^*(\xi_1), \dots, \mu^*(\xi_n)$ form a regular sequence at all points of $\mu^{-1}(0)^{\theta-ss}$. Let $\mathcal{D}_\hbar$ be a formal quantization of $\mathcal{O}_X$. Our goal is to define a (formal) quantization $\mathcal{D}_\hbar///^\theta_{\lambda(\hbar)}G$ of $X///^\theta_0G$ (so $\lambda(0) = 0$).

Recall that it is enough to define the following data:

(1) For an open affine covering $X///^\theta_\lambda G := \bigcup_i Y_i$, the algebras of sections $\Gamma(Y_i, \mathcal{D}_\hbar///^\theta_{\lambda(\hbar)}G)$ that quantize Y_i,
(2) and identifications $\Gamma(Y_i, \mathcal{D}_\hbar///^\theta_{\lambda(\hbar)}G)_{Y_i\cap Y_j} \cong \Gamma(Y_j, \mathcal{D}_\hbar///^\theta_{\lambda(\hbar)}G)_{Y_i\cap Y_j}$ satisfying cocycle conditions.

Recall that we can choose an open covering by setting $Y_i := X_{f_i}///_0G$, where polynomials $f_i \in \mathbb{C}[X]^{G,n_i\theta}$ are such that $X^{\theta-ss} = \bigcup_i X_{f_i}$. Then we set $\Gamma(Y_i, \mathcal{D}_\hbar///^\theta_{\lambda(\hbar)}G) := \Gamma(X_{f_i}, \mathcal{D}_\hbar)///_{\lambda(\hbar)}G$. The sections of the corresponding sheaf on $Y_i \cap Y_j$ are easily seen to be $\Gamma(X_{f_i} \cap X_{f_j}, \mathcal{D}_\hbar)///_{\lambda(\hbar)}G$ and this yields the gluing maps.

Now let us discuss the period map mentioned in Sect. 2.2.6. Suppose that the G-action on $\mu^{-1}(0)^{\theta-ss}$ is free so that $X///^\theta_0G$ is smooth and symplectic. In this case we have a period map associated to the quantization of $\mathcal{D}_\hbar///^\theta_{\lambda(\hbar)}G$. Assume, for simplicity, that $\lambda(\hbar) := \lambda\hbar$ for $\lambda \in \mathfrak{g}^{*G}$—this is the most interesting case, for example, it is the only case that appears when we work with the filtered setting. Further, assume that $\mathcal{D}_\hbar$ is canonical, i.e., has period 0. Recall that this means the existence of a parity anti-automorphism, let us denote it by ϱ. Finally, assume that Φ is *symmeterized*, meaning that $\varrho \circ \Phi = \Phi$, this can be achieved by modifying Φ. Then the period of $\mathcal{D}_\hbar///^\theta_{\lambda\hbar}G$ equals to the Chern class associated to λ (if λ integrates to a character of G, then it defines the line bundle on $X///^\theta_0 G$, in general, we extend the notion of a Chern class by linearity). This was essentially checked in [45, Sections 3.2, 5.4].

3.2.3 Algebra Versus Sheaf Level

We need to relate the sheaf $\mathcal{D}_\hbar///^\theta_{\lambda(\hbar)}G$ to the algebra $\mathcal{D}_\hbar///_{\lambda(\hbar)}G$. What one could expect is that the algebra is the global sections (or even better, the derived global sections) of the sheaf. Let us provide some sufficient conditions for this to hold.

Proposition 3.13 *Assume, for simplicity, that* $\lambda(0) = 0$. *Further, suppose that the following holds.*

(1) The moment map μ *is flat.*
(2) $X///_0 G$ *is a normal reduced scheme.*
(3) $X///^\theta_0 G \to X///_0 G$ *is a resolution of singularities.*

Then $R\Gamma(\mathcal{D}_\hbar///^\theta_{\lambda(\hbar)}G) \cong \mathcal{D}_\hbar///_{\lambda(\hbar)}G$.

Proof By (3) and the Grauert-Riemmenschneider theorem, the higher cohomology of $\mathcal{O}_{X///^\theta_0 G}$ vanish. This implies that the higher cohomology of $\mathcal{D}_\hbar///^\theta_{\lambda(\hbar)}G$ vanish. Moreover, $\Gamma(\mathcal{D}_\hbar///^\theta_{\lambda(\hbar)}G)/(\hbar) \cong \mathbb{C}[X///^\theta_0 G]$. By (2) and (3), the right hand side is naturally identified with $\mathbb{C}[X///_0 G]$. By (1), $(\mathcal{D}///_{\lambda(\hbar)}G)/(\hbar) = \mathbb{C}[X///_0 G]$. Besides, we have a natural homomorphism $\mathcal{D}///_{\lambda(\hbar)}G \to \Gamma(\mathcal{D}_\hbar///^\theta_{\lambda(\hbar)}G)$. Modulo $\hbar$, this homomorphism is the identity. The source algebra is complete and separated in the $\hbar$-adic topology, and the target algebra is flat over $\mathbb{C}[[\hbar]]$. It follows that the homomorphism $\mathcal{D}///_{\lambda(\hbar)}G \to \Gamma(\mathcal{D}_\hbar///^\theta_{\lambda(\hbar)}G)$ is an isomorphism. □

3.2.4 Isomorphism Theorem

Recall a $\mathbb{C}^\times$-equivariant isomorphism $\mathbb{C}^{2n}/\Gamma_n \cong \mathcal{M}^0(n\delta, \epsilon_0)$ of Poisson varieties. The left hand side admits a family of quantizations, $eH_{1,c}e$, and so does the right hand side, there quantizations are the quantum Hamiltonian reductions $D(R)///_\lambda G$, where we use the symmetrized quantum comoment map $\Phi(\xi) = \frac{1}{2}(\xi_R + \xi_{R^*})$. In fact, these two families are the same. Let us state a precise result to be proved in Sect. 4.3 (using Procesi bundles). We write $\mathbf{c}$ for

$$\frac{1}{|\Gamma_1|}\left(1 + \sum_{\gamma\in\Gamma_1\setminus\{1\}} c(\gamma)\gamma\right) \in \mathbb{C}\Gamma_n,$$

where $c(\gamma) := c_i$ for $\gamma \in S_i$ (recall that S_0 is the conjugacy class of a reflection in $\mathfrak{S}_n \subset \Gamma_n$ and $S_1, \ldots, S_r$ are conjugacy classes of elements of $\Gamma_1 \subset \Gamma_n$).

Theorem 3.14 *We have a filtered algebra isomorphism $eH_{1,c}e \cong D(R)///_\lambda G$ that is the identity on the level of associated graded algebras (we consider the filtration on $D(R)///_\lambda G$ induced from the Bernstein filtration on $D(R)$, where* $\deg R = \deg R^* = 1$*). Here $\lambda := \sum_{i=0}^r \lambda_i \operatorname{tr}_i$ is recovered from c by the following formulas:*

$$\lambda_i := \operatorname{tr}_{N_i} \mathbf{c},\ i = 1, \ldots, r, \quad \lambda_0 := \operatorname{tr}_{N_0} \mathbf{c} - \frac{1}{2}(c_0 + 1), \tag{5}$$

where in the $n = 1$ case one needs to put $c_0 = 1$.

For $n = 1$, this theorem was proved by Holland in [35]. The case of $\Gamma_1 = \{1\}$ was handled in [21, 23] ([21] proved a weaker statement and then in [23] the proof was completed). The case of cyclic Γ_1 was done in [27, 56]. In [20] the proof was completed: they considered the case when Q is a bipartite graph. Let us note that in these papers formulas look different from (5): they use the quantum comoment map $\Phi(\xi) = \xi_R$. A uniform and more conceptual proof was given in [45] using Procesi bundles, it will be sketched in Section 4.3.

Theorem 3.14 is of crucial importance for the representation theory of the algebras $H_{1,c}$. It turns out that the representation theory of the algebras $D(R)///_\lambda G$ (actually, of sheaves $D_R///^\theta_\lambda G$) is easier to study. The main ingredient here is the geometry of the quiver varieties $\mathcal{M}^\theta(v, \epsilon_0)$. Using this, in [12], the author and Bezrukavnikov have proved a conjecture of Etingof, [22], on the number of the finite dimensional irreducible representations of $H_{1,c}$.

3.2.5 Automorphisms

Here we are going to explain a quantum version of Namikawa's construction recalled in Sect. 3.1.11. In the complete generality this construction was given in [16, Section 3.3].

Let X be a conical symplectic resolution of X_0. Let $\tilde{X}$ be its universal deformation over $H^2_{DR}(X)$ and let $\tilde{\mathcal{D}}_\hbar$ be the canonical quantization of $\tilde{X}$. Let $\tilde{\mathcal{A}}_\hbar$ denote the $\mathbb{C}^\times$-finite part of $\Gamma(\tilde{\mathcal{D}}_\hbar)$. Then Namikawa's Weyl group W acts on $\tilde{\mathcal{A}}_\hbar$ by graded $\mathbb{C}[\hbar]$-algebra automorphisms preserving $H^2_{DR}(X)^*$. Moreover, the action on $H^2_{DR}(X)^*$ is as explained in Sect. 3.1.11.

3.3 Quantum Hamiltonian Reduction for Frobenius Constant Quantizations

In this section, we will consider the situation in characteristic p. Our main result is that a quantum GIT Hamitlonian reduction under a free Hamiltonian action is again Frobenius constant.

3.3.1 GIT in Characteristic p

The definition of a reductive group (one with trivial unipotent radical) makes sense in all characteristics. A crucial difficulty of dealing with reductive groups in positive characteristic is that their rational representations are no longer completely reducible, in general. The groups for which the complete reducibility holds are called *linearly reductive*. Tori are still linearly reductive independently of the characteristic. We need to deal with GIT for reductive groups (such as products of GL's) and so we need to explain how this works in positive characteristic.

It turns out that reductive groups satisfy a weaker condition than being linearly reductive, they are *geometrically reductive*. This was conjectured by Mumford and proved by Haboush, [32]. To state the condition of being geometrically reductive, let us reformulate the linear reductivity first: a group G is called linearly reductive, if, for any linear G-action on a vector space V and any fixed point $v \in V$, there is $f \in (V^*)^G$ with $f(v) \neq 0$. A group G is called *geometrically reductive* if instead of $f \in (V^*)^G$, one can find $f \in S^r(V^*)^G$ (for some $r > 0$) with $f(v) \neq 0$.

This condition is enough for many applications. For example, if X is an affine algebraic variety acted on by a reductive (and hence geometrically reductive) group G, then $\mathbb{F}[X]^G$ is finitely generated. So we can consider the quotient morphism $X \to X/\!/G$. This morphism is surjective and separates the closed orbits. Moreover, if $X' \subset X$ is a G-stable subvariety, then the natural morphism $X'/\!/G \to X/\!/G$ is injective with closed image.

The claim about the properties of the quotient morphism in the previous paragraph can be deduced from the following lemma, [50, Lemma A.1.2].

Lemma 3.15 *Let G be a geometrically reductive group acting on a finitely generated commutative $\mathbb{F}$-algebra R rationally and by algebra automorphisms. Let $I \subset R$ be a G-stable ideal and $f \in (R/I)^G$. Then there is n such that f^{p^n} lies in the image of R^G in $(R/I)^G$.*

In characteristic p, we can still speak about unstable and semistable points for reductive group actions on vector spaces, about GIT quotients, etc.

Another very useful and powerful result of Invariant theory in characteristic 0 is Luna's étale slice theorem, see, e.g., [57, Sect. 6.3]. There is a version of this theorem in characteristic p due to Bardsley and Richardson, see [1]. We will need a consequence of this theorem dealing with free actions.

Recall that, in characteristic 0, an action of an algebraic group G on a variety X is called free if the stabilizers of all points are trivial. In characteristic p one should give this definition more carefully: the stabilizer may be a nontrivial finite group scheme with a single point. An example is provided by the left action of G on $G^{(1)}$, we will discuss a closely related question in the next subsection. We have the following three equivalent definitions of a free action.

- For every $x \in X$, the stabilizer G_x equals $\{1\}$ as a group scheme.
- For every $x \in X$, the orbit map $G \to X$ corresponding to x is an isomorphism of algebraic varieties.
- For every $x \in X$, G_x coincides with $\{1\}$ as a set and the stabilizer of x in $\mathfrak{g}$ is trivial.

The following is a weak version of the slice theorem that we need.

Lemma 3.16 *Let X be a smooth affine variety equipped with a free action of a reductive algebraic group G. Then the quotient morphism $X \to X/G$ is a principal G-bundle in étale topology.*

3.3.2 Quiver Varieties

Let us now discuss Nakajima quiver varieties in characteristic $p \gg 0$. We have a finite localization $\mathfrak{R}$ of $\mathbb{Z}$ with the following properties:

(1) R together with the G-action and μ are defined over $\mathfrak{R}$.
(2) $\mu^{-1}(0)^{\theta-ss}$ and the G-bundle $\mu^{-1}(0)^{\theta-ss} \to \mu^{-1}(0)^{\theta-ss}/G$ are defined over $\mathfrak{R}$.

For an $\mathfrak{R}$-algebra $\mathfrak{R}'$, let $R_{\mathfrak{R}'}$, $G_{\mathfrak{R}'}$, $\mu_{\mathfrak{R}'}$ etc. denote the $\mathfrak{R}'$-forms of the corresponding objects. Let us write $X_{\mathfrak{R}}$ for an $\mathfrak{R}$-form of $\mu^{-1}(0)^{\theta-ss}/G$. After a finite localization of $\mathfrak{R}$, we can achieve that $X_{\mathfrak{R}}$ is a symplectic scheme over $\operatorname{Spec}(\mathfrak{R})$ with $\mathbb{C} \otimes_{\mathfrak{R}} \Gamma(X_{\mathfrak{R}}, \mathcal{O}_{X_{\mathfrak{R}}}) \xrightarrow{\sim} \mathbb{C}[X_{\mathbb{C}}]$ and $H^i(X_{\mathfrak{R}}, \mathcal{O}_{X_{\mathfrak{R}}}) = 0$ for $i > 0$.

For $\mathfrak{R}'$, we can take $\mathbb{F} := \overline{\mathbb{F}}_p$ when p is large enough. So we get a symplectic $\mathbb{F}$-variety $\mathcal{M}_0^\theta(n\delta, 1)_{\mathbb{F}}$ that is naturally identified with $T^*R_{\mathbb{F}}///_0^\theta G_{\mathbb{F}}$ as well as with $\operatorname{Spec}(\mathbb{F}) \times_{\operatorname{Spec}(\mathfrak{R})} X_{\mathfrak{R}}$. For $p \gg 0$, we get $\mathbb{F}[X_{\mathbb{F}}] = \mathbb{F} \otimes_{\mathfrak{R}} \Gamma(X_{\mathfrak{R}}, \mathcal{O}_{X_{\mathfrak{R}}})$ and $H^i(X_{\mathbb{F}}, \mathcal{O}_{X_{\mathbb{F}}}) = 0$.

We can take a finite algebraic extension of $\mathfrak{R}$ and assume that the Γ_n-module $\mathbb{C}^{2n}$ is defined over $\mathfrak{R}$. Now we claim that (again for $p \gg 0$) $\mathcal{M}_0^\theta(n\delta, 1)_{\mathbb{F}}$ is a symplectic resolution of $\mathbb{F}^{2n}/\Gamma_n$. This follows from the claim that both $\Gamma(X_{\mathfrak{R}}, \mathcal{O}_{X_{\mathfrak{R}}})$, $\mathfrak{R}[\underline{x}, \underline{y}]^{\Gamma_n}$ are $\mathfrak{R}$-forms of $\mathbb{C}[\underline{x}, \underline{y}]^{\Gamma_n}$ so they coincide after some finite localization of $\mathfrak{R}$.

3.3.3 Quantum Hamiltonian Reduction

Now suppose that R is a symplectic vector space over $\mathbb{F}$, G is a reductive group over $\mathbb{F}$ acting on R and θ is a character of G. We suppose that G acts freely on $\mu^{-1}(0)^{\theta-ss}$. We are going to define a Frobenius constant quantization $D_R///_\lambda^\theta G$ of $T^*R///_0^\theta G$, where $\lambda \in \operatorname{Hom}(G, \mathbb{F}^\times) \otimes_{\mathbb{Z}} \mathbb{F}_p \hookrightarrow \mathfrak{g}^{*G}$. The associated filtered quantization of $T^*R///_0^\theta G$

will be a quantization obtained by quantum Hamiltonian reduction, see Sect. 3.2.2. We note that for $\lambda \notin \mathrm{Hom}(G, \mathbb{F}^\times)$ we do not get a *Frobenius constant* quantization of $T^*R///_0^\theta G$.

Consider the Frobenius twist $G^{(1)}$. It is a group and the morphism $\mathrm{Fr}: G \to G^{(1)}$ is a group epimorphism. Its kernel (a.k.a. the Frobenius kernel) G_1 is a finite group scheme whose Lie algebra coincides with $\mathfrak{g}$.

The action of G on R induces an action of $G^{(1)}$ on $R^{(1)}$. The $G^{(1)}$-action on $T^*R^{(1)}$ is Hamiltonian with moment map $\mu^{(1)} : T^*R^{(1)} \to \mathfrak{g}^{(1)*}$ induced by μ. Consider the sheaf $D_R///_\lambda^\theta G_1$ (a subquotient of D_R) on $T^*R^{(1)\theta-ss}$. One can show, see [7, Section 3.6], that it is supported on $\left(\mu^{(1)}\right)^{-1}(0)$, here we use that $\lambda \in \mathrm{Hom}(G, \mathbb{F}^\times) \otimes_\mathbb{Z} \mathbb{F}_p$. Moreover, it is a $G^{(1)}$-equivariant Azumaya algebra on $\left(\mu^{(1)}\right)^{-1}(0)$. The descent of this algebra to $(T^*R///_0^\theta G)^{(1)} = T^*R^{(1)}///_0^\theta G^{(1)}$ is an Azumaya algebra with a filtration induced from that on D_R. We have a natural homomorphism $\mathrm{gr}(D_R///_\lambda^\theta G_1) \to \mathrm{Fr}_* \mathcal{O}_{T^*R///_0^\theta G_1}$. To show that it is an isomorphism one uses that the action of G_1 is free (that yields the required cohomology vanishing). This isomorphism implies $\mathrm{gr}(D_R///_\lambda^\theta G) \xrightarrow{\sim} \mathrm{Fr}_* \mathcal{O}_{T^*R///_0^\theta G}$. So $D_R///_\lambda^\theta G$ is indeed a Frobenius constant quantization.

Note that if $\lambda \notin \mathrm{Hom}(G, \mathbb{F}^\times) \otimes_\mathbb{Z} \mathbb{F}_p$, then $D_R///_\lambda^\theta G_1$ is supported on a nonzero fiber of $\mu^{(1)}$, see [7, Section 3.6] for details, and so $D_R///_\lambda^\theta G$ is no longer a Frobenius constant quantization of $X///_0^\theta G$.

4 Existence and classification of Procesi bundles

In this section we construct and classify Procesi bundles on $X = \mathcal{M}^\theta(n\delta, \epsilon_0)$ and also prove Theorem 3.14.

In Sect. 4.1 we construct a Procesi bundle on X. The case $n = 1$ is relatively easy, it was done in [38]. For $n > 1$, we follow [11]. A key step here is to construct a special Frobenius constant quantization of $X_\mathbb{F}$, where $\mathbb{F}$ is an algebraically closed field of large enough positive characteristic. This quantization provides a suitable version of derived McKay equivalence and using this equivalence we can produce a Procesi bundle over $\mathbb{F}$. Then we lift it to characteristic 0.

In Sect. 4.2 we prove that Symplectic reflection algebras satisfy PBW property and, in some sense, the family of SRA $H_{t,c}$ is universal with this property. The proof is based on computing relevant graded components in the Hochschild cohomology of $SV\#\Gamma$.

Theorem 3.14 is proved in Sect. 4.3. Using the Procesi bundle, we show that each algebra $D(R)///_\lambda G$ is isomorphic to some $eH_{1,c}e$. Then the task is to show that the correspondence between the parameters λ and the parameters c is as in Theorem 3.14. We first do this for $n = 1$. Then we reduce the case of $n > 1$ to $n = 1$ by studying completions of the algebras involved. This allows to show that the map between the parameters is conjugate to that in Theorem 3.14 up to a conjugation under an action of the group $W \times \mathbb{Z}/2\mathbb{Z}$, where W is the Weyl group of the finite part of the quiver

Q. But from Sect. 3.2.5 we know that this action lifts to an action on the universal reduction $D(R)///G$ by automorphisms. This completes the proof of Theorem 3.14.

Then, in Sect. 4.4, we classify Procesi bundles. Namely, we show that, when $n > 1$, there are $2|W|$ different Procesi bundles on X. For this, we use Theorem 3.14 to produce this number of bundles. And then we use techniques used in the proof to show that the number cannot exceed $2|W|$. Further, we show that each X carries a distinguished Procesi bundle.

4.1 Construction of Procesi Bundles

4.1.1 Baby Case: $n = 1$

In this case it is easy to construct a vector bundle of required rank on X. Namely, for $i = 0, \dots, r$, let U_i be the G-module $\mathbb{C}^{\delta_i}$ and let $\mathcal{U}_i$ be the corresponding vector bundle on X. We set $\mathcal{P} := \bigoplus_{i=0}^{r} \mathcal{U}_i^{\delta_i}$. It follows from results of Kapranov and Vasserot, [38], that this bundle satisfies the axioms of a Procesi bundle.

4.1.2 Procesi Bundles and Derived McKay Equivalence

Before we proceed to constructing Procesi bundles in general, let us explain their connection to derived Mckay equivalences, i.e., equivalences $D^b(\operatorname{Coh} X) \xrightarrow{\sim} D^b(\mathbb{K}[V_n]\#\Gamma_n)$, here $\mathbb{K}$ stands for the base field.

Proposition 4.1 *Let $\mathcal{P}$ be a Procesi bundle on X. Then the functor $R\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{P}, \bullet)$ is a derived equivalence $D^b(\operatorname{Coh} X) \to D^b(\mathbb{K}[V_n]\#\Gamma\text{-mod})$.*

The proof is based on the following more general result (Calabi-Yau trick) of (in this form) Bezrukavnikov and Kaledin.

Proposition 4.2 ([11, Proposition 2.2]). *Let X be a smooth variety, projective over an affine variety, with trivial canonical class. Furthermore, let $\mathcal{A}$ be an Azumaya algebra over X such that $\Gamma(\mathcal{A})$ has finite homological dimension and $H^i(X, \mathcal{A}) = 0$ for $i > 0$. Then the functor $R\Gamma : D^b(\operatorname{Coh}(X, \mathcal{A})) \to D^b(\Gamma(\mathcal{A})\text{-mod})$ is an equivalence.*

Proposition 4.1 follows from Proposition 4.2 with $\mathcal{A} = \mathcal{E}nd(\mathcal{P})$.

Now suppose that we have a derived equivalence $\iota : D^b(\operatorname{Coh}(X)) \xrightarrow{\sim} D^b(\mathbb{K}[V]\#\Gamma_n\text{-mod})$. Assume $\mathcal{P}' := \iota^{-1}(\mathbb{K}[V]\#\Gamma_n)$ is a vector bundle. Then $\operatorname{End}_{\mathcal{O}_X}(\mathcal{P}') = \mathbb{K}[V]\#\Gamma$ and $\operatorname{Ext}^i(\mathcal{P}', \mathcal{P}') = 0$ for $i > 0$. So $\mathcal{P}'$ is, basically, a Procesi bundle (it also needs to be $\mathbb{K}^\times$-equivariant, but we will see below that this always can be achieved). In fact, this is roughly, how the construction of a Procesi bundle will work, although it is more involved and technical.

4.1.3 Quantization of X

Here and in Sect. 4.1.4 everything is going to be over an algebraically closed field $\mathbb{F}$ of characteristic $p \gg 0$. The first step in the construction of a Procesi bundle is to produce a Frobenius constant quantization of X with special properties.

Proposition 4.3 *There is a Frobenius constant quantization $\mathcal{D}$ of X such that $\Gamma(\mathcal{D}) = \mathbf{A}(V_n)^{\Gamma_n}$ (an isomorphism of filtered algebras over $\mathbb{F}[X^{(1)}] = \mathbb{F}[V_n^{(1)}]^{\Gamma_n}$).*

Note that this proposition can be thought as a special case of the characteristic p version of Theorem 3.14. Here $\Gamma(\mathcal{D})$ is an analog of $D(R)///_\lambda G$ (indeed, the latter is the algebra of global sections of some filtered quantization of $X_{\mathbb{C}}$, see Proposition 3.13), while $\mathbf{A}(V_n)^{\Gamma_n}$ is the characteristic p analog of $eH_{1,0}e$.

In fact, the following is true.

Lemma 4.4 *Theorem 3.14 (for $c = 0$) implies Proposition 4.3.*

Proof First, let us see that we get an isomorphism $\Gamma(\mathcal{D}) \cong \mathbf{A}(V_n)^{\Gamma_n}$ of filtered algebras that is the identity on the associated graded algebra. Set $\mathcal{D} := D(R)///_\lambda^\theta G$, where λ is the parameter corresponding to $c = 0$.

The algebra $\mathbf{A}(V_{n,\mathbb{C}})^{\Gamma_n}$ is finitely generated and so an isomorphism in Theorem 3.14 is defined over some finitely generated subring $\mathfrak{R}$ of $\mathbb{C}$. We can enlarge $\mathfrak{R}$ and assume that we are in the situation described in Sect. 3.3.2. We can form filtered quantizations $\mathcal{D}'_{\mathbb{C}}, \mathcal{D}'_{\mathfrak{R}}, \mathcal{D}'_{\mathbb{F}}$ of $X_{\mathbb{C}}, X_{\mathfrak{R}}, X$. Both $\mathcal{D}'_{\mathbb{C}}, \mathcal{D}'_{\mathbb{F}}$ are obtained as suitable completions of base changes of $\mathcal{D}'_{\mathfrak{R}}$ (completions are necessary because of our condition on the filtration in the definition of a filtered quantization, see 2.2.4). In particular, $D(R_{\mathbb{C}})///_\lambda G_{\mathbb{C}} = (\Gamma(\mathcal{D}_{\mathbb{C}}) =)\mathbb{C} \otimes_{\mathfrak{R}} \Gamma(\mathcal{D}'_{\mathfrak{R}})$, while $\Gamma(\mathcal{D}) = (\Gamma(\mathcal{D}'_{\mathbb{F}}) =) \mathbb{F} \otimes_{\mathfrak{R}} \Gamma(\mathcal{D}'_{\mathfrak{R}})$.

So we can reduce an isomorphism from Theorem 3.14 (for $c = 0$) mod $p \gg 0$ and get an isomorphism $\Gamma(\mathcal{D}) \cong \mathbf{A}(V_n)^{\Gamma_n}$. What remains to show is that this isomorphism is $\mathbb{F}[V_n^{(1)}]^{\Gamma_n}$-linear. The first step here is to show that $\mathbb{F}[V_n^{(1)}]^{\Gamma_n}$ is the center of $\mathbf{A}(V_n)^{\Gamma_n}$. It is enough to check that $\mathbb{F}[V_n^{(1)}]^{\Gamma_n}$ coincides with the center of the Poisson algebra $\mathbb{F}[V_n]^{\Gamma_n}$. Here we just note that the Poisson center of $\mathbb{F}[V_n]^{\Gamma_n}$ is finite and birational over $\mathbb{F}[V_n^{(1)}]^{\Gamma_n}$ and use that the latter algebra is normal. So the isomorphism $\Gamma(\mathcal{D}) \cong \mathbf{A}(V_n)^{\Gamma_n}$ induces an automorphism of $\mathbb{F}[V_n^{(1)}]^{\Gamma_n}$. This isomorphism preserves the filtration and is trivial on the level of associated graded algebras.

The second step is to show that the algebra $\mathbb{F}[V_n^{(1)}]^{\Gamma_n}$ has no nontrivial automorphisms φ with such properties. Let us define a derivation ψ of $\mathbb{F}[V_n^{(1)}]^{\Gamma_n}$ that should be thought as $\ln \varphi$. The degrees of generators of $\mathbb{F}[V_n^{(1)}]^{\Gamma_n}$ are bounded from above for all $p \gg 0$ and so are degrees of relations between them. Observe that it is only enough to define a derivation on generators and it will be well-defined as long as it sends all relations to 0. Now to construct ψ we note that $\varphi - 1$ decreases degrees, and hence $\psi := \ln \varphi$ makes sense as long as p is sufficiently large. The derivation ψ lifts to $\mathbb{F}[V_n^{(1)}]$ because the quotient morphism $V_n^{(1)} \to V_n^{(1)}/\Gamma_n$ is ramified in codimension bigger than 1. Since it decreases degrees, we see that ψ has the form ∂_v for some $v \in \mathbb{F}^{2n(1)}$. But, if $\Gamma_1 \neq \{1\}$, the vector v cannot be Γ_n-equivariant and so ∂_v

does not preserve $\mathbb{F}[V_n^{(1)}]^{\Gamma_n}$. When $\Gamma_1 = \{1\}$, there is a Γ_n-invariant vector. However, in this case we can modify our construction: consider the reflection representation $\mathfrak{h}$ of $\mathfrak{S}_n$ instead of the permutation representation $\mathbb{C}^n$. We need to replace R with $\mathfrak{sl}_n \oplus \mathbb{C}^n$. Theorem 3.14 gets modified accordingly. □

However, the easiest way to prove Theorem 3.14 is by using Procesi bundles (at least for non-cyclic Γ_1 or general c, the case $c = 0$ may be easier). So we need some roundabout way to construct $\mathcal{D}$. In [11] the question of existence of $\mathcal{D}$ was reduced to $n = 1$. More precisely, let V^{sr} denote the set of all $v \in V_n$ such that $\dim V^{\Gamma_v} > 2$. Let us write $X_1 := \rho^{-1}(V_n^{sr}/\Gamma_n)$. This is an open subset in X with $\operatorname{codim}_X X \setminus X_1 > 1$. First, Bezrukavnikov and Kaledin produce a Frobenius constant quantization $\mathcal{D}_1$ of X_1 with $\Gamma(\mathcal{D}_1) = \mathbf{A}(V_n)^{\Gamma_n}$. This requires the existence of such a quantization in the case when $n = 1$. The latter case can be handled using Theorem 3.14 proved in this case by Holland (that can be alternatively proved using the existence of a Procesi bundle in the case $n = 1$). When $\mathcal{D}_1$ is constructed, Bezrukavnikov and Kaledin use the inequality $\operatorname{codim} X \setminus X_1 > 1$ to show that $\mathcal{D}_1$ uniquely extends to a Frobenius constant quantization $\mathcal{D}$ of X, automatically with $\Gamma(\mathcal{D}) = \mathbf{A}(V_n)^{\Gamma_n}$.

4.1.4 Construction of a Procesi Bundle: Characteristic *p*

Let $\mathcal{D}$ be as in the previous subsection. We will produce a Procesi bundle on $X^{(1)}$ starting from $\mathcal{D}$. Since $X^{(1)} \cong X$ (an isomorphism of $\mathbb{F}$-varieties), this will automatically establish a Procesi bundle on X. The isomorphism $X^{(1)} \cong X$ follows from the observation that X is defined over $\mathbb{F}_p$ and Fr is an isomorphism of $\mathbb{F}$ fixing $\mathbb{F}_p$.

By Proposition 4.2, we have a derived equivalence $D^b(\operatorname{Coh}(X^{(1)}, \mathcal{D})) \xrightarrow{\sim} D^b(\mathbf{A}(V_n)^{\Gamma_n}\text{-mod})$. Also we have an abelian equivalence $\mathbf{A}(V_n)^{\Gamma_n}\text{-mod} \xrightarrow{\sim} \mathbf{A}(V_n)\#\Gamma_n\text{-mod} = \mathbf{A}(V_n)\text{-mod}^{\Gamma_n}$. Composing the two equivalences, we get

$$D^b(\operatorname{Coh}(X, \mathcal{D})) \xrightarrow{\sim} D^b(\mathbf{A}(V_n)\text{-mod}\,\#\Gamma_n), \tag{6}$$

while what we need is a derived McKay equivalence

$$D^b(\operatorname{Coh} X^{(1)}) \xrightarrow{\sim} D^b(\mathbb{F}[V_n^{(1)}]\text{-mod}\,\#\Gamma_n). \tag{7}$$

Recall that $\mathcal{D}$ is an Azumaya algebra on X, while $\mathbf{A}(V_n)$ is a Γ_n-equivariant Azumaya algebra on $V_n^{(1)}$. If we had a splitting and a Γ_n-equivariant splitting, respectively, we would get (7) from (6). However, this is obviously not the case: $\mathbf{A}(V_n)$ admits no splitting at all.

This can be fixed by passing to completions at 0. Namely, let $X^{(1)\wedge_0}$ denote the formal neighborhood of $(\rho^{(1)})^{-1}(0)$ in $X^{(1)}$. It was checked in [11, Section 6.3] that the restriction of $\mathcal{D}$ to $X^{(1)\wedge_0}$ splits. Also it was checked that the restriction of $\mathbf{A}(V_n)$ to the formal neighborhood of 0 in $\mathbb{F}^{2n(1)\wedge_0}$ admits a Γ_n-equivariant splitting. So, we get an equivalence

$$\iota : D^b(\mathrm{Coh}(X^{(1)\wedge_0})) \xrightarrow{\sim} D^b(\mathbb{F}[V_n^{(1)}]^{\wedge_0}\#\Gamma_n\text{-mod})$$

that makes the following diagram commutative (all arrows are equivalences of triangulated categories and all arrows but $R\Gamma$ come from abelian equivalences):

$$\begin{array}{ccccc}
D^b(\mathrm{Coh}(X^{(1)\wedge_0}, \mathcal{D})) & \xrightarrow{R\Gamma} & D^b(\mathbf{A}(V_n)^{\wedge_0\Gamma_n}\text{-mod}) & \longrightarrow & D^b(\mathbf{A}(V_n)^{\wedge_0}\#\Gamma_n\text{-mod}) \\
\uparrow{\scriptstyle \mathcal{B}^*\otimes\bullet} & & & & \uparrow \\
D^b(\mathrm{Coh}(X^{(1)\wedge_0})) & & \longrightarrow & & D^b(\mathbb{F}[V_n^{(1)}]^{\wedge_0}\#\Gamma_n\text{-mod})
\end{array}$$

Here $\mathcal{B}$ denotes a splitting bundle for the restriction of $\mathcal{D}$ to $X^{(1)\wedge_0}$.

Set $\mathcal{P}' := \iota^{-1}(\mathbb{F}[V_n^{(1)}]^{\wedge_0}\#\Gamma_n)$. We claim that $\mathcal{P}'$ is a vector bundle on $X^{(1)\wedge_0}$. Indeed, the image of $\mathbb{F}[V_n^{(1)}]^{\wedge_0}\#\Gamma_n$ in $\mathbf{A}(V_n)^{\wedge_0\Gamma_n}$-mod is a projective generator and so is a direct summand in the sum of several copies of $\mathbf{A}(V_n)^{\wedge_0\Gamma_n}$. But $R\Gamma^{-1}(\mathbf{A}(V_n)^{\wedge_0\Gamma_n}) = \mathcal{B}^*$. So $\mathcal{P}'$ is a direct summand in a vector bundle (the sum of several copies of $\mathcal{B}^*$) and hence is a vector bundle itself.

So we get a vector bundle $\mathcal{P}'$ on $X^{(1)\wedge_0}$ that satisfies $\mathrm{End}(\mathcal{P}') \cong \mathbb{F}[V_n^{(1)}]^{\wedge_0}\#\Gamma_n$, $\mathrm{Ext}^i(\mathcal{P}', \mathcal{P}') = 0$ for $i > 0$. The latter vanishing implies that $\mathcal{P}'$ is equivariant with respect to the $\mathbb{F}^\times$-action on $X^{(1)\wedge_0}$, see [61]. From here it follows that $\mathcal{P}'$ can be extended to $X^{(1)}$ (this is because $\mathbb{F}^\times$ contracts $X^{(1)}$ to the zero fiber, see [11, Section 2.3]). Moreover, we can modify the equivariant structure on $\mathcal{P}'$ and achieve that the isomorphism $\mathrm{End}(\mathcal{P}') \cong \mathbb{F}[V_n^{(1)}]^{\wedge_0}\#\Gamma_n$ is $\mathbb{F}^\times$-equivariant, see [46, Section 3.1]. It follows that $\mathcal{P}$ is a Procesi bundle.

4.1.5 Construction of a Procesi Bundle: Lifting to Characteristic 0

Recall the $\mathfrak{R}$-scheme $X_{\mathfrak{R}}$ from Sect. 3.3.2. We may assume $\mathfrak{R}$ is regular. Taking an algebraic extension of $\mathfrak{R}$, we get a maximal ideal $\mathfrak{m}$ such that there is a Procesi bundle $\mathcal{P}_{\mathbb{F}}$ on $X_{\mathbb{F}}$, where $\mathbb{F}$ is an algebraic closure of $\mathbb{F}_0 := \mathfrak{R}/\mathfrak{m}$. We may assume that $\mathcal{P}_{\mathbb{F}}$ is defined over $\mathbb{F}_0$, let $\mathcal{P}_{\mathbb{F}_0}$ be the corresponding form. Let $\mathfrak{R}^\wedge$ be the $\mathfrak{m}$-adic completion of $\mathfrak{R}$. Since $\mathrm{Ext}^i(\mathcal{P}_{\mathbb{F}_0}, \mathcal{P}_{\mathbb{F}_0}) = 0$ for $i = 1, 2$, we see that $\mathcal{P}_{\mathbb{F}_0}$ uniquely deforms to a $\mathbb{G}_m$-equivariant vector bundle on the formal neighborhood of $X_{\mathbb{F}_0}$ in $X_{\mathfrak{R}^\wedge}$ (see [11, Section 2.3]).

Let us show that the $\mathbb{G}_m$-finite part of $\mathrm{End}(\mathcal{P}_{\mathfrak{R}^\wedge})$ is $\mathfrak{R}^\wedge[V_n]\#\Gamma_n$. Consider the formal neighborhood Z of $X^{reg}_{\mathbb{F}_0}$ in $X^{reg}_{\mathfrak{R}^\wedge}$. Note that $\mathrm{Ext}^1(\mathcal{P}_{\mathbb{F}_0}|_{X_0^{reg}}, \mathcal{P}_{\mathbb{F}_0}|_{X_0^{reg}}) = 0$, see, for example, [12, Appendix]. So the restriction of $\mathcal{P}_{\mathfrak{R}^\wedge}$ to Z coincides with $\eta_*\mathcal{O}_{(\mathfrak{R}^{\wedge 2n})^{reg}}$, where η denotes the quotient morphism $\mathfrak{R}^{\wedge 2n} \to \mathfrak{R}^{\wedge 2n}/\Gamma_n$. This implies the claim about endomorphisms.

Since $\mathcal{P}_{\mathfrak{R}^\wedge}$ is $\mathbb{G}_m$-equivariant and the $\mathbb{G}_m$-action is contracting, it extends from a formal neighborhood of $X_{\mathbb{F}_0}$ in $X_{\mathfrak{R}}$ to $X_{\mathfrak{R}^\wedge}$. So we get a Procesi bundle on X_K, where $K = \mathrm{Frac}(\mathfrak{R}^\wedge)$. But being a finite extension of the p-adic field, K embeds into $\mathbb{C}$ and so we get a Procesi bundle on X.

4.2 Symplectic Reflection Algebras

4.2.1 Flatness and Universality

Let V be a symplectic vector space with form Ω and $\Gamma \subset \mathrm{Sp}(V)$ be a finite group of symplectomorphisms. We write S for the set of symplectic reflections in Γ, it is a union of conjugacy classes: $S = S_0 \sqcup S_1 \sqcup \ldots \sqcup S_r$. We pick independent variables $t, c_0, \ldots, c_r$.

Recall the universal Symplectic reflection algebra $\mathbf{H}$, the quotient of $T(V)\#\Gamma[t, c_0, \ldots, c_r]$ by the relations (3). Let us write $\mathfrak{c}_{univ}$ for the vector space with basis $t, c_0, \ldots, c_r$ so that $\mathbf{H}$ is a graded $S(\mathfrak{c}_{univ})$-algebra.

Theorem 4.5 *The algebra* $\mathbf{H}$ *is a free graded* $S(\mathfrak{c}_{univ})$*-module. Moreover, assume that* Γ *is symplectically irreducible. Then* $\mathbf{H}$ *is universal with this property in the following sense. Let* $\mathfrak{c}'$ *be a vector space and* $\mathbf{H}'$ *be a graded* $S(\mathfrak{c}')$*-algebra (with* $\deg \mathfrak{c}' = 2$*) that is a free graded* $S(\mathfrak{c}')$*-module and* $\mathbf{H}'/(\mathfrak{c}') = S(V)\#\Gamma$*. Then there is a unique linear map* $\nu : \mathfrak{c}_{univ} \to \mathfrak{c}'$ *and unique isomorphism* $S(\mathfrak{c}') \otimes_{S(\mathfrak{c}_{univ})} \mathbf{H} \xrightarrow{\sim} \mathbf{H}'$ *of graded* $S(\mathfrak{c}')$*-algebras that induces the identity isomorphism of* $S(V)\#\Gamma_n$*.*

When $\Gamma_1 \neq \{1\}$, then the action of the group Γ_n on $V_n = \mathbb{C}^{2n}$ is symplectically irreducible. When $\Gamma_1 = \{1\}$, the module $\mathbb{C}^{2n}$ over Γ_n is not symplectically irreducible, so we replace $\mathbb{C}^{2n}$ with $V_n = \mathfrak{h} \oplus \mathfrak{h}^*$, where $\mathfrak{h}$ is the reflection representation of $\mathfrak{S}_n$. Note that we did the same in Sect. 4.1.3.

4.2.2 Hochschild Cohomology

Before we prove this theorem we will need to get some information about Hochshild cohomology of $S(V)\#\Gamma$. We need this because the Hochschild cohomology controls deformations of $S(V)\#\Gamma$.

Let A be a graded algebra. We want to describe graded deformations of A. The Hochschild cohomology group $\mathrm{HH}^i(A)$ inherits the grading from A, let $\mathrm{HH}^i(A)^j$ denote the jth graded component. The general deformation theory implies the following.

Lemma 4.6 *Assume that* $\dim \mathrm{HH}^2(A)^{-2} < \infty$ *and* $\mathrm{HH}^i(A)^j = 0$ *for* $i + j < 0$*. Set* $P_{univ} := (HH^2(A)^{-2})^*$*. Then there is a free graded* $S(P_{univ})$*-algebra* $\mathcal{A}_{univ}$ *(with* $\deg P_{univ} = 2$*) such that* $\mathcal{A}_{univ}/(P_{univ}) = A$ *that is a universal graded deformation of* A *in the same sense as in Theorem 4.5.*

What we are going to do is to compute the relevant graded components of $\mathrm{HH}^\bullet(SV\#\Gamma)$. The vanishing result is easy and the computation of P_{univ} is more subtle.

First, we use the fact that $\mathrm{HH}^i(A, M) = \mathrm{Ext}^i_{A\otimes A^{opp}}(A, M)$ (where M is an A-bimodule) to see that

$$\mathrm{HH}^i(S(V)\#\Gamma, S(V)\#\Gamma) = \mathrm{HH}^i(S(V), S(V)\#\Gamma)^\Gamma. \tag{8}$$

We have a Γ-action on $\mathrm{HH}^i(S(V), S(V)\#\Gamma)$ because both $S(V)$-bimodules $S(V)$, $S(V)\#\Gamma$ are Γ-equivariant. We have $S(V)\#\Gamma = \bigoplus_{\gamma\in\Gamma} S(V)\gamma$ of $S(V)$-bimodules, where $S(V)\gamma$ is identified with $S(V)$ as a left $S(V)$-module and the right action is given by $f \cdot a := f\gamma(a)$.

Let us compute $\mathrm{HH}^i(S(V), S(V)\gamma)$ in degrees we are interested in: $j < -i$ and also $j = -2$ for $i = 2$. We have $\gamma = \mathrm{diag}(\gamma_1, \ldots, \gamma_n)$, where we view γ_i as elements of cyclic groups acting on $\mathbb{C}$. Then we have an isomorphism of bigraded spaces

$$\bigoplus_i \mathrm{HH}^i(S(V), S(V)\gamma) \cong \bigotimes_{\ell=1}^n \bigoplus_i \mathrm{HH}^i(\mathbb{C}[x], \mathbb{C}[x]\gamma_\ell). \tag{9}$$

For an arbitrary γ_ℓ, we have $\mathrm{HH}^i(\mathbb{C}[x], \mathbb{C}[x]\gamma_\ell) = 0$ when $i > 1$. When $\gamma_\ell = 1$, we have $\mathrm{HH}^0(\mathbb{C}[x], \mathbb{C}[x]) = \mathbb{C}[x]$ and $\mathrm{HH}^1(\mathbb{C}[x], \mathbb{C}[x]) = \mathbb{C}[x]\{1\}$, where $\{1\}$ indicates the grading shift by 1 so that $\mathrm{HH}^1(\mathbb{C}[x], \mathbb{C}[x])$ is a free module generated in degree -1. When $\gamma_\ell \neq 1$, then $\mathrm{HH}^0(\mathbb{C}[x], \mathbb{C}[x]\gamma_\ell) = 0$ and $\mathrm{HH}^1(\mathbb{C}[x], \mathbb{C}[x]) = \mathbb{C}\{1\}$.

This computation easily implies that $\mathrm{HH}^i(S(V), S(V)\#\Gamma)^j = 0$ when $i + j < 0$. Now let explain how to compute $\left(\mathrm{HH}^2(S(V), S(V)\#\Gamma)^{-2}\right)^\Gamma$. If $\mathrm{HH}^2(S(V), S(V)\gamma)^{-2} \neq 0$, then either $\gamma = 1$ or γ is a symplectic reflection. When $\gamma = 1$, then $\mathrm{HH}^2(S(V), S(V)\gamma)^{-2} = \bigwedge^2 V$. When γ is a symplectic reflection, then $\mathrm{HH}^2(S(V), S(V)\gamma)^{-2} = \mathbb{C}$. An element $\gamma_1 \in \Gamma$ maps $S(V)\gamma$ to $S(V)(\gamma_1\gamma\gamma_1^{-1})$. The action of Γ on $\mathrm{HH}^2(S(V), S(V)\gamma)^{-2} = \bigwedge^2 V$ is a natural one. When γ is a symplectic reflection, then the action of $Z_\Gamma(\gamma)$ on $\mathrm{HH}^2(S(V), S(V)\gamma)^{-2} = \mathbb{C}$ is trivial. From here we deduce that

$$\dim \mathrm{HH}^2(S(V)\#\Gamma, S(V)\#\Gamma)^{-2} = r + 2,$$

as claimed.

4.2.3 Proof of Theorem 4.5

Let us write $\mathbf{H}_{univ}$ for the universal deformation, we need to prove that $\mathbf{H}_{univ} \xrightarrow{\sim} \mathbf{H}$.

First of all, note that degree 0 and 1 components of $\mathbf{H}_{univ}$ are the same as in $S(V)\#\Gamma$. So we have natural embeddings $\Gamma, V \hookrightarrow \mathbf{H}_{univ}$. It is easy to see that $\mathfrak{c}_{univ}, V, \Gamma$ generate $\mathbf{H}_{univ}$. This gives rise to an epimorphism $S(\mathfrak{c}_{univ}) \otimes T(V)\#\Gamma \twoheadrightarrow \mathbf{H}_{univ}$. Further, for $u, v \in V \subset \mathbf{H}_{univ}$, we have $[u, v] \in (\mathfrak{c}_{univ})$. The degree 2 of $(\mathfrak{c}_{univ})$ is $\mathfrak{c}_{univ} \otimes \mathbb{C}\Gamma$. So we get $[u, v] = \kappa(u, v)$ in $\mathbf{H}_{univ}$, where κ is a map $\bigwedge^2 V \to \mathfrak{c}_{univ} \otimes \mathbb{C}\Gamma$. A computation done in [21, Section 2] shows that, since $\mathbf{H}_{univ}$ is free over $S(\mathfrak{c}_{univ})$, we get

$$\kappa = t\Omega + \sum_{i=0}^{r} c_i \sum_{s\in S_i} \Omega_s(u, v)s.$$

This completes the proof of Theorem 4.5.

4.3 Proof of the Isomorphism Theorem

We will prove an isomorphism of $e\mathbf{H}e$ and the universal Hamiltonian reduction $\mathbf{A} := \mathbf{A}_\hbar(T^*R)///G$, where $\mathbf{A}_\hbar(T^*R)$ is the Rees algebra of $D(R)$ (with modified grading so that $\deg T^*R = 1$, $\deg \hbar = 2$). Here we take $R := R(Q, n\delta, \epsilon_0)$ for $n > 1$ and $R := R(Q, \delta, 0)$ for $n = 1$. In the case when $n > 1$, we take $G := \mathrm{GL}(n\delta)$. For $n = 1$, for G, we take the quotient of $\mathrm{GL}(\delta)$ by the one-dimensional central subgroup of constant elements.

Both $e\mathbf{H}e$, $\mathbf{A}$ are graded algebras. The algebra $e\mathbf{H}e$ is over $S(\mathfrak{c}_{univ})$ with $\deg \mathfrak{c}_{univ} = 2$. The algebra $\mathbf{A}$ is over $S(\mathfrak{c}_{red})$, where $\mathfrak{c}_{red} := \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \oplus \mathbb{C}\hbar$. We will prove that there is a graded algebra isomorphism $e\mathbf{H}e \xrightarrow{\sim} \mathbf{A}$ that maps $\mathfrak{c}_{univ}$ to $\mathfrak{c}_{red}$ and induces the identity automorphism $e\mathbf{H}e/(\mathfrak{c}_{univ}) = \mathbb{C}[V_n]^{\Gamma_n} = \mathbf{A}/(\mathfrak{c}_{red})$. Further, we will explain why the corresponding isomorphism $\mathfrak{c}_{univ} \cong \mathfrak{c}_{red}$ maps $\hbar$ to t and gives (5) on the hyperplanes $t = 1$ and $\hbar = 1$. In other words, the isomorphism $\nu : \mathfrak{c}_{univ} \to \mathfrak{c}_{red}$ is the inverse of the following map

$$\hbar \mapsto t, \quad \epsilon_i \mapsto \frac{1}{|\Gamma_1|}\mathrm{tr}_{N_i} \tilde{\mathbf{c}}, i \neq 0, \quad \epsilon_0 \mapsto \frac{1}{|\Gamma_1|}\mathrm{tr}_{N_0} \tilde{\mathbf{c}} - \frac{1}{2}(c_0 + t), \qquad (n > 1)$$
$$\hbar \mapsto t, \quad \epsilon_i \mapsto \frac{1}{|\Gamma_1|}\mathrm{tr}_{N_i} \tilde{\mathbf{c}}, i \neq 0, \quad \epsilon_0 \mapsto \frac{1}{|\Gamma_1|}\mathrm{tr}_{N_0} \tilde{\mathbf{c}} - t, \qquad (n = 1) \tag{10}$$

Here the notation is as follows. We write $\tilde{\mathbf{c}} := t + \sum_{i=1}^r c_i \sum_{\gamma\in S_i^0} \gamma$ and $\epsilon_i \in \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is specified by $\mathrm{tr}_i\, \epsilon_j := \delta_{ij}$.

4.3.1 Application of a Procesi Bundle

An isomorphism $e\mathbf{H}e \cong \mathbf{A}$ is produced as follows. The algebra $\mathbf{H}$ has better universality properties than $e\mathbf{H}e$ does[1]. We will produce a graded $S(\mathfrak{c}_{red})$-algebra $\tilde{\mathbf{A}}$ deforming $\mathbb{C}[V_n]\#\Gamma_n$ with $e\tilde{\mathbf{A}}e = \mathbf{A}$. This will give rise to a linear map $\nu : \mathfrak{c}_{univ} \to \mathfrak{c}_{red}$ and to an isomorphism $S(\mathfrak{c}_{red}) \otimes_{S(\mathfrak{c}_{univ})} \mathbf{H} \cong \tilde{\mathbf{A}}$ and hence also to an isomorphism $S(\mathfrak{c}_{red}) \otimes_{S(\mathfrak{c}_{univ})} e\mathbf{H}e \cong \mathbf{A}$. The algebra $\tilde{\mathbf{A}}$ will be constructed from a Procesi bundle $\mathcal{P}$ on $X = \mathcal{M}^\theta(n\delta, \epsilon_0)$.

[1] After this survey was written, I have proved that $e\mathbf{H}e$ is a universal graded deformation of $\mathbb{C}[V_n]^{\Gamma_n}$ compatible with the Poisson bracket in a suitable sense, which can be used to prove the isomorphism theorem without appealing to Procesi bundles, see [42, Section 3] for details.

First, let us produce a sheaf version of $\mathbf{A}$. Consider the variety $\widetilde{X} := T^*R///^{\theta}G$, this is a deformation of X over $\mathfrak{g}^{*G}$. Then we consider its formal quantization obtained by Hamiltonian reduction, the sheaf $\widetilde{\mathcal{D}}_{\hbar} := \mathbf{A}_{\hbar}(T^*R)^{\wedge_{\hbar}}///^{\theta}G$. The algebra $\mathbf{A}$ coincides with the $\mathbb{C}^{\times}$-finite part of $\Gamma(\widetilde{\mathcal{D}}_{\hbar})$. Now let us take a Procesi bundle $\mathcal{P}$ on X. Since $\operatorname{Ext}^i(\mathcal{P},\mathcal{P}) = 0$, the bundle $\mathcal{P}$ deforms to a unique $\mathbb{C}^{\times}$-equivariant vector bundle on the formal neighborhood of X in $\widetilde{X}$. But the $\mathbb{C}^{\times}$-action contracts $\widetilde{X}$ to X. So $\mathcal{P}$ extends to a unique $\mathbb{C}^{\times}$-equivariant bundle $\widetilde{\mathcal{P}}$ on $\widetilde{X}$. The extension $\widetilde{\mathcal{P}}$ again satisfies the Ext-vanishing conditions and so further extends to a unique $\mathbb{C}^{\times}$-equivariant right $\widetilde{\mathcal{D}}_{\hbar}$-module $\widetilde{\mathcal{P}}_{\hbar}$.

Consider the endomorphism algebra $\operatorname{End}_{\widetilde{\mathcal{D}}_{\hbar}^{opp}}(\widetilde{\mathcal{P}}_{\hbar})$. Modulo $(\mathfrak{c}_{red})$, this algebra coincides with $\operatorname{End}_{\mathcal{O}_X}(\mathcal{P}) = \mathbb{C}[V_n]\#\Gamma_n$. Let $\tilde{\mathbf{A}}$ be the $\mathbb{C}^{\times}$-finite part of $\operatorname{End}_{\widetilde{\mathcal{D}}_{\hbar}^{opp}}(\widetilde{\mathcal{P}}_{\hbar})$. It is the endomorphism algebra of the right $\widetilde{\mathcal{D}}_{\hbar,fin}$-module $\widetilde{\mathcal{P}}_{\hbar,fin}$. The algebra $\tilde{\mathbf{A}}$ is a graded $S(\mathfrak{c}_{red})$-algebra with $\tilde{\mathbf{A}}/(\mathfrak{c}_{red}) = \mathbb{C}[V_n]\#\Gamma_n$, where $\mathfrak{c}_{red}$ lives in degree 2. We conclude that there is a unique map $\nu_{\mathcal{P}} : \mathfrak{c}_{univ} \to \mathfrak{c}_{red}$ with $\tilde{\mathbf{A}} \cong S(\mathfrak{c}_{red}) \otimes_{S(\mathfrak{c}_{univ})} \mathbf{H}$. Then, automatically, we have

$$\mathbf{A}(= e\tilde{\mathbf{A}}e) \cong S(\mathfrak{c}_{red}) \otimes_{S(\mathfrak{c}_{univ})} e\mathbf{H}e. \tag{11}$$

We will study the linear maps $\nu : \mathfrak{c}_{univ} \to \mathfrak{c}_{red}$ such that (11) holds. We will see that

(a) any such ν is an isomorphism,
(b) that there are $|W|$ options for ν when $n = 1$ and $2|W|$ options else,
(c) and that one can choose ν as in (10).

(c) will complete the proof of Theorem 3.14, while (b) will be used to classify the Procesi bundles.

First of all, let us point out that $\nu(t) = \hbar$. Indeed, the Poisson bracket on $\mathbb{C}[\mathcal{M}_0^0(n\delta, \epsilon_0)]$ induced by the deformation $\mathbf{A}$ equals $\hbar\{\cdot,\cdot\}$, where $\{\cdot,\cdot\}$ is the standard bracket given by the Hamiltonian reduction (more precisely, if we specialize to $(\hbar', \lambda) \in \mathbb{C} \oplus \mathfrak{g}^{*G}$, then the bracket induced by the corresponding filtered deformation is $\hbar'\{\cdot,\cdot\}$). Similarly, the bracket on $\mathbb{C}[V_n]^{\Gamma_n}$ induced by $e\mathbf{H}e$ coincides with $t\{\cdot,\cdot\}$, see Example 2.4. Since the isomorphism $\mathcal{M}_0^0(n\delta, \epsilon_0) \cong V_n/\Gamma_n$ is Poisson, the equality $\nu(t) = \hbar$ follows.

4.3.2 Case $n = 1$

We start by proving (a)–(c) for $n = 1$.

Let us prove (c). First of all, recall that X can be constructed as the moduli space of the $\mathbb{C}[x, y]\#\Gamma_1$-modules isomorphic to $\mathbb{C}\Gamma_1$ as Γ_1-modules that admit a cyclic vector. The universal bundle on X is a Procesi bundle. Moreover, from [17, Section 8], it follows that $\widetilde{X}$ is the moduli space of the $\mathbf{H}/(t)$-modules isomorphic to $\mathbb{C}\Gamma_1$ and admitting a cyclic vector. The corresponding isomorphism $\mathfrak{c}_{red}/\mathbb{C}\hbar \cong \mathfrak{c}_{univ}/\mathbb{C}t$ is induced from ν.

To show that ν then is given by (10) we consider the loci of parameters λ and c where the homological dimensions of $A_{1,\lambda} := \mathbf{A}(T^*R)///_\lambda G$, $eH_{1,c}e$ are infinite. Both are given by the union of hyperplanes of the form $\lambda \cdot \beta = 0$, where β runs over the set of the roots of $Q \setminus \{0\}$ (when we speak of the parameter λ for the algebra $eH_{1,c}e$ we mean the parameter computed in Theorem 3.14). The claim for $eH_{1,c}e$ follows from [17, Theorem 0.4], and that on $A_{1,\lambda}$ then follows from [45, Section 5] (from an isomorphism of $A_{1,\lambda}$ with a central reduction of a suitable W-algebra) or from [14].

The same considerations as in the previous paragraph imply (a). To prove (b) one now needs to describe the group $\mathfrak{A}$ of the automorphisms of $\mathbf{A}(\cong e\mathbf{H}e)$ satisfying the following:

- they preserve the grading,
- they preserve $\mathfrak{c}_{red}$ as a subset of $\mathbf{A}$,
- they are the identity modulo $\mathfrak{c}_{red}$.

We have a natural homomorphism $\mathfrak{A} \to \mathrm{GL}(\mathfrak{c}_{red})$ that is easily seen to be injective. From the isomorphism with a W-algebra mentioned above, one sees that $W \subset \mathfrak{A}$ (recall that the W-action on $\mathfrak{g}^{*G}$ was described in Sect. 3.1.11). With some more work, see [45, Proposition 6.4.5], one shows that actually $W = \mathfrak{A}$. This implies (b).

4.3.3 Completions

The case of a general n is reduced to $n = 1$ using suitable completions of the algebras $\mathbf{A}, \mathbf{H}$. Let us explain what completions we use as well as general results on their structure.

First, let us describe completions of algebras of the form $\mathbf{A} := \mathbf{A}_\hbar(V)///G$, where V is a symplectic vector space and G is a reductive group acting on V by symplectomorphisms. Let $b \in V///_0G$. The point b defines a maximal ideal $\mathfrak{m} \subset \mathbf{A}$. So we can form the b-adic completion $\mathbf{A}^{\wedge_b} := \varprojlim_{n\to+\infty} \mathbf{A}/\mathfrak{m}^n$. Let $v \in V$ be a point with closed G-orbit mapping to b. Let us write $\mathbf{A}_\hbar(V)^{\wedge_{Gv}}$ for the completion of $\mathbf{A}_\hbar(V)$ with respect to the ideal of Gv. Then it is easy to see that $\mathbf{A}^{\wedge_b} \cong \mathbf{A}_\hbar(V)^{\wedge_{Gv}}///G$. The algebra $\mathbf{A}_\hbar(V)^{\wedge_{Gv}}$ can be described using a suitable version of the slice theorem. More precisely, it follows, for example, from [19, Section 4] that the formal neighborhood $V^{\wedge_{Gv}}$ is equivariantly symplectomorphic to the neighborhood of the base G/K in $(T^*G \times U)///_0K$, where $K := G_v$, $U := (T_vGv)^\perp/T_vGv$. This statement quantizes: $\mathbf{A}_\hbar(V)^{\wedge_{Gv}} \cong (D_\hbar(G) \otimes_{\mathbb{C}[\hbar]} \mathbf{A}_\hbar(U))///_0K$, this can be proved similarly to [47, Theorem 2.3.1]. From here one deduces that

$$\mathbf{A}^{\wedge_b} \cong \mathbb{C}[[\mathfrak{g}^{*G}]]\widehat{\otimes}_{\mathbb{C}[[\mathfrak{k}^{*K}]]} \left(\mathbf{A}_\hbar(U)^{\wedge_0}///K\right),$$

where a homomorphism $\mathbb{C}[[\mathfrak{k}^{*K}]] \to \mathbb{C}[[\mathfrak{g}^{*G}]]$ is induced from the restriction map $\mathfrak{g}^{*G} \to \mathfrak{k}^{*K}$.

On the other hand, take a symplectic vector space V' and a finite subgroup $\Gamma \subset \mathrm{Sp}(V)$. From these data we can form the symplectic reflection algebra $\mathbf{H}$. Pick $b \in$

V'/Γ. We can produce the completion $\mathbf{H}^{\wedge_b}$: the point b defines a natural maximal ideal in $\mathbb{C}[V']\#\Gamma$, we take its preimage in $\mathbf{H}$ and complete with respect to that preimage. The algebra $\mathbf{H}^{\wedge_b}$ can also be described in terms of a "smaller" algebra of the same type, [41, Theorem 1.2.1]. More precisely, let $\underline{\Gamma}$ be the stabilizer corresponding to b and let $\underline{\mathbf{H}}$ stand for the SRA corresponding to the pair $(\underline{\Gamma}, V')$, an algebra over $S(\mathfrak{c}_{univ})$. Then $\mathbf{H}^{\wedge_b} \cong Z(\Gamma, \underline{\Gamma}, \underline{\mathbf{H}}^{\wedge_0})$, where $Z(\Gamma, \underline{\Gamma}, \bullet)$ is the centralizer algebra from [6, Section 3.2], it is isomorphic to $\mathrm{Mat}_{|\Gamma/\underline{\Gamma}|}(\bullet)$. A consequence we need is that $e\mathbf{H}^{\wedge_b}e \cong \underline{e}\underline{\mathbf{H}}^{\wedge_0}\underline{e}$. The algebra $\underline{\mathbf{H}}$ can be described as follows. Let us write V^+ for a unique $\underline{\Gamma}$-stable complement to $V'^{\underline{\Gamma}}$ in V'. Consider the SRA $\underline{\mathbf{H}}^+$ over $S(\underline{\mathfrak{c}}_{univ})$, where $\underline{\mathfrak{c}}_{univ}$ is the parameter space for $\underline{\Gamma}$. The inclusion $\underline{\Gamma} \hookrightarrow \Gamma$ gives rise to a natural map $\underline{\mathfrak{c}}_{univ} \to \mathfrak{c}_{univ}$. Then $\underline{\mathbf{H}} = \mathbf{A}_t(V'^{\Gamma}) \otimes_{\mathbb{C}[t]} (S(\mathfrak{c}_{univ}) \otimes_{S(\underline{\mathfrak{c}}_{univ})} \underline{\mathbf{H}}^+)$.

4.3.4 Completions at Leaves of Codimension 2

We are going to use the completions of $\mathbf{A}$ and $e\mathbf{H}e$ at points lying in the codimension 2 symplectic leaves. Recall from Sect. 3.1.11 that when $n > 1$ and $\Gamma_1 \neq \{1\}$, we have two such leaves. One corresponds to $\underline{\Gamma} = \Gamma_1 \subset \Gamma_n$, the other to $\mathfrak{S}_2 \subset \Gamma_n$. Let $\underline{\mathbf{H}}^{1+}, \underline{\mathbf{H}}^{2+}$ be the corresponding SRA's. The corresponding parameter spaces are $\underline{\mathfrak{c}}^1_{univ} = \mathrm{Span}(c_1, \ldots, c_r, t)$ and $\underline{\mathfrak{c}}^2_{univ} = \mathrm{Span}(c_0, t)$. When $\Gamma_1 = \{1\}$, we have just one leaf of codimension 2, it corresponds to $\mathfrak{S}_2$.

Now let us describe the completions on the Hamiltonian reduction side. Let v^1, v^2 be elements from closed G-orbits in $\mu^{-1}(0) \in T^*R$ whose images b^1, b^2 in $\mathcal{M}^0_0(n\delta, \epsilon_0)$, V_n/Γ_n lie in the two leaves. We can take the points v^1, v^2 as follows. We have a natural embedding $\mu_1^{-1}(0)^n \hookrightarrow \mu^{-1}(0)$ from the proof of Proposition 3.10. Take pairwise different elements $v_1, \ldots, v_n \in \mu^{-1}(0)$ with closed $\mathrm{GL}(\delta)$-orbits. Then we can take $v^1 = (v_1, \ldots, v_{n-1}, 0) \in T^*R(n\delta, 0) \subset T^*R$ and $v^2 = (v_1, \ldots, v_{n-2}, v_{n-1}, v_{n-1})$.

Let us describe the completion $\mathbf{A}^{\wedge_{b^1}}$. We have $K_1 (= G_{v^1}) = (\mathbb{C}^\times)^{n-1} \times \mathrm{GL}(\delta)$. So the space $\mathfrak{k}_1^{*K_1}$ coincides $\mathbb{C}^{n-1} \oplus \mathbb{C}^{Q_0}$. The restriction map $\mathbb{C}^{Q_0} = \mathfrak{g}^{*G} \to \mathfrak{k}_1^{*K_1} = \mathbb{C}^{n-1} \oplus \mathbb{C}^{Q_0}$ sends λ to $(\lambda\cdot\delta, \ldots, \lambda\cdot\delta, \lambda)$. The symplectic part U of the normal space $T^*R/T_{v^1}Gv^1$ splits into the direct sum of the trivial module $\mathbb{C}^{2(n-1)}$, of the $(\mathbb{C}^\times)^{n-1}$-module $(T^*\mathbb{C})^{\oplus n-1}$, and of the $\mathrm{GL}(\delta)$-module $T^*R(\delta, \epsilon_0)$. So

$$\mathbf{A}_\hbar(U)/\!/\!/K_1 \cong \mathbb{C}[z_1, \ldots, z_{n-1}] \otimes \mathbf{A}_\hbar(\mathbb{C}^{2(n-1)}) \otimes_{\mathbb{C}[\hbar]} \mathbf{A}_\hbar(T^*R(\delta, \epsilon_0))/\!/\!/\,\mathrm{GL}(\delta),$$

where $z_1, \ldots, z_{n-1}$ are homogeneous elements of degree 2, the images of the natural basis in $\mathrm{Lie}(\mathbb{C}^{\times(n-1)})$ under the comoment map.

Let us write $\overline{\mathrm{GL}}(\delta)$ for the quotient of $\mathrm{GL}(\delta)$ by the one-dimensional torus of constant elements. Set $\mathfrak{g}_0^{*G} := \mathfrak{g}^{*G}/\mathbb{C}\delta$, clearly, $\mathfrak{g}_0^{*G} = \overline{\mathfrak{gl}}(\delta)^{*\overline{\mathrm{GL}}(\delta)}$. Set $\mathbf{A}^1 := \mathbf{A}_\hbar(T^*R(\delta, 0))/\!/\!/\overline{\mathrm{GL}}(\delta)$. It is easy to see that $\mathbf{A}_\hbar(T^*R(\delta, \epsilon_0))/\!/\!/\,\mathrm{GL}(\delta) = \mathbb{C}[\mathfrak{g}^{*G}] \otimes_{\mathbb{C}[\mathfrak{g}_0^{*G}]} \mathbf{A}^1$. From here and the description of the map $\mathfrak{k}_1^{*K_1} \to \mathfrak{g}^{*G}$ given above, we deduce that

$$\mathbb{C}[\mathfrak{g}^{*G}] \otimes_{\mathbb{C}[\mathfrak{k}_1^{*K_1}]} \mathbf{A}_\hbar(U)///K_1 \cong \mathbf{A}_\hbar(\mathbb{C}^{2n-2}) \otimes_{\mathbb{C}[\hbar]} (\mathbb{C}[\mathfrak{g}^{*G}] \otimes_{\mathbb{C}[\mathfrak{g}_0^{*G}]} \mathbf{A}^1).$$

It follows that

$$\mathbf{A}^{\wedge_{b^1}} \cong \mathbf{A}_\hbar^{\wedge_0}(\mathbb{C}^{2n-2})\widehat{\otimes}_{\mathbb{C}[[\hbar]]}(\mathbb{C}[[\mathfrak{c}_{red}^*]]\widehat{\otimes}_{\mathbb{C}[[\mathfrak{c}_{red}^{1*}]]}\mathbf{A}^{1\wedge_0}), \tag{12}$$

where we write $\mathfrak{c}_{red}^1$ for $\{\lambda \in \mathbb{C}^{Q_0} | \lambda \cdot \delta = 0\} \oplus \mathbb{C}\hbar$.

Let us now deal with $\mathbf{A}^{\wedge_{b^2}}$. We have $K_2 (= G_{v^2}) = (\mathbb{C}^\times)^{n-2} \times \mathrm{GL}(2)$. The map $\mathfrak{g}^{*G} \to \mathfrak{k}_2^{*K_2}$ sends λ to the $n-1$-tuple with equal coordinates $\lambda \cdot \delta$. The symplectic part U^2 of the normal space $T^*R/T_{v^2}Gv^2$ is the sum of the trivial module $\mathbb{C}^{2(n-1)}$, the $(\mathbb{C}^\times)^{n-2}$-module $(T^*\mathbb{C})^{\oplus 2}$ and the GL(2)-module $T^*(\mathfrak{sl}_2 \oplus \mathbb{C}^2)$. Let $\mathfrak{c}_{red}^2$ denote the span of $\sum_{i \in Q_0} \delta_i \epsilon_i$ and $\hbar$. Set $\mathbf{A}^2 := \mathbf{A}_\hbar(T^*(\mathfrak{sl}_2 \oplus \mathbb{C}^2))///\,\mathrm{GL}(2)$, we can view it as an algebra over $S(\mathfrak{c}_{red}^2)$ (where a natural generator of $\mathfrak{gl}_2/[\mathfrak{gl}_2, \mathfrak{gl}_2]$ corresponds to $\sum_{i \in Q_0} \delta_i \epsilon_i$). As above, we have

$$\mathbb{C}[\mathfrak{g}^{*G}] \otimes_{\mathbb{C}[\mathfrak{k}_2^{*K_2}]} \mathbf{A}_\hbar(U^2)///K_2 \cong S(\mathfrak{c}_{red}) \otimes_{S(\mathfrak{c}_{red}^2)} (\mathbf{A}_\hbar(\mathbb{C}^{2n-2}) \otimes_{\mathbb{C}[\hbar]} \mathbf{A}^2)$$

and we get the following description of $\mathbf{A}^{\wedge_{b^2}}$:

$$\mathbf{A}^{\wedge_{b^2}} \cong \mathbf{A}_\hbar^{\wedge_0}(\mathbb{C}^{2n-2})\widehat{\otimes}_{\mathbb{C}[[\hbar]]}(\mathbb{C}[[\mathfrak{c}_{red}^*]]\widehat{\otimes}_{\mathbb{C}[[\mathfrak{c}_{red}^{2*}]]}\mathbf{A}^{2\wedge_0}). \tag{13}$$

4.3.5 Reduction to $n = 1$

Using (12) we see that (11) yields an isomorphism of completions $\mathbf{A}^{\wedge_{b^1}} \cong e^1\mathbf{H}^{\wedge_{b^1}}e^1$ and hence an isomorphism

$$\begin{aligned}&\mathbf{A}_\hbar^{\wedge_0}(\mathbb{C}^{2(n-1)})\widehat{\otimes}_{\mathbb{C}[[\hbar]]}(\mathbb{C}[[\mathfrak{c}_{red}^*]]\widehat{\otimes}_{\mathbb{C}[[\mathfrak{c}_{red}^{1*}]]}\mathbf{A}^{1\wedge_0}) \cong \\ &\mathbf{A}_\hbar^{\wedge_0}(\mathbb{C}^{2(n-1)})\widehat{\otimes}_{\mathbb{C}[[\hbar]]}(\mathbb{C}[[\mathfrak{c}_{red}^*]] \otimes_{\mathbb{C}[[\mathfrak{c}_{univ}^*]]} \mathbb{C}[[\mathfrak{c}_{univ}^*]] \otimes_{\mathbb{C}[[\mathfrak{c}_{univ}^{1*}]]} e^1\mathbf{H}^{1\wedge_0}e^1).\end{aligned}$$

It was checked in [45, Section 6.5] that this isomorphism restricts to

$$S(\mathfrak{c}_{red}) \otimes_{S(\mathfrak{c}_{red}^1)} \mathbf{A}^1 \cong S(\mathfrak{c}_{red}) \otimes_{S(\mathfrak{c}_{univ}^1)} e^1\mathbf{H}^1e^1$$

that preserves the grading and is the identity modulo $(\mathfrak{c}_{red})$. From here it is easy to deduce that ν maps $\mathfrak{c}_{univ}^1$ to $\mathfrak{c}_{red}^1$ and restricts to one of W-conjugates of the map in (10) for $n = 1$.

Let us proceed to the second leaf. Similarly to Sect. 4.3.2, one can show that $\mathbf{A}_\hbar(T^*(\mathfrak{sl}_2 \oplus \mathbb{C}^2))///\,\mathrm{GL}(2) \cong e^2\mathbf{H}^2e^2$, where the isomorphism sends the element $\sum_{i \in Q_0} \delta_i \epsilon_i$ to $\pm(c_0 + t)/2$. It follows that ν maps $\mathfrak{c}_{univ}^2$ to $\mathfrak{c}_{red}^2$ and induces one of two maps in the previous sentence. It follows that ν is an isomorphism that is $W \times \mathbb{Z}/2\mathbb{Z}$-conjugate to the map given by (10) for $n > 1$. Since $W \times \mathbb{Z}/2\mathbb{Z}$-action comes from automorphisms, that preserve the grading, map $\mathfrak{c}_{red}$ to $\mathfrak{c}_{red}$, and are the identity modulo $(\mathfrak{c}_{red})$ (see Sect. 3.2.5), claims (b) and (c) follow. This completes the proof of Theorem 3.14.

4.4 Classification of Procesi bundles

Here we are going to prove that the number of different Procesi bundles on X equals $2|W|$ for $n > 1$ and $|W|$ for $n = 1$. Throughout the section we only consider normalized Procesi bundles.

4.4.1 Upper Bound

Recall that a Procesi bundle $\mathcal{P}$ on X defines a linear isomorphism $\nu_{\mathcal{P}} : \mathfrak{c}_{univ} \to \mathfrak{c}_{red}$. We claim that if $\nu_{\mathcal{P}^1} = \nu_{\mathcal{P}^2}$, then $\mathcal{P}^1 \cong \mathcal{P}^2$. Indeed, we have

$$\Gamma(\tilde{\mathcal{P}}^1_{\hbar,fin}) = \mathrm{End}(\tilde{\mathcal{P}}^1_{\hbar,fin})e \cong \mathrm{End}(\tilde{\mathcal{P}}^2_{\hbar,fin})e = \Gamma(\tilde{\mathcal{P}}^2{}_{\hbar,fin}) \tag{14}$$

(an isomorphism of graded right $\mathbf{H}$-modules). Note that $H^1(\tilde{X}, \tilde{\mathcal{P}}^i) = 0$ because $\tilde{\mathcal{P}}^i$ is a direct summand of $\mathcal{E}nd(\tilde{\mathcal{P}}^i)$ and the latter sheaf has no higher cohomology. It follows that $\Gamma(\tilde{\mathcal{P}}^i_\hbar)/\hbar\Gamma(\tilde{\mathcal{P}}^i_\hbar) \xrightarrow{\sim} \Gamma(\tilde{\mathcal{P}}^i)$. Taking the quotient of (14) by $\hbar$, we get an isomorphism $\Gamma(\tilde{\mathcal{P}}^1) \cong \Gamma(\tilde{\mathcal{P}}^2)$ of graded $\mathbb{C}[\tilde{X}]$-modules. We claim that this implies that the vector bundles $\tilde{\mathcal{P}}^1, \tilde{\mathcal{P}}^2$ are $\mathbb{C}^\times$-equivariantly isomorphic. Indeed, consider the resolution of singularities morphism $\tilde{\rho} : \tilde{X} \to \tilde{X}_0$. This morphism is birational over any $p \in \mathfrak{c}^*_{red}$. Moreover, for a Zariski generic p, the morphism ρ_p is an isomorphism, indeed, $\mu^{-1}(p)^{\theta-ss} = \mu^{-1}(p)$. It follows that the restrictions of bundles $\tilde{\mathcal{P}}^1, \tilde{\mathcal{P}}^2$ to some Zariski open subset in $\tilde{X}$ with codimension of complement bigger than 1 are isomorphic. It follows that $\tilde{\mathcal{P}}^1 \cong \tilde{\mathcal{P}}^2$ and hence $\mathcal{P}^1 \cong \mathcal{P}^2$.

We have seen above that $\nu_{\mathcal{P}}$ can only be one of $2|W|$ (for $n > 1$) or $|W|$ (for $n = 1$) maps. This implies the upper bound on the number of Procesi bundles.

4.4.2 Lower Bound

Let us show that there are $2|W|$ different Procesi bundles in the case of $n > 1$. Recall that one can construct a Procesi bundle $\mathcal{P}_{\mathcal{D}}$ once one has a Frobenius constant quantization $\mathcal{D}$ of $X_{\mathbb{F}}$ with $\Gamma(\mathcal{D}) = \mathbf{A}(V_{n,\mathbb{F}})^{\Gamma_n}$. Note that the action of $W \times \mathbb{Z}/2\mathbb{Z}$ on $\mathbf{A}$ is defined over some algebraic extension of $\mathbb{Z}$. So, as before, it can be reduced modulo q for $q = p^\ell$, $p \gg 0$. Let $\mathcal{D}_\lambda$ be the Frobenius constant quantization obtained by Hamiltonian reduction with parameter $\lambda \in \mathbb{F}_p^{Q_0}$. The parameter λ constructed from $c = 0$ belongs to $\mathbb{F}_p^{Q_0}$. Above, we have remarked that $\Gamma(\mathcal{D}_\lambda) \cong \mathbf{A}(V_{n,\mathbb{F}})^{\Gamma_n}$. Moreover, for $q \gg 0$, the stabilizer of this parameter in $W \times \mathbb{Z}/2\mathbb{Z}$ is trivial. So we get $2|W|$ different Frobenius constant quantizations with required global sections. Procesi bundles produced by them are different as well, as was checked in [46, Section 3.3].

4.4.3 Canonical Procesi Bundle

By a canonical Procesi bundle we mean $\mathcal{P}$ such that $\nu_{\mathcal{P}}$ is as in (10). According to [46, Section 4.2], this bundle has the following property: the subbundle $\mathcal{P}^{\Gamma_{n-1}}$ coincides with the rank $n|\Gamma_1|$ bundle $\mathcal{T}$ on $X = \mathcal{M}^\theta(n\delta, \epsilon_0)$ induced by the G-module $\bigoplus_{i\in Q_0}(\mathbb{C}^{n\delta_i})^{\oplus\delta_i}$. We will write $\mathcal{P}^\theta$ for this bundle. Recall that for $w \in W \times \mathbb{Z}/2\mathbb{Z}$ we get an isomorphism $\mathcal{M}^\theta(n\delta, \epsilon_0) \cong \mathcal{M}^{w\theta}(n\delta, \epsilon_0)$ that yields the map $\mathfrak{c}_{red} = H^2(\mathcal{M}^\theta(n\delta, \epsilon_0)) \oplus \mathbb{C} \to H^2(\mathcal{M}^{w\theta}(n\delta, \epsilon_0)) \oplus \mathbb{C} = \mathfrak{c}_{red}$ equal to w. It follows that $\nu_{w_*\mathcal{P}^\theta} = w\nu$. So every other Procesi bundle on $\mathcal{M}^\theta(n\delta, \epsilon_0)$ is obtained as a push-forward of the canonical Procesi bundle $\mathcal{P}^{w\theta}$ on $\mathcal{M}^{w\theta}(n\delta, \epsilon_0)$.

Note that when $\mathcal{P}$ is a Procesi bundle, then so is $\mathcal{P}^*$. Indeed, $\operatorname{End}_{\mathcal{O}_X}(\mathcal{P}^*, \mathcal{P}^*) \cong \operatorname{End}_{\mathcal{O}_X}(\mathcal{P}, \mathcal{P})^{opp}$. The algebra $\mathbb{C}[V_n]\#\Gamma_n$ is identified with its opposite via $v \mapsto v, \gamma \mapsto \gamma^{-1}, v \in V_n^*, \gamma \in \Gamma_n$ and this gives a Procesi bundle structure on $\mathcal{P}^*$. We have $\nu_{\mathcal{P}^{\theta*}} = w_0\sigma\nu_{\mathcal{P}^*}$, where w_0 is the longest element in W and σ is the image of 1 in $\mathbb{Z}/2\mathbb{Z}$, see [46, Remark 4.4].

5 Macdonald positivity and categories $\mathcal{O}$

In this section we provide some applications of results of Sect. 4.

In Sect. 5.1, we will produce equivalences between categories $D^b(H_{1,c})$ and $D^b(\operatorname{Coh}(\mathcal{D}_\lambda))$ (over $\mathbb{C}$). Here and below in this section we write $\mathcal{D}_\lambda$ for the quantum Hamiltonian reduction (in a filtered setting, see Sect. 3.2.2 for the formal setting)

$$\mathcal{D}_\lambda := D_R /\!/\!/^\theta_\lambda G,$$

of the microlocal sheaf of differential operators D_R on T^*R. Here we consider the conical topology for the dilation action of $\mathbb{C}^\times$ on R^* (so that R is fixed). So $\mathcal{D}_\lambda$ is a sheaf in conical topology on $\mathcal{M}^\theta(n\delta, \epsilon_0)$ whose global sections algebra is $D(R)/\!/\!/_\lambda G \cong eH_{1,c}e$, this follows from Proposition 3.13 combined with Theorem 3.14.

Starting from Sect. 5.2, we will only consider the groups Γ_n with cyclic Γ_1. Here Γ_n is a complex reflection group and the corresponding algebra $H_{t,c}$ (called a Rational Cherednik algebra) in this case admits a triangular decomposition. This decomposition allows to define Verma modules and, for $t = 1$, category $\mathcal{O}$ for $H_{1,c}$ that has a so called highest weight structure. We can also define the category $\mathcal{O}$ for $\mathcal{D}_\lambda$, this will be a subcategory in $\operatorname{Coh}(\mathcal{D}_\lambda)$. We will show that the derived equivalence $D^b(H_{1,c}\text{-mod}) \cong D^b(\operatorname{Coh}(\mathcal{D}_\lambda))$ restricts to categories $\mathcal{O}$. This was used in [25] to establish [58, Conjecture 5.6] for the groups Γ_n.

In Sect. 5.3 we prove Theorem 1.3 and also its generalization to the groups Γ_n due to Bezrukavnikov and Finkelberg. The proof is based on studying the algebras $H_{0,c}$ and their Verma modules.

Finally, in Sect. 5.4 we prove an analog of the Beilinson-Bernstein localization theorem, [3], for the Rational Cherednik algebras associated to the groups Γ_n. More precisely, we answer the question when the derived equivalence $D^b(\mathrm{Coh}(\mathcal{D}_\lambda) \to D^b(H_{1,c}\text{-mod})$ restricts to an equivalence $\mathrm{Coh}(\mathcal{D}_\lambda) \to H_{1,c}\text{-mod}$.

5.1 Derived equivalence

5.1.1 Deformed Derived Mckay Correspondence

Similarly to Sect. 4.1.2, the functor $R\Gamma(\mathcal{P} \otimes_{\mathcal{O}_X} \bullet)$ defines an equivalence

$D^b(\mathrm{Coh}\, X) \xrightarrow{\sim} D^b(\mathbb{C}[V_n]\#\Gamma_n\text{-mod})$ with quasi-inverse $\mathcal{P}^* \otimes^L_{\mathbb{C}[V_n]\#\Gamma_n} \bullet$. These equivalence automatically upgrade to the categories of $\mathbb{C}^\times$-equivariant objects: $D^b(\mathrm{Coh}^{\mathbb{C}^\times} X) \cong D^b(\mathbb{C}[V_n]\#\Gamma_n\text{-mod}^{\mathbb{C}^\times})$ defined in the same way.

Now let us consider the deformation $\tilde{\mathcal{P}}_\hbar$ of $\mathcal{P}$ to a right $\mathbb{C}^\times$-equivariant $\widetilde{\mathcal{D}}_\hbar$-module. It gives a functor $\tilde{\mathcal{F}} := R\Gamma(\tilde{\mathcal{P}}_{\hbar,fin} \otimes_{\tilde{\mathcal{D}}_{\hbar,fin}} \bullet) : D^b(\mathrm{Coh}^{\mathbb{C}^\times}(\tilde{\mathcal{D}}_{\hbar,fin})) \to D^b(\mathbf{H}\text{-mod}^{\mathbb{C}^\times})$. This functor has left adjoint and right inverse $\tilde{\mathcal{G}} = \tilde{\mathcal{P}}^*_{\hbar,fin} \otimes^L_{\mathbf{H}} \bullet$. So we get the adjunction morphism $\tilde{\mathcal{G}} \circ \tilde{\mathcal{F}} \to \mathrm{id}$. One can show (see [25, Section 5] for details) that since this morphism is an isomorphism modulo $\mathfrak{c}_{univ}$, it is an isomorphism itself.

5.1.2 Specialization

The equivalence $\tilde{\mathcal{F}}$ can be specialized to a numerical parameter. In particular, we get equivalences $D^b(\mathrm{Coh}(\mathcal{D}_\lambda)) \to D^b(H_{1,c}\text{-mod})$, where λ is recovered from c as in Theorem 3.14. This is done in two steps. First, one gets a derived equivalence between $\mathrm{Coh}^{\mathbb{C}^\times}(R_{\hbar^{1/2}}(\mathcal{D}_\lambda))$ and $R_{\hbar^{1/2}}(H_{1,c})\text{-mod}^{\mathbb{C}^\times}$, the corresponding sheaf and algebra are obtained from $\widetilde{\mathcal{D}}_{\hbar,fin}, \mathbf{H}$ by base change (and the equivalence we need comes from the corresponding base change of $\tilde{\mathcal{P}}_{\hbar,fin}$). To do the second step we recall that $H_{1,c}\text{-mod}$ is the quotient $R_{\hbar^{1/2}}(H_{1,c})\text{-mod}^{\mathbb{C}^\times}$ by the full subcategory of the $\mathbb{C}[\hbar]$-torsion modules and the similar claim holds for $\mathrm{Coh}(\mathcal{D}_\lambda)$, see Lemma 2.9. It follows that $D^b(H_{1,c}\text{-mod})$ is the quotient of $D^b(R_{\hbar^{1/2}}(H_{1,c})\text{-mod}^{\mathbb{C}^\times})$ by the category of all complexes whose homology are $\mathbb{C}[\hbar]$-torsion and a similar claim holds for $\mathcal{D}_\lambda$. Since the equivalence $D^b(R_{\hbar^{1/2}}(H_{1,c})\text{-mod}^{\mathbb{C}^\times}) \cong D^b(\mathrm{Coh}^{\mathbb{C}^\times}(R_{\hbar^{1/2}}(\mathcal{D}_\lambda)))$ is $\mathbb{C}[\hbar]$-linear by the construction, they induce

$$D^b(H_{1,c}\text{-mod}) \cong D^b(\mathrm{Coh}(\mathcal{D}_\lambda)). \tag{15}$$

5.1.3 Application: Shift Equivalences

The equivalences (15) can be applied to producing a result that only concerns the symplectic reflection algebras. Namely, we say that parameters c, c' for $H_?$ have integral difference if $\lambda - \lambda' \in \mathbb{Z}^{Q_0}$ for the corresponding parameters λ. Recall that we can view $\chi \in \mathbb{Z}^{Q_0}$ as a character of G. So χ defines a line bundle on X, explicitly, $\mathcal{O}_\chi = \pi_*(\mathcal{O}_{\mu^{-1}(0)^{\theta-ss}})^{G,\chi}$. This line bundle can be quantized to a $\mathcal{D}_{\lambda+\chi}$-$\mathcal{D}_\lambda$-bimodule to be denoted by $\mathcal{D}_{\lambda,\chi}$. Explicitly,

$$\mathcal{D}_{\lambda,\chi} := \pi_*(\mathcal{D}^{ss}/\mathcal{D}^{ss}\{\Phi(x) - \langle\lambda, x\rangle\})^{G,\chi}.$$

This bundle carries a natural filtration and an isomorphism $\operatorname{gr}\mathcal{D}_{\lambda,\chi} \cong \mathcal{O}_\chi$ follows from the flatness of the moment map.

Note that there is a natural (multiplication) homomorphism $\mathcal{D}_{\lambda+\chi,\chi'} \otimes_{\mathcal{D}_{\lambda+\chi}} \mathcal{D}_{\lambda,\chi} \to \mathcal{D}_{\lambda,\chi+\chi'}$ that becomes the isomorphism $\mathcal{O}_{\chi'} \otimes \mathcal{O}_\chi \to \mathcal{O}_{\chi+\chi'}$ after passing to the associated graded. So the multiplication homomorphism itself is an isomorphism. It follows that a functor $\mathcal{D}_{\lambda,\chi} \otimes_{\mathcal{D}_\lambda} \bullet : \operatorname{Coh}(\mathcal{D}_\lambda) \to \operatorname{Coh}(\mathcal{D}_{\lambda+\chi})$ is a category equivalence. We conclude that categories $D^b(H_{1,c}\text{-mod})$ and $D^b(H_{1,c'}\text{-mod})$ are equivalent provided c, c' have integral difference[2].

5.2 *Category $\mathcal{O}$*

Starting from now on, we assume that Γ_1 is a cyclic group $\mathbb{Z}/\ell\mathbb{Z}$. Recall that in this case the space V_n (equal to $\mathbb{C}^{2n}$ when $\ell > 1$ and $\mathbb{C}^{2(n-1)}$ when $\ell = 1$) splits as $\mathfrak{h} \oplus \mathfrak{h}^*$, where $\mathfrak{h}$ is a standard reflection representation of the group Γ_n. The embeddings $\mathfrak{h}, \mathfrak{h}^* \hookrightarrow \mathbf{H}$ extend to algebra embeddings $S(\mathfrak{h}), S(\mathfrak{h}^*) \hookrightarrow \mathbf{H}$. These embeddings give rise to the *triangular decomposition* $\mathbf{H} = S(\mathfrak{h}^*) \otimes S(\mathfrak{c}_{univ})\Gamma_n \otimes S(\mathfrak{h})$. We can also consider the specialization $H_{1,c} = S(\mathfrak{h}^*) \otimes \mathbb{C}\Gamma_n \otimes S(\mathfrak{h})$ (here and below c is a numerical parameter) of this decomposition.

5.2.1 Category $\mathcal{O}$ for $H_{1,c}$

By definition, the category $\mathcal{O}$ for $H_{1,c}$ consists of all $H_{1,c}$-modules M such that

(i) $\mathfrak{h}$ acts locally nilpotently on M.
(ii) M is finitely generated over $H_{1,c}$.

Note that, modulo (i), the condition (ii) is equivalent to

(ii′) M is finitely generated over $S(\mathfrak{h}^*)$.

An example of an object in the category $\mathcal{O}$ is a Verma module constructed as follows. Pick an irreducible representation τ of Γ_n and view it as a $S(\mathfrak{h})\#\Gamma_n$-module by making $\mathfrak{h}$ act by 0. Then set $\Delta_{1,c}(\tau) := H_{1,c} \otimes_{S(\mathfrak{h})\#\Gamma_n} \tau$. As a $S(\mathfrak{h}^*)\#W$-module, $\Delta_{1,c}(\tau)$ is naturally identified with $S(\mathfrak{h}^*) \otimes \tau$ (the algebra $S(\mathfrak{h}^*)$ acts by multiplications from the left, and W acts diagonally).

[2] After this survey was written, I have established a shift equivalence for general symplectic reflection groups, [44]. The proof follows the scheme outlined in this section: Procesi bundles on symplectic resolutions are replaced with their generalizations, *Procesi sheaves* on $\mathbb{Q}$-factorial terminalizations.

The algebra $H_{1,c}$ carries an *Euler grading* given by $\deg \mathfrak{h} = -1, \deg \mathfrak{h}^* = 1$, $\deg W = 0$. This grading is internal: we have an element $h \in H_{1,c}$ with $[h, a] = da$ for $a \in H_{1,c}$ of degree d. Explicitly, the element h is given by

$$\sum_{i=1}^{m} x_i y_i + \sum_{s \in S} \frac{c(s)}{1-\lambda_s} s.$$

Here the notation is as follows. We write $y_1, \ldots, y_m$ for a basis in $\mathfrak{h}$ (of course, $m = n$ for $\ell > 1$ and $m = n - 1$ for $\ell = 1$) and $x_1, \ldots, x_m$ for the dual basis in $\mathfrak{h}^*$. By S we, as usual, denote the set of reflections in Γ_n and $c(s)$ stands for c_i if $s \in S_i$ (note that the formula for h is different from the usual formula for the Euler element, see, e.g., [6, Section 2.1], because our $c(s)$ is rescaled). Finally, λ_s is the eigenvalue of s in $\mathfrak{h}^*$ different from 1.

Using the element h, we can show that every Verma module $\Delta_{1,c}(\tau)$ has a unique simple quotient. These quotients form a complete collection of the simple objects in $\mathcal{O}$. Also one can show that every object in $\mathcal{O}$ has finite length. These claims are left as exercises to the reader.

5.2.2 Category $\mathcal{O}$ for $\mathcal{D}_\lambda$

We have a $\mathbb{C}^\times$-action on $D(R)$ induced by the $\mathbb{C}^\times$-action on R given by $t.r := t^{-1}r$. This action is Hamiltonian, the corresponding quantum comoment map $\Phi : \mathbb{C} \to D(R)$ sends 1 to the Euler vector field. The action descends to a Hamiltonian $\mathbb{C}^\times$-action on $\mathcal{D}_\lambda$ for any λ.

Consider the corresponding Hamiltonian $\mathbb{C}^\times$-action on $X = \mathcal{M}_0^\theta(n\delta, \epsilon_0)$. Recall that the resolution of singularities morphism $X \to (\mathfrak{h} \oplus \mathfrak{h}^*)/\Gamma_n$ becomes $\mathbb{C}^\times$-equivariant if we equip the target variety with the $\mathbb{C}^\times$-action induced by $t.(a, b) = (t^{-1}a, tb)$, $a \in \mathfrak{h}$, $b \in \mathfrak{h}^*$. This action has finitely many fixed points that are in a natural bijection with the irreducible representations of Γ_n, see [30, Section 5.1]. Namely, $X^{\mathbb{C}^\times}$ is in a natural bijection with $\mathcal{M}_p^0(n\delta, \epsilon_0)^{\mathbb{C}^\times}$, where $p \in \mathfrak{g}^{*G}$ is generic. Indeed, $\mathcal{M}_p^0(n\delta, \epsilon_0) = \mathcal{M}_p^\theta(n\delta, \epsilon_0)$ and the sets $\mathcal{M}_p^\theta(n\delta, \epsilon_0)^{\mathbb{C}^\times}$ are identified for all p by continuity. Let c be a parameter corresponding to p (meaning that $\nu(0, c) = (0, p)$). Then we can consider the Verma module $\Delta_{0,c}(\tau) := H_{0,c} \otimes_{S(\mathfrak{h})\#\Gamma_n} \tau$. The subalgebra $S(\mathfrak{h}^*)^{\Gamma_n}$ is easily seen to be central. Let us write $S(\mathfrak{h}^*)_+^{\Gamma_n}$ for the augmentation ideal in $S(\mathfrak{h}^*)^{\Gamma_n}$. Following [28], consider the *baby Verma module* $\underline{\Delta}_{0,c}(\tau) := \Delta_{0,c}(\tau)/S(\mathfrak{h}^*)_+^{\Gamma_n}\Delta_{0,c}(\tau) \cong S(\mathfrak{h}^*)/(S(\mathfrak{h}^*)_+^{\Gamma_n}) \otimes \tau$ (the last isomorphism is that of $S(\mathfrak{h}^*)\#\Gamma_n$-modules). This module is easily seen to be indecomposable so it has a central character that is a point of $\operatorname{Spec}(Z(H_{0,c})) = \mathcal{M}_p^0(n\delta, \epsilon_0)$. Clearly, this point is fixed by $\mathbb{C}^\times$ and this defines a map $\operatorname{Irr}(\Gamma_n) \to \mathcal{M}_p^0(n\delta, \epsilon_0)^{\mathbb{C}^\times}$, $\tau \mapsto z_\tau$, that was shown to be a bijection in [30].

Fix some $p \in \mathfrak{g}^{*G}$. Consider the attracting locus $Y_p \subset \mathcal{M}_p^\theta(n\delta, \epsilon_0)$ for the $\mathbb{C}^\times$-action. Since this action has finitely many fixed points, we see that Y_p is a lagrangian subvariety with irreducible components indexed by $\mathrm{Irr}(\Gamma_n)$. Namely, to $\tau \in \mathrm{Irr}(\Gamma_n)$ we assign the attracting locus $Y_p(\tau) := \{z \in \mathcal{M}_p^\theta(n\delta, \epsilon_0) | \lim_{t\to 0} t.z = z_\tau\}$. The irreducible components of Y_p are the closures $\overline{Y_p(\tau)}$. When p is Zariski generic, the subvarietes $Y_p(\tau)$ are already closed.

By the category $\mathcal{O}^{loc}$ for $\mathcal{D}_\lambda$ we mean the full category of coherent $\mathcal{D}_\lambda$-modules that are supported on Y (see Sect. 2.3.4) and admit a $\mathbb{C}^\times$-equivariant structure compatible with the $\mathbb{C}^\times$-action on $\mathcal{D}_\lambda$. Such categories were systematically studied in [15]. In particular, it was shown that all modules in $\mathcal{O}^{loc}$ have finite length and are indexed by $\mathcal{M}_0^\theta(n\delta, \epsilon_0)^{\mathbb{C}^\times}$, see [15, Sections 3.3, 5.3].

5.2.3 Choice of Identification $X^{\mathbb{C}^\times} \cong \mathrm{Irr}(\Gamma_n)$

We note that despite our identification of $X^{\mathbb{C}^\times}$ with $\mathrm{Irr}(\Gamma_n)$ is natural, there are other natural choices as well. The choice we have made is good for working with the category $\mathcal{O}$. We could also consider the category $\mathcal{O}^*$, where the modules are locally nilpotent for $\mathfrak{h}^*$, not for $\mathfrak{h}$ (and are still finitely generated over $H_{1,c}$). Consequently, we need to use the opposite Hamiltonian $\mathbb{C}^\times$-action on X, $\mathcal{M}_p^0(n\delta, \epsilon_0)$ and Verma modules $\Delta_{0,c}^*(\tau) := H_{0,c} \otimes_{S(\mathfrak{h}^*)\#\Gamma_n} \tau$. Let us explain how the bijection $X^{\mathbb{C}^\times} \cong \mathrm{Irr}(\Gamma_n)$ changes.

All simple constituents of $\underline{\Delta}_{0,c}(\tau)$ are isomorphic modules of dimension $|\Gamma_n|$ (indeed, $H_{0,c}$ is the endomorphism algebra of the rank $|\Gamma_n|$ bundle $\tilde{\mathcal{P}}_p$ on $\mathcal{M}_p^0(n\delta, \epsilon_0)$). Let us denote this simple module by $L_{0,c}(\tau)$. This module is graded, the highest graded component is τ. Let us determine the lowest graded component in $L_{0,c}(\tau)$. This component coincides with the lowest graded component in $\Delta_{0,c}(\tau)$ that is the tensor product of τ with the lowest degree component in $\mathbb{C}[\mathfrak{h}]/(\mathbb{C}[\mathfrak{h}]^{\Gamma_n})_+$. It is easy to see that the latter is $\Lambda^{top}\mathfrak{h}$. Abusing the notation, we will denote $\tau \otimes \Lambda^{top}\mathfrak{h}$ by τ^t. When $\Gamma_1 = \{1\}$ we can use the standard identification of $\mathrm{Irr}(\mathfrak{S}_n)$ with the set of Young diagrams of n boxes. In this case, $\Lambda^{top}\mathfrak{h}$ is the sign representation of $\mathfrak{S}_n$ and τ^t indeed corresponds to the transposed Young diagram of τ.

The previous paragraph shows that there is an epimorphism $\Delta_{0,c}^*(\tau^t) \twoheadrightarrow L_p(\tau)$. So our new bijection sends the point $z_\tau \in X^{\mathbb{C}^\times}$ to τ^t.

We also note that the identification $X^{\mathbb{C}^\times} \cong \mathrm{Irr}(\Gamma_n)$, $\tau \mapsto z_\tau$, depends on the choice of a Procesi bundle $\mathcal{P}$ but we are not going to use this.

5.2.4 Highest Weight Structures

Let us recall the definition of a highest weight category. Let $\mathcal{C}$ be an abelian category that is equivalent to the category of modules over a finite dimensional algebra, equivalently, the category $\mathcal{C}$ has finitely many simples, enough projectives and finite dimensional Hom's (and hence every object has finite length). Let $\mathcal{T}$ denote an indexing set of the simple objects in $\mathcal{C}$, we write $L(\tau)$ for the simple object indexed by $\tau \in \mathcal{T}$ and $P(\tau)$ for its projective cover. The additional structure of a highest weight category is a partial order $\leqslant$ on $\mathcal{T}$ and a collection of so called *standard* objects $\Delta(\tau), \tau \in \mathcal{T}$, satisfying the following axioms:

(1) $\mathrm{Hom}_{\mathcal{C}}(\Delta(\tau), \Delta(\tau')) \neq 0$ implies $\tau \leqslant \tau'$,
(2) $\mathrm{End}_{\mathcal{C}}(\Delta(\tau)) = \mathbb{C}$.
(3) $P(\tau) \twoheadrightarrow \Delta(\tau)$ and the kernel admits a filtration with quotients $\Delta(\tau')$ for $\tau' > \tau$.

Remark 5.1 Let us point out that the standard objects are uniquely recovered from the partial order. Namely, consider the category $\mathcal{C}_{\leqslant\tau}$ that is the Serre span of the simples $L(\tau')$ with $\tau' \leqslant \tau$. Then $\Delta(\tau)$ is the projective cover of $L(\tau)$ in $\mathcal{C}_{\leqslant\tau}$.

Both categories $\mathcal{O}, \mathcal{O}^{loc}$ that were described above are highest weight, see [24, Sections 2.6,3.2] for $\mathcal{O}$ and [15, Section 5.3] for $\mathcal{O}^{loc}$. The standard objects $\Delta(\lambda)$ are the Verma modules. The order can be introduced as follows. Recall the element $h \in H_{1,c}$ introduced in Sect. 5.2.1. It acts on $\tau \subset \Delta(\tau)$ by $\sum_{s\in S} \frac{c(s)}{1-\lambda_s} s$. The latter element in $\mathbb{C}\Gamma_n$ is central and so acts on τ by a scalar, denote that scalar by c_τ. Then we set $\tau \leqslant \tau'$ if $c_\tau - c_{\tau'} \in \mathbb{Z}_{\geqslant 0}$.

Let us provide a formula for c_τ. We start with $\ell = 1$. Then a classical computation shows that $c_\tau = c_0 \operatorname{cont}(\tau)/2$, where the integer $\operatorname{cont}(\tau)$ is defined as follows. For the box $b \in \tau$ lying in xth column and yth row, we set $\operatorname{cont}(b) := x - y$. Then $\operatorname{cont}(\tau) := \sum_{b\in\tau} \operatorname{cont}(b)$. Now let us proceed to $\ell > 1$. In this case, the irreducible representations of Γ_n are parameterized by the ℓ-multipartitions $(\tau^{(1)}, \ldots, \tau^{(\ell)})$ of n. Define elements $\lambda_1, \ldots, \lambda_\ell$ by requiring that $\lambda_i, i = 1, \ldots, \ell - 1$, is recovered from c as in Theorem 3.14 and $\sum_{i=1}^{\ell} \lambda_i = 0$. For a box $b \in \tau^{(j)}$ set $d_c(b) := c_0 \ell \operatorname{cont}(b)/2 + \ell\lambda_j$. Then, up to a summand independent of τ, we have $c_\tau = \sum_{b\in\tau} d_c(b)$, see [58, Proposition 6.2] or [25, 2.3.5] (in both papers the notation is different from what we use).

In fact, one can take a weaker ordering on $\mathrm{Irr}(\Gamma_n)$ making $\mathcal{O}$ into a highest weight category. Namely, according to [31], for two boxes b, b' in jth and j'th diagrams respectively we say that $b \leqslant b'$ if $d_c(b) - d_c(b')$ is congruent to $j - j'$ modulo ℓ and is in $\mathbb{Z}_{\geqslant 0}$. Then $\lambda \leqslant \lambda'$ if one can order boxes $b_1, \ldots, b_n$ of λ and $b'_1, \ldots, b'_n$ of λ' in such a way that $b_i \leqslant b'_i$ for all i.

Let us proceed to the categories $\mathcal{O}^{loc}$. They are highest weight with respect to the order $\leqslant$ (we will often write $\leqslant^\theta$ to indicate the dependence on θ) defined as follows. We first define a pre-order $\leqslant'$ by setting $\tau \leqslant' \tau'$ if $z_\tau \in \overline{Y}_{\tau'}$ and then define $\leqslant$ as the transitive closure of $\leqslant'$.

Example 5.2 When $\ell = 1$ and $\theta < 0$, the bijection between the $\mathbb{C}_h^\times$-fixed points and partitions is the standard one. A combinatorial description of $\leqslant^\theta$ follows from [52, Section 4]: we have $\tau \leqslant^\theta \tau'$ if $\tau \leqslant \tau'$ as Young diagrams.

In the case when $\ell > 1$ an a priori stronger order (that automatically also makes $\mathcal{O}^{loc}$ into a highest weight category) was described by Gordon in [30, Section 7] in combinatorial terms. The standard modules are recovered from $\leqslant^\theta$ as before. Below we will see that they can be described using the deformations of the Procesi bundle.

5.2.5 Derived Equivalence

Here we are going to produce a derived equivalence $D^b(\mathcal{O}) \cong D^b(\mathcal{O}^{loc})$.

Inside $D^b(H_{1,c}\text{-mod})$ we can consider the full subcategory $D^b_{\mathcal{O}}(H_{1,c}\text{-mod})$ consisting of all complexes whose homology lie in the category $\mathcal{O}$. We then have a natural functor $D^b(\mathcal{O}) \to D^b_{\mathcal{O}}(H_{1,c}\text{-mod})$. This functor is an equivalence by [22, Proposition 4.4]. We can also consider the category $D^b_{\mathcal{O}}(\mathrm{Coh}(\mathcal{D}_\lambda))$, the functor $D^b(\mathcal{O}^{loc}) \to D^b_{\mathcal{O}}(\mathrm{Coh}(\mathcal{D}_\lambda))$ is an equivalence as well, this follows from [15, Corollary 5.13] and [16, Corollary 5.17].

The equivalence $D^b(H_{1,c}\text{-mod}) \xrightarrow{\sim} D^b(\mathrm{Coh}(\mathcal{D}_\lambda))$ is compatible with the supports in the following sense. Recall that we have two commuting $\mathbb{C}^\times$-actions. The Hamiltonian torus will be denoted by $\mathbb{C}^\times_h$, while, for the contracting torus (which is present even when Γ_1 is not cyclic), we will write $\mathbb{C}^\times_c$. Pick a closed subvariety $Y_0 \subset (\mathfrak{h} \oplus \mathfrak{h}^*)/\Gamma_n$ that is stable under the $\mathbb{C}^\times_c$-action. Consider the full subcategory $D^b_{Y_0}(H_{1,c}\text{-mod})$ in $D^b(H_{1,c})$ of all complexes with homology supported on Y_0. Set $Y := \rho^{-1}(Y_0)$, where, recall, ρ stands for the resolution of singularities morphism $\rho : X \to V_n/\Gamma_n$ and consider the subcategory $D^b_Y(\mathrm{Coh}(\mathcal{D}_\lambda)) \subset D^b(\mathrm{Coh}(\mathcal{D}_\lambda))$. Then the equivalence $D^b(\mathrm{Coh}(\mathcal{D}_\lambda)) \cong D^b(H_{1,c}\text{-mod})$ restricts to $D^b_Y(\mathrm{Coh}(\mathcal{D}_\lambda)) \cong D^b_{Y_0}(H_{1,c}\text{-mod})$.

Note that the bundle $\mathcal{P}$ on X is $(\mathbb{C}^\times)^2$-equivariant. Therefore the deformation $\tilde{\mathcal{P}}_\hbar$ is $(\mathbb{C}^\times)^2$-equivariant as well. It follows that the equivalence $D^b(\mathrm{Coh}(\mathcal{D}_\lambda)) \cong D^b(H_{1,c}\text{-mod})$ preserves complexes whose homology admit $\mathbb{C}^\times_h$-equivariant liftings. Combined with the previous paragraph, this means that we get an equivalence $D^b_{\mathcal{O}}(H_{1,c}\text{-mod}) \cong D^b_{\mathcal{O}}(\mathrm{Coh}(\mathcal{D}_\lambda))$ and hence an equivalence $D^b(\mathcal{O}) \cong D^b(\mathcal{O}^{loc})$.

This was used in [25, Section 5] to prove a conjecture of Rouquier, [58, Conjecture 5.6]. Namely, suppose that we have parameters c, c' such that the corresponding parameters λ, λ' have integral difference. Then we have an abelian equivalence $\mathrm{Coh}(\mathcal{D}_\lambda) \xrightarrow{\sim} \mathrm{Coh}(\mathcal{D}_{\lambda'})$, given by tensoring with the bimodule $\mathcal{D}_{\lambda,\lambda'-\lambda}$. This bimodule is $\mathbb{C}^\times_h$-equivariant, this follows from the construction. Also it is clear that tensoring with $\mathcal{D}_{\lambda,\lambda'-\lambda}$ preserves the supports. So we conclude that $\mathcal{O}^{loc}_\lambda \xrightarrow{\sim} \mathcal{O}^{loc}_{\lambda'}$. It follows that the categories $\mathcal{O}_c$ and $\mathcal{O}_{c'}$ are derived equivalent that was conjectured by Rouquier (in the generality of all Cherednik algebras)[3].

5.3 Macdonald positivity

Consider the $\mathbf{H}$-module $\Delta(\lambda) := \mathbf{H} \otimes_{S(\mathfrak{h})\#\Gamma_n} \lambda$. Recall the derived equivalence $D^b(\mathrm{Coh}(\widetilde{\mathcal{D}}_{\hbar,fin})) \xrightarrow{\sim} D^b(\mathbf{H}\text{-mod})$ given by

$$\mathcal{F} := \Gamma(\widetilde{\mathcal{P}}_{\hbar,fin} \otimes_{\widetilde{\mathcal{D}}_{\hbar,fin}} \bullet)$$

[3] The case of general complex reflection groups was done in [43] after this survey was written using different techniques.

and its inverse $\mathcal{G}$. It turns out that the study of the objects $\mathcal{G}(\Delta(\lambda))$ leads to the proof of the Macdonald positivity. The proof that we provide below is morally similar to but different from the original proof in [8].

5.3.1 Flatness

A key step in the proof is to establish the flatness over $\mathbb{C}[\mathfrak{h}]$ of an arbitrary Procesi bundle $\mathcal{P}$, where we view $\mathcal{P}$ ($\mathbb{C}[\mathfrak{h}]$ acts on $\mathcal{P}$ via the inclusion $\mathbb{C}[\mathfrak{h}] \hookrightarrow S(\mathfrak{h} \oplus \mathfrak{h}^*)\#\Gamma_n = \operatorname{End}_{\mathcal{O}_X}(\mathcal{P})$). This will imply that the Koszul complex

$$\mathcal{P} \leftarrow \mathfrak{h}^* \otimes \mathcal{P} \leftarrow \Lambda^2\mathfrak{h}^* \otimes \mathcal{P} \leftarrow \ldots \leftarrow \Lambda^n\mathfrak{h}^* \otimes \mathcal{P}$$

is a resolution of $\mathcal{P}/\mathfrak{h}^*\mathcal{P}$. The proof of the flatness is taken from the proof of [8, Lemma 3.7].

Note that, since Γ_n is a complex reflection group, $\mathbb{C}[\mathfrak{h}]$ is free over $\mathbb{C}[\mathfrak{h}]^{\Gamma_n}$. So it is enough to show that $\mathcal{P}$ is flat over $\mathbb{C}[\mathfrak{h}]^{\Gamma_n}$.

Let us recall how $\mathcal{P}$ was constructed, see Sect. 4.1.4 (construction of one Procesi bundle in characteristic $p \gg 0$), Sect. 4.1.5 (construction of one Procesi bundle in characteristic 0), Sect. 4.4.2 (construction of all Procesi bundles).

(1) We start with a suitable Frobenius constant quantization $\mathcal{D}$ of $X_{\mathbb{F}}$, where $\mathbb{F}$ is an algebraically closed field of characteristic 0.
(2) Then we take a splitting bundle $\mathcal{B}$ of $\mathcal{D}|_{X_{\mathbb{F}}^{(1)\wedge_0}}$.
(3) We form a bundle $\mathcal{P}'$ on $X_{\mathbb{F}}^{(1)\wedge_0}$ that is the sum of indecomposable summands of $\mathcal{S}^*$ with suitable multiplicities. Then we extend this bundle to $X_{\mathbb{F}}^{(1)}$ and get a Procesi bundle $\mathcal{P}_{\mathbb{F}}^{(1)}$ on $X_{\mathbb{F}}^{(1)}$.
(4) Since $X_{\mathbb{F}}^{(1)} \cong X_{\mathbb{F}}$ as $\mathbb{F}$-varieties, we can view $\mathcal{P}_{\mathbb{F}}^{(1)}$ as a bundle $\mathcal{P}_{\mathbb{F}}$ on X.
(5) Then we lift $\mathcal{P}_{\mathbb{F}}$ to characteristic 0.

The procedure in (5) implies that if $\mathcal{P}_{\mathbb{F}}$ is flat over $\mathbb{F}[\mathfrak{h}]^{\Gamma_n}$, then $\mathcal{P}$ is flat over $\mathbb{C}[\mathfrak{h}]^{\Gamma_n}$ (the reader is welcome to verify the technical details). Obviously, $\mathcal{P}_{\mathbb{F}}$ is flat over $\mathbb{F}[\mathfrak{h}]^{\Gamma_n}$ if and only if $\mathcal{P}_{\mathbb{F}}^{(1)}$ is flat over $\mathbb{F}[\mathfrak{h}^{(1)}]^{\Gamma_n}$. The latter is equivalent to $\mathcal{B}^*$ being flat over $\mathbb{F}[[\mathfrak{h}^{(1)}]]^{\Gamma_n}$, which, in turn, is equivalent to the claim that $\mathcal{D}$ is a flat $\mathbb{F}[\mathfrak{h}^{(1)}]^{\Gamma_n}$-module. But $\operatorname{gr}\mathcal{D} \cong \operatorname{Fr}_*^X \mathcal{O}_{X_{\mathbb{F}}}$. So it is enough to verify that $\mathcal{O}_{X_{\mathbb{F}}}$ is flat over $\mathbb{F}[\mathfrak{h}^{(1)}]^{\Gamma_n}$. Since $\mathbb{F}[\mathfrak{h}]^{\Gamma_n}$ is flat over $\mathbb{F}[\mathfrak{h}^{(1)}]^{\Gamma_n}$, we reduce to proving that $X_{\mathbb{F}}$ is flat over $\mathfrak{h}_{\mathbb{F}}/\Gamma_n$, equivalently, all fibers of $X_{\mathbb{F}} \to \mathfrak{h}_{\mathbb{F}}/\Gamma_n$ have the same dimension, equivalently, the zero fiber has dimension $\dim \mathfrak{h}$. But the zero fiber of this map is precisely the contracting variety for the Hamiltonian $\mathbb{F}^\times$-action and so is lagrangian. This completes the proof.

Similarly, $\mathcal{P}$ is flat over $\mathbb{C}[\mathfrak{h}^*]$. Also let us recall, see 4.4.3, that $\mathcal{P}^*$ can be equipped with a structure of the Procesi bundle, for which we need to convert the right $S(\mathfrak{h} \oplus \mathfrak{h}^*)\#\Gamma_n$-module into a left $S(\mathfrak{h} \oplus \mathfrak{h}^*)\#\Gamma_n$ using a natural anti-automorphism of $S(\mathfrak{h} \oplus \mathfrak{h}^*)\#\Gamma_n$. This shows that $\mathcal{P}^*$ is a flat *right* module over both $\mathbb{C}[\mathfrak{h}]$ and $\mathbb{C}[\mathfrak{h}^*]$. This is what we are going to use below.

5.3.2 Upper Triangularity

Let θ be a generic stability condition and take $X = X^\theta$. This gives rise to the partial order $\leqslant^\theta$ on the set $\mathrm{Irr}(\Gamma_n)$ described in Sect. 5.2.2. Recall that we write z_τ for the $\mathbb{C}_h^\times$-fixed point in X corresponding to τ as explained in Sect. 5.2.2. We write Y_τ for the $\mathbb{C}_h^\times$-contracting component of z_τ, a lagrangian subvariety in X^θ. Further, write e_τ for a primitive idempotent in $\mathbb{C}\Gamma_n$ corresponding to τ so that $\tau \cong (\mathbb{C}\Gamma_n)e_\tau$.

Proposition 5.3 *Let $\mathcal{P}$ be the canonical Procesi bundle on X^θ. Then the sheaf $(\mathcal{P}^*/\mathcal{P}^*\mathfrak{h})e_\tau$ is supported on $\bigcup_{\tau' \leqslant^\theta \tau} Y_{\tau'}$.*

Proof Consider the deformation $\widetilde{\mathcal{P}}^*$ of $\mathcal{P}^*$ to $\widetilde{X}$. It is flat over $\mathbb{C}[\mathfrak{g}^{*G}, \mathfrak{h}^*]$. Therefore $\widetilde{\mathcal{P}}^*/\widetilde{\mathcal{P}}^*\mathfrak{h}$ is flat over $\mathbb{C}[\mathfrak{g}^{*G}]$. It follows that $\mathrm{Supp}((\mathcal{P}^*/\mathcal{P}^*\mathfrak{h})e_\tau) \subset \overline{\mathbb{C}_c^\times \mathrm{Supp}(\mathcal{P}_p^*/\mathcal{P}_p^*\mathfrak{h})e_\tau}$ for a generic $p \in \mathfrak{g}^{*G}$. But $(\mathcal{P}_p^*/\mathcal{P}_p^*\mathfrak{h})e_\tau$ is nothing else but $e\Delta_{0,c}(\tau)$. We claim that $\mathrm{Supp}\,\Delta_{0,c}(\tau) \subset Y_{p,\tau}$. Indeed, we have shown in Sect. 5.2.2 that $\Delta_{0,c}(\tau)/S(\mathfrak{h}^*)_+^{\Gamma_n}\Delta_{0,c}(\tau)$ is supported in $z_{p,\tau}$, the point in $\mathcal{M}_p^0(n\delta, \epsilon_0)^{\mathbb{C}_h^\times}$ indexed by τ. If $\mathrm{Supp}\,\Delta_{0,c}(\tau) \not\subset Y_{p,\tau}$, then there is $\tau' \neq \tau$ with $z_{p,\tau'} \in \mathrm{Supp}\,\Delta_{0,c}(\tau)$ (because the latter is closed and contained in Y_p). The support of $\Delta_{0,c}(\tau)$ is disconnected and so the module $\Delta_{0,c}(\tau)$ is indecomposable. From here one deduces that $z_{p,\tau'}$ lies in the support of $\Delta_{0,c}(\tau)/S(\mathfrak{h}^*)_+^{\Gamma_n}\Delta_{0,c}(\tau)$, contradiction.

Now the inclusion

$$\mathrm{Supp}\left((\mathcal{P}^*/\mathcal{P}^*\mathfrak{h})e_\tau\right) \subset \bigcup_{\tau' \leqslant^\theta \tau} Y_{\tau'}$$

follows from

$$\overline{\mathbb{C}^\times Y_{p,\tau}} \cap X^\theta \subset \bigcup_{\tau' \leqslant^\theta \tau} Y_{\tau'},$$

see [8, Lemma 3.8]. □

In fact, $e\Delta_{0,c}(\tau) = \mathbb{C}[Y_{p,\tau}]$ but we do not need this fact.

5.3.3 Wreath-Macdonald Positivity

Now we are ready to prove the Macdonald positivity theorem, Theorem 1.3, and its "wreath-generalization" due to Bezrukavnikov and Finkelberg.

First of all, Proposition 5.3 implies that if the fiber of $[\mathcal{P}^*/\mathcal{P}^*\mathfrak{h}]e_\tau$ in $z_{\tau'}$ is nonzero, then $\tau' \leqslant^\theta \tau$. It follows that if τ^* is a constituent of the fiber $(\mathcal{P}^*/\mathcal{P}^*\mathfrak{h})_{z_{\tau'}}$, then $\tau \geqslant^\theta \tau'$. But since $\mathcal{P}^*$ is a flat right $\mathbb{C}[\mathfrak{h}]$-module, we see that the class of $[\mathcal{P}^*/\mathcal{P}^*\mathfrak{h}^*]_{z_{\tau'}}$ in the K_0 of bigraded Γ_n-modules coincides with that of the Koszul resolution

$$\mathcal{P}^*_{z_{\tau'}} \leftarrow \mathcal{P}^*_{z_{\tau'}} \otimes \mathfrak{h} \leftarrow \dots$$

Taking the duals, we see that if τ occurs in the class

$$\mathcal{P}_{z_{\tau'}} \otimes \sum_{i=0}^{\dim \mathfrak{h}} (-1)^i \Lambda^i \mathfrak{h}^*,$$

then $\tau' \leqslant^\theta \tau$. When $\Gamma_1 = \{1\}$, this yields (a) from Definition 1.2.

To get (b) in that definition (and its wreath-generalization), we consider $[\mathcal{P}^*/\mathcal{P}^*\mathfrak{h}^*]e_\tau$. This sheaf is supported on the union of repelling components for $\mathbb{C}^\times_\hbar$ and can have nonzero fibers only in the fixed points $z_{\tau'}$ with $z_{\tau'} \geqslant^\theta z_{\tau^t}$ meaning $\tau^t \leqslant^\theta \tau'$. In other words, if τ appears in

$$\mathcal{P}_{z_{\tau'}} \otimes \sum_{i=0}^{\dim \mathfrak{h}} (-1)^i \Lambda^i \mathfrak{h},$$

then $\tau^t \leqslant^\theta \tau'$. When $\Gamma_1 = \{1\}$, this yields (b) in Definition 1.2. (c) there follows because $\mathcal{P}$ is normalized.

5.4 Localization theorem

Let $\mathcal{P}_{1,\lambda}$ denote the the right $\mathcal{D}_\lambda$-module obtained by specializing $\widetilde{\mathcal{P}}_\hbar$. One can ask when (i.e., for which λ) the functor $\Gamma(\mathcal{P}_{1,\lambda} \otimes_{\mathcal{D}_\lambda} \bullet) : \mathcal{O}^{loc}_\lambda \to \mathcal{O}_c$ is a category equivalence. The following result answers this question.

Theorem 5.4 *Suppose that there is an order $\leqslant$ on* $\operatorname{Irr}(\Gamma_n)$ *refining $\leqslant^\theta$ and making* <u>*both*</u> $\mathcal{O}^{loc}_\lambda$, $\mathcal{O}_c$ *into highest weight categories. Then* $\Gamma : \operatorname{Coh}(\mathcal{D}_\lambda) \to H_{1,c}$ -mod, $\mathcal{O}^{loc}_\lambda \to \mathcal{O}_c$ *are equivalences of categories.*

This theorem can be viewed as an analog of the Beilinson-Bernstein localization theorem, [3], from the Lie representation theory.

(*Sketch of proof*). It is enough to prove that Γ gives an equivalence between the categories $\mathcal{O}$, see [40, Section 3.3]. So below in the proof we only deal with the categories $\mathcal{O}$.

Set $\Delta^{loc}(\lambda) := [\mathcal{P}^*_{1,\lambda}/\mathcal{P}^*_{1,\lambda}\mathfrak{h}]e_\lambda$. Further, let $\mathcal{F}$ stand for $R\Gamma(\mathcal{P}_{1,\lambda} \otimes_{\mathcal{D}_\lambda} \bullet)$. The flatness of $\mathcal{P}$ over $S(\mathfrak{h})$ from the previous subsection implies that

$$\mathcal{F}\Delta^{loc}_\lambda(\tau) = \Delta_c(\tau). \tag{16}$$

We have $\Delta^{loc}_\lambda(\tau) \in \mathcal{O}^{loc}_{\leqslant^\theta\lambda}$. The condition on the orders implies that $\Delta^{loc}_\lambda(\tau)$ is the standard object in $\mathcal{O}^{loc}_\lambda$. Now the claim of Theorem 5.4 follows from the next general claim. □

Lemma 5.5 *Let $\mathcal{C}^1, \mathcal{C}^2$ be two highest weight categories with the same indexing poset $\mathcal{T}$. Suppose that $\mathcal{F} : D^b(\mathcal{C}^1) \to D^b(\mathcal{C}^2)$ is a derived equivalence mapping $\Delta^1(\tau)$ to $\Delta^2(\tau)$ for any $\tau \in \mathcal{T}$. Then $\mathcal{F}$ is induced from an abelian equivalence $\mathcal{C}^1 \to \mathcal{C}^2$.*

Theorem 5.4 generalizes results of [26, 39] for $\Gamma_1 = \{1\}$ to the case of general cyclic Γ_1.

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Three Lectures on Algebraic Microlocal Analysis

Pierre Schapira

Abstract This is a survey talk with some historical comments. I will first explain the notions of Sato's hyperfunctions and microfunctions, at the origin of the story, and I will describe the Sato's microlocalization functor which was first motivated by problems of analysis. Then I will briefly recall the main features of the microlocal theory of sheaves with emphasize on the functor μhom which will be an essential tool in the sequel. Then, I will construct the microlocal Euler class associated with trace kernels. This construction applies in particular to constructible sheaves on real manifolds and $\mathscr{D}$-modules (or more generally, elliptic pairs) on complex manifolds. Finally, I will first recall the construction of the sheaves of holomorphic functions with temperate growth or with exponential decay. These are not sheaves on the usual topology, but ind-sheaves, or else, sheaves on the subanalytic site. I will explain how these objects appear naturally in the study of irregular holonomic $\mathscr{D}$-modules.

Keywords Microlocal sheaf theory · $\mathscr{D}$-modules · Hyperfunctions · Index theorem · Hochschild homology
MSC14F05, 35A27, 53D37

1 Lecture 1: Microlocalization of Sheaves

Abstract This first talk is a survey talk with some historical comments and I refer to [54] for a more detailed overview.

I will first explain the notions of Sato's hyperfunctions and microfunctions, at the origin of the story, and I will describe the Sato's microlocalization functor which was first motivated by problems of Analysis (see [52]). Then I will briefly recall the main

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P. Schapira (✉)
Institut de Mathématiques de Jussieu, Sorbonne Universités, UPMC Univ Paris 6, Paris, France
e-mail: pierre.schapira@imj-prg.fr
URL: http://webusers.imj-prg.fr/~pierre.schapira/

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features of the microlocal theory of sheaves of [24] with emphasize on the functor μhom which will be the main tool for the second talk.

1.1 Generalized Functions

In the sixties, people were used to work with various spaces of generalized functions constructed with the tools of functional analysis. Sato's construction of hyperfunctions in 59–60 is at the opposite of this practice: he uses purely algebraic tools and complex analysis. The importance of Sato's definition is twofold: first, it is purely algebraic (starting with the analytic object $\mathscr{O}_X$), and second it highlights the link between real and complex geometry. (See [50] and see [53] for an exposition of Sato's work.)

Consider first the case where M is an open subset of the real line $\mathbb{R}$ and let X be an open neighborhood of M in the complex line $\mathbb{C}$ satisfying $X \cap \mathbb{R} = M$. The space $\mathscr{B}(M)$ of hyperfunctions on M is given by

$$\mathscr{B}(M) = \mathscr{O}(X \setminus M)/\mathscr{O}(X).$$

It is easily proved, using the solution of the Cousin problem, that this space depends only on M, not on the choice of X, and that the correspondence $U \mapsto \mathscr{B}(U)$ (U open in M) defines a flabby sheaf $\mathscr{B}_M$ on M.

With Sato's definition, the boundary values always exist and are no more a limit in any classical sense.

Example 1.1 (i) The Dirac function at 0 is

$$\delta(0) = \frac{1}{2i\pi}\left(\frac{1}{x - i0} - \frac{1}{x + i0}\right).$$

Indeed, if φ is a C^0-function on $\mathbb{R}$ with compact support, one has

$$\varphi(0) = \lim_{\varepsilon \xrightarrow{>} 0} \frac{1}{2i\pi} \int_{\mathbb{R}} \left(\frac{\varphi(x)}{x - i\varepsilon} - \frac{\varphi(x)}{x + i\varepsilon}\right) dx.$$

(ii) The holomorphic function $\exp(1/z)$ defined on $\mathbb{C} \setminus \{0\}$ has a boundary value as a hyperfunction (supported by $\{0\}$) not as a distribution.

On a real analytic manifold M of dimension n, the sheaf $\mathscr{B}_M$ was originally defined as

$$\mathscr{B}_M = H^n_M(\mathscr{O}_X) \otimes \mathrm{or}_M$$

where X is a complexification of M and or_M is the orientation sheaf on M. Since X is oriented, Poincaré's duality gives the isomorphism $\mathrm{D}'_X(\mathbb{C}_M) \simeq \mathrm{or}_M\,[-n]$ (see (1.3) below for the definition of D'_M). On the other hand, it is shown (by Sato) that $\mathrm{R}\Gamma_M(\mathscr{O}_X)\,[n]$ is concentrated in degree 0. Hence, an equivalent definition of hyperfunctions is given by

$$\mathscr{B}_M = \mathrm{R}\mathscr{H}om_{\mathbb{C}_X}(\mathrm{D}'_X(\mathbb{C}_M), \mathscr{O}_X). \tag{1.1}$$

Let us define the notion of "boundary value" in this settings. Consider a subanalytic open subset Ω of X and denote by $\overline{\Omega}$ its closure. Assume that:

$$\begin{cases} \mathrm{D}'_X(\mathbb{C}_\Omega) \simeq \mathbb{C}_{\overline{\Omega}}, \\ M \subset \overline{\Omega}. \end{cases}$$

The morphism $\mathbb{C}_{\overline{\Omega}} \to \mathbb{C}_M$ defines by duality the morphism $\mathrm{D}'_X(\mathbb{C}_M) \to \mathrm{D}'_X(\mathbb{C}_{\overline{\Omega}}) \simeq \mathbb{C}_\Omega$. Applying the functor $\mathrm{RHom}\,(\bullet\,, \mathscr{O}_X)$, we get the boundary value morphism

$$\mathrm{b}\colon \mathscr{O}(\Omega) \to \mathscr{B}(M). \tag{1.2}$$

When considering operations on hyperfunctions such as integral transforms, one is naturally lead to consider more general sheaves of generalized functions such as $\mathrm{R}\mathscr{H}om\,(G, \mathscr{O}_X)$ where G is a constructible sheaf. We shall come back on this point later.

Similarly as in dimension one, we can represent the sheaf $\mathscr{B}_M$ by using Čech cohomology of coverings of $X \setminus M$. For example, let X be a Stein open subset of $\mathbb{C}^n$ and set $M = \mathbb{R}^n \cap X$. Denote by x the coordinates on $\mathbb{R}^n$ and by $x + iy$ the coordinates on $\mathbb{C}^n$. One can cover $\mathbb{C}^n \setminus \mathbb{R}^n$ by $n+1$ open half-spaces $V_i = \langle y, \xi_i \rangle > 0$ $(i = 1, \dots, n+1)$. For $J \subset \{1, \dots, n+1\}$ set $V_J = \bigcap_{j\in J} V_j$. Assuming $n > 1$, we have the isomorphism $H^n_M(X; \mathscr{O}_X) \simeq H^{n-1}(X \setminus M; \mathscr{O}_X)$. Therefore, setting $U_J = V_J \cap X$, one has

$$\mathscr{B}(M) \simeq \sum_{|J|=n} \mathscr{O}_X(U_J) / \sum_{|K|=n-1} \mathscr{O}_X(U_K).$$

On a real analytic manifold M, any hyperfunction $u \in \Gamma(M; \mathscr{B})$ is a (non unique) sum of boundary values of holomorphic functions defined in tubes with edge M. Such a decomposition leads to the so-called Edge of the Wedge theorem and was intensively studied in the seventies (see [4, 39]).

Then comes naturally the following problem: how to recognize the directions associated with these tubes? The answer is given by Sato's microlocalization functor.

1.2 Microlocalization

Unless otherwise specified, all manifolds are real, say of class C^∞ and $\mathbf{k}$ denotes a commutative unital ring with finite global homological dimension.

We denote by $\mathbf{k}_M$ the constant sheaf on M with stalk $\mathbf{k}$, by $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ the bounded derived category of sheaves of $\mathbf{k}$-modules on M and by $\mathsf{D}^{\mathrm{b}}_{\mathrm{cc}}(\mathbf{k}_M)$ the full triangulated subcategory of $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ consisting of cohomologically constructible objects. If M is real analytic, we denote by $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_M)$ the triangulated category of $\mathbb{R}$-constructible sheaves.

We denote by ω_M the dualizing complex on M. Then $\omega_M \simeq \mathrm{or}_M\,[\dim M]$ where or_M is the orientation sheaf and $\dim M$ the dimension of M. We shall use the duality functors

$$\mathrm{D}'_M F = \mathrm{R}\mathscr{H}om\,(F, \mathbf{k}_M), \quad \mathrm{D}_M F = \mathrm{R}\mathscr{H}om\,(F, \omega_M). \tag{1.3}$$

For a locally closed subset A of M, we denote by $\mathbf{k}_{MA}$ the sheaf which is the constant sheaf on A with stalk $\mathbf{k}$ and which is 0 on $M \setminus A$. If there is no risk of confusion, we simply denote it by $\mathbf{k}_A$.

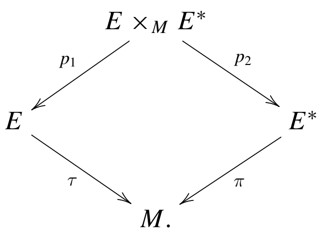
Fourier-Sato Transform

The classical Fourier transform interchanges (generalized) functions on a vector space V and (generalized) functions on the dual vector space V^*. The idea of extending this formalism to sheaves, hence to replacing an isomorphism of spaces with an equivalence of categories, seems to have appeared first in Mikio Sato's construction of microfunctions in the 70s.

Let $\tau : E \to M$ be a finite dimensional real vector bundle over a real manifold M with fiber dimension n and let $\pi : E^* \to M$ be the dual vector bundle. Denote by p_1 and p_2 the first and second projection defined on $E \times_M E^*$, and define:

$$P = \{(x, y) \in E \times_M E^*;\ \langle x, y\rangle \geq 0\},$$
$$P' = \{(x, y) \in E \times_M E^*;\ \langle x, y\rangle \leq 0\}.$$

Consider the diagram:



Denote by $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbf{k}_E)$ the full triangulated subcategory of $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_E)$ consisting of conic sheaves, that is, objects with locally constant cohomology on the orbits of $\mathbb{R}^+$.

Definition 1.2 Let $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbf{k}_E)$, $G \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbf{k}_{E^*})$. One sets:

$$\begin{aligned} F^{\wedge} &= \mathrm{R}p_{2!}(p_1^{-1}F)_{P'} \simeq \mathrm{R}p_{2*}(\mathrm{R}\Gamma_P\, p_1^{-1}F), \\ G^{\vee} &= \mathrm{R}p_{1*}(\mathrm{R}\Gamma_{P'}\, p_2^{!}G) \simeq \mathrm{R}p_{1!}(p_2^{!}G)_P. \end{aligned}$$

The main result of the theory is the following.

Theorem 1.3 *The two functors* $(\cdot)^{\wedge}$ *and* $(\cdot)^{\vee}$ *are inverse to each other, hence define an equivalence of categories* $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbf{k}_E) \simeq \mathsf{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbf{k}_{E^*})$.

Example 1.4 (i) Let γ be a closed proper convex cone in E with $M \subset \gamma$. Then:

$$(\mathbf{k}_{\gamma})^{\wedge} \simeq \mathbf{k}_{\mathrm{Int}\gamma^{\circ}}.$$

Here γ° is the polar cone to γ, a closed convex cone in E^* and $\mathrm{Int}\gamma^{\circ}$ denotes its interior.
(ii) Let γ be an open convex cone in E. Then:

$$(\mathbf{k}_{\gamma})^{\wedge} \simeq \mathbf{k}_{\gamma^{\circ a}} \otimes \mathrm{or}_{E^*/M}[-n].$$

Here $\lambda^a = -\lambda$, the image of λ by the antipodal map.
(iii) Let $(x) = (x', x'')$ be coordinates on $\mathbb{R}^n$ with $(x') = (x_1, \ldots, x_p)$ and $(x'') = (x_{p+1}, \ldots, x_n)$. Denote by $(y) = (y', y'')$ the dual coordinates on $(\mathbb{R}^n)^*$. Set

$$\gamma = \{x; x'^2 - x''^2 \geq 0\}, \quad \lambda = \{y; y'^2 - y''^2 \leq 0\}.$$

Then $(\mathbf{k}_{\gamma})^{\wedge} \simeq \mathbf{k}_{\lambda}[-p]$. (See [26].)

Specialization

Let $\iota\colon N \hookrightarrow M$ be the embedding of a closed submanifold N of M. Denote by $\tau_M\colon T_NM \to N$ the normal bundle to N.

If F is a sheaf on M, its restriction to N, denoted $F|_N$, may be viewed as a global object, namely the direct image by τ_M of a sheaf $\nu_M F$ on T_NM, called the specialization of F along N. Intuitively, T_NM is the set of light rays issued from N in M and the germ of $\nu_N F$ at a normal vector $(x; v) \in T_NM$ is the germ at x of the restriction of F along the light ray v.

One constructs a new manifold $\widetilde{M}_N$, called the normal deformation of M along N, together with the maps

$$\begin{array}{ccccc} T_NM & \overset{s}{\hookrightarrow} & \widetilde{M}_N & \overset{j}{\hookleftarrow} & \Omega \\ {\scriptstyle \tau_M}\downarrow & & \downarrow{\scriptstyle p} & \swarrow{\scriptstyle \tilde{p}} & \\ N & \underset{\iota}{\hookrightarrow} & M & & \end{array}, \qquad \begin{array}{l} t\colon \widetilde{M}_N \to \mathbb{R}, \\ \Omega = t^{-1}(\mathbb{R}_{>0}), \\ T_NM \simeq t^{-1}(0). \end{array} \tag{1.4}$$

Locally, after choosing a local coordinate system (x', x'') on M such that $N = \{x' = 0\}$, we have $\widetilde{M}_N = M \times \mathbb{R}$, $t\colon \widetilde{M}_N \to \mathbb{R}$ is the projection, $p(x', x'', t) = (tx', x'')$.

Let $S \subset M$ be a locally closed subset. The Whitney normal cone $C_N(S)$ is a closed conic subset of $T_N M$ given by

$$C_N(S) = \overline{\widetilde{p}^{-1}(S)} \cap T_N M$$

where, for a set A, $\overline{A}$ denotes the closure of A. One defines the specialization functor

$$\nu_N\colon \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M) \to \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{T_N M})$$

by a similar formula, namely:

$$\nu_N F := s^{-1} j_* \widetilde{p}^{-1} F.$$

Clearly, $\nu_N F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbf{k}_{T_N M})$, that is, $\nu_N F$ is a conic sheaf for the $\mathbb{R}^+$-action on $T_N M$. Moreover,

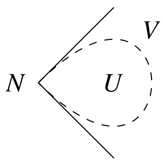
$$\mathrm{R}\tau_{M*}\nu_N F \simeq \nu_N F|_N \simeq F|_N.$$

For an open cone $V \subset T_N M$, one has

$$H^j(V; \nu_N F) \simeq \varinjlim_U H^j(U; F)$$

where U ranges through the family of open subsets of M such that

$$C_N(M \setminus U) \cap V = \emptyset.$$



Microlocalization

Denote by $\pi_M\colon T^*_N M \to N$ the conormal bundle to N in M, that is, the dual bundle to $\tau_M\colon T_N M \to N$.

If F is a sheaf on M, the sheaf of sections of F supported by N, denoted $\mathrm{R}\Gamma_N F$, may be viewed as a global object, namely the direct image by π_M of a sheaf $\mu_M F$ on $T^*_N M$. Intuitively, $T^*_N M$ is the set of "walls" (half-spaces) in M passing through N and the germ of $\mu_N F$ at a conormal vector $(x; \xi) \in T^*_N M$ is the germ at x of the sheaf of sections of F supported by closed tubes with edge N and which are almost the half-space associated with ξ.

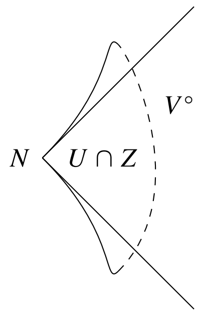
More precisely, the microlocalization of F along N, denoted $\mu_N F$, is the Fourier-Sato transform of $\nu_N F$, hence is an object of $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbf{k}_{T^*_N M})$. It satisfies:

$$\mathrm{R}\pi_{M*}\mu_N F \simeq \mu_N F|_N \simeq \mathrm{R}\Gamma_N F.$$

For a convex open cone $V \subset T^*_N M$, one has

$$H^j(V; \mu_N F) \simeq \varinjlim_{U,Z} H^j_{U\cap Z}(U; F),$$

where U ranges over the family of open subsets of M such that $U \cap N = \pi_M(V)$ and Z ranges over the family of closed subsets of M such that $C_M(Z) \subset V^\circ$ where V° is the polar cone to V.



Back to Hyperfunctions

Assume now that M is a real analytic manifold and X is a complexification of M. First notice the isomorphisms

$$M \times_X T^*X \simeq \mathbb{C} \otimes_{\mathbb{R}} T^*M \simeq T^*M \oplus \sqrt{-1}T^*M.$$

One deduces the isomorphism

$$T^*_M X \simeq \sqrt{-1}T^*M. \tag{1.5}$$

The sheaf $\mathscr{C}_M$ on $T^*_M X$ of Sato's microfunctions (see [52]) is defined as

$$\mathscr{C}_M := \mu_M(\mathscr{O}_X) \otimes \pi_M^{-1}\omega_M.$$

It is shown that this object is concentrated in degree 0. Therefore, we have an isomorphism

$$\mathrm{spec}\colon \mathscr{B}_M \xrightarrow{\sim} \pi_{M*}\mathscr{C}_M$$

and Sato defines the analytic wave front set of a hyperfunction $u \in \Gamma(M; \mathscr{B}_M)$ as the support of $\mathrm{spec}(u) \in \Gamma(T^*_M X; \mathscr{C}_M)$.

Consider a closed convex proper cone $Z \subset T^*_M X$ which contains the zero-section M. Then $\mathrm{spec}(u) \subset Z$ if and only if u is the boundary value of a holomorphic function defined in a tuboid U with profile the interior of the polar tube to Z^a (where Z^a is the image of Z by the antipodal map), that is, satisfying

$$C_M(X \setminus U) \cap \mathrm{Int} Z^{\circ a} = \emptyset.$$

Moreover, the sheaf $\mathscr{C}_M$ is conically flabby. Therefore, any hyperfunction may be decomposed as a sum of boundary values of holomorphic functions f_i's defined in suitable tuboids U_i and if we have hyperfunctions u_i $(i = 1, \dots N)$ satisfying $\sum_j u_j = 0$, there exist hyperfunctions u_{ij} $(i, j = 1, \dots N)$ such that

$$u_{ij} = -u_{ji}, \quad u_i = \sum_j u_{ij} \text{ and } \mathrm{spec}(u_{ij}) \subset \mathrm{spec}(u_i) \cap \mathrm{spec}(u_j).$$

When translating this result in terms of boundary values of holomorphic functions, we get the so-called "Edge of the wedge theorem", already mentioned.

Sato's introduction of the sheaf $\mathscr{C}_M$ was the starting point of an intense activity in the domain of linear partial differential equations after Hörmander adapted Sato's ideas to classical analysis with the help of the (usual) Fourier transform. See [14] and also [4, 58] for related constructions. Note that the appearance of $\sqrt{-1}$ in the usual Fourier transform may be understood as following from the isomorphism (1.5).

1.3 Microsupport

The microsupport of sheaves (also called "singular support") has been introduced in [22] and developed in [23, 24]. Roughly speaking, the microsupport of F describes the codirections of non propagation of F. The idea of microsupport takes its origin in the study of linear PDE and particularly in the study of hyperbolic systems.

Definition 1.5 Let $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ and let $p \in T^*M$. One says that $p \notin \mathrm{SS}(F)$ if there exists an open neighborhood U of p such that for any $x_0 \in M$ and any real C^1-function φ on M defined in a neighborhood of x_0 with $(x_0; d\varphi(x_0)) \in U$, one has $(\mathrm{R}\Gamma_{\{x;\varphi(x)\geq\varphi(x_0)\}}F)_{x_0} \simeq 0$.

In other words, $p \notin \mathrm{SS}(F)$ if the sheaf F has no cohomology supported by "half-spaces" whose conormals are contained in a neighborhood of p.

- By its construction, the microsupport is $\mathbb{R}^+$-conic, that is, invariant by the action of $\mathbb{R}^+$ on T^*M.
- $\mathrm{SS}(F) \cap T^*_M M = \pi_M(\mathrm{SS}(F)) = \mathrm{Supp}(F)$.

- The microsupport satisfies the triangular inequality: if $F_1 \to F_2 \to F_3 \xrightarrow{+1}$ is a distinguished triangle in $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$, then $\mathrm{SS}(F_i) \subset \mathrm{SS}(F_j) \cup \mathrm{SS}(F_k)$ for all $i, j, k \in \{1, 2, 3\}$ with $j \neq k$.

Example 1.6 (i) If F is a non-zero local system on M and M is connected, then $\mathrm{SS}(F) = T^*_M M$.
(ii) If N is a closed submanifold of M and $F = \mathbf{k}_N$, then $\mathrm{SS}(F) = T^*_N M$, the conormal bundle to N in M.
(iii) Let φ be a C^1-function such that $d\varphi(x) \neq 0$ whenever $\varphi(x) = 0$. Let $U = \{x \in M; \varphi(x) > 0\}$ and let $Z = \{x \in M; \varphi(x) \geq 0\}$. Then

$$\mathrm{SS}(\mathbf{k}_U) = U \times_M T^*_M M \cup \{(x; \lambda d\varphi(x)); \varphi(x) = 0, \lambda \leq 0\},$$
$$\mathrm{SS}(\mathbf{k}_Z) = Z \times_M T^*_M M \cup \{(x; \lambda d\varphi(x)); \varphi(x) = 0, \lambda \geq 0\}.$$

For a precise definition of being co-isotropic (one also says involutive), we refer to [24, Def. 6.5.1].

Theorem 1.7 *Let $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$. Then its microsupport* $\mathrm{SS}(F)$ *is co-isotropic.*

Assume now that $(X, \mathscr{O}_X)$ is a complex manifold and denote as usual by $\mathscr{D}_X$ the sheaf of rings of finite order differential operators on X. For a coherent $\mathscr{D}_X$-module $\mathscr{M}$, one denotes by $\mathrm{char}(\mathscr{M})$ its characteristic variety, a closed conic complex analytic subvariety of T^*X. One also sets for short

$$\mathcal{S}ol(\mathscr{M}) := \mathrm{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{M}, \mathscr{O}_X).$$

After identifying X with its real underlying manifold, the link between the microsupport of sheaves and the characteristic variety of coherent $\mathscr{D}$-modules is given by

Theorem 1.8 *Let $\mathscr{M}$ be a coherent $\mathscr{D}$-module. Then* $\mathrm{SS}(\mathcal{S}ol(\mathscr{M})) = \mathrm{char}(\mathscr{M})$.

The inclusion $\mathrm{SS}(\mathcal{S}ol(\mathscr{M})) \subset \mathrm{char}(\mathscr{M})$ is the most useful in practice. Its proof only makes use of the Cauchy-Kowalevsky theorem in its precise form given by Petrovsky and Leray (see [14, § 9.4]) and of purely algebraic arguments. As a corollary of Theorems 1.7 and 1.8, one recovers the fact that the characteristic variety of a coherent $\mathscr{D}_X$-module is co-isotropic, a theorem of [52] which also have a purely algebraic proof due to Gabber [10].

1.4 The Functor μhom

We denote by $\delta\colon M \to M \times M$ the diagonal embedding and we set $\Delta = \delta(M)$. For short, we also denote by δ the isomorphism

$$\delta\colon T^*M \xrightarrow{\sim} T^*_\Delta(M\times M),\quad (x;\xi)\mapsto(x,x;\xi,-\xi).$$

Let us briefly recall the main properties of the functor μhom, a variant of Sato's microlocalization functor.

$$\mu hom\colon \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)^{\mathrm{op}}\times\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)\to\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{T^*M}),$$
$$\mu hom(G,F) := \delta^{-1}\mu_\Delta \mathrm{R}\mathscr{H}om\,(q_2^{-1}G, q_1^!F)$$

where q_i $(i=1,2)$ denotes the ith projection on $M\times M$. Note that

$$\mathrm{R}\pi_{M*}\mu hom(G,F)\simeq \mathrm{R}\mathscr{H}om\,(G,F),$$
$$\mu hom(\mathbf{k}_N,F)\simeq\mu_N(F)\text{ for } N \text{ a closed submanifold of } M,$$
$$\operatorname{supp}\mu hom(G,F)\subset \mathrm{SS}(G)\cap\mathrm{SS}(F),$$
$$\mu hom(G,F)\simeq\mu_\Delta(F\overset{\mathrm{L}}{\boxtimes}\mathrm{D}_M G)\text{ if } G \text{ is constructible.}$$

In some sense, μhom is the sheaf of microlocal morphisms. More precisely, for $p\in T^*M$, we have;

$$H^0\mu hom(G,F)_p\simeq \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M;p)}(G,F)$$

where the category $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M;p)$ is the localization of $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ by the subcategory of sheaves whose microsupport does not contain p.

There is an interesting phenomena which holds with μhom and not with $\mathrm{R}\mathscr{H}om$. Indeed, assume M is real analytic. Then, although the category $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_M)$ of constructible sheaves does not admit a Serre functor, it admits a kind of microlocal Serre functor, as shown by the isomorphism, functorial with respect to F and G (see [24, Prop. 8.4.14]):

$$\mathrm{D}_{T^*M}\mu hom(F,G)\simeq\mu hom(G,F)\otimes\pi_M^{-1}\omega_M.$$

This confirms the fact that to fully understand constructible sheaves, it is natural to look at them microlocally, that is, in T^*M. This is also in accordance with the "philosophy" of Mirror Symmetry which interchanges the category of coherent $\mathscr{O}_X$-modules on a complex manifold X with the Fukaya category on a symplectic manifold Y. In case of $Y=T^*M$, the Fukaya category is equivalent to the category of $\mathbb{R}$-constructible sheaves on M, according to Nadler-Zaslow [43, 44] (see also [9] for related results.)

1.5 An Application: Elliptic Pairs

Denote by $\dot{T}^*M$ the set $T^*M\setminus T^*_MM$ and denote by $\dot\pi_M$ the restriction of $\pi_M\colon T^*M\to M$ to $\dot{T}^*M$. If $H\in\mathsf{D}^{\mathrm{b}}_{\mathbb{R}^+}(\mathbf{k}_{T^*M})$ is a conic sheaf on T^*M, there is the Sato's distinguished triangle

$$\mathrm{R}\pi_{M!}H \to \mathrm{R}\pi_* H \to R\dot{\pi}_{M*}H \xrightarrow{+1} .$$

Applying this result with $H = \mu hom(G, F)$ and assuming G is constructible, we get the distinguished triangle

$$\mathrm{D}'_M G \overset{\mathrm{L}}{\otimes} F \to \mathrm{R}\mathscr{H}om\,(G, F) \to R\dot{\pi}_{M*}\mu hom(G, F).$$

Theorem 1.9 (The Petrovsky theorem for sheaves.) *Assume that G is constructible and* $\mathrm{SS}(G) \cap \mathrm{SS}(F) \subset T^*_M M$. *Then the natural morphism*

$$\mathrm{R}\mathscr{H}om\,(G, \mathbf{k}_M) \overset{\mathrm{L}}{\otimes} F \to \mathrm{R}\mathscr{H}om\,(G, F)$$

is an isomorphism.

Let us apply this result when X is a complex manifold and $\mathbf{k} = \mathbb{C}$. For $G \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$, set

$$\mathscr{A}_G = \mathscr{O}_X \otimes G, \quad \mathscr{B}_G := \mathrm{R}\mathscr{H}om\,(\mathrm{D}'_X G, \mathscr{O}_X).$$

Note that if X is the complexification of a real analytic manifold M and we choose $G = \mathbb{C}_M$, we recover the sheaf of real analytic functions and the sheaf of hyperunctions:

$$\mathscr{A}_{\mathbb{C}_M} = \mathscr{A}_M, \quad \mathscr{B}_{\mathbb{C}_M} = \mathscr{B}_M.$$

Now let $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$. According to [55], one says that the pair $(G, \mathscr{M})$ is elliptic if $\mathrm{char}(\mathscr{M}) \cap \mathrm{SS}(G) \subset T^*_X X$.

Corollary 1.10 [55] *Let $(\mathscr{M}, G)$ be an elliptic pair.*

(a) *We have the canonical isomorphism:*

$$\mathrm{R}\mathscr{H}om_{\,\mathscr{D}_X}(\mathscr{M}, \mathscr{A}_G) \xrightarrow{\sim} \mathrm{R}\mathscr{H}om_{\,\mathscr{D}_X}(\mathscr{M}, \mathscr{B}_G). \tag{1.6}$$

(b) *Assume moreover that* $\mathrm{Supp}(\mathscr{M}) \cap \mathrm{Supp}(G)$ *is compact and $\mathscr{M}$ admits a good filtration. Then the cohomology of the complex* $\mathrm{RHom}_{\mathscr{D}_X}(\mathscr{M}, \mathscr{A}_G)$ *is finite dimensional.*

To prove the part (b) of the corollary, one represents the left hand side of the global sections of (1.6) by a complex of topological vector spaces of type DFN and the right hand side by a complex of topological vector spaces of type FN.

2 Lecture 2: Microlocal Euler Classes and Hochschild Homology

Abstract This is a joint work with Masaki Kashiwara (see [31]). On a complex manifold $(X, \mathscr{O}_X)$, the Hochschild homology is a powerful tool to construct characteristic classes of coherent modules and to get index theorems. Here, I will show how to adapt this formalism to a wide class of sheaves on a real manifold M by using the functor μhom of microlocalization. This construction applies in particular to constructible sheaves on real manifolds and $\mathscr{D}$-modules on complex manifolds, or more generally to elliptic pairs.

2.1 Hochschild Homology on Complex Manifolds

Hochschild homology of $\mathscr{O}$-modules has given rise to a vast literature. Let us quote in particular [6, 7, 15, 47].

Consider a complex manifold $(X, \mathscr{O}_X)$ and denote by ω_X^{hol} the dualizing complex in the category of $\mathscr{O}_X$-modules, that is, $\omega_X^{\mathrm{hol}} = \Omega_X\,[d_X]$, where d_X is the complex dimension of X and Ω_X is the sheaf of holomorphic forms of degree d_X. We shall use the classical six operations for $\mathscr{O}$-modules, $f^*, \mathrm{R}f_*, f^!, \mathrm{R}f_!, \overset{\mathrm{L}}{\otimes}_{\mathscr{O}}$ and $\mathrm{R}\mathscr{H}om_{\mathscr{O}}$. In particular we have the two duality functors

$$\mathrm{D}'_{\mathscr{O}}(\,\cdot\,) = \mathrm{R}\mathscr{H}om_{\mathscr{O}_X}(\,\cdot\,, \mathscr{O}_X),$$
$$\mathrm{D}_{\mathscr{O}}(\,\cdot\,) = \mathrm{R}\mathscr{H}om_{\mathscr{O}_X}(\,\cdot\,, \omega_X^{\mathrm{hol}})$$

as well as the external product that we denote by $\overset{\mathrm{L}}{\boxtimes}_{\mathscr{O}}$. Denote by $\delta\colon X \hookrightarrow X \times X$ the diagonal embedding and let $\Delta = \delta(X)$. We set

$$\mathscr{O}_\Delta := \delta_*\mathscr{O}_X, \quad \omega_X^{\mathrm{hol},\otimes-1} := \mathrm{D}'_{\mathscr{O}}\omega_X^{\mathrm{hol}}, \quad \omega_\Delta^{\mathrm{hol},\otimes-1} := \delta_*\omega_X^{\mathrm{hol},\otimes-1}. \tag{2.1}$$

It is well-known that

$$\omega_\Delta^{\mathrm{hol},\otimes-1} \simeq \mathrm{R}\mathscr{H}om_{\mathscr{O}_{X\times X}}(\mathscr{O}_\Delta, \mathscr{O}_{X\times X}). \tag{2.2}$$

The Hochschild homology of $\mathscr{O}_X$ is usually defined by

$$\mathscr{HH}(\mathscr{O}_X) = \delta^{-1}\big(\mathscr{O}_\Delta \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_{X\times X}} \mathscr{O}_\Delta\big). \tag{2.3}$$

Note the isomorphisms

$$\begin{array}{ccc} & \mathscr{HH}(\mathscr{O}_X) & \\ \swarrow\sim & & \sim\searrow \\ \delta^*\delta_*\mathscr{O}_X & \xrightarrow{\sim} & \delta^!\delta_!\omega_X^{\mathrm{hol}} \end{array}$$

and the canonical isomorphisms

$$\delta^*\delta_*\mathscr{O}_X \simeq \delta^{-1}\mathrm{R}\mathscr{H}om_{\mathscr{O}_{X\times X}}\big(\omega_\Delta^{\mathrm{hol},\otimes-1}, \mathscr{O}_\Delta\big),$$
$$\delta^!\delta_!\omega_X^{\mathrm{hol}} \simeq \delta^{-1}\mathrm{R}\mathscr{H}om_{\mathscr{O}_{X\times X}}\big(\mathscr{O}_\Delta, \omega_\Delta^{\mathrm{hol}}\big).$$

For a closed subset S of X, we set:

$$\mathbb{HH}^0_S(\mathscr{O}_X) = H^0(X; \mathrm{R}\Gamma_S\mathscr{HH}(\mathscr{O}_X)). \tag{2.4}$$

Let $\mathscr{F} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X)$. The morphisms $\mathrm{D}'_{\mathscr{O}}\mathscr{F} \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X} \mathscr{F} \to \mathscr{O}_X$ and $\mathrm{D}_{\mathscr{O}}\mathscr{F} \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X} \mathscr{F} \to \omega_X^{\mathrm{hol}}$ give by adjunction the morphisms

$$\mathrm{D}'_{\mathscr{O}}\mathscr{F}\overset{\mathrm{L}}{\boxtimes}_{\mathscr{O}_{X\times X}}\mathscr{F} \to \mathscr{O}_\Delta, \quad \mathrm{D}_{\mathscr{O}}\mathscr{F}\overset{\mathrm{L}}{\boxtimes}_{\mathscr{O}_{X\times X}}\mathscr{F} \to \omega_\Delta^{\mathrm{hol}}$$

and then by duality the morphisms

$$\omega_\Delta^{\mathrm{hol},\otimes-1} \to \mathrm{D}'_{\mathscr{O}}\mathscr{F}\overset{\mathrm{L}}{\boxtimes}_{\mathscr{O}_{X\times X}}\mathscr{F} \to \mathscr{O}_\Delta, \quad \mathscr{O}_\Delta \to \mathrm{D}_{\mathscr{O}}\mathscr{F}\overset{\mathrm{L}}{\boxtimes}_{\mathscr{O}_{X\times X}}\mathscr{F} \to \omega_\Delta^{\mathrm{hol}}$$

and the composition defines the Hochschild classes of $\mathscr{F}$:

$$\mathrm{hh}_{\mathscr{O}}(\mathscr{F}) \in H^0_{\mathrm{supp}(\mathscr{F})}(X; \delta^{-1}\delta_*\mathscr{O}_X), \quad \widetilde{\mathrm{hh}}_{\mathscr{O}}(\mathscr{F}) \in H^0_{\mathrm{supp}(\mathscr{F})}(X; \delta^!\delta_!\omega_X). \tag{2.5}$$

One can compose Hochschild homology and the Hochschild class commutes with the composition of kernels. More precisely, consider complex manifolds X_i $(i = 1, 2, 3)$.

- We write $X_{ij} := X_i \times X_j$ $(1 \le i, j \le 3)$, $X_{123} = X_1 \times X_2 \times X_3$, $X_{1223} = X_1 \times X_2 \times X_2 \times X_3$, etc.
- We denote by q_i the projection $X_{ij} \to X_i$ or the projection $X_{123} \to X_i$ and by q_{ij} the projection $X_{123} \to X_{ij}$.

Let $K_{ij} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_{X_{ij}})$ $(i = 1, 2,\ j = i + 1)$. One sets

$$K_{12} \underset{2}{\circ} K_{23} = \mathrm{R}q_{13!}(q_{12}^*K_{12} \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_{X_{123}}} q_{23}^*K_{23}).$$

Theorem 2.1 (a) *There is a natural morphism*

$$\mathscr{HH}(\mathscr{O}_{X_{12}}) \underset{2}{\circ} \mathscr{HH}(\mathscr{O}_{X_{23}}) \to \mathscr{HH}(\mathscr{O}_{X_{13}}).$$

(b) *Let* $S_{ij} \subset X_{ij}$ *be a closed subset* $(i = 1, 2,\ j = i + 1)$. *Assume that* q_{13} *is proper over* $S_{12} \times_{X_2} S_{23}$ *and set* $S_{13} = q_{13}(S_{12} \times_{X_2} S_{23})$. *Then the morphism above induces a map*

$$\underset{2}{\circ}\colon \mathbb{HH}^0_{S_{12}}(\mathscr{O}_{X_{12}}) \otimes \mathbb{HH}^0_{S_{23}}(\mathscr{O}_{X_{23}}) \to \mathbb{HH}^0_{S_{13}}(\mathscr{O}_{X_{13}}).$$

(c) *Let* K_{ij} *be as above and assume that* $\mathrm{supp}(K_{ij}) \subset S_{ij}$. *Set* $K_{13} = K_{12} \underset{2}{\circ} K_{23}$ *and* $\widetilde{K}_{13} = (K_{12} \otimes \omega_2^{\mathrm{hol} \otimes -1}) \underset{2}{\circ} K_{23}$. *Then* K_{13} *and* $\widetilde{K}_{13}$ *belong to* $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_{X_{13}})$ *and we have the equalities in* $\mathbb{HH}^0_{S_{13}}(\mathscr{O}_{X_{13}})$:

$$\mathrm{hh}_{\mathscr{O}}(K_{13}) = \mathrm{hh}_{\mathscr{O}}(K_{12}) \underset{2}{\circ} \mathrm{hh}_{\mathscr{O}}(K_{23}), \quad \widetilde{\mathrm{hh}}_{\mathscr{O}}(\widetilde{K}_{13}) = \widetilde{\mathrm{hh}}_{\mathscr{O}}(K_{12}) \underset{2}{\circ} \widetilde{\mathrm{hh}}_{\mathscr{O}}(K_{23}).$$

This theorem shows in particular that the Hochschild class commutes with external product, inverse image and proper direct image.

Theorem 2.1 seems to be well-known from the specialists although it is difficult to find a precise statement (see however [7, 48]). The construction of the Hochschild homology as well as Theorem 2.1 (including complete proofs) have been extended when replacing $\mathscr{O}_X$ with a so-called DQ-algebroid stack $\mathscr{A}_X$ in [30].

Coming back to $\mathscr{O}_X$-modules, the Hodge cohomology of $\mathscr{O}_X$ is given by:

$$\mathcal{HD}(\mathscr{O}_X) := \bigoplus_{i=0}^{d_X} \Omega^i_X\,[i], \text{ an object of } \mathsf{D}^{\mathrm{b}}(\mathscr{O}_X). \tag{2.6}$$

There is a commutative diagram constructed by Kashiwara in [20] in which α_X is the HKR (Hochschild-Kostant-Rosenberg) isomorphism and β_X is a kind of dual HKR isomorphism:

$$\begin{array}{ccc} \delta^*\delta_*\mathscr{O}_X & \xrightarrow[\mathrm{td}]{\sim} & \delta^!\delta_!\omega_X^{\mathrm{hol}} \\ {\scriptstyle \alpha_X}\downarrow{\scriptstyle\sim} & & {\scriptstyle\sim}\uparrow{\scriptstyle \beta_X} \\ \mathcal{HD}(\mathscr{O}_X) & \xrightarrow[\tau]{\sim} & \mathcal{HD}(\mathscr{O}_X). \end{array} \tag{2.7}$$

If $\mathscr{F} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X)$, the Chern character of $\mathscr{F}$ is the image by α_X of $\mathrm{hh}_{\mathscr{O}}(\mathscr{F})$.

In [20] Kashiwara made the conjecture that the arrow τ making the diagram commutative is given by the cup product by the Todd class of X. This conjecture has recently been proved by Ramadoss [47] in the algebraic case (after preliminary important results by Markarian) and Grivaux [11] in the analytic case (and with a very simple proof). Since the morphism β_X commutes with proper direct images, we get a new and functorial approach to the Riemann-Roch-Hirzebruch-Grothendieck theorem.

2.2 Microlocal Homology

We keep the notations of Lecture I. In particular ω_M denotes the dualizing complex on M and D'_M is the duality functor. We set

$$\omega_\Delta := \delta_*\omega_M, \quad \omega_M^{\otimes -1} := \mathrm{D}'_M\omega_M, \quad \omega_\Delta^{\otimes -1} := \delta_*\omega_M^{\otimes -1}. \tag{2.8}$$

Let M_i $(i = 1, 2, 3)$ be manifolds.

- For short, we write as above $M_{ij} := M_i \times M_j$ $(1 \le i, j \le 3)$, $M_{123} = M_1 \times M_2 \times M_3$, etc.
- We will often write for short $\mathbf{k}_i$ instead of $\mathbf{k}_{M_i}$ and $\mathbf{k}_{\Delta_i}$ instead of $\mathbf{k}_{\Delta_{M_i}}$, π_i instead of π_{M_i}, etc.
- We denote by q_i the projection $M_{ij} \to M_i$ or the projection $M_{123} \to M_i$ and by q_{ij} the projection $M_{123} \to M_{ij}$. Similarly, we denote by p_i the projection $T^*M_{ij} \to T^*M_i$ or the projection $T^*M_{123} \to T^*M_i$ and by p_{ij} the projection $T^*M_{123} \to T^*M_{ij}$.
- We also need to introduce the maps p_{j^a} or p_{ij^a}, the composition of p_j or p_{ij} and the antipodal map on T^*M_j.

We consider the operations of composition of kernels. For $K_{ij} \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M_{ij}})$ $(i = 1, 2, j = i + 1)$, we set

$$\begin{aligned} K_1 \underset{2}{\circ} K_2 &:= \mathrm{R}q_{13!}\delta_2^{-1}(K_1 \overset{\mathrm{L}}{\boxtimes} K_2) \simeq \mathrm{R}q_{13!}(q_{12}^{-1}K_1 \overset{\mathrm{L}}{\otimes} q_{23}^{-1}K_2), \\ K_1 \underset{2}{*} K_2 &:= \mathrm{R}q_{13*}\big(\delta_2^!(K_1 \overset{\mathrm{L}}{\boxtimes} K_2) \otimes q_2^{-1}\omega_2\big). \end{aligned}$$

We have a natural morphism $K_1 \circ K_2 \to K_1 * K_2$. It is an isomorphism if $p_{12^a}^{-1}\mathrm{SS}(K_1) \cap p_{23^a}^{-1}\mathrm{SS}(K_2) \to T^*M_{13}$ is proper.

We also define the composition of kernels on cotangent bundles. For $L_i \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{T^*_{M_{ij}}})$ $(i = 1, 2, j = i + 1)$, we set

$$L_1 \overset{a}{\underset{2}{\circ}} L_2 := \mathrm{R}p_{13^a!}(p_{12^a}^{-1}L_1 \overset{\mathrm{L}}{\otimes} p_{23^a}^{-1}L_2).$$

For $K_1, F_1 \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M_{12}})$ and $K_2, F_2 \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M_{23}})$ there exists a canonical morphism:

$$\mu hom(K_1, F_1) \overset{a}{\underset{2}{\circ}} \mu hom(K_2, F_2) \to \mu hom(K_1 \underset{2}{*} K_2, F_1 \underset{2}{\circ} F_2). \tag{2.9}$$

We also define the corresponding operations for subsets of cotangent bundles. Let $A \subset T^*M_{12}$ and $B \subset T^*M_{23}$. We set $A \overset{a}{\underset{2}{\circ}} B = p_{13}(A \overset{a}{\underset{2}{\times}} B)$ where $A \overset{a}{\underset{2}{\times}} B = p_{12^a}^{-1}(A) \cap p_{23}^{-1}(B)$.

If there is no risk of confusion, we simply denote by δ^a the map:

$$\delta^a\colon\ T^*M \hookrightarrow T^*(M \times M)\ ,\quad (x;\xi) \mapsto (x, x; \xi, -\xi).$$

Definition 2.2 Let Λ be a closed conic subset of T^*M. We set

$$\begin{aligned} \mathscr{MH}(\mathbf{k}_M) &:= (\delta^a)^{-1}\mu hom(\mathbf{k}_{\Delta_M}, \omega_{\Delta_M}), \\ \mathbb{MH}^0_\Lambda(\mathbf{k}_M) &:= H^0_\Lambda(T^*M; \mathscr{MH}(\mathbf{k}_M)). \end{aligned}$$

We call $\mathscr{MH}(\mathbf{k}_M)$ the microlocal homology of M.

We have isomorphisms

$$\mathscr{MH}(\mathbf{k}_M) \simeq (\delta^a)^{-1}\mu_\Delta(\omega_\Delta) \simeq \pi_M^{-1}\omega_M$$

and the isomorphism $\mathscr{MH}(\mathbf{k}_M) \simeq \pi_M^{-1}\omega_M$ plays the role of the HKR isomorphism in the complex case.

We have the analogue of Theorem 2.1 (a) and (b). (For the part (c), see Theorem 2.6 below.)

Let $i = 1, 2$, $j = i + 1$ and let Λ_{ij} be a closed conic subset of T^*M_{ij}. Assume that

$$\Lambda_{12} \overset{a}{\underset{2}{\times}} \Lambda_{23} \text{ is proper over } T^*M_{13}. \tag{2.10}$$

Note that this hypothesis is equivalent to

$$\begin{cases} p_{12^a}^{-1}(\Lambda_{12}) \cap p_{23^a}^{-1}(\Lambda_{23}) \cap (T^*_{M_1}M_1 \times T^*M_2 \times T^*_{M_3}M_3) \subset T^*_{M_{123}}M_{123}, \\ q_{13} \text{ is proper on } \pi_{12}(\Lambda_{12}) \times_{M_2} \pi_{23}(\Lambda_{23}). \end{cases}$$

Set

$$\Lambda_{13} = \Lambda_{12} \overset{a}{\underset{2}{\circ}} \Lambda_{23}. \tag{2.11}$$

Theorem 2.3 (a) *There is a natural morphism*

$$\mathscr{MH}(\mathbf{k}_{M_{12}}) \underset{2}{\circ} \mathscr{MH}(\mathbf{k}_{M_{23}}) \to \mathscr{MH}(\mathbf{k}_{M_{13}}). \tag{2.12}$$

(b) *Let $\Lambda_{ij} \subset T^*M_{ij}$ be as above and assume* (2.10)*. Then the morphism* (2.12) *induces a map*

$$\underset{2}{\circ}\colon \mathbb{MH}^0_{\Lambda_{12}}(\mathbf{k}_{M_{12}}) \otimes \mathbb{MH}^0_{\Lambda_{23}}(\mathbf{k}_{M_{23}}) \to \mathbb{MH}^0_{\Lambda_{13}}(\mathbf{k}_{M_{13}}). \tag{2.13}$$

The construction of the morphism (2.12) uses (2.9), which makes the computations not easy. Fortunately, we have the following result.

Proposition 2.4 *Let M_i $(i = 1, 2, 3)$ be manifolds and let Λ_{ij} be a closed conic subset of T^*M_{ij} $(ij = 12, 13, 23)$. We have a commutative diagram*

$$\begin{array}{ccc} \mathscr{MH}(\mathbf{k}_{12}) \overset{a}{\underset{2}{\circ}} \mathscr{MH}(\mathbf{k}_{23}) & \longrightarrow & \mathscr{MH}(\mathbf{k}_{13}) \\ \downarrow\wr & & \downarrow\wr \\ \pi_{12}^{-1}\omega_{12} \overset{a}{\underset{2}{\circ}} \pi_{23}^{-1}\omega_{23} & \longrightarrow & \pi_{13}^{-1}\omega_{13}. \end{array} \tag{2.14}$$

Here the bottom horizontal arrow is induced by

$$p_{12^a}^{-1}\pi_{12}^{-1}\omega_{12} \otimes p_{23^a}^{-1}\pi_{23^a}^{-1}\omega_{23} \simeq \pi_1^{-1}\omega_1 \boxtimes \omega_{T^*M_2} \boxtimes \pi_3^{-1}\omega_3$$

and

$$\mathrm{R}p_{13^a\,!}\big(\pi_1^{-1}\omega_1 \boxtimes \omega_{T^*M_2} \boxtimes \pi_{M_3}^{-1}\omega_3\big) \longrightarrow \pi_1^{-1}\omega_1 \boxtimes \pi_3^{-1}\omega_3.$$

Remark 2.5 (i) If we consider that the isomorphism $\mathscr{MH}(\mathbf{k}_M) \simeq \pi^{-1}\omega_M$ is a real analogue of the Hochschild-Kostant-Rosenberg isomorphism, then the commutativity of Diagram (2.14) says that, contrarily to the complex case, the real HKR isomorphism commutes with inverse and direct images.
(ii) As a particular case of Proposition 2.4, we get canonical isomorphisms

$$\mathscr{MH}(\mathbf{k}_M) \otimes \mathscr{MH}(\mathbf{k}_M) \simeq \pi^{-1}\omega_M \otimes \pi^{-1}\omega_M \simeq \omega_{T^*M}.$$

Hence, $\mathscr{MH}(\mathbf{k}_M)$ behaves as a "square root" of the dualizing complex.

2.3 Trace Kernels and Microlocal Euler Classes

A trace kernel (K, u, v) on M is the data of $K \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M\times M})$ together with morphisms (u, v)

$$\mathbf{k}_\Delta \xrightarrow{u} K \xrightarrow{v} \omega_\Delta.$$

Setting $\mathrm{SS}_\Delta(K) := \mathrm{SS}(K) \cap T^*_\Delta(M \times M)$, the morphism u gives an element of $H^0_{\mathrm{SS}_\Delta(K)}(T^*M; \mu hom(\mathbf{k}_\Delta, K))$ whose image by v is the microlocal Euler class of K

$$\mu\mathrm{eu}_M(K) \in \mathbb{MH}^0_{\mathrm{SS}_\Delta(K)}(\mathbf{k}_M)) \simeq H^0_{\mathrm{SS}_\Delta(K)}(T^*M;\pi^{-1}\omega_M).$$

If $M = \mathrm{pt}$, a trace kernel K is nothing but an object of $\mathsf{D}^{\mathrm{b}}(\mathbf{k})$ together with linear maps $\mathbf{k} \to K \to \mathbf{k}$. The composition gives the element $\mu\mathrm{eu}(K)$ of $\mathbf{k}$. If $\mathbf{k}$ is a field of characteristic zero and $K = L \otimes L^*$ where L is a bounded complex of $\mathbf{k}$-modules with finite dimensional cohomology and L^* is its dual, one recovers the classical Euler-Poincaré index of L, that is, $\mu\mathrm{eu}(K) = \chi(L)$.

Let $i = 1, 2$, $j = i + 1$ and let Λ_{iijj} be a closed conic subset of T^*M_{iijj}. Assume that

$$\Lambda_{1122} \overset{a}{\underset{22}{\times}} \Lambda_{2233} \text{ is proper over } T^*M_{1133}. \tag{2.15}$$

Set $\Lambda_{1133} = \Lambda_{1122} \overset{a}{\underset{22}{\circ}} \Lambda_{2233}$ and $\Lambda_{ij} = \Lambda_{iijj} \cap T^*_{\Delta_{ij}} M_{iijj}$.

Theorem 2.6 *Let K_{ij} be a trace kernel on M_{ij} with $\mathrm{SS}(K_{ij}) \subset \Lambda_{iijj}$. Assume (2.15), set $\widetilde{K}_{23} = \omega^{\otimes -1}_{\Delta_2} \underset{2}{\circ} K_{23} \simeq (\omega_2^{\otimes -1} \boxtimes \mathbf{k}_{233}) \overset{\mathrm{L}}{\otimes} K_{23}$ and set $K_{13} = K_{12} \underset{22}{\circ} \widetilde{K}_{23}$. Then*

(a) *K_{13} is a trace kernel on M_{13},*
(b) *$\mu\mathrm{eu}_{M_{13}}(K_{13}) = \mu\mathrm{eu}_{M_{12}}(K_{12}) \overset{a}{\underset{2}{\circ}} \mu\mathrm{eu}_{M_{23}}(K_{23})$ as elements of $\mathbb{MH}^0_{\Lambda_{13}}(\mathbf{k}_{13})$.*

As an application, one can perform the external product, the proper direct image and the non characteristic inverse image of trace kernels and compute their microlocal Euler classes.

Consider in particular the case where Λ_1 and Λ_2 are two closed conic subsets of T^*M satisfying the transversality condition

$$\Lambda_1 \cap \Lambda_2^a \subset T^*_M M. \tag{2.16}$$

Then applying Theorem 2.6 and composing the external product with the restriction to the diagonal, we get a convolution map:

$$\star\colon \mathbb{MH}_{\Lambda_1}(\mathbf{k}_M) \times \mathbb{MH}_{\Lambda_2}(\mathbf{k}_M) \to \mathbb{MH}_{\Lambda_1+\Lambda_2}(\mathbf{k}_M).$$

Proposition 2.7 *Let K_i be a trace kernel with $\mathrm{SS}_\Delta(K_i) \subset \Lambda_i$ $(i = 1, 2)$ and assume (2.15). Then the object $K_1 \overset{\mathrm{L}}{\otimes} (\mathbf{k}_M \overset{\mathrm{L}}{\boxtimes} \omega_M^{\otimes -1}) \overset{\mathrm{L}}{\otimes} K_2$ is a trace kernel on M and*

$$\mu\mathrm{eu}_M(K_1 \overset{\mathrm{L}}{\otimes} (\mathbf{k}_M \boxtimes \omega_M^{\otimes -1}) \overset{\mathrm{L}}{\otimes} K_2) = \mu\mathrm{eu}_M(K_1) \star \mu\mathrm{eu}_M(K_2).$$

In particular if $\mathrm{supp} K_1 \cap \mathrm{supp} K_2$ *is compact, we have*

$$\mu\mathrm{eu}\big(\mathrm{R}\Gamma(M \times M; K_1 \overset{\mathrm{L}}{\otimes} (\mathbf{k}_M \boxtimes \omega_M^{\otimes -1}) \overset{\mathrm{L}}{\otimes} K_2)\big) = \int_M (\mu\mathrm{eu}(K_1) \star \mu\mathrm{eu}(K_2))|_M$$
$$= \int_{T^*M} \mu\mathrm{eu}(K_1) \cup \mu\mathrm{eu}(K_2).$$

We shall apply this result to elliptic pairs.

2.4 Microlocal Euler Class of Constructible Sheaves

Let us denote by $\mathsf{D}^{\mathrm{b}}_{\mathrm{cc}}(\mathbf{k}_M)$ the full triangulated subcategory of $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ consisting of cohomologically constructible sheaves and let $G \in \mathsf{D}^{\mathrm{b}}_{\mathrm{cc}}(\mathbf{k}_M)$.

The evaluation morphism $G \overset{\mathrm{L}}{\otimes} \mathrm{D}_M G \to \omega_M$ gives by adjunction the morphism $G \overset{\mathrm{L}}{\boxtimes} \mathrm{D}_M G \to \omega_\Delta$. By duality, one gets the morphism $\mathbf{k}_\Delta \to G \overset{\mathrm{L}}{\boxtimes} \mathrm{D}_M G$. To summarize, we have the morphisms in $\mathsf{D}^{\mathrm{b}}_{\mathrm{cc}}(\mathbf{k}_{M\times M})$:

$$\mathbf{k}_\Delta \to G \overset{\mathrm{L}}{\boxtimes} \mathrm{D}G \to \omega_\Delta. \tag{2.17}$$

Denote by $\mathrm{TK}(G)$ the trace kernel so constructed. If G is $\mathbb{R}$-constructible, the class $\mu\mathrm{eu}_M(\mathrm{TK}(G))$ is nothing but the Lagrangian cycle of G constructed by Kashiwara [19]. In the sequel, if there is no risk of confusion, we simply denote this class by $\mu\mathrm{eu}_M(G)$.

One recovers the classical functorial properties of Lagrangian cycles. Let $f\colon M \to N$ be a morphism of manifolds. To f one associates the maps

$$T^*M \xleftarrow{f_d} M \times_N T^*N \xrightarrow{f_\pi} T^*N.$$

There are natural morphisms

$$f_\mu\colon f_{\pi !} f_d^{-1} \pi_M^{-1} \omega_M \to \pi_N^{-1} \omega_N,$$
$$f^\mu\colon f_{d !} f_\pi^{-1} \pi_N^{-1} \omega_N \to \pi_M^{-1} \omega_M.$$

- Let $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_M)$ and assume f is proper on $\mathrm{supp}(F)$, or equivalently, f_π is proper on $f_d^{-1}\mathrm{SS}(F)$. Then $\mu\mathrm{eu}(\mathrm{R}f_* F) = f_\mu \mu\mathrm{eu}(F)$,
- Let $G \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_N)$ and assume that f is non characteristic for G, that is, f_d is proper on $f_\pi^{-1}\mathrm{SS}(G)$. Then $\mu\mathrm{eu}(f^{-1}G) = f^\mu \mu\mathrm{eu}(G)$.

2.5 Microlocal Euler Class of $\mathscr{D}$-Modules

In this section, we denote by X a complex manifold of complex dimension d_X and the base ring $\mathbf{k}$ is the field $\mathbb{C}$. One denotes by $\mathscr{D}_X$ the sheaf of $\mathbb{C}_X$-algebras of (finite order) holomorphic differential operators on X and refer to [21] for a detailed exposition of the theory of $\mathscr{D}$-modules.

We also denote by $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$ the full triangulated subcategory of $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)$ consisting of objects with coherent cohomology. We denote by $\mathrm{D}_{\mathscr{D}}\colon \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X) \to \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)$ the duality functor for left $\mathscr{D}$-modules:

$$\mathrm{D}_{\mathscr{D}}\mathscr{M} := \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X) \otimes_{\mathscr{O}_X} \omega_X^{\mathrm{hol},\otimes -1}.$$

We denote by $\cdot \underline{\boxtimes} \cdot$ the external product for $\mathscr{D}$-modules:

$$\mathscr{M}\underline{\boxtimes}\mathscr{N} := \mathscr{D}_{X\times X} \otimes_{\mathscr{D}_X \boxtimes \mathscr{D}_X} (\mathscr{M}\boxtimes\mathscr{N}).$$

Let Δ be the diagonal of $X \times X$. The left $\mathscr{D}_{X\times X}$-module $H^{d_X}_{[\Delta]}(\mathscr{O}_{X\times X})$ (the algebraic cohomology with support in Δ) is denoted as usual by $\mathscr{B}_\Delta$. We also introduce $\mathscr{B}_\Delta^\vee := \mathscr{B}_\Delta\,[2d_X]$. For a coherent $\mathscr{D}_X$-module $\mathscr{M}$, we have the isomorphism

$$\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{M}) \simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}_{X\times X}}(\mathscr{B}_\Delta, \mathscr{M}\underline{\boxtimes}\mathrm{D}_{\mathscr{D}}\mathscr{M})\,[d_X].$$

We get the morphisms

$$\mathscr{B}_\Delta \to \mathscr{M}\underline{\boxtimes}\mathrm{D}_{\mathscr{D}}\mathscr{M}\,[d_X] \to \mathscr{B}_\Delta^\vee \tag{2.18}$$

where the second morphism is deduced by duality.

Denote by $\mathscr{E}_{T^*X}$ the sheaf on T^*X of microdifferential operators of [52]. For a coherent $\mathscr{D}_X$-module $\mathscr{M}$ set

$$\mathscr{M}^E := \mathscr{E}_{T^*X} \otimes_{\pi^{-1}\mathscr{D}_X} \pi^{-1}\mathscr{M}.$$

Recall that, denoting by $\mathrm{char}(\mathscr{M})$ the characteristic variety of $\mathscr{M}$, we have $\mathrm{char}(\mathscr{M}) = \mathrm{supp}(\mathscr{M}^E)$. Set

$$\mathscr{C}_\Delta := \mathscr{B}_\Delta^E, \quad \mathscr{C}_\Delta^\vee := (\mathscr{B}_\Delta^\vee)^E.$$

Let Λ be a closed conic subset of T^*X. One sets

$$\mathscr{H}\mathscr{H}(\mathscr{E}_{T^*X}) = (\delta^a)^{-1}\mathrm{R}\mathscr{H}om_{\mathscr{E}_{X\times X}}(\mathscr{C}_\Delta, \mathscr{C}_\Delta^\vee),$$
$$\mathbb{HH}^0_\Lambda(\mathscr{E}_{T^*X}) = H^0_\Lambda(T^*X; \mathscr{H}\mathscr{H}(\mathscr{E}_{T^*X})).$$

One calls $\mathscr{H}\mathscr{H}(\mathscr{E}_{T^*X})$ the Hochschild homology of $\mathscr{E}_{T^*X}$.

We deduce from (2.18) the morphisms

$$\mathscr{C}_\Delta \to (\mathscr{M}\underline{\boxtimes}\mathrm{D}_{\mathscr{D}}\mathscr{M})^E\,[d_X] \to \mathscr{C}_\Delta^\vee \tag{2.19}$$

which define the Hochschild class of $\mathscr{M}$:

$$\mathrm{hh}_{\mathscr{E}}(\mathscr{M}) \in \mathbb{HH}^0_{\mathrm{char}(\mathscr{M})}(\mathscr{E}_{T^*X}). \tag{2.20}$$

We shall make a link between the Hochschild class of $\mathscr{M}$ and the microlocal Euler class of a trace kernel attached to the sheaf of holomorphic solutions of $\mathscr{M}$. We have

$$\Omega_{X\times X}\,[-d_X] \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{X\times X}} \mathscr{B}_\Delta \simeq \mathbb{C}_\Delta,$$
$$\Omega_{X\times X}\,[-d_X] \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{X\times X}} \mathscr{B}_\Delta^\vee \simeq \omega_\Delta.$$

Now remark that for $\mathscr{N}_1, \mathscr{N}_2 \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$, we have a natural morphism

$$\mathrm{R}\mathscr{H}om_{\,\pi^{-1}\mathscr{D}_X}(\pi^{-1}\mathscr{N}_1, \mathscr{N}_2^E) \to \mu hom(\Omega_X \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{N}_1, \Omega_X \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{N}_2).$$

One deduces the morphisms

$$\begin{aligned}\mathrm{R}\mathscr{H}om_{\,\mathscr{E}_{X\times X}}(\mathscr{C}_\Delta, \mathscr{C}_\Delta^\vee) &\simeq \mathrm{R}\mathscr{H}om_{\,\pi^{-1}\mathscr{D}_{X\times X}}(\pi^{-1}\mathscr{B}_\Delta, (\mathscr{B}_\Delta^\vee)^E)\\ &\to \mu hom(\Omega_{X\times X} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{X\times X}} \mathscr{B}_\Delta{}^{\otimes -1}, \Omega_{X\times X} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{X\times X}} \mathscr{B}_\Delta)\\ &\simeq \mu hom(\mathbb{C}_\Delta, \omega_\Delta).\end{aligned}$$

Since all the arrows above are isomorphisms, we get

$$\mathscr{H}\mathscr{H}(\mathscr{E}_{T^*X}) \simeq \mathscr{M}\mathscr{H}(\mathbb{C}_X).$$

Recall that the Hochschild homology of $\mathscr{E}_{T^*X}$ has been already calculated in [5].

By this isomorphism, $\mathrm{hh}_{\mathscr{E}}(\mathscr{M})$ belongs to $\mathbb{MH}^0_{\mathrm{char}(\mathscr{M})}(\mathbb{C}_X)$ and this class coincides with that already introduced in [55].

Applying the functor $\Omega_{X\times X}[-d_X] \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{X\times X}} \bullet$ to the morphisms in (2.18) we get the morphisms

$$\mathbb{C}_\Delta \to \Omega_{X\times X} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{X\times X}} (\mathscr{M}\underline{\boxtimes}\mathrm{D}_{\mathscr{D}}\mathscr{M}) \to \omega_\Delta. \tag{2.21}$$

For $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$, we set

$$\mathrm{TK}(\mathscr{M}) := \Omega_{X\times X} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{X\times X}} (\mathscr{M}\underline{\boxtimes}\mathrm{D}_{\mathscr{D}}\mathscr{M}).$$

Then $\mathrm{TK}(\mathscr{M})$ is a trace kernel by (2.21) and $\mu\mathrm{eu}_M(\mathrm{TK}(\mathscr{M}))$ is supported by $\mathrm{char}(\mathscr{M})$ by Theorem 1.8.

Proposition 2.8 *The Hochschild class of $\mathscr{M}$ is the microlocal Euler class of the trace kernel associated to $\mathscr{M}$, that is,* $\mathrm{hh}_{\mathscr{E}}(\mathscr{M}) = \mu\mathrm{eu}_X(\mathrm{TK}(\mathscr{M}))$ *in* $H^0_{\mathrm{char}(\mathscr{M})}(T^*X;\pi^{-1}\omega_X)$.

2.6 Microlocal Euler Class of Elliptic Pairs

Let X be a complex manifold, $\mathscr{M}$ an object of $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$ and G an object of $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$. The pair $(\mathscr{M}, G)$ is called an elliptic pair in [55] if $\mathrm{char}(\mathscr{M}) \cap \mathrm{SS}(G) \subset T^*_X X$. From now on, we assume that $(\mathscr{M}, G)$ is an elliptic pair. We set

$$\mathrm{TK}(\mathscr{M}, G) := \Omega_{X\times X} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{X\times X}} \big((\mathscr{M}\otimes G)\underline{\boxtimes}(\mathrm{D}_{\mathscr{D}}\mathscr{M}\otimes \mathrm{D}'_X G)\big). \tag{2.22}$$

It follows from the preceding results that $\mathrm{TK}(\mathscr{M}, G)$ is a trace kernel and

$$\mu\mathrm{eu}_X\big(\mathrm{TK}(\mathscr{M}, G)\big) = \mu\mathrm{eu}_X(\mathscr{M}) \star \mu\mathrm{eu}_X(G). \tag{2.23}$$

Applying Corollary 1.10(a), we get the natural isomorphism

$$\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathrm{D}'_X G\otimes \mathscr{O}_X) \xrightarrow{\sim} \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}\otimes G, \mathscr{O}_X). \tag{2.24}$$

Assume moreover that $\mathrm{Supp}(\mathscr{M}) \cap \mathrm{Supp}(G)$ is compact. Applying Corollary 1.10(b), we get that the cohomology of the complex

$$\mathrm{Sol}(\mathscr{M}\otimes G) := \mathrm{RHom}_{\mathscr{D}_X}(\mathscr{M}\otimes G, \mathscr{O}_X)$$

is finite dimensional. Moreover

$$\mathrm{R}\Gamma(X\times X; \mathrm{TK}(\mathscr{M}, G)) \simeq \mathrm{Sol}(\mathscr{M}\otimes G)\otimes \mathrm{Sol}(\mathscr{M}\otimes G)^*.$$

Applying Proposition 2.7, we get

$$\begin{aligned}\chi\big(\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}\otimes G, \mathscr{O}_X)\big) &= \int_X (\mathrm{hh}_{\mathscr{E}}(\mathscr{M}) \star \mu\mathrm{eu}_X(G))|_X \\ &= \int_{T^*X} (\mathrm{hh}_{\mathscr{E}}(\mathscr{M}) \cup \mu\mathrm{eu}_X(G)).\end{aligned}$$

This formula has many applications, as far as one is able to calculate $\mu\mathrm{eu}_X(\mathscr{M})$.

Assume that $\mathscr{M}$ is endowed with a good filtration and $\mathrm{char}(\mathscr{M}) \subset \Lambda$. Set

$$\widetilde{\mathrm{gr}}\mathscr{M} := \mathscr{O}_{T^*X} \otimes_{\pi^{-1}\mathrm{gr}\mathscr{D}_X} \pi^{-1}\mathrm{gr}\mathscr{M}$$
$$\sigma_\Lambda(\mathscr{M}) = \mathrm{Ch}_\Lambda(\widetilde{\mathrm{gr}}\mathscr{M}) \in \bigoplus_j H^{2j}_\Lambda(T^*X; \mathbb{C}_{T^*X}),$$
$$\mu\mathrm{Ch}_\Lambda(\mathscr{M}) = \sigma_\Lambda(\mathscr{M}) \cup \pi^*\mathrm{Td}_X(T^*X) \text{ for a left } \mathscr{D}\text{-module},$$
$$\mu\mathrm{Ch}_\Lambda(\mathscr{M}) = \sigma_\Lambda(\mathscr{M}) \cup \pi^*\mathrm{Td}_X(TX) \text{ for a right } \mathscr{D}\text{-module},$$

where Ch is the Chern character and Td is the Todd class. Note that μCh commutes with proper direct images (Laumon's version of the RR theorem for $\mathscr{D}$-modules [36]) and non characteristic inverse images. In [55] we made the conjecture that

$$\mu\mathrm{eu}_\Lambda(\mathscr{M}) = [\mu\mathrm{Ch}_\Lambda(\mathscr{M})]^{2d_X}$$

This conjecture has been proved in [3] by Bressler-Nest-Tsygan and generalized in [2].

Example 2.9 (i) If X is a complex compact manifold, one recovers the Riemann-Roch theorem: one takes $G = \mathbb{C}_X$ and if $\mathscr{F}$ is a coherent $\mathscr{O}_X$-module, one sets $\mathscr{M} = \mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{F}$.
(ii) If M is a compact real analytic manifold and X is a complexification of M, one recovers the Atiyah-Singer theorem by choosing $G = \mathrm{D}'_X\mathbb{C}_M$.

3 Lecture 3: Ind-Sheaves and Applications to $\mathscr{D}$-Modules

Abstract I will first recall the constructions of [25, 27] of the sheaves of temperate or Whitney holomorphic functions. These are not sheaves on the usual topology, but sheaves on the subanalytic site or better, ind-sheaves. Then I will explain how these objects appear naturally in the study of irregular holonomic $\mathscr{D}$-modules.

3.1 Ind-Sheaves

Ind-objects

References are made to [51] or to [29] for an exposition. We keep the notations of the preceding lectures.

Let $\mathscr{C}$ be an abelian category (in a given universe $\mathscr{U}$). One denotes by $\mathscr{C}^{\wedge,add}$ the big category of additive functors from $\mathscr{C}^{\mathrm{op}}$ to $\mathrm{Mod}(\mathbb{Z})$. This big category is abelian and the functor $h^\wedge\colon \mathscr{C} \to \mathscr{C}^\wedge$ makes $\mathscr{C}$ a full abelian subcategory of $\mathscr{C}^{\wedge,add}$. This functor is left exact, but not exact in general.

An ind-object in $\mathscr{C}$ is an object $A \in \mathscr{C}^\wedge$ which is isomorphic to "$\varinjlim$" α for some functor $\alpha\colon I \to \mathscr{C}$ with I filtrant and small. One denotes by $\mathrm{Ind}(\mathscr{C})$ the full additive subcategory of $\mathscr{C}^{\wedge,add}$ consisting of ind-objects.

Theorem 3.1 (i) *The category* $\mathrm{Ind}(\mathscr{C})$ *is abelian.*
(ii) *The natural functor* $\mathscr{C} \to \mathrm{Ind}(\mathscr{C})$ *is fully faithful and exact and the natural functor* $\mathrm{Ind}(\mathscr{C}) \to \mathscr{C}^{\wedge,add}$ *is fully faithful and left exact.*
(iii) *The category* $\mathrm{Ind}(\mathscr{C})$ *admits exact small filtrant inductive limits and the functor* $\mathrm{Ind}(\mathscr{C}) \to \mathscr{C}^{\wedge,add}$ *commutes with such limits.*
(iv) *Assume that* $\mathscr{C}$ *admits small projective limits. Then the category* $\mathrm{Ind}(\mathscr{C})$ *admits small projective limits, and the functor* $\mathscr{C} \to \mathrm{Ind}(\mathscr{C})$ *commutes with such limits.*

Example 3.2 Assume that $\mathbf{k}$ is a field and denote by $\mathrm{Mod}^{\mathrm{f}}(\mathbf{k})$ the category of finite dimensional $\mathbf{k}$-vector spaces. Let $\mathrm{I}(\mathbf{k})$ denote the category of ind-objects of $\mathrm{Mod}(\mathbf{k})$. Define $\beta\colon \mathrm{Mod}(\mathbf{k}) \to \mathrm{I}(\mathbf{k})$ by setting $\beta(V) =$ "$\varinjlim$" W, where W ranges over the family of finite-dimensional vector subspaces of V. In other words, $\beta(V)$ is the functor from $\mathrm{Mod}(\mathbf{k})^{\mathrm{op}}$ to $\mathrm{Mod}(\mathbb{Z})$ given by $M \mapsto \varinjlim_W \mathrm{Hom}_k(M, W)$. Therefore,

$$\varinjlim_{W\subset V,\, W\in \mathrm{Mod}^{\mathrm{f}}(\mathbf{k})} \mathrm{Hom}_{\mathbf{k}}(L, W) \simeq \mathrm{Hom}_{\mathrm{I}(\mathbf{k})}(L, \text{“}\varinjlim_{W\subset V,\, W\in \mathrm{Mod}^{\mathrm{f}}(\mathbf{k})}\text{”} W)$$
$$= \mathrm{Hom}_{\mathrm{I}(\mathbf{k})}(L, \beta(V)).$$

If V is infinite-dimensional, $\beta(V)$ is not representable in $\mathrm{Mod}(\mathbf{k})$. Moreover, $\mathrm{Hom}_{\mathrm{I}(\mathbf{k})}(\mathbf{k}, V/\beta(V)) \simeq 0$.

It is proved in [29] that the category $\mathrm{Ind}(\mathscr{C})$ for $\mathscr{C} = \mathrm{Mod}(\mathbf{k})$ does not have enough injectives.

Definition 3.3 An object $A \in \mathrm{Ind}(\mathscr{C})$ is quasi-injective if the functor $\mathrm{Hom}_{\mathrm{Ind}(\mathscr{C})}(\,\bullet\,, A)$ is exact on the category $\mathscr{C}$.

It is proved in loc. cit. that if $\mathscr{C}$ has enough injectives, then $\mathrm{Ind}(\mathscr{C})$ has enough quasi-injectives.

Ind-Sheaves

References are made to [27].

Let X be a locally compact space countable at infinity. Recall that $\mathrm{Mod}(\mathbf{k}_X)$ denotes the abelian category of sheaves of $\mathbf{k}$-modules on X. We denote by $\mathrm{Mod}^{\mathrm{c}}(\mathbf{k}_X)$ the full subcategory consisting of sheaves with compact support. We set for short:

$$\mathrm{I}(\mathbf{k}_X) := \mathrm{Ind}(\mathrm{Mod}^{\mathrm{c}}(\mathbf{k}_X))$$

and call an object of this category an indsheaf on X.

Theorem 3.4 *The prestack* $U \mapsto \mathrm{I}(\mathbf{k}_U)$*,* U *open in* X*, is a stack.*

The following example explains why we have considered sheaves with compact supports.

Example 3.5 Let $X = \mathbb{R}$, let $F = \mathbf{k}_X$, $G_n = \mathbf{k}_{[n,+\infty[}$, $G = \text{“}\varinjlim_n\text{”}\, G_n$. Then $G|_U = 0$ in $\mathrm{Ind}(\mathrm{Mod}(\mathbf{k}_U))$ for any relatively compact open subset U of X. On the other hand, $\mathrm{Hom}_{\mathrm{Ind}(\mathrm{Mod}(\mathbf{k}_X))}(\mathbf{k}_X, G) \simeq \varinjlim_n \mathrm{Hom}_{\mathbf{k}_X}(\mathbf{k}_X, G_n) \simeq \mathbf{k}$.

We have two pairs (α_X, ι_X) and (β_X, α_X) of adjoint functors

$$\mathrm{Mod}(\mathbf{k}_X) \underset{\beta_X}{\overset{\iota_X}{\rightrightarrows}} \mathrm{I}(\mathbf{k}_X), \qquad \mathrm{Mod}(\mathbf{k}_X) \xleftarrow{\ \alpha_X\ } \mathrm{I}(\mathbf{k}_X).$$

The functor ι_X is the natural one. If F has compact support, $\iota_X(F) = F$ after identifying a category $\mathscr{C}$ to a full subcategory of $\mathrm{Ind}(\mathscr{C})$. The functor α_X associates $\varinjlim_i F_i$ ($F_i \in \mathrm{Mod}^{\mathrm{c}}(\mathbf{k}_X)$, $i \in I$, I small and filtrant) to the object “$\varinjlim_i$” F_i. If $\mathbf{k}$ is a field, $\beta_X(F)$ is the functor $G \mapsto \Gamma(X; H^0(\mathrm{D}'_X G) \otimes F)$.

- ι_X is exact, fully faithful, and commutes with $\varprojlim$,
- α_X is exact and commutes with $\varprojlim$ and $\varinjlim$,
- β_X is right exact, fully faithful and commutes with $\varinjlim$,
- α_X is left adjoint to ι_X,
- α_X is right adjoint to β_X,
- $\alpha_X \circ \iota_X \simeq \mathrm{id}_{\mathrm{Mod}(\mathbf{k}_X)}$ and $\alpha_X \circ \beta_X \simeq \mathrm{id}_{\mathrm{Mod}(\mathbf{k}_X)}$.

Example 3.6 Let $U \subset X$ be an open subset, $S \subset X$ a closed subset. Then

$$\beta_X(\mathbf{k}_U) \simeq \text{“}\varinjlim_V\text{”}\, \mathbf{k}_V,\ V \text{ open },\ V \subset\subset U,$$
$$\beta_X(\mathbf{k}_S) \simeq \text{“}\varinjlim_V\text{”}\, \mathbf{k}_{\overline{V}},\ V \text{ open },\ S \subset V.$$

Let $a \in X$ and consider the skyscraper sheaf $\mathbf{k}_{\{a\}}$. Then $\beta_X(\mathbf{k}_{\{a\}}) \to \mathbf{k}_{\{a\}}$ is an epimorphism in $\mathrm{I}(\mathbf{k}_X)$ and defining N_a by the exact sequence:

$$0 \to N_a \to \beta_X(\mathbf{k}_{\{a\}}) \to \mathbf{k}_{\{a\}} \to 0$$

we get that $\mathrm{Hom}_{\mathrm{I}(\mathbf{k}_X)}(\mathbf{k}_U, N_a) \simeq 0$ for all open neighborhood U of a.

We shall not recall here the construction of the derived category of indsheaves, nor the six operations on such “sheaves”.

3.2 Sheaves on the Subanalytic Site

The subanalytic site was introduced in [27, Chapt 7] and the results on sheaves on this site were obtained as particular cases of more general results on indsheaves, which makes the reading not so easy. A direct and more elementary study of sheaves on the subanalytic site is performed in [45, 46].

Let M be a real analytic manifold. One denotes by $\mathbb{R}\text{-C}(\mathbf{k}_M)$ the abelian category of $\mathbb{R}$-constructible sheaves on M and by $\mathbb{R}\text{-C}^c(\mathbf{k}_M)$ the full subcategory consisting of sheaves with compact support. There is an equivalence $\mathsf{D}^{\mathrm{b}}(\mathbb{R}\text{-C}(\mathbf{k}_M)) \simeq \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_M)$ where this last category is the full triangulated subcategory of $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ consisting of $\mathbb{R}$-constructible sheaves. (This classical result has first been proved by Kashiwara [18].)

We denote by Op_M the category whose objects are the open subsets of M and the morphisms are the inclusions of open subsets. One defines a Grothendieck topology on Op_M by deciding that a family $\{U_i\}_{i\in I}$ of subobjects of $U \in \mathrm{Op}_M$ is a covering of U if it is a covering in the usual sense.

Definition 3.7 Denote by $\mathrm{Op}_{M_{\mathrm{sa}}}$ the full subcategory of Op_M consisting of subanalytic and relatively compact open subsets. The site M_{sa} is obtained by deciding that a family $\{U_i\}_{i\in I}$ of subobjects of $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$ is a covering of U if there exists a finite subset $J \subset I$ such that $\bigcup_{j\in J} U_j = U$.

Let us denote by

$$\rho_{\mathrm{sa}} \colon M \to M_{\mathrm{sa}} \tag{3.1}$$

the natural morphism of sites. Here again, we have two pairs of adjoint functors $(\rho_{\mathrm{sa}}^{-1}, \rho_{\mathrm{sa}*})$ and $(\rho_{\mathrm{sa}!}, \rho_{\mathrm{sa}}^{-1})$:

$$\mathrm{Mod}(\mathbf{k}_M) \underset{\rho_{\mathrm{sa}!}}{\overset{\rho_{\mathrm{sa}*}}{\underset{\xleftarrow{\ \rho_{\mathrm{sa}}^{-1}\ }}{\rightrightarrows}}} \mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}}).$$

For $F \in \mathrm{Mod}(\mathbf{k}_M)$, $\rho_{\mathrm{sa}!}F$ is the sheaf associated to the presheaf $U \mapsto F(\overline{U})$, $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$.

Proposition 3.8 *The restriction of the functor $\rho_{\mathrm{sa}*}$ to the category $\mathbb{R}\text{-C}(\mathbf{k}_M)$ is exact and fully faithful.*

By this result, we shall consider the category $\mathbb{R}\text{-C}(\mathbf{k}_M)$ as a full subcategory of $\mathrm{Mod}(\mathbf{k}_M)$ as well as a full subcategory of $\mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}})$. Set

$$\mathrm{I}_{\mathbb{R}-\mathrm{c}}(\mathbf{k}_M) = \mathrm{Ind}(\mathbb{R}\text{-C}^c(\mathbf{k}_M)).$$

Theorem 3.9 *The natural functor $\alpha_{M_{\mathrm{sa}}} \colon \mathrm{I}_{\mathbb{R}-\mathrm{c}}(\mathbf{k}_M) \to \mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}})$ is an equivalence of categories.*

In other words, ind-$\mathbb{R}$-constructible sheaves are "usual sheaves" on the subanalytic site. By this result, the embedding $\mathbb{R}\text{-C}^c(\mathbf{k}_M) \hookrightarrow \mathrm{Mod}^c(\mathbf{k}_M)$ gives a functor $I_M\colon \mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}}) \to \mathrm{I}(\mathbf{k}_M)$. Hence, we have a quasi-commutative diagram of categories

$$\begin{array}{ccc} \mathrm{Mod}(\mathbf{k}_M) & \xrightarrow{\iota_M} & \mathrm{I}(\mathbf{k}_M) \\ \uparrow & & \uparrow{\scriptstyle I_M} \\ \mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbf{k}_M) & \xrightarrow{\rho_{\mathrm{sa}*}} & \mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}}) \end{array} \tag{3.2}$$

in which all arrows are exact and fully faithful. One shall be aware that the diagram:

$$\begin{array}{ccc} \mathrm{Mod}(\mathbf{k}_M) & \xrightarrow{\iota_M} & \mathrm{I}(\mathbf{k}_M) \\ & {\scriptstyle\rho_{\mathrm{sa}*}}\searrow \quad NC & \uparrow{\scriptstyle I_M} \\ & & \mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}}) \end{array} \tag{3.3}$$

is not commutative. Moreover, ι_M is exact and $\rho_{\mathrm{sa}*}$ is not right exact in general.

One denotes by "$\varinjlim$" the inductive limit in the category $\mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}})$. One shall be aware that the functor I_M commutes with inductive limits but $\rho_{\mathrm{sa}*}$ does not.

3.3 Moderate and Formal Cohomology

From now on, $\mathbf{k} = \mathbb{C}$. As usual, we denote by $\mathscr{C}_M^\infty$ (resp. $\mathscr{C}_M^\omega$) the sheaf of complex functions of class C^∞ (resp. real analytic), by $\mathscr{D}b_M$ (resp. $\mathscr{B}_M$) the sheaf of Schwartz's distributions (resp. Sato's hyperfunctions), and by $\mathscr{D}_M$ the sheaf of analytic finite-order differential operators. We also use the notation $\mathscr{A}_M = \mathscr{C}_M^\omega$.

Definition 3.10 Let $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$ and let $f \in \mathscr{C}_M^\infty(U)$. One says that f has *polynomial growth* at $p \in M$ if it satisfies the following condition. For a local coordinate system $(x_1, \dots, x_n)$ around p, there exist a sufficiently small compact neighborhood K of p and a positive integer N such that

$$\sup_{x \in K \cap U} \big(\mathrm{dist}(x, K \setminus U)\big)^N |f(x)| < \infty\,. \tag{3.4}$$

It is obvious that f has polynomial growth at any point of U. We say that f is temperate at p if all its derivatives have polynomial growth at p. We say that f is temperate if it is temperate at any point.

For $U \in \mathrm{Op}_{M_{\mathrm{sa}}}$, denote by $\mathcal{C}_M^{\infty,\mathrm{tp}}(U)$ the subspace of $\mathscr{C}_M^\infty(U)$ consisting of tempered functions.

Denote by $\mathcal{D}b^{\rm tp}_M(U)$ the space of tempered distributions on U, defined by the exact sequence

$$0 \to \Gamma_{M\setminus U}(M;\mathcal{D}b_M) \to \Gamma(M;\mathscr{D}b_M) \to \mathcal{D}b^{\rm tp}_M(U) \to 0.$$

Using Lojasiewicz's inequalities [37] (see also [38]), one easily proves that

- the presheaf $U \mapsto \mathcal{C}^{\infty,{\rm tp}}_M(U)$ is a sheaf on $M_{\rm sa}$,
- the presheaf $U \mapsto \mathcal{D}b^{\rm tp}_M(U)$ is a sheaf on $M_{\rm sa}$.

One denotes by $\mathcal{C}^{\infty,{\rm tp}}_{M_{\rm sa}}$ the first one and calls it the sheaf of temperate C^∞-functions. One denotes by $\mathcal{D}b^{\rm tp}_{M_{\rm sa}}$ the second one and calls it the sheaf of temperate distributions. Let $F \in \mathsf{D}^{\rm b}_{\mathbb{R}\text{-c}}(\mathbb{C}_M)$. One has the isomorphism

$$\rho_{\rm sa}^{-1}{\rm R}\mathscr{H}om\,(F,\mathcal{D}b^{\rm tp}_{M_{\rm sa}}) \simeq thom(F,\mathcal{D}b_M) \tag{3.5}$$

where the right-hand side was defined by Kashiwara as the main tool for his proof of the Riemann-Hilbert correspondence in [17, 18].

For a closed subanalytic subset S in M, denote by $\mathscr{I}^\infty_{M,S}$ the subsheaf of $\mathscr{C}^\infty_M$ consisting of functions which vanish up to infinite order on S. In [25], one introduces the sheaf:

$$\mathbb{C}_U \overset{\rm w}{\otimes} \mathscr{C}^\infty_M := V \mapsto \Gamma(V;\mathscr{I}^\infty_{V,V\setminus U})$$

and shows how to extend this construction and define an exact functor $\bullet \overset{\rm w}{\otimes} \mathscr{C}^\infty_M$ on ${\rm Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_M)$. One denotes by $\mathscr{C}^{\infty,{\rm w}}_M$ the sheaf on $M_{\rm sa}$ given by

$$\mathscr{C}^{\infty,{\rm w}}_M(U) = \Gamma(M;H^0({\rm D}'_M\mathbf{k}_U)\overset{\rm w}{\otimes}\mathscr{C}^\infty_M),\, U \in {\rm Op}_{M_{\rm sa}}.$$

If ${\rm D}'_M\mathbb{C}_U \simeq \mathbb{C}_{\overline{U}}$, $\mathscr{C}^{\infty,{\rm w}}_M(U)$ is the space of Whitney functions on U, that is the quotient $\mathscr{C}^\infty(M)/\mathscr{I}^\infty_{M,M\setminus U}$. It is thus natural to call $\mathscr{C}^{\infty,{\rm w}}_M$ the sheaf of Whitney C^∞-functions on $M_{\rm sa}$.

Note that the sheaf $\rho_{{\rm sa}*}\mathscr{D}_M$ does not operate on the sheaves $\mathcal{C}^{\infty,{\rm tp}}_M$, $\mathscr{D}b^t_M$, $\mathbb{C}^{\infty,{\rm w}}_M$ but $\rho_{{\rm sa}!}\mathscr{D}_M$ does.

Now let X be a *complex* manifold. We still denote by X the real underlying manifold and we denote by $\overline{X}$ the complex manifold conjugate to X. One defines the sheaf of temperate holomorphic functions $\mathcal{O}^{tp}_{X_{\rm sa}}$ as the Dolbeault complex with coefficients in $\mathcal{C}^{\infty,{\rm tp}}_{X_{\rm sa}}$. More precisely

$$\mathcal{O}^{tp}_{X_{\rm sa}} = {\rm R}\mathscr{H}om_{\,\rho_{{\rm sa}!}\mathscr{D}_{\overline{X}}}(\rho_{{\rm sa}!}\mathscr{O}_{\overline{X}},\mathcal{C}^{\infty,{\rm tp}}_{X_{\rm sa}}). \tag{3.6}$$

One proves the isomorphism

$$\mathcal{O}^{tp}_{X_{\rm sa}} \simeq {\rm R}\mathscr{H}om_{\,\rho_{{\rm sa}!}\mathscr{D}_{\overline{X}}}(\rho_{{\rm sa}!}\mathscr{O}_{\overline{X}},\mathcal{D}b^{\rm tp}_{X_{\rm sa}}). \tag{3.7}$$

Similarly, one defines the sheaf

$$\mathcal{O}^{\mathrm{w}}_{X_{\mathrm{sa}}} = \mathrm{R}\mathcal{H}om_{\,\rho_{\mathrm{sa}!}\mathcal{D}_{\overline{X}}}(\rho_{\mathrm{sa}!}\mathcal{O}_{\overline{X}}, \mathcal{C}^{\infty,\mathrm{w}}_{X_{\mathrm{sa}}}). \tag{3.8}$$

Note that the object $\mathcal{O}^{\mathrm{t}p}_{X_{\mathrm{sa}}}$, $\mathcal{O}^{\mathrm{w}}_{X_{\mathrm{sa}}}$ and $\mathrm{R}\rho_{\mathrm{sa}*}\mathcal{O}_X$ are not concentrated in degree zero in dimension > 1. Indeed, with the subanalytic topology, only finite coverings are allowed. If one considers for example the open set $U \subset \mathbb{C}^n$, the difference of an open ball of radius $R > 0$ and a closed ball of radius r with $0 < r < R$, then the Dolbeault complex will not be exact after any finite covering.

In order that $\mathcal{O}_X$ remains concentrated in degree 0, we shall better consider ind-sheaves and we shall embed the category $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_{X_{\mathrm{sa}}})$ into the category $\mathsf{D}^{\mathrm{b}}(\mathrm{I}(\mathbb{C}_X))$ by the exact functor I_X. (Recall that Diagram (3.3) is not commutative.) Hence we consider subanalytic sheaves as indsheaves. In the category $\mathsf{D}^{\mathrm{b}}(\mathrm{I}(\mathbb{C}_X))$ we have thus the morphisms of sheaves

$$\mathcal{O}^{\omega}_X \to \mathcal{O}^{\mathrm{w}}_X \to \mathcal{O}^{\mathrm{t}p}_X \to \mathcal{O}_X.$$

Here $\mathcal{O}^{\mathrm{w}}_X$ and $\mathcal{O}^{\mathrm{t}p}_X$ are the images of $\mathcal{O}^{\mathrm{w}}_{X_{\mathrm{sa}}}$ and $\mathcal{O}^{\mathrm{t}p}_{X_{\mathrm{sa}}}$ by the functor I_X (there are still not concentrated in degree 0), we have kept the same notation for $\mathcal{O}_X$ and its image in $\mathrm{Mod}(\mathrm{I}(\mathbb{C}_X))$ by the functor ι_X, and we have set

$$\mathcal{O}^{\omega}_X := \beta_X(\mathcal{O}_X).$$

We call $\mathcal{O}^{\mathrm{w}}_X$ and $\mathcal{O}^{\mathrm{t}p}_X$ the sheaves of temperate and Whitney holomorphic functions, respectively.

Example 3.11 Let Z be a closed complex analytic subset of the complex manifold X. We have the isomorphisms

$$\begin{aligned}
&\alpha_X \mathrm{R}\mathcal{H}om_{\,\mathrm{I}(\mathbb{C}_X)}(\mathrm{D}'\mathbb{C}_Z, \mathcal{O}^{\omega}_X) \simeq \mathcal{O}_X|_Z,\\
&\alpha_X \mathrm{R}\mathcal{H}om_{\,\mathrm{I}(\mathbb{C}_X)}(\mathrm{D}'\mathbb{C}_Z, \mathcal{O}^{\mathrm{w}}_X) \simeq \mathcal{O}_X\widehat{|}_Z \text{ (formal completion along } Z),\\
&\alpha_X \mathrm{R}\mathcal{H}om_{\,\mathrm{I}(\mathbb{C}_X)}(\mathbb{C}_Z, \mathcal{O}^{\mathrm{t}p}_X) \simeq \mathrm{R}\Gamma_{[Z]}(\mathcal{O}_X) \text{ (algebraic cohomology)},\\
&\alpha_X \mathrm{R}\mathcal{H}om_{\,\mathrm{I}(\mathbb{C}_X)}(\mathbb{C}_Z, \mathcal{O}_X) \simeq \mathrm{R}\Gamma_Z(\mathcal{O}_X).
\end{aligned}$$

Example 3.12 let M be a real analytic manifold and X a complexification of M. We have the isomorphisms

$$\begin{aligned}
&\alpha_X \mathrm{R}\mathcal{H}om_{\,\mathrm{I}(\mathbb{C}_X)}(\mathrm{D}'\mathbb{C}_M, \mathcal{O}^{\omega}_X) \simeq \mathcal{A}_M,\\
&\alpha_X \mathrm{R}\mathcal{H}om_{\,\mathrm{I}(\mathbb{C}_X)}(\mathrm{D}'\mathbb{C}_M, \mathcal{O}^{\mathrm{w}}_X) \simeq \mathcal{C}^{\infty}_M,\\
&\alpha_X \mathrm{R}\mathcal{H}om_{\,\mathrm{I}(\mathbb{C}_X)}(\mathrm{D}'\mathbb{C}_M, \mathcal{O}^{\mathrm{t}p}_X) \simeq \mathcal{D}b_M,\\
&\alpha_X \mathrm{R}\mathcal{H}om_{\,\mathrm{I}(\mathbb{C}_X)}(\mathrm{D}'\mathbb{C}_M, \mathcal{O}_X) \simeq \mathcal{B}_M.
\end{aligned}$$

Notice that with this approach, the sheaf $\mathcal{D}b_M$ of Schwartz's distributions is constructed similarly as the sheaf of Sato's hyperfunctions. In particular, functional analysis is not used in the construction.

Remark 3.13 The subanalytic topology allows us to define functions whose growth at the boundary is bounded by some power of the inverse of the distance to the boundary, but not to make precise this power. In order to define such sheaves, we have recently defined with S. Guillermou in [12] the linear subanalytic topology M_{sal} on a real analytic manifold M. The open sets of this topology are those of M_{sa}, namely $\mathrm{Op}_{M_{\mathrm{sa}}}$, but there are less coverings. Roughly speaking, a finite covering $\{U_i\}_{i\in I}$ is a linear covering of $U=\bigcup_i U_i$ if there is a constant C such that for any $x\in M$

$$d(x, M\setminus\bigcup_{i\in I}U_i)\leq C\cdot\max_{i\in I}d(x,M\setminus U_i). \tag{3.9}$$

Here d is a distance on M which is locally equivalent to the Euclidian distance on $\mathbb{R}^n$. One proves that the family of linear coverings satisfies the axioms of Grothendieck topologies. One denotes by M_{sal} the site so defined and by $\rho_{\mathrm{sal}}\colon M_{\mathrm{sa}}\to M_{\mathrm{sal}}$ the natural morphism of sites. One of the main results of the theory is that the functor $\mathrm{R}\rho_{\mathrm{sal}*}\colon \mathrm{D}^+(\mathbf{k}_{M_{\mathrm{sa}}})\to\mathrm{D}^+(\mathbf{k}_{M_{\mathrm{sal}}})$ admits a right adjoint $\rho_{\mathrm{sal}}^!\colon\mathrm{D}^+(\mathbf{k}_{M_{\mathrm{sal}}})\to\mathrm{D}^+(\mathbf{k}_{M_{\mathrm{sa}}})$. Moreover, if $U\in\mathrm{Op}_{M_{\mathrm{sa}}}$ has Lipschitz boundary, then $\mathrm{R}\rho_{\mathrm{sal}*}\mathbb{C}_U$ is concentrated in degree 0. It follows that if F is a presheaf on M_{sa} such that the sequence $0\to F(U_1\cup U_2)\to F(U_1)\oplus F(U_2)\to F(U_1\cap U_2)\to 0$ is exact for any linear covering (U_1,U_2) of $U_1\cup U_2$, then there exists $F\in\mathrm{D}^+(\mathbf{k}_{M_{\mathrm{sa}}})$ such that $\mathrm{R}\Gamma(U;F)\simeq F(U)$ for all $U\in\mathrm{Op}_{M_{\mathrm{sa}}}$ with Lipschitz boundaries.

This topology allows us to define the subsheaf $\mathcal{C}^{\infty,s}_{M_{\mathrm{sal}}}$ of $\mathcal{C}^{\infty}_{M_{\mathrm{sal}}}$ consisting of functions tempered of order s. On a complex manifold X we may thus endow the sheaf $\mathcal{O}^{\mathrm{t}p}_{X_{\mathrm{sa}}}$ with a natural filtration (in the derived sense). We refer to loc. cit. for more details.

3.4 Applications to $\mathscr{D}$-Modules I

Let us show on an example extracted of [28] the possible role of the sheaf $\mathscr{O}^t_X$ in the study of irregular holonomic $\mathscr{D}$-modules.

Let X be a complex manifold and let $\mathscr{M}$ be a holonomic $\mathscr{D}$-module. We set for short

$$\mathcal{S}ol^0(\mathscr{M})=\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{O}_X),$$
$$\mathcal{S}ol^{0,t}(\mathscr{M})=\mathscr{H}om_{\beta_X\mathscr{D}_X}(\beta_X\mathscr{M},\mathscr{O}^t_X).$$

We shall compare these two objects in a simple example in which $\mathscr{M}$ is not regular. Let $X=\mathbb{C}$ endowed with the holomorphic coordinate z and let $P=z^2\partial_z+1$. We consider the $\mathscr{D}_X$-module $\mathscr{M}:=\mathscr{D}_X\exp(1/z)\simeq\mathscr{D}_X/\mathscr{D}_X\cdot P$.

Notice first that $\mathcal{O}^t_X$ is concentrated in degree 0 (since dim $X = 1$) and it is a sub-indsheaf of $\mathcal{O}_X$. It follows that the morphism $\mathcal{S}ol^{0,t}(\mathcal{M}) \to \mathcal{S}ol^0(\mathcal{M})$ is a monomorphism. Moreover,

$$\mathcal{S}ol^0(\mathcal{M}) \simeq \mathbb{C}_{X,X\setminus\{0\}} \cdot \exp(1/z).$$

It follows that for $V \subset X$ a connected open subset, we have $\Gamma(V; \mathcal{S}ol^{0,t}(\mathcal{M})) \neq 0$ if and only if $V \subset X \setminus \{0\}$ and $\exp(1/z)|_V$ is tempered.

Let $\bar{B}_\varepsilon$ denote the closed ball with center $(\varepsilon, 0)$ and radius ε and set $U_\varepsilon = X \setminus \bar{B}_\varepsilon$.

Then one proves that $\exp(1/z)$ is temperate (in a neighborhood of 0) on an open subanalytic subset $V \subset X \setminus \{0\}$ if and only if $\mathrm{Re}(1/z)$ is bounded on V, that is, if and only if $V \subset U_\varepsilon$ for some $\varepsilon > 0$. We get

Proposition 3.14 *One has the isomorphism*

$$\text{“}\varinjlim_{\varepsilon>0}\text{”}\, \mathbb{C}_{XU_\varepsilon} \xrightarrow{\sim} \mathcal{S}ol^{0,t}(\mathcal{M}). \tag{3.10}$$

Unfortunately, the functor $\mathcal{S}ol^t$ (as well as its derived functor) is not fully faithful since the $\mathcal{D}$-modules $\mathcal{M} := \mathcal{D}_X \exp(1/z)$ and $\mathcal{N} := \mathcal{D}_X \exp(2/z)$ have the same indsheaves of temperate holomorphic solutions although they are not isomorphic.[1]

Proposition 3.14 has been generalized to the study of holonomic modules in dimension one in [41].

3.5 Applications to $\mathcal{D}$-Modules II

For $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$, set (see (3.5)):

$$F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X := \mathrm{R}\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, F \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X),$$
$$thom(F, \mathcal{O}_X) := \mathrm{R}\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, thom(F, \mathcal{D}b_X)).$$

Let $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$ and $\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D})$. Recall that we have set $\omega^{\mathrm{hol}}_X := \Omega_X[d_X]$. Set for short

$$W(\mathcal{M}, F) := \mathrm{R}\mathcal{H}om_{\mathcal{D}}(\mathcal{M}, F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X),$$
$$T(F, \mathcal{M}) := thom(F, \omega^{\mathrm{hol}}_X) \overset{\mathrm{L}}{\otimes}_{\mathcal{D}} \mathcal{M}.$$

There is a natural morphism

$$W(\mathcal{M}, F) \otimes T(F, \mathcal{M}) \to \omega^{\mathrm{hol}}_X, \tag{3.11}$$

[1]This difficulty is overcome in [8] by adding a variable.

functorial in F and $\mathscr{M}$. For $G \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$ one gets a pairing

$$\begin{aligned} \mathrm{RHom}\,(G, W(\mathscr{M}, F)) \otimes \mathrm{R}\Gamma_c(X; G \otimes T(F, \mathscr{M})) & \\ \to \mathrm{R}\Gamma_c(X; W(\mathscr{M}, F) \otimes T(F, \mathscr{M})) & \\ \to \mathrm{R}\Gamma_c(X; \omega_X^{\mathrm{hol}}) \to \mathbb{C}. & \end{aligned} \tag{3.12}$$

Denote by $\mathsf{D}^{\mathrm{b}}(FN)$ the derived category of the quasi-abelian category of Fréchet nuclear $\mathbb{C}$-vector spaces and define similarly the category $\mathsf{D}^{\mathrm{b}}(DFN)$, where now DFN stands for "dual of Fréchet nuclear".

Theorem 3.15 ([25, Theorem 6.1]) *Let $F, G \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathbb{C}_X)$ and $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D})$. Then the two complexes*

$$\mathrm{RHom}\,(G, W(\mathscr{M}, F)) \in \mathsf{D}^{\mathrm{b}}(FN) \textit{ and } \mathrm{R}\Gamma_c(X; G \otimes T(F, \mathscr{M})) \in \mathsf{D}^{\mathrm{b}}(DFN)$$

are dual to each other through (3.12)*, functorially in F, G and $\mathscr{M}$.*

Now we assume that $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X)$ and we consider the following assertions.

(a) $W(\mathscr{M}, F) = \mathrm{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{M}, F \overset{\mathrm{w}}{\otimes} \mathscr{O}_X)$ is $\mathbb{R}$-constructible,
(b) $T(F, \mathscr{M}) = thom(F, \omega_X^{\mathrm{hol}}) \overset{\mathrm{L}}{\otimes}_{\mathscr{D}} \mathscr{M}$ is $\mathbb{R}$-constructible,
(c) the two complexes in (a) and (b) are dual to each other in the category $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$, that is, $W(\mathscr{M}, F) \simeq \mathrm{D}_X T(F, \mathscr{M})$.

It was conjectured[2] in [28] that (b) is always satisfied. Based on the work of Mochizuki [40] (see also [34, 35, 49]), partial results in this direction have been obtained in [42].

On the other hand, one deduces easily from Theorem 3.15 that (a) and (b) are equivalent and imply (c). Finally, it follows immediately from [16, 18] that (b), hence (a) and (c), are true when $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)$.

Corollary 3.16 *Assume that $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(\mathbb{C}_X)$ and X is compact. Then the complexes $\mathrm{R}\Gamma(X; W(\mathscr{M}, F))$ and $\mathrm{R}\Gamma(X; T(F, \mathscr{M}))$ have finite-dimensional cohomology and* (3.12) *induces a perfect pairing for all $i \in \mathbb{Z}$*

$$H^{-i}\mathrm{R}\Gamma(X; W(\mathscr{M}, F)) \otimes H^i\mathrm{R}\Gamma(X; T(F, \mathscr{M})) \to \mathbb{C},$$

functorial in F and $\mathscr{M}$.

In [1], S. Bloch and H. Esnault prove directly a similar result on an algebraic curve X when assuming that $\mathscr{M}$ is a meromorphic connection with poles on a divisor D. They interpret the duality pairing by considering sections of the type $\gamma \otimes \varepsilon$, where γ is a cycle with boundary on D and ε is a horizontal section of the connection on γ

[2]This result is now proved in [33, Th. 2.5.13].

with exponential decay on D. Their work has been extended to higher dimension by M. Hien [13].

It would be interesting to make a link with these results and Corollary 3.16.

Remark 3.17 After this paper has been written, important progress have been made in the study of irregular holonomic $\mathscr{D}$-modules. See [8, 32] and see [33] for a survey.

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Microlocal Condition for Non-displaceability

Dmitry Tamarkin

To Boris Tsygan on his 50th birthday

Abstract We formulate a sufficient condition for non-displaceability (by Hamiltonian symplectomorphisms which are identity outside of a compact) of a pair of subsets in a cotangent bundle. This condition is based on micro-local analysis of sheaves on manifolds by Kashiwara–Schapira. This condition is used to prove that the real projective space and the Clifford torus inside the complex projective space are mutually non-displaceable.

1 Introduction

Let M be a symplectic manifold and $A, B \subset M$ its compact subsets. A and B are called non-displaceable if $A \cap X(B) \neq \emptyset$, where X is any Hamiltonian symplectomorphism of M which is identity outside of a compact. Given such A and B, it is, in general, a non-trivial problem to decide, whether they are displaceable or not (see, for example, [3] and the literature therein). In non-trivial cases (when, say, A and B can be displaced by a diffeomorphism), all the methods known so far use different versions of Floer cohomology.

In this paper we introduce a sufficient condition for non-displaceability in the case when $M = T^*X$ with the standard symplectic structure. Our approach is based on Kashiwara–Shapira's microlocal theory of sheaves on manifolds and is independent

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D. Tamarkin (✉)
Northwestern University, Evanston, IL, USA
e-mail: tamarkin@math.northwestern.edu

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of Floer's theory. We apply our condition in the following setting. Let our symplectic manifold be $\mathbb{CP}^N$ with the standard symplectic structure and let our subsets be $\mathbb{RP}^N \subset \mathbb{CP}^N$ and $\mathbb{T}^N \subset \mathbb{CP}^N$, where $\mathbb{T}^N$ is the Clifford torus consisting of all points $(z_0 : z_1 : \cdots : z_N)$ such that $|z_0| = |z_1| = \cdots = |z_N|$. Let A and B be arbitrarily chosen from the two subsets specified, we show that such A and B are non-displaceable. Same result has been proven in [3] using Hamiltonian Floer theory. Non-displaceability of Clifford tori has been proven in [4] via computing Floer cohomology.

Observe that our condition applies despite $\mathbb{CP}^N \neq T^*X$. We use a certain Lagrangian correspondence between $T^*\mathrm{SU}(N)$ and $\mathbb{CP}^N \times (\mathbb{CP}^N)^{\mathrm{opp}}$, where the symplectic form on $(\mathbb{CP}^N)^{\mathrm{opp}}$ equals the opposite to that on $\mathbb{CP}^N$, see Sect. 4.0.1. This way our original problem gets reduced to non-displaceability of certain subsets in $T^*\mathrm{SU}(N)$.

Let us now get back to the non-displaceability condition for subsets in a symplectic manifold T^*X, where X is a smooth manifold. Fix a ground field $\mathbb{K}$. We start with a category $\mathcal{D}(X)$ which is defined as a full subcategory of the unbounded derived category of sheaves of $\mathbb{K}$-vector spaces on $X \times \mathbb{R}$, consisting of all objects $F \in D(X \times \mathbb{R})$ satisfying the following condition: for any open $U \subset X$ and any $c \in \mathbb{R} \cup \{\infty\}$, $R\Gamma_c(U \times (-\infty, c); F) = 0$. The category $\mathcal{D}(X)$ admits a microlocal definition. Let ∂_t be the vector field on $X \times \mathbb{R}$ corresponding to the infinitesimal shifts along $\mathbb{R}$. Let $\Omega_{\leq 0} \subset T^*(X \times \mathbb{R})$ be the subset consisting of all 1-forms η satisfying $i_{\partial_t}\eta \leq 0$. Let $\mathcal{C}_{\leq 0} \subset D(X \times \mathbb{R})$ be the full subcategory consisting of all objects microsupported on $\Omega_{\leq 0}$. One can show that $\mathcal{D}(X)$ is the left orthogonal complement to $\mathcal{C}_{\leq 0}$.

One can show that the embedding $\mathcal{C}_{\leq 0} \subset D(X \times \mathbb{R})$ admits a left adjoint. Therefore, $\mathcal{D}(X)$ can be identified with a quotient $D(X \times \mathbb{R})/\mathcal{C}_{\leq 0}$. This motivates us to define microsupports of objects from $\mathcal{D}(X)$ as conic closed subsets of $\Omega_{>0} := T^*(X \times \mathbb{R}) \backslash \Omega_{\leq 0}$. Thus, we set $\mathrm{SS}_{\mathcal{D}}(F) := \mathrm{SS}(F) \cap \Omega_{>0}$ for any $F \in \mathcal{D}(X)$.

Let us identify $T^*(X \times \mathbb{R}) = T^*X \times T^*\mathbb{R}$. Let $A \subset T^*X$ be a subset. Define $\mathrm{Cone}(A) \subset \Omega_{>0}$ to consist of all points $(\eta, \alpha) \in T^*X \times T^*\mathbb{R}$ such that $i_{\partial_t}\alpha > 0$ (meaning that $(\eta, \alpha) \in \Omega_{>0}$) and

$$\frac{\eta}{i_{\partial_t}\alpha} \in A.$$

Let $\mathcal{D}_A(X) \subset \mathcal{D}(X)$ be the full subcategory consisting of all $F \in \mathcal{D}(X)$ such that $\mathrm{SS}_{\mathcal{D}}(F) \subset \mathrm{Cone}(A)$. This way we can link subsets of T^*X with the category $\mathcal{D}(X)$.

Let $c \in \mathbb{R}$, let $T_c : X \times \mathbb{R} \to X \times \mathbb{R}$ be the shift by c: $T_c(x, t) = (x, t + c)$. One sees that $T_c(\mathrm{Cone}(A)) = \mathrm{Cone}(A)$. Therefore, the endofunctor $T_{c*} : D(X \times \mathbb{R}) \to D(X \times \mathbb{R})$ preserves $\mathcal{D}_A(X)$ for all A. For any $c > 0$, one can construct a natural transformation $\tau_c : \mathrm{Id} \to T_{c*}$ of endofunctors on $\mathcal{D}_A(X)$ for any A, see Sect. 2.2.2.

We can now formulate the non-displaceablity condition (Theorem 3.1).

Let $A, B \subset T^*X$ *be compact subsets. Suppose there exist* $F_A \in \mathcal{D}_A(X)$; $F_B \in \mathcal{D}_B(X)$ *such that for any* $c \geq 0$, *the natural map* $R\hom(F_A; F_B) \to R\hom(F_A; T_{c*}F_B)$, *induced by* τ_c, *does not vanish. Then* A *and* B *are non-displaceable.*

Remark. For $c \in \mathbb{R}$ set $H_c(F_A, F_B) := H_c := R\hom(F_A; T_{c*}F_B)$. For any $d \geq 0$, the natural transformation τ_d induces a map $\tau_{c,c+d} : H_c \to H_{c+d}$.

Let $H(F_A, F_B) := H \subset \prod_{c\in\mathbb{R}} H_c$ be defined as a subset consisting of all collections $h_c \in H_c$ such that there exists a sequence $c_1 < c_2 < \cdots < c_n < \cdots ; c_n \to \infty$ such that $h_c = 0$ for all $c \notin \{c_1, c_2, \ldots, c_n, \ldots\}$. The maps $\tau_{c,c+d}$ induce maps $\tau_d : H \to H$ for all $d \geq 0$. This way we get an action of the semigroup $\mathbb{R}_{\geq 0}$ on H. This implies that Novikov's ring, which is a group ring of $\mathbb{R}_{\geq 0}$, acts on H. There are indications that thus defined module over Novikov's ring H is related to Floer cohomology of the pair A, B. In this language, our nondisplaceability condition means that $H(F_A, F_B)$ has a non-trivial non-torsion part.

Remark. It seems likely that under an appropriate version of Riemann–Hilbert correspondence our picture should become similar to the setting of [9]. This paper can be considered as an attempt to translate [9] into the language of constructible sheaves.

Remark. There is some similarity between our theory and the approach from [7, 8] where the authors identify the derived category of constructible sheaves on X with a certain version of the Fukaya category on T^*X. The authors use exact Lagrangian submanifolds of T^*X which are close to being conic, whereas we work with compact subsets of T^*X, some of them being non-exact Lagrangian submanifolds.

Let us now briefly describe the way our non-displaceability condition is applied to the above mentioned example $\mathbb{RP}^N, \mathbb{T}^N \subset \mathbb{CP}^N$. As was explained, the problem can be reduced to proving non-displaceability of certain subsets of $T^*\mathrm{SU}(N)$. Given such a subset, say A, it is, in general, a non-trivial problem to construct a non-zero object $F \in \mathcal{D}_A(\mathrm{SU}(N))$. Our major tool here is a certain object $\mathfrak{S} \in D(G \times \mathfrak{h})$ which is defined uniquely up-to a unique isomorphism by certain microlocal conditions to be now specified. Here $G = \mathrm{SU}(N)$ and $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}$, the Lie algebra of $\mathrm{SU}(N)$.

Let $C_+ \subset \mathfrak{h}$ be the positive Weyl chamber. For every $A \in \mathfrak{g}$ there exists a unique element $\|A\| \in \mathfrak{h}$ such that $\|A\|$ is conjugated with A. Let us identify $T^*(G \times \mathfrak{h}) = G \times \mathfrak{h} \times \mathfrak{g}^* \times \mathfrak{h}^*$ (via interpreting $\mathfrak{g}^*$ as the space of right-invariant 1-forms on G). Let us identify $\mathfrak{g}^* = \mathfrak{g}$, $\mathfrak{h}^* = \mathfrak{h}$ by means of the Killing form. Let $\Omega_{\mathfrak{S}} \subset G \times \mathfrak{h} \times \mathfrak{g} \times \mathfrak{h} = \Omega_{\mathfrak{S}} \subset G \times \mathfrak{h} \times \mathfrak{g}^* \times \mathfrak{h}^*$ consist of all points of the form (g, X, ω, η), where $\eta = \|\omega\|$. Let also $i_0 : G \to G \times \mathfrak{h}$ be the embedding $i_0(g) = (g, 0)$. We then define $\mathfrak{S}$ as an object of $D(G \times \mathfrak{h})$ such that $\mathrm{SS}(\mathfrak{S}) \subset \Omega_{\mathfrak{S}}$ and $i_0^{-1}\mathfrak{S} \cong \mathbb{K}_e$, where $\mathbb{K}_e$ is the skyscraper at the unit $e \in G$. One can show that this way $\mathfrak{S}$ is determined uniquely up-to a unique isomorphism. It turns out that the required objects $F_A \in \mathcal{D}_A(X)$, $F_B \in \mathcal{D}_B(X), \ldots,$ can be easily expressed in terms of $\mathfrak{S}$.

Our next task is to compute the graded vector spaces $R\hom(F_A, T_{c*}F_B)$ and to make sure that the maps $\tau_c : R\hom(F_A, F_B) \to R\hom(F_A, T_{c*}F_B)$ are not zero for all $c \geq 0$. This problem gets gradually reduced to finding an explicit description of the restriction $i_e^{-1}\mathfrak{S} \in D(\mathfrak{h})$, where $i_e : \mathfrak{h} \to G \times \mathfrak{h}$, $i_e(X) = (e, X)$, and $e \in G$ is the unit.

Remark. Let $C_- := -C_+$, let $C_-^\circ \subset C_-$ be the interior. It turns out that the stalks of $(i_e^{-1}\mathfrak{S}|_{C_-})$ have a transparent topological meaning (however, this meaning won't be used in our proofs). Let $X \in C_-$; let $O(X) := \mathfrak{S}|_{e\times X}[-\dim \mathfrak{h}]$.

On the other hand, let us consider the smooth loop space ΩG. For $\gamma : [0, 1] \to G$ being a smooth loop, we set $\|\gamma\| \in C_+$,

$$\|\gamma\| := \int_0^1 \|\gamma'(t)\|dt,$$

where $\gamma'(t) \in \mathfrak{g}$ is the t-derivative of γ. Let $\Omega_X \subset \Omega(G)$ be the subspace consisting of all loops γ such that $\|\gamma\| \leq -X$ (here $Y \leq -X$ means $< Y + X, C_+ > \leq 0$, where $<, >$ is the restriction of the positive definite invariant form on $\mathfrak{g}$ onto $\mathfrak{h}$). It can be shown that $\mathcal{O}(X) \cong H_\bullet(\Omega_X)$.

In regard with this setting, one can ask the following question (which will be probably discussed in a subsequent paper). We have an obvious concatenation map $\Omega_X \times \Omega_Y \to \Omega_{X+Y}$ whence a product $\mathcal{O}(X) \otimes \mathcal{O}(Y) \to \mathcal{O}(X+Y)$. One can show that this product is commutative so that the spaces $\mathcal{O}()$ form a filtered commutative algebra. It can be shown that this algebra can be obtained in the following algebro-geometric way. Let $\mathcal{FL}$ be the projective $\mathbb{K}$-variety of complete flags in $\mathbb{K}^N$. Fix a regular nilpotent operator $n : \mathbb{K}^N \to \mathbb{K}^N$ (that is, n consists of one Jordan block). Let **Pet** $\subset \mathcal{FL}(N)$ be the closed subvariety consisting of all flags $0 = V_0 \subset V_1 \subset \cdots V_N = \mathbb{K}^N$ satisfying $nV_i \subset V_{i+1}$ for all $i < N$. This variety was discovered by Peterson, see e.g. [6].

Let $\mathbb{L} \in \mathfrak{h}$ be the lattice formed by all elements X such that e^X is in the center of G. Given $l \in \mathbb{L}$ we canonically have a line bubble L_l on $\mathcal{FL}$. It turns out that for all $l \in \mathbb{L} \cap C_+$ we have an isomorphism $O(l) = \Gamma(\mathbf{Pet}; L_l|_{\mathbf{Pet}})$, and this isomorphism is compatible with the natural product on both sides.

A related result is proven in [5], where, among other interesting results, the authors identify $H_\bullet(\Omega(G))$ with the algebra of functions on a certain affine open subset of **Pet**.

Let us now go over the content of the paper. In Sects. 2, 3 we formulate and prove the non-displaceability condition.

In Sect. 4 we start applying the non-displaceability condition to $\mathbb{RP}^N$, $\mathbb{T}^N \subset \mathbb{CP}^N$. Finally, the problem is reduced to the existence of an object $u_\mathcal{O} \in \mathcal{D}(G)$ satisfying certain properties (see Proposition 4.4).

In Sect. 5 the object $u_\mathcal{O}$ gets constructed out of $\mathfrak{S}$ (where we use certain properties of $\mathfrak{S}$ to be proven in the subsequent sections).

The rest of the paper is devoted to constructing and studying $\mathfrak{S}$. In Sect. 6 we construct an object $\mathfrak{S}$ and prove its uniqueness.

In Sect. 7 we compute an isomorphism type of $\mathfrak{S}|_{z \times C_-^\circ}$ where z is any element in the center of G. In essense, the computation is a version of Bott's computation of $H_\bullet(\Omega(G))$ using Morse theory.

The goal of Sect. 8 is to extend the result of the previous section to $z \times \mathfrak{h}$. This is done by means of establishing a certain periodicity property of $\mathfrak{S}$ with respect to shifts along $\mathfrak{h}$ by elements of the lattice $\mathbb{L} = \{X \in \mathfrak{h} | e^X \in \mathbf{Z}\}$, where $\mathbf{Z} \subset G$ is the center. Namely, we show that $\mathfrak{S}$ is, what we call, *a strict B-sheaf.* (see Sect. 8.2). We show that any strict B-sheaf can be recovered from its restriction onto $\mathbf{Z} \times C_-^\circ$. By virtue of this statement we are able to identify the isomorphism type of $\mathfrak{S}|_{\mathbf{Z} \times G}$.

There are two appendices. In the first one we introduce the notation used when working with $\mathrm{SU}(N)$ and its Lie algebra. We also included a couple of useful Lemmas (which, most likely, can be found elsewhere in the literature). These Lemmas are mainly used when constructing and studying $\mathfrak{S}$. The notation is used systematically starting from Sect. 5.

In the second appendix we list, for the reader's convenience, the rules for computing the microsupport from [1]. These rules are used throughout the paper.

Strictly speaking, these rules are proved in [1] for the bounded derived category. However, one sees that they carry over directly to the unbounded derived category, in which case we use them.

2 Generalities

2.1 Unbounded Derived Category

2.1.1

Fix a ground field $\mathbb{K}$. The Abelian category Sh_M of sheaves of $\mathbb{K}$-vector spaces on a smooth manifold M is of finite injective dimension. Therefore, one has a simple model of the unbounded derived category $D(M)$, namely one can take unbounded complexes of injective sheaves on M; given two such complexes, we define $\hom_{D(M)}(I_1, I_2) := H^0 \hom^\bullet(I_1, I_2)$. This definition is stable under quasi-isomorphisms precisely because of finite injective dimension of Sh_M. The main results of the formalism of 6 functors remain valid for $D(X)$ (excluding the Verdier duality).

2.1.2

We still have a notion of singular support of an object of $D(M)$ and it is defined in the same way as in [1] The results on functorial properties of singular support from Chaps. 5, 6 of [1] are still valid for the unbounded derived category, and we will freely use them. For the convenience of the reader the results from [1] used in this paper are listed in Sect. 11

2.2 Sheaves on $X \times \mathbb{R}$

Let X be a smooth manifold. We will work with the manifold $X \times \mathbb{R}$. Let t be the coordinate on $\mathbb{R}$ and let $V = \partial/\partial t$ be the vector field corresponding to the infinitesimal shift along $\mathbb{R}$. Let $\Omega_{\leq 0} \subset T^*(X \times \mathbb{R})$ be the closed subset consisting of all 1-forms ω with $(\omega, V) \leq 0$. Let $\Omega_{>0} \subset T^*(X \times \mathbb{R})$ be the complement to $\Omega_{\leq 0}$, i.e. the set of all 1-forms ω such that $(\omega, V) > 0$.

Let $C_{\leq 0}(X) \subset D(X \times \mathbb{R})$ be the full subcategory of objects microsupported on $\Omega_{\leq 0}$. Let $\mathcal{D}(X) := D(X \times \mathbb{R})/C_{\leq 0}(X)$.

Proposition 2.1 *The embedding $C_{\leq 0}(X) \to D(X \times \mathbb{R})$ has a left adjoint. Therefore, $\mathcal{D}(X)$ is equivalent to the left orthogonal complement to $C_{\leq 0}(X)$ in $D(X \times \mathbb{R})$.*

Proof Let $p_1 : X \times \mathbb{R} \times \mathbb{R} \to X \times \mathbb{R}$; $p_2 : X \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$; $a : X \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be given by $p_1(x, t_1, t_2) = (x, t_1)$; $p_2(x, t_1, t_2) = t_2$; $a(x, t_1, t_2) = t_1 + t_2$. For $F \in D(X \times \mathbb{R})$ and $S \in D(\mathbb{R})$ set $F *_{\mathbb{R}} S := Ra_!(p_1^{-1} F \otimes p_2^{-1} S)$.

It is clear that $F *_{\mathbb{R}} \mathbb{K}_0 \cong F$ where $\mathbb{K}_0$ is the skyscraper at $0 \in \mathbb{R}$.

We have a natural map $\mathbb{K}_{[0,\infty)} \to \mathbb{K}_0$ in $D(\mathbb{R})$.

For an $F \in D(X \times \mathbb{R})$, consider the induced map

$$F *_{\mathbb{R}} \mathbb{K}_{[0,\infty)} \to F *_{\mathbb{R}} \mathbb{K}_0 = F. \tag{1}$$

(1) Let us show that *$F *_{\mathbb{R}} \mathbb{K}_{[0,\infty)}$ is in the left orthogonal complement to $C_{\leq 0}(X)$.*

Indeed, let $G \in C_{\leq 0}(X)$. Let $U \subset X$ be an open subset and let $(a, b) \subset \mathbb{R}$. Any object $F \in D(X \times \mathbb{R})$ can be produced from objects of the type $\mathbb{K}_{U \times (a,b)}$ for various U and (a, b) by taking direct limit. Therefore, without loss of generality, one can assume $F = \mathbb{K}_{U \times (a,b)}$. One then has

$$R\operatorname{hom}_{X \times \mathbb{R}}(F *_{\mathbb{R}} \mathbb{K}_{[0,\infty)}; G) =$$

$$= R\operatorname{hom}_{X \times \mathbb{R}}(\mathbb{K}_{U \times (a,b)} *_{\mathbb{R}} \mathbb{K}_{[0,\infty)}; G)$$

$$= R\operatorname{hom}_{X \times \mathbb{R}}(\mathbb{K}_{U \times [a,\infty)}[-1]; G)$$

$$= \operatorname{Cone}(R\Gamma(U \times \mathbb{R}; G) \xrightarrow{r} R\Gamma(U \times (-\infty; a); G)).$$

The map r is an isomorphism because $G \in C_{\leq 0}$. Therefore, $\operatorname{Cone}(r) = 0$, whence the statement.

(2) *Cone of the map* (1) *is in $C_{\leq 0}(X)$.* Indeed, consider the cone of the map $\mathbb{K}_{[0,\infty)} \to \mathbb{K}_0$. This cone is isomorphic to $\mathbb{K}_{(0,\infty)}[1]$. One then has to check that $F *_{\mathbb{R}} \mathbb{K}_{(0,\infty)} \in C_{\leq 0}(X)$. One can represent F as an inductive limit of compactly supported objects. Therefore, without loss of generality, one can assume F is compactly supported. One then can estimate the microsupport of $F *_{\mathbb{R}} \mathbb{K}_{[0,\infty)}$ using functorial properties of microsupport. Indeed, let us identify

$$T^*(X \times \mathbb{R} \times \mathbb{R}) = T^*X \times T^*(\mathbb{R} \times \mathbb{R}).$$

Let us also identify $T^*(\mathbb{R} \times \mathbb{R}) = \mathbb{R}^4$ so that a point $(t_1, t_2, k_1, k_2) \in \mathbb{R}^4$ corresponds to the 1-form $k_1 dt_1 + k_2 dt_2$ at the point $(t_1, t_2) \in \mathbb{R} \times \mathbb{R}$. We then have

$$p_1^{-1} F \otimes p_2^{-1} \mathbb{K}_{(0,\infty)} = F \boxtimes \mathbb{K}_{(0,\infty)};$$

$$\mathrm{SS}(F \boxtimes \mathbb{K}_{(0,\infty)})$$

$$\subset \{(\omega, t_1, t_2, k_1, k_2) \in T^*X \times \mathbb{R}^4 | (t_2, k_2) \in \mathrm{SS}(\mathbb{K}_{(0,\infty)})\}.$$

This means that either $t_2 = 0$ and $k_2 \leq 0$ or $t_2 > 0$ and $k_2 = 0$.

As F is compactly supported, it follows that the map a is proper on the support of $F \boxtimes \mathbb{K}_{(0,\infty)}$.

Therefore, $\mathrm{SS}Ra_!(F \boxtimes \mathbb{K}_{(0,\infty)})$ is contained in the set of all points $(\omega, t, k) \in T^*X \times \mathbb{R}^2$ such that there exists a point $(\omega, t_1, t_2, k_1, k_2) \in \mathrm{SS}(F \boxtimes \mathbb{K}_{(0,\infty)})$ such that $t = t_1 + t_2$; $k_1 = k_2 = k$. This implies that $k \leq 0$, therefore,

$$F *_{\mathbb{R}} \mathbb{K}_{(0,\infty)} = Ra_!(F \boxtimes \mathbb{K}_{(0,\infty)}) \in C_{\leq 0}(X),$$

as was required.

The statements (1) and (2) imply that we have an exact triangle

$$\to F *_{\mathbb{R}} \mathbb{K}_{(0,\infty)} \to F *_{\mathbb{R}} \mathbb{K}_{[0,\infty)} \to F \to F *_{\mathbb{R}} \mathbb{K}_{(0,\infty)}[1] \to \cdots,$$

where $F *_{\mathbb{R}} \mathbb{K}_{(0,\infty)}[1]$ is in $C_{\leq 0}(X)$ and $F *_{\mathbb{R}} \mathbb{K}_{[0,\infty)}$ is in the left orthogonal complement to $C_{\leq 0}(X)$. Therefore, $F \mapsto F *_{\mathbb{R}} \mathbb{K}_{(0,\infty)}[1]$ is the left adjoint functor to the embedding $C_{\leq 0}(X) \to D(X \times \mathbb{R})$. □

Thus, we have proven

Proposition 2.2 *An object $F \in D(X \times \mathbb{R})$ is in the left orthogonal complement to $C_{\leq 0}(X)$ iff the map (1) is an isomorphism.*

2.2.1

From now on we identify $\mathcal{D}(X)$ with a full subcategory of $D(X \times \mathbb{R})$ which is the left orthogonal complement to $C_{\leq 0}(X)$. Thus, the arrow (1) is an isomorphism for any $F \in \mathcal{D}(X) \subset D(X \times \mathbb{R})$ (and only for objects from $\mathcal{D}(X)$).

2.2.2

Let $T_c : X \times \mathbb{R} \to X \times \mathbb{R}$ be the shift along $\mathbb{R}$ by c: $T_c(x, t) = (x, t + c)$. We have $T_{c*}F = F *_{\mathbb{R}} \mathbb{K}_c$. If $F \in \mathcal{D}$, we have

$$T_{c*}F \cong F *_{\mathbb{R}} \mathbb{K}_{[0,\infty)} *_{\mathbb{R}} \mathbb{K}_c \cong F *_{\mathbb{R}} \mathbb{K}_{[c,\infty)}. \tag{2}$$

One can easily check that $T_{c*}F \in \mathcal{D}(X)$; for example, this follows from an isomorphism

$$F *_{\mathbb{R}} \mathbb{K}_{[c,\infty)} \cong T_{c*}F *_{\mathbb{R}} \mathbb{K}_{[0,\infty)},$$

which is the case for any $F \in D(X \times \mathbb{R})$.

For all $c \geq d$ we then have a natural map $T_{d*}F \to T_{c*}F$ which is induced by the embedding $[c, \infty) \subset [d, \infty)$ and we use the identification (2). This implies that we have natural transformations $\tau_{dc} : T_{d*} \to T_{c*}$ of endofunctors on $\mathcal{D}(X)$ for all $d \leq c$. It is clear that $\tau_{dc}\tau_{ed} = \tau_{ec}$ for all $e \leq d \leq c$.

2.2.3

Call an object $F \in \mathcal{D}(X)$ *a torsion object* if there exists $c > 0$ such that the natural map $\tau_{0c} : F \to T_{c*}F$ is zero in $\mathcal{D}(X)$.

2.2.4

Still thinking of $\mathcal{D}(X)$ as a quotient $D(X \times \mathbb{R})/C_{\leq 0}(X)$, the microsupport of an object $F \in \mathcal{D}(X)$ is naturally defined as a closed subset of $\Omega_{>0} \subset T^*(X \times \mathbb{R})$. Denote this microsupport by $\mathrm{SS}_{\mathcal{D}}(F) \subset \Omega_{>0}$.

Let us see what this means in terms of the identification of $\mathcal{D}(X)$ with a full subcategory of $\mathcal{D}(X)$ which is the left orthogonal complement to $C_{\leq}(X)$. Let $F \in \mathcal{D}(X) \subset D(X \times \mathbb{R})$. We then have $\mathrm{SS}_{\mathcal{D}}(F) = \mathrm{SS}(F) \cap \Omega_{>0}$, where $\mathrm{SS}(F)$ is the microsupport of F which is viewed as an object of $D(X \times \mathbb{R})$.

2.2.5

Let us identify $T^*\mathbb{R} = \mathbb{R} \times \mathbb{R}$ so that $(t_0, k) \in \mathbb{R} \times \mathbb{R}$ corresponds to the 1-form kdt at the point $t_0 \in \mathbb{R}$. We then have an induced identification $T^*(X \times \mathbb{R}) = T^*X \times \mathbb{R} \times \mathbb{R}$.

Let $A \subset T^*X$ be a subset. Define the conification $\mathrm{Cone}(A) \subset \Omega_{>0}$ to consist of all points $(\omega, t, k) \in T^*X \times \mathbb{R} \times \mathbb{R}$ such that $k > 0, (x, \omega/k) \in A$. Let $\mathcal{D}_A(X) \subset \mathcal{D}(X)$ be the full subcategory consisting of all objects $F \in \mathcal{D}(X)$ such that $\mathrm{SS}_{\mathcal{D}}(F) \subset \mathrm{Cone}(A)$.

3 Non-displaceability Condition

Let X be a compact manifold. Let $L_1, L_2 \subset T^*X$ be compact subsets. Call L_1, L_2 mutually non-displaceable if for every Hamiltonian symplectomorphism Φ of T^*X which is identity outside of a compact, $\Phi(L_1) \cap L_2 \neq \emptyset$. Our goal is to prove

Theorem 3.1 *Suppose there exist objects $F_i \in \mathcal{D}_{L_i}(X)$, $i = 1, 2$ such that for all $c > 0$ the natural map*

$$\tau_c : R\,\mathrm{hom}(F_1, F_2) \to R\,\mathrm{hom}(F_1, T_cF_2)$$

is not zero. Then L_1 and L_2 are mutually non-displaceable.

The proof will occupy the whole section.

3.1 Disjoint Supports

Our goal is to prove:

Theorem 3.2 *let $F_i \in \mathcal{D}_{A_i}(X)$, where $i = 1, 2$, $A_i \subset T^*X$ are compact sets and $A_1 \cap A_2 = \emptyset$. We then have $R\hom_{\mathcal{D}(X)}(F_1, F_2) = 0$.*

3.1.1 Lemma

Let M be a smooth manifold let E be a finite-dimensional real vector space of dimension ≥ 1. Let $p : M \times E \to M$ be the projection. Let $F \in D(M \times E)$. Let $\omega \in T^*M, \omega \neq 0$. Let $U \subset T^*M$ be a neighborhood of ω. Let $V \subset E^*$ be a neighborhood of 0 in the dual vector space. Let us identify $T^*(M \times E) = T^*M \times E \times E^*$.

Lemma 3.3 *Suppose that*
F is non-singular on the set

$$U \times E \times V \subset T^*M \times E \times E^* = T^*(M \times E)$$

Then $Rp_!F$ and Rp_*F are non-singular at ω.

Proof We will only prove Lemma for $Rp_!F$; the proof for Rp_*F is similar.

Fix a Euclidean inner product $<,>$ on E. Without loss of generality one can assume that $V = B \subset E^*$ is an open unit ball.

Let $\theta : [0, \infty) \to [0, 1)$ be a function such that:

- $\theta'(x) > 0$ for all $x \geq 0$;
- there exists an $\varepsilon > 0$ such that for all $x \in [0, \varepsilon]$ we have $\theta(x) = x$.
- there exists an $M > 0$ such that for all $x > M$, $\theta(x) = 1 - 1/x$.

Let $B := \{v \in E |\ |v| < 1\}$. Let $Z : E \to B$ be the embedding given by

$$Z(v) = \frac{\theta(|v|)}{|v|} v.$$

It follows that Z is a diffeomorphism. Let $J : B \to E$ be the open embedding. Let us split $p : M \times E \to M$ as

$$M \times E \stackrel{\mathrm{Id}\times Z}{\to} M \times B \stackrel{\mathrm{Id}\times j}{\to} M \times E \stackrel{p}{\to} X.$$

Denote $z := \mathrm{Id} \times Z$; $j := \mathrm{Id} \times J$. We have $Rp_!F = Rp_!j_!z_!F$. We then see that p is proper on the support of $j_!z_!F$. Let us estimate $\mathrm{SS}(z_!F)$.

Let $a(x) : [0, 1) \to [0, \infty)$ be the inverse function to θ. It follows that $a(x) = x$ for $x < \varepsilon$ and there exists $\delta > 0$ such that for all $x \in (1 - \delta; 1)$, $a(x) = 1/(1 - x)$. We then get $Z^{-1}v = (a(|v|)/|v|)v$. The condition of Lemma implies that for all $\omega \in U$ and for all $c \in B$ we have $(\omega, v, c) \notin \mathrm{SS}(F)$, where $v \in E$. Let

$$S_{UB} = \{(\omega, v, c)|\omega \in U; v \in E; c \in B\} \subset T^*(M \times E).$$

We then see that the set $(Z^{-1})^* S_{UB} \subset T^*(X \times B)$;

$$(Z^{-1})^* S_{UB} = \{(\omega, v, \sum_j c_j d((a(|v|)/|v|)v^j)\},$$

where $\omega \in U$, $v \in B$, and $|c| < 1$. Let us now estimate $\mathrm{SS}(j_!z_!F)$. According to Sect. 11.0.7, we have

$$\mathrm{SS}(j_!z_!F) \subset \mathrm{SS}(z_!F)\hat{+}N^*(X \times B)^a,$$

where on the RHS we have a Witney sum of the following conic subsets of $T^*(M \times E)$:

- we identify $\mathrm{SS}(z_!F)$ with a conic subset of $T^*(M \times E)$ as follows: $\mathrm{SS}(z_!F) \subset T^*(M \times B) \subset T^*(M \times E)$;
- $N^*(M \times B)^a$ is the exterior conormal cone to the boundary of $M \times B \subset M \times E$. We have

$$N^*(M \times B)^a = \{(\omega, b, tb) \in T^*M \times E \times E||b| = 1; t \geq 0\},$$

where we identify $T^*M \times E \times E = T^*M \times E \times E^*$.

By definition one has:

$$\mathrm{SS}(z_!F)\hat{+}N^*(M \times B)^a = \mathrm{SS}(z_!F) \cup \Lambda,$$

where Λ consists of all points of the form $(\omega, b, \eta) \in T^*M \times E \times E$ where

- $\omega \in T^*_{x_0}M$; so let us choose a neighborhood U_{x_0} of x_0 in M and identify $T^*U = U \times \mathbb{R}^{\dim M}$; let us denote points of T^*U by (x, ζ), $x \in U$; $\zeta \in \mathbb{R}^{\dim X}$;
- $b \in \partial B$ and there exists a sequence of points $(x_k, \omega_k, b_k, \eta_k) \in \mathrm{SS}(Z_!F) \cap T^*(U_{x_0} \times B)$; $(\beta_k; t_k) \in \partial B \times \mathbb{R}_{\geq 0}$ where $x_k \to x_0$; $b_k \to b$; $\beta_k \to b$; $\omega_k \to \omega$; $\eta_k + 2t_k \sum_j \beta_j dv_j \to \eta$; $t_k(|\beta_k - b_k| + |x_k - x_0|) \to 0$.

We will show that $(x_0, b, \omega, 0) \notin \Lambda$ for any $b \in \partial B$. Let us prove the statement by contradiction. Indeed, without loss of generality, one can assume that $(x_k, \omega_k) \in U$, therefore, $(b_k, \eta_k) \notin Z^{-1*}(E \times V)$. As $V \subset E^*$ is an open unit ball, this means that (b_k, η_k) is of the form

$$\eta_k = \sum_j c_k^j d(a(|b_k|)b_k^j/|b_k|)$$

and $|c_k| \geq 1$. as $b_k \to b$, $|b_k| \to 1$ and without loss of generality one can assume $|b_k| > 1 - \delta$ so that $a(|b_k|) = 1/(1 - |b_k|)$. Thus

$$\eta_k = \sum_j c_k^j d(b_k^j/(|b_k|(1 - |b_k|))$$

Let $R_k = |b_k|$. We then have

$$\eta_k =< c_k, db_k > /(R_k(1 - R_k)) + < c_k, b_k > \frac{2R_k - 1}{R_k^3(1 - R_k)^2} < b_k, db_k >$$

so that

$$< \eta_k, \eta_k >=< c_k, c_k > /(R_k^2(1 - R_k)^2) + < c_k, b_k >^2 \frac{(2R_k - 1)^2}{R_k^4(1 - R_k)^4}$$

$$+2 < c_k, b_k >^2 \frac{2R_k - 1}{R_k^4(1 - R_k)^3}$$

$$>< c_k, c_k > /(R_k^2(1 - R_k)^2) > 1/(1 - R_k)^2$$

as long as $R_k > 1/2$ which is the case for all k large enough, without loss of generality we can assume that $R_k > 1/2$ for all k. Thus, $|\eta_k| > 1/(1 - R_k)$.

Therefore,

$$|\eta_k + 2t_k \sum_j \beta_k^j dv_k^j| \geq |\eta_k| - 2|t_k||\beta| > 1/(1 - R_k) - 2t_k$$

By assumption $|\eta_k + 2t_k \sum_j \beta_k^j dv_k^j| \to 0$, hence

$$1/(1 - R_k) - 2t_k \to 0$$

and $2t_k(1 - R_k) \to 1$. On the other hand, we have

$$t_k(|b_k - \beta_k|) \geq t_k(1 - R_k),$$

because $|\beta_k| = 1$ and $|b_k| = R_k$. Therefore, $t_k(1 - R_k) \to 0$. We have a contradiction which shows that as long as $(x, \omega) \in U$, $(x, \omega, e, 0) \notin \mathrm{SS}(j_! Z_! F)$. Since the map

$p : X \times E \to X$ is proper on the support of $j_! Z_! F$ (i.e. $X \times \overline{B}$) we know that $(x, \omega) \notin SS(Rp_! j_! Z_! F)$ which proves Lemma □

Corollary 3.4 *Let $F \in D(X \times E)$ and let $p : X \times E \to X$, $\kappa : T^*X \times E \times E^* \to T^*X \times E^*$ be the projections. Let $\mathcal{I} : T^*X \to T^*X \times E^*$ be the embedding given by $I(x, \omega) = (x, \omega, 0)$. We then have*

$$SS(Rp_! F), SS(Rp_* F) \subset \mathcal{I}^{-1}\overline{\kappa(SS(F))},$$

where the bar means the closure.

Proof Clear. □

3.1.2 Kernels and Convolutions

Let X_1, X_2, X_3 be manifolds. We are going to define a functor

$$D(X_1 \times X_2 \times \mathbb{R}) \times D(X_2 \times X_3 \times \mathbb{R}) \to D(X_1 \times X_3 \times \mathbb{R}).$$

Let

$$p_{ij} : X_1 \times X_2 \times X_3 \times \mathbb{R} \times \mathbb{R} \to X_i \times X_j \times \mathbb{R} \tag{3}$$

be the following maps

$$p_{12}(x_1, x_2, x_3, t_1, t_2) = (x_1, x_2, t_1);$$

$$p_{23}(x_1, x_2, x_3, t_1, t_2) = (x_2, x_3, t_2);$$

$$p_{13}(x_1, x_2, x_3, t_1, t_2) = (x_1, x_3, t_1 + t_2).$$

Let $A \in D(X_1 \times X_2 \times \mathbb{R})$ and $B \in D(X_2 \times X_3 \times \mathbb{R})$. Set

$$A \bullet_{X_2} B := Rp_{13!}(p_{12}^{-1} A \otimes p_{23}^{-1} B),$$

$A \bullet_{X_2} B \in D(X_1 \times X_3 \times \mathbb{R})$.

Let now X_k, $k = 1, 2, 3, 4$, are manifolds and let $A_k \in D(X_k \times X_{k+1} \times \mathbb{R})$, $k = 1, 2, 3$. We then have a natural isomorphism

$$(A_1 \bullet_{X_2} A_2) \bullet_{X_3} A_3 \cong A_1 \bullet_{X_2} (A_2 \bullet_{X_3} A_3).$$

Let $A \in D(X \times \mathbb{R})$ and $S \in D(\mathbb{R})$. Let **pt** be a point. We then have $A *_{\mathbb{R}} S \cong A \bullet_{\mathbf{pt}} S \cong S \bullet_{\mathbf{pt}} A$.

Let $A \in \mathcal{D}(X_1 \times X_2)$ and $B \in D(X_2 \times X_3 \times \mathbb{R})$. Then $A \bullet_{X_2} B \in \mathcal{D}(X_1 \times X_3 \times \mathbb{R})$. Indeed, according to Proposition 2.2, we need to check that the natural map

$$\mathbb{K}_{[0,\infty)} *_{\mathbb{R}} (A \bullet_{X_2} B) \to \mathbb{K}_0 *_{\mathbb{R}} (A \bullet_{X_2} B)$$

is an isomorphism.

It follows that this map is isomorphic to a map

$$(\mathbb{K}_{[0,\infty)} \bullet_{\mathbf{pt}} A) \bullet_{X_2} B \to (\mathbb{K}_0 \bullet_{\mathbf{pt}} A) \bullet_{X_2} B$$

which is, in turn, induced by the natural map

$$\mathbb{K}_{[0,\infty)} \bullet_{\mathbf{pt}} A \to \mathbb{K}_0 \bullet_{\mathbf{pt}} A$$

which is an isomorphism because $A \in \mathcal{D}(X_1 \times X_2)$.

In particular, it follows that

$$\bullet_{X_2} : \mathcal{D}(X_1 \times X_2) \times \mathcal{D}(X_2 \times X_3) \to \mathcal{D}(X_1 \times X_3).$$

3.1.3 Fourier Transform

Let $E = \mathbb{R}^n$ be a real vector space and let E^* be the dual space. Let $G \subset E \times E^* \times \mathbb{R}$ be a closed subset $G = \{(X, P, t) | < X, P > +t \geq 0\}$, where $<,>: E \times E^* \to \mathbb{R}$ is the pairing. One sees that $\mathbb{K}_G \in \mathcal{D}(E \times E^*)$. Let $\Gamma \subset E^* \times E \times \mathbb{R}$ be a closed subset $G = \{(P, X, t)| - < P, X > +t \geq 0\}$. Again, we have $\mathbb{K}_\Gamma \in \mathcal{D}(E^* \times E \times \mathbb{R})$.

Define functors $F : \mathcal{D}(E) \to \mathcal{D}(E^*)$; $\Phi : \mathcal{D}(E^*) \to \mathcal{D}(E)$ as follows. Set

$$F(A) := A \bullet_E \mathbb{K}_G;$$

$$\Phi(B) := B \bullet_{E^*} \mathbb{K}_\Gamma.$$

F, Φ are called 'Fourier transform'.

Let us study the composition $\Phi \circ F : \mathcal{D}(E) \to \mathcal{D}(E)$. We have an isomorphism

$$\Phi \circ F(A) \cong A \bullet_E (\mathbb{K}_G \bullet_{E^*} \mathbb{K}_\Gamma).$$

Let us compute $\mathbb{K}_G \bullet_{E^*} \mathbb{K}_\Gamma$. Let

$$q : E \times E^* \times E \times \mathbb{R} \times \mathbb{R} \to E \times E \times \mathbb{R}$$

be given by $q(X_1, P, X_2, t_1, t_2) = (X_1, X_2, t_1 + t_2)$. By definition, we have

$$\mathbb{K}_G \bullet_{E^*} \mathbb{K}_\Gamma = Rq_! \mathbb{K}_K,$$

where

$$K = \{(X_1, P, X_2, t_1, t_2) | t_1 + < X_1, P > \geq 0; t_2 - < X_2, P > \geq 0\}$$

Let us decompose $q = q_1 q_2$, where

$$q_2 : E \times E^* \times E \times \mathbb{R} \times \mathbb{R} \to E \times E^* \times E \times \mathbb{R}$$

$q_2(X_1, P, X_2, t_1, t_2) = (X_1, P, X_2, t_1 + t_2)$; and

$$q_1 : E \times E^* \times E \times \mathbb{R} \to E \times E \times \mathbb{R},$$

$$q_1(X_1, P, X_2, t) = (X_1, X_2, t).$$

We see that $q_2(K) = L := \{(X_1, P, X_2, t) | t + < X_1 - X_2, P > \geq 0\}$. Furthermore, the map $q_2|_K : K \to L$ is proper; it is also a Serre fibration with a contractible fiber. Therefore, we have an isomorphism $Rq_{2!}\mathbb{K}_K \cong \mathbb{K}_L$.

Let us now compute $Rq_{1!}\mathbb{K}_L$. Let $\Delta \subset E \times E^* \times E \times \mathbb{R}$ be given by

$$\Delta = \{(X_1, P, X_2, t) | X_1 = X_2; t \geq 0\}.$$

We have $\Delta \subset L$ so that we have an induced map

$$\mathbb{K}_L \to \mathbb{K}_\Delta.$$

It is easy to check that the induced map

$$Rq_{1!}\mathbb{K}_L \to Rq_{1!}\mathbb{K}_\Delta$$

is an isomorphism.

We also have an isomorphism $Rq_{1!}\mathbb{K}_\Delta \cong \mathbb{K}_{\{(X_1,X_2,t)|X_1=X_2;t\geq 0\}}[-n]$.

Thus, we have an isomorphism

$$Rq_!\mathbb{K}_K = \mathbb{K}_{\{(X_1,X_2,t)|X_1=X_2;t\geq 0\}}[-n]$$

For any $A \in D(E \times \mathbb{R})$, we have an isomorphism

$$A \bullet_E \mathbb{K}_{\{(X_1,X_2,t)|X_1=X_2;t\geq 0\}} \cong A *_\mathbb{R} \mathbb{K}_{[0,\infty)}.$$

Thus we have an isomorphism of functors

$$\Phi(F(\cdot)) \cong (\cdot) *_\mathbb{R} \mathbb{K}_{[0,\infty)}[-n]$$

The functor on the RHS acts on $\mathcal{D}(E)$ as the shift by $-n$. Thus we have established an isomorphism of functors $\Phi \circ F \cong \mathrm{Id}[-n]$. Analogously, we can prove $F \circ \Phi \cong \mathrm{Id}[-n]$. We have proven:

Theorem 3.5 *$\Phi[n]$ and F are mutually inverse equivalences of $\mathcal{D}(E)$ and $\mathcal{D}(E^*)$.*

3.1.4

Let us now study the effect of the Fourier transform on the microsupports. Let

$$a : T^*E = E \times E^* \to T^*E^* = E^* \times E$$

be given by $a(X, P) = (-P, X)$. It is clear that a is a symplectomorphism.

Theorem 3.6 *Let $A \subset T^*E$ be a closed subset and $S \in \mathcal{D}_A(E)$. Then $F(S) \in \mathcal{D}_{a(A)}(E^*)$.*

Let $B \in T^*E^*$ be a closed subset and $S \in \mathcal{D}_B(E^*)$. Then $\Phi(S) \in \mathcal{D}_{a^{-1}(B)}(E)$.

Proof By definition, we have

$$F(S) = Rp_{13!}(p_{12}^{-1}S \otimes p_{23}^{-1}\mathbb{K}_G).$$

Here the maps p_{ij} are the same as in (3) for $X_1 = \mathbf{pt}$; $X_2 = E$; $X_3 = E^*$.

The condition $S \in \mathcal{D}_A(E)$ means that $\mathrm{SS}(S)$ is contained in the set Ω_0 of all points

$$(x, t, \omega, k) \in E \times \mathbb{R} \times E^* \times \mathbb{R} = T^*(E \times \mathbb{R}),$$

where either $k \leq 0$ or $k > 0$ and $(x, \omega/k) \in A$.

Therefore,

$$\mathrm{SS}(p_{12}^{-1}S) \subset \Omega_1 := \{(X, P, t_1, t_2, \omega, 0, k, 0)|(X, t, \omega, k) \in \Omega_0\}.$$

As $G \subset E \times E^* \times \mathbb{R}$ is defined by the equation $t+ < X, P > \geq 0$, we know that $\mathrm{SS}(\mathbb{K}_G)$ consists of all points of the form

$$(X, P, t, kP, kX, k) \in E \times E^* \times \mathbb{R} \times E^* \times E \times \mathbb{R}$$

where $t+ < X, P > \geq 0$, $k \geq 0$ and $k > 0$ implies $t+ < X, P >= 0$.

Therefore

$$\mathrm{SS}(p_{23}^{-1}\mathbb{K}_G) = \Omega_2 := \{(X, P, t_1, t_2, k_1P, k_1X, 0, k_1)|(X, P, t_2, k_1P, k_1X, k_1) \in \mathrm{SS}(\mathbb{K}_G)\}.$$

We see that $\Omega_1 \cap -\Omega_2$ is contained in the zero section of $T^*(E \times E^* \times \mathbb{R} \times \mathbb{R})$. Therefore,

$$\mathrm{SS}((p_{12}^{-1}S) \otimes (p_{23}^{-1}\mathbb{K}_G))$$

is contained in the set of all poits of the form $\omega_1 + \omega_2$ where $\omega_i \in \Omega_i$ and ω_1, ω_2 are in the same fiber of $T^*(E \times E^* \times \mathbb{R} \times \mathbb{R})$.

We have

$$\mathrm{SS}(p_{12}^{-1}S \otimes p_{23}^{-1}\mathbb{K}_G) \subset \Omega_3,$$

where Ω_3 consists of all points of the form

$$(X, P, t_1, t_2, \omega + k_1 P, k_1 X, k, k_1)$$

where:

– if $k > 0$, then $(X, \omega/k) \in A$;
– $t_2 + < X, P > \geq 0$;
– $k_1 \geq 0$;
– if $k_1 > 0$, then $t_2 + < X, P >= 0$.

Let $I : E \times E^* \times \mathbb{R} \times \mathbb{R} \to E \times E^* \times \mathbb{R} \times \mathbb{R}$ be given by

$$I(X, P, t_1, t_2) = (X, P, t_1 + t_2; t_2).$$

Let $\pi : E \times E^* \times \mathbb{R} \times \mathbb{R} \to E^* \times \mathbb{R}$ be given by $\pi(X, P, t_1, t_2) = (P, t_1)$. We then have $p_{13} = \pi I$;

$$Rp_{13!}(p_{12}^{-1}S \otimes p_{23}^{-1}\mathbb{K}_G) \cong R\pi_! I_!(p_{12}^{-1}S \otimes p_{23}^{-1}\mathbb{K}_G).$$

It is easy to see that

$$\mathrm{SS}I_!(p_{12}^{-1}S \otimes p_{23}^{-1}\mathbb{K}_G)$$

is contained in the set Ω_4 of of all points $(X, P, t_1 + t_2, t_2, \omega, \eta, k, k_1 - k)$ where $(X, P, t_1, t_2, \omega, \eta, k, k_1) \in \Omega_3$.

Suppose that a point $(P, \xi) \in E^* \times E = T^*E^*$ does not belong to $a(A)$, that is $(-\xi, P) \notin A$. We will prove that $R\pi_! I_!(p_{12}^{-1}S \otimes p_{23}^{-1}\mathbb{K}_G)$ is non-singular at any point of the form $(P, t, \xi, 1) \in E^* \times \mathbb{R} \times E \times \mathbb{R} = T^*(E^* \times \mathbb{R})$ (this means precisely that $R\pi_! I_!(p_{12}^{-1}S \otimes p_{23}^{-1}\mathbb{K}_G) \in \mathcal{D}_{a(A)}(E^*)$.)

According to Lemma 3.3, it suffices to find an $\varepsilon > 0$ such that any point of the form

$$(X, P', t_1, t_2, \omega, \eta, k', k_1')$$

with $|P' - P| < \varepsilon$; $|\omega| < \varepsilon$; $|\eta - \xi| < \varepsilon$; $|k' - 1| < \varepsilon$; $|k_1'| < \varepsilon$ is not in Ω_4. Assume it is, then there should exist a point $(X, P', t_1, t_2, \omega + k_1 P', k_1 X, k, k_1) \in \Omega_3$ such that $|P' - P| < \varepsilon$; $|\omega + k_1 P'| < \varepsilon$; $|k_1 X - \xi| < \varepsilon$; $|k - 1| < \varepsilon$; $|k' - k| < \varepsilon$. If ε is small enough, we have $k, k_1 > 0$ and $(X, \omega/k) \in A$. For any $\delta > 0$, there exists a $\varepsilon > 0$ such that these conditions imply:

$$|\omega + P| < \delta;\ |X - \xi| < \delta. \tag{4}$$

However,we know that $(\xi; -P) = a^{-1}(P, \xi) \notin A$. As A is closed, for δ small enough, there will be no points in A satisfying (4).

The proof of Part 2 is similar. □

3.1.5 Lemma

Lemma 3.7 *Let $S \in \mathcal{D}_A(X)$ where A is a compact. Then $SS(S) \cap \Omega_{\le 0}(X) \subset T^*_{X\times\mathbb{R}}(X \times \mathbb{R})$. That is S is non-singular at every point of the form (x, t, ω, kdt), where either $k \le 0$ and $\omega \neq 0$ or $k < 0$.*

Proof Choose a point $x_0 \in X$, coordinates x^i near x_0 so that x_0 has zero coordinates and let U be a small neighborhood of x_0 given by $|x^i| < 1$ for all i. Consider the set $A \cap T^*U$. This set is contained in the set $B := \{(x, \sum a_i dx^i) | |a_i| \le M\}$, for some $M > 0$ large enough. Let $\psi : \mathbb{R} \to (-1, 1)$ be an increasing surjective smooth function whose derivative is bounded (say $\psi(x) = (2/\pi)\arctan(x)$). Fix a constant $C > 0$ such that $0 < \psi'(x) \le C$ for all x.

We then have a diffeomorphism $\Psi : E := \mathbb{R}^n \to U$, $\Psi(X^1, X^2, \ldots, X^n) = (\psi(X^1), \psi(X^2), \ldots, \psi(X^n))$. It then follows that the set $\Psi^{-1}B$ consists of all points $(X, \sum a_i d\psi(X^i))$, where $|a_i| < M$. But $\sum a_i d\psi(X_i) = \sum a_i \psi'(X_i) dX_i$. We know that $|a_i \psi'(X_i)| < CM =: M_1$. Let $V \subset E^*$ be given by $\{\sum_i b_i dX^i | |b_i| \le M_1\}$ so that $\Psi^{-1}B$ is contained in the set $E \times V \subset E \times E^* = T^*E$.

Let $S \in \mathcal{D}_A(X)$. It follows that $G := \Psi^{-1}(S|_{U\times\mathbb{R}}) \in \mathcal{D}_{E\times V}(E)$. Our task now reduces to showing: *let $G \in \mathcal{D}_{E\times V}(E)$. Then G is nonsingular at a point $(X, t, \omega, kdt) \in E \times \mathbb{R} \times E^* \times \mathbb{R}$ if either $k < 0$ or $k = 0$ and $\omega \neq 0$.*

The statement will be proven using the Fourier transform.

First, we have an isomorphism $G = \Phi(F(G))[n]$. Next, Theorem 3.6 implies that $H := F(G) \in \mathcal{D}_{V\times E}(E^*)$. Let $W \subset E^* \backslash V$ be an open subset such that its closure is also a subset of $E^* \backslash V$. We then see that the restriction $H|_{W\times\mathbb{R}}$ is both in $C_{\le 0}(U)$ (clear) and in the left orthogonal complement to $C_{\le 0}(U)$ (follows from (Proposition 2.2)). Therefore, $H|_{W\times\mathbb{R}} = 0$. Hence H is supported on $V \times \mathbb{R} \subset E^* \times \mathbb{R}$. Let us now study $\Phi(H)[n] = G$. We have

$$\Phi(H) = Rp_{13!}(p_{12}^{-1} H \otimes p_{23}^{-1} \mathbb{K}_{\{(P,X,t)|t-<X,P>\ge 0\}}),$$

where p_{ij} are the same as in (3) with $X_1 = \mathbf{pt}$; $X_2 = E^*$; $X_3 = E$. We need to show that *if* $(X, t, \omega, k) \in \mathrm{SS}(\Phi(H))$ *and* $k \le 0$, *then* $k = 0$ *and* $\omega = 0$.

We have

$$\mathrm{SS}(p_{12}^{-1} H) \subset \Omega_1 = \{(P, X, t_1, t_2, \pi, 0, k_1, 0) | P \in V\};$$

$$\mathrm{SS}(p_{23}^{-1} \mathbb{K}_{\{(P,X,t)|t-<X,P>\ge 0\}}) \subset \Omega_2 = \{(P, X, t_1, t_2, -kX, -kP, 0, k) | k \ge 0\}$$

Let $\omega_i \in \Omega_i$ belong to the fiber of $T^*(E^* \times E \times \mathbb{R} \times \mathbb{R})$ over a point (P, X, t_1, t_2). It is clear that $\omega_1 + \omega_2 = 0$ implies that $\omega_2 = \omega_1 = 0$. Therefore, we have

$$\mathrm{SS}(p_{12}^{-1} H \otimes p_{23}^{-1} \mathbb{K}_{\{(P,X,t)|t-<X,P>\ge 0\}})$$

$$\subset \Omega_3 = \{(P, X, t_1, t_2, \pi - kX, -kP, k_1, k) | k \geq 0;\ P \in V\}$$

Let us decompose $p_{13} = pI$, where $I : E^* \times E \times \mathbb{R} \times \mathbb{R} \to E^* \times E \times \mathbb{R} \times \mathbb{R}$ is given by $I(P, X, t_1, t_2) = (P, X, t_1 + t_2, t_2)$ and $p(P, X, T_1, T_2) = (P, T_1)$. We then see that

$$\mathrm{SS}(I_!(p_{12}^{-1} H \otimes p_{23}^{-1} \mathbb{K}_{\{(P,X,t)|t-<X,P>\geq 0\}})) \subset \Omega_4,$$

where Ω_4 consists of all points of the form $(P, X, t_1, t_2, \pi - kX, -kP, k_1, k - k_1)$, where $(P, X, t_1, t_2, \pi - kX, -kP, k_1, k) \in \Omega_3$.

Assume

$$(X', t, \omega, k') \in \mathrm{SS}(Rp_! I_!(p_{12}^{-1} H \otimes p_{23}^{-1} \mathbb{K}_{\{(P,X,t)|t-<X,P>\geq 0\}}))$$

and $k' \leq 0$. We are to show $k' = 0$, $\omega = 0$.

According to Lemma 3.3 for any $\varepsilon > 0$ there should exist a point $(P, X, t_1, t_2, \pi - kX, -kP, k_1, k - k_1) \in \Omega_4$ such that $| - kP - \omega| < \varepsilon$; $|k_1 - k'| < \varepsilon$; $|k - k_1| < \varepsilon$, $P \in V$, $k \geq 0$, $k' \leq 0$. Therefore, $-k' \leq k - k' = |k - k'| \leq |k - k_1| + |k_1 - k'| < 2\varepsilon$. Similarly, $k < 2\varepsilon$. Since ε can be made arbitrarily small, $k' = 0$. Next, $|\omega| < \varepsilon + |k||P|$. As V is bounded, there exists $D > 0$ such that $|P| < D$. Thus, $|\omega| < \varepsilon(1 + 2D)$ for any $\varepsilon > 0$. Therefore, $\omega = 0$. □

3.1.6

Choose F_1, F_2 in the left orthogonal complement to $C_{\leq 0}(X)$.

Consider the following sheaf on $X \times \mathbb{R}$:

$$H := Rp_{2*} R\underline{\mathrm{Hom}}(p_1^{-1} F_1; a^! F_2),$$

where $p_1, p_2, a : X \times \mathbb{R} \times \mathbb{R} \to X \times \mathbb{R}$ are given by: $p_i(x, t_1, t_2) = (x, t_i)$; $a(x, t_1, t_2) = (x, t_1 + t_2)$.

Let $q : X \times \mathbb{R} \to \mathbb{R}$ be the projection.

Lemma 3.8 *One has (1)* $R\,\mathrm{hom}(F_1, F_2) = R\,\mathrm{hom}_{\mathbb{R}}(\mathbb{K}_0; Rq_* H)$*;*

(2) $R\,\mathrm{hom}_{\mathbb{R}}(\mathbb{K}_{\mathbb{R}}; Rq_* H) = 0$*;*

(3) $Rq_* H$ *is locally constant along* $\mathbb{R}$*.*

Proof Let $S \in D(\mathbb{R})$. We have

$$R\,\mathrm{hom}_{\mathbb{R}}(S; Rq_* H) = R\,\mathrm{hom}_{\mathbb{R}}(S; R\pi_* \underline{\mathrm{Hom}}(p_1^{-1} F_1; a^! F_2)),$$

where $\pi = qp_2 : X \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$; $\pi(x, t_1, t_2) = t_2$.

Next,

$$R\,\mathrm{hom}_{\mathbb{R}}(S; R\pi_* \underline{\mathrm{Hom}}(p_1^{-1} F_1; a^! F_2))$$

$$\cong R\,\mathrm{hom}_{X \times \mathbb{R}}(Ra_!(\pi^{-1} S \otimes p_1^{-1} F_1); F_2)$$

$$\cong R\operatorname{hom}_{X\times\mathbb{R}}(F_1 *_{\mathbb{R}} S; F_2).$$

Thus,

$$R\operatorname{hom}(S; Rq_* H) \cong R\operatorname{hom}_{X\times\mathbb{R}}(F_1 *_{\mathbb{R}} S; F_2).$$

Let us now prove (1)
We have:

$$R\operatorname{hom}_{\mathbb{R}}(\mathbb{K}_0; Rq_* H) = R\operatorname{hom}_{X\times\mathbb{R}}(F_1 *_{\mathbb{R}} \mathbb{K}_0; F_2) = R\operatorname{hom}(F_1, F_2).$$

(2) We have

$$R\operatorname{hom}_{\mathbb{R}}(\mathbb{K}_{\mathbb{R}}; Rq_* H) = R\operatorname{hom}_{X\times\mathbb{R}}(F_1 *_{\mathbb{R}} \mathbb{K}_{\mathbb{R}}; F_2)$$

As $F_1 \in \mathcal{D}(X)$, we have an isomorphism

$$F_1 *_{\mathbb{R}} \mathbb{K}_{[0,\infty)} *_{\mathbb{R}} \mathbb{K}_{\mathbb{R}} \to F_1 *_{\mathbb{R}} \mathbb{K}_{\mathbb{R}}.$$

However, one can easily check that $\mathbb{K}_{[0,\infty)} *_{\mathbb{R}} \mathbb{K}_{\mathbb{R}} = 0$.Therefore,

$$F_1 *_{\mathbb{R}} \mathbb{K}_{[0,\infty)} *_{\mathbb{R}} \mathbb{K}_{\mathbb{R}} = 0,$$

whence the statement.

(3) Let us identify $T^*(X \times \mathbb{R}) = T^*X \times \mathbb{R}^2$; $T^*(X \times \mathbb{R} \times \mathbb{R}) = T^*X \times \mathbb{R}^4$ so that $(\omega, t, k) \in T^*X \times \mathbb{R}^2$ corresponds to a point $(\omega, \eta) \in T^*X \times T^*\mathbb{R}$, where η is a 1-form kdt at the point $t \in \mathbb{R}$; analogously, we let $(\omega, t_1, t_2, k_1, k_2)$ correspond to a point $(\omega, \zeta) \in T^*X \times T^*(\mathbb{R} \times \mathbb{R})$ where $\zeta = k_1 dt_1 + k_2 dt_2$ is a 1-form at the point $(t_1, t_2) \in \mathbb{R}^2$.

According to Lemma 3.7, We know that

$$\mathrm{SS}(F_1) \cap \{(\omega, t, k) | k \leq 0\} \subset T^*_{X\times\mathbb{R}}(X \times \mathbb{R}).$$

Since $F_1 \in \mathcal{D}_{A_1}(X)$, we have

$$\mathrm{SS}(F_1) \cap \{(\omega, t, k) | k > 0\} \subset \{(\omega, t, k) | k > 0; (x, \omega/k) \in A_1\}.$$

Thus,

$$\mathrm{SS}(F_1) \subset \{(k\omega, t, k) | k \geq 0; \omega \in A_1\}$$

Analogously,

$$\mathrm{SS}(F_2) \subset \{(k\omega, t, k) | k \geq 0; \omega \in A_2\}.$$

Therefore,

$$\mathrm{SS}(p_1^{-1} F_1) \subset \{(k_1\omega_1, t_1, t_2, k_1, 0) | k_1 \geq 0; \omega_1 \in A_1\};$$

$$SS(a^!F_2) \subset \{(k_2\omega_2, t_1, t_2, k_2, k_2)|k_2 \geq 0; \omega_2 \in A_2\}.$$

In order to estimate $SS R\underline{Hom}(p_1^{-1}F_1; a^!F_2)$ one should first check that $SS(p_1^{-1}F_1) \cap SS(a^!F_2) \subset T^*_{X\times\mathbb{R}\times\mathbb{R}}(X \times \mathbb{R} \times \mathbb{R})$. This is indeed so, because every point p in $SS(p_1^{-1}F_1) \cap SS(a^!F_2)$ is of the form

$$p = (k_1\omega_1, t_1, t_2, k_1, 0) = (k_2\omega_2, t_1, t_2, k_2, k_2).$$

which implies $k_1 = k_2 = 0$, hence $k_1\omega_1 = k_2\omega_2 = 0$. Therefore, one has

$$SS R\underline{Hom}(p_1^{-1}F_1; a^!F_2) \subset \{(k_2\omega_1 - k_1\omega_2, t_1, t_2; k_2 - k_1; k_2)|k_1, k_2 \geq 0; \omega_1 \in A_1; \omega_2 \in A_2\},$$

where it is also assumed that ω_1, ω_2 belong to the same fiber of T^*X. Let $q' : X \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ be the projection. Consider an object

$$G := Rq'_* R\underline{Hom}(p_1^{-1}F_1; a^!F_2)$$

so that $Rq_*H = Rq_*Rp_{2*}R\underline{Hom}(p_1^{-1}F_1; a^!F_2) = Rp'_{2*}G$, where $p'_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the projection along the first factor; $p'_1(t_1, t_2) = t_2$.

As the map q' is proper, the microsupport of G can be estimated as

$$SS(G) \subset \{(t_1, t_2, k_2 - k_1, k_2)|k_1, k_2 \geq 0; \exists\omega_i \in A_i : k_1\omega_1 = k_2\omega_2\},$$

where again it is assumed that ω_i are in the same fiber of T^*X. Denote the set on the RHS by $\Gamma \subset \mathbb{R}^4 = T^*(\mathbb{R} \times \mathbb{R})$. Let us now estimate $SS(Rq_*H) = Rp'_{2*}G$ using Corollary 3.4.

Let us first prove that $(t, 1) \notin SS(q_*H)$, where we identify $T^*\mathbb{R} = \mathbb{R} \times \mathbb{R}$. Assuming the opposite implies that for any $\varepsilon > 0$ there should exist $(t_1, t_2, k_2 - k_1, k_2) \in \Gamma$ such that $|k_2 - k_1| < \varepsilon$; $|k_2 - 1| < \varepsilon$. As A_1, A_2 are compact and do not intersect, it is clear that for ε small enough we have $k_1A_1 \cap k_2A_2 = \emptyset$ which contradicts to $(t_1, t_2, k_2 - k_1, k_2) \in \Gamma$.

Let us now show that Rq_*H is non-singular at any point $(t, -1)$. Similar to above, assuming the contrary implies that for any $\varepsilon > 0$ there should exist $(t_1, t_2, k_2 - k_1, k_2) \in \Gamma$ such that $|k_2 + 1| < \varepsilon$. As $k_2 \geq 0$, this leads to contradiction. □

3.1.7 Proof of Theorem 3.2

It now follows that Rq_*H is a constant sheaf on $\mathbb{R}$ with $R\Gamma(\mathbb{R}, Rq_*H) = 0$, i.e. $Rq_*H = 0$. Hence $R\operatorname{hom}(F_1, F_2) = 0$ by Lemma 3.8 (1).

This proves Theorem 3.2.

3.2 Hamiltonian Shifts

Let $\Phi : T^*X \to T^*X$ be a Hamiltonian symplectomorphism which is equal to identity outside of a compact. Let $L \subset T^*X$ be a compact subset.

Theorem 3.9 *There exist:*

a collection of endofunctors $T_n : \mathcal{D}(X) \to \mathcal{D}(X),$, $1 \le n \le N$ for some N, and a collection of transformations of functors $t_k : T_{2k} \to T_{2k+1}$ (for all k with $2k+1 \le N$). $s_k : T_{2k+2} \to T_{2k+1}$ (for all k with $2k+2 \le N$);

Such that

(1) $T_N = Id$;

(2) $T_1(\mathcal{D}_L(X)) \subset \mathcal{D}_{\Phi(L)}(X)$;

(3) For all k and for all $F \in \mathcal{D}(X)$, we have $Cone(t_k(F))$ and $Cone(s_k(F))$ are torsion sheaves (see Sect. 2.2.3)

3.2.1 Singular Support of Convolutions

Let $A \in T^*X$ and $B \subset T^*(X \times Y) = T^*X \times T^*Y$ be compact subsets. Let $C \subset T^*Y$;

$$C := A \bullet B = \{p \in T^*Y | \exists q \in A : (-q, p) \in B\}.$$

3.2.2 Lemma

Lemma 3.10 *Let $A_1 \supset A_2 \supset \cdots \supset A_n \supset \cdots \supset A$ be a collection of compact sets such that $\bigcap_i A_i = A$. Let $U \supset C$ be an open neighborhood. There exists an $N > 0$ such that for all $n > N$, $A_n \bullet B \subset U$.*

Proof Assume not and pick points $b_n \in (A_n \bullet B)\backslash U$. One then has points $a_n \in A_n$ such that $(-a_n, b_n) \in B$. As B is compact, one can choose a convergent subsequence $a_{n_k} \to a$ and $b_{n_k} \to b$. It follows that $(-a, b) \in B$. We see that $a \in A_{n_k}$ for all k, hence $a \in A$. Therefore, $b \in C$. On the other hand, as $b_{n_k} \notin U$, $b \notin U$, we have a contradiction. □

3.2.3

Let A, B, C are compact sets as above.

Proposition 3.11 *Let $F \in \mathcal{D}_A(X)$; $K \in \mathcal{D}_B(X \times Y)$. Then $F \bullet K \in \mathcal{D}_C(Y)$.*

Proof It suffices to prove: *let $(y_0, \eta_0) \notin C$. Then $F \bullet K$ is nonsingular at $(y_0, t, \eta_0, 1)$*

for all $t \in \mathbb{R}$.

Let us identify of $T^*(X \times Y \times \mathbb{R} \times \mathbb{R}) = T^*X \times T^*Y \times T^*(\mathbb{R} \times \mathbb{R}) = T^*X \times T^*Y \times \mathbb{R}^4$, where we identify $T^*(\mathbb{R} \times \mathbb{R}) = \mathbb{R}^4$ in the same way as above: a point $(t_1, t_2, k_1, k_2) \in \mathbb{R}^4$ corresponds to a 1-form $k_1 dt_1 + k_2 dt_2$ at the point $(t_1, t_2) \in \mathbb{R}^2$.

Let us estimate the microsupport of $F \bullet K := p_{13!}(p_{12}^{-1} F \otimes p_{23}^{-1} K)$, where p_{ij} are the same as in (3) with $X_1 = \mathbf{pt}$; $X_2 = X$; $X_3 = Y$. We have $p_{12}^{-1} F$ is microsupported within the set S_F consisting of all points of the form

$$(k_1\omega_1, 0_y, t_1, t_2, k_1, 0),$$

where $0_y \in T_Y^*Y$, $(x, \omega_1) \in A$; $k_1 \geq 0$ (as follows from Lemma 3.7). Analogously, The sheaf $p_{23}^{-1} K$ is microsupported on the set S_K consisting of all points of the form

$$(k_2\omega_2, k_2\eta_2, t_1, t_2, 0, k_2),$$

where $k_2 \geq 0$, $(\omega_2, \eta_2) \in B$.

One sees that $S_K \cap -S_F \subset T^*_{X\times Y\times\mathbb{R}\times\mathbb{R}}(X \times Y \times \mathbb{R} \times \mathbb{R})$. Therefore, $p_{12}^{-1} F \otimes p_{23}^{-1} K$ is microsupported within the set of all points of the form

$$(k_1\omega_1 + k_2\omega_2, k_2\eta_2, t_1, t_2, k_1, k_2),$$

where $k_1, k_2 \geq 0$; $\omega_1 \in A$; $(\omega_2, \eta_2) \in B$.

Let $Q : X \times Y \times \mathbb{R} \times \mathbb{R} \to Y \times \mathbb{R} \times \mathbb{R}$, $a : Y \times \mathbb{R} \times \mathbb{R} \to Y \times \mathbb{R}$ be given by

$$Q(x, y, t_1, t_2) = (y, t_1, t_2);$$

$$a(y, t_1, t_2) = (y, t_1 + t_2)$$

so that $p_{13} = aQ$.

We see that the map Q is proper on the support of $p_{12}^{-1} F \otimes p_{23}^{-1} K$. It then follows that the sheaf $\Psi := RQ_!(p_{12}^{-1} F \otimes p_{23}^{-1} K)$ is microsupported on the set S_Q of all points

$$(k_2\eta_2, t_1, t_2, k_1, k_2)$$

such that $k_1, k_2 \geq 0$ and there exist $\omega_1 \in A$, $(\omega_2, \eta_2) \in B$ such that ω_1 and ω_2 are in the same fiber of T^*X and $k_1\omega_1 + k_2\omega_2 = 0$

Let us now estimate the microsupport of $Ra_!\Psi$. We will use Corollary 3.4. Let us use an isomorphism $I : Y \times \mathbb{R} \times \mathbb{R} \to Y \times \mathbb{R} \times \mathbb{R}$, where

$$I(y, t_1, t_2) = (y, t_1 + t_2; t_2).$$

Let $p_2 : Y \times \mathbb{R} \times \mathbb{R} \to Y \times \mathbb{R}$ be given by $p_2(y, t_1, t_2) = (y, t_1)$ so that we have

$$a = p_2 I$$

and $Ra_!\Psi = Rp_{2!}I_!\Psi$. We see that the sheaf $I_!\Psi$ is microsupported on the set Γ_1 consisting of all points of the form

$$(k_2\eta_2, t_1, t_2, k_1, k_2 - k_1)$$

such that $k_1, k_2 \geq 0$ and there exist $\omega_1 \in A$, $(\omega_2, \eta_2) \in B$ such that ω_1 and ω_2 are in the same fiber of T^*X and $k_1\omega_1 + k_2\omega_2 = 0$. Let us now use Corollary 3.4 in order to estimate $\mathrm{SS}Rp_{2!}I_!\Psi$. Let $\eta \in T^*Y$; $\eta \notin C$. We need to show that $Rp_{2!}I_!\Psi$ is non-singular at any point of the form

$$(\eta, t, 1) \in T^*Y \times \mathbb{R} \times \mathbb{R} = T^*Y \times T^*\mathbb{R}.$$

Assuming the contrary, for any $\delta > 0$ there should exist a point $(k_2\eta_2, T_1, T_2, k_1, k_2 - k_1) \in \Gamma_1$ such that $|\eta - k_2\eta_2| < \delta$ and $|k_1 - 1|, |k_2 - k_1| < \delta$. Given $\varepsilon > 0$, one can choose $\delta > 0$ such that under the conditions specified, $|1 - k_1/k_2| < \varepsilon$. Let $A_\varepsilon = [1-\varepsilon, 1+\varepsilon].A$'. We then see that there should exist $\omega_2 \in T^*X$ such that $(\omega_2, \eta_2) \in B$ and $-\omega_2 \in A_\varepsilon$ (because $-\omega_2 = k_1/k_2\omega_1$ and $\omega_1 \in A$). Thus, $\eta_2 \in A_\varepsilon \bullet B$. We see that the sets $A_{1/n}$, $n = 1, 2, \ldots$ are compact and $\bigcap_n A_{1/n} = A$. Let $U \supset C$ be an open neighborhood.

By Lemma 3.10, there exists an N such that $A_{1/N} \bullet B \subset U$ i.e. for all $\varepsilon \leq 1/N$ we have $\eta_2 \in U$. Taking into account the inequality $|\eta - k_2\eta_2| < \delta$ and letting δ arbitrarily small, we see that $\eta \in U$. As U is any open neighborhood of C, we conclude $\eta \in C$. We get a contradiction. □

3.2.4

If $\Phi = \Phi_1\Phi_2\cdots\Phi_N$ and the statement of the Theorem is true for each Φ_k, it is true for Φ. In other words, if Z is the set of Hamiltonian symplectomorphisms of T^*X which are identity outside of a compact and if Z generates the whole group of Hamiltonian symplectomorphisms of T^*X which are identity outside a compact, then it suffices to prove Theorem for all $\Phi \in Z$.

Let us now choose an appropriate Z. Call a symplectomorphism $\Phi : X \to X$ *small* if

(1) There exists a Darboux chart $U \subset T^*X$ with Darboux coordinates x, P, where x^i are local coordinates on x, $P^i = \partial/\partial x^i$, $|x^i| < 1$ and for some fixed

$$\pi \in \mathbb{R}^n, \tag{5}$$

$|P^i - \pi^i| < 1$ for all i. Let $p^i := P^i - \pi^i$. For $x \in \mathbb{R}^n$ we set $|x| := \max_i |x_i|$. We demand that Φ should be identity outside a subset $V \subset U$, $|x| < 1/2$, $|p| < 1/2$.

(2) Let $(x', p') = \Phi(x, p)$. Then (x, p') form a non-degenerate coordinate system on U so that (x, p') map U diffeomorphically onto a domain $W \subset \mathbb{R}^{2n}$.

It is well known that the set Z formed by small symplectomorphisms satisfies the conditions.

3.2.5 Small Symplectomorphisms in Terms of Generating Functions

The coordinates (x, p) define an embedding $U \subset \mathbb{R}^{2n}$. Let us extend $\Phi|_U$ to a map $\overline{\Phi} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ by setting $\overline{\Phi}(x, p_0) = (x, p_0)$ for all $(x, p_0) \notin U$. We see that $\overline{\Phi}$ is a diffeomorphism because it maps U diffeomorphically to itself, as well as the complement to U. Hence, $\overline{\Phi} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a symplectomorphism with respect to the standard symplectic structure.

As above let $\overline{\Phi}(x, p) = (x', p')$. Let $\Psi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ where $\Psi(x, p) = (x, p')$.

Lemma 3.12 *Ψ is a diffeomorphism.*

Proof (a) Ψ has a non-zero Jacobian everywhere. Indeed, if $|x| < 1, |p| < 1$ this is postulated by (2); otherwise $\Psi = \mathrm{Id}$ in a neighborhood of (x, p).

(b) Φ is an injection. Suppose $\Psi(x_1, p_1) = \Psi(x_2, p_2)$. Then $x_1 = x_2 = x$ and $p'(x, p_1) = p'(x, p_2)$. Consider several cases:

(1) $|x| \geq 1$, then $p'(x, p_1) = p_1$; $p'(x, p_2) = p_2$ and $p_1 = p_2$;

(2) $|x| < 1$; $|p_1| < 1$. If $|p_2| < 1$, then $p_1 = p_2$ by Condition (2). If $|p_2| \geq 1$, then $p'(x, p_2) = p_2$; $|p'(x, p_2)| \geq 1$ and $|p'(x, p_1)| < 1$ because $\overline{\Phi}$ preserves U, so $p'(x_1, p_1) \neq p'(x_2, p_2)$;

(3) $|x| < 1$ and $|p_2| < 1$— similar to (2);

4) $|x| < 1$ and $|p_1|, |p_2| = 1$. Then $p'(x, p_i) = p_i$, therefore $p_1 = p_2$.

(c) Ψ is surjective. We know that $\Psi(x, p) = (x, p)$ if $|x| > 1$ or $|p| > 1$. Assume that, on the contrary, (x_0, p_0) does not belong to the image of Ψ. It follows that $|x_0| < 1$; $|p_0| < 1$. For $R > 0$ consider the sphere S_R given by the equation $\sum_i (x^i)^2 + \sum_i (p^i)^2 = R^2$. Choose R so large that $(x, p) \in S_R$ implies $|x| > 1$ or $|p| > 1$. We then have $\Psi|_{S_R} = \mathrm{Id}$. It also follows S_R cannot be homotopized to a point in $\mathbb{R}^{2n} \backslash (x_0, p_0)$ (because (x_0, p_0) is inside the open ball bounded by S_R). On the other hand it can: Let $\gamma : S_R \times [0, 1] \to \mathbb{R}^{2n}$ be any homotopy which contracts S_R to a point. Then $\Psi \circ \gamma$ is a required homotopy. This is a contradiction. □

Lemma 3.13 *There exists a smooth function $S(x, p')$ on $\mathbb{R}^{2n}$ such that*

(1) $(x', p') = \overline{\Phi}(x, p)$ iff for all i:

$$p^i = (p')^i + \frac{\partial S}{\partial x^i};$$

$$(x')^i = x^i + \frac{\partial S}{\partial (p')^i};$$

(2) $S = 0$ if $|x| \geq 1/2$ or $|p'| \geq 1/2$;

(3)

$$\max_{|x| \leq 1/2, |p'| \leq 1/2} |x^i + \partial S/\partial (p')^i| \leq 1/2$$

Proof Consider the following 1-form on $\mathbb{R}^{2n}$: $\sum p^i dx^i + \sum (x')^i d(p')^i$. This form is closed, hence exact. So one can write

$$\sum p^i dx^i + \sum (x')^i d(p')^i = d(S(x, p') + < x, p' >)$$

by virtue of Lemma 3.12. This equation is equivalent to the part (1) of this Lemma.

We know that $\overline{\Phi} = \mathrm{Id}$ if $|x| \geq 1/2$ or $|p| \geq 1/2$. Therefore, $\overline{\Phi}$, being bijective, preserves the region $\{(x, p) | |x|, |p| < 1/2\}$. Therefore, if $|p'(x, p)| \geq 1/2$, then either $|x| \geq 1/2$ or $|p| \geq 1/2$, hence $p'(x, p) = p; x'(x, p) = x$ and $dS(x, p') = 0$ as soon as $|x| \geq 1/2$ or $|p'| \geq 1/2$. As the specified region is connected, S is a constant in this region, and one can choose S to be 0 as long as $|x| \geq 1/2$ or $|p'| \geq 1/2$. This proves (2).

It also follows that if $|x| \leq 1/2$ and $|p'(x, p)| \leq 1/2$ then $|p| \leq 1/2$, because otherwise $\Phi(x, p) = (x, p)$ and $p' = p$, which is a contradiction. This implies (3).

□

3.2.6

Let J be the set of all smooth functions $S(x, p')$ on $\mathbb{R}^{2n}$ such that S is supported on the set $\{(x, p') | |x| \leq 1/2, |p'| \leq 1/2\}$ and the inequality (3) from Lemma 3.13 is satisfied. Our ultimate goal is: given such an S, we would like to construct certain kernels in $\mathcal{D}(\mathbb{R}^n \times \mathbb{R}^n)$ and then $\mathcal{D}(X \times X)$.

Let $\pi \in \mathbb{R}^n$ (this parameter has the same meaning as in (5). Let $S \in J$. We will start with constructing an appropriate object $\Lambda_{S,\pi} \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R}^n)$ and estimating its microsupport.

Let $\Sigma_\pi(x_1, x_2, p') := -S(x_1, p') - < x_1 - x_2, p' + \pi >$. We can decompose

$$d\Sigma_\pi = d_{x_1}\Sigma_\pi + d_{x_2}\Sigma_\pi + d_{p'}\Sigma_\pi.$$

Let $\Gamma_\pi(S) \subset T^*\mathbb{R}^n \times T^*\mathbb{R}^n$ consist of all points (x_1, p_1, x_2, p_2) satisfying: there exists p' such that $d_{p'}\Sigma_\pi(x_1, x_2, p') = 0$ and $p_1 = d_{x_1}\Sigma_\pi(x_1, x_2, p')$; $p_2 = d_{x_2}\Sigma_\pi(x_1, x_2, p')$.

Remark. Let us take S as in Lemma 3.13. The set $\Gamma_\pi(S)$ then consists of all points (x_1, P_1, x_2, P_2) such that $\overline{\Phi}(x_1; -P_1 - \pi) = (x_2, P_2 - \pi)$. That is, if $|P_1 + \pi| < 1$, then $(x_2, P_2) = \Phi(x_1, -P_1)$; if $|P_1 + \pi| \geq 1$, then $x_2 = x_1$, $P_2 = -P_1$, where we use notation from Sect. 3.2.4.

We are now passing to constructing an object $\Lambda_{S,\pi} \in \mathcal{D}(\Gamma_\pi(S))$. Consider the following subset $C_{S,\pi} \subset \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$;

$$\{(x_1, x_2, p', t,) | t + \Sigma_\pi \geq 0\},$$

Let $q : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ be given by

$$q(x_1, x_2, p', t) = (x_1, x_2, t).$$

Set $\Lambda_{S,\pi} := Rq_!\mathbb{K}_{C_{S,\pi}}$.

Lemma 3.14 *Assume $S \in J$. Then $\Lambda_{S,\pi} \in \mathcal{D}_{\Gamma_\pi(S)}(\mathbb{R}^n \times \mathbb{R}^n)$.*

Proof It is straightforward to check that $\Lambda_{S,\pi}$ is in the left orthogonal complement to $C_{\leq 0}(\mathbb{R}^n \times \mathbb{R}^n)$.

Let us now estimate the microsupport of $\Lambda_{S,\pi}$. Let us choose a large positive number C and consider objects

$$F_C := Rq_!\mathbb{K}_{\{(x_1,x_2,p',t)|t+\Sigma_\pi(x_1,x_2,p')\geq 0;|p'|<C\}}$$

so that $\Lambda_{S,\pi} = L\varinjlim_{C\to\infty} F_C$.

We will prove: *let $(x_1, x_2, t, \omega_1, \omega_2, k) \in T^*(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R})$ be a singular point of F_C. Then one of the following 3 statements is true:*

- $k = \omega_1 = \omega_2 = 0$;
- $k > 0$ and $|\omega_i/k| \geq C - |\pi|$, $i = 1, 2$.
- $(x_1, x_2, \omega_1, \omega_2) \in \Gamma_\pi(S)$

This implies Lemma, as C can be chosen arbitrarily large.

Let us estimate the microsupport of the sheaf

$$\mathbb{K}_{\{(x_1,x_2,p',t)|t+\Sigma_\pi(x_1,x_2,p')\geq 0;|p'|<C\}}$$

we see that it is contained within the set of all points of the form

$$(x_1, x_2, p', t, kd\Sigma_\pi(x_1, x_2, p') + \sum a_i d(p')^i),$$

where $|p'| \leq C$ and $a_i \leq 0$ if $(p')^i = -C$; $a_i = 0$ if $|(p')^i| < C$, and $a_i \geq 0$ if $(p')^i = C$; also, $k \geq 0$ and if $k > 0$, then $t + \Sigma_\pi(x_1, x_2, p') = 0$.

Let us now estimate the singular support of the sheaf

$$Rq_!\mathbb{K}_{\{(x,x',p,t)|t+\Sigma_\pi(x_1,x_2,p')\geq 0;|p'|<C\}}.$$

As q is proper on the support of this sheaf, we see that

$$Rq_!\mathbb{K}_{\{(x,x',p,t)|t+\Sigma_\pi(x_1,x_2,p');|p'|<C\}}$$

is microsupported on the set of points

$$(x_1, x_2, t, \omega_1, \omega_2, k),$$

where there exists p', $|p'| \leq C$ such that

$$\omega_i = kd_{x_i}\Sigma_\pi(x_1, x_2, p') \tag{6}$$

where $k \geq 0$ and if $k > 0$ then there exists p' such that

$$\frac{\partial \Sigma_\pi}{\partial (p')^i}(x_1, x_2, p') \geq 0 \text{ if } (p')^i = -C;$$

$$\frac{\partial \Sigma_\pi}{\partial (p')^i}(x_1, x_2, p') = 0 \text{ if } |(p')^i| < C; \tag{7}$$

$$\frac{\partial \Sigma_\pi}{\partial (p')^i}(x_1, x_2, p') \leq 0 \text{ if } (p')^i = C.$$

Let us first consider the case $C > 1/2$, $|p'| = C$, and $k > 0$. Observe that if $|p'| > 1/2$, then $S(x, p') = 0$; $\Sigma_\pi = - < x_1 - x_2, p' + \pi >$. Eq. (6) then implies: If $C > 1/2$ and $|p'| = C$, then $\omega_1 = -k(p' + \pi)$; $\omega_2 = k(p' + \pi)$. Hence: if $k > 0$ and $|p| = C$, then $|\omega_1|/k, |\omega_2|/k \geq C - |\pi|$.

If $k > 0$ and $|p| < C$, then $(x_1, x_2, \omega_1, \omega_2) \in \Gamma_\pi(S)$ by (6) and (7).

If $k = 0$, then $\omega_1 = \omega_2 = 0$. Finally, k is always non-negative. This proves the statement. □

3.2.7

Let $A, B \subset \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ be the following open subsets:

$$A = \{(x_1, x_2, t') | |x_1| > 1/2\}$$

$$B = \{(x_1, x_2, t') | |x_1| < 3/5; |x_2| > 4/5\}$$

Lemma 3.15 *For every $S \in J$ we have: (1) $\Lambda_{S,\pi}|_A \cong \mathbb{K}_{\{x_1 = x_2; t \geq 0\}}[-n]$; (2) $\Lambda_{S,\pi}|_B = 0$.*

Proof (1) We have $S(x_1, p') = 0$ for all $x_1 > 1/2$. Therefore,

$$\Lambda_S|_A = Rq_! \mathbb{K}_{\{t - \langle x_1 - x_2, p' + \pi \rangle \geq 0\}} \cong \mathbb{K}_{\{x_1 = x_2; t \geq 0\}}[-n].$$

The last isomorphism has been established in Sect. 3.1.3.

(2) Let $|x_1| < 3/5$, $|x_2| > 4/5$, and consider the equation

$$\partial_{p'} \Sigma_\pi(x_1, x_2, p') = 0.$$

We have

$$\partial_{p'}(-S(x_1, p') - < x_1 - x_2, p' + \pi >) = -x_1 - \partial_{p'} S(x_1, p') + x_2 = x_2 - y,$$

where

$$y = x_1 + \partial_{p'} S(x_1, p')$$

if $|p'| \leq 1/2$ then $|y| \leq 1/2$ as $S \in J$. If $|p'| \geq 1/2$, then $y = x$ and $|y| < 3/5$. Thus, in any case $|y| < 3/5$, therefore,$x_2 - y \neq 0$ because $|x_2| > 4/5$.

Thus for all p',

$$\partial_{p'} \Sigma_\pi(x_1, x_2, p') \neq 0.$$

(2) Fix $(x_1, x_2) \in B$. Set $G(p') := \Sigma_\pi(x_1, x_2, p')$. We know that $dG(p') \neq 0$ for all p'. For $|p'| > 1/2$, $G(p') = - < x_1 - x_2, p' + \pi > = < c, p' > + K$ for some constants $c \neq 0$ and K.

We need to show that given a function G satisfying these conditions, one has: $Rq_! \mathbb{K}_{\{(t,p):t+G(p)\geq 0\}} = 0$, where $q : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is the projection.

Let $Y \subset \mathbb{R}^n$ be the hyperplane $< c, p > + K = -M$ for $M >> 0$. Let F_τ be the flow of the gradient vector field of G. We then get a map

$$\Gamma : Y \times \mathbb{R} \to \mathbb{R}^n,$$

$$\Gamma(y, \tau) = F_\tau(y)$$

The map Γ is clearly a diffeomorphism and $G(\Gamma(y, \tau)) = \tau - M$. Thus, under diffeomorphism Γ, the function $G(p')$ gets transformed into $\tau - M$. Therefore it suffices to show the statement for G being a linear function on $\mathbb{R}^n$, in which case the statement is clear. □

3.2.8

Using the above Lemma we will now construct a kernel in $\mathcal{D}(X \times X)$ where X is as in Sect. 3.2.4. Observe that $A \cup B$ contains the set $C := \{(x, x', t)||x| > 4/5 \text{ or } |x'| > 4/5\}$ and the above Lemma implies that $\Lambda_{S,\pi}|_C \cong \mathbb{K}_{\{x=x',t\geq 0\}}$.

Recall (Sect. 3.2.4) that we have a Darboux chart $U \subset T^*X$. Let U_1 be the projection of U onto X. U_1 is identified with the cube $|x| < 1$ in $\mathbb{R}^n$. Let $V \subset U_1 \subset X$ be given by the equation $|x| < 1$ and $K \subset V$ by the equation $|x| < 4/5$. We then have a sheaf $\Lambda_{S,\pi}|_{V\times V\times\mathbb{R}}$ and a compact $K \subset V$ such that on $W := V \times V \times \mathbb{R} \backslash (K \times K \times \mathbb{R})$ we have an identification $\Lambda_{S,\pi}|_W = \mathbb{K}_{\{(x_1,x_2,t)\in W|x_1=x_2;t\geq 0\}}[-n]$

One can now extend $\Lambda_{S,\pi}$ to a sheaf on $X \times \times X \times \mathbb{R}$ by setting $L_S|_{X\times\mathbb{R}\times X\times\mathbb{R}\backslash W} = \mathbb{K}_{\{(x,x',t)|x=x';t\geq 0\}}$. Denote thus obtained sheaf by L_S. Let $\Gamma_\Phi = \{(-\omega, \Phi(\omega)\} \subset T^*X \times T^*X$.

Proposition 3.16 *We have $L_S \in \mathcal{D}_{\Gamma_\Phi}(X \times X)$.*

Proof Follows easily from Lemma 3.14 and Remark before this Lemma. □

3.2.9

Let $S_+(x, p)$ be a function on $\mathbb{R}^{2n}$ defined as follows: if $S(x, p) \leq 0$, then we set $S_+(x, p) = 0$; if $S(x, p) \geq 0$, then we set $S_+(x, p) = S(x, p)$.

Lemma 3.17 *For every $S \in J$ and any $\pi \in \mathbb{R}^n$ we have: (1)* $\Lambda_{S_+,\pi}|_A \cong \mathbb{K}_{\{x_1=x_2;t\geq 0\}}[-n]$*;*

(2) $\Lambda_{S_+,\pi}|_B = 0$.

Proof There exists a sequence of smooth functions $g_n(x)$ on $\mathbb{R}$ with the following properties: (1) each function $g_n(x)$ is non-decreasing; furthermore, $0 \leq g_n'(x) \leq 1$ for all n and x;

(2) for every x, the sequence $g_n(x)$ is non-decreasing;

(3) for $x \leq 0$, $g_n(x) = 0$;

(4) for $x \geq 1/n$, $g_n'(x) = 1$.

Fix such a sequence of functions.

For $S \in J$ consider functions $S_n(x, p) = g_n(S(x, p))$. Let us check that $S_n \in J$. Indeed, S_n are supported on the set $|x| \leq 1/2$, $|p| \leq 1/2$ because $g_n(0) = 0$. Next, we have $|x^i + \partial S/\partial p^i| \leq 1/2$ for all x with $|x| \leq 1/2$, i.e

$$\partial S/\partial p^i \in [-x_i - 1/2; -x_i + 1/2]$$

The interval on the RHS contains zero, therefore is closed under multiplication by any number $\lambda \in [0, 1]$.

We have

$$\partial S_n/\partial p^i = g_n'(S)\partial S/\partial p^i \in [-x_i - 1/2; -x_i + 1/2]$$

precisely because $0 \leq g_n' < 1$. Thus, $S_n \in J$.

Next, we see that $S_1(x) \leq S_2(x) \leq \cdots \leq S_n(x) \leq \cdots$ and that $S_n(x)$ converges uniformly to $S_+(x)$. It then follows that we have induced maps $\Lambda_{S_1,\pi} \to \Lambda_{S_2,\pi} \to \cdots \Lambda_{S_n,\pi} \to \cdots$ and we have an isomorphism

$$L\varinjlim_n \Lambda_{S_n,\pi} \to \Lambda_{S_+,\pi}.$$

Since the sheaves $\Lambda_{S_n,\pi}$ satisfy the Lemma, so does $\Lambda_{S_+,\pi}$. □

This implies that in the same way as above, $\Lambda_{S_+,\pi}$ can be extended to $X \times X \times \mathbb{R}$ in the same way as $\Lambda_{S,\pi}$ and we denote thus obtained sheaf by $L_{S_+,\pi}$.

3.2.10 Proof of the Theorem 3.9

We will prove an equivalent statement as in Sect. 3.2.4

Define a functor $T : \mathcal{D}(X \times \mathbb{R}) \to \mathcal{D}(X \times \mathbb{R})$ by setting $T(F) = F \bullet L_S$ (see Sect. 3.1.2). Because of Lemma 3.16 and Proposition 3.11 we see that if $F \in \mathcal{D}_L(X)$, then $TF \in \mathcal{D}_{\Phi(L)}(X)$.

Next, we have natural maps

$$L_{S,\pi} \xrightarrow{i} L_{S_+,\pi} \xleftarrow{j} L_{0,\pi}$$

Note that $L_{0,\pi} = \mathbb{K}_{\{(x_1,x_2,t)|x_1=x_2,t\geq 0\}}$. In order to finish the proof of the theorem, it suffices to show that the cones of the induced maps $F \bullet L_{S,\pi} \to F \bullet L_{S_+,\pi}$ and $F = F \bullet L_{0,\pi} \to F \bullet L_{S_+,\pi}$ are torsion sheaves for all $F \in \mathcal{D}(X)$. This easily follows from the fact that the cones of the maps $L_{S,\pi} \to L_{S_+,\pi}$ and $L_{0,\pi} \to L_{S_+,\pi}$ are torsion objects in $\mathcal{D}(X \times X \times \mathbb{R})$. This fact can be seen from the following: each of the cones in question is supported on the set $\{(x_1, x_2, t)|m \leq t \leq M\}$ where m is the minimum of S and M is the maximum of S. Any sheaf G with such a property is necessarily torsion, because the supports of G and $T_{c*}G$ are disjoint for $c >> 0$ and $R\hom(G, T_{c*}G) = 0$. This proves Theorem 3.9.

3.2.11 Proof of Theorem 3.1

Let $F_1, F_2 \in \mathcal{D}(X)$ and let $f : F_1 \to F_2$. Call f *an isomorphism up-to torsion* if the cone of f is a torsion object. Call F_1 and F_2 *isomorphic up-to torsion* if they can be connected by a chain of isomorphisms up-to torsion.

It is easy to see that if F_1 and F_2 are isomorphic up-to torsion and for some $G \in \mathcal{D}(X)$, the natural map $R\hom(G, F_1) \to R\hom(G, T_{c*}F_1)$ is zero for some $c > 0$, then the map $R\hom(G, F_2) \to R\hom(G, T_{d*}F_2)$ is zero for some $d > 0$.

Suppose L_1 and L_2 are displaceable compact Lagrangians in T^*X, i.e. for some symplectomorphism Φ of T^*X such that Φ is identity outside of a compact, we have $L_1 \cap \Phi(L_2) = \emptyset$. Let $F_i \in \mathcal{D}_{L_i}(X)$. Theorem 3.1 is equivalent to the statement: *for some* $c > 0$, *the natural map* $R\hom(F_1, F_2) \to R\hom(F_1, T_{c*}F_2)$ *is zero.*

This statement can be proven as follows. By Theorem 3.9, there exists an object $F_3 \in \mathcal{D}_{\Phi(L_2)}(X)$ such that F_3 and F_2 are isomorphic up-to torsion. Therefore, it suffices to show that the natural map

$$R\hom(F_1, F_3) \to R\hom(F_1, T_{c*}F_3)$$

is zero for some $c > 0$. But Theorem 3.2 asserts that $R\hom(F_1, F_3) = R\hom(F_1, T_cF_3) = 0$, whence the statement.

4 Non-dispaceability of Certain Lagrangian Submanifolds in $\mathbb{CP}^n$

Consider $\mathbb{CP}^N$ with the standard symplectic structure. We have the following standard Lagrangian subvarieties in $\mathbb{CP}^N$: the Clifford torus $\mathbb{T} \subset \mathbb{CP}^N$ consisting of all points with homogeneous coordinates $(z_0 : z_1 : z_2 : \cdots : z_N)$ such that $|z_0| = |z_1| = \cdots = |z_N| > 0$. Another Lagrangian subvariety we will consider is $\mathbb{RP}^N \subset \mathbb{CP}^N$. Our main goal is to prove

Theorem 4.1 *(1)* $\mathbb{T}$ *is non-displaceable from itself;*
(2) $\mathbb{RP}^N$ *is non-displaceable from itself;*
(3) $\mathbb{T}$ *and* $\mathbb{RP}^N$ *are non-displaceable from one another.*

4.0.1

Let us first of all explain how Theorem 3.1 can be applied.

Let $G = \mathrm{SU}(N)$ Realize $\mathbb{CP}^{N-1}$ as a coadjoint orbit $\mathbb{CP}^N = \mathcal{O} \subset \mathfrak{g}^*$, where $\mathfrak{g} = \mathfrak{su}(N)$ is the Lie algebra of G. We identify $\mathfrak{g}$ with the real vector space of $N \times N$ skew-hermitian matrices. We have an invariant positive definite inner product on $\mathfrak{g}$ by the formula $< A, B >= -\mathrm{Tr}(AB)$. This way we get an identification $\mathfrak{g} \cong \mathfrak{g}^*$.

The orbit $\mathcal{O} \subset \mathfrak{g}^* \cong \mathfrak{g}$ is the orbit of the following diagonal skew-hermitian matrix

$$i\lambda(P_V - (1/N)I) \in \mathfrak{g}$$

where $V \subset \mathbb{C}^N$ is a one-dimensional sub-space, P_V is the orthogonal projector onto V, $\lambda \in \mathbb{R}$, $\lambda \neq 0$ is a fixed real number. For simplicity we will only work with $\lambda > 0$. However, the case $\lambda < 0$ is absolutely similar.

Consider T^*G. We have a diffeomorphism $I_R : T^*G \to G \times \mathfrak{g}^*$ where we identify $\mathfrak{g}^*$ with right-invariant forms on G. Any element $X \in \mathfrak{g}$ gives rise to a function on f_X on $\mathfrak{g}^*$. We have a standard Poisson structure on $\mathfrak{g}^*$ determined by the condition $\{f_X, f_Y\} = f_{[X,Y]}$. The canonical projection $p_R : T^*G \xrightarrow{I_R} G \times \mathfrak{g}^* \to \mathfrak{g}^*$ is then a Poisson map.

Let $\mathfrak{g}^{op}$ be the Lie algebra whose underlying vector space is $\mathfrak{g}$ but $[X, Y]_{\mathfrak{g}^{\mathrm{op}}} = -[X, Y]_{\mathfrak{g}}$, We then have an identification $I_L : T^*G \to G \times (\mathfrak{g}^{\mathrm{op}})^*$, where we identify $(\mathfrak{g}^{\mathrm{op}})^*$ with left-invariant forms on G. The composition $I_R I_L^{-1} : G \times (\mathfrak{g}^{\mathrm{op}})^* \to G \times \mathfrak{g}^*$ is as follows: $I_R I_L^{-1}(g, A) = (g, \mathrm{Ad}^*_{g^{-1}}(A))$.

Indeed, the conjugate map $(I_R I_L^{-1})^* : G \times \mathfrak{g} \to G \times \mathfrak{g}^{\mathrm{op}}$ is given by $(I_R I_L^{-1})^*(g, X) = (g, \mathrm{Ad}_{g^{-1}} X)$.

Respectively, $I_L I_R^{-1} : G \times \mathfrak{g}^* \to G \times (\mathfrak{g}^{\mathrm{op}})^*$ is given by $I_L I_R^{-1}(g, A) = (g, \mathrm{Ad}^*_g A)$.

One can easily check that the product $p_L \times p_R : T^*G \to (\mathfrak{g}^{\mathrm{op}})^* \times \mathfrak{g}^*$ is a Poisson map.

We know that $\mathcal{O}^{\mathrm{op}} \subset \mathfrak{g}^*$ is a symplectic leaf, hence a co-isotropic sub-variety. Therefore, so is $M := p_R^{-1}\mathcal{O} \subset T^*G$.

Let $\mathcal{O}^{\mathrm{op}} \subset (\mathfrak{g}^*)^{\mathrm{op}}$ be the image of $\mathcal{O} \subset \mathfrak{g}^*$ under the identification of vector spaces $\mathfrak{g}^* = (\mathfrak{g}^{\mathrm{op}})^*$.

We then see that $M = p_L^{-1}\mathcal{O}^{\mathrm{op}} = p_R^{-1}\mathcal{O}$. Indeed, we know that $I_L I_R^{-1}(g, A) = (g, \mathrm{Ad}_g A)$ and $A \in \mathcal{O}$ iff $\mathrm{Ad}_g A \in \mathcal{O}$.

Hence, we have $M = (p_L \times p_R)^{-1}(\mathcal{O}^{\mathrm{op}} \times \mathcal{O})$. Given any Poisson fibration $f : X \to Y$ and a coisotropic subvariety $N \subset Y$, the subvariety $f^{-1}N \subset X$ is also co-isotropic. Let $n \in f^{-1}N$ and let $V \in T_n f^{-1}N$ be a co-isotropic vector (i.e $V = X_H$ where H is a function in a neighborhood of n and $H|_{f^{-1}N} = 0$), we then see that $f_* V \in T_{f(n)}N$ is also a co-isotropic vector.

Let us apply this observation to our case. We see that $\mathcal{O}^{\mathrm{op}} \times \mathcal{O}$ has only zero co-isotropic vectors. Therefore, all co-isotropic vectors in TM are tangent to fibers of the map $p_L \times p_R : M \to \mathcal{O}^{\mathrm{op}} \times \mathcal{O}$. Comparison of dimensions shows that the inverse is also true: co-isotropic vectors in TM are precisely those tangent to the fibers of the map $p_L \times p_R$. Thus, co-isotropic foliation to M is the tangent foliation to $p_L \times p_R$.

We know that this implies an induced symplectic structure on $\mathcal{O}^{\mathrm{op}} \times \mathcal{O}$. As the map $p_L \times p_R$ is Poisson, it follows that the induced Poisson structure coincides with that induced by the inclusion $\mathcal{O}^{\mathrm{op}} \times \mathcal{O} \hookrightarrow \mathfrak{g}^{\mathrm{op}} \times \mathfrak{g}$. The corresponding symplectic 2 form is equal to $(-\omega; \omega)$ where ω is Kirillov's symplectic form on $\mathcal{O}$ and we use the identification of manifolds $\mathcal{O}^{\mathrm{op}} = \mathcal{O}$.

Let $I : M \to T^*G$ be the inclusion and $P = p_L \times p_R : M \to \mathcal{O}^{\mathrm{op}} \times \mathcal{O}$. It then follows that $I^*\omega_{T^*G} = P^*\omega_{\mathcal{O}^{\mathrm{op}}\times\mathcal{O}}$.

It follows that if $L \subset \mathcal{O}^{\mathrm{op}} \times \mathcal{O}$ is a Lagrangian manifold, then so is $IP^{-1}L \subset T^*G$.

Another important observation: let H be a function on $\mathcal{O}^{\mathrm{op}} \times \mathcal{O}$ and let H' be a function on T^*G such that $H'|_M = P^{-1}H$.

(1) Then the Hamiltonian vector field $X_{H'}$ is tangent to M;

(2) given any function f on $\mathcal{O}^{\mathrm{op}} \times \mathcal{O}$ we have

$$X_{H'}P^{-1}f = P^{-1}X_H f.$$

Let $e^{tX_{H'}}$ be the Hamiltonian flow of H' and e^{tX_H} the Hamiltonian flow of H. It then follows that for any point $m \in M$, $Pe^{tX_{H'}}(m) = e^{tX_H}(P(m))$.

These observations imply:

Proposition 4.2 *Let $L_1, L_2 \subset \mathcal{O}^{op} \times \mathcal{O}$ be subsets such that $IP^{-1}L_1, IP^{-1}L_2 \subset T^*G$ are non-displaceable. Then so are L_1, L_2.*

Proof Suppose L_1 and L_2 are displaceable. Then there exist functions $H_1, \ldots, H_k$ on $\mathcal{O}^{\mathrm{op}} \times \mathcal{O}$ such that $e^{X_{H_1}} \cdots e^{X_{H_k}} L_1 \cap L_2 = \emptyset$. Choose compactly supported functions $H_1', \ldots, H_k'$ on T^*G such that $H_i'|_M = P^{-1}H_i$. One then has

$$Pe^{X_{H_1'}} \cdots e^{X_{H_k'}} m = e^{X_{H_1}} \cdots e^{X_{H_k}} Pm$$

for every $m \in M$. Therefore,

$$IP^{-1}L_1 \cap e^{X_{H_1'}} \cdots e^{X_{H_m'}} P^{-1}L_2 = \emptyset,$$

i.e the Lagrangians $IP^{-1}L_1$ and $IP^{-1}L_2$ are displaceable, whence the statement □

Let $\Delta \subset \mathcal{O}^{\mathrm{op}} \times \mathcal{O}$ be the diagonal. Δ is clearly Lagrangian.

It then follows that Theorem 4.1 follows from the following one:

Theorem 4.3 *(1) $IP^{-1}\Delta$ and $IP^{-1}(\mathbb{T} \times \mathbb{T})$ are non-displaceble;*

(2) $IP^{-1}\Delta$ and $IP^{-1}(\mathbb{RP}^N \times \mathbb{RP}^N)$ are non-displaceable;

(3) $IP^{-1}(\mathbb{RP}^N \times \mathbb{RP}^N)$ and $IP^{-1}(\mathbb{T} \times \mathbb{T})$ are non-displaceable

4.0.2

We will prove Theorem 4.3 using Theorem 3.1.

Our main tool will be a certain object $u_{\mathcal{O}} \in \mathcal{D}(G)$ which will be now introduced.

We need a notation. Let $S \in \mathcal{D}(G)$. Let $F \in D(G)$. Let $m : G \times G \times \mathbb{R} \to G \times \mathbb{R}$ be the map induced by the product on G. Set $F *_G S := Rm_!(F \boxtimes S)$ (this is nothing else but a convolution). One can easily check that $F *_G S \in \mathcal{D}(G)$ (use Proposition 2.2).

Proposition 4.4 *There exists an object $u_{\mathcal{O}} \in \mathcal{D}_{IP^{-1}\Delta}(G)$ with the following properties:*

(1) there exists a neighborhood of the unit $U \subset G$; $e \in U$ with the following property:

for every $g \in G$ and every object $F \in D(G)$ such that F is supported on gU and $R\Gamma(G, F) = 0$, the object $F *_G u_{\mathcal{O}}$ is a torsion object;

(2) The object $u_{\mathcal{O}}$ is not a torsion object.

The proof of this Proposition is rather long, so we will first show how this Proposition (along with Theorem 3.1) implies Theorem 4.3.

4.0.3

Lemma 4.5 *Let $\mathfrak{h} \subset \mathfrak{g}$ be the standard Cartan subalgebra consisting of the diagonal traceless skew-hermitian matrices. Let $\mathfrak{k} := \mathfrak{so}(N) \subset \mathfrak{su}(N)$. We then have $\mathbb{T} = (\mathfrak{g}/\mathfrak{h})^* \cap \mathcal{O}$; $\mathbb{RP}^N = (\mathfrak{g}/\mathfrak{k})^* \cap \mathcal{O}$.*

Proof The symplectomorphism $f : \mathbb{CP}^N \to \mathcal{O}$ is as follows. Given a line $l \in \mathbb{C}^N$ we set $f(l) := i(\lambda P_L - \lambda/NI)$, where $\lambda > 0$ is a fixed positive real number. Let $v = (v_1, v_2, \ldots, v_N) \in l$; $v \neq 0$. We then have

$$f(l)_{pq} = (i\lambda/|v|^2)v_p\overline{v_q} - i\lambda/N\delta_{pq},$$

where δ_{pq} is the Kronecker symbol.

Thus, $f(l) \in \mathcal{O} \cap (\mathfrak{g}/\mathfrak{h})^*$ iff $f(l)_{pp} = 0$ for all p, i.e. $|v_p|^2/|v|^2 = 1/N$, i.e. $|v_1|^2 = |v_2|^2 = \cdots |v_N|^2$, i.e. $l \in \mathbb{T}$.

Analogously, $f(l) \in (\mathfrak{g}/\mathfrak{k})^*$ iff $f(l)_{pq} \in i\mathbb{R}$ for all p, q, i.e $v_p\overline{v_q} \in \mathbb{R}$ for all p, q. Let $v_{p_0} \neq 0$. Then $v_q = t_q/\overline{v_{p_0}}$for some $t_q \in \mathbb{R}$ and for all q. Let $t = (t_1, t_2, \ldots, t_N)$ then $v = t/\overline{v_{p_0}}$ and $l \in \mathbb{RP}^N \subset \mathbb{CP}^N$. The inverse can be easily checked as well. □

Proposition 4.6 *Let $T \subset SU(N)$ be the subgroup of diagonal matrices and let $SO(N) \subset SU(N)$ be the subgroup of special orthogonal matrices.*

We then have

(1) $\mathbb{K}_T *_G u_{\mathcal{O}} \in \mathcal{D}_{IP^{-1}(\mathbb{T}\times\mathbb{T})}(G)$;

(2) $\mathbb{K}_{SO(N)} *_G u_{\mathcal{O}} \in \mathcal{D}_{IP^{-1}(\mathbb{RP}^N\times\mathbb{RP}^N)}(G)$.

Proof Let us prove (1). First of all, one can easily check that $\mathbb{K}_T *_G u_{\mathcal{O}} \in \mathcal{D}(G)$ using Proposition 2.2. It only remains to show that $\mathbb{K}_T *_G u_{\mathcal{O}}$ *is microsupported on the set* $\{(g, t, k\omega, k)|k \geq 0; \omega \in IP^{-1}\mathbb{T} \times \mathbb{T}\}$. We have $\mathbb{K}_T *_G u_{\mathcal{O}} = Rm_!(\mathbb{K}_T \boxtimes u_{\mathcal{O}})$, where $m : G \times G \times \mathbb{R} \to G \times \mathbb{R}$ is induced by the product on G. Let also $M : G \times G \to G$ be the product on G Let $g_1, g_2 \in G$. We then have an induced map

$$M_{g_1,g_2*} : T_{(g_1,g_2)}G \times G \to T_{g_1g_2}G$$

Let $(g_1, g_2, X_1, X_2) \in G \times G \times \mathfrak{g} \times \mathfrak{g} = T(G \times G)$. One then has $M_{g_1,g_2*}(g_1, g_2, X_1, X_2) = (g_1g_2, X_1 + \mathrm{Ad}_{g_1} X_2)$. The dual map

$$M^*_{g_1,g_2} : T^*_{g_1g_2}G \to T^*_{(g_1,g_2)}G \times G$$

is as follows

$$M^*_{g_1,g_2}(g_1g_2, \omega) = (g_1, g_2, \omega; \mathrm{Ad}^*_{g_1}\omega).$$

Finally, the map

$$m^*_{g_1,g_2,t} : T^*_{(g_1g_2,t)}(G \times \mathbb{R}) \to T^*_{(g_1,g_2,t)}(G \times G \times \mathbb{R})$$

is given by

$$m^*_{g_1,g_2,t}(g_1g_2, t, \omega, k) = (g_1, g_2, t, \omega, \mathrm{Ad}^*_{g_1}\omega, k). \tag{8}$$

The map m being proper, we know that the object $Rm_!(\mathbb{K}_T \boxtimes u_{\mathcal{O}})$ is microsupported on the set of all points of the form

$$(g_1g_2, t, \omega, k) \tag{9}$$

where

$$m^*_{g_1,g_2,t}(g_1g_2, t, \omega, k) \in \mathrm{SS}(\mathbb{K}_T \boxtimes u_{\mathcal{O}}),$$

i.e

$$(g_1, \omega) \in \mathrm{SS}(\mathbb{K}_T); \tag{10}$$

$$(g_2, t, \mathrm{Ad}^*_{g_1}\omega, k) \in \mathrm{SS}(u_{\mathcal{O}}). \tag{11}$$

We have,

$$\mathrm{SS}(\mathbb{K}_T) \subset \{(g, \omega_1) \in G \times \mathfrak{g}^* | g \in T; \omega_1 \in (\mathfrak{g}/\mathfrak{t})^*\}; \tag{12}$$

$$\mathrm{SS}(u_{\mathcal{O}}) \subset \{(g, t, k\omega_2, k) | k \geq 0; (g, \omega_2) \in IP^{-1}\Delta\}, \tag{13}$$

as follows from Lemma 3.7. The condition $(g, \omega_2) \in IP^{-1}\Delta$ means that $\omega_2 \in \mathcal{O}$ and $P_L\omega_2 = P_R\omega_2$, i.e $\omega_2 = \mathrm{Ad}^*_g\omega_2$.

Therefore

$$(g_1g_2, t, \omega, k) \in \mathrm{SS}Rm_!(\mathbb{K}_T \boxtimes u_{\mathcal{O}})$$

only if (compare (11) and (13)):

$$k \geq 0 \tag{14}$$

$$\mathrm{Ad}^*_{g_1}\omega = k\omega_2, \tag{15}$$

where

$$\omega_2 \in \mathcal{O} \tag{16}$$

and

$$\mathrm{Ad}^*_{g_2}\omega_2 = \omega_2. \tag{17}$$

We should also have (compare (10) and (12)):

$$g_1 \in T \tag{18}$$

and

$$\omega \in (\mathfrak{g}/\mathfrak{h})^*. \tag{19}$$

Let us now show that (g_1g_2, ω, k) is of the form $(g_1g_2, k\omega^1, k)$, where $k \geq 0$ and $(g_1g_2, \omega^1) \in IP^{-1}(\mathbb{T} \times \mathbb{T})$. The latter means that $(P_L \times P_R)(g_1g_2, \omega^1) \in \mathcal{O}^{\mathrm{op}} \times \mathcal{O}$ i.e. *both* ω^1 *and* $\mathrm{Ad}^*_{g_1g_2}\omega^1$ *belong to* $\mathbb{T} = \mathcal{O} \cap (\mathfrak{g}/\mathfrak{h})^*$. We have $k \geq 0$ (see (14). If $k = 0$, then $\omega = \mathrm{Ad}^*_{g_1^{-1}}k\omega_2 = 0$ and $(g_1g_2, \omega, k) = (g_1g_2, 0, 0)$, the condition is fulfilled.

Let now $k > 0$. We have $\omega = k\mathrm{Ad}^*_{g_1^{-1}}\omega_2$ (see (15)) so that $\omega^1 = \mathrm{Ad}^*_{g_1^{-1}}\omega_2$.

As $\omega_2 \in \mathcal{O}$ (see (16)), it follows that $\omega^1 = \mathrm{Ad}^*_{g_1^{-1}}\omega_2 \in \mathcal{O}$. We also have $\omega_2 = \omega/k \in (\mathfrak{g}/\mathfrak{h})^*$ (see (19).

Next, let us consider

$$\mathrm{Ad}^*_{g_1g_2}\omega^1 = \mathrm{Ad}^*_{g_1g_2}\mathrm{Ad}^*_{g_1^{-1}}\omega_2$$

$$= \mathrm{Ad}^*_{g_2}\omega_2 = \omega_2$$

(the latter equality comes from (17), and we have already shown that $\omega_2 \in \mathcal{O} \cap (\mathfrak{g}/\mathfrak{t})^*$.

This proves the statement (1). The statement (2) can be proven in precisely the same way. □

4.1

Our goal is to prove the following statements

Proposition 4.7 *The object* $\mathbb{K}_T *_G u_{\mathcal{O}} \in \mathcal{D}(G)$ *is isomorphic up to torsion to the object* $u_{\mathcal{O}} \otimes_{\mathbb{K}} H^*(T, \mathbb{K})$.

Proposition 4.8 *Suppose that* $\mathbb{K}$ *is a field of characteristic 2. The object* $\mathbb{K}_{SO(N)} *_G u_{\mathcal{O}} \in \mathcal{D}(G)$ *is isomorphic up-to torsion to* $u_{\mathcal{O}} \otimes_{\mathbb{K}} H^*(SO(N), \mathbb{K})$.

4.1.1

These Propositions imply Theorem 4.3. Let $\mathbb{K}$ have characteristic 2 and let each of objects F_1 and F_2 be either $\mathbb{K}_T *_G u_{\mathcal{O}}$ or $\mathbb{K}_{SO(N)} *_G u_{\mathcal{O}}$. Taking into account Proposition 4.6 and Theorem 3.1, it suffices to show that for any $c > 0$, the induced map $R\,\mathrm{hom}(F_1, F_2) \to R\,\mathrm{hom}(F_1; T_{c*}F_2)$ does not vanish (for all choices of F_1 and F_2). By virtue of the just formulated Propositions, this follows from $u_{\mathcal{O}}$ being non-torsion which is promised in Proposition 4.4. Thus, Theorem 4.3 is now reduced to Propositions 4.4, 4.7, and 4.8. We will first deduced the last two Propositions from the first one, and, finally, we will prove Proposition 4.4.

4.1.2

In order to prove Propositions 4.7 and 4.8 we need to develop corollaries from Proposition 4.4(1).

Let C_U be the full subcategory of $D(G)$ generated by all objects F as in Proposition 4.4(1) and their finite extensions.

Lemma 4.9 *Let $Q := [0,1]^M$, $M \geq 0$. Let $\pi : Q \to G$ be any continuous map. Let $F \in D(Q)$, $R\Gamma(Q, F) = 0$. Then $R\pi_! F \in C_U$.*

Proof The case $M = 0$ is obvious. Let $M > 0$. Let $Q_0 := [0, 1/2] \times [0,1]^{M-1}$ and let $Q_1 = [1/2, 1] \times [0,1]^{M-1}$.

(1) We will first prove that *F can be obtained by a finite number of extensions from objects $X_1, X_2, \ldots, X_m$, where each X_i is supported on either Q_0 or Q_1 and* $R\Gamma(Q, X_i) = 0$. Call such objects and their extensions *admissible*. Thus, we are to show that F is admissible.

Let $I := Q_1 \cap Q_2$. Let $i_k : Q_k \to Q$ and $i : I \to Q$ be inclusions. Realize F as a complex of soft sheaves on Q Let $F_k := F|_{G_k}$ and $F_I := F|_I$. Each of these objects is also a complex of soft sheaves.

We then have an isomorphism

$$F \to \mathrm{Cone}(i_{1*}F_1 \oplus i_{2*}F_2 \to i_* F_I)$$

Let $p_k : Q_k \to \mathbf{pt}$ and $p_I : I \to \mathbf{pt}$ be the natural projections. Let $V_k := p_{k*}F_k = p_{k!}F_k$; let $V_I := p_{I*}F_I = p_{I!}F_I$. V_k and V_I are just complexes of $\mathbb{K}$-vector spaces. We then have maps

$$a_k : p_k^{-1}V_k \to F_k; a_I : \; p_I^{-1}V_I \to F_I;$$

$$b_k : i_{k*}p_k^{-1}V_k \to i_{I*}p_I^{-1}V_I$$

We then have the following commutative diagram of complexes of sheaves

$$\begin{array}{ccc} i_{1*}F_1 \oplus i_{2*}F_2 & \longrightarrow & i_{I*}F_I \\ \uparrow{\scriptstyle a_1\oplus a_2} & & \uparrow \\ i_{1*}p_1^{-1}V_1 \oplus i_{2*}p_2^{-1}V_2 & \xrightarrow{b_1\oplus b_2} & i_{I*}p_I^{-1}V_I \end{array}$$

Let Φ be the total complex of this diagram. Φ can be obtained by successive extensions from the following objects

$$\mathrm{Cone}(i_{k*}p_k^{-1}V_k \to i_{k*}F_k);$$

$$\mathrm{Cone}(i_{I*}p_I^{-1}V_I \to i_{I*}F_I);$$

each of these objects is admissible. Hence Φ is admissible.

Next, we have a natural map

$$\Phi \to \mathrm{Cone}(i_{1*}p_1^{-1}V_1 \oplus i_{2*}p_2^{-1}V_2 \to i_{I*}p_I^{-1}V_I)$$

The cone of this map is quasi-isomorphic to F. Thus, in order to show that F is admissible, it suffices to show that

$$\mathrm{Cone}(i_{1*}p_1^{-1}V_1 \oplus i_{2*}p_2^{-1}V_2 \to i_{I*}p_I^{-1}V_I)$$

is admissible.

Let us study the arrow $i_{1*}p_1^{-1}V_1 \oplus i_{2*}p_2^{-1}V_2 \to i_{I*}p_I^{-1}V_I$. This arrow is induced by the natural maps $V_1 \to V_I$ and $V_2 \to V_I$. The cone of the induced map $f : V_1 \oplus V_2 \to V_I$ is quasi-isomorphic to $R\Gamma(Q, F) = 0$. Therefore, f is a quasi-isomorphism and we have an induced quasi-isomorphism

$$\mathrm{Cone}(i_{1*}p_1^{-1}V_1 \oplus i_{2*}p_2^{-1}V_2 \to i_{I*}p_I^{-1}(V_1 \oplus V_2)) \to \mathrm{Cone}(i_{1*}p_1^{-1}V_1 \oplus i_{2*}p_2^{-1}V_2 \to i_{I*}p_I^{-1}V_I).$$

The object on the left hand side is isomorphic to

$$\mathrm{Cone}(i_{1*}p_1^{-1}V_1 \to i_{I*}p_I^{-1}V_1) \oplus \mathrm{Cone}(i_{2*}p_2^{-1}V_2 \to i_{I*}p_I^{-1}V_1).$$

We see that this object is a direct sum of admissible objects, hence is itself admissible, therefore the object

$$\mathrm{Cone}(i_{1*}p_1^{-1}V_1 \oplus i_{2*}p_2^{-1}V_2 \to i_{I*}p_I^{-1}V_I)$$

is also admissible, whence the statement.

(2) Choose a positive integer M and subdivide Q into 2^M small cubes, denote these small cubes by $\mathbf{q}_i$, $i = 1, \dots 2^M$. Call an object $X \in D(Q)$ *M-admissible* if either

(a) X is supported on one of $\mathbf{q}_i$ and $R\Gamma(Q, X) = 0$ or (b) X can be obtained from objects as in a) by a finite number of extensions.

By repeatedly applying the statement from (1) we see that *every object* $F \in D(Q)$ *such that* $R\Gamma(Q, F) = 0$ *is* M*-admissible.*

(3) For M large enough one has: for every i there exists $g_i \in G$ such that $\pi(\mathbf{q}_i) \subset g_i U$. This implies that given any object $X \in D(Q)$ supported on $\mathbf{q}_i$ and satisfying $R\Gamma(Q, X) = 0$, one has $R\pi_! X \in C_U$. Therefore, every M-admissible object is in C_U, including F. □

Corollary 4.10 *Let* U *be a neighborhood of unit in* G *such that* U *is diffeomorphic to an open ball. Then* $C_U = C$*, where* $C \subset D(G)$ *is the full subcategory formed by finite extensions of objects of the form* $R\pi_! X$*, where* $\pi : Q \to G$*,* $X \in D(Q)$*,* $R\Gamma(Q, X) = 0$.

Corollary 4.11 *Let* $F \in C$ *and* $X \in D(G)$*. One then has* $F *_G X \in C$*;* $X *_G F \in C$.

Proof Choose a small open ball $U \in G, e \in U$, small means that there exists another open ball $V \subset G$ such that $U \cdot U \subset V$. It is not hard to see that any $X \in D(G)$ can be realized as a finite extension of objects X_i, where each X_i is supported on $g_i U$ for some U. Without loss of generality, one then can assume that $X = X_i$. Therefore, $X *_G F$ is supported on $g_i U^2 \subset g_i V$. One also sees that $R\Gamma(G, X *_G F) = R\Gamma(G, X) \otimes R\Gamma(G, F) = 0$. Thus, $X *_G F \in C_V$.

The case of $F *_G X$ can be proven in a similar way. □

4.1.3

Call a map $f : F \to H$ in $D(G)$ a C-isomorphism if the cone of f is in C. Call two objects $F, H \in D(G)$ C*-isomorphic* if they can be joined by a chain of C-isomorphisms.

Corollary 4.12 *if* F_1 *and* F_2 *are* C*-isomorphic and* H_1 *and* H_2 *are* C*-isomorphic, then* $F_1 *_G H_1$ *and* $F_2 *_G H_2$ *are* C*-isomorphic*

4.1.4

We have

Claim 4.13 *If* F *and* H *are* C*-isomorphic, then* $F *_G u_{\mathcal{O}}$ *and* $H *_G u_{\mathcal{O}}$ *are isomoprhic up-to torsion*

Proof Indeed, $C = C_U$, where U is the same as in Proposition 4.4. The statement follows immediately from part (1) of this Proposition. □

4.2 Proof of Proposition 4.7

Let $S_k \subset \mathrm{SU}(N)$ be the one-parametric subgroup consisting of all matrices of the form $\mathrm{diag}(1, 1, \ldots, e^{i\phi}; e^{-i\phi}, 1, \ldots, 1)$, where $e^{i\phi}$ is at the kth position. We then have $T = S_1 S_2 \cdots S_{N-1}$; $\mathbb{K}_T = \mathbb{K}_{S_1} *_G \mathbb{K}_{S_2} *_G \cdots *_G \mathbb{K}_{S_{N-1}}$.

It is clear that the statement of Proposition follows from

Lemma 4.14 *For any k, $\mathbb{K}_{S_k}$ is C-isomorphic to $\mathbb{K}_e \oplus \mathbb{K}_e[-1]$*

Indeed, Corollary 4.12 will then imply that $\mathbb{K}_T$ is C-isomorphic to $(\mathbb{K}_e \oplus \mathbb{K}_e[-1])^{*N-1} = \mathbb{K}_e \otimes_{\mathbb{K}} H^\bullet(T, \mathbb{K})$. Therefore, by Claim 4.13, the objects $\mathbb{K}_T * u_{\mathcal{O}}$ and $(\mathbb{K}_e \otimes_{\mathbb{K}} H^\bullet(T, \mathbb{K}))) * u_{\mathcal{O}} = u_{\mathcal{O}} \otimes_{\mathbb{K}} H^\bullet(T, \mathbb{K})$ are isomorphic up-to torsion.

It now remains to prove Lemma.

4.2.1 Proof of Lemma 4.14

As all subgroups S_k are conjugated in G, it suffices to prove Lemma for S_1. One then has $S_1 \subset \mathrm{SU}(2) \subset \mathrm{SU}(N)$, where the embedding $\mathrm{SU}(2) \subset \mathrm{SU}(N)$ is induced by the standard decomposition $\mathbb{C}^N = \mathbb{C}^2 \oplus \mathbb{C}^{N-2}$. Let U be an open neighborhood of unit in $\mathrm{SU}(N)$ and let $U' := U \cap \mathrm{SU}(2)$. Let $\iota : \mathrm{SU}(2) \subset \mathrm{SU}(N)$ be the inclusion. It is clear that $i_* C_{U'} \subset C_U$, hence if two objects $F_1, F_2 \in D(\mathrm{SU}(2))$ are C-isomorphic, then so are $i_* F_1$ and $i_* F_2$. Therefore, in order to prove Lemma, it suffices to show that $\mathbb{K}_{S_1}$ *and* $\mathbb{K}_e \oplus \mathbb{K}_e[-1]$ *viewed as objects of* $D(\mathrm{SU}(2))$ *are* C *-isomorphic.*

Let $B \subset \mathfrak{su}(2)$ consist of all matrices of the form iM, where M is a Hermitian matrix whose eigenvalues have absolute value of at most π. Let $B_\pi \subset B$ be the subset of all matrices iM, where the eigenvalues of M are precisely π and $-\pi$. It is clear the B is diffeomorphic to a 3-dimensional closed ball and $B_\pi \subset B$ is the boundary 2-sphere.

Let $I : [-\pi; \pi] \to B$ be given by $I(\phi) = i\mathrm{diag}(\phi; -\phi)$.

We then have a diagram

$$\begin{array}{ccccc} [-\pi; \pi] & \xrightarrow{i_1} & B & \xrightarrow{i_2} & \mathrm{SU}(2) \\ \uparrow{\scriptstyle a_1} & & \uparrow{\scriptstyle a_2} & & \uparrow{\scriptstyle a_3} \\ \{-\pi, \pi\} & \xrightarrow{k_1} & B_\pi & \xrightarrow{k_3} & \{-I\} \end{array}$$

where i_2 is induced by the exponential map $\mathfrak{su}(2) \to \mathrm{SU}(2)$; a_1, k_1, a_2, a_3 are obvious inclusions; k_3 is the projection. We then have

$$\mathbb{K}_{S_1} = \mathrm{Cone}(R(i_2 i_1)_! \mathbb{K}_{[-\pi;-\pi]} \oplus a_{3!} \mathbb{K}_{-I} \to R(i_2 i_1 a_1)_! \mathbb{K}_{\{-\pi,\pi\}}). \qquad (20)$$

The arrow in this equation is induced by natural maps

$$\alpha : R(i_2 i_1)_! \mathbb{K}_{[-\pi;-\pi]} \to R(i_2 i_1 a_1)_! \mathbb{K}_{\{-\pi,\pi\}})$$

and

$$\beta : a_{3!}\mathbb{K}_{-I} \to R(i_2 i_1 a_1)_! \mathbb{K}_{\{-\pi,\pi\}} = R(a_3 k_3 k_1)_! \mathbb{K}_{\{-\pi,\pi\}}$$

where α is induced by the natural map

$$\mathbb{K}_{[-\pi;-\pi]} \to a_{1!}\mathbb{K}_{\{-\pi,\pi\}}$$

induced by the embedding $\{-\pi, \pi\} \subset [-\pi, \pi]$.

The map β is induced by the natural map

$$\mathbb{K}_{-I} \to (k_3 k_1)_! \mathbb{K}_{\{-\pi,\pi\}} = (k_3 k_1)_* (k_3 k_1)^{-1} \mathbb{K}_{\{-\pi,\pi\}}.$$

We have a C-isomorphism

$$\gamma : R i_{2!} \mathbb{K}_B \to R(i_2 i_1 a_1)_! \mathbb{K}_{[-\pi,\pi]}$$

Therefore the object in (20) is C-isomorphic to

$$\mathrm{Cone}(R i_{2!} \mathbb{K}_B \oplus \mathbb{K}_{-I} \stackrel{\alpha_1 \oplus \beta}{\to} R(i_2 i_1 a_1)_! \mathbb{K}_{\{-\pi,\pi\}}) \qquad (21)$$

where $\alpha_1 = \alpha\gamma$.

The map $\alpha_1 : R i_{2!} \mathbb{K}_B \to R(i_2 i_1 a_1)_! \mathbb{K}_{\{-\pi,\pi\}}$ can be factored as

$$R i_{2!} \mathbb{K}_B \to R(i_2 a_2)_! \mathbb{K}_{B_\pi} \to R(i_2 i_1 a_1)_! \mathbb{K}_{\{-\pi,\pi\}}.$$

Observe that $B_\pi = \mathbb{CP}^1$ and that $R(i_2 a_2)_! \mathbb{K}_{B_\pi} \cong H^*(B_\pi, \mathbb{K}) \otimes_{\mathbb{K}} \mathbb{K}_{-I}$. Next $R(i_2 i_1 a_1)_! \mathbb{K}_{\{-\pi,\pi\}} = \mathbb{K}_{-I} \oplus \mathbb{K}_{-I}$. The map $R(i_2 a_2)_! \mathbb{K}_{B_\pi} \to R(i_2 i_1 a_1)_! \mathbb{K}_{\{-\pi,\pi\}}$ factors as

$$R(i_2 a_2)_! \mathbb{K}_{B_\pi} = a_{3!}\mathbb{K}_{-I} \otimes_{\mathbb{K}} H^*(\mathbb{CP}^1) \to a_{3!}\mathbb{K}_{-I} \stackrel{\beta}{\to} a_{3!}(\mathbb{K}_{-I} \oplus \mathbb{K}_{-I}) = R(i_2 i_1 a_1)_! \mathbb{K}_{\{-\pi,\pi\}}$$

Thus we see that α_1 factors as $\alpha_1 = \beta u$. It is well known that in this case we have a quasi-isomorphism

$$\mathrm{Cone}(\alpha_1 \oplus \beta) \cong \mathrm{Cone}(0 \oplus \beta).$$

meaning that the object in (21) is isomorphic to $R i_{2!} \mathbb{K}_B \oplus \mathbb{K}_{-I}[-1]$ (because $\mathrm{Cone}(\beta) \cong \mathbb{K}_{-I}[-1]$).

Let $\varepsilon : 0 \in B$ be the zero matrix. one then has a C-isomorphism $R i_{2!} \mathbb{K}_B \to R i_{2!} \varepsilon_! \mathbb{K}_0 = \mathbb{K}_e$. Analogously, by choosing a point $0' \in B_\pi$, one gets a C-isomorphism $R i_{2!} \mathbb{K}_B \to \mathbb{K}_{-I}$. Therefore, the object in (21) is C-isomorphic to $\mathbb{K}_e \oplus \mathbb{K}_{-I}[-1]$ and $\mathbb{K}_{-I}$ is C-isomorphic with $\mathbb{K}_e$ (via $R i_{2!} \mathbb{K}_B$). Thus, the object in (21), hence $\mathbb{K}_{S_1}$ is C-isomorphic to $\mathbb{K}_e \oplus \mathbb{K}_e[-1]$. Lemma is proven.

4.3 Proof of Proposition 4.8

In this subsection we fix char $\mathbb{K} = 2$.

We have standard embeddings

$$\mathrm{SO}(2) \subset \mathrm{SO}(3) \subset \cdots \subset \cdots \mathrm{SO}(N) \subset \mathrm{SU}(N)$$

where the embedding $\mathrm{SO}(k) \subset \mathrm{SO}(N)$ is induced by the embedding $\mathbb{R}^k \hookrightarrow \mathbb{R}^N$; $(x_1, x_2, \ldots, x_k) \mapsto (x_1, x_2, \ldots, x_k, 0, \ldots, 0)$.

We will prove the following statement.

Lemma 4.15 *The sheaf* $\mathbb{K}_{SO(k)} \in D(SU(N))$ *is* C*-isomorphic to* $\mathbb{K}_{SO(k-1)} \oplus \mathbb{K}_{SO(k-1)}[1-k]$*, for all* $k \geq 2$.

It is clear that this Lemma implies the Proposition. Let us now prove Lemma.

4.3.1

We have an embedding $\mathrm{SO}(k) \subset \mathrm{SU}(k) \subset \mathrm{SU}(N)$ and, in the same way as in the proof of Lemma 4.14, it suffices to prove that $\mathbb{K}_{\mathrm{SO}(k)}$ is C-isomorphic to $\mathbb{K}_{\mathrm{SO}(k-1)} \oplus \mathbb{K}_{\mathrm{SO}(k-1)}[1-k]$ in $D(\mathrm{SU}(k))$.

4.3.2

Let $M := \mathrm{SU}(k)/\mathrm{SO}(k-1)$, let $\Pi : \mathrm{SU}(k) \to M$ be the canonical projection.

For any smooth manifold Y, let $C(Y) \subset D(Y)$ be the full subcategory formed by finite extensions of objects of the form $Rp_!X$ where $p : Q \to Y$ is a continuous map, $Q = [0, 1]^M$, $M \geq 0$, $X \in D(Q)$; $R\Gamma(Q, X) = 0$.

Lemma 4.16 *If* $F \in C(M)$*, then* $\Pi^{-1}F \in C(SU(k))$.

Proof Let $p : Q \to M$ be a continuous map. Π is a locally trivial fibration with fiber $\mathrm{SO}(k-1)$, let $\Pi_Q : \mathrm{SU}(k) \times_M Q \to Q$ be the pull-back of this fibration with respect to the map $p : Q \to M$. The fibration Π_Q is trivial, hence we have a homeomorphism

$$\mathrm{SO}(k-1) \times Q \cong \mathrm{SU}(k) \times_M Q.$$

We then have natural maps

$$\pi : \mathrm{SO}(k-1) \times Q \cong \mathrm{SU}(k) \times_M Q \to \mathrm{SU}(k);$$

Let $q' : \mathrm{SO}(k-1) \times Q \to \mathrm{SO}(k-1)$, $q : \mathrm{SO}(k-1) \times Q \to Q$, be projections. Let $X \in D(Q)$, $R\Gamma(Q, X) = 0$. We then have $\Pi^{-1}Rp_!X = R\pi_!q^{-1}X$.

Let us cover

$$\mathrm{SO}(k-1) = \bigcup_{i=1}^{n} Q_i,$$

where each $Q_i \subset \mathrm{SO}(k-1)$ is a closed subset homeomorphic to a cube. One then can represent the sheaf $\mathbb{K}_{\mathrm{SU}(k-1)}$ (actually any object of $D(\mathrm{SU}(k-1))$ as a finite extension formed by objects $Y_i \in D(\mathrm{SU}(k-1))$ such that each Y_i is supported on Q_{l_i} for some l_i. Let $Z_i \in D(Q_{l_i})$, $Z_i = Y_i|_{Q_{l_i}}$. The object $q^{-1}X$ is then a finite extension of objects of the form

$$q^{-1}X \otimes (q')^{-1}Y_i$$

Let $\pi_i : Q_{l_i} \times Q \to \mathrm{SO}(k-1) \times Q \to \mathrm{SU}(k)$ be the through map. Let $q_i : Q_{l_i} \times Q \to Q$, $p_i : Q_{l_i} \times Q \to Q_{l_i}$ be projections.

We then have $R\pi_! q^{-1}X$ is a finite extension formed by objects

$$R\pi_!(q^{-1}X \otimes (q')^{-1}Y_i) \cong R\pi_{i!}(q_i^{-1}X \otimes p_i^{-1}Z_i) \in C(\mathrm{SU}(k)).$$

Therefore, $\Pi^{-1}R\pi_! X \in C(\mathrm{SU}(k))$, whence the statement. □

4.3.3

We have an identification $\mathrm{SO}(k)/\mathrm{SO}(k-1) = S^k$. We have the natural map $S^k = \mathrm{SO}(k)/\mathrm{SO}(k-1) \to \mathrm{SU}(k)/\mathrm{SO}(k-1) = M$.

This map is an embedding; denote the image of this embedding $S \subset M$. Let $\overline{e} \in S^{k-1}$ be the image of the unit of $\mathrm{SO}(k)$. Fix the standard basis $(e^1, e^2, \ldots, e^k)$ in $\mathbb{R}^k$. Then S^k gets identified with the unit sphere in $\mathbb{R}^k$ and $\overline{e} = e^k$. The point $\overline{e}$ determines a point on S, to be also denoted by $\overline{e}$.

Lemma 4.16 implies that Lemma 4.15 follows from the following statement:

Lemma 4.17 *The object $\mathbb{K}_S$ is $C(M)$-equivalent to $\mathbb{K}_{\overline{e}} \oplus \mathbb{K}_{\overline{e}}[1-k]$.*

Proof As was explained above, S is identified with the unit sphere in $\mathbb{R}^k$. Let $V \subset \mathbb{R}^k$ be an orthogonal complement to e_k. Let us denote $e := e_k$ and $\varepsilon = -e$. Let $B \subset V$ be the ball of radius π. We have a surjective map $P : B \to S$: let $f = \phi n \in B$, where $0 \le \phi \le \pi$ and $n \in B$. Set $P(\phi n) = \cos(\phi)e + \sin(\phi)n$. It follows that P is 1-to 1 on the interior of B and that P takes the boundary of B to the point $\varepsilon \in S$. Let $c : B \xrightarrow{P} S \to M$ be the through map Let $\partial B \subset B$ be the boundary. We have a commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{c} & M \\ \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle \iota} \\ \partial B & \xrightarrow{p} & \varepsilon \end{array} \tag{22}$$

One has

$$\mathbb{K}_S \cong \mathrm{Cone}(Rc_!\mathbb{K}_B \oplus \iota_!\mathbb{K}_\varepsilon \xrightarrow{f_0} \iota_! Rp_! \mathbb{K}_{\partial B}), \tag{23}$$

where $f_0 = \alpha \oplus \beta$; the map $\alpha : Rc_!\mathbb{K}_B \to \iota_! Rp_!\mathbb{K}_{\partial B} = Rc_! i_* \mathbb{K}_{\partial B}$ is induced by the canonical map

$$\mathbb{K}_B \to i_* \mathbb{K}_{\partial B},$$

and the map

$$\beta : \iota_! \mathbb{K}_e \to \iota_! Rp_! \mathbb{K}_{\partial B}$$

is induced by the canonical map

$$\mathbb{K}_\varepsilon \to Rp_* \mathbb{K}_{\partial B} = Rp_! \mathbb{K}_{\partial B}.$$

Let $M : B \to \mathrm{SO}(k)$ as follows:

- $M(\phi n)$ is identity on any vector which is orthogonal to both n and e;
- $M(\phi n)e = \cos(\phi)e + \sin(\phi)n$;
- $M(\phi n)n = -\sin(\phi)e + \cos(\phi)n$.

One then sees that the composition

$$B \xrightarrow{M} \mathrm{SO}(k) \xrightarrow{\Pi} S$$

equals $P : B \to S$. Thus, $P = \Pi M$. One can also rewrite:

$$M(\phi n) = I + (e^{i\phi} - 1)\mathbf{pr}_{(e+in)/\sqrt{2}} + (e^{-i\phi} - 1)\mathbf{pr}_{(e-in)/\sqrt{2}},$$

where **pr** is the orthogonal projector.

For $0 \le \alpha \le \pi/4$, set

$$\mu(\alpha, \phi n) = I + (e^{i\phi} - 1)P_{(\cos\alpha e + i\sin\alpha n)} + (e^{-i\phi} - 1)P_{(\sin\alpha e - \cos\alpha i n)}$$

One sees that:

$\mu : [0, \pi/4] \times B \to \mathrm{SU}(k)$;

$$\mu(\alpha, 0) = I;$$

$$\mu(\alpha, \pi n) \in \mathrm{SO}(k);$$

$$\mu(\pi/4, \phi n) = M(\phi n);$$

$$\mu(\alpha, \pi n)e = -e.$$

Let $\nu : [0; \pi/4] \times B \xrightarrow{\mu} \mathrm{SU}(k) \to M$ be the through map. It then follows that $\nu(\alpha, \pi n) = \varepsilon$.

We have a commutative diagram

$$\begin{array}{ccccc} B & \xrightarrow{i} & [0;\pi/4]\times B & \xrightarrow{\nu} & M \\ \uparrow{\scriptstyle k_0} & & \uparrow{\scriptstyle k_1} & & \uparrow{\scriptstyle \iota} \\ \partial B & \xrightarrow{i_0} & [0,\pi/4]\times \partial B & \xrightarrow{\pi} & \varepsilon \end{array} \tag{24}$$

Here $i(b) = (\pi/4, b)$ for all $b \in B$; $i_0(b) = (\pi/4, b)$ for all $b \in \partial B$.
We have $c = \nu i$; $\pi i_0 = p$ (where p is as in diagram (22)).
In a way similar to above we can construct a map

$$f : R\nu_! \mathbb{K}_{[0;\pi/4]\times B} \oplus i_! \mathbb{K}_\varepsilon \to \iota_! R\pi_! \mathbb{K}_{[0;\pi/4]\times \partial B}$$

The diagram (24) gives rise to a commutative diagram in $D(M)$:

$$\begin{array}{ccc} Rc_!\mathbb{K}_B \oplus \iota_!\mathbb{K}_\varepsilon & \xrightarrow{f_0} & \iota_! Rp_! \mathbb{K}_{\partial B} \\ \uparrow{\scriptstyle a} & & \uparrow \\ R\nu_!\mathbb{K}_{[0;\pi/4]\times B} \oplus \iota_!\mathbb{K}_\varepsilon & \xrightarrow{f} & \iota_! R\pi_! \mathbb{K}_{[0;\pi/4]\times \partial B} \end{array} \tag{25}$$

in which the right vertical arrow is an isomorphism; the left vertical arrow is a direct sum of the identity arrow $\iota_!\mathbb{K}_e$ and the natural arrow

$$a : R\nu_!\mathbb{K}_{[0;\pi/4]\times B} \to R\nu_! Ri_! \mathbb{K}_B = Rc_! \mathbb{K}_B.$$

This diagram defines uniquely a map $A : \mathrm{Cone}(f) \to \mathrm{Cone}(f_0)$ (because the rightmost arrow in diagram (25) is an isomorphism) the cone of this map is isomorphic to the cone of the map a. It easily follows that $\mathrm{Cone}(a) \in C(M)$, therefore, A is a $C(M)$-isomorphism.

Consider now the diagram (25) where all ingredients are the same except that the map $i : B \to [0;\pi/4] \times B$ gets replaced with the map $i_1 : B \to [0;\pi/4] \times B$, where $i_1(b) = (0, b)$. Let us compute $c_1 := \nu i_1 : B \to M$. We have

$$\mu(0, \phi n) = I + (1 - e^{i\phi})\mathbf{pr}_e + (1 - e^{-i\phi})\mathbf{pr}_n; \tag{26}$$

$$c_1(\phi n) = P\mu(0, \phi n). \tag{27}$$

We then have a commutative diagram obtained from diagram (22) by replacement c with c_1. Hence we have a map

$$\mathbb{K}_S \cong \mathrm{Cone}(Rc_{1!}\mathbb{K}_B \oplus \iota_!\mathbb{K}_e \xrightarrow{f_1} \iota_! Rp_! \mathbb{K}_{\partial B}), \tag{28}$$

constructed in the same way as the map f_0 in (23).

In the same way as above one can show that $\mathrm{Cone}(f_1)$ is $C(M)$-isomorphic to $\mathrm{Cone}(f)$, hence to $\mathrm{Cone}(f_0)$, hence to $\mathbb{K}_S$.

Let us now work with Cone(f_1).

(1) Equations (26) and (27) imply that $c_1(rn) = c_1(-rn)$ for any $rn \in B$. Let $B/2$ be the quotient of B in which $b \in B$ gets identified with $-b$. Let $\delta : B \to B/2$ be the projection. We then have a unique map $c_2 : B/2 \to M$ such that $c_1 = \delta c_2$. Let $\partial B/2$ is the image of ∂B in $B/2$. Of course, $\partial B \cong S^{k-2}$ and $\partial B/2 \cong \mathbb{RP}^{k-2}$. We have a natural quotient map $\delta_1 : \partial B \to \partial B/2$. These maps fit into the following commutative diagram:

$$\begin{array}{ccccc} B & \xrightarrow{\delta} & B/2 & \xrightarrow{c_2} & M \\ {\scriptstyle i}\uparrow & & {\scriptstyle i_1}\uparrow & & {\scriptstyle \iota}\uparrow \\ \partial B & \xrightarrow{\delta_1} & \partial B/2 & \xrightarrow{p_1} & \varepsilon \end{array}$$

One then can construct an arrow

$$f_2 : Rc_{2!}\mathbb{K}_{B/2} \oplus \mathbb{K}_\varepsilon \to \iota_!(p_1\delta_1)_!\mathbb{K}_{\partial B}$$

in the same way as above. Similar to above, there exists a natural map

$$\mathrm{Cone}(f_2) \to \mathrm{Cone}(f_1)$$

whose cone is isomorphic to the cone of the natural map

$$Rc_{2!}\mathbb{K}_{B/2} \to Rc_{2!}R\delta_!\mathbb{K}_B. \tag{29}$$

Let us show that *the cone of this map is in* $C(M)$.

Indeed, choose a covering $\partial B = \bigcup_{k=1}^m C_k$ where C_k, and all non-empty intresections of these sets are closed sets homeomorphic to the closed disk of the same dimension as dimension of ∂B and $C_k \cap -C_k = \emptyset$.

Consider the set of all multiple non-empty intersections of the sets C_k and denote elements of this set by $C'_1, C'_2, \ldots, C'_M$. Each of these sets is homeomorphic to a closed disk of the same dimension as dimension of ∂B and for each i, $C'_i \cap -C'_i = \emptyset$.

Let $B_k \subset B$ be the cones of C'_k:

$$B_k = \{rn | 0 \le r \le \pi; n \in C'_k\}.$$

It is clear that B_k cover B and that $B_k \cap -B_k = \{0\}$.

Let $B_k/2$ be the images of B_k in $B/2$. The map $\delta|_{B_k} : B_k \to B_k/2$ is a homeomorphism. It follows that $\mathbb{K}_{B/2}$ is a finite extension of objects, each of them being of the form $\mathbb{K}_{B_k/2}$. It then suffices to show that the cone of the natural map

$$c_{2!}\delta_!\mathbb{K}_{B_k \cup -B_k} = c_{2!}\delta_!\delta^{-1}\mathbb{K}_{B_k/2} \to c_{2!}\mathbb{K}_{B_k/2} \in C(M)$$

We have

$$\delta_! \mathbb{K}_{B_k \cup -B_k} = \delta_!(\mathrm{Cone}(\mathbb{K}_{B_k} \oplus \mathbb{K}_{B_k} \to \mathbb{K}_0))$$

$$= \mathrm{Cone}(\mathbb{K}_{B_k/2} \oplus \mathbb{K}_{B_k/2} \to \mathbb{K}_0)$$

The natural map $\delta_! \mathbb{K}_{B_k \cup -B_k} \to \mathbb{K}_{B_k/2}$ is given by the natural map

$$\mathrm{Cone}(\mathbb{K}_{B_k/2} \oplus \mathbb{K}_{B_k/2} \to \mathbb{K}_0) \to \mathbb{K}_{B_k/2} \tag{30}$$

induced by

$$\mathrm{Id} \oplus \mathrm{Id} : \mathbb{K}_{B_k/2} \oplus \mathbb{K}_{B_k/2} \to \mathbb{K}_{B_k/2}$$

Therefore, the cone of the map (30) is isomorphic to the cone of the natural map

$$\mathbb{K}_{B_k/2} \to \mathbb{K}_0$$

Denote this cone by F' and let $F := F'|_{B_k/2}$. It follows that $R\Gamma(B_k/2, F) = 0$. Let $P : B_k/2 \to B/2 \to M$ be the trough map. Our task is now reduced to showing that $RP_! F \in C(M)$. This follows from the fact that $B_k/2$ is homeomorphic to a unit cube.

Thus, the cone of the map (29) is in $C(M)$, therefore $\mathrm{Cone}(f_1)$ and $\mathrm{Cone}(f_2)$ are $C(M)$ isomorphic.

Let us now study $\mathrm{Cone}(f_2)$. The map f_2 is a direct sum of two maps: one of them is the natural map $g : Rc_{2!}\mathbb{K}_{B/2} \to \iota_!(p_1\delta_1)_! \mathbb{K}_{\partial B} = Rc_{2!} i_{1!} \delta_{1!} \mathbb{K}_{\partial B}$ and the other is the natural map

$$h : \mathbb{K}_\varepsilon \to \iota_!(p_1\delta_1)_! \mathbb{K}_{\partial B} \tag{31}$$

The map g factors as

$$Rc_{2!}\mathbb{K}_{B/2} \xrightarrow{g_1} Rc_{2!} i_{1!} \mathbb{K}_{\partial B/2} \xrightarrow{l} Rc_{2!} i_{1!} \delta_{1!} \mathbb{K}_{\partial B} \tag{32}$$

We have $Rc_{2!}\mathbb{K}_{\partial B/2} = \iota_! Rp_{1!} \mathbb{K}_{\partial B/2} \cong H^*(\partial B/2, \mathbb{K}) \otimes_{\mathbb{K}} \iota_! \mathbb{K}_\varepsilon$;

$$Rc_{2!} i_{1!} \delta_! \mathbb{K}_{\partial B} = \iota_! Rp_! \mathbb{K}_{\partial B} = H^*(\partial B; \mathbb{K}) \otimes_{\mathbb{K}} \iota_! \mathbb{K}_\varepsilon.$$

The map l in (32) is induced by the map

$$\delta_1^* : H^*(\partial B/2; \mathbb{K}) \to H^*(\partial B; \mathbb{K}).$$

Recall that $\partial B \cong S^{k-2}$; $\partial B/2 \cong \mathbb{RP}^{k-2}$ and δ_1 is the quotient map. As char $\mathbb{K} = 2$, it follows that the map δ_1^* factors as

$$H^*(\partial B/2; \mathbb{K}) \xrightarrow{n_1} \mathbb{K} \xrightarrow{n_2} H^*(\partial B; \mathbb{K}),$$

where the arrow n_1 is induced by any embedding $\mathrm{pt} \to \partial B/2$ and the arrow n_2 is induced by the projection $\partial B \to \mathrm{pt}$. This means that $l = l_2 l_1$, where

$$l_1 : Rc_{2!}\mathbb{K}_{\partial B/2} \to \iota_!\mathbb{K}_\varepsilon$$

is induced by n_1, and

$$l_2 : \mathbb{K}_\varepsilon \to Rc_{2!}i_{1!}\delta_{1!}\mathbb{K}_{\partial B}$$

is induced by n_2. Let us now consider the map h in (31). As was explained above, $\iota_!(p_1\delta_1)_!\mathbb{K}_{\partial B} \cong H^*(\partial B; \mathbb{K}) \otimes_{\mathbb{K}} \iota_!\mathbb{K}_\varepsilon$ and the map h is induced by the map $\mathbb{K} \to H^*(\partial B; \mathbb{K})$ induced by the projection $\partial B \to \mathbf{pt}$. That is $h = l_2$

These observations show that the map $g = l_2 l_1 g_1 = h l_1 g_1$ factors through h. This implies that

$$\mathrm{Cone}(f_2) = \mathrm{Cone}(g \oplus h) = \mathrm{Cone}(0 \oplus h) = Rc_{2!}\mathbb{K}_{B/2} \oplus \mathrm{Cone}(h) = Rc_{2!}\mathbb{K}_{B/2} \oplus \iota_!\mathbb{K}_\varepsilon[1-k]$$

As was explained above, $Rc_{2!}\mathbb{K}_{B/2}$ is C-isomorphic to $Rc_!\mathbb{K}_B$. Let $x \in \partial B$. We then have natural C-isomorphisms

$$Rc_!\mathbb{K}_B \to Rc_!\mathbb{K}_0 = \mathbb{K}_e$$

and

$$Rc_!\mathbb{K}_B \to Rc_!\mathbb{K}_{0'} = \mathbb{K}_\varepsilon$$

hence, $Rc_{2!}\mathbb{K}_{B/2}$ is C-isomorphic with both $\mathbb{K}_e$ and $\mathbb{K}_\varepsilon$, as well as with $Rc_{2!}\mathbb{K}_{B/2}$.

Thus,

$$Rc_{2!}\mathbb{K}_{B/2} \oplus \iota_!\mathbb{K}_\varepsilon[1-k]$$

is C-isomorphic with $\mathbb{K}_e \oplus \mathbb{K}_e[1-k]$, hence so is $\mathrm{Cone}(f_2)$. This proves Lemma. □

5 Proof of Proposition 4.4: Constructing $u_{\mathcal{O}}$

The rest of this paper will be devoted to proving Proposition 4.4. In this section we will construct the object $u_{\mathcal{O}}$. In the subsequent sections we will check it satisfies all the required properties.

5.1 *Constructing $u_{\mathcal{O}}$*

Our construction is based on a certain object $\mathfrak{S} \in D(G \times \mathfrak{h})$. This object is introduced and studied in the subsequent Sect. 6. It is defined as any object satisfying the conditions in Theorem 6.1.

5.1.1 Convolution on $\mathfrak{h}$

Let X, Y are manifolds. Let $a : X \times \mathfrak{h} \times Y \times \mathfrak{h} \to X \times Y \times \mathfrak{h}$ be given by $a(x, A_1, y, A_2) = (x, y, A_1 + A_2)$. Let $F \in D(X \times \mathfrak{h})$ and $G \in D(Y \times \mathfrak{h})$. Set $F *_{\mathfrak{h}} G := Ra_!(F \times G)$.

5.1.2

Let $L := \mathcal{O} \cap C_+$. We have $L = \lambda e_1$, where $\lambda > 0$.

Let $\gamma_L \in D(\mathfrak{h} \times \mathbb{R})$ be given by $\gamma_L = \mathbb{K}_{\{(A,t)|t+<A,L>\geq 0\}}$. Let $I_0 : G \times \mathbb{R} \to G \times \mathfrak{h} \times \mathbb{R}$ be given by $I_0(g, t) = (g, 0, t)$. Set

$$u_{\mathcal{O}} = I_0^{-1}(\mathfrak{S} *_{\mathfrak{h}} \gamma_L).$$

Let us first of all prove that $u_{\mathcal{O}} \in \mathcal{D}_{IP^{-1}\Delta}(G)$. Using Proposition 2.2 it is easy to show that $u_{\mathcal{O}}$ is in the left orthogonal complement to $C_{\leq 0}(G)$. Let us now estimate $\mathrm{SS}(u_{\mathcal{O}})$.

Let $p_3 : G \times \mathfrak{h} \times \mathbb{R} \to G \times \mathfrak{h}$; $p_1 : G \times \mathfrak{h} \times \mathbb{R} \to \mathfrak{h} \times \mathbb{R}$; $p_2 : G \times \mathfrak{h} \times \mathbb{R} \to G \times \mathbb{R}$ be the projections.

One can show that

$$u_{\mathcal{O}} = Rp_{2!}(p_1^{-1}\mathbb{K}_{\{(A,t)|t\geq <A,L>\}} \otimes p_3^{-1}\mathfrak{S})$$

As usual let us identify

$$T^*(G \times \mathfrak{h} \times \mathbb{R}) = G \times \mathfrak{h} \times \mathbb{R} \times \mathfrak{g}^* \times \mathfrak{h}^* \times \mathbb{R}$$

We see that $p_1^{-1}\mathbb{K}_{\{(A,t)|t\geq <A,L>\}}$ is microsupported on the set

$$\Omega_1 := \{(g, A, t, 0, -kL; k)|k \geq 0\}.$$

The object $p_2^{-1}\mathfrak{S}$ is microsupported on the set

$$\Omega_2 := \{(g, A, t, \omega, \eta, 0)\},$$

where $(g, A, \omega, \eta) \in \Omega_{\mathfrak{S}}$ (See Eq. 51 for the definition of $\Omega_{\mathfrak{S}}$).

One sees that if $\zeta_j \in \Omega_j \cap T^*_{(g,A,t)}(G \times \mathfrak{h} \times \mathbb{R})$ and $\zeta_1 + \zeta_2 = 0$, then $k = 0$ and $\zeta_1 = 0$, hence $\zeta_2 = 0$. Therefore, the object

$$\Psi := p_1^{-1}\mathbb{K}_{\{(A,t)|t\geq <A,L>\}} \otimes p_3^{-1}\mathfrak{S}$$

is microsupported on the set

$$\Omega_3 := \{(g, A, t, \omega_1 + \omega_2; \eta_1 + \omega_2; k_1 + k_2)|(g, A, t, \omega_j; \eta_j; k_j) \in \Omega_j\}$$

We have

$$\Omega_3 = \{(g, A, t, \omega; \eta - kL; k)|k \geq 0; (g, A, \omega, \eta) \in \Omega\}$$

Let us now apply Corollary 3.4 to the projection p_2 (so that $E = \mathfrak{h}$).
Let

$$\pi : G \times \mathfrak{h} \times \mathbb{R} \times \mathfrak{g}^* \times \mathfrak{h}^* \times \mathbb{R} \to G \times \mathbb{R} \times \mathfrak{g}^* \times \mathfrak{h}^* \times \mathbb{R}.$$

Let us find $\pi(\Omega_3)$ We see that

$$\pi(\Omega_3) \subset \{(g, t, \omega, \eta - kL; k)|k \geq 0, \mathrm{Ad}_g\omega = \omega; \eta = |\omega|\} =: \Omega_4.$$

The set Ω_4 is closed. Therefore, $\mathrm{SS}(Rp_{2!}\Psi)$ is confined within the set of all points of the form $\{(g, t, \omega, k)|(g, t, \omega, 0, k) \in \Omega_4\}$ Thus $\|\omega\| = kL$, $\mathrm{Ad}_g\omega = \omega$ and $k \geq 0$. If $k = 0$. then $\omega = 0$ and we have $(g, t, 0, 0) \in \mathrm{SS}(Rp_{2!}\Psi)$. If $k > 0$, then set $\omega = k\zeta$. We then have $|\zeta| = L$ (which means that $\zeta \in \mathcal{O}$) and $\mathrm{Ad}_g\zeta = \zeta$. This is the same as to say $(g, \zeta) \in IP^{-1}\mathcal{O}$. This proves the statement

5.2 *Proof of Proposition 4.4 (1)*

5.2.1 The Map $\tau_c : u_{\mathcal{O}} \to T_{c*}u_{\mathcal{O}}$

We will rewrite this map in a way more convenient to us.

Let $c > 0$. We then have an obvious map $\tau_c^\gamma : \gamma_L \to T_{c*}\gamma_L$;

$$T_{c*}u_{\mathcal{O}} = I_0^{-1}(\mathfrak{S} *_{\mathfrak{h}} T_{c*}\gamma_L)$$

The natural map $\tau_c : u_{\mathcal{O}} \to T_{c*}u_{\mathcal{O}}$ (coming from the fact that $u_{\mathcal{O}} \in \mathcal{D}(G)$), in terms of the above identifications, is given by the map

$$I_0^{-1}(\mathfrak{S} *_{\mathfrak{h}} \gamma_L) \to I_0^{-1}(\mathfrak{S} *_{\mathfrak{h}} T_{c*}\gamma_L)$$

which is induced by the map τ_L^γ.

Let $A_1, A_2 \in \mathfrak{h}$. For $A \in \mathfrak{h}$ set $U_A = \{A_1 \in \mathfrak{h}|A_1 << A\}$. Set $V_A := \mathbb{K}_{U_A}[\dim \mathfrak{h}]$. We then have a natural map

$$e_A : \mathbb{K}_A \to V_A. \tag{33}$$

It is defined as follows. Let us identify $\mathbb{R}^{N-1} = \mathfrak{h}$, where

$$(x_1, x_2, \ldots, x_{N-1}) \mapsto \sum x_k f_k.$$

Let $A=\sum t_k f_k$ Upon this identification, $U_A=\{(x_1,x_2,\ldots,x_{N-1})|x_k<t_k\}$ and $V_A=\boxtimes_k(\mathbb{K}_{(-\infty,t_k)}[1])$; $\mathbb{K}_A=\boxtimes_k\mathbb{K}_{t_k}$. The map e_A is defined as a product of maps $\varepsilon_k:\mathbb{K}_{t_k}\to\mathbb{K}_{(-\infty;t_k)}[1]$ which represents the class of the extension

$$0\to\mathbb{K}_{(-\infty;t_k)}\to\mathbb{K}_{(-\infty,t_k]}\to\mathbb{K}_{t_k}\to 0.$$

Let $A\in C_+$, we then have $c=<A,L>\geq 0$ because $A,L\in C_+$.

Lemma 5.1 *Let $A\in\mathfrak{h}$ be such that $<A,L>=c$*
The natural map

$$\mathbb{K}_A*_{\mathfrak{h}}\gamma_L\stackrel{e_A}{\to}V_A*_{\mathfrak{h}}\gamma_L$$

is an isomorphism.

Proof Clear □

Let now $A\in C_+$. Since $A,L\in C_+$, it follows that $c=<A,L>\geq 0$. We also have a natural isomorphism

$$\mathbb{K}_A*_{\mathfrak{h}}\gamma_L\cong T_{c*}\gamma_L.$$

Let us combine this isomorphism with that of the Lemma, we will get an isomorphism

$$V_A*\gamma_L\cong T_{c*}\gamma_L$$

By substituting $A=0$, we get an isomorphism

$$V_0*\gamma_L\cong\gamma_L.$$

Upon these identifications, the map τ_c^γ corresponds to a map

$$\tau_A^V:V_0\to V_A$$

induced by the inclusion $U_0\subset U_A$.

Thus, the map $\tau_c:u_{\mathcal{O}}\to T_c u_{\mathcal{O}}$ is isomorphic to the map

$$I_0^{-1}(\mathfrak{S}*_{\mathfrak{h}}V_0*_{\mathfrak{h}}\gamma_L)\to I_0^{-1}(\mathfrak{S}*_{\mathfrak{h}}V_A*_{\mathfrak{h}}\gamma_L)$$

induced by the natural map $\tau_A^V:V_0\to V_A$. As $\mathfrak{h}$ is an abelian Lie group, we can rewrite the above map as

$$I_0^{-1}(V_0*_{\mathfrak{h}}\mathfrak{S}*_{\mathfrak{h}}\gamma_L)\to I_0^{-1}(V_A*_{\mathfrak{h}}\mathfrak{S}*_{\mathfrak{h}}\gamma_L).\tag{34}$$

5.2.2

Let $*_{G\times\mathfrak{h}}$ denote the convolution on $D(G\times\mathfrak{h})$.

Taking into account the expression (34) for τ_c, the Proposition 4.4(1) can be deduced from the following Proposition:

Proposition 5.2 *Let U and $F \in D(G)$ be as in Proposition 4.4(1). Then there exists $A \in C_+$ such that the natural map*

$$(F \boxtimes V_0) *_{G\times\mathfrak{h}} \mathfrak{S} \to (F \boxtimes V_A) *_{G\times\mathfrak{h}} \mathfrak{S} \tag{35}$$

induced by the map $\tau_A^V : V_0 \to V_A$, is zero in $D(G \times \mathfrak{h})$

Thus, Proposition 4.4(1) is now reduced to Proposition 5.2

5.3 *Proof of Proposition 5.2*

Let H be any sheaf on $\mathfrak{h}$. Let $\alpha : \mathfrak{h} \to \mathfrak{h}$ be the antipode map. We then have $H *_{\mathfrak{h}} \mathfrak{S} = Rp_{2!}(p_1^{-1}\alpha_* H \otimes a^{-1}\mathfrak{S})$, where as usual $p_1 : G \times \mathfrak{h} \times \mathfrak{h} \to \mathfrak{h}$ is given by

$$p_1(g, A_1, A_2) = A_1;$$

and $p_2 : G \times \mathfrak{h} \times \mathfrak{h} \to G \times \mathfrak{h}$ is given by $p_2(g, A_1, A_2) = (g, A_2)$. Set $H^\alpha := \alpha_* H$. We then have

$$Rp_{2!}(p_1^{-1}H^\alpha \otimes a^{-1}\mathfrak{S}) = Rp_{2!}((p_1^{-1}H^\alpha) \otimes (\mathfrak{S} *_G \mathfrak{S})),$$

where we have used the isomorphism (64). Next,

$$Rp_{2!}((p_1^{-1}H^\alpha) \otimes (\mathfrak{S} *_G \mathfrak{S})) \cong [Rp_!(\pi^{-1}H^\alpha \otimes \mathfrak{S})] *_G \mathfrak{S},$$

where $\pi : G \times \mathfrak{h} \to \mathfrak{h}$; $p : G \times \mathfrak{h} \to G$ are projections.

One then has

$$Rp_!(\pi^{-1}H^\alpha \otimes \mathfrak{S}) = I_0^{-1}(H *_{\mathfrak{h}} \mathfrak{S}).$$

Let $S_A := I_0^{-1}(V_A *_{\mathfrak{h}} \mathfrak{S})$. We then have a natural map $\tau_A^S : S_0 \to S_A$.

We have $V_A *_{\mathfrak{h}} \mathfrak{S} \cong S_A *_G \mathfrak{S}$ and

$$(F \boxtimes V_A) *_{G\times\mathfrak{h}} \mathfrak{S} \cong F *_G (S_A *_G \mathfrak{S}) = (F *_G S_A) *_G \mathfrak{S}$$

The map (35) is then induced by the map τ_A^S.

Thus, Proposition 5.2 is now reduced to

Proposition 5.3 *There exist: a neighborhood $U \subset G$ of the unit $e \in G$ and $A \in C_+$ such that the natural map*

$$F *_G S_0 \to F *_G S_A$$

induced by τ_A^S *is zero for any* $F \in D(G)$ *which is supported on* gU *for some* $g \in G$ *and satisfies* $R\Gamma(G, F) = 0$.

Proof We have a natural map $\mathbb{K}_A \to V_A$, as in (33). Hence, we have an induced map

$$I_0^{-1}(\mathbb{K}_A *_{\mathfrak{h}} \mathfrak{S}) \to I_0^{-1}(V_A *_{\mathfrak{h}} \mathfrak{S}) =: S_A. \tag{36}$$

One sees that *this map is actually an isomorphism.* Indeed, one can easily show that for any object $F \in D(G \times \mathfrak{h})$ such that $\mathrm{SS}(F) \subset T^*G \times \mathfrak{h} \times C_+ \subset T^*G \times T^*\mathfrak{h}$, the map

$$I_0^{-1}(\mathbb{K}_A *_{\mathfrak{h}} F) \to I_0^{-1}(V_A *_{\mathfrak{h}} F)$$

induced by the map (33), is an isomorphism, and $\mathfrak{S}$ is of this type by virtue of Theorem 6.1.

One also sees that $I_0^{-1}(\mathbb{K}_A *_{\mathfrak{h}} \mathfrak{S}) = I_{-A}^{-1}\mathfrak{S}$, where $I_{-A} : G \to G \times \mathfrak{h}$; $I_{-A}g = (G, -A)$. Taking into account (36), we obtain an isomorphism

$$S_A \cong I_{-A}^{-1}\mathfrak{S}.$$

Let us choose a small A, $A >> 0$.

As was shown in the course of proving Theorem 6.1, for $0 << A << b$ we have

$$S_A = I_{-A}^{-1}\mathfrak{S} \cong \mathbb{K}_{\mathcal{U}_A}.$$

where $\mathcal{U}_A = \{e^X | \|X\| << A\} \subset G$. We also know that $S_0 = \mathbb{K}_e$.

Without loss of generality one can assume that for some $A \in C_+$; $A << b$, $U \subset \mathcal{U}_{A/10}$. Let $h \in U$ so that $h = e^X$, where $\|X\| << A/10$. We have $(F *_G S_A)|_{gh} = R\Gamma_c(\{ghr^{-1} | r \in \mathcal{U}_A\}; F)[\dim G]$. It follows that $gU \subset \{ghr^{-1} | r \in \mathcal{U}_A\}$ (Indeed, let $gh' \in gU$ so that $h' = e^{X'}$, $\|X'\| << A/10$. We have $h' = hr^{-1}$, $r = (h')^{-1}h$. By Lemma 10.4, $r = e^Z$, where $\|Z\| \le \| - X'\| + \|X\| << A$. So $r \in \mathcal{U}_A$). Therefore, $(F *_G S_A)|_{gh} = R\Gamma(gU, F)[\dim G] = 0$. Thus, $F *_G S_A$ is supported away from gU. But $F *_G S_0 = F$ is supported on gU. Therefore, $R\hom(F *_G S_0; F *_G S_A) = 0$ which proves the statement. □

Thus, we have proven Proposition 4.4 (1). The rest of the paper is devoted to proving the second part of the Proposition.

5.4

Recall that we have a sheaf $\gamma_L := \mathbb{K}_{\{(A,t)|t+<A,L>\ge 0\}}$ on $\mathfrak{h} \times \mathbb{R}$. Let $\iota : \mathbb{R} \to \mathfrak{h} \times \mathbb{R}$ be given by $\iota(t) = (0, t)$. We have a natural isomorphism

$$\mathbb{K}_{[0,\infty]}[-\dim \mathfrak{h}] = \iota^! \gamma_L$$

hence a natural map

$$\mathbb{K}_{0\times[0,\infty)}[-\dim\mathfrak{h}] \to \gamma_L. \tag{37}$$

This map induces a map

$$I_0^{-1}(\mathfrak{S} *_{\mathfrak{h}} \mathbb{K}_{0\times[0,\infty)})[-\dim\mathfrak{h}] \to \mathbb{K}_{I_0^{-1}(\mathfrak{S}*_{\mathfrak{h}}\gamma_L)} = u_{\mathcal{O}} \tag{38}$$

where $I_0 : G \times \mathbb{R} \to G \times \mathfrak{h} \times \mathbb{R}$, $I_0(g,t) = (g,0,t)$ Next, one has

$$I_0^{-1}(\mathfrak{S} *_{\mathfrak{h}} \mathbb{K}_{0\times[0,\infty)}) = i_0^{-1}\mathfrak{S} \boxtimes \mathbb{K}_{[0,\infty)}$$

where $i_0 : G \to G \times \mathfrak{h}$, $i_0(g) = (g,0)$. We know that $i_0^{-1}\mathfrak{S} = \mathbb{K}_e$, thus we have an isomorphism

$$I_0^{-1}(\mathfrak{S} *_{\mathfrak{h}} \mathbb{K}_{0\times[0,\infty)}) = \mathbb{K}_{e\times[0,\infty)}$$

The map (38) then can be rewritten as:

$$\mathbb{K}_{e\times[0,\infty]}[-\dim\mathfrak{h}] \to u_{\mathcal{O}} \tag{39}$$

Proposition 5.4 *Let $\Phi \in \mathcal{D}_{G\times\mathcal{O}}(G)$ The natural map*

$$\hom_{G\times\mathbb{R}}(u_{\mathcal{O}};\Phi) \to R\hom_{G\times\mathbb{R}}(\mathbb{K}_{(e,0)}[-\dim\mathfrak{h}];\Phi)$$

induced by the map (39) is an isomorphism.

Proof We have

$$u_{\mathcal{O}} = Rp_{2!}(p_3^{-1}\mathfrak{S} \otimes p_1^{-1}\mathbb{K}_{\{(A,t)|t\ge(A,L)\}});$$

$$\mathbb{K}_{e\times[0,\infty)} = I_0^{-1}(\mathfrak{S} *_{\mathfrak{h}} \mathbb{K}_{0\times[0,\infty)})$$

$$= Rp_{2!}(p_3^{-1}\mathfrak{S} \otimes p_1^{-1}\mathbb{K}_{0\times[0,\infty)})$$

where $p_1 : G \times \mathfrak{h} \times \mathbb{R} \to \mathfrak{h} \times \mathbb{R}$; $p_2 : G \times \mathfrak{h} \times \mathbb{R} \to G \times \mathbb{R}$; $p_3 : G \times \mathfrak{h} \times \mathbb{R} \to G \times \mathfrak{h}$ are projections.

Let $X \in D(\mathfrak{h} \times \mathbb{R})$. We then have

$$R\hom(Rp_{2!}(p_3^{-1}\mathfrak{S} \otimes p_1^{-1}X);\Phi)$$

$$= R\hom(p_3^{-1}\mathfrak{S} \otimes p_1^{-1}X;\, p_2^!\Phi)$$

$$= R\hom(p_1^{-1}X;\, R\underline{\mathrm{Hom}}(p_3^{-1}\mathfrak{S};\, p_2^!\Phi)$$

$$= R\hom_{\mathfrak{h}\times\mathbb{R}}(X;\, Rp_{1*}R\underline{\mathrm{Hom}}(p_3^{-1}\mathfrak{S};\, p_2^!\Phi)). \tag{40}$$

Let us estimate the microsupport of the sheaf

$$Rp_{1*}R\underline{\mathrm{Hom}}(p_3^{-1}\mathfrak{S};\, p_2^!\Phi).$$

We know that $\mathrm{SS}(\mathfrak{S}) \subset \{(g, A, \omega, |\omega|) \in G \times \mathfrak{h} \times \mathfrak{g}^* \times \mathfrak{h}^*\}$. Therefore,

$$\mathrm{SS}(p_3^{-1}\mathfrak{S}) \subset \Omega_1 := \{(g, A, t, \omega_1, |\omega_1|, 0)\} \subset G \times \mathfrak{h} \times \mathbb{R} \times \mathfrak{g}^* \times \mathfrak{h}^* \times \mathbb{R}.$$

Analogously,

$$\mathrm{SS}(p_2^!\Phi) \subset \Omega_2 = \{(g, A, t, k\omega, 0, k) | k \geq 0, \omega \in \mathcal{O}\}$$

One sees that if $(g, A, t, \omega_i, \eta_i, k_i) \in \Omega_i$ and $\omega_2 = \omega_1, \eta_2 = \eta_1, k_1 = k_2$, then $0 = k_1 = k_2$, hence $\omega_2 = \omega_1 = 0$; also $0 = \eta_2 = \eta_1$. Therefore,

$$\mathrm{SS}(R\underline{\mathrm{Hom}}(p_3^{-1}\mathfrak{S};\, p_2^!\Phi)) \subset \Omega_3 := \{(g, A, t, \omega_2 - \omega_1; \eta_2 - \eta_1; k_2 - k_1) | (g, A, t, \omega_i, \eta_i, k_i) \in \Omega_i\}$$

$$= \{(g, A, t, k\omega - \omega_1; -|\omega_1|; k) | k \geq 0; \omega \in \mathcal{O}\}$$

As the map p_1 is proper, one has

$$\mathrm{SS}(Rp_{1*}R\underline{\mathrm{Hom}}(p_3^{-1}\mathfrak{S};\, p_2^!\Phi)) \subset \Omega_4 := \{(A, t, \eta, k) | \exists g \in G : (g, A, t, 0, \eta, k) \in \Omega_3\}.$$

We see that

$$\Omega_4 = \{(A, t, -k|\omega|, k)\} = \{(A, t, -kL, k)\}.$$

Let $\pi : \mathfrak{h} \times \mathbb{R} \to \mathbb{R};\ \pi(A, t) = t - <A, L>$. It then follows that $Rp_{1*}R\underline{\mathrm{Hom}}$ $(p_3^{-1}\mathfrak{S};\, p_2^!\Phi)$ is locally constant along the fibers of π i.e. there exists a sheaf Γ on $\mathbb{R}$ such that

$$Rp_{1*}R\underline{\mathrm{Hom}}(p_3^{-1}\mathfrak{S};\, p_2^!\Phi) = \pi^!\Gamma$$

Taking into account (40) the statement is reduced to showing that the natural map

$$R\,\mathrm{hom}_{\mathfrak{h}\times\mathbb{R}}(\mathbb{K}_{\{(A,t)|t\geq <A,L>\}};\, \pi^!\Gamma) \to R\,\mathrm{hom}_{\mathfrak{h}\times\mathbb{R}}(\mathbb{K}_{0\times[0,\infty)};\, \pi^!\Gamma)$$

is an isomorphism for any sheaf $\Gamma \in D(\mathbb{R})$. This is equivalent to showing that the map

$$R\pi_!\mathbb{K}_{0\times[0,\infty)}[-\dim\mathfrak{h}] \to R\pi_!\mathbb{K}_{\{(A,t)|t\geq <A,L>\}}$$

induced by the map (37) is an isomorphism, which is easy. □

It then follows that for all $c \in \mathbb{R}$, we have an isomorphism

$$R\,\mathrm{hom}(u_{\mathcal{O}};\, T_{c*}u_{\mathcal{O}}) \cong R\,\mathrm{hom}(\mathbb{K}_{e\times[0,\infty)}[-\dim\mathfrak{h}];\, T_{c*}u_{\mathcal{O}})$$

Let $i : \mathbb{R} \to G \times \mathbb{R}$; $i(t) = (e, t)$. We then have

$$R\operatorname{hom}(\mathbb{K}_{e\times[0,\infty)}; T_{c*}u_{\mathcal{O}}) = R\operatorname{hom}(\mathbb{K}_{[0,\infty)}; i^! T_{c*}u_{\mathcal{O}}).$$

One sees that the submanifold $i(\mathbb{R}) \subset G \times \mathbb{R}$ is non-characteristic for $T_{c*}u_{\mathcal{O}}$ (because $\mathrm{SS}(T_{c*}u_{\mathcal{O}}) \subset \{(g, t, k\omega, k), k \geq 0; \omega \in \mathcal{O}\}$). Therefore, according to Sect. 11.0.5, we have an isomorphism

$$i^{-1}T_{c*}u_{\mathcal{O}}[-\dim G] \cong i^! T_{c*}u_{\mathcal{O}}.$$

Thus, we have an isomorphism

$$\rho : R\operatorname{hom}(u_{\mathcal{O}}; T_{c*}u_{\mathcal{O}}) \cong R\operatorname{hom}(\mathbb{K}_{[0,\infty)}[-\dim \mathfrak{h}]; i^{-1}T_{c*}u_{\mathcal{O}}[-\dim G])$$

For $c > 0$ the natural maps

$$R\operatorname{hom}(u_{\mathcal{O}}; u_{\mathcal{O}}) \to R\operatorname{hom}(u_{\mathcal{O}}; T_{c*}u_{\mathcal{O}})$$

and

$$R\operatorname{hom}(\mathbb{K}_{[0,\infty)}[-\dim \mathfrak{h}]; i^{-1}u_{\mathcal{O}}[-\dim G]) \to R\operatorname{hom}(\mathbb{K}_{[0,\infty)}[-\dim \mathfrak{h}]; i^{-1}T_{c*}u_{\mathcal{O}}[-\dim G])$$

commute with our isomorphism.

Proposition (4.4) (2) reduces to

Proposition 5.5 *For any $c > 0$, the natural map*

$$R\operatorname{hom}(\mathbb{K}_{[0,\infty)}[-\dim \mathfrak{h}]; i^{-1}u_{\mathcal{O}}[-\dim G]) \to R\operatorname{hom}(\mathbb{K}_{[0,\infty)}[-\dim \mathfrak{h}]; i^{-1}T_c u_{\mathcal{O}}[-\dim G]) \tag{41}$$

is non-zero.

5.4.1

Let $I : \mathfrak{h} \to G \times \mathfrak{h}$ be given by $I(A) = (e, A)$. Let $\mathcal{S}_e := I^! \mathfrak{S} = I^{-1}\mathfrak{S}[-\dim G]$. We then have

$$i^{-1}u_{\mathcal{O}} = I_0^{-1}(\mathcal{S}_e *_{\mathfrak{h}} \gamma_L)[\dim G] \tag{42}$$

This equation dictates us to find an explicit expression for $\mathcal{S}_e$. It turns out to be more convenient to work with a slightly different object. Namely, let $\mathbf{Z} \subset G$ be the center of G. Let $I_{\mathbf{Z}} : \mathbf{Z} \times \mathfrak{h} \to G \times \mathfrak{h}$ be the obvious embedding. Set $\mathcal{S} := I_{\mathbf{Z}}^! \mathfrak{S} = I_{\mathbf{Z}}^{-1}\mathfrak{S}[-\dim G]$. We will identify this object up-to an isomorphism.

5.5 Identifying $\mathcal{S}$

We will now give an explicit description of the object $\mathcal{S}$ up-to isomorphism. The proof of this result will be given in the subsequent sections of the paper.

5.5.1 Object $\mathcal{Y}$

We first define an object $\mathcal{Y} \in D(\mathbf{Z} \times \mathfrak{h})$ as follows. Let $\mathbb{L} \subset \mathfrak{h}$ be the lattice consisting of all $A \in \mathfrak{h}$ such that $e^A \in \mathbf{Z}$.

For a subset $J \subset \{1, 2, \ldots, N-1\}$ and $l \in \mathbb{L}$ let $K(J, l) \subset e^l \times \mathfrak{h} \subset \mathbf{Z} \times h$ be defined as follows:

$$K(J, l) := \{(e^l, x) \in \mathbf{Z} \times \mathfrak{h} | \forall j \in J :< x - l, e_j >\geq 0\}.$$

Let $V(J, l) := \mathbb{K}_{K(J,l)}[D(l)]$, where $D(l)$ is an integer defined in (Sect. 7.5.5). That is, decompose $l = \sum l_k e_k$, where $e^1, e^2, \ldots, e^n$ is a basis in $\mathfrak{h}$ as in (96). Then $D(l) = \sum l_k D_k$, where $D_k = k(N-k)$ and $N = \dim \mathfrak{h} + 1$. Let $\mathbb{L}_J = \{l \in \mathbb{L} | \forall i \notin J :< l, f_j >\leq 0\}$ Let $\Psi^J := \bigoplus_{l \in \mathbb{L}_J} V(J, l)$.

Let $J_1 \subset J_2$. Construct a map

$$I_{J_1 J_2} : \Psi^{J_1} \to \Psi^{J_2}.$$

It is defined as the direct sum of the natural maps

$$V(J_1, l) \to V(J_2, l)$$

for all $l \in \mathbb{L}_{J_1} \subset \mathbb{L}_{J_2}$. These maps come from the closed embeddings $K(J_2, l) \subset K(J_1, l)$.

Let Subsets be the poset (hence the category) of all subsets of $\{1, 2, \ldots, N-1\}$. We then see that Ψ is a functor from Subsets to the category of sheaves on $\mathbf{Z} \times \mathfrak{h}$. We then construct the standard complex $\mathcal{Y}^\bullet$ such that

$$\mathcal{Y}^k := \bigoplus_{I, |I|=k} \Psi^I \tag{43}$$

and the differential $d_k : \mathcal{Y}^k \to \mathcal{Y}^{k+1}$ is given by

$$d_k = \sum (-1)^{\sigma(J_1, J_2)} I_{J_1 J_2}, \tag{44}$$

where the sum is taken over all pairs $J_1 \subset J_2$ such that $|J_1| = k$ and $|J_2| = k+1$. The set $J_2 \backslash J_1$ then consists of a single element e and $\sigma(J_1 J_2)$ is defined as the number of elements in J_2 which are less than e.

5.5.2 Object $\mathcal{S}$

Let $I \subset \{1, 2, \ldots, N-1\}$ be a subset. Denote $e_I := \sum_{i \in I} e_i \in \mathfrak{h}$. Let also $G(I)$ be a graded vector space as in Lemma 8.2.

For any $l \in \mathfrak{h}$, let $T_l : \mathbf{Z} \times \mathfrak{h} \to \mathbf{Z} \times \mathfrak{h}$ be the shift by l: $T_l(z, A) := (z, A + l)$

Theorem 5.6 *We have an isomorphism*

$$\mathcal{S} \cong \bigoplus_I G_I[D(-2\pi e_I)] \otimes T_{-2\pi e_I *}\mathcal{Y}, \tag{45}$$

Proof of this theorem is obtained as a result of a study of the object $\mathcal{S}$ in Sects. 6–9.

Given this description of $\mathcal{S}$, we can now compute $i^{-1}u_{\mathcal{O}}$.

5.6 *Computing* $i^{-1}u_{\mathcal{O}}$

Let $\mathcal{O}$ be the orbit of $L \in \mathfrak{g}^*$, where $L = \lambda e_1$, $\lambda > 0$. For each $z \in \mathbf{Z}$, let us define objects $\mathcal{V}_z \in D(\mathbb{R})$ by the formula:

$$\mathcal{V}_z := \bigoplus_{l \in \mathbb{L}^z; \forall j \neq 1: <l, f_j> \leq 0} \mathbb{K}_{[<l,L>;\infty)}[D(l) - \dim \mathfrak{h}], \tag{46}$$

where $\mathbb{L}^z := \{l \in \mathbb{L} | e^l = z\}$. For every $d > 0$ we have natural maps $\tau_d : \mathcal{V}_z \to T_{d*}\mathcal{V}_z$, where T_d is the shift by d. The map τ_d is induced by the obvious maps

$$\mathbb{K}_{[<l,L>;\infty)} \to \mathbb{K}_{[<l,L>+d;\infty)} = T_{d*}\mathbb{K}_{[<l,L>,\infty)}.$$

Theorem 5.7 *(1) We have an isomorphism*

$$i^{-1}u_{\mathcal{O}} \cong \bigoplus_I G_I[D(-2\pi e_I)] \otimes T_{<-2\pi e_I,L>*}\mathcal{V}_{e^{2\pi e_I}}[\dim G] \tag{47}$$

(2) The natural map $i^{-1}u_{\mathcal{O}} \to i^{-1}T_{d*}u_{\mathcal{O}}$ *is induced by the maps* τ_d.

Proof Let $\mathbb{L}^c = \{l \in L; e^l = c\}$. Let $\mathbb{L}^c_J = \mathbb{L}^c \cap \mathbb{L}_J$. Let $\mathcal{Y}_c \in D(\mathfrak{h})$; $\mathcal{Y}_c = \mathcal{Y}|_{c \times \mathfrak{h}}$.

It follows from (45) and (42) that we have an isomorphism

$$i^{-1}u_{\mathcal{O}} \cong \bigoplus_I G_I[D(-2\pi e_I)] \otimes I_0^{-1}(T_{-2\pi e_I *}\mathcal{Y}|_{e \times \mathfrak{h}} *_{\mathfrak{h}} \gamma_L)[\dim G].$$

Let $\mathcal{U}_z := I_0^{-1}\mathcal{Y}|_{z \times \mathfrak{h}} *_{\mathfrak{h}} \gamma_L$. We then have

$$I_0^{-1}[T_{-2\pi e_I *}\mathcal{Y}|_{e\times\mathfrak{h}} *_{\mathfrak{h}} \gamma_l] = I_0^{-1}[\mathcal{Y}_{e^{2\pi e_I}} *_{\mathfrak{h}} T_{-2\pi e_I *}\gamma_l]$$

$$= I_0^{-1}[\mathcal{Y}_{e^{2\pi e_I}} *_{\mathfrak{h}} T_{<-2\pi e_I, L>*}\gamma_L]$$

$$= T_{<-2\pi e_I, L>*}\mathcal{U}_{e^{2\pi e_I}},$$

where for a real number t, we define a map $T_t : G \times \mathbb{R} \to G \times \mathbb{R}$ to be the shift along $\mathbb{R}$ by t, whereas for $A \in \mathfrak{h}$, T_A is the shift by A along $\mathfrak{h}$ in $G \times \mathfrak{h}$.

We then have an isomorphism

$$i^{-1}u_{\mathcal{O}} \cong \bigoplus_I G_I[D(-2\pi e_I)] \otimes T_{<-2\pi e_I, L>*}\mathcal{U}_{e^{2\pi e_I}}[\dim G] \tag{48}$$

One also sees that the natural map

$$i^{-1}u_{\mathcal{O}} \to i^{-1}T_{d*}u_{\mathcal{O}}$$

for $d > 0$ corresponds under this isomorphism to the natural map induced by the maps

$$\tau_d : \mathcal{U}_c \to T_{d*}\mathcal{U}_c, \tag{49}$$

in turn induced by the natural map $\gamma_L \to T_{d*}\gamma_L$ coming from the embedding

$$\{(t, A)|t \geq - < A, L > +d\} \subset \{(t, A)|t \geq - < A, L >\}$$

(we have $\gamma_L = \mathbb{K}_{\{t\geq -<A,L>\}}$ and $T_{d*}\gamma_L = \mathbb{K}_{\{(t,A)|t\geq -<A,L>+d\}}$)).

Let us compute $\mathcal{U}_z$ for $z \in \mathbf{Z}$. We will actually see that $\mathcal{U}_z \cong \mathcal{V}_z$.

Lemma 5.8 *We have* $I_0^{-1}((V(J,l)|_{e^l\times\mathfrak{h}}) *_{\mathfrak{h}} \gamma_L) = 0$ *for all* $J \neq \{1\}$.

Proof Let $V'(J,l) := V(J,l)|_{e^l\times\mathfrak{h}}$.

We have $\gamma_L = \mathbb{K}_{\{(A,t)|t+<A,L>\geq 0\}}$. The inequality $t+ < A, L >\geq 0$ is equivalent to $t/\lambda+ < A, e_1 >\geq 0$. Set $T = t/\lambda$. Then our statement can be reformulated as:

$$V'(J,l) *_{\mathfrak{h}} \mathbb{K}_{\{(A,T)|T+<A,e_1>\geq 0\}} = RP_!((V'(J,l) \boxtimes \mathbb{K}_{\mathbb{R}}) \otimes \mathbb{K}_{\{(A,T)|T\geq <A,e_1>\}}) = 0,$$

where $P : \mathfrak{h} \times \mathbb{R} \to \mathbb{R}$ is the projection. This is equivalent to showing that for any $T \in \mathbb{R}$,

$$R\Gamma_c(\mathfrak{h}; V'(J,l) \otimes \mathbb{K}_{\{A\in\mathfrak{h}|T\geq (A,e_1)\}}) = 0.$$

Let $x_j : \mathfrak{h} \to \mathbb{R}; x_j =< A, e_j >$. We then have

$$V'(J,l) \otimes \mathbb{K}_{\{A\in\mathfrak{h}|T\geq <A,e_1>\}} = \mathbb{K}_S[D(l)],$$

where $S = \{A \in \mathfrak{h}|x_1(A) \leq T; \forall j \in J : x_j(A) \geq x_j(l)\}$.

Suppose there exists $j \in J$, $j \neq 1$. Decompose $\mathfrak{h} = \mathbb{R}.f_j \times E$, where E is the span of all f_i, $i \neq j$ (recall that f_j form the basis dual to $e_1, e_2, \ldots, e_{N-1}$). Thus, $\mathfrak{h} = \mathbb{R} \times E$. Then $\mathbb{K}_S[D(l)] = \mathbb{K}_{[0,\infty)} \boxtimes A$ for some $A \in D(E)$. Let $\pi : \mathfrak{h} \to E$ be the projection. Then $R\pi_! \mathbb{K}_S[D(l)] = 0$ because $R\Gamma_c(\mathbb{R}, \mathbb{K}_{[0,\infty)}) = 0$. If $J = \emptyset$, then $S = \{A \in \mathfrak{h} | x_1(A) \leq T\}$. It is easy to see that $R\Gamma_c(\mathfrak{h}, \mathbb{K}_S[D(l)]) = 0$. This exhausts all subsets $J \neq \{1\}$. □

It now follows that $I_0^{-1}(\Psi^J *_{\mathfrak{h}} \gamma_L) = 0$ for all $J \neq \{1\}$ Therefore, we have an isomorphism

$$\mathcal{U}_z = I_0^{-1}(\Phi_z *_{\mathfrak{h}} \gamma_L)[\dim G] \cong I_0^{-1}(\Psi_z^{\{1\}} *_{\mathfrak{h}} \gamma_L)[-1][\dim G]$$

$$\cong \bigoplus_{l \in \mathbb{L}_{\{1\}}^z} I_0^{-1}[V(\{1\}; l))_z *_{\mathfrak{h}} \gamma_L][-1][\dim G],$$

where the subscript z hear and below means the restriction onto $z \times \mathfrak{h} \subset \mathbf{Z} \times \mathfrak{h}$. Let us compute

$$I_0^{-1}[V(\{1\}; l)_z *_{\mathfrak{h}} \gamma_L] = RP_!(\mathbb{K}_{(A,t);x_1(A) \geq x_1(l)} \otimes \mathbb{K}_{\{(A,t)|\lambda x_1(A) \leq t\}})[D(l)],$$

where $P : \mathfrak{h} \times \mathbb{R} \to \mathbf{Z} \times \mathbb{R}$ is the projection. We have

$$RP_!(\mathbb{K}_{\{(A,t);x_1(A) \geq x_1(l)\}} \otimes \mathbb{K}_{\{(A,t)|\lambda x_1(A) \leq t\}})$$

$$= RP_!(\mathbb{K}_{\{(A,t);x_1(l) \leq x_1(A) \leq t/\lambda\}}) = \mathbb{K}_{[\lambda x_1(l),\infty)}[1 - \dim \mathfrak{h}]$$

Thus,

$$I_0^{-1}[V(\{1\}; l)_c *_{\mathfrak{h}} \gamma_L] \cong \mathbb{K}_{[\lambda x_1(l),\infty)}[1 - \dim \mathfrak{h}][D(l)]$$

Let $d \geq 0$. We need to compute the map

$$\tau_d : I_0^{-1}[V(\{1\}; l)_c *_{\mathfrak{h}} \gamma_L] \to T_{d*} I_0^{-1}[V(\{1\}; l)_c *_{\mathfrak{h}} \gamma_L]$$

induced by the natural map

$$\gamma_L \to T_{d*}\gamma_L.$$

It is easy to see that the map τ_d is isomorphic to the natural map

$$\mathbb{K}_{[\lambda x_1(l),\infty)}[1 - \dim \mathfrak{h}] \to T_{d*}\mathbb{K}_{[\lambda x_1(l),\infty)}[1 - \dim \mathfrak{h}]$$

$$= \mathbb{K}_{[\lambda x_1(l)+d,\infty)}[1 - \dim \mathfrak{h}],$$

induced by the embedding

$$[\lambda x_1(l)+d,\infty)\subset[\lambda x_1(l),\infty).$$

Thus, we have,

$$\mathcal{U}_z=\bigoplus_{l\in\mathbb{L}^z_{\{1\}}}\mathbb{K}_{[\lambda x_1(l),\infty)\}}[D(l)][-\dim\mathfrak{h}]$$

$$=\bigoplus_{l\in\mathbb{L}^z;\forall j\neq 1:<l,f_j>\ \leq 0}\mathbb{K}_{[\lambda<l,e_1>,\infty)}[D(l)-\dim\mathfrak{h}].$$

Thus, we see that $\mathcal{U}_z\cong\mathcal{V}_z$. It is now straightforward to check that the maps τ_d on both sides do match □

Let us substitute (46) into (47). We will get

$$i^{-1}u_{\mathcal{O}}\cong\bigoplus_I G(I)\otimes\upsilon(I)[-\dim\mathfrak{h}+\dim G],$$

where

$$\upsilon(I)=\bigoplus_{l\in\mathbb{L}^{e^{2\pi e_I}};\forall j\neq 1:<l,f_j>\leq 0}\mathbb{K}_{[<l-2\pi e_I,L>;\infty)}[D(l-2\pi e_I)].$$

Let us replace l with $l+2\pi e_I$. We will get an ultimate formula

$$\upsilon_I=\bigoplus_{l\in\mathbb{L}^0;\forall j\neq 1:<l+2\pi e_I,f_j>\leq 0}\mathbb{K}_{[<l,L>;\infty)}[D(l)].\tag{50}$$

The map $\tau_d:i^{-1}u_{\mathcal{O}}\to T_{d*}i^{-1}u_{\mathcal{O}}$, $d\leq 0$ is induced by natural maps $\tau_d:\upsilon_I\to T_{d*}\upsilon_I$ which are produced by the embeddings $T_d[<l,L>;\infty))\subset[<l,L>;\infty)$.

5.6.1 Proof of Proposition 5.5

We have

$$R\hom(\mathbb{K}_{[0,\infty)}[-\dim\mathfrak{h}];T_{d*}i^{-1}u_{\mathcal{O}}[-\dim G])=\bigoplus_I G(I)\otimes H_I(d),$$

where

$$H_I(d):=R\hom(\mathbb{K}_{[0,\infty)};T_{d*}\upsilon_I)\cong\bigoplus_{l\in S_I(d)}\mathbb{K}[D(l)],$$

and

$$S_I(d):=\{l\in L^0|\forall j\neq 1:<l+2\pi e_I,f_j>\leq 0;<l,L>+d\geq 0\}.$$

The map (41) is induced by maps $\tau_d : H_I(0) \to H_i(d)$, which are in turn induced by the maps $\tau_d : \upsilon \to T_{d*}\upsilon$. It is not hard to see that the map $\tau_d : H_I(0) \to H_I(d)$ is induced by the inclusion $S_I(0) \subset S_I(d)$. As $S_I(0)$ is not empty, the maps $\tau_d : H_I(0) \to H_I(d)$ do not vanish for any $d \geq 0$, which proves the Proposition.

6 An Object $\mathfrak{S}$

We will freely use notations from Sect. 10.

The object $\mathfrak{S}$ will be characterized microlocally. Let us first define a subset

$$\Omega_{\mathfrak{S}} \in T^*(G \times \mathfrak{h}) \tag{51}$$

which will serve as a microsupport of $\mathfrak{S}$. Define $\Omega_{\mathfrak{S}}$ as a set of all points

$$(g, A, \omega, \eta) \in G \times \mathfrak{h} \times \mathfrak{g} \times \mathfrak{h} = T^*(G \times \mathfrak{h})$$

satisfying:

(1) $g(V_k(\omega)) \subset V_k(\omega)$, that is $\mathrm{Ad}_g\omega = \omega$;

(2) $\det g|_{V_k(\omega)} = e^{-i<e_k,A>}$;

(3) $\eta = \|\omega\|$. The notation $V_k(\omega)$ is introduced in the beginning of Sect. 10, see (97).

Finally, let us denote for $A \in \mathfrak{h}$, $I_A : G \to G \times \mathfrak{h}$ the embedding $I_A(g) = (g, A)$.

We now formulate

Theorem 6.1 *There exists an object $\mathfrak{S} \in D(G \times \mathfrak{h})$ such that*

(1) $SS(\mathfrak{S}) \subset \Omega_{\mathfrak{S}}$;

(2) $I_0^{-1}\mathfrak{S} = \mathbb{K}_{e_G}$.

6.1 Proof of Theorem 6.1

6.1.1

Let $U_1, U_2 \subset \mathfrak{h}$ be open convex sets. Let $a : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{h}$ be addition. The map a induces a map $U_1 \times U_2 \to U_1 + U_2$ which is well known to be a trivial smooth fibration whose fiber and base are diffeomorphic to $\mathfrak{h}$.

Let $F_k \in D(G \times U_k)$, $k = 1, 2$. Let $M : G \times U_1 \times G \times U_2 \to G \times U_1 \times U_2$ be the map induced by the product on G. Set $F_1 *_G F_2 := RM_!(F_1 \boxtimes F_2)$.

Let $a : G \times U_1 \times U_2 \to G \times (U_1 + U_2)$ be induced by the addition on $\mathfrak{h}$.

Lemma 6.2 *Suppose that $SS(F_k) \subset \Omega_{\mathfrak{S}} \cap T^*(G \times U_k)$. Then (1) The natural map*

$$a^{-1}Ra_*(F_1 *_G F_2) \to F_1 *_G F_2$$

is an isomorphism;

(2) $SS(Ra_*(F_1 *_G F_2)) \subset \Omega_{\mathfrak{S}} \cap T^*(G \times (U_1 + U_2))$.

Proof Let us first estimate the microsupport of $F_1 *_G F_2 = RM_!(F_1 \boxtimes F_2)$. Since the map M is proper, we know that a point

$$\zeta := (g, A_1, A_2, \omega, \eta_1, \eta_2) \in G \times U_1 \times U_2 \times \mathfrak{g}^* \times \mathfrak{h}^* \times \mathfrak{h}^* = T^*(G \times U_1 \times U_2)$$

belongs to $\mathrm{SS}RM_!(F_1 \boxtimes F_2)$ only if there exist $g_1, g_2 \in G$ such that $M(g_1, A_1, g_2, A_2) = (g, A_1, A_2)$ (i.e. $g = g_1 g_2$) and

$$M^*\zeta|_{(g_1,A_1,g_2,A_2)} \in \mathrm{SS}(F_1 \boxtimes F_2).$$

We have

$$M^*\zeta|_{(g_1,A_1,g_2,A_2)} = (g_1, A_1, \omega, \eta_1, g_2, A_2, \mathrm{Ad}^*_{g_1}\omega, \eta_2).$$

We then have $(g_1, A_1, \omega, \eta_1), (g_2, A_2, \mathrm{Ad}^*_{g_1}\omega, \eta_2) \in \Omega_{\mathfrak{S}}$. Therefore, $\mathrm{Ad}^*_{g_1}\omega = \omega$, and we have

$$(g_k, A_k, \omega, \eta_k) \in \Omega_{\mathfrak{S}}.$$

This implies $\eta_1 = \eta_2 = \|\omega\|$. This means that any 1-form in $\mathrm{SS}(RM_!(F_1 \boxtimes F_2))$ vanishes on fibers of a. This proves part 1).

Let us now estimate $\mathrm{SS}Ra_*(F_1 *_G F_2)$. We know that $\zeta \in \mathrm{SS}Ra_*(F_1 *_G F_2)$, where $\zeta \in T^*_{(g,A)}(G \times (U_1 + U_2))$, iff for every point $(g, A_1, A_2) \in G \times U_1 \times U_2$ such that $A_1 + A_2 = A$, we have

$$a^*\zeta|_{(g,A_1,A_2)} \in \mathrm{SS}(a^{-1}Ra_*(F_1 *_G F_2)).$$

Let $\zeta = (g, A, \omega, \eta)$, then $a^*\zeta|_{(g,A_1,A_2)} = (g, A_1, A_2, \omega, \eta, \eta)$. Using the isomorphism $a^{-1}Ra_*(F_1 *_G F_2) \to F_1 *_G F_2$. and the above estimate for $\mathrm{SS}(F_1 *_G F_2)$, we get: there exist $g_1, g_2 \in G$ such that $g = g_1 g_2$ and

$$(g_k, A_k, \omega, \eta) \in \Omega_{\mathfrak{S}}.$$

It remains to show that $(g_1 g_2, A_1 + A_2, \omega, \eta) \in \Omega_{\mathfrak{S}}$. Indeed, we have $\eta = \|\omega\|$. Next, $Ad^*_{g_k}\omega = \omega$, therefore, $\mathrm{Ad}^*_{g_1 g_2}\omega = \omega$.

Finally,

$$\det g_1 g_2|_{V_k(\omega)} = \det g_1|_{V_k(\omega)} \det g_2|_{V_k(\omega)}$$

$$= e^{-i<A_1,e_{d_k(\omega)}>} e^{-i<A_2,e_{d_k(\omega)}>}$$

$$e^{-i<A_1+A_2,e_{d_k(\omega)}>}.$$

□

6.1.2

Let $b \in C_+^\circ$; $b \le e_1/100$. Let $V_b^- := \{A \in C_-^\circ | -A << b\}$, where C_+° is the interior of the positive Weyl chamber and $C_-^\circ = -C_+^\circ$, see Sect. 10. let $W_b^- \subset G \times V_b^-$;

$$W_b^- := \{(e^X, A);\ A \in V_b^-;\ \|X\| << -A\}.$$

Set $F^- \in D(G \times V_b^-)$;

$$F^- := \mathbb{K}_{W_b^-}[\dim G].$$

6.1.3

We will identify $TG = G \times \mathfrak{g}$; $T^*G = G \times \mathfrak{g}^* = G \times \mathfrak{g}$ via identifying $\mathfrak{g}$ with the space of all right invariant vector fields on G and $\mathfrak{g}^* = \mathfrak{g}$ with the space of all right invariant 1-forms on G. Analogously, we will identify $T(G \times \mathfrak{h}) = G \times \mathfrak{h} \times \mathfrak{g} \times \mathfrak{h}$ and $T^*(G \times \mathfrak{h}) = G \times \mathfrak{h} \times \mathfrak{g}^* \times \mathfrak{h}^* == G \times \mathfrak{h} \times \mathfrak{g} \times \mathfrak{h}$.

Lemma 6.3 *The microsupport of F^- is contained in the set of all points* $(e^X, A, \omega, \eta) \in G \times V_b^{-1} \times \mathfrak{g}^* \times \mathfrak{h}^*$*, where*

(1) $\|X\| \le -A$*;*
(2) $[X, \omega] = 0$*;*
(3) $TrX|_{V_k(\omega)} = -i < A, e_{d_k} >$*;*
(4) $\eta = \|\omega\|$

Proof Let $U \subset \mathfrak{g} \times V_b^-$; $U = \{(X, A) | \|X\| << -A\}$. Let

$$\exp : \mathfrak{g} \times V_b^{-1} \to G \times V_b^{-1}$$

be the exponential map. We see that exp maps U diffeomorphically onto W_b^-, hence we have an induced diffeomorphism $\exp : T^*U \to T^*W_b^-$. It also follows that $F^- = \exp_* \mathbb{K}_U[\dim G]$ and that $SS(F^-) = \exp(SS(\mathbb{K}_U))$.

Let us estimate $SS(\mathbb{K}_U)$. $U \subset \mathfrak{g} \times V_b^-$ is an open convex subset. It follows that a point $(X, A, \omega, \eta) \in \mathfrak{g} \times V_b^- \times \mathfrak{g}^* \times \mathfrak{h}^*$ is in the microsupport of $\mathbb{K}_U$ iff (1) $\|X\| \le -A$;

(2) for all $(X', A') \in U$, $< X', \omega > + < A', \eta > < < X, \omega > + < A, \eta >$.

Fix A', then $X' \in \mathfrak{g}$ is an arbitrary element such that $\|X'\| << -A$. Lemma 10.1 implies that

$$\sup < X', \omega > = < -A'; \|\omega\| >$$

Thus, Condition (2) is equivalent to

$$< -A', \|\omega\| > + < A', \eta > \le < X, \omega > + < A, \eta > \tag{52}$$

for all $A' \in V_b^-$. Plug $A' = A$. We will get

$$< -A, \|\omega\| > \leq < X, \omega > .$$

On the other hand $< X, \omega > \leq < \|X\|, \|\omega\| > \leq < -A, \|\omega\| >$. This implies that

$$< -A, \|\omega\| > = < X, \omega > . \tag{53}$$

According to Lemma 10.1, for all k,

$$\mathrm{Tr} X|_{V_k(\omega)} = -i < A, e_{d_k(\omega)} > .$$

Let us plug (53) into (52). We will get

$$< -A', \|\omega\| > + < A', \eta > \leq < -A, |\omega| > + < A, \eta >$$

for all $A' \in V_b^-$. As $A \in V_b^-$ and V_b^- is open, this is only possible if $\eta = \|\omega\|$. □

Corollary 6.4 *We have* $SS(F^-) \subset \Omega_{\mathfrak{G}} \cap T^*(G \times V_b^-)$.

6.1.4

Let $U \subset G \times V_b^- \times V_b^-$ be given by

$$U := \{(e^X, A_1, A_2) | \|X\| << -A_1 - A_2\}$$

Lemma 6.5 *We have an isomorphism*

$$F^- *_G F^- \cong \mathbb{K}_U[\dim G].$$

Proof Let $j_{U_1} : U_1 \hookrightarrow G \times V_b^- \times V_b^-$ be an open set defined by $U_1 = M(W_b^- \times W_b^-)$. It follows that we have an isomorphism

$$j_{U_1 !}((F^- *_G F^-)|_{U_1}) \to F^- *_G F^-.$$

We have

$$U_1 = \{(e^{X_1} e^{X_2}, A_1, A_2) | A_k \in V_b^-; \|X_k\| << -A_k\}.$$

According to Lemma 10.4, we have $e^{X_1} e^{X_2} = e^Y$, where $\|Y\| \leq \|X_1\| + \|X_2\| << -A_1 - A_2$. Thus $U_1 \subset U$.

Let $j_U : U \to G \times V_b^- \times V_b^-$ be the open embedding. We then have an isomorphism

$$j_{U!}((F^- *_G F^-)|_U) \to F^- *_G F^- \tag{54}$$

Let us now study $F^- *_G F^-|_U$. Let us estimate the microsupport of this object. Similar to proof of Lemma 6.2, we see that a point

$$(g, A_1, A_2, \omega, \eta_1, \eta_2) \in G \times V_b^- \times V_b^- \times \mathfrak{g}^* \times \mathfrak{h}^* \times \mathfrak{h}^* = T^*(G \times V_b^- \times V_b^-) \tag{55}$$

is in $\mathrm{SS}(F^- *_G F^-|_U)$ iff

(1) $(g, A_1, A_2) \in U$;

(2) there exist $X_1, X_2 \in \mathfrak{g}$ such that $g = e^{X_1}e^{X_2}$ and $(e^{X_k}, A_k, \omega, \eta_k) \in \mathrm{SS}(F^-)$ for $k = 1, 2$.

According to Lemma 6.3, we have

$$\|X_k\| \leq -A_k;$$

$$\mathrm{Tr} X_k|_{V_l(\omega)} = -i < A_l, e_{d_l} > .$$

Hence, $e^Y = e^{X_1}e^{X_2}$ preserves the spaces $V_l(\omega)$. As $\|Y\| << \|X_1\| + \|X_2\| \leq e_1/(50N)$, it follows that all eigenvalues of $-iY$ have absolute value of less than $1/(50N)$. It then follows that Y does preserve the spaces $V_l(\omega)$ as well, and $\mathrm{Tr} Y|_{V_l(\omega)}$ has absolute value of at most $1/50$.

We also have

$$\det e^Y|_{V_l(\omega)} = \det e^{X_1}|_{V_l(\omega)} e^{X_2}|_{V_l(\omega)} = e^{-i<A_1+A_2, e_{d_l(\omega)}>}.$$

We have $| < A_1 + A_2, e_{d_l(\omega)} > | \leq 1/50$, therefore,

$$\mathrm{Tr} Y|_{V_l(\omega)} = -i < A_1 + A_2, e_{d_l(\omega)} > . \tag{56}$$

Assume $\omega \neq 0$. Then there exists a subspace $V_l(\omega)$ which is proper, i.e. $0 < d_l(\omega) < N$. On the other hand, we have $(e^Y, A_1, A_2) \in U$, meaning that $e^Y = e^{Y'}$, where $\|Y'\| << -A_1 - A_2$. We then have $\|Y\|, \|Y'\| < e_1/(50N)$ which implies $Y = Y'$ and $\|Y\| << A_1 + A_2$. This clearly contradicts to (56). Therefore, it is impossible that $\omega \neq 0$, hence $\omega = 0$. It then follows that in (55), $\eta_1 = \eta_2 = \|\omega\| = 0$. Thus, we have proven that $F^- *_G F^-|_U$ is microsupported on the zero-section, hence is locally constant. However, under the exponential map, U is a diffeomorphic image of an open convex set $\{(X, A_1, A_2)|A_k \in V_b^-; \|X\| << -A_1 + A_2\} \subset \mathfrak{g} \times V_b^- \times V_b^-$. Therefore, U is diffeomorphic to $\mathbb{R}^{\dim U}$ and $F^- *_G F^-$ is constant on U.

Let $Z := R\Gamma_c(U; F^- *_G F^-)$. We then have a natural isomorphism $F^- *_G F^-|_U \cong Z_U[\dim U]$. Because of an isomorphism (54), we have an induced isomorphism

$$R\Gamma_c(U; F^- *_G F^-) \to R\Gamma_c(G \times V_b^- \times V_b^-; F^- *_G F^-)$$

$$\cong R\Gamma_c(G \times V_b; F^-) \otimes R\Gamma_c(G \times V_b; F^-)$$

$$\cong \mathbb{K}[-\dim G \times V_b^-] \otimes \mathbb{K}[-\dim G \times V_b^-][2\dim G] = \mathbb{K}[-\dim U + \dim G].$$

This implies the statement. □

Let $a : G \times V_b^- \times V_b^- \to G \times 2V_b^-$ be the addition map. The just proven Lemma as well as Lemma 6.2 imply that the natural map $a^{-1}Ra_*(F^- *_G F^-) \to F^- *_G F^-$ is an isomorphism and that

$$Ra_*(F^- *_G F^-) \cong \mathbb{K}_{\{(e^X,A)|A\in 2V_b^-;\|X\|<<-A\}}[\dim G].$$

We then have an induced isomorphism

$$\iota : Ra_*F^- *_G F^-|_{G\times V_b^-} \cong F^-. \tag{57}$$

6.1.5

Let $M > 0$ and let $F_M^- \in D(G \times (V_b^-)^M)$;

$$F_M^- := F^- *_G F^- *_G \cdots *_G F^-,$$

where F^- occurs M times.

Let $a_M : G \times (V_b^-)^M) \to G \times MV_b^-$ be the addition map. Lemma 6.2 implies that the natural map

$$a_M^{-1}Ra_{M*}F_M^- \to F_M^-$$

is an isomorphism.

Let $\Phi_M^- := Ra_{M*}F_M^-$.

Let us construct a map

$$I_M : \Phi_M^-|_{G\times(M-1)V_b^-} \to \Phi_{M-1}^-,$$

where $M \geq 2$, as follows.

Let $W \subset (V_b^-)^2$ be an open convex subset consisting of all points of the form (v_1, v_2), where $v_1 + v_2 \in V_b^-$. Let $W_M := (V_b^-)^{M-2} \times W \subset (V_b^-)^M$.

Let us decompose

$$\alpha_M := a_M|_{G\times W_M} : G \times W_M = G \times (V_b^-)^{M-2} \times W \stackrel{a_2}{\to} G \times (V_b^-)^{M-2} \times V_b^-$$

$$\stackrel{a_{M-1}}{\to} (M-1)V_b^-.$$

It follows that $\alpha_M(W_M) = G \times (M-1)V_b^-$. We have a natural isomorphism

$$Ra_{M*}F_M^-|_{G\times(V_b^-)^{M-1}} = R\alpha_{M*}F_M^-|_{G\times W_M} \cong Ra_{M-1*}Ra_{2*}F_M^-|_{G\times W_M}.$$

We have

$$Ra_{2*}F_M^-|_{G\times W_M} \cong F_{M-2}^- *_G (R\alpha_*F^- *_G F^-|_W),$$

where $\alpha : G \times W \to G \times V_b^-$ is the addition map. We have an isomorphism (see (57))

$$R\alpha_*(F^- *_G F^-|_W) \cong (Ra_*(F^- *_G F^- *))|_{V_b^-} \overset{\iota}{\cong} F^-.$$

Hence, we have isomorphisms

$$Ra_{2*}(F_M^-|_{G\times W_M}) \cong F_{M-1}^-$$

$$I_M : Ra_{M*}F_M^-|_{G\times(V_b^-)^{M-1}} \cong Ra_{M-1*}F_{M-1}^-.$$

Thus, we have objects $\Phi_M^- \in D(G \times MV_b^-)$ and isomorphisms

$$I_M : \Phi_M^-|_{(M-1)V_b^-} \to \Phi_{M-1}^-.$$

It then follows that there exists an object $\Phi^- \in D(G \times C_-^\circ)$ (note that $C_-^\circ = \bigcup_M MV_b^-$) along with isomorphisms

$$J_M : \Phi^-|_{MV_b^-} \to \Phi_M^-$$

which are compatible with I_M in the obvious way.

Let $\Psi^- \in D(G \times C_-^\circ)$ be another object endowed with isomorphisms $J'_M : \Psi^-|_{MV_b^-} \to \Phi_M^-$ so that J'_M are compatible with I_M. Then there exists a (non-canonical) isomorphism $\Phi^- \to \Psi^-$ which is compatible with the isomorphisms J_M, J'_M.

Lemma 6.2 implies that $\mathrm{SS}(\Phi_M^-) \subset \Omega_{\mathfrak{S}} \cap T^*(G \times MV_b^-)$. Therefore,

$$\mathrm{SS}(\Phi^-) \subset \Omega_{\mathfrak{S}} \cap T^*(G \times C_-^\circ).$$

6.1.6

Lemma 6.2 implies that we have an isomorphism

$$A^{-1}RA_*(\Phi^- *_G \Phi^-) \to \Phi^- *_G \Phi^-$$

where $A : G \times C_-^\circ \times C_-^\circ \to G \times C_-^\circ$ is the addition.

Let us restrict this isomorphism to $G \times MV_b^- \times V_b^-$. We will then get an isomorphism

$$A^{-1}(RA_*\Phi^- *_G \Phi^-|_{(M+1)V_b^-}) \to \Phi_M^- *_G F^- = A^{-1}\Phi_{M+1}^-.$$

Thus, we have an isomorphism

$$J'_{M+1} : RA_*\Phi^- *_G \Phi^-|_{(M+1)V_b^-} \cong \Phi_{M+1}^-$$

One can easily check that these isomorphisms are compatible with I_M hence, there exists an isomorphism

$$J : RA_*(\Phi^- *_G \Phi^-) \cong \Phi^-$$

which is compatible with isomorphisms J_M, J'_M. Therefore, we have an isomorphism

$$I : \Phi^- *_G \Phi^- \cong A^{-1}\Phi^-. \tag{58}$$

6.1.7

Let $X^\pm \in D(G)$, $X^+ := \mathbb{K}_{\{e^{-X}|\|X\|\leq b/2\}}$; $X^- := \mathbb{K}_{\{e^X|\|X\|<<b/2\}}[\dim G]$. We have an isomorphism $X^- \cong \Phi^-|_{G\times(-b/2)}$.

Lemma 6.6 *We have an isomorphism* $X^- *_G X^+ \cong \mathbb{K}_e$.

Proof Let us first compute the microsupport of $X^- *_G X^+$. We will prove the following: $SS(X^- *_G X^+)$ *is contained in the set of all points of the form*$(e^Y, \omega) \in G \times \mathfrak{g}^*$,*where* $Y \in \mathfrak{g}$; $\|Y\| \leq (e_1 + e_{N-1})/200$, $[Y, \omega] = 0$ *and* $< Y, \omega >= 0$.

Let us first estimate $SS(X^-)$. Let $\exp : \mathfrak{g} \times G$ be the exponential map. We then see that $X^- = \exp_* \mathbb{K}_U[\dim G]$, where $U \subset \mathfrak{g}$ is an open convex subset $U = \{X|\|X\| << b/2\}$.

We know that $SS(\mathbb{K}_U)$ consists of all points of the form $(X, \omega) \subset \mathfrak{g} \times \mathfrak{g}^*$, where $\|X\| \leq b/2$ and

$$< X', \omega > < < X, \omega >$$

for all $X' << b/2$. Lemma 10.1 implies that this is equivalent to $< b/2, \|\omega\| >\leq< X, \omega >$ (because

$$\sup_{X'<<b/2} < X', \omega >=< b/2, \|\omega\| >);$$

on the other hand

$$< X, \omega >\leq< \|b/2\|, \|\omega\| >$$

by the same Lemma 10.1. Therefore $< X, \omega >=< \|b/2\|, \|\omega\| >$. As $\|X\| \leq b/2$ this implies $[X, \omega] = 0$;

$$\mathrm{Tr} X|_{V_k(\omega)} = i < b, e_{d_k(\omega)} > /2. \tag{59}$$

As $[X, \omega] = 0$, we see that $SS(X^-)$ consists of all points (e^X, ω), where $\|X\| \leq b/2$ and $[X, \omega] = 0$ and we have (59).

Analogously, $X^+ = \exp_* \mathbb{K}_K$, where $K \subset \mathfrak{g}$ is a convex compact $K = \{X|\| - X\| \leq b/2\}$. Therefore, $SS(\mathbb{K}_K)$ consists of all points (X_1, ω_1), where $\| - X_1\| \leq b/2$ and $< X', \omega_1 >\geq< X_1, \omega_1 >$ for all $X' \in K$. I.e. $< -X', \omega_1 >\leq< -X_1, \omega_1 >$. In the same way as above, we conclude that this is equivalent to $< -X_1, \omega_1 >=< b/2, \|\omega_1\| >$ which in turn is equivalent to $\| - X_1\| \leq b/2$; $[X_1, \omega_1] = 0$;

$$\mathrm{Tr}(-X_1)|_{V_k(\omega_1)} = i < b/2, e_{d_k(\omega_1)} > . \tag{60}$$

Thus, $\mathrm{SS}(X^+)$ consists of all points of the form (e^{X_1}, ω_1), where $[X_1, \omega_1] = 0$; $\| - X_1\| \leq b/2$ and (60) is the case. Observe that we have $\| - X_1\| \leq e_1/200$ which means $\|X_1\| \leq e_{N-1}/200$.

We know that the microsupport of $X^- *_G X^+ = Rm_!(X^- \boxtimes X^+)$ is contained in the set of all points of the form $(g_1 g_2, \omega)$ where $g_1, g_2 \in G$; $(g_1, \omega) \in \mathrm{SS}(X^-)$; $(g_2, \mathrm{Ad}^*_{g_1}\omega) \in \mathrm{SS}(X^+)$. This means that $\mathrm{SS}(X^- *_G X^+)$ consists of all points of the form

$$(e^X e^{X_1}, \omega),$$

where $(e^X, \omega) \in \mathrm{SS}(X^-)$ and $(e^{X_1}, \omega) \in \mathrm{SS}(X^+)$ (because $[X, \omega] = [X_1, \omega] = 0$). According to Lemma 10.4, $e^X e^{X_1} = e^Y$, where $\|Y\| \leq \|X\| + \|X_1\| \leq (e_1 + e_{N-1})/200$. It follows that $e^Y V_k(\omega) \subset V_k(\omega)$ and

$$\det e^Y|_{V_k(\omega)} = e^{i<b/2-b/2,\|\omega\|>} = 1,$$

see (59), (60). As $\|Y\| \leq b$, this implies $\mathrm{Tr}Y|_{V_k(\omega)} = 0$. This in turn implies that $< Y, \omega >= 0$, which we wanted.

Let $c := (e_1 + e_{N-1})/200$. Let $W := \{X \in g; \|X\| << 2c\}$. The exponential map gives rise to an open embedding $\exp : W \to G$. The object $X^- *_G X^+$ is supported within $\exp(W)$. Consider $E \in D(W)$; $E := \exp^{-1}(X^- *_G) X^+)$. It suffices to show that $E \cong \mathbb{K}_0$.

We see that E is microsupported within the set (X, ω), where $\|X\| \leq c$, $[\omega, X] = 0$, $(\omega, X) = 0$. Let D be the dilation vector field on $\mathfrak{g}$. That is D is a section of $T\mathfrak{g} = \mathfrak{g} \times \mathfrak{g}$; $D : \mathfrak{g} \to \mathfrak{g} \times \mathfrak{g}$; $D(X) = (X, X)$. It then follows that every point $(x, \omega) \in \mathrm{SS}(E)$ satisfies $i_{D_x}(\omega) = 0$. Let $X \in \mathfrak{g}$; $X \neq 0$. Let $R_X := (\mathbb{R}_{>0}.X) \cap W$ be an open segment. It then follows that $E|_{R_X}$ is a constant sheaf. However, R_X does necessarily contain points $Y \in R_X$ such that $\|Y\| >> c$, meaning that $E|_Y = 0$, and $E|_{R_X} = 0$. Hence E is supported at 0 and it suffices to show that $E|_0 \cong \mathbb{K}$.

We have

$$E|_0 = (X^- *_G X^+)|_e = R\Gamma_c(G; \mathbb{K}_{\{e^X|\|X\|<<b/2\}} \otimes \mathbb{K}_{\{e^X|\|X\|\leq b/2\}})[\dim G]$$

$$= R\Gamma_c(G; \mathbb{K}_{\{e^X|\|X\|<<b/2\}})[\dim G] = \mathbb{K},$$

because the open set $\{e^X|\|X\| << b/2\}$ is diffeomorphic to an open ball. □

6.1.8

Let $T : C^\circ_- \to C^\circ_-$ be the shift by $-b/2$. $T(l) = l - b/2$.

Let us restrict the isomorphism (58) onto $G \times V_b^- \times (-b/2)$. We will get an isomorphism

$$\Phi^- *_G X^- \cong A^{-1}\Phi^-|_{G\times V_b^- \times(-b/2)}$$

$$= T^{-1}\Phi^-.$$

Taking the convolution with X^+ and using the previous Lemma, we will get an isomorphism

$$\Phi^- \cong (T^{-1}\Phi) *_G X^+. \tag{61}$$

Let $\mathbf{T}_M : (C_-^\circ + Mb/2) \to C_-^\circ$ be the shift by $-Mb/2$. Set

$$\Psi_M := \mathbf{T}_M^{-1}\Phi *_G (X^+)^{*_G^M},$$

$$\Psi_M \in D(G \times (C_-^\circ + Mb/2)).$$

We have an isomorphism

$$i_M : \Psi_M|_{C_-^\circ+(M-1)b/2} \cong \mathbf{T}_{M-1}^{-1}[(T^{-1}\Phi^- *_G X^+) *_G (X^+)^{*_G^{M-1}}]$$

$$\cong \mathbf{T}_{M-1}^{-1}((\Phi^- *_G (X^+)^{*_G^{M-1}}) = \Psi_{M-1}$$

where on the last step we have used the isomorphism (61). Similar to above, there exists an object $\mathfrak{S} \in D(G \times \mathfrak{h})$ and isomorphisms $\mathfrak{S}|_{C_-+Mb/2} \to \Psi_M$ which are compatible with isomorphisms i_M. Lemma 6.2 readily implies that $SS(\mathfrak{S}) \subset \Omega_{\mathfrak{S}}$. Let us compute $\mathfrak{S}|_{G\times 0}$. We have $0 \in C_-^\circ - b/2$. Therefore, we have an isomorphism

$$\mathfrak{S}|_{G\times 0} \cong \Psi_1^-|_{G\times 0} \cong \Phi^-|_{G\times -b/2} *_G X^+ \cong X^- *_G X^+ \cong \mathbb{K}_e.$$

This proves that the object $\mathfrak{S}$ satisfies all the conditions of Theorem 6.1.

6.1.9 Uniqueness

Theorem 6.7 *Let $\mathfrak{S}_1, \mathfrak{S}_2$ satisfy the conditions of Theorem 6.1. Then $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are canonically isomorphic.*

Proof According to Lemma 6.2 (take $U_1 = U_2 = \mathfrak{h}$), we have an isomorphism

$$a^{-1}Ra_*(\mathfrak{S}_1 *_G \mathfrak{S}_2) \to \mathfrak{S}_1 *_G \mathfrak{S}_2. \tag{62}$$

Let $I_1, I_2 : \mathfrak{h} \to \mathfrak{h} \times \mathfrak{h}$ be as follows: $I_1(A) = (A, 0)$; $I_2(A) = (0, A)$. Applying functors I_1^{-1}, I_2^{-1} to (62) and taking into account the isomorphisms $I_0^{-1}\mathfrak{S}_i \cong \mathbb{K}_{e_G}$, we will get the following isomorphisms

$$Ra_*(\mathfrak{S}_1 *_G \mathfrak{S}_2) \to \mathfrak{S}_2; \quad Ra_*(\mathfrak{S}_1 *_G \mathfrak{S}_2) \to \mathfrak{S}_1, \tag{63}$$

whence an isomorphism $\mathfrak{S}_1 \to \mathfrak{S}_2$. □

From now on we will denote by $\mathfrak{S}$ any object satisfying Theorem 6.1 (they are all canonically isomorphic).

Equations (62), (63) imply that we have an isomorphism

$$\mathfrak{S} *_G \mathfrak{S} \to a^{-1}\mathfrak{S}. \tag{64}$$

6.1.10

One can prove even more general result. Let $\Upsilon \subset T^*(G \times \mathfrak{h})$ consist of all points $(G, A, \omega, \|\omega\|) \in G \times \mathfrak{h} \times \mathfrak{g}^{\times}\mathfrak{h} = T^*(G \times \mathfrak{h})$. Of course, $\Omega_{\mathfrak{S}} \subset \Upsilon$. Let $C_\Upsilon \subset D(G \times \mathfrak{h})$ be the full subcategory consisting of all objects F microsupported on Υ. Let $i_0 : G \to G \times \mathfrak{h}$ be the embedding $i_0(g) = (g, 0)$. We have a functor $i_0^{-1} : C_\Upsilon \to D(G)$. We also have a functor $\Sigma : D(G) \to C_\Upsilon$; $\Sigma(F) = F *_G \mathfrak{S}$ (it is easy to show that $\mathrm{SS}(F *_G \mathfrak{S}) \subset \Upsilon$).

Theorem 6.8 *The functors i_0^{-1} anb Σ are mutually quasi-inverse equivalences.*

Proof Let $F \in C_\Upsilon$ and consider $F *_G \mathfrak{S} \in D(G \times \mathfrak{h} \times \mathfrak{h})$. As above, let $a : G \times \mathfrak{h} \times \mathfrak{h} \to G \times \mathfrak{h}$ be the addition. Similar to above, one can show that the natural map

$$a^{-1}Ra_*(F *_G \mathfrak{S}) \to F *_G \mathfrak{S} \tag{65}$$

is an isomorphism. Let $i_1, i_2 : G \times \mathfrak{h} \to G \times \mathfrak{h} \times \mathfrak{h}$ be given by $i_1(g, A) = (g, A, 0)$; $i_2(g, A) = (g, 0, A)$. In the same spirit as above, we can apply i_1^{-1}, i_2^{-1} to (65). We will get functorial isomorphisms

$$Ra_*(F *_G \mathfrak{S}) \to F = \mathrm{Id}(F), \quad Ra_*(F *_G \mathfrak{S}) \to i_0^{-1}F *_G \mathfrak{S} = \Sigma i_0^{-1}F,$$

whence an isomorphism of functors $\mathrm{Id}_{D(G)} \cong \Sigma i_0^{-1}$.

Let us consider the composition in the opposite order:

$$i_0^{-1}\Sigma(F) = F *_G (\mathfrak{S}|_{G\times 0}) = F *_G \mathbb{K}_{e_G} = F.$$

This way we get an isomorphism $\mathrm{Id}_{D(G\times\mathfrak{h})} \cong i_0^{-1}\Sigma$. □

6.1.11 Lemma

These Lemma will be used in the sequel. Let $A \in \mathfrak{h}$. Let $I_A : G \to G \times \mathfrak{h}$ be given by $I_A(g) = (g, A)$. Let $S_A := I_A^{-1}\mathfrak{S}$. Let $T_A : \mathfrak{h} \to \mathfrak{h}$; $T_A(A_1) = A + A_1$ be the shift by A.

Lemma 6.9 *We have an isomorphism $T_A^{-1}\mathfrak{S} \cong S_A *_G \mathfrak{S}$.*

Proof Apply the functor I_A^{-1} to (62). □

7 Study of $\mathfrak{S}|_{\mathbf{Z}\times C_-^\circ}$

Let $\mathbf{Z} \subset \mathrm{SU}(N)$ be the center consisting of all matrices of the form $\zeta.I$, where ζ is an Nth root of unit.

We denote by $j_{C_-^\circ} : C_-^\circ \to \mathfrak{h}$ the open embedding. We will denote by the same symbol the induced embeddings $\mathbf{Z} \times C_-^\circ \to \mathbf{Z} \times \mathfrak{h}$; $G \times C_-^\circ \to G \times \mathfrak{h}$.

We start with studying the object $j_{C_-^\circ}^{-1}\mathfrak{S}$.

7.1 Microsupport of $j_{C_-^\circ}^{-1}\mathfrak{S}$

One has the following improvement on Theorem 6.1.

Lemma 7.1 *The object $j_{C_-^\circ}^{-1}\mathfrak{S}$ is microsupported within the set of points of the form $(g, A, \omega, \eta) \in G \times C_-^\circ \times \mathfrak{g}^* \times \mathfrak{h}^*$ such that there exists an $X \in \mathfrak{g}$ satisfying:*

(1) $g = e^X$;
(2) $\|X\| \leq -A$;
(3) $[X, \omega] = 0$;
(4) $TrX|_{V_k(\omega)} = -i < A, e_{d_k} >$;
(5) $\eta = \|\omega\|$.

Proof As was shown in the proof of Theorem 6.1, we have $C_-^\circ = \bigcup_M MV_b$ and

$$\mathfrak{S}|_{G\times MV_b} \cong \Phi_M^-.$$

The object Φ_M^- is defined by

$$a_M^{-1}\Phi_M^- \cong F^- *_G F^- *_G \cdots *_G F^-$$

(total M copies of F^- and we use the same notation as in Sect. 6.1.)

The object $F^- *_G F^- *_G \cdots *_G F^-$ (M times) is the same as

$$Rm_!(F^-)^{\boxtimes M},$$

where $m : (G \times V_b^-)^M \to G \times (V_b^-)^M$ is induced by the product on G. The map m is proper, so we can estimate the microsupport of $Rm_!(F^-)^{\boxtimes M}$ in the standard way. Using Lemma 6.3, we conclude that $Rm_!(F^-)^{\boxtimes M}$ is microsupported within the set of points of the form

$$(e^{X_1}e^{X_2}\cdots e^{X_M}; A_1, A_2, \ldots, A_M; \omega, \eta_1, \eta_2, \ldots, \eta_M),$$

where $(X_k, A_k, \omega, \eta_k) \in \mathrm{SS}(F^-)$ (we use the the equality $[X_k, \omega] = 0$).

By Lemma 6.3, $\eta_1 = \eta_2 = \cdots = \eta_M = \|\omega\|$. This implies that the object Φ_M^- is microsupported within the set of all points of the form

$$(e^{X_1}e^{X_2}\cdots e^{X_M}; A_1 + A_2 + \cdots + A_M; \omega, \|\omega\|),$$

where $(e^{X_k}; A_k; \omega; \|\omega\|) \in SS(F^-)$ for all k.

Lemma 6.3 says that $[X_k, \omega] = 0$. By Lemma 10.5, there exists $X \in \mathfrak{g}$ such that

$$e^X = e^{X_1}e^{X_2}\cdots e^{X_M}; \quad [X, \omega] = 0;$$

$$\mathrm{Tr}X|_{V_k(\omega)} = \sum_k \mathrm{Tr}X_k|_{V_k(\omega)};$$

$$\|X\| \leq \|X_1\| + \|X_2\| + \cdots + \|X_M\|$$

According to Lemma 6.3,

$$\sum_k \mathrm{Tr}X_k|_{V_k(\omega)} = -i < \sum_k A_k; e_{d_k} >;$$

$$\sum_k \|X_k\| \leq -\sum_k A_k.$$

This implies the statement. □

Let $\mathcal{S} := \mathfrak{S}|_{\mathbf{Z}\times\mathfrak{h}^*}$. Let us estimate the microsupport of $j_{C_-^\circ}^{-1}\mathcal{S}$ using the above Lemma.

Proposition 7.2 *The object $j_{C_-^\circ}^{-1}\mathcal{S}$ is microsupported within the set of all points of the form $(z; A; \eta) \subset \mathbf{Z} \times C_-^\circ \times \mathfrak{h}^*$, where there exists $B \in C_-$ such that*

(1) $e^{-B} = z$;

(2) $B \geq A$ (i.e. $\forall k :< B - A, e_k >\geq 0$);

(3) $\eta \in C_+$ (i.e. $\forall k :< \eta, f_k >\geq 0$); if $< \eta, f_k > > 0$, then $< B - A, e_k >= 0$.

Proof Let $I : \mathbf{Z} \times C_-^\circ \hookrightarrow G \times C_-^\circ$ be the embedding. We have $j_{C_-^\circ}^{-1}\mathcal{S} = I^{-1}j_{C_-^\circ}^{-1}\mathfrak{S}[-\dim G]$. The just proven Lemma implies that $j_{C_-^\circ}^{-1}\mathfrak{S}$ is non-singular with respect to the embedding I (i.e. given a point $\zeta \in SS(j_{C_-^\circ}^{-1}\mathfrak{S})$ where $\zeta \in T_x^*(G \times C_-^\circ)$, $x \in \mathbf{Z} \times C_-^\circ$, and $I^*\zeta = 0$, one then has $\zeta = 0$).

Therefore, the microsupport $I^{-1}j_{C_-^\circ}^{-1}\mathfrak{S}[-\dim G]$ consists of all points of the form $I^*\zeta$, where $\zeta \in SS(j_{C_-^\circ}^{-1}\mathfrak{S})$, $\zeta \in T_x^*(G \times C_-^\circ)$, $x \in \mathbf{Z} \times C_-^\circ$.

Thus the microsupport $I^{-1}j_{C_-^\circ}^{-1}\mathfrak{S}$ is contained in the set of all points of the form

$$(e^X, A, \eta) \in \mathbf{Z} \times C_-^\circ \times \mathfrak{h}^*,$$

where there exists $\omega \in \mathfrak{g}^*$ such that $(e^X, A, \omega, \eta) \in \mathrm{SS} j_{C_-^\circ}^{-1}\mathfrak{S}$. According to the previous Lemma, this implies that $\|X\| \leq -A$; $\|\omega\| = \eta$ (so $\eta \in C_+$).

This means that $\eta = i(\lambda^1(\omega), \lambda^2(\omega), \ldots, \lambda^N(\omega))$, where $\lambda^1(\omega) \geq \lambda^2(\omega) \geq \cdots \geq \lambda^N(\omega)$ is the spectrum of $-i\omega$ (with multiplicities).

It is clear that the flag $V_\bullet(\omega)$ contains a k-dimensional subspace iff $\lambda^k(\omega) > \lambda^{k+1}(\omega)$ which is the same as $< \eta, f_k > > 0$.

Denote this k-dimensional subspace by V^k. We then know that $XV^k \subset V^k$ and $\mathrm{Tr} X|_{V^k} = -i < A, e_k >$. On the other hand we know that

$$\mathrm{Tr} - iX|_{V^k} \leq < \|X\|, e_k >,$$

for any $X \in \mathfrak{g}$. Hence,

$$< -A, e_k > \leq < \|X\|, e_k > .$$

As $\|X\| \leq -A$, this means that $< -A, e_k > = < \|X\|, e_k >$.

Let us now set $B := -\|X\|$. We see that thus defined B satisfies all the conditions. □

7.1.1

Let us reformulate the just proven Proposition.

Let $\Lambda \subset C_-$ be a discrete subset.

Let $X(\Lambda) \subset C_-^\circ \times C_+ \subset T^*C_-^\circ$ consist of all points (A, η) such that there exists a $B \in \Lambda$ satisfying:

(1) $B \geq A$;

(2) If $< \eta, f_k > > 0$, then $< B - A, e_k > = 0$.

For $z \in \mathbf{Z}$ let $S_z \in D(C_-^\circ)$ be the restriction

$$S_z := j_{C_-^\circ}^{-1}(\mathcal{S}|_{z \times C_-^\circ}) = \mathfrak{S}|_{z \times C_-^\circ}.$$

Let $\mathbb{L}_z^- := \{B \in C_- | e^{-B} = z\}$. $\mathbb{L}_z^-$ is an intersection of a discrete lattice in $\mathfrak{h}$ with C_-, hence is itself discrete.

Proposition 7.2 can be now reformulated as:

Proposition 7.3 *We have $SS(\mathcal{S}_z) \subset X(\mathbb{L}_z^-)$.*

7.2 *Sheaves with Microsupport of the Form $X(\Lambda)$*

Fix a discrete subset $\Lambda \subset C_-$. One can number elements of Λ in such a way that $\Lambda = \{m_1, m_2, \ldots, m_n, \ldots\}$ and m_n is a maximum of $\Lambda \backslash \{m_1, m_2, \ldots, m_{n-1}\}$ with respect to the partial order on C_-.

For $x \in C_-$ we set $U_x^- \subset C_-^\circ$ to consist of all $y \in C_-^\circ$ such that $y << x$.

Proposition 7.4 *Let $F \in D(C_-^\circ)$ be microsupported on the set $X(\Lambda)$. Then there exists an inductive system of objects in $D(C_-^\circ)$:*

$$F = F_0 \to F_1 \to F_2 \to \cdots F_n \to \cdots,$$

such that (1) $L \varinjlim_n F_n = 0$;

(2) We have isomorphisms

$$M_n \otimes_{\mathbb{K}} \mathbb{K}_{U_{mn}^-} \to Cone(F_{n-1} \to F_n),$$

for certain graded vector spaces M_n.

7.2.1 Lemma

Lemma 7.5 *Let $U \subset V \subset \mathbb{R}^n$ be open convex sets. Let $\gamma \subset \mathbb{R}^n$ be an open proper cone. Let $\gamma^\circ \subset \mathbb{R}^n$ be the dual closed cone; $\gamma^\circ = \{v| < \gamma, v > \geq 0\}$. Suppose that $V \subset U - \gamma$. Let $F \in D(V)$ be such that $SS(F) \subset V \times \gamma^\circ$. Then the restriction map $R\Gamma(V, F) \to R\Gamma(U, F)$ is an isomorphism*

Proof Let $X \subset U \times V$ to consists of all pairs $(u, v) \in U \times V$ such that $v - u \in -\gamma$.

Let $\phi : X \times (0, 1) \to V$; $F(u, v) = (1 - t)u + tv$. We see that ϕ is a smooth fibration with contractible fiber of dimension $n + 1$. Therefore, the object $\phi^{-1}F$ is microsupported on the set of those 1-forms which are ϕ-pullbacks of 1-forms in the microsupport of F. Let $E := \mathbb{R}^n$. Identify $T^*V = V \times E^*$;

$$T^*(X \times (0, 1)) = X \times (0, 1) \times E^* \times E^* \times \mathbb{R}.$$

We then have $\mathrm{SS}(F) \subset V \times \gamma^\circ$;

$$\mathrm{SS}(\phi^{-1}F) \subset \{(u, v, t, (1 - t)\eta; t\eta; < v - u, \eta >)\},$$

where $\eta \in \gamma^\circ$.

Here we have used the formula

$$< \eta, d((1 - t)u + tv) = (1 - t) < \eta, du > +t < \eta, dv > + < \eta, (v - u) > dt.$$

As $v - u \in -\gamma, \eta \in \gamma^\circ$, we see that

$$\mathrm{SS}(\phi^{-1}F) \subset \{(u, v, t, \eta_1, \eta_2, k)|k \leq 0\}.$$

Let $S \subset X \times (0, 1)$ be any open subset such that for any $(u, v) \in X$, the set of all $t \in (0, 1)$ such that $(u, v, t) \in S$ is of the form $(0, T(u, v))$ for some $T(u, v) > 0$. It then follows that the restriction map

$$R\Gamma(X, \phi^{-1}F) \to R\Gamma(S; \phi^{-1}F)$$

is an isomorphism.

Let now $S := \phi^{-1}U$. It is easy to see that all the conditions are satisfied. It also follows that the projection $\phi_U : S \to U$ induced by ϕ is a smooth fibration with contractible fiber.

We have a commutative diagram

$$\begin{array}{ccc} R\Gamma(V, F) & \longrightarrow & R\Gamma(U, F) \\ \downarrow & & \downarrow \\ R\Gamma(X \times (0, 1); \phi^{-1}F) & \longrightarrow & R\Gamma(S; \phi^{-1}F) \end{array} \tag{66}$$

Coming from the Cartesian square

$$\begin{array}{ccc} S & \longrightarrow & X \times (0, 1) \\ \downarrow & & \downarrow \\ U & \longrightarrow & V \end{array} .$$

As the fibrations $S \to U$ and $X \times (0, 1) \to V$ have contractible fibers, the vertical arrows in (66) are isomorphisms. So is the low horizontal arrow. Hence the upper vertical arrow is also an isomorphism. □

7.2.2

Lemma 7.6 *We have*

$$R\operatorname{hom}(\mathbb{K}_{U_x^-}; \mathbb{K}_{U_y^-}) \cong \mathbb{K}$$

if $x \leq y$. *Othewise* $R\operatorname{hom}(\mathbb{K}_{U_x^-}; \mathbb{K}_{U_y^-}) = 0$.

Proof If $x \leq y$, we have an isomorphism

$$R\operatorname{hom}(\mathbb{K}_{U_x^-}; \mathbb{K}_{U_y^-}) = R\operatorname{hom}(\mathbb{K}_{U_x^-}; \mathbb{K}_{U_x^-}) = \mathbb{K}$$

because U_x^- is a convex hence contractible set.

If it is not true that $x \leq y$, then x does not belong to the closure of U_y^- and there exists a convex neighborhood W of x in $\mathfrak{h}$ such that W still does not intersect the closure of U_y^-. Let $V := U_x^- \cap W$. According to the previous Lemma, we have an isomorphism

$$R\operatorname{hom}(\mathbb{K}_{U_x^-}; \mathbb{K}_{U_y^-}) \to R\operatorname{hom}(\mathbb{K}_V; \mathbb{K}_{U_y^-}) = 0.$$

Indeed, $\mathbb{K}_{U_y^-}$ is microsupported within the set $C_-^\circ \times C_+$. The dual cone to C_+ is $\gamma := \{x | x \geq 0\}$ and $U_x^- = V - \gamma$. □

7.2.3 Lemma

Let E_1, E be real finitely-dimensional vector spaces and let $U \subset E_1 \times E$ be an open convex set. Let $\gamma \subset E^*$ be a closed proper cone such that γ is the closure of its interior $\operatorname{Int}\gamma$. Let $\delta \subset E$ be the dual closed cone. Let $x, y \in E$, $y - x \in \operatorname{Int}\delta$. Let $V \subset E_1$ be an open subset such that $V \times ((x + \delta) \cap (y - \operatorname{Int}\delta)) \subset U$. Let $H := V \times ((x + \delta) \cap (y - \operatorname{Int}\delta))$.

Let us identify $T^*U = U \times E_1^* \times E^*$. Let $F \in D(U)$ be such that $\mathrm{SS}(F) \subset U \times E_1^* \times \gamma$.

Lemma 7.7 *We have* $Rhom(\mathbb{K}_H; F) = 0$.

Proof Choose vectors $e \in \operatorname{Int}\gamma$ and $f \in \operatorname{Int}\delta$. We have $< e, f > > 0$. Let $E' := \operatorname{Ker} e$. We have $E = \mathbb{R}.f \oplus E'$; $E^* = \mathbb{R}.e \oplus (E')^*$. Let $\varepsilon > 0$. Let $T_\varepsilon : E \to E$ be given by $T_\varepsilon|_{E'} = \mathrm{Id}$; $T_\varepsilon(f) = \varepsilon f$. Let $\delta_\varepsilon := T_\varepsilon \delta$.

There exists a sequence of points $y_n \in E_2$, $\varepsilon_n \in (0, 1)$ such that

$$(x + \delta) \cap (y_n - \delta_{\varepsilon_n}) \subset (x + \delta) \cap (y_m - \operatorname{Int}\delta_{\varepsilon_m})$$

for all $n < m$ and

$$\bigcup_n (x + \delta) \cap (y_n - \operatorname{Int}\delta_{\varepsilon_n}) = (x + \delta) \cap (y - \operatorname{Int}\delta).$$

We then have

$$\mathbb{K}_{(x+\delta)\cap(y-\operatorname{Int}\delta)} = \varinjlim_n \mathbb{K}_{(x+\delta)\cap(y_n-\operatorname{Int}\delta_{\varepsilon_n})}.$$

Therefore, it suffices to show that

$$R\hom(\mathbb{K}_{V\times((x+\delta)\cap(y_n-\operatorname{Int}\delta_{\varepsilon_n}))}; F) = 0.$$

More precisely, given $z \in E$, $\varepsilon \in (0, 1)$, and any open $W_1 \subset V$ such that the closure of W_1 is contained in V and $(x + \delta) \cap (z - \delta_\varepsilon) \subset (x + \delta) \cap (y - \operatorname{Int}\delta)$, we will show

$$R\hom(\mathbb{K}_{W_1\times(x+\delta\cap z-\operatorname{Int}\delta_\varepsilon)}; F) = 0$$

It follows that there exists an open convex $W_2 \subset E$ such that $W_1 \times W_2 \subset U$ and

$$(x + \delta) \cap (z - \delta_\varepsilon) \subset W_2.$$

Indeed, let $\overline{V}$ be the closure of V. Then $\overline{V} \times ((x + \delta) \cap (z - \delta_\varepsilon)) \subset U$. As both sets in this product are compact and U is open, there exists a neighborhood W_2 of $(x + \delta) \cap (z - \delta_\varepsilon)$ such that $\overline{V} \times W_2 \subset U$.

There exists $z' \in (z + \operatorname{Int}\delta_\varepsilon) \cap W_2$ such that $(x + \delta) \cap (z' - \delta_\varepsilon) \subset W_2$ Let $Z := (z' - \operatorname{Int}\delta_\varepsilon) \cap W_2$ so that $z \in Z$ and for any $u \in Z$, $(x + \delta) \cap (u - \operatorname{Int}\delta_\varepsilon) \subset W_2$ (because $(u - \operatorname{Int}\delta_\varepsilon) \subset (z' - \operatorname{Int}\delta_\varepsilon)$).

Let $G \subset W_2 \times Z$ be the following locally closed subset:

$$G = \{(w, u) | w \in x + \delta \cap u - \mathrm{Int}\, \delta_\varepsilon\}.$$

Let $p : W_1 \times W_2 \times Z \to W_1 \times W_2$ and $q : W_1 \times W_2 \times Z \to W_1 \times Z$.
Let $\Phi := F|_{W_1 \times W_2}$.
We will show $Rq_* \underline{\mathrm{Hom}}(\mathbb{K}_{W_1 \times G}; p^! \Phi) = 0$. by computing microsupports.
Let us first study $\mathrm{SS}(\mathbb{K}_G)$, where $\mathbb{K}_G \in D(W_2 \times Z)$.
We have

$$\mathbb{K}_G = \mathbb{K}_{G_1} \otimes \mathbb{K}_{G_2},$$

where $G_1, G_2 \subset W_2 \times Z$, $G_1 = (x + \delta \cap W_2) \times Z$; $G_2 = \{(w, u)|w - u \in -\mathrm{Int}\, \delta_\varepsilon\}$.

We have $\mathrm{SS}(\mathbb{K}_{G_1})$ is contained within the set of all points $(w, u, \eta, 0) \in W_2 \times Z \times E_2^* \times E_2^*$, where $\eta \in \gamma$.

Similarly, $\mathrm{SS}(\mathbb{K}_{G_2})$ is contained within the set of all points $(w, u, \zeta, -\zeta)$, where $\zeta \in \gamma_{1/\varepsilon}$ ($\gamma_{1/\varepsilon} := T_{1/ve}\gamma$ is the dual cone to δ_ε).

Therefore, $\mathbb{K}_G$ is microsupported within the set of all points of the form

$$(w, u, \eta + \zeta, -\zeta),$$

where w, u, η, ζ are as above.

Hence $\mathrm{SS}(\mathbb{K}_{W_1 \times G})$ is contained within the set of all points of the form

$$(w_1, w_2, u, 0, \eta + \zeta, -\zeta) \in W_1 \times W_2 \times Z \times E_1^* \times E_2^* \times E_2^*.$$

The object $p^! \Phi$ is microsupported within the set of all points of the form

$$(w_1, w_2, u, \alpha, \kappa, 0) \in W_1 \times W_2 \times Z \times E_1^* \times E_2^* \times E_2^*,$$

where $\alpha \in E_1^*$ is arbitrary and $\kappa \notin \mathrm{Int}\, \gamma$.

It follows that $\underline{\mathrm{Hom}}(\mathbb{K}_{W_1 \times G}; p^! \Phi)$ is microsupported within the set of all points of the form

$$(w_1, w_2, u, \alpha, \kappa - \eta - \zeta; \zeta),$$

where η, ζ, κ are same as before.

The map q is proper on the support of $\underline{\mathrm{Hom}}(\mathbb{K}_{W_1 \times G}; p^! \Phi)$, because the latter is contained within the set

$$W_1 \times ((x + \delta) \cap (z' - \delta_\varepsilon)) \times Z,$$

and $(x + \delta) \cap (z' - \delta_\varepsilon) \subset W_2$ is compact. Therefore, $Rq_* \underline{\mathrm{Hom}}(\mathbb{K}_{W_1 \times G}; p^! \Phi)$ is contained within the set of all points of the form

$$(w, u, \alpha, \zeta) \in W_1 \times Z \times E_1^* \times E_2^*,$$

where α is arbitrary, $\zeta \in \gamma_{1/\varepsilon}$, and there exist κ, η as above, such that $\kappa - \eta - \zeta = 0$. The latter is only possible if $\zeta = 0$ (otherwise $\zeta + \eta \in \operatorname{Int}\gamma$ because

$$\gamma_{1/\varepsilon} \subset \{0\} \cup \operatorname{Int}\gamma.$$

Thus, $Rq_*\underline{\mathrm{Hom}}(\mathbb{K}_{W_1\times G}; p^!\Phi)$ is microsupported within the set of all points of the form $(w, u, \alpha, 0)$, i.e. is locally constant along Z. There exists a convex open subset $U_0 \subset Z$, $U_0 \subset x - \delta$. It follows that $G \cap (W_2 \times U_0) = \emptyset$. Therefore,

$$Rq_*\underline{\mathrm{Hom}}(\mathbb{K}_{W_1\times G}; p^!\Phi)|_{W_1\times U_0} = 0.$$

This implies that

$$Rq_*\underline{\mathrm{Hom}}(\mathbb{K}_{W_1\times G}; p^!\Phi) = 0,$$

because Z is convex, and our object is locally constant along Z.

Therefore,

$$0 = R\hom(\mathbb{K}_{W_1\times z}; Rq_*\underline{\mathrm{Hom}}(\mathbb{K}_{W_1\times G}; p^!\Phi)$$

$$= R\hom(\mathbb{K}_{W_1\times G} \otimes \mathbb{K}_{W_1\times W_2\times z}; p^!\Phi)$$

$$= R\hom(\mathbb{K}_{W_1\times(x+\delta\cap z-\operatorname{Int}\delta_\varepsilon)\times z}; p^!\Phi)$$

$$= R\hom(\mathbb{K}_{W_1\times(x+\delta\cap z-\operatorname{Int}\delta_\varepsilon)}; \Phi),$$

as was required. □

7.2.4 Lemma

Lemma 7.8 *Let $x, y \in C_-$, $y > x$. Let $I_x := \{k| < x, f_k >< 0\}$. There exists $k \in I_x$ such that $< y - x, e_k >> 0$.*

Proof Assume the contrary, i.e. $< y - x, e_k >= 0$ for all $k \in I_x$. Let $z = y - x$ and let $z_k =< z, e_k >$ so that $z_k = 0$ for all $k \in I_x$. If $l \notin I_x$, then $z_l \geq 0$ and $< z, f_l >=< y, f_l >\leq 0$. On the other hand, $< z, f_l >= 2z_l - z_{l-1} - z_{l+1}$ (we set $z_0 = z_N = 0$). For $l \notin I_x$ let $[a, b] \subset [1, N - 1]$ be the largest interval containing l and not intersecting with I_x. We then have $z_{A-1} = z_{B+1} = 0$;

$$0 \geq -z_A \geq z_A - z_{A+1} \geq z_{A+1} - z_{A+2} \geq \cdots \geq z_B \geq 0$$

(because for any $l \notin I_x, 2z_l - z_{l-1} - z_{l+1} \leq 0$). This implies that $z_A = z_{A+1} = \cdots = z_B = 0$. Hence, $z_l = 0$ for all l, $z = 0$, and $y = x$, which contradicts to $y > x$. □

7.2.5 Lemma

Lemma 7.9 *Let $F \in D(C_-^\circ)$ be such that $SS(F) \subset X(\Lambda)$ and assume that for all $l \in \Lambda$, $R\Gamma(U_l^-; F) = 0$. Then $F = 0$.*

Proof Consider open subsets of C_-° of the form $U \cap U_x^-$ where U is open and convex and $x \in C_-$. These sets form a base of topology of C_-°. Thus, it suffices to show $R\Gamma(U \cap U_x^-; F) = 0$ for all such U, U_x^-. By Lemma 7.5, we have an isomorphism

$$R\Gamma(U_x^-; F) \to R\Gamma(U \cap U_x^-; F).$$

Thus, it suffices to show that $R\Gamma(U_x^-; F) = 0$ for all x.

Given $x \in C_-$, let $\Lambda_x := \{l \in \Lambda | l \geq x\}$. Let $N_x = |\Lambda_x|$. Let us prove the statement by induction with respect to N_x.

If $N_x = 0$, then there are no points in $X(\Lambda)$ which project to x. Hence $x \notin \mathrm{Supp} F$. Therefore, there exists a convex neighborhood of U of x such that $F|_U = 0$. Therefore, we have an isomorphism

$$R\Gamma(U_x^-; F) \tilde{\to} R\Gamma(U \cap U_x^-; F) = 0.$$

Suppose now that $R\Gamma(U_x^-; F)$ for all x with $N_x < n$. Prove that the same is true for all x with $N_x \leq n$. Let $S \subset C_-$ be the set of all points y such that $\Lambda_y = \Lambda_x$. Let $t_k := \sup_{y \in S} < y, e_k >$. As $S \in C_-$, $t_k \geq 0$. Let

$$x' := \sum_{k=1}^{N-1} t_k f_k.$$

Let us show $x' \in C_-$. This is equivalent to $< x'; f_l > \leq 0$ for all l. We have $< x', f_l >= 2 < x', e_l > - < x', e_{l-1} > - < x'; e_{l+1} >$. We have

$$2 < x', e_l >= \sup_{y \in S} 2 < y, e_l > \leq \sup_{y \in S} < y, e_{l-1} > + < y, e_{l+1} >$$

$$\leq < x', e_{l-1} > + < x', e_{l+1} > .$$

Thus, $x' \in C_-$. It then easily follows that $x' \in S$.

It is clear that $x' \geq x$. Let us show that the restriction map $R\Gamma(U_{x'}^-; F) \to R\Gamma(U_x; F)$ is an isomorphism. If $x' = x$, there is nothing to prove, so assume $x' > x$. Let $I := \{tx + (1-t)x' | 0 \leq t < 1\}$. Let $K := \{k| < x' - x, e_k >> 0\}$. Ler U' be a convex neighborhood of 0 in $\mathfrak{h}$. Let $U := C_-^\circ \cap U'$. We then see that

(1) $I + U \subset C_-^\circ$ is convex and open;

(2) For U' small enough the following is true. Given any $y \in I + U$, we have $\Lambda_y = \Lambda_x$; for any $l \in \Lambda_y$ and for any $k \in K$, $< l - y, e_k >> 0$.

The restriction maps

$$R\Gamma(U_{x'}^-; F) \to R\Gamma(I + U; F);$$

$$R\Gamma(U_x^-; F) \to R\Gamma(x + U; F)$$

are isomorphisms by Lemma 7.5. Hence it suffices to show that the restriction map

$$R\Gamma(I + U; F) \to R\Gamma(x + U; F) \quad (67)$$

is an isomorphism.

It follows from the definition of $X(\Lambda)$ that $F|_{I+U}$ is microsupported within the set of all points $(y, \eta) \in (I + U) \times \mathfrak{h}^*$ such that $< \eta, f_k >= 0$ for all $k \in K$. Hence, $< \eta, x' - x >= 0$. This implies that that (67) is an isomorphism.

We can now assume $x = x'$. By the construction of $x = x'$, given any point $y \in C_-$, $y > x$, the set Λ_y is a proper subset of Λ_x. If $x \in \Lambda$ there is nothing to prove. Assume $x \notin \Lambda$. Let $I_x := \{k| < x, f_k >< 0\}$. By Lemma 7.8 for any $l \in \Lambda_x$ there exists $k \in I_x$ such that $< l - x, e_k >> 0$. It follows that there exists a neighborhood U' of x in C_- such that for all $y \in U'$, $\Lambda_y \subset \Lambda_x$ and for all $l \in \Lambda_y$ there exists $k \in I_x$ such that $< l - y, e_k >> 0$. Let $U = U' \cap C_-^\circ$. It follows that $F|_U$ is microsupported within the set of all points of the form

$$(u, \eta) \in U \times \mathfrak{h}^*,$$

where $< \eta, f_k >= 0$ for some $k \in I_x$.

Let $\mathcal{V} \subset \mathfrak{h}$ be the $\mathbb{R}$-span of all f_k, $k \in K$.

It follows that there exists $\varepsilon > 0$ such that $x + \sum_{k\in I_x} t_k f_k \in U'$ if for all $k \in I_x$, $t_k \in [0, \varepsilon]$. Indeed, let $U' = W \cap C_-$, where W is a neighborhood of x in $\mathfrak{h}$. It is clear that for ε small enough, $x + \sum_{k\in I_x} t_k f_k \in W$. As $< x, f_k >< 0$ for all $k \in I_x$, for all ε small enough and for all $k' \in I_x$ we have: $< x + \sum_{k\in I_x} t_k f_k, f_{k'} >< 0$. If $\lambda \notin I_x$, then $< x + \sum_{k\in I_x} t_k f_k, f_\lambda >= \sum_{k\in I_x} t_k < f_k, f_\lambda >\leq 0$, because $< f_k, f_\lambda >\leq 0$ for all $k \neq \lambda$. Thus,

$$x + \sum_{k\in I_x} t_k f_k \in C_-.$$

Fix $\varepsilon > 0$ as above. There also exists $\varepsilon_1 > 0$ such that

$$x + \sum_{k\in I_x} t_k f_k + \sum_{\lambda=1}^{N-1} a_\lambda e_\lambda \in C_-^\circ$$

as long as $t_k \in [0, \varepsilon]$ and $0 > a_\lambda > -\varepsilon_1$.

Let $U_{\varepsilon_1} := \{x + \sum a_\lambda e_\lambda | 0 > a_\lambda > -\varepsilon_1\} \subset \mathfrak{h}$;

$$M_\varepsilon := \{\sum_{k\in I_x} t_k f_k | 0 < t_k < \varepsilon\} \subset \mathcal{V}.$$

Let $A : \mathfrak{h} \times \mathcal{V} \to \mathfrak{h}$ be the addition map. There exists an open convex neighborhood $\mathcal{U} \in \mathfrak{h} \times \mathcal{V}$ of $U_{\varepsilon_1} \times M_\varepsilon$ such that $A(\mathcal{U}) \subset U$. Let $\alpha : \mathcal{U} \to A(\mathcal{U}) \subset U$ be the map induced by A. As $\mathcal{U}$ is convex, $\alpha : \mathcal{U} \to A(\mathcal{U})$ is a smooth fibration. Let $\Phi := \alpha^!(F|_{A(\mathcal{U})})$. It follows that $\mathrm{SS}(\Phi)$ consists of pull-backs of 1-forms from $\mathrm{SS}(F)$. Thus, $\mathrm{SS}(\Phi)$ is contained in the set of all points of the form

$$(A, u, \eta, \kappa) \in \mathfrak{h} \times \mathcal{V} \times \mathfrak{h}^* \times \mathcal{V}^*,$$

where $(A, u) \in \mathcal{U}$ and there exists $k \in I_x$ such that $< \kappa, f_k >= 0$. By Lemma 7.7, we have

$$R\hom(\mathbb{K}_{U_{\varepsilon_1} \times G}; \Phi) = 0,$$

where $G = \{\sum_{k \in K} t_k f_k | 0 \leq t_k < \varepsilon\}$.

For $L \subset I_x$, let $G_L := \{\sum_{l \in L} t_l f_l | 0 < t_l < \varepsilon\}$. Set $G_\emptyset := \{0\}$. We have a natural map

$$\mathbb{K}_{U_{\varepsilon_1} \times G} \to \mathbb{K}_{U_{\varepsilon_1} \times G_\emptyset}.$$

The cone of this map is obtained from sheaves $\mathbb{K}_{U_{\varepsilon_1} \times G_L}$, $L \neq \emptyset$, by means of successive extensions.

We also have

$$R\hom(\mathbb{K}_{U_{\varepsilon_1} \times G_L}; \Phi) = R\Gamma(A(U_{\varepsilon_1} \times G_L); \Phi).$$

We have

$$A(U_{\varepsilon_1} \times G_L) \subset U^-_{x + \sum_{l \in L} \varepsilon f_l}.$$

By Lemma 7.5 the restriction map

$$R\Gamma(U^-_{x + \sum_{l \in L} \varepsilon f_l}; F) \to R\Gamma(A(U_{\varepsilon_1}, G_L); F)$$

is an isomorphism. As $x + \sum_{l \in L} \varepsilon f_l > x$ for $L \neq \emptyset$, we have

$$R\Gamma(U^-_{x + \sum_{l \in L} \varepsilon f_l}; F) = 0$$

and

$$R\hom(\mathbb{K}_{U_{\varepsilon_1} \times G_L}; \Phi) = 0$$

for all $L \neq \emptyset$. Therefore

$$R\hom(\mathrm{Cone}(\mathbb{K}_{U_{\varepsilon_1} \times G} \to \mathbb{K}_{U_{\varepsilon_1} \times G_\emptyset}); \Phi) = 0.$$

Therefore

$$0 = R\hom(\mathbb{K}_{U_{\varepsilon_1} \times G_\emptyset}; \Phi) = R\hom(U^-_x; F)$$

7.2.6 Proof of Proposition 7.4

□

Let us construct objects $F_n \in D(C_-^\circ)$ by induction. Set $F_0 = F$. Set $M_n := R\Gamma(U_{m_n}^-; F_{n-1})$ and

$$F_n := \mathrm{Cone}(\alpha_n : M_n \otimes \mathbb{K}_{U_{m_n}^-} \to F_{n-1}),$$

where α_n is the natural map.

We have structure maps $i_n : F_{n-1} \to F_n$ so that the sheaves F_n form an inductive system. This system stabilizes on any compact $K \subset C_-^\circ$ because for n large enough, $K \cap U_{m_n}^- = \emptyset$.

Let $G := L\varinjlim_n F_n$. It follows that $\mathrm{SS}(G) \subset X(\Lambda)$ (because $\mathrm{SS}(F_n) \subset X(\Lambda)$).

Let U_n be a neighborhood of m_n in C_-° such that the closure of U_n in C_-° is compact. We have

$$R\Gamma(U_{m_n}^-; G) \cong R\Gamma(U_{m_n}^- \cap U_n; G)$$

$$\cong R\Gamma(U_{m_n}^- \cap U_n; F_N) \cong R\Gamma(U_{m_n}^-; F_N)$$

for N large enough.

Let $S^i := R\Gamma(U_{m_n}^-; F_i)$ As follows from Lemma 7.6, $S^i = S^{i+1}$ for all $i \geq n$; also, by construction, $S^n = 0$. Thus $S^N = 0$ for $N \geq n$. Therefore,

$$R\Gamma(U_{m_n}^-; G) = 0$$

for all n and $G = 0$ by Lemma 7.9.

Next, $\mathrm{Cone}(F_n \to F_{n-1})$ is isomorphic to $M_n \otimes \mathbb{K}_{U_{m_n}^-}$. This proves the proposition.

7.3 Invariant Definition of the Spaces M_n

The goal of this section is to define spaces M_n from Proposition 7.4 in a more invariant way.

7.3.1 Lemma

As in the previous Lemma, let $x \in C_-$ and let $I_x := \{k| < x, f_k >< 0\}$. As was shown in the previous Lemma, there exists $\varepsilon > 0$ such that $x + \sum_{k \in I_x} t_k f_k \in C_-$ as long as all $t_k \in [0, \varepsilon]$. Fix such a $\varepsilon > 0$.

Set

$$V := V(x, \varepsilon) := \{y \in C_-^\circ | \forall k \in I_x :< y - x, e_k > \in [0, \varepsilon); \forall l \notin I_x :< y - x, e_l >< 0.\}$$

Lemma 7.10 *(1) We have*

$$R\,\hom(\mathbb{K}_V; \mathbb{K}_{U_x^-}) \cong \mathbb{K}[-|I_x|].$$

(2) Let $y \in C_-$. Suppose there exists $k \in \{1, 2, \ldots, N-1\}$ such that either $k \in I_x$ and $< y - x, e_k > \notin [0, \varepsilon]$ or $k \notin I_x$ and $< y, e_k > < < x, e_k >$. Then

$$R\,\hom(\mathbb{K}_V; \mathbb{K}_{U_y^-}) = 0.$$

Proof For $L \subset I_x$ set

$$f_L := \varepsilon \sum_{l \in L} f_l.$$

For every $k \in I_x$ we have a natural map

$$\mathbb{K}_{U^-_{x+f_{I_x - \{k\}}}} \to \mathbb{K}_{U^-_{x+f_{I_x}}}.$$

Let C_k be the corresponding 2-term complex, we put $\mathbb{K}_{U^-_{x+f_{I_x}}}$ into degree 0.

Consider the complex

$$D := \bigotimes_{k \in I_x} C_k$$

We have

$$D^{-i} = \bigoplus_L \mathbb{K}_{U^-_{x+f_L}},$$

where the sum is taken over all $|I_x| - i$-element subsets L of I_x.

In particular $D^0 = \mathbb{K}_{U^-_{x+f_{I_x}}}$. As $V \subset U^-_{x+f_{I_x}}$ is a closed subset, we have a natural map

$$\mathbb{K}_{U^-_{x+f_{I_x}}} \to \mathbb{K}_V.$$

This map defines a map of complexes $D \to \mathbb{K}_V$ which is a quasi-isomrorphism. Therefore, we have an isomorphism

$$R\,\hom(\mathbb{K}_V; \mathbb{K}_{U_y^-}) \to R\,\hom(D; \mathbb{K}_{U_y^-}).$$

Let $y = x$, then, according to Lemma 7.6, $R\,\hom(\mathbb{K}_{U^-_{x+f_L}}; \mathbb{K}_{U_x^-}) = 0$ for all $L \neq \emptyset$. For $L = \emptyset$, we have

$$R\,\hom(\mathbb{K}_{U_x^-}; \mathbb{K}_{U_x^-}) = \mathbb{K}.$$

Therefore, we have an isomorphism

$$R\,\hom(D, \mathbb{K}_{U_x^-}) \cong \mathbb{K}[-|I_x|].$$

Let now $y \in C_-$ and $k \in I_x$ be such that $< y - x, e_k > \notin [0, \varepsilon]$.

Let $D_k := \bigotimes_{l \neq k} C_l$ so that we have $D = D_k \otimes C_k$. I.e

$$D \cong \mathrm{Cone}(D_k \otimes \mathbb{K}_{U^-_{x+f_{I_x-\{k\}}}} \to D_k \otimes \mathbb{K}_{U^-_{x+f_{I_x}}}), \tag{68}$$

where the map is induced by the natural map

$$\mathbb{K}_{U^-_{x+f_{I_x-\{k\}}}} \to \mathbb{K}_{U^-_{x+f_{I_x}}}.$$

We have

$$D_k^{-i} \otimes \mathbb{K}_{U^-_{x+f_{I_x-\{k\}}}} = \bigoplus_L \mathbb{K}_{U^-_{x+f_L}},$$

where the sum is taken over all $|I_x| - i - 1$-element subsets $L \subset I_x - \{k\}$.

Analogously,

$$D_k^{-i} \otimes \mathbb{K}_{U^-_{x+f_{I_x}}} = \bigoplus_L \mathbb{K}_{U^-_{x+f_L}}$$

where the sum is taken over all $|I_x| - i$-element subsets $L \subset I_x$ such that $k \in L$.

In view of these identifications, the $-i$th degree component of the map in (68) is induced by the natural maps

$$\mathbb{K}_{U_{x+f_L}} \to \mathbb{K}_{U_{x+f_{L\cup\{k\}}}}.$$

If $< y - x, e_k > \notin [0, \varepsilon]$, then these maps induce isomorphism

$$R\,\mathrm{hom}(\mathbb{K}_{U_{x+f_{L\cup\{k\}}}}; \mathbb{K}_{U_y^-}) \to R\,\mathrm{hom}(\mathbb{K}_{U_{x+f_L}}; \mathbb{K}_{U_y^-})$$

Hence, the map in (68) induces an isomorphism

$$R\,\mathrm{hom}(D_k \otimes \mathbb{K}_{U^-_{x+f_{I_x}}}; \mathbb{K}_{U_y^-}) \to R\,\mathrm{hom}(D_k \otimes \mathbb{K}_{U^-_{x+f_{I_x-\{k\}}}}; \mathbb{K}_{U_y^-})$$

Therefore,

$$R\,\mathrm{hom}(D, \mathbb{K}_{U_y^-}) = 0,$$

as was stated.

If there exists $k \notin I_x$ such that $< y, e_k > << x, e_k >$, then it follows that $R\,\mathrm{hom}(\mathbb{K}_{U^-_{x+f_L}}; \mathbb{K}_y) = 0$ for all L (because it is not true that $x + f_L \leq y$). □

7.3.2

Lemma 7.11 *Let $l \in \Lambda$. There exists $\varepsilon > 0$ such that for any $l' \in \Lambda$, $l' \neq l$:*

– *either there exists* $k \in I_l$ *such that* $< l' - l, e_k > \notin [0, \varepsilon]$
– *or there exists* $k \notin I_l$ *such that* $< l', e_k > << l, e_k >$.

Proof If there exists $k \in \{1, 2, \ldots, N-1\}$ such that $< l' - l, e_k >< 0$, then one of the conditions is satisfied. If such a k does not exist, then $l' \geq l$. There are only finitely many $l' \in \Lambda$ with this property. Hence, the statement follows from Lemma 7.8. □

7.3.3

Let m_n be a numbering of Λ as in Proposition 7.4. Let ε be as in the proof of the previous Lemma.

Lemma 7.12 *Let* $\varepsilon' \in (0, \varepsilon)$. *We have*

$$M_n \cong R\hom(\mathbb{K}_{V(m_n,\varepsilon')}; F)[|I_{m_n}|]$$

Proof Follows from Proposition 7.4 and two previous Lemmas. □

7.4 The Sheaf $\mathcal{S}_z$

Proposition 7.4 and Lemma 7.12 applies to $j_{C_-^\circ}^{-1}\mathcal{S}_z$ with $\Lambda = \mathbb{L}_z^-$. We would like to rewrite the expression from Lemma 7.12 in a more convenient way.

Let $x \in \mathfrak{h}$ and $I \subset \{1, 2, \ldots, N-1\}$. let $W(I, x) \subset \mathfrak{h}$ be given by

$$W(I, x, \varepsilon) = \{y : \forall k \in I :< y - x, e_k > \in [0, \varepsilon); \forall k \notin I :< y - x, e_k >< 0\}.$$

For $x \in C_-$ and ε as in Sect. 7.3.1, we have

$$V := V(x, \varepsilon) = W(I_x, x, \varepsilon) \cap C_-^\circ,$$

Set $W := W(I_x, x, \varepsilon)$. Set $I := I_x$.

For any $F \in D(\mathfrak{h})$ we have an induced map of sheaves

$$R\hom_{\mathfrak{h}}(\mathbb{K}_W; F) \to R\hom_{\mathfrak{h}}(\mathbb{K}_V; F) = R\hom_{C_-^\circ}(\mathbb{K}_V; j_{C_-^\circ}^{-1}F). \quad (69)$$

Lemma 7.13 *Suppose that* $SS(F) \subset \mathfrak{h} \times C_+$. *Then the map (69) is an isomorphism*

Proof For $z \in \mathfrak{h}$ set $U_z = \{y \in \mathfrak{h} | y << z\}$. Lemma 7.5 implies that for any $z \in C_-$, the restriction map

$$R\hom(\mathbb{K}_{U_z}; F) \to R\hom(\mathbb{K}_{U_z^-}; F)$$

is an isomorphism.

For $k \in I$ consider the following 2-term complex C'_k

$$\mathbb{K}_{U_{x+f_{I-\{k\}}}} \to \mathbb{K}_{U_{x+f_I}},$$

where we use the notation from proof of Lemma 7.10. Let

$$D' := \bigotimes_{k \in I} C'_k. \tag{70}$$

Similar to D, we have a quasi-isomorphism

$$D' \to \mathbb{K}_W.$$

We also have

$$(D')^{-i} = \bigoplus_L \mathbb{K}_{U_{x+f_L}},$$

where the sum is taken over all $|I| - i$-element subsets of I. We have natural maps $C_k \to C'_k$ which induce maps $D \to D'$. The latter map is induced by maps

$$\mathbb{K}_{U^-_{x+f_L}} \to \mathbb{K}_{U_{x+f_L}}$$

According to Lemma 7.5, the induced map

$$R\,\hom(\mathbb{K}_{U_{x+f_L}}; F) \to R\,\hom(\mathbb{K}_{U^-_{x+f_L}}; F)$$

is an isomorphism for all F such that $\mathrm{SS}(F) \subset \mathfrak{h} \times C_+$. This implies the statement. □

7.4.1

Lemma 7.14 *Let $F \in D(\mathfrak{h})$ be constant along fibers of projection $\mathfrak{h} \to \mathfrak{h}/\mathbb{R}.f_k$ for some k. Then for all $I \subset \{1, 2, \ldots, N-1\}$ such that $k \in I$ and for all $\varepsilon > 0$, we have*

$$R\,\hom(\mathbb{K}_{W(I,x,\varepsilon)}; F) = 0$$

Proof Follows easily from the quasi-isomorphism $D' \to \mathbb{K}_{W(I,x,\varepsilon)}$. □

7.5 Periodicity

Let us get back to the object $j_{C^\circ_-}^{-1}\mathcal{S}_z$. In this case $\Lambda = \mathbb{L}_z^-$. There exists $\varepsilon > 0$ such that the condition of Lemma 7.11 is satisfied for all $l \in \mathbb{L}_z^-$. Fix such a ε throughout. Proposition 7.4 applies to $F = \mathcal{S}_z$. by Lemma 7.12 and (69) we have an isomorphism

$$M_n = R\hom(\mathbb{K}_{V(m_n,\varepsilon)}; \mathcal{S}_z)[-|I_{m_n}|] = R\hom(\mathbb{K}_{W(I_{m_n},m_n,\varepsilon)}; \mathcal{S}_z)[-|I_{m_n}|].$$

For $z \in \mathfrak{h}$ and $I \subset \{1, 2, \ldots, N-1\}$ and $F \in D(\mathfrak{h})$

$$\Delta_{I;z}(F) := R\hom(\mathbb{K}_{W(I,z,\varepsilon)}; F)[|I|]$$

Our goal is to prove the following theorem

Theorem 7.15 *For any $m \in \mathfrak{h}$, any $I \subset \{1, 2, \ldots, N-1\}$ and any $k \in I$ there exists a quasi-isomorphism*

$$\Delta_{I;m}\mathcal{S}_z \to \Delta_{I;m-2\pi e_k}\mathcal{S}_{ze^{-2\pi e_k}}[-D_k]$$

where $D_k = 2k(N-k)$.

The rest of the current subsection will be devoted to proving this Theorem.

In the next two subsections we will prove the main auxiliary result towards the proof.

7.5.1 Sheaves $\mathfrak{S}|_{G\times -2\pi e_k}$

Recall that $\mathfrak{S} \in D(G \times \mathfrak{h})$. Let $\Sigma_k := \mathfrak{S}|_{G\times -2\pi e_k}$, so that $\Sigma_k \in D(G)$.

Lemma 7.16 *We have an isomorphism $\Sigma_k = \mathbb{K}_{W_k}$, where $W_k \subset G$ is an open subset consisting of all points of the form*

$$W_k = \{e^{-Y} | \|Y\| < 2\pi e_k\}.$$

Proof As follows from the proof of Theorem 6.1 Σ_k can be constructed as follows. Let us decompose $-2\pi e_k = A_1 + A_2 + \cdots A_M$, where $A_i \in V_b^-$. For $A \in C^\circ_-$ set $U(A) \subset G$; $U(A) := \{e^X | X \in \mathfrak{g}; \|X\| << -A\}$. One then has

$$\Sigma_k \cong \mathbb{K}_{U_{A_1}} *_G \mathbb{K}_{U_{A_2}} *_G \cdots *_G \mathbb{K}_{U_{A_M}}[M \dim \mathfrak{g}]$$

Let $g \in G$. It follows that $\Sigma_k|_g \neq 0$ only if there exist $X_k \in \mathfrak{g}$; $\|X_k\| << -A_k$ such that $g = e^{X_1}e^{X_2}\cdots e^{X_M}$. According to Lemma 10.4, this implies that $g = e^Y$, where $\|Y\| << -(A_1 + \cdots + A_M) = 2\pi e_k$. Thus, fibers of Σ_k at any point outside of W_k are zeros.

Let $H := \Sigma_k|_{W_k}$. It then suffices to prove that $H \cong \mathbb{K}[\dim \mathfrak{g}]$.

Let us find $SS(H)$. Observe that the exponential map identifies W_k with $\{X \in \mathfrak{g} | \|X\| << 2\pi e_k\}$. Lemma 7.1 implies that $(g, \omega) \in SS(H)$ only if there exists $X \in \mathfrak{g}$ such that $g = e^X$; $\|X\| \leq 2\pi e_k$, $[X, \omega] = 0$; $< \|X\| - 2\pi e_k, e_{d_r(\omega)} >= 0$ for all r. As $g \in U_{-2\pi e_k}$ and $\|X\| \leq 2\pi e_k$ we must have $\|X\| << 2\pi e_k$, so that $< \|X\| - 2\pi e_k, e_l >< 0$ for all l. This means that $\omega = 0$.

Thus, *H is a constant sheaf.*

Let us now find $H|_e$. We have $H|_e = \mathcal{S}_e|_{-2\pi e_k}$.

However, as follows from Proposition 7.2, $\mathcal{S}_e$ is constant in the domain consisting of all $A \in C_-$ such that there is no $l \in \mathbb{L}_0^-$, $l \neq 0$, $A \geq l$. Both $-2\pi e_k$ and $-e_1/100$ lie in this domain. Thus we have an isomorphism

$$\mathcal{S}_e|_{-2\pi e_k} = \mathcal{S}_e|_{-e_1/100} = \mathbb{K}[\dim \mathfrak{g}].$$

This finishes the proof. □

Let us compute $H_k := R\,\hom(\mathbb{K}_{e^{-2\pi e_k}}; \Sigma_k)$.

Let us choose a small neighborhood U of $e^{-2\pi e_k}$ in G so that $U = \{e^{-X} e^{-2\pi e_k} | \|X\| << b\}$. Let us describe the set $U_k := U \cap W_k$. Let $g \in U \cap W_k$. As $g \in W_k$, we have $g = e^{-Y}$ where $\|Y\| << 2\pi e_k$ which simply means that $\lambda_1(Y) < 2\pi(N-k)/N$; $\lambda_N(Y) > -2\pi k/N$, where

$$\lambda_1(Y) \geq \lambda_2(Y) \geq \cdots \geq \lambda_N(Y)$$

is the spectrum of a Hermitian matrix Y/i.

As $g \in U$, there must exist X, $\|X\| << b$ such that $e^{-Y} = e^{-X} e^{-2\pi e_k}$, or

$$e^Y = e^{2\pi e_k} e^X.$$

Observe that $e^{2\pi e_k} = e^{-2\pi k/N}\mathrm{Id}$. Therefore, one can number the spectrum of X/i in such a way that $\lambda^j(X) - 2\pi k/N - \lambda_j(Y) \in 2\pi\mathbb{Z}$, $j = 1 \ldots N$. In other words, there exist integers m_i such that $\lambda_j(Y) = -2\pi k/N + \lambda^j(X) + 2\pi m_i$, where m_i are integers.

As $-2\pi k/N < \lambda_j(Y) < 2\pi(N - k/N)$ and $\lambda^j(X)$ are small we see that $m_j = 0$ or $m_j = 1$. Since $\mathrm{Tr}(Y) = \mathrm{Tr}(X) = 0$, $\sum m_j = k$. Since $\lambda_1(Y) \geq \lambda_2(Y) \geq \cdots$, we conclude that $m_1 = \cdots = m_k = 1$; $m_{k+1} = m_{k+2} = \cdots m_N = 0$. We then see that

$$0 > \lambda^1(X) \geq \lambda^2(X) \geq \cdots \geq \lambda^k(X);$$

$$\lambda^{k+1}(X) \geq \cdots \geq \lambda^N(X) > 0.$$

In other words, the set W_k consists of all elements of the form $e^{-2\pi e_k} e^{-X}$ where$\|X\| << b$ and X/i has k negative eigenvalues and $N-k$ positive eigenvalues (and no 0 eigenvalues). Let $H_k \subset \mathfrak{g}$ be an open subset consisting of all matrices A such that A/i has k negative and $N-k$ positive eigenvalues. It now follows that

$$R\,\hom_G(\mathbb{K}_{e^{-2\pi e_k}};\Sigma_k)\cong R\,\hom_{\mathfrak{g}}(\mathbb{K}_0;\mathbb{K}_{H_k}[\dim\mathfrak{g}]).$$

Let $M^\circ\subset M\subset E\subset G(k,N)\times\mathfrak{g}$ be defined as follows:

$$E=\{(V,X)|XV\subset V\};$$

$$M=\{(V,X)|XV\subset V;X/i|_V\geq 0;X/i|_{V^\perp}\leq 0\};$$

$$M^\circ=\{(V,X)|XV\subset V;X/i|_V>0;X/i|_{V^\perp}<0\}.$$

It follows that $M\subset E\subset G(k,N)\times\mathfrak{g}$ are closed embeddings and that $M^\circ\subset M$ is an open embedding. The projection $\pi:E\to\mathfrak{g}$ is proper. The natural projection $p_E:E\to G(k,N)$ is a complex unitary bundle; $E=S\otimes\overline{S}\oplus S^\perp\otimes\overline{S^\perp}$, where S is the k-dimensional tautological bundle over $G(k,N)$.

Let $j:M^\circ\to E$ be the open inclusion. Then $k_{H_k}=R\pi_!j_!\mathbb{K}_{M^\circ}=R\pi_*j_!\mathbb{K}_{M^\circ}$. Therefore,

$$R\,\hom_{\mathfrak{g}}(\mathbb{K}_0;\mathbb{K}_{H_k}[\dim\mathfrak{g}])=R\,\hom_{\mathfrak{g}}(\mathbb{K}_0;R\pi_*j_!\mathbb{K}_{M^\circ}[\dim\mathfrak{g}])$$

$$=R\,\hom_M(\pi^{-1}\mathbb{K}_0;j_!\mathbb{K}_{M^\circ}[\dim\mathfrak{g}])$$

Let $i:G(k,N)\to E;i(V)=(V,0)$ be the zero section. We then have

$$R\,\hom_{\mathfrak{g}}(\mathbb{K}_0;\mathbb{K}_{H_k}[\dim\mathfrak{g}])=R\,\hom_M(i_*\mathbb{K}_{G(k,N)};j_!\mathbb{K}_{M^\circ}[\dim\mathfrak{g}]).$$

It is easy to see that the natural map

$$R\,\hom_M(i_*\mathbb{K}_{G(k,N)};j_!\mathbb{K}_{M^\circ}[\dim\mathfrak{g}])\to R\,\hom_M(i_*\mathbb{K}_{G(k,N)};\mathbb{K}_E[\dim\mathfrak{g}])=R\Gamma(G(k,N);i^!\mathbb{K}_E)[\dim\mathfrak{g}]$$

is a quasi-isomorphism. We have a natural isomorphism $i^!\mathbb{K}_E\cong\mathrm{or}_E[-\dim_{\mathbb{R}}E]$ where or_E is the sheaf of orientations on E which is canonically trivial on every complex bundle. Thus $i^!k_E[\dim\mathfrak{g}]=\mathbb{K}_{G(k,N)}[-\dim E+\dim\mathfrak{g}]=\mathbb{K}_{G(k,N)}[\dim G(k,N)]\cong D_{G(k,N)}$, where $D_{G(k,N)}$ is the dualizing sheaf on $G(k,N)$. Finally we have $R\Gamma(G(k,N);D)\cong H_*(G(k,N);k)$. Thus we have established

Proposition 7.17 *There is a natural isomorphism*

$$R^{-\bullet}\hom(\mathbb{K}_{e^{-2\pi e_k}};\Sigma_k)\cong H_\bullet G(k,N).$$

7.5.2

Let $D_k:=\dim_{\mathbb{R}}G(k,N)=2k(N-k)$. Let $\beta\in H_{D_k}(G(k,N))$ be the fundamental class.

According to the previous Proposition, the element β defines a map $B_k : \mathbb{K}_{e^{-2\pi e_k}} \to \Sigma_k[-D_k]$ in $D(G)$. Let $C_k := \mathrm{Cone} B_k$.

Proposition 7.18 *The singular support of the sheaf C_k is confined within the set*

$$\{(g,\omega)| < |\omega|, f_k >= 0\}$$

Proof First, consider the case $g \neq e^{-2\pi e_k}$.

Then $(g,\omega) \in \mathrm{SS}(C_k)$ iff $(g,\omega) \in \mathrm{SS}(\Sigma_k)$. The sheaf Σ_k is microsupported within the set

$$(e^X, \omega),$$

where $\| - X\| \leq 2\pi e_k$ and if $< \omega, f_j > \neq 0$, then $< \| - X\|, e_j >= 2\pi < e_k, e_j >$ for all j.

Therefore, it suffices to show that $< \| - X\|/2\pi, e_k > < < e_k, e_k >$. Assume the contrary and let $\eta := \| - X\|/2\pi$. Let $\eta_l :=< \eta, e_l >$; $\varepsilon_l =< e_k, e_l >$. Set $\eta_0 = \eta_N = \varepsilon_0 = \varepsilon_N = 0$. We have $0 \leq < \eta, f_l >= 2\eta_l - \eta_{l-1} - \eta_{l+1}$. Therefore, $\eta_l - \eta_{l-1} \geq \eta_{l+1} - \eta_l$. These convexity inequalities imply

$$\eta_l \geq l/k\eta_k$$

for all $l \leq k$;

$$\eta_l \geq (N-l)/(N-k)\eta_k,$$

for all $l \geq k$.

If $\eta_k = \varepsilon_k$, these inequalities mean that $\eta_l \geq \varepsilon_l$ for all l. However, we know that $\eta \leq \varepsilon$. Hence, $\eta = \varepsilon$ and $\| - X\| = 2\pi e_k$, hence $e^X = e^{-2\pi e_k}$ which is a contradiction.

Thus, $< \| - X\|, e_k > < < 2\pi e_k, e_k >$, therefore, $< \eta, f_k >= 0$.

Let us now consider the case $g = e^{-2\pi e_k}$. It suffices to consider the restriction $\Sigma_k|_{U \cap W_k}$ as in the previous theorem. Let $V := \{X \in \mathfrak{g} || X| << b\}$ We then have an identification $I : V \to U$; $X \mapsto e^{-X} e^{-2\pi e_k}$. we know that $I^{-1}\Sigma_k \cong R\pi_* \mathbb{K}_{M^\circ}$ [dim $\mathfrak{g}$]$|_{V_k}$ an the map $\mathbb{K}_{e^{-2\pi e_k}} \to \Sigma_k[d_k]$ is induced by a certain map $\mathbb{K}_0 \to R\pi_* \mathbb{K}_{M^\circ}$ [dim $\mathfrak{g}$]. Namely, this map comes from the identification

$$\hom(\mathbb{K}_0; R\pi_*\mathbb{K}_{M^\circ}[\dim \mathfrak{g}]) \to \hom(\mathbb{K}_{G(k,N)}; \mathbb{K}_{M^\circ}[\dim \mathfrak{g}]) \to \hom(\mathbb{K}_{G(k,N)}; \mathbb{K}_E[\dim \mathfrak{g}])$$

$$= H_*(G(k,N)).$$

Note that the sheaves $\mathbb{K}_0$ and $R\pi_*\mathbb{K}_{M^\circ}$ are dilation invariant, so we may study their Fourier-Sato transforms. Let us find $(R\pi_*\mathbb{K}_{M^\circ})^\vee$. Let $E^* \cong E$ be the dual bundle over $G(k,N)$; let $M^* \subset E^*$ be the closed cone dual to the open convex cone $M^\circ \subset E$. Upon the identification $E^* = E$ by means of the scalar product, we identify M^* with the set of all pairs $(X, V) \in \mathfrak{g} \times G(k,N)$ such that $XV = V$ and the smallest eigenvalue of $X/i|_V$ is greater or equal to the largest eigenvalues of $X/i|_{V^\perp}$.

Let $P : \mathfrak{g} \times G(k, N) \to E^*$ be the map dual to $\pi : E \to \mathfrak{g}$. Let $p_{\mathfrak{g}} : \mathfrak{g} \times G(k, N) \to \mathfrak{g}$ be the projection. We then have

$$R\pi_* \mathbb{K}^{\vee}_{M^\circ} = p_{\mathfrak{g}!} P^{-1} \mathbb{K}_{M^*}[-\dim_{\mathbb{R}} E/G(k, N)]$$

We then see that

$$P^{-1}\mathbb{K}_{M^*} = \mathbb{K}_Z,$$

where $Z \subset \mathfrak{g} \times G(k, N)$;

$$Z = \{(X, V)|XV \subset V; \lambda_{\min} X|_V \geq \lambda_{\max} X|_{V^\perp}\}.$$

Thus, $R\pi_* \mathbb{K}_{M^\circ}[\dim \mathfrak{g}]^\vee = Rp_{\mathfrak{g}!}\mathbb{K}_Z[\dim \mathfrak{g} - \dim E/G(k, N)] = Rp_{\mathfrak{g}!}\mathbb{K}_Z[\dim G(k, N)]$. Next, $\mathbb{K}_0^\vee = \mathbb{K}_{\mathfrak{g}}$. The map B_k induces a map of Fourier-Sato transforms:

$$B^\vee : \mathbb{K}_{\mathfrak{g}} \to Rp_{\mathfrak{g}!}\mathbb{K}_Z$$

Let us specify this map. By the conjugacy (since $p_{\mathfrak{g}}$ is proper), one can instead specify a map

$$B^\vee_{\text{conj}} : p_{\mathfrak{g}}^{-1}\mathbb{K}_{\mathfrak{g}} = \mathbb{K}_{\mathfrak{g}\times G(k,N)} \to \mathbb{K}_Z.$$

One can show that this map is simply the natural map induced by the closed embedding $Z \subset \mathfrak{g} \times G(k, N)$.

Let us now consider an open set $U \subset \mathfrak{g}$ consisting of all $X \in \mathfrak{g}$ such that $\lambda_k(X) > \lambda_{k+1}(X)$. We then see that the projection $Z \times_{\mathfrak{g}} U \to U$ is a homeomorphism. Therefore, $\text{Cone}B^\vee|_U = 0$ that is $\text{Cone}B^\vee = (\text{Cone}B)^\vee$ is supported on the complement of U which is precisely the set of all $X \in \mathfrak{g}$ such that $< \|X\|, f_k >= 0$. This proves the statement. □

7.5.3

Let $l \in \mathfrak{h}$. Let $T_l : G \times \mathfrak{h} \to G \times \mathfrak{h}$ be the shift in l: $T_l(g, X) = (g, X + l)$. We know that $T_l^{-1}\mathfrak{S} = \mathfrak{S}|_{G\times l} *_G \mathfrak{S}$ (Lemma 6.9). Therefore, the maps B_k induce maps

$$B'_k : \mathbb{K}_{e^{-2\pi e_k}} *_G \mathfrak{S} \to \mathfrak{S}|_{G\times e^{-2\pi e_k}} *_G \mathfrak{S}[-D_k] = T^{-1}_{-2\pi e_k}\mathfrak{S}[-D_k], \tag{71}$$

where $D_k = \dim G(k, N)$. The previous Proposition implies that

Corollary 7.19 *$\text{Cone}B_{k'}$ is locally constant on the fibers of the projection $G \times \mathfrak{h} \to G \times \mathfrak{h}/f_k$.*

Proof We have

$$\text{Cone}B_{k'} \cong C_{k'} *_G \mathfrak{S}.$$

Using the previous Proposition as well as Theorem 6.1 one can easily show that 1-forms from $\mathrm{SS}(C_{k'} *_G \mathfrak{S})$ do vanish on the fibers of the projection $G \times \mathfrak{h} \to G \times \mathfrak{h}/f_k$. □

Let $z \in \mathbf{Z}$ and restrict (71) onto $ze^{-2\pi e_k} \in G$. We will get a map

$$B_k^g : \mathcal{S}_z \to T^{-1}_{-2\pi e_k} \mathcal{S}_{ze^{-2\pi e_k}}[-D_k].$$

It follows that $\mathrm{Cone} B_k^g$ is also constant along the fibers of the projection $\mathfrak{h} \to \mathfrak{h}/f_k$.

7.5.4

The map B_k^g induces a map

$$\Delta_{I,m}(\mathcal{S}_z) \to \Delta_{I,m} T^{-1}_{-2\pi e_k} \mathcal{S}_{ze^{-2\pi e_k}}[-D_k].$$

for all I and m. This is the same as a map

$$\Delta_{I,m}\mathcal{S}_z \to \Delta_{I,m-2\pi e_k} \mathcal{S}_{ze^{-2\pi e_k}}[-D_k]. \tag{72}$$

Proposition 7.20 *If $k \in I$, the above map is a quasi-isomorphism.*

Proof Follows from Lemma 7.14. □

Theorem 7.15 now follows directly from the previous Proposition.

7.5.5 Corollary from Theorem 7.15

Let $u \in \mathfrak{h}$, $u = 2\pi \sum x_i e_i$, set $D(u) := -\sum x_k D_k$.
We then see:

Corollary 7.21 *Let $z \in \mathbf{Z}$, $m \in \mathbb{L}_z \cap C_-$. Then there exists an isomorphism*

$$\Delta_{m,I_m}\mathcal{S}_z \cong \Delta_{0,I_m}\mathcal{S}_e[D(m)]. \tag{73}$$

Proof Follows directly from Theorem 7.15. □

7.6 Computing $\Delta_{0,I}\mathcal{S}_e$

Let $I := \{j_1 < j_2 < \cdots < j_r\}$. Let $\mathcal{FL}(I)$ be the partial flag manifold with dimensions of the subspaces being $j_1, j_2, \ldots, j_r$. We will show

Proposition 7.22

$$\Delta_{0,I}\mathcal{S}_e \cong H^\bullet(\mathcal{FL}(I)).$$

Proof Let b be as in Sect. 6. Let $Z \in C_-^\circ$; $-Z << b$. One can choose ε so small that $Z + \sum_k a_k f_k \in C_-^\circ$ if $0 \leq a_k \leq \varepsilon$.

For $A \in \mathfrak{h}$, set $S_A := \mathfrak{S}|_{G\times A}$. We have

$$\Delta_{0,I}\mathcal{S}_e = R\hom_{\mathfrak{h}}(\mathbb{K}_{W(I,0,\varepsilon)}; \mathcal{S}_e)[|I|]$$

For $\delta > 0$, let

$$W(I, 0, \varepsilon, \delta) \subset \mathfrak{h}$$

be the set of all points A such that for all $k \in I$, $< A, e_k >\in [0, \varepsilon)$; for all $k \notin I$, $-\delta << A, e_k >< 0$.

We have a natural map $\mathbb{K}_{W(I,0,\varepsilon,\delta)} \to \mathbb{K}_{W(I,0,\varepsilon)}$. Using the complex D' from 70 one can easily prove that for any object in $D(\mathfrak{h})$ whose microsupport is contained within $\mathfrak{h} \times C_+$, in particular, for $\mathcal{S}_e$, the natural map

$$R\hom_{\mathfrak{h}}(\mathbb{K}_{W(I,0,\varepsilon)}; \mathcal{S}_e) \to R\hom_{\mathfrak{h}}(\mathbb{K}_{W(I,0,\varepsilon,\delta)}; \mathcal{S}_e)$$

is an isomorphism.

One can choose ε, δ so small that $Z + W(I, 0, \varepsilon, \delta) \subset V_b \cap C_-^\circ$. Set $W := W(I, 0, \varepsilon, \delta)$.

By definition, we have:

$$R\hom(\mathbb{K}_W; \mathcal{S}_e)$$

$$= R\hom_{G\times\mathfrak{h}}(\mathbb{K}_e \boxtimes \mathbb{K}_W; \mathfrak{S}).$$

We have a endofunctors on $D(G \times \mathfrak{h})$: $E_\pm : F \mapsto S_{\pm Z} *_G F$. The composition

$$E_+E_-(F) = S_Z *_G S_{-Z} *_G F = S_{Z-Z} *_G F = S_0 *_G F = F$$

is isomorphic to the identity (we have use an isomorphism $S_{Z_1} *_G S_{Z_2} = S_{Z_1+Z_2}$ which follows directly from (64).) Thus, $E_+E_- \cong \mathrm{Id}$; likewise $E_-E_+ \cong \mathrm{Id}$, so $E_\pm$ are quasi-inverse autoequivalences of $D(G \times \mathfrak{h})$. Hence, we have

$$R\hom_{G\times\mathfrak{h}}(\mathbb{K}_e \boxtimes \mathbb{K}_W; \mathfrak{S}) = R\hom_{G\times\mathfrak{h}}(S_Z \boxtimes \mathbb{K}_W; T_Z^{-1}\mathfrak{S})$$

$$= R\hom_{G\times\mathfrak{h}}(S_Z \boxtimes \mathbb{K}_{Z+W}; \mathfrak{S}),$$

where the last equality follows from Lemma 6.9. As $Z + W \subset C_-^\circ \cap V_b$, we have:

$$R\hom(\mathbb{K}_W; \mathcal{S}_e) = R\hom_{G\times(C_-^\circ\cap V_b)}(S_Z \boxtimes \mathbb{K}_{Z+W}; \mathfrak{S}|_{G\times(C_-^\circ\cap V_b)})$$

Let $V := C_-^\circ \cap V_b$. As follows from the proof of Theorem 6.1, we have

$$\mathfrak{S}|_{G\times V} = \mathbb{K}_{\{(e^X,v)|\|X\|<<-v\}}[\dim \mathfrak{g}],$$

Analogously, $S_Z := \{e^X|\|X\| << -Z\}[\dim \mathfrak{g}]$.

Let $V_Z \subset \mathfrak{g}$, $V_Z := \{X|\|X\| << -Z\}$. Let $\Omega \subset \mathfrak{g}\times\mathfrak{h}$;

$$\Omega := \{(X, A)|\|X\| << -A\}.$$

We then have

$$\Delta_{I,0}\mathcal{S}_e = R\,\mathrm{hom}_{\mathfrak{g}\times\mathfrak{h}}(\mathbb{K}_{V_Z}\boxtimes\mathbb{K}_{Z+W};\mathbb{K}_\Omega)[\|I\|]$$

Let $\mathcal{O}$ be the closure of Ω in $\mathfrak{g}\times\mathfrak{h}$. As Ω is an open proper cone, we have

$$\mathbb{K}_\Omega = R\underline{\mathrm{Hom}}(\mathbb{K}_\mathcal{O};\mathbb{K}_{\mathfrak{g}\times\mathfrak{h}}).$$

Therefore,

$$\Delta_{I,0}\mathcal{S}_e = R\,\mathrm{hom}_{\mathfrak{g}\times\mathfrak{h}}((\mathbb{K}_{V_Z}\boxtimes\mathbb{K}_{Z+W})\otimes\mathbb{K}_\mathcal{O};\mathbb{K}_{\mathfrak{g}\times\mathfrak{h}})[|I|]$$

Let $A := (V_Z\times(Z+W))\cap\mathcal{O}$ so that

$$(\mathbb{K}_{V_Z}\boxtimes\mathbb{K}_{Z+W})\otimes\mathbb{K}_\mathcal{O} = \mathbb{K}_A.$$

Let $p:\mathfrak{g}\times\mathfrak{h}\to\mathfrak{g}$ be the projection. We have $\mathbb{K}_{\mathfrak{g}\times\mathfrak{h}} = p^!\mathbb{K}_\mathfrak{g}[-\dim\mathfrak{h}]$. Hence, by the conjugacy,

$$\Delta_{I,0}\mathcal{S}_e = R\,\mathrm{hom}_\mathfrak{g}(Rp_!\mathbb{K}_A;\mathbb{K}_\mathfrak{g})[-\dim\mathfrak{h}+|I|]. \tag{74}$$

Let $X\in\mathfrak{g}$ and consider

$$(Rp_!\mathbb{K}_A)|_X = R\Gamma_c(\mathfrak{h};\mathbb{K}_{A\cap X\times\mathfrak{h}}).$$

Let $X_k =<\|X\|, e_k>$; $Z_k =< Z, e_k >$. We see that $A\cap X\times\mathfrak{h}$ is non-empty only if $X\in V_Z$, i.e. $X_k+Z_k<0$ for all k. In this case we see that $A\cap X\times\mathfrak{h}$ consists of all points of the form $(X, Z+\sum_{k=1}^N t_k f_k)$, where $0\le t_k<\varepsilon$ for all $k\in I$; $-\delta<t_k<0$ for all $k\notin I$; $Z_k+t_k+X_k\le 0$ for all k. Let L be the set of all $k\in I$ such that $Z_k+X_k>-\varepsilon$. One then sees that these conditions are equivalent to

$$0\le t_k\le -X_k-Z_k$$

for all $k\in L$;

$$0\le t_k<\varepsilon$$

for all $k \in I\backslash L$;

$$-\delta < t_k < 0.$$

for all $k \notin I$.

It follows that $R\Gamma_c(\mathfrak{h}; \mathbb{K}_{A\cap X\times\mathfrak{h}}) = 0$ if $L \neq I$. Thus, the object $Rp_!\mathbb{K}_A$ is supported on an open subset $E_\varepsilon \subset \mathfrak{g}$ consisting of all points X such that $X_k + Z_k < 0$ for all $k \notin I$ and $-\varepsilon < X_k + Z_k < 0$ for all $k \in I$.

Let $F_\varepsilon := E_\varepsilon \times \mathfrak{h} \cap A$. It follows that the natural map $Rp_!\mathbb{K}_{F_\varepsilon} \to Rp_!\mathbb{K}_A$ is an isomorphism.

One also has a natural isomorphism

$$Rp_!\mathbb{K}_{F_\varepsilon} = \mathbb{K}_{E_\varepsilon}[|I| - N + 1] = \mathbb{K}_{E_\varepsilon}[|I| - \dim \mathfrak{h}].$$

We can substitute this into (74):

$$\Delta_{I,0}\mathcal{S}_e = R\hom_{\mathfrak{g}}(\mathbb{K}_{E_\varepsilon}; \mathbb{K}_{\mathfrak{g}})$$

which can be rewritten as

$$\Delta_{I,0}\mathcal{S}_e[\|I\|] = H^\bullet(E_\varepsilon),$$

because $E_\varepsilon \subset \mathfrak{g}$ is an open subset.

Lemma 7.23 *For $\varepsilon > 0$ small enough, we get:*

$$\forall Y \in E_\varepsilon; \forall i \notin I :< Y, f_i >< 0.$$

Proof We have $< Y, f_i > = < X + \sum t_j f_j, f_i > << X, f_i > + t_i < f_i, f_i >$, because $< f_i, f_j > \leq 0$ for all $i \neq j$. Next,

$$< X, f_i > + t_i < f_i, f_i > \leq < X, f_i > + 2\varepsilon < 0$$

for ε small enough. □

This implies that for any $X \in E_\varepsilon$ and for every $k \in I$, we have a well-defined k-dimensional eigenspace space $V^k(X)$ spanned by the eigenvectors of X/i with top k eigenvalues. The spaces $V^\bullet(X)$ form a flag from $\mathcal{FL}(I)$. Thus we have a map $P : E_\varepsilon \to \mathcal{FL}(I)$; $P(X) := V^\bullet(X)$.

Let $\mathcal{E} \to \mathcal{FL}(I)$ be the vector bundle whose fiber at $V^\bullet \in \mathcal{FL}(I)$ consists of all unitary matrices preserving $V^\bullet$. One can easily check that $E_\varepsilon \subset \mathcal{E}$ is an open convex subset. Therefore, P induces an isomorphism $H^\bullet(E_\varepsilon) = H^\bullet(\mathcal{FL}(I))$ so that

$$\Delta_{I,0}\mathcal{S}_e[|I|] \cong H^\bullet(\mathcal{FL}(I)).$$

□

7.6.1 The Sheaf $j_{C_-^\circ}^{-1}\mathcal{S}$, up to an Isomorphism

Let us combine Proposition 7.4, Corollary 7.21, and Proposition 7.22. We will then get the following statement:

Proposition 7.24 *Let $g \in \mathbf{Z}$. There exists an inductive system of sheaves on C_-:*

$$j_{C_-^\circ}^{-1}\mathcal{S}_g = F^0 \to F^1 \to \cdots \to F^n \to \cdots$$

such that

$$L\varinjlim_n F_n = 0;$$

$$Cone(F_{n-1} \to F_n) \cong \mathbb{K}_{U^-{}_{m_n}} \otimes H^*(\mathcal{FL}(I_{m_n}))[D(m_n)],$$

where the sequence $m_1, m_2, \ldots, m_n, \ldots$ consists of all elements of $\mathbb{L}_g \cap C_-$, each term occurring once.

It turns out that this Proposition allows us to recover the isomorpism type of $j_{C_-^\circ}^{-1}\mathcal{S}_g$.

Let $A_n := \mathbb{K}_{U^-{}_{m_n}} \otimes H^*(\mathcal{FL}(I^c_{m_n}))[D(m_n)]$.

Lemma 7.25 *There exist maps*

$$i_n : A_n \to F_0$$

such that for every n the triangle

$$\bigoplus_{n' \leq n} A_{n'} \to F_0 \to F_n \tag{75}$$

is exact.

Proof Let us prove the statement by induction in n.For $n = 1$ we have a natural map $i_1 : A_1 \to j_{C_-^\circ}^{-1}\mathcal{S}_g = F_0$ whose cone is F_1; this proves the base.

Let us now proceed to the induction step.

Suppose we have aready constructed an exact triangle as in (75) for some n. Let us apply to this triangle the functor $R\,\hom(A_{n+1}, \cdot)$.

We will then get an exact sequence

$$R^0\hom(A_{n+1}; j_{C_-^\circ}^{-1}\mathcal{S}_g) \to R^0\hom(A_{n+1}; F_n) \to \bigoplus_{n' \leq n} R^1\hom(A_{n+1}; A_{n'}). \tag{76}$$

Observe that the last arrow in this sequence is 0: because of Lemma 7.6 and because all the spaces M_i are concentrated in the even degrees, therefore, $R^{\text{odd}}\hom(A_i, A_j) = 0$ for all i, j.

Therefore, the left arrow in (76) is surjective. Next, we have a map $E_{n+1} : A_{n+1} = \mathrm{Cone}(F_n \to F_{n+1}) \to F_n$. Let $i_{n+1} : A_{n+1} \to \mathcal{S}_g$ be the lifting of E_{n+1} (which exists precisely because of surjectivity of the left arrow in (76). It is straightforward to see that so chosen i_{n+1} satisfies the conditions □

Theorem 7.26 *There exists an isomorphism*

$$\bigoplus_{l \in \mathbb{L}_g \cap C_-} A_l \to j_{C_-^\circ}^{-1} \mathcal{S}_g,$$

where $A_l := \mathbb{K}_{U^-_l} \otimes H^*(\mathcal{FL}(I_l))[D(l)]$.

Proof Indeed, the previous Lemma implies that the map $\bigoplus_n i_n : \bigoplus_n A_n \to j_{C_-}^{-1} \mathcal{S}_g$ is an isomorphism, whence the statement. □

8 *B*-Sheaves

For a manifold X let $\mathbf{Complexes}_X$ be the dg-category of complexes of sheaves on X.

Suppose X is equipped with an action of the monoid $\mathbb{L}_-$. Let $T_l : X \to X$ be the translation by $l \in \mathbb{L}_-$. In all our examples all T_l will be open embeddings.

Let $F \in \mathbf{Complexes}_X$ and $l \in \mathbb{L}_-$. Set $A(l) := A_F(l) := \mathrm{hom}(F, T_l^{-1} F)$. These complexes obviously form a $\mathbb{L}_-$-graded dg-algebra to be denoted by $A = A_F$.

Let B be another $\mathbb{L}_-$-graded dg-algebra. We define *a B-sheaf structure on F* as a $\mathbb{L}_-$-graded dg-algebra homomorphism $B \to A_F$. B-sheaves form a triangulated dg-category in the obvious way.

We will only use algebras B of a special type. Namely, We will assume that:

- $B(l)$ is concentrated in degrees $\leq -D(l)$;
- the cohomology $H^\bullet(B(l))$ is concentrated in degree $-D(l)$ and is one dimensional;
- one can choose generators $b_l \in H^{-D(l)}(B(l))$ which are stable under the product induced by the product on B.

Call such a B *homotopically standard.*

Let $\mathbf{b}$ be a $\mathbb{L}_-$-graded dg-algebra defined by setting $\mathbf{b}(l) = k[D(l)]$. Let $1_l := 1 \in k[D(l)]^{-D(l)}$ be generators.

We then define the product on $\mathbf{b}$ by setting $1_l 1_m = 1_{l+m}$. It follows that we have a unique $\mathbb{L}_-$-graded dg-algebra homomorphism $B \to \mathbf{b}$ such that the induced map $H^\bullet(B) \to H^\bullet(\mathbf{b}) = \mathbf{b}$ sends b_l to 1_l.

We call a B-sheaf F *acyclic* if it is acyclic as a complex of sheaves on X (i.e. for each $x \in X$ the complex of fibers F_x is acyclic).

Following [2] we can produce the derived dg-category by taking the quotient with respect to the full subcategory of acyclic objects.

However, in our situation one can prove

Proposition 8.1 *The category of B-sheaves has enough injective objects.*

Remark. By an injective object we mean any B-sheaf X such that for any acyclic B-sheaf Z, the complex $\hom(Z, X)$ is acyclic.

Proof Let A be a B-sheaf. Let β_A be another B-sheaf such that $\beta_A := \prod_{l \in \mathbb{L}_-} \hom(B(l); T_l^{-1} A)$ We then get a B-structure on β_A and a natural map of B-sheaves $A \to \beta_A$. Let now $A \to A'$ be a termwise injective map in the category of complexes of sheaves on X (we forget the B-structure) such that A' is injective. We then have a termwise injective map of B-sheaves

$$A \hookrightarrow \beta(A) \hookrightarrow \beta_{A'}$$

One sees that $\beta_{A'}$ is injective: given any B- sheaf T on X we have

$$\hom(T, \beta_{A'}) = \hom(T, A'),$$

where hom on the RHS is in the category of complexes of sheaves on X. As A' is injective, we see that $\hom(T, A') \sim 0$ as long as T is acyclic. □

As we know, in this case, the derived category is equivalent to the full subcategory of injective objects.

We will only need the homotopy category of the derived cateogory of B-sheaves. Denote this category by DBSh_X.

Let $f : X_1 \to X_2$ be a $\mathbb{L}_-$-equivariant map We then have a right derived functor of f_*: $Rf_* : \mathrm{DBSh}_{X_1} \to \mathrm{DBSh}_{X_2}$: if we choose the category of injective B-sheaves on X_1 as a model for DBSh_{X_2} then Rf_* is given by the termwise application of the functor f_*. Similarly, one defines functors $Rf_!$, f^{-1}. One can also define a functor $f^!$ as a right adjoint to $Rf_!$, but we won't need this functor.

Recall that we have a natural map $p : B \to \mathbf{b}$. This map induces an obvious functor p^{-1} from the category of $\mathbf{b}$-sheaves to the category of B-sheaves on X and one sees that this map has a right adjoint p_*. This functor admits a right derived $\pi := Rp_* : \mathrm{DBSh}_X \to D\mathbf{b}\mathrm{Sh}_X$. This functor is an equivalence.

8.0.1 A B-Sheaf Structure on the Sheaves $\mathfrak{S}$ and $\mathcal{S}$

Let $\mathfrak{S} \in D(G \times \mathfrak{h})$ be as in Theorem 6.1. Choose an injective representative for $\mathfrak{S}$, to be denoted by the same symbol $\mathfrak{S}$. Define a diagonal $\mathbb{L}_-$-action on $G \times \mathfrak{h}$ by setting $l.(g, A) := (e^l g, A + l)$. For $l \in \mathbb{L}_-$ consider the complex $B'(l) := \hom_{G\times\mathfrak{h}}(\mathfrak{S}; T_l^{-1}\mathfrak{S})$ and compute its cohomology:

$$H^\bullet(B'(l)) = R^\bullet \hom(\mathfrak{S}; T_l^{-1}\mathfrak{S}).$$

Let $i_0 : G \to G \times \mathfrak{h}$, $i_0(g) = (g, 0)$. By Theorem (6.8) we have

$$R^\bullet \hom(\mathfrak{S}; T_l^{-1}\mathfrak{S}) = R^\bullet \hom_G(i_0^{-1}\mathfrak{S}; i_0^{-1}T_l^{-1}\mathfrak{S}).$$

We know that $i_0^{-1}\mathfrak{S} = \mathbb{K}_{e_G}$. Thus,

$$R^\bullet \hom_G(\mathfrak{S}; T_l^{-1}\mathfrak{S}) = R^\bullet \hom_G(\mathbb{K}_e; i_0^{-1}T_l^{-1}\mathfrak{S}).$$

As $T_l^{-1}\mathfrak{S}$ is non-singular along $i_0(G) \subset G \times \mathfrak{h}$, we have an isomoprphism $i_0^{-1}T_l^{-1}\mathfrak{S} \cong i_0^! T_l^{-1}\mathfrak{S}[\dim \mathfrak{h}] = i_0^! T_l^! \mathfrak{S}[\dim \mathfrak{h}]$. Thus,

$$R^\bullet \hom_G(\mathbb{K}_e; i_0^{-1}T_l^{-1}\mathfrak{S}[\dim \mathfrak{h}]) = R^\bullet \hom_G(\mathbb{K}_e; i_0^! T_l^! \mathfrak{S}[\dim \mathfrak{h}])$$

$$= i^!_{(e,0)} T_l^! \mathfrak{S}[\dim \mathfrak{h}] = i^!_{(e^l,l)} \mathfrak{S}[\dim \mathfrak{h}]$$

$$= i_l^! \mathcal{S}_l[\dim \mathfrak{h}].$$

Here $i_{(e,0)}; i_{(e^l,l)}$ denote embeddings of the points specified into $G \times \mathfrak{h}$, and, likewise, i_l is the embedding of the point l into $\mathfrak{h}$.

Theorem 7.26 implies that $H^{<D(l)} i_l^! \mathcal{S}_{e^l} = 0$ and $H^{D(l)} i_l^! \mathcal{S}_{e^l}$ is one dimensional. Indeed, one sees that given $l' \in C_-$, we have: $i_l^! \mathbb{K}_{U^-_{l'}} \cong \mathbb{K}[-\dim \mathfrak{h}]$ for all $l' \geq l$; otherwise $i_l^! \mathbb{K}_{U^-_{l'}} = 0$. Therefore,

$$i_l^! \mathcal{S}_{e^l}[\dim \mathfrak{h}] \cong \bigoplus_{l' \in \mathbb{L}_{e^l}, l' \geq l} H^\bullet(\mathcal{FL}(I_{l'}))[D(l')],$$

and the lowest degree contribution comes from $H^0(\mathcal{FL}(I_l))[D(l)] = \mathbb{K}[D(l)]$.

Set $B(l) := \tau_{\leq -D(l)} B'(l)$. It then follows that B is a homotopically standard $\mathbb{L}_-$-graded algebra. We thus automatically get a B-sheaf structure on $\mathfrak{S}$. Let $I_Z : \mathbf{Z} \times \mathfrak{h} \to G \times \mathfrak{h}$ be the embedding. This embedding is $\mathbb{L}_-$-equivariant, where $\mathbb{L}_-$-action on $\mathbf{Z} \times \mathfrak{h}$ is defined by

$$T_l(c, A) = (e^l c; A + c).$$

Hence we get a B-sheaf structure on $\mathcal{S} := I_Z^! \mathfrak{S}$ (as $\mathfrak{S}$ is injective and I_Z is a closed embedding one can compute $I_Z^!$ by taking sections supported on $\mathbf{Z} \times \mathfrak{h} \subset G \times \mathfrak{h}$.

8.1 A B-Sheaf $j_{C_-^\circ}^{-1}\mathcal{S}$ on $\mathbf{Z} \times C_-^\circ$

Let $j_{C_-^\circ} : \mathbf{Z} \times C_-^\circ \to \mathbf{Z} \times \mathfrak{h}$ be the open embedding. The $\mathbb{L}_-$-action on $\mathbf{Z} \times \mathfrak{h}$ preserves $\mathbf{Z} \times C_-^\circ$, thus making the embedding $j_{C_-^\circ}$ to be $\mathbb{L}_-$-equivariant.

We then have a B-sheaf $j_{C_-^\circ}^{-1}\mathcal{S}$. Let $p : B \to \mathbf{b}$ be the canonical map. Let us choose an injective model for $Rp_*\mathcal{S}$, to be still denoted by $\mathcal{S}$.

Let us study the $\mathbf{b}$-structure on $j_{C_-^\circ}^{-1}\mathcal{S}$. Let $I \subset \{1, 2, \ldots, N-1\}$. Let $e_I := \sum_{i \in I} e_i$.

According to Theorem 7.26 we have a map

$$i_I : H^*(\mathcal{FL}(I))[D(-2\pi e_I)] \otimes \mathbb{K}_{e^{-2\pi e_I} \times U^-_{-2\pi e_I}} \to j^{-1}_{C^\circ_-}\mathcal{S}$$

Set $H(I) := H^\bullet(\mathcal{FL}(I))$. For $I \subset J$ we have a tautological projection

$$\mathcal{FL}(J) \to \mathcal{FL}(I)$$

hence an induced map $H(I) \to H(J)$. Hence H is a functor from the poset of subsets of $\{1, 2, \ldots, N-1\}$ to the category of graded vector spaces.

One can show that this functor is actually free, i.e:

Lemma 8.2 *There exist graded vector spaces $G(I)$, where $I \subset \{1, 2, \ldots, N-1\}$ such that we have decompositions*

$$H(I) = \bigoplus_{J \subset I} G(J), \tag{77}$$

which are compatible with the structure maps $H(I_1) \to H(I_2)$, $I_1 \subset I_2$ in the obvious way.

Proof Let us use Schubert cellular decomposition of partial flag varieties $\mathcal{FL}(I)$. Let $\mathbf{f} \subset \mathcal{FL}(I)$ be the flag such that $\mathbf{f}^r \subset \mathbb{C}^N$ consists of all points $(v_1, v_2, \ldots, v_N) \in \mathbb{C}^N$ such that $v_k = 0$ for all $k > i_r$.

Let $H := \mathrm{GL}_N(\mathbb{C})$. Let $P(I) \subset H$ be the standard parabolic subgroup, namely the stabilizer of $\mathbf{f}$. We have $\mathcal{FL}(I) = H/P(I)$. Let $W \subset G$ be the standard Weyl group. For any $w \in W/W \cap P(I)$ let $[w] \in H/P(I)$ be the image of $[w]$ and let $C_{I,w} := C_w := B.[w]$ where $B \subset H$ is the standard Borel subgroup of upper-triangular matrices. It is well known that the cells C_w, $w \in W/W \cap P(I)$ form a cellular decomposition of $\mathcal{FL}(I)$. We have $\dim_{\mathbb{R}} C_w = 2D_I(w)$, where $D_I(w)$ is defined as follows. Let $\pi_I : \{1, 2, \ldots, N\} \to \{1, 2, \ldots, |I|\}$ be defined by letting $\pi_I(k)$ be the minimal number r such that $i_r \geq k$.

In particular, for any $M \in P(I)$, we have $M_{ij} = 0$ as long as $\pi_I(i) > \pi_I(j)$. Let $w' \in W$ be any representative of $w \in W/W \cap P(I)$.

One then has that $D_I(w)$ is equal to the number of all pairs (i, j) such that $i, j \in \{1, 2, \ldots, N\}$, $i < j$ and $\pi_I(w^{-1}(i)) > \pi_I(w^{-1}j)$.

Thus we have a basis of $H_*(\mathcal{FL}(I))$ labelled by the cells C_w. Let $c_w \in H_{2D_I(w)}$ $\mathcal{FL}(I)$ be the class corresponding to C_w.

We see that the map $p_{IJ} : \mathcal{FL}(I) \to \mathcal{FL}(J)$ is cellular. We have $p_{IJ}C_w \subset C_{w'}$ where w' is the image of $w \in W/W \cap P(I)$ in $W/W \cap P(J)$. One sees that $\dim C_{w'} \leq \dim C_w$. It then follows that $p_{IJ*}c_w = c_{w'}$ is $D_I(w) = D_J(w')$. Otherwise $p_{IJ*}(c_w) = 0$.

Let us describe the dual map p^*_{IJ}. Let $c^w \in H^\bullet(\mathcal{FL}(I)$ be the element dual to c_w. Let us identify $W/W \cap P(I)$ with the set $V(I)$ of partitions $\{1, 2, \ldots, N\} = A_1 \sqcup A_2 \sqcup A_{|I|}$ where $|A_r| = i_r - i_{r-1}$ and we assume $i_0 = 0$, $i_{|I|} = N$. We have a map $Q^{JI}: V(J) \to V(I)$ defined as follows. Pick $t \le N$. Let $i_m = j_{t-1}$; $i_M = j_t$. Order A_t and subdivide it into several subsets, such that the first subset consist of the first $i_{m+1} - i_m$ elements of A_t; the second subset consists of the next $i_{m+2} - i_{m+1}$ elements of A_t, etc. This way we get a partition $Q^{JI}A$. One sees that $Q^{JI} = p^*_{IJ}$. For $A \in V(I)$ let $\sim_A$ be an equivalence relation on I given by $i_1 \sim_A i_2$ if for all $j_1 < j_2$, $j_1, j_2 \in [i_1, i_2]$, $A_{j_1} < A_{j_2}$. Call $A \in V(I)$ elementary if $\sim_A$ is trivial. One then can set $G(I)$ to be the span of all elementary $A \in V(I)$. □

Let us now consider through maps

$$j_I : G(I) \otimes \mathbb{K}_{e^{-2\pi e_I} \times U^-_{-2\pi e_I}}[D(-2\pi e_I)] \to H(I) \otimes \mathbb{K}_{e^{-2\pi e_I} \times U^-_{-2\pi e_I}}[D(-2\pi e_I)] \to j^{-1}_{C^\circ_-}\mathcal{S}.$$

Introduce a notation: for $l \in \mathbb{L}_-$, set $\mathcal{U}_l := e^l \times U^-{}_l \subset \mathbf{Z} \times C^\circ_-$. Denote $\mathcal{G}_I := G(I) \otimes \mathbb{K}_{e^{-2\pi e_I} \times \mathcal{U}_{-2\pi e_I}}[D(-2\pi e_I)]$. The **b**-structure on $j^{-1}_{C^\circ_-}\mathcal{S}$ gives rise to maps

$$T_{l*}\mathcal{G}_I \otimes \mathbf{b}(l) \to T_{l*}j^{-1}_{C^\circ_-}\mathcal{S} \otimes \mathbf{b}(l) \to T_{l*}T_l^{-1}j^{-1}_{C^\circ_-}\mathcal{S} = j^{-1}_{C^\circ_-}\mathcal{S}$$

for all $l \in \mathbb{L}_-$. Take the direct sum:

$$\iota : \bigoplus_{I \subset \{1,2,\ldots,N-1\}; l \in \mathbb{L}_-} T_{l*}\mathcal{G}_I[D(l)] \to j^{-1}_{C^\circ_-}\mathcal{S} \tag{78}$$

(we have replaced $\mathbf{b}(l) = k[D(l)]$). The sheaf on the LHS has an obvious structure of a **b**-sheaf and the map ι is a map of **b**-sheaves.

Furthermore the **b**-sheaf on the LHS splits into a direct sum of **b**-sheaves

$$\mathbb{S}^I := \bigoplus_{l \in \mathbb{L}_-} T_{l*}\mathcal{G}_I[D(l)] \tag{79}$$

thus we have a map of **b**-sheaves

$$\iota : \bigoplus_{I \subset \{1,2,\ldots,N-1\}} \mathbb{S}^I \to \mathcal{S} \tag{80}$$

For future purposes, let us rewrite the definition of $\mathbb{S}^I$. We have

$$\begin{aligned}\mathbb{S}^I &:= \bigoplus_{l \in \mathbb{L}_-} T_{l*}\mathcal{G}_I[D(l)] \\ &= G_I[D(-2\pi e_I)] \otimes [\bigoplus_{l \in \mathbb{L}_-} \mathbb{K}_{\mathcal{U}_{-2\pi e_I + l}}[D(l)]] \\ &= G_I[D(-2\pi e_I)] \otimes T_{-2\pi e_I *}[\bigoplus_{l \in \mathbb{L}_-} \mathbb{K}_{\mathcal{U}_l}[D(l)]]\end{aligned}$$

Let

$$\mathcal{X} := \bigoplus_{l \in \mathbb{L}_-} \mathbb{K}_{\mathcal{U}_l}[D(l)] \tag{81}$$

with the obvious **b**-structure. We then have an isomorphism of **b**-sheaves:

$$\mathbb{S}^I \cong G_I[D(-2\pi e_I)] \otimes T_{-2\pi e_I *}\mathcal{X}. \tag{82}$$

Proposition 8.3 *The map (80) is a quasi-isomorphism.*

Proof For any $z \in \mathbf{Z}$ and any $F \in D(\mathbf{Z} \times C_-^\circ)$ we set $F_z \in D(C_-^\circ)$; $F_z := F|_{z \times C_-^\circ}$. We have induced maps

$$\iota_z : \bigoplus_I \mathbb{S}_z^I \to j_{C_-^\circ}^{-1}\mathcal{S}_z,$$

and it suffices to show that these maps are isomorphisms for all $z \in \mathbf{Z}$. We know (Proposition 7.3) that $\mathrm{SS}(j_{C_-^\circ}^{-1}\mathcal{S}_z) \subset X(\mathbb{L}_-^z)$. One can easily check that $\mathbb{S}_z^I \in X(\mathbb{L}_-^z)$ for all I. As follows from Proposition 7.4 and Lemma 7.12, it suffices to show that the induced maps

$$R\,\mathrm{hom}_{C_-^\circ}(\mathbb{K}_{V_{x,\varepsilon}}; \bigoplus_I \mathbb{S}_z^I) \to R\,\mathrm{hom}_{C_-^\circ}(\mathbb{K}_{V_{x,\varepsilon}}; j_{C_-^\circ}^{-1}\mathcal{S}_z) \tag{83}$$

are isomorphisms for $\varepsilon > 0$ small enough and for all $x \in \mathbb{L}_-^z$. Let $F \in D(\mathbf{Z} \times C_-^\circ)$ and $x \in \mathbb{L}_-$. Set

$$\Delta_x(F) := R\,\mathrm{hom}(\mathbb{K}_{V_{x,\varepsilon}}; F|_{e^x}).$$

Let now F be a **b**-sheaf on $\mathbf{Z} \times \mathfrak{h}$. The **b**-structure gives rise to maps

$$\Delta_x(F) \to \Delta_{x+l}(F)[-D(l)],$$

for all $l \in \mathbb{L}_-$. Set $\delta_F(x) := \Delta_x(F)[-D(x)]$. Introduce a partial order $\preceq$ on $\mathbb{L}_-$ by setting $l_1 \preceq l_2$ if $l_2 - l_1 \in \mathbb{L}_-$. We see that δ_F is a functor from this poset, viewed as a category, to the category of graded $\mathbb{K}$-vector spaces. As follows from Corollary 7.21 and Proposition 7.22, we have $\delta_{\mathcal{S}}(l) = H^\bullet(\mathcal{FL}(I_l))$. Let $l_1 \preceq l_2$. As follows from the proof of Proposition 7.22, the induced map $\delta_{\mathcal{S}}(l_1) \to \delta_{\mathcal{S}}(l_2)$ is induced by the projection $\mathcal{FL}(I_{l_2}) \to \mathcal{FL}(I_{l_1})$ coming from the embedding $I_{l_1} \subset I_{l_2}$. It then follows from Lemma 8.2 that $\delta_{\mathcal{S}}$, as a functor, is freely generated by subspaces

$$G(I) \subset H^\bullet(\mathcal{FL}(I)) = \delta_{\mathcal{S}}(-2\pi e_I) \tag{84}$$

for all $I \subset \{1, 2, \ldots, N-1\}$.

One can easily check that $\delta_{\oplus_I \mathbb{S}^I}$ is freely generated by the subspaces

$$G(I)=\delta_{\mathbb{S}(I)}(-2\pi e_I)\subset \delta_{\oplus_I \mathbb{S}^I}(-2\pi e_I). \qquad (85)$$

The map ι preserves the generating subspaces (84), (85). Hence, the maps (83) are isomorphisms, which proves the Proposition. □

8.2 Strict B-Sheaves

Let F be a B-sheaf on $\mathfrak{h}$. Let $v_k \in B(-e_k)$ be a representative of $u_k \in H^{D(-e_k)}B(-e_k)$. We then have induced maps

$$a_k : F \to T_{-e_k}^{-1} F[D(-e_k)]$$

induced by v_k.

Let

$$\mathbf{Con}_k := \text{Cone } a_k; \qquad (86)$$

let $p_k : \mathfrak{h} \to \mathfrak{h}/\mathbb{R} f_k$. We call F *strict* if

(1) for all k, the natural map $p_k^{-1} Rp_{k*}\mathbf{Con}_k \to \mathbf{Con}_k$ is an isomorphism in $D(\mathfrak{h})$ (that is, $\mathbf{Con}_k$ is constant along fibers of p_k);

(2) F is microsupported on $\mathfrak{h} \times C_+ \subset \mathfrak{h} \times \mathfrak{h}^*$.

Denote the full subcategory of $\text{DBSh}_{\mathfrak{h}}$ consisting of all strict B-sheaves on $\mathfrak{h}$ by $\text{DBSh}_{\mathfrak{h}}^{\text{strict}}$.

Analogously, let F be a sheaf on C_-°. Let us define a_k and $\mathbf{Con}_k$ in the same way as above.

Let $C_-^\circ/\mathbb{R} f_k$ be the image of C_-° under the map $C_-^\circ \to \mathfrak{h} \to \mathfrak{h}/\mathbb{R} f_k$. Let $p_k : C_-^\circ \to C_-^\circ/\mathbb{R} f_k$ be the projection.

As above, let us call F *strict* if

(1) the natural map $p_k^{-1} Rp_{k*}\mathbf{Con}_k \to \mathbf{Con}_k$ is an isomorphism in $D(C_-^\circ)$ for all k;

(2) F is microsupported on $C_-^\circ \times C_+ \subset C_-^\circ \times \mathfrak{h}^*$.

Denote the full subcategory of $\text{DBSh}_{C_-^\circ}$ consisting of all strict B-sheaves on C_-° by $\text{DBSh}_{C_-^\circ}^{\text{strict}}$.

Let $\lambda \in \mathfrak{h}$ and consider a shifted open set $C_-^\circ + \lambda \subset \mathfrak{h}$. We then have a notion of a B-sheaf and of a strict B-sheaf on $C_-^\circ + \lambda$ via an identification $C_-^\circ + \lambda \cong C_-^\circ$ via the shift T_λ. Hence we have categories $\text{DBSh}_{C_-^\circ+\lambda}$; $\text{DBSh}_{C_-^\circ+\lambda}^{\text{strict}}$.

8.2.1

Let $\lambda \in \mathfrak{h}$ and let $j_\lambda : C_-^\circ + \lambda \to \mathfrak{h}$ be an open embedding. We then see that the functor j_λ^{-1} transforms strict sheaves on $\mathfrak{h}$ into strict sheaves on $C_-^\circ + \lambda$

Theorem 8.4 *The functor*

$$j_\lambda^{-1} : DBSh_{\mathfrak{h}}^{strict} \to DBSh_{C_-^\circ+\lambda}^{strict}$$

is an equivalence.

8.3 Proof of the Theorem

8.3.1 First Reductions

Without loss of generality one can put $\lambda = 0$. We also set $j := j_0$.

Let $\pi : B \to \mathbf{b}$ be the projection. As the functor $R\pi_*$ is an equivalence, without loss of generality, one can assume $B = \mathbf{b}$.

Let $I \subset \{1, 2, \ldots, N-1\}$. Let $\mathcal{C}(I, \mathfrak{h}) \subset D\mathbf{b}\mathrm{Sh}_{\mathfrak{h}}$ be the full subcategory consisting of all sheaves F satisfying:

(1) for all $i \in I$, we have: $\mathbf{Con}_i(F) = 0$;

(2) for all $i \notin I$ the natural map $p_i^{-1} Rp_{i*}F \to F$ is an isomorphism.

It is clear that every object of $\mathcal{C}(I, \mathfrak{h})$ is strict.

Let us define the category $\mathcal{C}(I, C_-^\circ)$ in a similar way.

Lemma 8.5 *Every strict* $\mathbf{b}$*-sheaf on* C_- *(resp.* $\mathfrak{h}$*) is quasi-isomorphic to a complex of objects from* $\bigsqcup_I \mathcal{C}(I, \mathfrak{h})$ *(resp.* $\bigsqcup_I \mathcal{C}(I, C_-^\circ)$*).*

Proof We will prove Lemma for strict sheaves on C_-°. The proof for $\mathfrak{h}$ is similar.

Let us first consider a through map

$$\pi_I : C_-^\circ \to \mathfrak{h} \to \mathfrak{h}/(\mathbb{R} < f_j >_{j \notin I})$$

let C_I be the image of π_I.

We also have a through map

$$\sigma_I : \mathbb{R}_{<0} < e_i >_{i \in I} \hookrightarrow \mathfrak{h} \to \mathfrak{h}/(\mathbb{R} < f_j >_{j \notin I})$$

Sublemma 8.6 *The map* σ_I *is an open embedding whose image is the same as the image of* π_I

Proof (of sublemma) It is easy to see that the vectors f_j, $j \notin I$; e_i; $i \in I$ form a basis of $\mathfrak{h}$. Therefore, the vectors e_i; $i \in I$ (more precisely, their images) form a basis of $\mathfrak{h}/(\mathbb{R} < f_j >_{j \notin I})$. Let $x \in C_-^\circ$. Let us expand

$$x = \sum_{i \in I} a_i e_i + \sum_{j \notin I} b_j f_j$$

Then $p_I(x) = \sum_{i \in I} a_i e_i$.

We have: for all $j \notin I$:

$$< x, f_j >= \sum_{k \notin I} b_k < f_j, f_k > \leq 0.$$

Let $J := \{1, 2, \ldots, N-1\} \backslash I$ and let us decompose J into intervals as follows: $J = J_1 \sqcup J_2 \sqcup \cdots \sqcup J_s$ where each $J_t = [k_t; l_t]$, $k_t \leq l_t < k_{t+1} - 1$. Set $b_k^t = b_k$ if $k \in J_t$; otherwise set $b_k^t = 0$. We then have $2b_k^t - b_{k-1}^t - b_{k+1}^t \leq 0$ for all $k \in J_t$. Let $D_k^t := b_k^t - b_{k-1}^t$. We then know that $D_{k+1}^t \geq D_k^t$ if $k, k+1 \in J_t$. We then have $b_k = D_{k_l}^t + \cdots + D_k^t$. Assume $b_k > 0$. Then $D_k^t > 0$ (because $D_{k_l}^t \leq D_{k_l+1}^t \leq \cdots \leq D_k^t$). Hence, $0 \leq D_k^t \leq D_{k+1}^t \leq \cdots$ and $0 < b_k^t < b_{k+1}^t < \cdots < b_{l_t+1}^k = 0$. Contradiction. Thus, $b_k^t \leq 0$ for all k. Therefore, for all k, $b_k \leq 0$.

For every $i \in I$ we have

$$0 >< x, f_i >= a_i + \sum_{j \notin I} b_j < f_i, f_j > .$$

Hence,

$$< x, f_i > - \sum_{i \in I} b_i < f_i, f_j >= a_j.$$

For $i \in I$, $j \notin I$, we have $i \neq j$ and $< f_i; f_j > \leq 0$. As $b_i \leq 0$, we see that $0 >< x, f_j > \geq a_j$. Hence, Image $(\pi_k) \subset$ Image (σ_k). Let us prove the inverse inclusion. Let $g := \sum_{i \in I} a_i e_i - b \sum_{j \notin I} f_j$ We see that for $a_j > 0$ and $0 < b << 1$,we have $g \in C_-^\circ$ and $\pi_k(g) = \sum_{i \in I} a_i e_i$. □

Let $\Gamma_I := R_{<0} < e_i >_{i \in I}$.

We then have a surjection $P_I : C_-^\circ \to \Gamma_I$. It is easy to see that P_I is a trivial bundle whose fiber is homeomorphic to $\mathbb{R}^{N-1-|I|}$. Let $J \subset I$. It follows that we have projections $P_{IJ} : \Gamma_I \to \Gamma_J$ so that $P_J = P_{IJ} P_I$.

Let F be a strict sheaf on C_-°. Let $F_J := P_J^{-1} P_{J*} F$. It is easy to see that F_J is a strict sheaf on C_-°. For $J \subset I$ we have a natural map $F_J \to F_I$. Let Subsets be the poset of all subsets of $\{1, 2, \ldots, N-1\}$; view this subset as a category. We then see that $I \mapsto F_I$ is a functor from Subsets to the dg category of B-sheaves whose image lies in the full subcategory of strict B-sheaves.

For a subset $I \subset \{1, 2, \ldots, N-1\}$ consider the standard complex

$$K(I, F) = \bigoplus_{J : J \subset I} F_J \otimes \Lambda^{\mathrm{top}}(\mathbb{K}[I \backslash J])[|I \backslash J|]$$

with the standard differential. We then see that (1) $K(I, F)$ is a strict B-sheaf on C_-°;

(2) $p_I^{-1} R p_{I*} K(I, F) \to K(I, F)$ is an isomorphism;

(3) Let $J \subset I$ and $J \neq I$. Then $R p_{J*} K(I, F) = 0$.

(2) and (3) imply that

(4) for any J which intersects I, $R p_{J*} K(I, F) = 0$.

Let $k \in I$. Then we know that $p_k^{-1} R p_{k*} \mathbf{Con}_k \to \mathbf{Con}_k$ is an isomorphism. On the other hand, 4) implies that $R p_{k*} \mathbf{Con}_k(K(I, F)) = 0$ Hence,

(5) $\mathbf{Con}_k K(I, F) = 0$ for all $k \notin I$.

Thus, $K(I, F) \in \mathcal{C}(I, C_-^\circ)$, which proves Lemma for C_-°. The proof for $\mathfrak{h}$ is similar. □

8.3.2

It is clear that the functor j^{-1} takes $\mathcal{C}(I, \mathfrak{h})$ to $\mathcal{C}(I, C_-^\circ)$ for all I.

We will prove:

Lemma 8.7 *Let X be a* $\mathbf{b}$*-sheaf on* $\mathfrak{h}$ *and let* $Y \in \mathcal{C}(\mathfrak{h}; I)$ *for some* $I \subset \{1, 2, \ldots, N-1\}$. *Then the natural map*

$$R\hom_{D\mathbf{b}Sh_{\mathfrak{h}}}(X, Y) \to R\hom_{D\mathbf{b}Sh_{C_-^\circ}}(j^{-1}X; j^{-1}Y)$$

is an isomorphism

Proof We see that $j_! j^{-1} X$ is a $\mathbf{b}$-sheaf on $\mathfrak{h}$ and that

$$R\hom_{D\mathbf{b}\mathrm{Sh}_{C_-^\circ}}(j^{-1}X; j^{-1}Y) = R\hom_{D\mathbf{b}\mathrm{Sh}_{\mathfrak{h}}}(j_! j^{-1}X; Y).$$

We also have a natural map $j_! j^{-1} X \to X$ of $\mathbf{b}$-sheaves on $\mathfrak{h}$. Let Z be the cone of this map. The statement of the Lemma is equivalent to $R\hom_{D\mathbf{b}\mathrm{Sh}_{\mathfrak{h}}}(Z, Y) = 0$

For every $k \in I$ we have a structure map

$$Z \to T_{-2\pi e_k}^{-1} Z[D(-2\pi e_k)] \tag{87}$$

Sublemma 8.8 *The natural map*

$$R\hom_{D\mathbf{b}Sh_{\mathfrak{h}}}(T_{-2\pi e_k}^{-1} Z[D(-2\pi e_k)]; Y) \to R\hom_{D\mathbf{b}Sh_{\mathfrak{h}}}(Z; Y)$$

is an isomorphism.

Proof (of Sublemma) Let W be the cone of the map (87). We are to show that $R\hom_{D\mathbf{b}\mathrm{Sh}_{\mathfrak{h}}}(W, Y) = 0$. It follows that the structure map $W \to T_{-e_k}^{-1} W[D(-e_k)]$ is homotopy equivalent to 0.

Choose an injective representative of Y and consider a $\mathbb{L}_-$- graded complex $H(l) := \hom(W; T_l^{-1} Y)$. This complex is a $\mathbb{L}_-$-graded $\mathbf{b}$-bimodule. We also have a $\mathbb{L}_-$-graded $\mathbf{b}$-bimodule structure on $\mathbf{b}$. We then have

$$R\hom_{D\mathbf{b}\mathrm{Sh}_{\mathfrak{h}}}(W, Y) = R\hom_{\mathbf{b}-\mathrm{bimod}}(\mathbf{b}; H).$$

Let $R_k := 1 \otimes 1_{-e_k} \in \mathbf{b} \otimes \mathbf{b}$; $L_k = 1_{-e_k} \otimes 1 \in \mathbf{b} \otimes \mathbf{b}$. We then see that the action of R_k on H is a quasi-isomorphism, whereas the action of L_k is homotopy equivalent to 0. Hence the action of $R_k - L_k$ on H is a quasi-isomorphism. The action of $R_k - L_k$ on $\mathbf{b}$ is zero. Hence an induced action of $R_k - L_k$ on $R\,\mathrm{hom}_{\mathbf{b}-\mathrm{bimod}}(\mathbf{b}; H)$ is simultaneously 0 and an isomorphism, meaning that $R\,\mathrm{hom}_{\mathbf{b}-\mathrm{bimod}}(\mathbf{b}; H) = 0$, whence the statement □

Let $g = -\sum_{i\in I} 2\pi e_i$. Consider an inductive system of $\mathbf{b}$-sheaves on $\mathfrak{h}$:

$$Z \to T_g^{-1}[D(g)]Z \to T_{2g}^{-1}[D(2g)]Z \to \cdots \to T_{ng}^{-1}[D(ng)]Z \to \cdots$$

and let $L(Z)$ be the derived direct limit of this system. We have a natural map $Z \to L(Z)$. The previous Lemma easily implies that the induced map

$$R\,\mathrm{hom}_{D\mathbf{b}\mathrm{Sh}_{\mathfrak{h}}}(L(Z); Y) \to R\,\mathrm{hom}_{D\mathbf{b}\mathrm{Sh}_{\mathfrak{h}}}(Z; Y)$$

is an isomorphism.

It also follows that the natural map

$$R\,\mathrm{hom}(L(Z); Y) \to R\,\mathrm{hom}(Rp_{I!}L(Z); Rp_{I!}Y)$$

is an isomorphism (because Y is locally constant along fibers of p_I). Thus, the statement of our Lemma reduces to showing that

$$Rp_{I!}L(Z) = 0$$

Let $x \in \mathfrak{h}_I$ and show that $Rp_{I!}L(Z)|_x = 0$. We have

$$Rp_{I!}L(Z)|_x = R\Gamma_c(p_I^{-1}x; L(Z)|_{p_I^{-1}x})$$

Let $U_x \subset \mathfrak{h}$; $U_x := p_I^{-1}x$. By definition, we have

$$R\Gamma_c(p_I^{-1}x; L(Z)|_{p_I^{-1}x}) = \varinjlim_n R\Gamma_c(U_{x+ng}; Z|_{U_{x+ng}}[D(ng)])$$

where the spaces $R\Gamma_c(U_{x+ng}; Z|_{U_{x+ng}})$ form an inductive system by means of the structure maps $Z \to T_g^{-1}Z[D(g)]$. Next, we have

$$R\Gamma_c(U_{x+ng}; Z|_{U_{x+ng}}) = \mathrm{Cone}[R\Gamma_c(U_{x+ng} \cap C_-^\circ; X|_{U_{x+ng}}) \to R\Gamma_c(U_{x+ng}; X|_{U_{x+ng}})].$$

We have maps

$$U_x \to U_{x+g} \to U_{x+2g} \to \cdots \to U_{x+ng} \to \cdots$$

induced by the shifts T_g. Let $U := \bigcup_n T_{ng}^{-1}(U_{x+ng} \cap C_-^\circ)$. We then have

$$U_{x+ng} \cap C_-^\circ \subset T_{ng}U \subset U_{x+ng}.$$

It follows that U consists of all vectors $v = \sum_{i\in I} x_i e_i + \sum_{j\notin I} y_j f_j$, where

$$x = \sum_{i\in I} x_i e_i \tag{88}$$

and for all $l \notin I$,

$$< \sum_{j\notin I} y_j f_j, f_l >< 0. \tag{89}$$

It also follows that the natural maps

$$\varinjlim_n R\Gamma_c(U_{x+ng} \cap C_-^\circ; X|_{U_{x+ng}}][D(ng)) \to \varinjlim_n R\Gamma_c(T_{ng}U; X|_{U_{x+ng}}[D(ng)]) \tag{90}$$

is an isomorphism. Indeed, set $Z_n := T_{ng}^{-1} Z|_{T_{ng}U}[D(ng)]$, $Z_n \in D(U)$. The objects Z_n form an inductive system. Set $U_n := T_{ng}^{-1}(U_{x+ng} \cap C_-^\circ) \subset U$. We see that $U_0 \subset U_1 \subset U_2 \subset \cdots$ and $\bigcup_n U_n = U$ We then see that our inductive systems and their map can be rewritten as

$$\varinjlim_n R\Gamma_c(U_n; Z_n) \to \varinjlim_n R\Gamma_c(U; Z_n)$$

Let $K_n := U \backslash U_n$. We then see that $\cap_n K_n = 0$ and that the cone of the above map is isomorphic to

$$\varinjlim_n R\Gamma_c(K_n; Z_n|_{K_n}). \tag{91}$$

We see that for each m, the natural map

$$R\Gamma_c(K_m; Z_m|_{K_m}) \to \varinjlim_n R\Gamma_c(K_n; Z_n|_{K_n}).$$

factors as

$$R\Gamma_c(K_m; Z_m|_{K_m}) = \varinjlim_{n>m} R\Gamma_c(K_m \backslash K_n; Z_m|_{K_m}) \to \varinjlim_n R\Gamma_c(K_n; Z_n|_{K_n})$$

hence it is 0, which means that the space (91) is 0 and the map (90) is an isomorphism.

Therefore, our original statement now reduces to showing that

$$\mathrm{Cone}(R\Gamma_c(T_{ng}U; X|_{U_{x+ng}}) \to R\Gamma_c(U_{x+ng} : X|_{U_{x+ng}})) = 0 \tag{92}$$

for all $n > 0$.

let $A := \mathbb{R} < f_j >_{j\neq I}$. We have an identification

$$\alpha : A \to U_{x+ng}, \quad a \mapsto \sum_{i\in I} x_i e_i + ng + a,$$

where x_i are the same as in (88). Let $B \subset A$ be an open subset specified by the condition (89). It follows that $\alpha(B) = T_{ng}U$. Let $Y \in D(A)$, $Y := \alpha^{-1} X|_{U_{x+ng}}$. We can rewrite (92) as

$$\mathrm{Cone}(R\Gamma_c(B, Y) \to R\Gamma_c(A, Y))$$

Let us estimate the microsupport of Y. We know that $SS(X) \subset \mathfrak{h} \times C_+$. Using Proposition (11.8) one can show that Y is microsupported on the set $A \times \beta^*(C_+)$, where $\beta^* : \mathfrak{h}^* \to A^*$ is dual to the embedding $\beta : A \to \mathfrak{h}$; $\beta(f_j) = f_j$. let $\varepsilon^j \in A^*$ be the basis dual to f^j. One sees that

$$\beta^*(C_+) = \mathbb{R}_{\geq} 0 < \varepsilon^j >_{j \notin I} .$$

Let $\gamma \subset A$ be the dual cone to $\beta^*(C_+)$; $\gamma = \mathbb{R}_{>0} < f_j >_{j \notin I}$. One can check $B + \gamma = A$. As $SS(Y) \subset A \times \beta^*(C_+)$, the Lemma follows. □

It now follows that the functor $j^{-1} : D\mathbf{b}\mathrm{Sh}_{\mathfrak{h}} \to D\mathbf{b}\mathrm{Sh}_{C^\circ_-}$ is conservative (the natural map $R\hom(F, G) \to R\hom(j^{-1}F; j^{-1}G)$ is an isomorphism). We only need to check the essential surjectivity of j^{-1}. It suffices to check that for each $I \subset \{1, 2, \ldots, N-1\}$, the functor $j^{-1} : \mathcal{C}(I, \mathfrak{h}) \to \mathcal{C}(I, C^\circ_-)$ is essentially surjective. Let $F \in \mathcal{C}(I, C^\circ_-)$ and consider a $\mathbf{b}$-sheaf $G := Rp_I^! Rp_{I!} L(j_! F) := Rp_I^{-1} Rp_{I!} L(j_! F)$ $[N-1-|I|]$, where L is the same as in the proof of Lemma. One easily checks that $j^{-1}G \cong F$. This completes the proof of the theorem.

8.3.3

Let us check that the $\mathbf{b}$- sheaf $\mathcal{S}$ is strict. Indeed, it follows that the structure map

$$b_{-2\pi e_k} : \mathcal{S} \to T^{-1}_{-2\pi e_k}\mathcal{S}$$

is induced by the correponding map

$$b^{\mathfrak{S}}_{-2\pi e_k} : \mathfrak{S} \to T^{-1}_{-2\pi e_k}\mathfrak{S} = \mathfrak{S}|_{G \times -2\pi e_k} *_G \mathfrak{S}$$

which is in turn induced by the map

$$\beta_{-e_k} : \mathbb{K}_e \to \mathfrak{S}|_{G \times -e_k}$$

as in Proposition 7.18. let $B_k := \mathrm{Cone} b_k$. We then get

$$\mathrm{Cone} b^{\mathfrak{S}}_{-2\pi e_k} = B_k * \mathfrak{S}.$$

According to Proposition 7.18, $SS(B_k) \subset \{(g, \omega)| :< \|\omega\|, f_k >= 0\}$ Standard computation shows that the sheaf $B_k * \mathfrak{S}$ is microsupported on the set

$$\{(g, A, \omega, \eta)|(\eta, f_k) = 0\}$$

meaning that $\mathrm{Cone} b^{\mathfrak{S}}_{-2\pi e_k} = B_k * \mathfrak{S}$ is constant along the fibers of the projection $G \times \mathfrak{h} \to G \times (\mathfrak{h}/f_k)$. Hence, $\mathrm{Cone} b_{-2\pi e_k} = i^{-1} b^{\mathfrak{S}}_{-2\pi e_k}$ is constant along the fibers of the projection

$$\mathbf{Z} \times \mathfrak{h} \to \mathbf{Z} \times \mathfrak{h}/f_k$$

It then follows that the sheaf $j^{-1}_{C^\circ_-}\mathcal{S}$ is a strict **b**-sheaf on C°_-. We know (see (80) that $j^{-1}_{C^\circ_-} S \cong \bigoplus_{I \subset \{1,2,\ldots,N-1\}} \mathbb{S}_I|_{C^\circ_-}$. It then easily follows that each $\mathbb{S}_I$ is a strict **b**-sheaf on C°_-. Indeed, $\mathrm{Cone} b^{\mathcal{S}}_k = \bigoplus_I \mathrm{Cone} b^{\mathbb{S}_I}_k$. Let $C := \mathrm{Cone} b^{\mathcal{S}}_k$ and $C_I := \mathrm{Cone} b^{\mathcal{S}'_I}_k$. Let $p_k : \mathbf{Z} \times C^\circ_- \to \mathbf{Z} \times C^\circ_-/f_k$. One then sees that the natural map

$$p_k^{-1} R p_{k*} C \to C$$

is isomorphic to the direct sum of natural maps

$$p_k^{-1} p_{k*} C_I \to C_I$$

As the map $p_k^{-1} R p_{k*} C \to C$ is an isomorphism, so is each of its direct summands, i.e. all maps $p_k^{-1} p_{k*} C_I \to C_I$ are isomorphisms meaning that all sheaves $\mathcal{S}'_I$ are strict.

Remark. One can also prove that the sheaves $\mathbb{S}_I$ are strict directly from the definition (79).

According to Theorem 8.4, there exist strict **b**-sheaves on $\mathbf{Z} \times \mathfrak{h}$, to be denoted by $\mathcal{S}_I$ such that $i^!_{C^\circ_-} \mathcal{S}_I \cong \mathbb{S}_I$ and the sheaves $\mathcal{S}_I$ are unique up-to a unique isomorphism. Same theorem implies that we should have an isomorphism

$$\mathcal{S} \cong \bigoplus_I \mathcal{S}_I.$$

9 Identifying the Sheaf $\mathcal{S}$

One can check that the **b**- sheaf $\mathcal{X}$ on $\mathbf{Z} \times C^\circ_-$ as in (81) is strict. Indeed, this follows from the fact that the **b**- sheaf $\mathcal{S}_\emptyset = G_\emptyset \otimes \mathcal{X}$ is strict, or it can be checked directly.

It then follows that there exists a strict **b**-sheaf $\mathcal{Y}$ on $\mathbf{Z} \times \mathfrak{h}$ such that $j^{-1}_{C^\circ_-}\mathcal{Y} = \mathcal{X}$. As $\mathbb{S}_I \cong G_I[D(-2\pi e_I)] \otimes T_{-2\pi e_I *}\mathcal{X}$, it then follows that we have an isomorphism $\mathcal{S}_I \cong G_I[D(-2\pi e_I)] \otimes T_{-2\pi e_I *}\mathcal{Y}$ which is induced by the obvious isomorphism

$$G_I[D(-2\pi e_I)] \otimes T_{-2\pi e_I *}\mathcal{X}|_{(C^\circ_- - 2\pi e_I)} = G_I[D(-2\pi e_I)] \otimes T_{-2\pi e_I *}\mathcal{Y}|_{(C^\circ_- - 2\pi e_I)}.$$

Thus, we have an isomorphism

$$\mathcal{S} \cong \bigoplus_I G_I[D(-2\pi e_I)] \otimes T_{-2\pi e_I *}\mathcal{Y}. \tag{93}$$

It now remains to identify the **b**-sheaf $\mathcal{Y}$.

9.1 Identifying $\mathcal{Y}$

9.1.1

For a subset $J \subset \{1, 2, \ldots, N-1\}$ and $l \in L$ let $K(J,l) \subset e^l \times \mathfrak{h} \subset \mathbf{Z} \times h$ be defined as follows:

$$K(J,l) := \{(e^l, x) \in \mathbf{Z} \times \mathfrak{h} | \forall j \in J :< x - l, e_j >\geq 0\}.$$

Let $V(J,l) := \mathbb{K}_{K(J,l)}[D(l)]$. Let $\mathbb{L}_J = \{l \in \mathbb{L} | \forall i \notin J :< l, f_j >\leq 0\}$ Let $\Psi^J := \bigoplus_{l \in \mathbb{L}_J} V(J,l)$. Let us endow Ψ^J with a **b**-structure. Let $\lambda \in \mathbb{L}_-$. We have

$$T_\lambda^{-1} V(J,l) = \mathbb{K}_{T_\lambda^{-1} K(J,l)}[D(l)];$$

$$T_\lambda^{-1} K(J,l') = \{(e^{l'}, x) | \forall j \in J : e^\lambda e^{l'} = e^l; < x + \lambda - l, e_j >\geq 0\}$$

$$= K(J, l - \lambda).$$

Thus,

$$T_\lambda^{-1} V(J,l) = \mathbb{K}_{K(J,l-\lambda)}[D(l)] = V(J, l-\lambda)[D(\lambda)].$$

It is clear that if $l \in \mathbb{L}_J$, then $l + \lambda \in \mathbb{L}_J$. We then can define the map $b_\lambda : \Psi^J \otimes \mathbf{b}(\lambda) \to T_\lambda^{-1} \Psi^J$ as a direct sum of maps

$$V(J,l) \otimes \mathbf{b}(\lambda) = V(J,l)[D(\lambda)] = T_\lambda^{-1} V(J, l + \lambda).$$

Let us now check that Ψ^J are strict **b**-sheaves.

Let $j \notin J$. Then it is clear that Ψ^J is constant along the fibers of the map $p_j : \mathbf{Z} \times \mathfrak{h} \to \mathbf{Z} \times \mathfrak{h}/f_j$. Therefore so is the cone of b_{-e_j}. Let $j \in J$. it is then easy to see that the map b_{-e_j} is an isomorphism, whence the statement.

Let $J_1 \subset J_2$. Construct a map of **b**-sheaves

$$I_{J_1 J_2} : \Psi^{J_1} \to \Psi^{J_2}.$$

It is defined as the direct sum of the natural maps

$$V(J_1, l) \to V(J_2, l)$$

for all $l \in \mathbb{L}_{J_1} \subset \mathbb{L}_{J_2}$. These maps come from the closed embeddings $K(J_2, l) \subset K(J_1, l)$.

Let Subsets be the poset (hence the category) of all subsets of $\{1, 2, \ldots, N-1\}$. We then see that Ψ is a functor from Subsets to the category of **b**-sheaves on $\mathbf{Z} \times \mathfrak{h}$. We then construct the standard complex $\Phi^\bullet$ such that

$$\Phi^k := \bigoplus_{I, |I|=k} \Psi^I \tag{94}$$

and the differential $d_k : \Phi^k \to \Phi^{k+1}$ is given by

$$d_k = \sum (-1)^{\sigma(J_1, J_2)} I_{J_1 J_2}, \tag{95}$$

where the sum is taken over all pairs $J_1 \subset J_2$ such that $|J_1| = k$ and $|J_2| = k+1$. The set $J_2 \backslash J_1$ then consists of a single element e and $\sigma(J_1 J_2)$ is defined as the number of elements in J_2 which are less than e.

The constructed complex defines an object in $\mathrm{DBSh}^{\mathrm{strict}}_{\mathbf{Z}\times\mathfrak{h}}$, to be denoted by Φ.

We will show $\Phi \cong \mathcal{Y}$. To this end it suffices to prove:

Lemma 9.1 *We have* $j^{-1}_{C^\circ_-} \Psi \cong \mathcal{X}$.

Proof We have a natural map $\iota : \mathcal{X} \to j^{-1}_{C^\circ_-} \Phi^0 = j^{-1}_{C^\circ_-} \Psi^\emptyset$. Indeed,

$$\mathcal{X} = \bigoplus_{l \in \mathbb{L}_-} \mathbb{K}_{\mathcal{U}_l}[D(l)]$$

and

$$j^{-1}_{C^\circ_-} \Psi^\emptyset = \bigoplus_{l \in \mathbb{L}_-} \mathbb{K}_{e^l \times C^\circ_-}[D(l)].$$

The map ι is defined as a direct sum of the obvious maps

$$\mathbb{K}_{\mathcal{U}_l}[D(l)] \to \mathbb{K}_{e^l \times C^\circ_-}[D(l)]$$

coming from the open embeddings $\mathcal{U}_l \subset e^l \times C^\circ_-$.

It is clear that $I_{\emptyset, J}\iota = 0$ for all nonempty J. Hence the map ι defines a map $\mathcal{X} \to j^{-1}_{C^\circ_-}\Phi$. Let us show that this map is an isomorphism.

For each $l \in L$ set

$$\Phi^n_l := \bigoplus_{J | l \in \mathbb{L}_J; |J|=n} V(J, l)[D(l)].$$

It is clear that for each l, $\Phi^\bullet_l \subset \Phi$ is a subcomplex (in the category of complexes of sheaves on $\mathbf{Z} \times \mathfrak{h}$) and

$$\Phi = \bigoplus_{l \in \mathbb{L}} \Phi_l$$

The map ι takes values in $\bigoplus_{l\in\mathbb{L}_-} j_{C_-^\circ}^{-1}\Phi_l$ and splits into a direct sum of maps $\iota_l : \mathbb{K}_{\mathcal{U}_l} \to j_{C_-^\circ}^{-1}\Phi_l$.

We thus need to show that (1) complexes $j_{C_-^\circ}^{-1}\Phi_l$ are acyclic for all $l \notin \mathbb{L}_-$;
(2) the maps ι_l are quasi-isomorphisms.

Let us first study the complexes Φ_l. Let us identify $\mathfrak{h} = \mathbb{R}^{N-1}$ by means of the basis $f_1, f_2, \ldots, f_{N-1}$. Let $X_j : \mathbf{Z} \times \mathfrak{h} \to \mathbf{Z} \times \mathbb{R}$ be defined by

$$X_j(c, A) = (c, x_j(A)),$$

where $A = \sum_j x_j(A) f_j$. Let $l_i =< l, f_i >$. Let $J_l := \{i | l_i > 0\}$. It follows that $l \in \mathbb{L}_J$ iff $J \supset J_l$. We also have

$$V(J, l) = T_{l*}(\bigotimes_{j\in J} X_j^{-1}\mathbb{K}_{e\times[0,\infty)} \otimes \bigotimes_{i\notin J} X_j^{-1}\mathbb{K}_{e\times\mathbb{R}})[D(l)],$$

where $e \in \mathbf{Z}$ is the unit. Let E be the following complex of sheaves on $\mathbf{Z} \times \mathbb{R}$:

$$\mathbb{K}_{e\times\mathbb{R}} \to \mathbb{K}_{e\times[0,\infty)}.$$

We then have an isomorphism of complexes

$$\Phi_l = (T_{l*} \bigotimes_{j\in J_l} X_j^{-1}\mathbb{K}_{e\times[0,\infty)} \otimes \bigotimes_{i\notin J_l} X_i^{-1} E)[D(l) + |J_l|].$$

We have a quasi-isomorphism $\mathbb{K}_{e\times(-\infty,0)} \to E$ which induces a quasi-isomorphism

$$\Phi_l \cong (T_{l*} \bigotimes_{j\in J_l} X_j^{-1}\mathbb{K}_{e\times[0,\infty)} \otimes \bigotimes_{i\notin J_l} X_i^{-1}\mathbb{K}_{e\times(-\infty,0)})[D(l) + |J_l|]$$

$$= T_{l*}\mathbb{K}_{W_J}[D(l) + |J_l|],$$

where

$$W_J = \{(e, A) \in \mathbf{Z} \times \mathfrak{h} | j \in J \Rightarrow x_j(A) \geq 0; i \notin J \Rightarrow x_i(A) < 0\}$$

Let us now prove (1) It follows that Φ_l is supported on the set $\overline{T_l(W_{J_l})} = \overline{W_{J_l}} + (e^l, l)$. It suffices to prove that $\overline{T_l(W_{J_l})} \cap \mathbf{Z} \times C_-^\circ = 0$. Suppose $z' \in \overline{T_l(W_{J_l})} \cap \mathbf{Z} \times C_-^\circ$. Let $z' = (e^l, z)$, $z \in \mathfrak{h}$.

Let $z = A + l, (e^l, A) \in W_{J_l}$. Let $A_j = (A, f_j)$ and $l_j = (l, f_j)$. We also set $A_0 = A_N = l_0 = l_N = 0$. Set $x_j := x_j(A)$. We then know that $l_j > 0$ for all $j \in J_l$; $l_j \leq 0$ otherwise. We also have

$$A_j =< A, f_j >=< A, 2e_j - e_{j-1} - e_{j+1} >= 2x_j - x_{j-1} - x_{j+1}.$$

As $A + l \in C_-^\circ$, we have $A_j + l_j < 0$. Therefore, if $j \in J$, then $A_j < 0$, thus $2x_j - x_{j-1} - x_{j+1} < 0$. We also know that if $j \in J$, then $x_j \geq 0$.

If $j \notin J$, then we know that $x_j \leq 0$. For $j \in J$ let $j_1 < j$ be the largest number such that $j_1 \notin J$, if it does not exist, set $j_1 = 0$. Similarly, let $j_2 > j$ be the smallest number such that $j_2 \notin J$, if it does not exist set $j_2 = N$.

We then have $x_{j_1} \leq 0$; $x_{j_2} \leq 0$; for all j such that $j_1 < j < j_2$; $2x_j - x_{j-1} - x_{j+1} < 0$, hence $x_j - x_{j-1} < x_{j+1} - x_j$, and $x_j \geq 0$.

Therefore, we have

$$0 \leq x_{j_1+1} - x_{j_1} < x_{j_1+2} - x_{j_1+1} < \cdots < x_{j_2} - x_{j_2-1} \leq 0.$$

Observe that $j_2 - j_1 \geq 2$, therefore, we get $0 < 0$, which is a contradiction. Thus, indeed, $j_{C_-^\circ}^{-1}\Phi_l \cong 0$ for all $l \notin \mathbb{L}_-$.

(2) If $l \in \mathbb{L}_-$, then $J_l = \emptyset$ and we have a quasi-isomorphism $\mathbb{K}_{(e^l,x)|x<<l}[D(l)] \to \Phi_l$. Therefore we have an induced quasi-isomorphism $\mathbb{K}_{\mathcal{U}_l}[D(l)] \to j_{C_-^\circ}^{-1}\Phi_l$. One can easily check that this map is isomorphic to ι, whence the statement. □

From now on we set $\mathcal{Y} = \Phi$.

Let us summarize our results:

Theorem 9.2 *Let $\mathcal{Y} = \Phi$, where Φ is as in (94), (95). Then we have an isomorphism (93)*

This theorem is equivalent to Theorem 5.6.

10 Appendix 1: SU(N) and its Lie Algebra: Notations and a Couple of Lemmas

Let us introduce notation we will use when working with $G = \mathrm{SU}(N)$. Let $\mathfrak{g}$ be the Lie algebra of G; it is naturally identified with the space of all skew-hermitian traceless $N \times N$ matrices. Let $\mathfrak{h} \subset \mathfrak{g}$ be the Cartan subalgebra of $\mathfrak{g}$ consisting of all diagonal matrices from $\mathfrak{g}$. The abelian Lie algebra $\mathfrak{h}$ consists of all matrices of the form $i\,\mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N)$, where $\lambda_i \in \mathbb{R}$ and $\sum_i \lambda_i = 0$.

Let $C_+ \subset \mathfrak{h}$ be the positive Weyl chamber consisting of all matrices $i\,\mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$. For every $X \in \mathfrak{g}$ there exists a unique element $\|X\| \in C_+$ such that X is conjugate with $\|X\|$.

We have an invariant positive definite inner product $<,>$ on $\mathfrak{g}$ such that $<X, Y> = -\mathrm{Tr}(XY)$. By means of this product we identify $\mathfrak{g} = \mathfrak{g}^*$, $\mathfrak{h} = \mathfrak{h}^*$.

We will use the basis of roots in $\mathfrak{h}^*$ which consists of vectors $f_1, f_2, \ldots, f_{N-1}$, where

$$f_k(i\,\mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N)) = \lambda_k - \lambda_{k+1}.$$

Via identification $\mathfrak{h} = \mathfrak{h}^*$, the vector $f_k \in \mathfrak{h}^*$ corresponds to a vector in $\mathfrak{h}$ denoted by the same symbol, and we have

$$f_k = i\text{diag}(0, 0, \dots, 0, 1, -1, 0, \dots, 0)$$

where 1 is at the kth position.

We also have the dual basis of coroots $e_1, e_2, \dots, e_N$ determined by $< f_k, e_l >= \delta_{kl}$. One has

$$e_k = i\text{diag}((N-k)/N, (N-k)/N, \dots, (N-k)/N, -k/N, -k/N, \dots, -k/N) \tag{96}$$

where there are total k entries equal to $(N-k)/N$. One can check that $f_k = 2e_k - e_{k-1} - e_{k+1}$ for $k = 1, 2, \dots, N-1$ and we assume $e_0 = e_N = 0$.

One can rewrite $e_k = i\text{diag}(1, 1, \dots, 1, 0, 0, \dots, 0) - ik/N\text{diag}(1, 1, 1, \dots, 1)$, where it is assumed that we have k entries of 1 in $\text{diag}(1, 1, \dots, 1, 0, 0, \dots, 0)$. In particular, we have

$$< e_k, i\text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) >= \lambda_1 + \lambda_2 + \dots + \lambda_k.$$

One also sees that C_+ consists of all $X \in \mathfrak{h}$ such that $< X, f_k >\geq 0$. Therefore,

$$C_+ = \{\sum_{k=1}^{N} L_k e_k | L_k \geq 0\}.$$

We have a partial order on $\mathfrak{h}$: $X \geq Y$ means $< X - Y, e_k >\geq 0$ for all k.

We also write $X >> Y$ if $< X - Y, e_k >> 0$ for all k.

Let $\omega \in \mathfrak{g}$. The matrix $-i\omega$ is hermitian and let $\lambda_1(\omega) > \lambda_2(\omega) > \dots > \lambda_r(\omega)$ be eigenvalues of $-i\omega$. Let $V^k(\omega)$ be the eigenspace of $-i\omega$ of eigenvalue λ_k. Let

$$V_k(\omega) = V^1(\omega) \oplus V^2(\omega) \oplus \dots \oplus V^k(\omega).$$

We then get a partial flag

$$0 \subset V_1(\omega) \subset \dots \subset V_r(\omega) = \mathbb{C}^N. \tag{97}$$

Let $d_k(\omega) := \dim V_k(\omega)$.

10.0.1

In the future, we will need

Lemma 10.1 *Let $X, \omega \in \mathfrak{g}$. Let $\|X\| = idiag(A_1, A_2, \dots, A_N) \in C_+$ and let*

$$0 \subset V_1(\omega) \subset \dots \subset V_r(\omega) = \mathbb{C}^N$$

be the flag as in (97).

Then

$$< \omega, X > \leq (\|\omega\|, \|X\|).$$

The equality takes place iff

(a) $[X, \omega] = 0$ *(hence* $XV_k(\omega) \subset V_k(\omega)$ *for all k, and*

(b)$TrX|_{V_k} = i(A_1 + A_2 + \cdots A_{d_k(\omega)}) = i < e_{d_k(\omega)}; \|X\| >$.

Proof Let $\mu_k = \lambda_k(\omega) - \lambda_{k+1}(\omega)$; $k < r$. Let us also set $\mu_r = \lambda_r(\omega)$. We then have

$$\omega = i \sum_{k=1}^{r} \mu_k \mathbf{pr}_{V_k(\omega)},$$

where **pr** denotes the orthogonal projector;

$$< \omega, X >= \sum_{k=1}^{r-1} \mu_k \mathrm{Tr}(-iX\mathbf{pr}_{V_k(\omega)}).$$

We know that $\mathrm{Tr}(-iX\mathbf{pr}_{V_k(\omega)}) \leq A_1 + A_2 + \cdots + A_{d_k(\omega)}$ (this is a particular case of the general fact: given an hermitian matrix Y on $\mathbb{C}^N$ (in our case $-iX$) and a vector subspace $V \subset \mathbb{C}^N$ of dimension n the value of $\mathrm{Tr}(Y\mathbf{pr}_V)$ does not exceed the sum of top n eigenvalues of Y).

Hence

$$< \omega, X > \leq \sum_{k=1}^{r-1} \mu_k (A_1 + \cdots + A_{d_k(\omega)}) = \sum_{j=1}^{r} A_j \sum_{k | j \leq d_k(\omega)} \mu_k$$

$$= \sum_{j=1}^{r} A_j \lambda_j(\omega) =< \|\omega\|, \|X\| > .$$

The equality is only possible if for all k $\mathrm{Tr}(-iX\mathbf{pr}_{V_k(\omega)}) = A_1 + \cdots A_{d_k(\omega)}$. As $A_1, \ldots, A_{d_k(\omega)}$ are top $d_k(\omega)$ eigenvalues of $-iX$, the equality occurs iff $V_k(\omega)$ is the span of eigenvectors of $-iX$ with eigenvalues $A_1, \ldots, A_{d_k(\omega)}$, which implies the statement b) of Lemma. □

10.0.2

Lemma 10.2 *Let* $X, Y \in \mathfrak{g}$. *We have* $\|X + Y\| \leq \|X\| + \|Y\|$.

Proof We need to show that for every k,

$$< \|X + Y\|, e_k > \leq < \|X\|, e_k > + < \|Y\|, e_k > .$$

For a Hermitian operator A on a finite-dimensional Hermitian vector space V we set $\mathbf{n}(A) := \max_{|v|=1} < Av, v >$, where $<,>$ is the hermitian inner product on V. We see that

$$\mathbf{n}(A+B) \le \mathbf{n}(A) + \mathbf{n}(B) \tag{98}$$

and that $\mathbf{n}(A)$ equals the maximal eigenvalue of A.

Let ε_k be the standard representation of $\mathfrak{g}$ on $\Lambda^k \mathbb{C}^N$. Let $X \in \mathfrak{g}$ and let $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_N$ be the spectrum of a Hermitian matrix $-iX$. This means that $\|X\| = i\,\mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N)$.

Eigenvalues of $-i\varepsilon_k(X)$ are of the form $\lambda_{i_1} + \lambda_{i_2} + \cdots + \lambda_{i_k}$ where $i_1 < i_2 < \ldots < i_k$. Therefore, the maximal eigenvalue of $-i\varepsilon_k(X)$ is $\lambda_1 + \lambda_2 + \ldots + \lambda_k$, i.e.

$$\mathbf{n}(-i\varepsilon_k(X)) =< \|X\|, e_k > .$$

As follows from (98),

$$\mathbf{n}(-i\varepsilon_k(X+Y)) \le \mathbf{n}(-i\varepsilon_k(X)) + \mathbf{n}(-i\varepsilon_k(Y)),$$

hence

$$< \|X+Y\|, e_k > \le < \|X\|, e_k > + < \|Y\|, e_k >,$$

as was required. □

10.0.3

Let $[a,b] \subset \mathbb{R}, a \le b$, be a segment. Let $g \in \mathrm{SU}(N)$. Write $g \sim [a,b]$ if every eigenvalue of g is of the form $e^{i\phi}$, where $\phi \in [a,b]$.

Lemma 10.3 *Let $g_k \sim [a_k, b_k]$, $k = 1, 2$. Then $g_1 g_2 \sim [a_1 + a_2, b_1 + b_2]$.*

Proof If $b_1 + b_2 - (a_1 + a_2) \ge 2\pi$, there is nothing to prove, because $x \sim [a_1 + a_2, b_1 + b_2]$ for any element $x \in \mathrm{SU}(N)$. Let now $b_1 + b_2 - (a_1 + a_2) < 2\pi$. Let $c_k = (a_k + b_k)/2$ and $d_k = (b_k - a_k)/2$. We have $d_1 + d_2 < \pi$, hence $d_k < \pi$, $k = 1, 2$.

Let $h_k = e^{-ic_k} g_k$. We have $h_k \sim [-d_k, d_k]$. Let $S \subset \mathbb{C}^N$ be the unit sphere. Let ρ be the standard metric on S; $\rho(v,w) = \arccos \mathrm{Re} < v, w >$; $\rho(v,w) \in [0,\pi]$. For $g \in \mathrm{SU}(N)$, set

$$\mathbf{N}(g) := \max_{v \in S} \rho(gv, v).$$

It follows $\mathbf{N}(g_1 g_2) \le \mathbf{N}(g_1) + \mathbf{N}(g_2)$ for all $g_1, g_2 \in \mathrm{SU}(N)$.

Let us estimate $\mathbf{N}(h_k)$. Let $e_1, e_2, \ldots, e_N$ be an eigenbasis of h_k. We have $h_k(e_s) = e^{i\alpha_{ks}} e_s$, where $\alpha_{ks} \in [-d_k, d_k]$. Let $v = \sum_s v_s e_s$, $v \in S$, so that $1 = \sum_s |v_s|^2$. We have

$$h_k v = \sum_s v_s e^{i\alpha_{ks}} e_s;$$

$$< h_k v, v >= \sum_s |v_s|^2 e^{i\alpha_{ks}};$$

$$\text{Re} < h_k v, v >= \sum_s |v_s|^2 \cos \alpha_{ks}.$$

As $\alpha_{ks} \in [-d_k, d_k]$ and $0 \le d_k < \pi$, we have $\cos \alpha_{ks} \ge \cos d_k$. Therefore,

$$\text{Re} < h_k v, v >\ge \sum_s |v_s|^2 \cos d_k = \cos d_k.$$

Therefore,

$$\mathbf{N}(h_k) = \rho(h_k v, v) = \arccos \text{Re} < h_k v, v >\le d_k.$$

Therefore, $\mathbf{N}(h_1 h_2) \le \mathbf{N}(h_1) + \mathbf{N}(h_2) \le d_1 + d_2$. It then follows that $h_1 h_2 \sim [-d_1 - d_2; d_1 + d_2]$. Indeed, assuming the contrary, we have an eigenvalue $e^{i\phi}$ of $h_1 h_2$, where $d_1 + d_2 < |\phi| \le \pi$. let $h_1 h_2 v = e^{i\phi} v$, $|v| = 1$. We then have $\rho(h_1 h_2 v, v) = |\phi| > d_1 + d_2$, which is a contradiction.

Finally, we have $g_1 g_2 = e^{c_1+c_2} h_1 h_2$, which implies that

$$g_1 g_2 \sim [c_1 + c_2 - d_1 - d_2; c_1 + c_2 + d_1 + d_2] = [a_1 + a_2; b_1 + b_2].$$

□

10.0.4

Fix $b \in C_+^\circ$; $b < e_1/(100N)$. Here and below $\circ$ means the interior.

Lemma 10.4 *Let $X, Y \in \mathfrak{g}$ and $\|X\|, \|Y\| \le b$. Then $e^X e^Y = e^Z$, where $\|Z\| \le \|X\| + \|Y\|$.*

Proof We have

$$e_1 = ((N-1)/N, -1/N, -1/N, \dots, -1/N) = (1, 0, 0, \dots, 0) - 1/N(1, 1, \dots, 1).$$

Let $b = i\text{diag}(b_1, b_2, \dots, b_N)$. Since $b \in C_+^\circ$, we have $b_1 > b_2 > \cdots > b_N$. We have $< b, e_k ><< e_1/(100N), e_k >$ for all k. In particular, $b_1 =< b, e_1 >\le (N - 1/N) \cdot (1/100N) < 1/(100N)$; $b_1 + b_2 + \cdots + b_{N-1} = (b, e_{N-1}) \le< e_1, e_{N-1} > /(100N) = 1/(100N^2) < 1/(100N)$. As $\sum_k b_k = 0$, we have $b_N > -1/(100N)$. Thus, $\forall k$, $|b_k| \le 1/(100N)$. Let $\|X\| = i\text{diag}(X_1, X_2, \dots, X_N)$. As $\|X\| \le b$, $|X_k| \le 1/(100N)$.

Therefore, one has $e^X \sim [-1/(100N); 1/(100N)]$. Analogously, $e^Y \sim [-1/(100N); 1/(100N)]$. Lemma 10.3 implies that

$$e^X e^Y = [-2/(100N); 2/(100N)] = [-1/(50N); 1/(50N)].$$

Let $u_1, u_2, \dots, u_N$ be the eigenbasis for $e^X e^Y$. It then follows that $e^X e^Y = e^{i\phi_s} u_s$, where $|\phi_s| \le 1/(50N)$. We have $1 = \det(e^X e^Y) = e^{i\sum_s \phi_s}$. Therefore $\sum_s \phi_s = 2\pi n$, $n \in \mathbb{Z}$. However, $|\sum_s \phi_s| \le 1/50 < 2\pi$. Hence, $n = 0$ and $\sum_s \phi_s = 0$. Let Z be a skew-hermitian matrix defined by $Zu_s = i\phi_s u_s$. As $\sum_s \phi_s = 0$, $Z \in \mathfrak{su}(N) = \mathfrak{g}$. We also have $e^X e^Y = e^Z$. Let us prove that $\|Z\| \le \|X\| + \|Y\|$.

Let Λ_k (resp. ε_k) be the standard representation of $G = \mathrm{SU}(N)$ (resp. $\mathfrak{g} = \mathfrak{su}(N)$) on $\Lambda^k \mathbb{C}^N$. We then have

$$e^{\varepsilon_k(Z)} = e^{\varepsilon_k(X)} e^{\varepsilon_k(Y)}.$$

Let $\|Z\| = i\mathrm{diag}(Z_1, Z_2, \dots, Z_N)$. As was shown above, we have $|Z_j| \le 1/(50N)$.

We then see that the spectrum of $\varepsilon_k(Z)$ consists of all numbers of the form

$$i(Z_{j_1} + Z_{j_2} + \cdots Z_{j_k}),$$

where $j_1 < j_2 < \cdots < j_k$. We have

$$|Z_{j_1} + Z_{j_2} + \cdots Z_{j_k}| \le k/(50N) \le 1/50. \tag{99}$$

Let $\|X\| = i\mathrm{diag}(X_1, X_2, \dots, X_N)$. the spectrum of $e^{\lambda_k(X)}$ consists of numbers of the form

$$e^{i(X_{j_1} + X_{j_2} + \cdots + X_{j_k})},$$

where $j_1 < j_2 < \ldots < j_k$. Therefore

$$e^{\lambda_k(X)} \sim [X_{N-k+1} + X_{N-k+2} + \cdots + X_N; X_1 + X_2 + \cdots + X_k].$$

We have $X_{N-k+1} + X_{N-k+2} + \cdots + X_N = -(X_1 + X_2 + \cdots + X_{N-k}) = - < X, e_{N-k} >$. Therefore,

$$e^{\lambda_k(X)} \sim [- < \|X\|, e_{N-k} >; < \|X\|, e_k >].$$

Analogously,

$$e^{\lambda_k(Y)} \sim [- < \|Y\|, e_{N-k} >; < \|Y\|, e_k >].$$

By Lemma 10.3, we have

$$e^{\lambda_k(Z)} = e^{\lambda_k(X)} e^{\lambda_k(Y)} \sim [- < \|X\| + \|Y\|, e_{N-k} >; < \|X\| + \|Y\|, e_k >].$$

As was shown above, we have $|X_j|, |Y_j| \le 1/(100N)$ for all j. Therefore, $| < \|X\|, e_{N-k} > | \le (N-k)/(100N) < 1/100$. Analogously

$$| < \|X\|, e_k > |, | < \|Y\|, e_k > |, < \|Y\|, e_{N-k} < 1/100.$$

Therefore

$$[- < \|X\| + \|Y\|, e_{N-k}; < \|X\| + \|Y\|, e_k >] \subset [-1/50; 1/50].$$

According to (99), all eigenvalues of $\lambda_k(Z)$ are of the form it, $|t| \leq 1/50$. It now follows that all eigenvalues of $\lambda_k(Z)$ are of the form it, where

$$t \in [- < \|X\| + \|Y\|, e_{N-k} >; < \|X\| + \|Y\|, e_k >].$$

(otherwise, $e^{i\lambda_k(Z)}$ is not of the form e^{it}, where $t \in [- < \|X\| + \|Y\|, e_{N-k} >; < \|X\| + \|Y\|, e_k >]$, as follows from our estimates). In particular,

$$< \|Z\|, e_k >= Z_1 + Z_2 + \cdots + Z_k \in [- < \|X\| + \|Y\|, e_{N-k}; < \|X\| + \|Y\|, e_k >],$$

whence

$$< \|Z\|, e_k > \leq < \|X\| + \|Y\|, e_k > .$$

As k is arbitrary, it follows that $\|Z\| \leq \|X\| + \|Y\|$. □

For our future purposes we will need a stronger result.

10.0.5

Lemma 10.5 *Let $X_1, X_2, \ldots X_n \in \mathfrak{g}$; $\|X_i\| \leq b$. Let $V_1 \subset V_2 \subset \cdots V_r = \mathbb{C}^N$ be a flag which is preserved by all X_i. Then there exists an $X \in \mathfrak{g}$ such that:*
(1) $e^{X_1}e^{X_2} \cdots e^{X_n} = e^X$;
(2) $XV_k \subset V_k$ and $TrX|_{V_k} = \sum_k TrX_k|_{V_k}$ for all k;
(3) $\|X\| \leq \sum_k \|X_i\|$

Proof (1) Fix an Ad- invariant Hilbert norm $\mathbf{N}$ on $\mathfrak{g}$ (such an $\mathbf{N}$ is unique up-to a scalar multiple). It follows that $\mathbf{N}(Z) \leq \mathbf{N}(Y_1) + \mathbf{N}(Y_2)$, the equality being possible only if Y_1 and Y_2 are proportional with non-negative coefficient (indeed: $\mathbf{N}(Z)$ is the length of the geodesic from the unit to e^Z; $\mathbf{N}(Y_1) + \mathbf{N}(Y_2)$ is the length of a broken line, the equality is possible only if this broken line is actually a geodesic).

(2) Suppose $Y_1, Y_2 \in \mathfrak{g}$; $\|Y_1\|, \|Y_2\| \leq b$. According to Lemma 10.4 there exists a unique $Z := Z(Y_1, Y_2) \in \mathfrak{g}$; $\|Z\| \leq \|Y_1\| + \|Y_2\|$ such that $e^Z = e^{Y_1}e^{Y_2}$. We see that $e^Z V_k = V_k$, hence $(e^Z - \mathrm{Id})V_k \subset V_k$. We can express Z as a convergent series in powers of $e^Z - \mathrm{Id}$, therefore, $ZV_k \subset V_k$. The equality

$$\det e^Z|_{V_k} = \det e^{Y_1}|_{V_k} \det e^{Y_2}|_{V_k}$$

implies that $e^{\mathrm{Tr}Z|_{V_k}} = e^{\mathrm{Tr}(Y_1+Y_2)|_{V_k}}$. As $\|Z\| \leq 2b$, this implies that $\mathrm{Tr}Z|_{V_k} = \mathrm{Tr}(Y_1 + Y_2)|_{V_k}$.

(3) Let $(Y_1, Y_2, \cdots Y_n)$ be a sequence of elements $Y_i \in \mathfrak{g}$; $|Y_i| \leq b$. Let

$$S_k(Y_1, Y_2, \ldots, Y_n) := (Y_1, \ldots, Y_{k-1}, Z/2, Z/2, Y_{k+2}, \ldots, Y_n),$$

where $k = 1, 2, \ldots, n-1$, $Z = Z(Y_k, Y_{k+1})$ is as explained above.

Let $\mathcal{X} \subset \mathfrak{g}^n$ be the set consisting of all sequences of the form

$$S_{k_1} S_{k_2} \cdots S_{k_R}(X_1, X_2, \ldots, X_n)$$

for all R and all $k_1, k_2, \ldots, k_R$. Let μ be the infimum of $\mathbf{N}(Y_1) + \mathbf{N}(Y_2) + \cdots \mathbf{N}(Y_n)$ where $(Y_1, Y_2, \ldots, Y_n) \in \mathcal{X}$.

Let $(Y_1(k), Y_2(k), \ldots, Y_n(k)) \in \mathcal{X}$, $k = 1, 2, \ldots,$ be a sequence such that $\mathbf{N}(Y_1(k)) + \cdots \mathbf{N}(Y_n(k)) \to \mu$ as $k \to \infty$. As $|Y_i(k)| \leq b$, one can choose a convergent subsequence, hence without loss of generality, one can assume that our sequence converges:

$$\lim_{k \to \infty} Y_i(k) = Z_i.$$

Then for all $(Y_1, Y_2, \ldots, Y_n) \in X$,

$$\mathbf{N}(Y_1) + \cdots \mathbf{N}(Y_n) \geq \mathbf{N}(Z_1) + \cdots \mathbf{N}(Z_n).$$

Let us show that there exists $Z \in \mathfrak{g}$ such that each Z_i is proportional to Z with a non-negative coefficient. If not then there are $i < j$ such that

(1) for all $i < k < j$, $Z_k = 0$;

(2) Z_i and Z_j are not proportional to each other with a non-negative coefficient. Let $(Z_1', \ldots, Z_n') = T_{j-1} \cdots T_{i+1} T_i(Z_1, Z_2, \ldots, Z_n)$. We then have $\mathbf{N}(Z_1') + \cdots \mathbf{N}(Z_n') < \mathbf{N}(Z_1) + \cdots \mathbf{N}(Z_n)$. Hence there exists a k such that

$$\mathbf{N}(Y_1') + \cdots + \mathbf{N}(Y_n') < \mathbf{N}(Z_1) + \mathbf{N}(Z_2) + \cdots + \mathbf{N}(Z_n),$$

where

$$(Y_1', Y_2', \ldots, Y_n') = T_{j-1} \cdots T_i(Y_1(k), Y_2(k), \ldots, Y_n(k)).$$

But $(Y_1', Y_2', \ldots, Y_n') \in \mathcal{X}$, so we get a contradiction.

Thus all Z_i are proportional with non-negative coefficients. Let us now set $X = Z_1 + Z_2 + \ldots Z_n$. Such an X satisfies all the conditions. □

11 Appendix 2: Results From [1] on Functorial Properties of Microsupport

Although the results to be quoted here are proved in [1] for the bounded derived category, the same arguments work for the unbounded derived category, the proofs are therefore omitted.

11.0.1

Let $S \subset X$ be a subset and $x \in X$. Following [1] Definition 5.3.6, one can define subsets $N(S) \subset TX$ and $N^*(S) \subset T^*X$. As explained on p 228, these subsets can be characterized as follows. Let $x \in X$. A non-zero vector $\theta \in T_xX$ belongs to $N_x(S)$ if and only if, in a local chart near x, there exists an open cone γ containing θ and a neighborhood U of x such that $U \cap ((S \cap U) + \gamma) \subset S$.

One then defines $N_x^*(S) \subset T_x^*X$ as the dual cone to $N_x(S)$. Finally one sets $N(S) = \cup_x N_x S$; $N^*(S) = \cup_x N_x^*(S)$. If $S \subset X$ is a closed submanifold, then $N^*(S) = T_S^*(X)$.

Let now $x \in X$ and let U be a neighborhood of x. Suppose that $S \cap U$ is defined by an inequality $f > 0$ (or $f \geq 0$), where $f : U \to \mathbb{R}$ is a smooth function and $d_x f \neq 0$. In this case $N_x^*(S) = \mathbb{R}_{\geq 0} \cdot d_x f$.

For a subset $K \subset T^*X$ we set $K^a \subset T^*X$ to consist of all vectors ω such that $-\omega \in K$.

Proposition 11.1 ([1], Proposition 5.3.8) *Let X be a manifold, Ω an open subset and Z a closed subsets. Then $SS(\mathbb{K}_\Omega) = N^*(\Omega)^a$; $SS(\mathbb{K}_Z) = N^*(Z)$*

11.0.2

Proposition 11.2 ([1], Proposition 5.4.1) *Let $F \in D(X)$ and $G \in D(Y)$. Then*

$$SS(F \boxtimes G) \subset SS(F) \times SS(G).$$

(Note that since our ground ring is a field $\mathbb{K}$, the bifunctor $\boxtimes$ is exact).

11.0.3

Let $q_1 : X \times Y \to X$; $q_2 : X \times Y \to Y$ be the projections.

Proposition 11.3 ([1], Proposition 5.4.2) *Let $F \in D(X)$; $G \in D(Y)$. Then:*

$$SSR\underline{Hom}(q_2^{-1}G; q_1^{-1}F) \subset SS(F) \times SS(G)^a,$$

where $SS(G)^a \subset T^*Y$ consists of all points ω such that $-\omega \in SS(G)$.

11.0.4

Let $f : Y \to X$ be a morphism of manifolds. We have natural maps

$$(f^t) : T^*X \times_X Y \to T^*Y$$

and $f_\pi : T^*x \times_X Y \to T^*X$.

Thus, $T^*X \times_X Y$ is a correspondence between T^*X and T^*Y. Using this correspondence, one can transport sets from T^*Y to T^*X and vice versa. Indeed, given a subset $A \subset T^*Y$ one has a subset $f_\pi(f^t)^{-1}A \subset T^*X$. Given a subset $B \subset T^*X$, one has a subset $(f^t)f_\pi^{-1}(B) \subset T^*Y$.

Proposition 11.4 ([1], Proposition 5.4.4) *Let $f : Y \to X$ be a morphism of manifolds, $G \in D(Y)$, and assume f is proper on Supp(G). Then*

$$SS(Rf_*G) \subset f_\pi((f^t)^{-1}(SS(G))).$$

Observe that under the hypothesis of this Proposition, the natural map $Rf_!G \to Rf_*G$ is an isomorphism. Therefore, the Proposition remains true upon replacement of Rf_* with $Rf_!$.

11.0.5

Let $f : Y \to X$ be a morphism of manifolds and $A \subset T^*X$ a closed conic subset. We say that f is non-characteristic for A if $f_\pi^{-1}A \cap T_Y^*X \subset Y \times_X T_X^*X$. Here $T_Y^*X \subset T^*X \times XY$ is the kernel of (f^t) viewed as a linear map of vector bundles.

Proposition 11.5 ([1], Proposition 5.4.13) *Let $F \in D(X)$ and assume $f : Y \to X$ is non-characteristic for $SS(F)$. Then*

$$\textit{(i)}\quad SS(f^{-1}F) \subset (f^t)(f_\pi^{-1}(SS(F)));$$

(ii) The natural morphism $f^{-1}F \otimes \omega_{Y/X} \to f^!F$ is an isomorphism.

11.0.6

Proposition 11.6 ([1], Proposition 5.4.14) *Let $F, G \in D(X)$.*

(i) Assume $SS(F) \cap SS(G)^a \subset T_X^*X$. Then $SS(F \otimes G) \subset SS(F) + SS(G)$;

(ii) Assume $SS(F) \cap SS(G) \subset T_X^*X$. Then $SS(R\underline{Hom}(G, F) \subset SS(F) - SS(G)$.

11.0.7

We need a notion of Witney sum of two conic closed subsets $A, B \subset T^*X$. We will reproduce a definition in terms of local coordinates from [1] Remark 6.2.8 (ii).

Let (x) be a system of local coordinates on X, (x, ξ) the associated coordinates on T^*X. Then $x_o, \xi_o \in A\hat{+}B$ iff there exist sequences $\{(x_n, \xi_n)\}$ in A and $\{(y_n, \eta_n)\}$ in B such that $x_n \to x_o$, $y_n \to y_o$, $\xi_n + \eta_n \to \xi_o$, and $|x_n - y_n||\xi_n| \to 0$.

Proposition 11.7 ([1], Theorem 6.3.1) *Let* Ω *be an open subset of* X *and* $j : \Omega \to X$ *the embedding. Let* $F \in D(X)$. *Then* $SS(Rj_*F) = SS(F)\hat{+}N^*(\Omega)$; $SS(j_!F) \subset SS(F)\hat{+}N^*(\Omega)^a$.

11.0.8

Let $f : Y \to X$ be a morphism of manifolds and $A \subset T^*X$ be a closed conic subset. One can define a closed conic subset $f^\#(A) \subset T^*M$ ([1], Definition 6.2.3 (iv)).

Proposition 11.8 ([1], Corollary 6.4.4) *Let* $F \in D(X)$. *Then* $SS(f^{-1}F) \subset f^\#(SS(F))$.

In a particular case when f is a closed embedding, the set $f^\#(A)$ admits an explicit description in local coordinates [1], Remark 6.2.8, (i). That's the only case we will need.

Let (x', x'') be a system of local coordinates on X such that $Y = \{(x', 0)\}$. Then $(x_o''; x_o'') \in f^\#(A)$ iff there exists a sequence of points $(x_n', x_n'', \xi_n', \xi_n'') \in A$ such that $x_n' \to 0$; $x_n'' \to x_o''$; $\xi_n'' \to \xi_o''$, and $|x_n'||\xi_n'| \to 0$.

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A Microlocal Category Associated to a Symplectic Manifold

Boris Tsygan

In memory of Boris Vasilievich Fedosov and Moshé Flato

Abstract For a symplectic manifold subject to certain topological conditions a category enriched in A_∞ local systems of modules over the Novikov ring is constructed. The construction is based on the category of modules over Fedosov's deformation quantization algebra that have an additional structure, namely an action of the fundamental groupoid up to inner automorphisms. Based in large part on the ideas of Bressler-Soibelman, Feigin, and Karabegov, it motivated by the theory of Lagrangian distributions and is related to other microlocal constructions of a category starting from a symplectic manifold, such as those due to Nadler-Zaslow and Tamarkin. In the case when our manifold is a flat two-torus, the answer is very close to both the Tamarkin microlocal category and the Fukaya category as computed by Polishchuk and Zaslow.

1 Introduction

There are several ways to construct a category which is an invariant of a symplectic manifold. One is due to Fukaya and is based on Floer cohomology [11, 12]. A connection between the Fukaya theory and theory of constructible sheaves was established by Nadler and Zaslow [29, 30]. Another construction of a category starting from a symplectic manifold was carried out by Tamarkin [37, 38]. It is based on microlocal theory of sheaves on manifolds developed by Kashiwara and Schapira in [21].

B. Tsygan (✉)
Northwestern University, 2033 Sheridan Rd, B4, Evanston, IL 60208, USA
e-mail: b-tsygan@northwestern.edu

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In this paper we describe yet another construction. It is based on microlocal objects, as [37, 38] are. But instead of microlocal theory of sheaves we use asymptotics of pseudodifferential operators and Lagrangian distributions [15, 16], or rather their algebraic version described by deformation quantization [1, 9, 31, 32].

1.1 Motivation from Morse Theory

1.1.1 The Classical Morse Filtration

First recall that, given a function f on a C^∞ manifold X, one can study De Rham cohomology of X using a filtration of the sheaf $\mathbb{C}_X$ by subsheaves $\mathbb{C}_{X,t} = \mathbb{C}_{\{f(x)\geq t\}}$ for any real t. If f is a Morse function, the cohomology $H^\bullet(X, \mathbb{C}_{X,t}/\mathbb{C}_{X,t'})$ is described in terms of critical points of f.

1.1.2 The Filtered Local System of $\mathbb{K}$-Modules

The above can be interpreted as follows. Let

$$\Lambda = \left\{ \sum_{k=0}^{\infty} a_k \exp\left(\frac{1}{i\hbar} c_k\right) | a_k \in \mathbb{C};\, c_k \geq 0;\, c_k \to \infty \right\}$$

be the Novikov ring. Let $\mathbb{K}$ be its field of quotients which is defined the same way as Λ, with the condition $c_k \geq 0$ replaced by $c_k \in \mathbb{R}$. Consider the trivial $\mathbb{K}$-module of rank one and the corresponding constant sheaf $\mathbb{K}_X$ on X. Given a function f, consider the action of the fundamental groupoid $\pi_1(X)$ on $\mathbb{K}_X$ such that any class of a path $x_0 \to x_1$ acts by multiplication by $\exp(\frac{1}{i\hbar}(f(x_0) - f(x_1)))$.

For any real number t, denote by $C^\infty_{\Lambda^t, X}$ the sheaf associated to the presheaf of formal expressions

$$\left\{ \sum_{k=0}^{\infty} a_k \exp\left(\frac{1}{i\hbar} \varphi_k\right) | a_k \in C^\infty_X((\hbar));\, \varphi_k \in C^\infty_X;\, \varphi_k \geq t;\, \varphi_k \to \infty \right\} \quad (1.1.1)$$

Define $C^\infty_{\mathbb{K},X}$ the same way but without the condition $\varphi_k \geq t$. When $t = 0$, we denote $C^\infty_{\Lambda^t,X}$ by $C^\infty_{\Lambda,X}$.

The fundamental groupoid $\pi_1(X)$ acts on $C^\infty_{\mathbb{K},X}$ (the simple exact meaning of this statement is explained in Definition 6.18). Horizontal sections are of the form

$$\sum_k a_k \exp\left(\frac{1}{i\hbar}(c_k + f(x))\right) \quad (1.1.2)$$

where $a_k \in \mathbb{C}((\hbar))$, $c_k \in \mathbb{R}$, and $c_k \to \infty$. Now consider the sheaf $\mathcal{F}^t(f)$ of horizontal sections that are in $C^\infty_{\Lambda^t,X}$. Note that $\exp(\frac{1}{i\hbar}(c+f))$ is in $\mathcal{F}^t(f)$ on an open set if and only if $c \geq t - f$ on this open set. Therefore

$$H^\bullet(X, \mathcal{F}^t(f)) = \widehat{\oplus}_c H^\bullet(U_{c,t})((\hbar)) \tag{1.1.3}$$

where $U_{c,t}$ is the biggest open subset on which $c \geq t - f$. We see that this cohomology essentially contains all the information about the cohomology of X_t for various t. The symbol $\widehat{\oplus}$ denotes the completed direct sum, i.e. the space of infinite sums

$$\sum_{k=1}^{\infty} A_k, \;\; A_k \in H^\bullet(U_{c_k,t})((\hbar)), \;\; c_k \to \infty \tag{1.1.4}$$

1.1.3 The Twisted De Rham Complex

The language of local systems and of actions of the fundamental groupoid makes it natural to look at flat connections.

Definition 1.1 Denote by $\Omega^\bullet_{\mathbb{K},X}$, resp. $\Omega^\bullet_{\Lambda^t,X}$, resp. $\Omega^\bullet_{\Lambda,X}$, the sheaf of differential forms with coefficients in $C^\infty_{\mathbb{K},X}$, resp. $C^\infty_{\Lambda^t,X}$, resp. $C^\infty_{\Lambda,X}$.

Consider the twisted De Rham complex

$$(\Omega^\bullet_{\mathbb{K},X}, \; i\hbar d_{\mathrm{DR}} + df\wedge) \tag{1.1.5}$$

This complex is filtered by subcomplexes $\Omega^\bullet_{\Lambda^t,X}$. The fundamental groupoid acts on it preserving the differential (again, see Definition 6.18 for the exact meaning of this).

Now, for traditional local systems of finite dimensional vector spaces, locally, the cohomology of the De Rham complex is the same as the space of horizontal sections. The latter is (again, locally) the same as the derived space of horizontal sections, which is by definition the cohomology of the fundamental groupoid with coefficients in functions. In the context of $C^\infty_{\mathbb{K},M}$-valued forms, the first of these statements is false. In fact, the cohomology of the complex (1.1.5) is huge: regardless of f, it is the sum of cohomologies of $d_{\mathrm{DR}} + d\varphi\wedge$ for *all* φ. But if we consider the local double complex of cochains of the fundamental groupoid with coefficients in (1.1.5), we get the cohomology isomorphic to $\mathbb{K}$. This is easy to see. In fact, we can replace f by 0 in (1.1.5), since the two complexes are isomorphic by means of multiplication by $\exp(\frac{1}{i\hbar}f)$. The value of the local double complex on a coordinate chart U becomes

$$\mathcal{C}^{p,q} = \Omega^p_{\mathbb{K}}(U^{q+1})$$

for $p, q \geq 0$. There are two differentials: one is $d_{\mathrm{DR}} : \mathcal{C}^{p,q} \to \mathcal{C}^{p+1,q}$; the other is $\delta : \mathcal{C}^{p,q} \to \mathcal{C}^{p,q+1}$ where for $\omega \in \mathcal{C}^{p,q}$

$$\delta\omega = \sum_{j=0}^{q}(-1)^j p_j^*\omega \tag{1.1.6}$$

Here p_j is the projection $X^{q+1} \to X^q$ along the jth factor. But the differential δ admits a contracting homotopy

$$h\omega = i_0^*\omega \tag{1.1.7}$$

where $i_0(x_0, \ldots, x_{q-1}) = (0, x_0, \ldots, x_{q-1})$. More precisely, $[\delta, h] = \mathtt{Id} - r_0$ where $r_0 = 0$ for $q > 0$ or $p > 0$, and $r_0 a = a(0)$ for $p = q = 0$.

The sheaf associated to the presheaf of local complexes $\mathcal{C}^{\bullet,\bullet}$ inherits the action of the fundamental groupoid. The easiest way to express this is to say that, if

$$\mathcal{C}_x^{p,q} = \varinjlim_{x\in U} \mathcal{C}^{p,q}(U), \tag{1.1.8}$$

then there are operators

$$\pi_1(x, y) \times \mathcal{C}_y^{p,q} \to \mathcal{C}_x^{p,q} \tag{1.1.9}$$

that define an action. In a more general situation, when we start with a differential graded module $\mathcal{E}^\bullet$ over $\Omega^\bullet_{\mathbb{K},X}$ with a compatible action of $\pi_1(X)$, they define an A_∞ action. This is more or less the same for all practical purposes (cf. Sect. 6.1).

We summarize the above as follows. Starting from a function f we constructed a filtered differential graded module $\mathcal{E}^\bullet$ over $\Omega^\bullet_{\mathbb{K},X}$ with a compatible action of $\pi_1(X)$, namely the twisted De Rham complex (1.1.5). From that we passed to a filtered $\mathbb{K}_X$-module with an (a priori A_∞) action of $\pi_1(X)$. It is natural to call it an infinity local system of $\mathbb{K}$-modules. (Note that the complex is filtered but π_1 does not preserve the filtration). The goal of this paper is to generalize large parts of the above in the way that we explain next.

1.2 Lagrangian Submanifolds

1.2.1 Review of the Results

Let M be a symplectic manifold and L_0, L_1 its Lagrangian submanifolds. Under some topological assumptions that we will list below, we will construct an infinity-local system of $\mathbb{K}$-modules $\mathcal{C}^\bullet(L_0, L_1)$ on M. In examples, this infinity local system is often filtered. The precise topological conditions that guarantee it being filtered are given by Proposition 9.8. Complexes $\mathcal{C}^\bullet(L_0, L_1)$ have a structure of an A_∞ -category enriched in A_∞ local systems of $\mathbb{K}$-modules (we will develop this in detail in a subsequent work). When $M = T^*X$, $L_0 = \mathrm{graph}(0)$, and $L_1 = \mathrm{graph}(df)$, we recover the construction we discussed above (with some modification).

The topological conditions, most probably much too conservative for large parts of the construction, are as follows.

(1) The manifold M has an Sp^4-structure (cf. Sect. 12.3). In other words, for an almost complex structure compatible with ω, consider the first Chern class $c_1(M)$ of the tangent bundle viewed as a complex vector bundle. Then $2c_1(M)$ must be trivial in $H^2(M, \mathbb{Z}/4\mathbb{Z})$. An Sp^4 structure is a trivialization of $2c_1(M)$.

(2) The image of the pairing of the class of the symplectic form with the image of the Hurewicz morphism is zero: $\langle \pi_2(M), [\omega] \rangle = 0$.

(The properties of Lagrangian submanifolds that are usually considered in Fukaya theory, such as exactness, grading, and existence of a Spin structure, all make their appearance in our considerations, as well as in [38]. Their exact role will be discussed in a subsequent work).

The infinity local system will be constructed in several steps indicated below. The meaning of all the terms used will be explained later in the introduction and/or in the rest of the article. All steps are possible under some additional conditions.

(a) We will introduce a sheaf of algebras $\mathcal{A}_M$ with a flat connection on M. On this sheaf, the fundamental groupoid $\pi_1(M)$ will act up to inner automorphisms. Denote by $\mathcal{A}_M^\bullet$ the differential graded algebra of $\mathcal{A}_M$-valued forms, with the differential given by the connection.

(b) Consider two modules $\mathcal{V}$ and $\mathcal{W}$ over $\mathcal{A}_M$ with a compatible action of $\pi_1(M)$ and a compatible connection. Denote by $\mathcal{V}^\bullet$, $\mathcal{W}^\bullet$ the differential graded modules of forms with values in $\mathcal{V}$ or $\mathcal{W}$. Then the standard complex computing their Ext over $\mathcal{A}_M^\bullet$ has a structure of a $\Omega^\bullet_{\mathbb{K},M}$-module with a (twisted) A_∞ action of $\pi_1(M)$.

(c) Given an $\Omega^\bullet_{\mathbb{K},M}$-module with a (twisted) A_∞ action of $\pi_1(M)$, we will construct an infinity local system as in (1.1.8).

(d) To construct modules $\mathcal{V}$ as in b), note that we can start with an $\mathcal{A}_M$-module with a compatible connection and a compatible action of a bigger groupoid $\widetilde{\mathbf{G}}_M$ that maps onto $\pi_1(M)$ in such a way that the kernel of this map acts by inner automorphisms.

(e) Given a Lagrangian submanifold L, we notice that there exists a subgroupoid of $\widetilde{\mathbf{G}}_M|L$ on L, as well as an $\mathcal{A}_M|L$-module with a compatible connection and a compatible action of this subgroupoid. Now we can get an object as in (d) by an induction procedure.

We will now outline the steps (a)–(e) in more detail.

1.3 Deformation Quantization

1.3.1 The Twisted De Rham Complex, Deformation Quantization, and Ext Functors

The fact that the twisted De Rham complex can be interpreted in terms of homological algebra had been known for a long time. Namely, let $\mathcal{D}_\hbar(X)$ be the ring of C^∞ $\hbar$-differential operators, i.e. the subalgebra of all differential operators which is generated, in any local coordinate system, by $F(x_1, \ldots, x_n)$ for all functions F

and by $i\hbar\frac{\partial}{\partial x_j}$ for all j. Here $\hbar$ can be any nonzero number, but it is easy to modify this construction to make $\hbar$ a formal parameter (in which case $\mathcal{D}_\hbar(X)$ is the Rees ring [2]). The algebra $\mathcal{D}_\hbar(X)$ acts on the space of functions on X. Denote the corresponding module by V_0. Now note that a function f defines an automorphism of $\mathcal{D}_\hbar(X)$, namely the conjugation with $\exp(\frac{1}{i\hbar}f)$. When $\hbar$ is not a number but a formal parameter, it is not clear how to define $\exp(\frac{1}{i\hbar}f)$ but conjugation by it makes perfect sense. Namely, in any coordinate system it sends $F(x_1, \ldots, x_n)$ to itself for all F and $i\hbar\frac{\partial}{\partial x_j}$ to $i\hbar\frac{\partial}{\partial x_j} + \frac{\partial f}{\partial x_j}$ for all j. It can be easily shown that $\mathrm{Ext}^{\bullet}_{\mathcal{D}_\hbar}(V_0, V_f)$ can be computed by the twisted De Rham complex. When $\hbar$ is a nonzero number, this complex is of course isomorphic to the standard De Rham complex. When $\hbar$ is a formal parameter, this complex is

$$(\Omega^{\bullet}(X)[\hbar], i\hbar d_{\mathrm{DR}} + df\wedge) \tag{1.3.1}$$

When we formally invert $\hbar$ the cohomology of this differential becomes easier to compute because we can use the spectral sequence associated to the filtration by powers of $\hbar$. The first differential in this spectral sequence is $df\wedge$. When f has isolated nondegenerate critical points, the cohomology of this differential, and therefore the cohomology of the twisted De Rham complex, is concentrated in the top degree n and its dimension over the field $\mathbb{C}((\hbar))$ of Laurent series is equal to the number of critical points.

Now let $\mathbb{A}_M$ be a deformation quantization of $C^\infty(M)$ (cf. [1]; we recall the definitions in Sect. 3.2). When $M = T^*X$, there is the canonical deformation quantization that is a certain completion of $\mathcal{D}_\hbar(X)$. (Another, arguably more correct, deformation is a completion of the algebra of $\hbar$-differential operators on half-forms). The algebra $\mathbb{A}_M$ is a reasonable replacement of $\mathcal{D}_\hbar(X)$, although it is no longer an algebra over $\mathbb{C}[\hbar]$ but only over $\mathbb{C}[[\hbar]]$. In particular it does not allow any specialization at a nonzero number $\hbar$.

In mid-eighties, Boris Feigin suggested an idea based on the intuition from algebraic theory of $\mathcal{D}$-modules [2]. According to this idea, and to a subsequent work [3] of Bressler and Soibelman, one should associate to a Lagrangian submanifold L a sheaf of $\mathbb{A}_M$-modules $\mathbb{V}_L$ supported on L. Then $\mathrm{Ext}^{\bullet}(\mathbb{V}_{L_0}, \mathbb{V}_{L_1})$ should somehow be a first approximation for a more interesting theory, namely the Floer cohomology. The latter also sees intersection points of transversal Lagrangian submanifolds, but in a much subtler way. Those intersection points define cochains (not necessarily cocycles) of the Floer complex that are not of the same but of different degrees (given by the Maslov index). Furthermore, the differential in the Floer complex may send one such cochain to a linear combination of other points (in other words, there may be instanton corrections). The standard homological algebra seems to be unable to catch these effects.

Below we will outline several tools that, combined, seem to allow to construct a category some (but not all) of whose objects come from Lagrangian submanifolds and which is much closer to the Fukaya category than the bare category of $\mathbb{A}_M$-modules.

1.3.2 The Fedosov Construction

The work of Fedosov [9] provided a simple and very efficient tool for working with deformation quantization of symplectic manifolds. Recall that a local model for deformation quantization is the Weyl algebra $C^\infty(M)[[\hbar]]$ with the Moyal–Weyl product $*$. The key properties of this product are that it is $\mathrm{Sp}(2n, \mathbb{R})$-invariant and that

$$[\xi_j, x_k] = i\hbar\delta_{jk};\ [x_j, x_k] = [\xi_j, \xi_k] = 0.$$

The local model for the Fedosov construction is as follows. Start with the space $\widehat{\mathbb{A}}$ of power series in formal variables $\widehat{x}_j$, $\widehat{\xi}_j$, and $\hbar$, $1 \le j \le n$. Turn it into an algebra by introducing the Moyal–Weyl product. Now consider the algebra of $\widehat{\mathbb{A}}$-valued differential forms on the Darboux chart with coordinates x_j, ξ_j. This algebra is equipped with the differential given by formula

$$\nabla_{\mathbb{A}} = \sum_{j=1}^{n} \left(\left(\frac{\partial}{\partial x_j} - \frac{\partial}{\partial \widehat{x}_j} \right) dx_j + \left(\frac{\partial}{\partial \xi_j} - \frac{\partial}{\partial \widehat{\xi}_j} \right) d\xi_j \right) \tag{1.3.2}$$

(cf. also (3.1.1)). The cohomology algebra of this differential is the usual deformation quantization.

For a general symplectic manifold M, one replaces a deformation $\mathbb{A}_M$ with the algebra $\Omega^\bullet(M, \widehat{\mathbb{A}}_M)$ of $\widehat{\mathbb{A}}_M$-valued differential forms on M. Here $\widehat{\mathbb{A}}_M$ is the bundle of algebras with fiber $\widehat{\mathbb{A}}$. The differential on the algebra $\Omega^\bullet(M, \widehat{\mathbb{A}}_M)$ is a chosen Fedosov connection. On any local Darboux chart, this algebra is isomorphic to the one discussed in the previous paragraph.

Note that the usual intuition about flat connections does not work here. Namely, there is no action of the fundamental groupoid (monodromy) preserving this flat connection. In fact, even locally, the algebra of horizontal sections is not at all isomorphic to the fiber. This feature will change rather radically after a modification that we introduce next. Much of what follows is based on the idea suggested to the author by Alexander Karabegov: extend the work of Fedosov so that it will describe an asymptotic version of Maslov's theory of canonical operators and of Hörmander's theory of Lagrangian distributions (cf. [15, 16, 26]). Actually, the constructions below require nothing but a systematic introduction into deformation quantization of quantities of the form (1.3.3) below. They do however have very strong connections to [15, 16, 26]. We discuss these connections in Appendices (Sects. 12, 13, 15, and 17). Note that exponentials (1.3.3) were considered in deformation quantization since the introduction of the subject, in particular in [1, 7, 10].

1.3.3 The Extended Fedosov Construction

Let us start with a remark about what happens when one tries systematically to introduce into deformation quantization quantities of the form

$$\exp\left(\frac{1}{i\hbar}\varphi\right). \tag{1.3.3}$$

Let us do this at the level of the algebra of formal series $\widehat{\mathbb{A}}$. All such quantities where φ are power series starting with cubic terms become elements of a new algebra automatically as soon as one replaces $\widehat{\mathbb{A}}$ by a completion $\widehat{\widehat{\mathbb{A}}}$ (cf. Sect. 4.1). We interpret quantities (1.3.3) where φ are quadratic as elements of the 4-fold covering group $\mathrm{Sp}^4(2n, \mathbb{R})$ (see the remark below). To add elements (1.3.3) where φ is constant, we tensor our algebra by the Novikov field $\mathbb{K}$ (as in Sect. 1.1.2).

Remark 1.2 Here is an explanation of the presence of Sp^4 (cf. Sect. 12 for definitions). The Lie algebra of derivations of the algebra $\widehat{\mathbb{A}}$ has a subalgebra consisting of elements $\frac{1}{i\hbar}\,\mathrm{ad}(q(\widehat{x}, \widehat{\xi}))$ where q is a quadratic function. This Lie subalgebra is isomorphic to $\mathfrak{sp}(2n)$, and its action is the standard action by linear coordinate changes. Consider the $\widehat{\mathbb{A}}$-module $\mathbb{C}[[\widehat{x}, \hbar]][\hbar^{-1}]$ on which $\widehat{x}$ acts by multiplication and $\widehat{\xi}$ by $i\hbar\frac{\partial}{\partial \widehat{x}}$. On it, $\frac{1}{i\hbar}\widehat{x}_j\widehat{\xi}_k$ acts by $\widehat{x}_j\frac{\partial}{\partial \widehat{x}_k} + \frac{1}{2}\delta_{jk}$. Note that $\mathrm{ad}(\frac{1}{i\hbar}\widehat{x}_j\widehat{\xi}_k)$ form a basis of the subalgebra $\mathfrak{gl}(n)$ inside $\mathfrak{sp}(2n)$. We see that one can integrate the action of this Lie subalgebra on the module to an action of the group, put the most natural way to do this is to pass to the two-fold cover $\mathrm{ML}(n, \mathbb{R})$ consisting of pairs $\{(g, \zeta)|\,\det(g) = \zeta^2\}$. One cannot extend this group action to the full symplectic group. To achieve that, we will have to extend the module considerably. But the group containing $\mathrm{ML}(n)$ is not $\mathrm{Sp}(2n)$ but its universal two-fold cover $\mathrm{Mp}(2n)$. The group Sp^4 contains $\mathrm{Mp}(2n)$ as a normal subgroup with quotient $\mathbb{Z}/2\mathbb{Z}$. We pass to this bigger group because it behaves better with respect to Lagrangian subspaces. For example, if a symplectic manifold M has a real polarization, then M has an $\mathrm{Sp}^4(2n)$-structure but not necessarily an $\mathrm{Mp}(2n)$-structure. On a more basic level, the pre-image of $\mathrm{GL}(n, \mathbb{R})$ in $\mathrm{Sp}^4(2n, \mathbb{R})$ splits, i.e. is isomorphic to $\mathrm{GL}(n, \mathbb{R}) \times \mathbb{Z}/4\mathbb{Z}$.

Finally, we do not add elements (1.3.3) where φ are linear, for the following reason. Note that $\mathrm{ad}(\frac{1}{i\hbar}\widehat{\xi}_j) = \frac{\partial}{\partial \widehat{x}_j}$ and $\mathrm{ad}(\frac{1}{i\hbar}\widehat{x}_j) = -\frac{\partial}{\partial \widehat{\xi}_j}$. Exponentials of these operators should be shifts in formal variables $\widehat{x}_j$ and $\widehat{\xi}_j$. But such shifts do not act on power series. Instead, they should correspond to shifts acting from one fiber of the associated bundle of algebras to another. These shifts will be discussed in Sect. 1.4 below. One does not need to add them, they will act automatically as long as topological conditions (1), (2) from Sect. 1.2.1 are satisfied.

We get an algebra $\mathcal{A}$ containing $\widehat{\widehat{\mathbb{A}}}$, $\mathbb{C}[\mathrm{Sp}^4(2n)]$, and $\mathbb{K}$ as subalgebras. The associated bundle of algebras $\mathcal{A}_M$ carries a Fedosov connection $\nabla_{\mathcal{A}}$ that extends the one on $\widehat{\mathbb{A}}_M$. For all we know, the cohomology of the De Rham complex of this connection is huge. But the bundle of algebras $\mathcal{A}_M$ carries another structure that we are going to discuss next.

1.4 The Action of π_1 up to Inner Automorphisms

It turns out that, if conditions (1) and (2) from Sect. 1.2.1 are satisfied, the fundamental groupoid $\pi_1(M)$ *acts on the bundle of algebras* $\mathcal{A}_M$ *up to inner automorphisms*. The notion of such an action is defined in Sect. 5. Moreover, the Fedosov connection $\nabla_{\mathcal{A}}$ extends to a flat connection up to inner derivations compatible with this action (cf. Sect. 5.7.2).

All the requisite notions are well-known and go back to Grothendieck. The version that suits our purposes is developed here in Sect. 5. For the readers convenience we introduce these notions gradually, starting with the case of a group acting on an algebra, though the generality we need is that of a Lie groupoid acting on a sheaf of algebras. The Lie groupoid in question will be the fundamental groupoid or its extension by a bundle of Lie groups.

1.5 From an Action up to Inner Automorphisms to an A_∞ Local System

In Sect. 6 we explain that, given an action of a groupoid $\mathcal{G}$ on a sheaf of algebras $\mathcal{A}$ up to inner automorphisms and given two $\mathcal{A}$-modules $\mathcal{V}$ and $\mathcal{W}$ with a compatible action of the groupoid, the standard complex $\mathcal{C}^\bullet(\mathcal{V}, \mathcal{A}, \mathcal{W})$ that computes $\mathrm{Ext}^\bullet_{\mathcal{A}}(\mathcal{V}, \mathcal{W})$ carries a (*twisted*) A_∞ action of $\mathcal{G}$. We make a similar argument when $\mathcal{A}$ carries a flat connection up to inner derivations. (Twisted A_∞ actions are discussed in Sect. 16. They are needed because the action in Sect. 1.4 is continuous only locally).

Let $\mathcal{A}^\bullet_M$ be the sheaf of $\mathcal{A}_M$-valued forms on M. The above procedure starts with two differential graded $\mathcal{A}$-modules $\mathcal{V}^\bullet$, $\mathcal{W}^\bullet$ with compatible actions of $\pi_1(M)$ and produces the standard complex $\mathcal{C}^\bullet(\mathcal{V}^\bullet, \mathcal{A}^\bullet, \mathcal{W}^\bullet)$ which is a sheaf of $\Omega^\bullet_{\mathbb{K},M}$-modules with a compatible *twisted* A_∞ action of $\pi_1(M)$. Finally, for an open chart U in M, consider the double complex $\mathcal{C}^{\bullet,\bullet}(\mathcal{V}^\bullet, \mathcal{W}^\bullet)(U)$ where $\mathcal{C}^{p,q}(U)$ is the space of q-cochains of $\pi_1(U)$ with coefficients in the graded component $\mathcal{C}^p(\mathcal{V}^\bullet, \mathcal{A}^\bullet, \mathcal{W}^\bullet)$, as in the second part of Sect. 1.1.3. Let

$$\mathcal{C}^{\bullet,\bullet}_x = \varinjlim_{x\in U} \mathcal{C}^{\bullet,\bullet}(\mathcal{V}^\bullet, \mathcal{W}^\bullet)(U)$$

be the stalk at a point x. As we indicated in Sect. 1.1.3 (after (1.1.8)), these complexes form an A_∞ local system of $\mathbb{K}$-modules. We denote this local system by $\underline{\mathbb{R}\,\mathrm{HOM}}(\mathcal{V}^\bullet, \mathcal{W}^\bullet)$.

We sum up the construction up to this point in Sect. 8.

1.6 Objects Constructed from Lagrangian Submanifolds

We proceed to construct a differential graded module $\mathcal{V}_L$ as in Sect. 1.5 starting from a Lagrangian submanifold L. This is done using an induction procedure that is explained in Sect. 9, in particular in Sect. 9.2. In Sect. 10, we prove that the general construction, when applied to $M = \mathbb{R}^{2n}$, $L_0 = \text{graph}(0)$, and $L_1 = \text{graph}(df)$, reproduces the one in Sect. 1.1.2, with the one important distinction. Namely, the filtered A_∞ local system $\underline{\mathbb{R}\,\text{HOM}}(\mathcal{V}^\bullet_{L_0}, \mathcal{V}^\bullet_{L_1})$ whose construction is outlined above is a module over a trivial local system of differential graded algebras whose fiber is the algebra

$$\mathcal{S}^\bullet = C^\bullet(\text{MPar}(n), \mathbb{K}) \tag{1.6.1}$$

of cochains of the group $\text{MPar}(n)$ with coefficients in the Novikov field $\mathbb{K}$. Here $\text{MPar}(n)$ is the parabolic subgroup of the group $\text{Sp}^4(2n)$ which is the pre-image of the stabilizer of the Lagrangian submanifold $\xi_1 = \cdots = \xi_n = 0$ in $\text{Sp}(2n)$. We prove that the general construction outlined in Sect. 1.5 is the tensor product of $\mathcal{S}^\bullet$ by the filtered local system described in Sect. 1.1.2.

Remark 1.3 There probably exists a correct way of factoring out the maximal ideal of $\mathcal{S}^\bullet$ and in particular recovering the exact answer as in Sect. 1.1.2. Note that the algebra $\mathcal{S}^\bullet$ plays a vital role in the computation in Sect. 10. Namely, the vanishing of the cohomology of $\text{MPar}(n)$ with coefficients in a certain class of modules leads to a vanishing result for all components involving a factor $\exp(\frac{1}{i\hbar}\varphi(x, \widehat{x}))$ where the quadratic part of φ with respect to $\widehat{x}$ is nonzero. Cf. Lemma 10.7, Corollary 10.8 (which we interpret as stationary phase statements of some sort).

1.6.1 The Example of a Two-Dimensional Torus

In Sect. 11, we compute $\underline{\mathbb{R}\,\text{HOM}}(\mathcal{V}^\bullet_{L_0}, \mathcal{V}^\bullet_{L_m})$ where $M = \mathbb{R}^2/\mathbb{Z}^2$, $L_0 = \{\xi = 0\}$, and $L_m = \{\xi = mx\}$. The answer is the trivial bundle whose fiber is the space of matrices indexed by $k, \ell \in \mathbb{Z}$ with coefficients in $\mathcal{S}^\bullet$. If γ_1, γ_2 are the two generators of the fundamental group $\pi_1(M) \xrightarrow{\sim} \mathbb{Z}^2$, then the action of $\pi_1(M)$ on the matrix units $\mathbf{E}_{k\ell}$ is given by

$$\gamma_1^q \gamma_2^p : \mathbf{E}_{k\ell} \mapsto \exp\left(\frac{1}{i\hbar}\left(\frac{mq^2}{2} + q(\ell - k)\right)\right)\mathbf{E}_{k+p,\ell+p-mq}$$

As a consequence (Corollary 11.3), horizontal sections of this local system have the same algebraic expression as theta functions. This agrees with the computation of the Fukaya category of M given by Polishchuk and Zaslow in [34].

1.7 *Microlocal Category of Sheaves*

1.7.1 The Microlocal Category of Tamarkin

In [37], Tamarkin defined the category $D(T^*X)$ for a manifold X. This is a full subcategory of the differential graded category of complexes of sheaves on $X \times \mathbb{R}$. Below are the key properties of the differential graded category $D(T^*X)$.

(1) For $c \geq 0$, there is a natural transformation $\tau_c : \mathtt{Id} \to (T_c)_*$ where, for $(x, t) \in X \times \mathbb{R}$, $T_c(x, t) = (x, t + c)$. One has $\tau_c \tau_{c'} = \tau_{c+c'}$. Define

$$\mathrm{HOM}(\mathcal{F}, \mathcal{G}) = \prod_{c \geq 0}{}' \mathbb{R}\,\mathrm{Hom}(\mathcal{F}, (T_c)_* \mathcal{G})$$

where $\prod'$ is the subset of the direct product consisting of all elements (v_c) such that $v_c = 0$ for all but countably many $c_k, k = 1, 2, \ldots$, satisfying $c_k \to \infty$. Then $\mathrm{HOM}(\mathcal{F}, \mathcal{G})$ is a complex of modules over the Novikov ring $\Lambda_{\mathbb{Z}} = \{\sum_{k=0}^{\infty} a_k e^{-\frac{c_k}{i\hbar}}\}$ where $a_k \in \mathbb{Z}$, $c_k \in \mathbb{R}$, $c_k \geq 0$, and $c_k \to \infty$.

Remark 1.4 For a general sheaf $\mathcal{F}$ there is no relation between its behavior on an open subset U and on the shift of U by c in the t direction. But Tamarkin's subcategory has a remarkable property that the natural transformation τ_c exists. A key example is provided by sheaves $\mathcal{F}_f$ defined in the paragraph below.

(2) For every object $\mathcal{F}$ of $D(T^*X)$, a closed subset $\mu S(\mathcal{F})$ is defined, called the microsupport of $\mathcal{F}$. Let f be a smooth function on X. Denote $\mathcal{F}_f = \mathbb{Z}_{\{t+f(x) \geq 0\}}$. Then $\mu S(\mathcal{F}_f) = \mathrm{graph}(df)$. (Observe that $T_c^* \mathcal{F}_f = \mathcal{F}_{f-c}$; the morphism $\tau_c : \mathcal{F} \to T_{c*}\mathcal{F}$ is the restriction to the subset $\{t - f - c \geq 0\}$ of $\mathbb{Z}_{\{t-f \geq 0\}}$).

(3) For a Morse function f, the complex $HOM(\mathcal{F}_0, \mathcal{F}_f)$ is quasi-isomorphic to the Morse complex of f.

(4) Let $\mathbf{T}^2$ be the standard 2-torus with the flat symplectic structure. One defines the category $D(\mathbf{T}^2)$ of objects of $D(T^*\mathbb{R}^1)$ equivariant under certain projective action of $\mathbb{Z}^2$. For every Lagrangian submanifold of $\mathbf{T}^2$ of the form $a\xi + bx = c$, a, b, c being integers, one constructs an object $\mathcal{F}_{a,b,c}$ of $D(\mathbf{T}^2)$. The full subcategory generated by these objects is isomorphic to the full subcategory of the Fukaya category generated by Lagrangian submanifolds $a\xi + bx = c$ as computed by Polishchuk–Zaslow in [34].

Remark 1.5 The category $D(\mathbf{T}^2)$ can be defined either as a partial case of the general construction [38] or by an explicit procedure that we recall in Sect. 11.3.

(5) *Theorem B*. Let Φ be a Hamiltonian symplectomorphism of T^*X which is equal to identity outside a compact subset. There exists a functor $T_\Phi : D(T^*X) \to D(T^*X)$ such that, if $\mu S(\mathcal{F})$ is compact, $\mu S(T_\Phi(\mathcal{F})) \subset \Phi(\mu S(\mathcal{F}))$. For every $\mathcal{F}$ and $\mathcal{G}$, $\mathrm{HOM}(\mathcal{F}, \mathcal{G})$ and $\mathrm{HOM}(\mathcal{F}, T_\Phi(\mathcal{G}))$ are isomorphic modulo $\Lambda_{\mathbb{Z}}$-torsion. Similarly for $\mathrm{HOM}(\mathcal{F}, \mathcal{G})$ and $\mathrm{HOM}(T_\Phi(\mathcal{F}), \mathcal{G})$.

(6) *Theorem A*. Let $\mathcal{F}$ and $\mathcal{G}$ be objects of $D(T^*X)$ such that $\mu S(\mathcal{F})$ and $\mu S(\mathcal{G})$ are compact and do not intersect. Then $\mathrm{HOM}(\mathcal{F}, \mathcal{G}) = 0$ modulo $\Lambda_{\mathbb{Z}}$-torsion.

For the sake of completeness, let us indicate how some of the above constructions are carried out. For a sheaf $\mathcal{F}$ on $X \times \mathbb{R}$, let $\mathrm{SS}(\mathcal{F})$ be its singular support as defined in [21]. Let $D(T^*X)$ be the left orthogonal complement to the subcategory of sheaves $\mathcal{G}$ such that $\mathrm{SS}(\mathcal{G})$ is contained in $\{\tau \leq 0\}$, where τ is the variable dual to the coordinate t on $\mathbb{R}$. The microsupport of an object $\mathcal{F}$ is defined by $\mu S(\mathcal{F}) = \{(x, \xi) \in T^*X | (x, \xi, t, 1) \in \mathrm{SS}(\mathcal{F}) \text{ for some } t \in \mathbb{R}\}$.

Tamarkin's current work [38] generalizes the construction of $D(T^*X)$ to any symplectic manifold M.

1.7.2 Comparisons Between the Categories

As we can see, many properties of the category $D(T^*X)$ are parallel to those of categories such as $\mathcal{A}_M^\bullet$-modules with an A_∞ action of $\pi_1(M)$. These include (1) (the second half), (3), and (4). Property (5) is very likely to hold. Properties (2) and (6) need further study (see next remark).

The following idea probably allows to construct a functor from $(\mathcal{A}_M, \pi_1(M))$-modules on T^*X satisfying some conditions to sheaves on $X \times \mathbb{R}$. For such a module $\mathcal{V}^\bullet$, assume that $\underline{\mathbb{R}\,\mathrm{HOM}}(\mathcal{V}_0^\bullet, \mathcal{V}^\bullet)$ is a *filtered* infinity local system as, for example, in Proposition 9.8 if the latter is true. Denote the filtration by Filt_a, $a \in \mathbb{R}$. Then the stalk at (x, t) of the sheaf corresponding to $\mathcal{V}^\bullet$ should be the Filt_t part of the complex that computes local cohomology of this infinity local system at x (cf. [20]).

Remark 1.6 Our source of defining $(\mathcal{A}_M, \pi_1(M))$-modules are *oscillatory modules*. (Their original version was defined in [40]). Oscillatory modules as defined here in Sect. 8.2 are actually complexes of sheaves. It is possible to relax the definition somewhat and only require them to carry a differential $\nabla_\mathcal{V}$ satisfying $\nabla_\mathcal{V}^2 = \frac{1}{i\hbar}\omega$ where ω is the symplectic form. (In other words, we can use the groupoid $\tilde{\mathbf{G}}_M$ as defined in Sect. 7.2.1 and not in Sect. 7.2.2). If we allow this, we seem to gain much more generality. For example, it will be much easier to construct an oscillatory module not only from a Lagrangian but from a coisotropic submanifold (as discussed in [18]) and maybe for more general submanifolds. On the other hand, it seems that the condition $\nabla_\mathcal{V}^2 = 0$ is indispensable (cf. Sect. 9.3.1) if one wants to define the microlocal support $\mu S(\mathcal{V}^\bullet)$ (the latter is a version of the support of the differential $\nabla_\mathcal{V}$). Cf., for example, an explicit formula for $\nabla_\mathcal{V}$ given by (9.4.5).

Remark 1.7 Much of the motivation behind our approach came from [43]. We do not know any rigorous link between the two works. It would be very interesting to relate our methods to the study of asymptotics of eigenvalues of the Schrödinger operator.

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2 $\mathbb{R}$ Hom and the Twisted De Rham Complex

2.1 Deformation Quantization Algebra

Put

$$\mathbb{A} = C^{\infty}(\mathbb{R}^{2n})[[\hbar]]$$

with the Moyal–Weyl product

$$(f * g)(x, \xi) = \exp\left(\frac{i\hbar}{2}\left(\frac{\partial}{\partial \xi}\frac{\partial}{\partial y} - \frac{\partial}{\partial x}\frac{\partial}{\partial \eta}\right)\right)(f(x, \xi)g(y, \eta))|_{x=y,\xi=\eta}$$

For a function $f(x)$ denote

$$\mathbb{V}_f = \mathbb{A}\Big/\sum_j \mathbb{A}\left(\xi_j - \frac{\partial f}{\partial x_j}\right)$$

or, in a simplified notation,

$$\mathbb{V}_f = \mathbb{A}/\mathbb{A}(\xi - f'(x))$$

Lemma 2.1 *As a* $\mathbb{C}[[\hbar]]$*-module,* $\mathbb{V}_f$ *is isomorphic to* $C^{\infty}(\mathbb{R}^n)[[\hbar]]$ *on which* x_j *acts by multiplication and* ξ_j *by* $i\hbar\frac{\partial}{\partial x_j} + \frac{\partial f}{\partial x_j}$.

2.2 The Complex Computing $\mathbb{R}$ Hom($\mathbb{V}_0$, $\mathbb{V}_f$)

Lemma 2.2 *The complex* $(\Omega^{\bullet}(\mathbb{R}^n)[[\hbar]], i\hbar d_{\mathrm{DR}} + df\wedge)$ *computes* $\mathrm{Ext}^{\bullet}_{\mathbb{A}}(\mathbb{V}_0, \mathbb{V}_f)$

Proof Fix a basis $e_1, \ldots, e_n$ of $\mathbb{C}^n$. Let $e^1, \ldots, e^n$ be the dual basis of $(\mathbb{C}^n)^*$. Let $\mathcal{R}_k = \mathbb{A} \otimes \wedge^k(\mathbb{C}^n)$. Define the differential

$$\partial(a \otimes e_{j_1} \wedge \ldots \wedge e_{j_k}) = \sum_{p=1}^{k}(-1)^p a\xi_{j_p} \otimes e_{j_1} \wedge \ldots \wedge \widehat{e_{j_p}} \wedge \ldots \wedge e_{j_k} \quad (2.2.1)$$

The complex $(\mathcal{R}_{\bullet}, \partial)$ is a free resolution of the module $\mathbb{V}_0$. The complex $\mathrm{Hom}_{\mathbb{A}}(\mathcal{R}_{\bullet}, \mathbb{V}_f))$ becomes

$$C^k = \wedge^k(\mathbb{C}^n)^* \otimes \mathbb{V}_f; \quad (2.2.2)$$

$$d(e^{j_1} \wedge \ldots \wedge e^{j_k} \otimes v) = \sum_{p=1}^{k} e^{j_1} \wedge \ldots \wedge e^{j_k} \wedge e_p \otimes \xi_p v \quad (2.2.3)$$

which is isomorphic to $(\Omega^\bullet(\mathbb{R}^n)[[\hbar]], i\hbar d_{\mathrm{DR}} + df\wedge)$ because of Lemma 2.1. □

3 The Weyl Algebra and the Fedosov Connection

3.1 The Case of $\mathbb{R}^{2n}$

Set

$$\widehat{\mathbb{A}} = \mathbb{C}[[\widehat{x}_1, \ldots, \widehat{x}_n, \widehat{\xi}_1, \ldots, \widehat{\xi}_n, \hbar]]$$

with the Moyal–Weyl product

$$(f * g)(\widehat{x}, \widehat{\xi}) = \exp\left(\frac{i\hbar}{2}\left(\frac{\partial}{\partial\widehat{\xi}}\frac{\partial}{\partial\widehat{y}} - \frac{\partial}{\partial\widehat{x}}\frac{\partial}{\partial\widehat{\eta}}\right)\right)(f(\widehat{x}, \widehat{\xi})g(\widehat{y}, \widehat{\eta}))|_{\widehat{x}=\widehat{y},\widehat{\xi}=\widehat{\eta}}$$

Define the operator on $\widehat{\mathbb{A}}$-valued forms by

$$\nabla_{\mathbb{A}} = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial\widehat{x}}\right)dx + \left(\frac{\partial}{\partial\xi} - \frac{\partial}{\partial\widehat{\xi}}\right)d\xi \tag{3.1.1}$$

This is the Fedosov connection (in the partial case of a flat space). One has $\nabla_{\mathbb{A}}^2 = 0$; the complex $(\Omega^\bullet(\mathbb{R}^{2n}, \widehat{\mathbb{A}}), \nabla_{\mathbb{A}})$ is quasi-isomorphic to $C^\infty(\mathbb{R}^{2n})[[\hbar]]$. The latter embeds quasi-isomorphically to the former by means of

$$f \mapsto f(x + \widehat{x}, \xi + \widehat{\xi}). \tag{3.1.2}$$

3.1.1 Infinitesimal Symmetries of the Deformation Quantization Algebra on a Formal Neighborhood

Let $\widehat{\mathbb{A}} = \mathbb{C}[[\widehat{x}_1, \ldots, \widehat{x}_n, \widehat{\xi}_1, \ldots, \widehat{\xi}_n, \hbar]]$ with the Moyal–Weyl product as in Sect. 2.1. Put

$$\mathfrak{g} = \mathtt{Der}_{\mathrm{cont}}(\widehat{\mathbb{A}}) = \frac{1}{i\hbar}\widehat{\mathbb{A}}/\frac{1}{i\hbar}\mathbb{C}[[\hbar]]; \quad \widetilde{\mathfrak{g}} = \frac{1}{i\hbar}\widehat{\mathbb{A}}$$

viewed as Lie algebras with the bracket $a * b - b * a$.

Introduce the grading

$$|\widehat{x}_i| = |\widehat{\xi}_i| = 1; \ |\hbar| = 2. \tag{3.1.3}$$

One has a central extension

$$0 \to \frac{1}{i\hbar}\mathbb{C}[[\hbar]] \to \widetilde{\mathfrak{g}} \to \mathfrak{g} \to 0, \tag{3.1.4}$$

as well as

$$\mathfrak{g} = \prod_{i=-1}^{\infty} \mathfrak{g}_i; \ \widetilde{\mathfrak{g}} = \prod_{i=-2}^{\infty} \widetilde{\mathfrak{g}}_i. \tag{3.1.5}$$

We will use the notation

$$\mathfrak{g}_{\geq 0} = \prod_{i=0}^{\infty} \mathfrak{g}_i; \ \widetilde{g}_{\geq 0} = \prod_{i=0}^{\infty} \widetilde{\mathfrak{g}}_i. \tag{3.1.6}$$

Note that

$$\mathfrak{g}_0 \xrightarrow{\sim} \mathfrak{sp}(2n) \tag{3.1.7}$$

and the action of this Lie algebra on $\widehat{\mathbb{A}}$ is the standard action of $\mathfrak{sp}$ by infinitesimal linear coordinate changes.

3.1.2 DG Model for $\mathbb{R}\,\mathrm{Hom}(\mathbb{V}_0, \mathbb{V}_f)$

Though this is not needed for the sequel, let us explain how modules $\mathbb{V}_f$ can be replaced by their DG analogs. Define

$$\Omega^\bullet(\mathbb{R}^{2n}, \widehat{\mathbb{V}}_f) = \Omega^\bullet(\mathbb{R}^n)[[\widehat{x}, \hbar]] \tag{3.1.8}$$

with the differential

$$\nabla_{\mathbb{V}} = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial \widehat{x}}\right) dx \tag{3.1.9}$$

and the action of $\Omega^\bullet(\mathbb{R}^{2n}, \widehat{\mathbb{A}})$ defined as follows: x and $\widehat{x}$ act by multiplication; ξ acts by multiplication by $f'(x)$; $\widehat{\xi}$ acts by $i\hbar\frac{\partial}{\partial \widehat{x}} + f'(x+\widehat{x}) - f'(x)$; $d\xi$ acts by $df'(x) = f''(x)dx$.

It is easy to see that $\Omega^\bullet(\mathbb{R}^{2n}, \widehat{\mathbb{A}})$ is the space of global sections of a sheaf of differential graded algebras, and $\Omega^\bullet(\mathbb{R}^{2n}, \widehat{\mathbb{V}}_f)$ is the space of global sections of a sheaf of differential graded modules supported on the Lagrangian submanifold $L_f = \{\xi = f'(x)\}$. The formula $v \mapsto v(x + \widehat{x})$ defines a quasi-isomorphic embedding

$$\mathbb{V}_f \to \Omega(\mathbb{R}^n, \widehat{\mathbb{V}}_f)$$

compatible with the embedding of algebras $C^\infty(\mathbb{R}^{2n})[[\hbar]] \to \Omega^\bullet(\mathbb{R}^{2n}, \widehat{\mathbb{A}})$ defined in (3.1.2).

Lemma 3.1 *Let e^*, $\widehat{e}^*$ and a^* be three free graded commutative variables of degrees* 1, 1, *and* 0 *respectively. The cohomology*

$$\mathbb{R}\,\mathrm{Hom}_{\Omega^\bullet(\mathbb{R}^{2n}, \widehat{\mathbb{A}})}(\Omega^\bullet(\mathbb{R}^{2n}, \widehat{\mathbb{V}}_0), \Omega^\bullet(\mathbb{R}^{2n}, \widehat{\mathbb{V}}_f))$$

is computed by the complex

$$\Omega^\bullet(\mathbb{R}^n, \widehat{\mathbb{V}}_f)[e^*, \widehat{e}^*][[a^*]], \nabla_{\mathbb{V}} + e^*\xi + \widehat{e}^*\widehat{\xi} + a^* d\xi + (e^* - \widehat{e}^*)\frac{\partial}{\partial a^*}$$

which is isomorphic to $\Omega^\bullet(\mathbb{R}^n)[[\widehat{x}, \hbar]][e^*, \widehat{e}^*][[a^*]]$ *with the differential*

$$\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial \widehat{x}}\right)dx + e^* f'(x) + a^* f''(x)dx + \widehat{e}^*\left(i\hbar\frac{\partial}{\partial \widehat{x}} + f'(x+\widehat{x}) - f'(x)\right) + (e^* - \widehat{e}^*)\frac{\partial}{\partial a^*}$$

The latter complex is quasi-isomorphic to the one in Lemma 2.2.

Proof The DG module $\Omega^\bullet(\mathbb{R}^n, \widehat{\mathbb{V}}_0)$ is the quotient of the free DG module $\Omega^\bullet(\mathbb{R}^{2n}, \widehat{\mathbb{A}})$ by the differential graded submodule generated by ξ, $d\xi$, and $\widehat{\xi}$. A Koszul complex $\mathcal{P} = \Omega^\bullet(\mathbb{R}^{2n}, \widehat{\mathbb{A}})[e, \widehat{e}, a]$ is a semi-free resolution of this quotient. The differential extends $\nabla_{\mathbb{A}}$, sends ev to $\xi v + av$, $\widehat{e}v$ to $-\widehat{\xi}v + av$, av to $d\xi \cdot v$, and is a coderivation with respect to the action of $\mathbb{C}[e, \widehat{e}, a]$. The complex $\mathrm{Hom}_{\Omega^\bullet(\mathbb{R}^{2n}, \widehat{\mathbb{A}})}(\mathcal{R}, \Omega^\bullet(\mathbb{R}^n, \widehat{\mathbb{V}}_f))$ is isomorphic to both complexes above. It remains to show that the latter of those complexes is quasi-isomorphic to $(\Omega^\bullet(\mathbb{R}^{2n})[[\hbar]], i\hbar d_{\mathrm{DR}} + df\wedge)$. To this end, consider the second complex in the statement of the lemma. Change the odd variables to e^* and $e^* - \widehat{e}^*$; note that we can factor out all positive powers of a^* and $e^* - \widehat{e}^*$. This is because the differential $(e^* - \widehat{e}^*)\frac{\partial}{\partial a^*}$ is acyclic. We are left with the complex $\Omega^\bullet(\mathbb{R}^n)[[\widehat{x}, \hbar]][e^*]$ with differential

$$\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial \widehat{x}}\right)dx + e^*\left(i\hbar\frac{\partial}{\partial \widehat{x}} + f'(x+\widehat{x})\right)$$

Now change the even variables. Put $y = x + \widehat{x}$ and keep $\widehat{x}$ as the second variable. As for the odd variables, put $Dx = dx - i\hbar e^*$ and keep e^* as the second variable. The differential becomes

$$\left(i\hbar\frac{\partial}{\partial y} + f'(y)\right)e^* - \frac{\partial}{\partial \widehat{x}}Dx.$$

We can factor out all positive powers of $\widehat{x}$ and of Dx because the differential $\frac{\partial}{\partial \widehat{x}}Dx$ is acyclic. □

3.2 Deformation Quantization of Symplectic Manifolds

We recall from [1] that a deformation quantization of a symplectic manifold M is a formal product

$$f * g = fg + \sum_{k=1}^{\infty}(i\hbar)^k P_k(f, g)$$

where $P : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ are bilinear bidifferential operators, $f * (g * h) = (f * g) * h$ in $C^\infty(M)[[\hbar]]$, $1 * f = f * 1 = f$, and

$$P_1(f, g) - P_1(g, f) = \{f, g\}.$$

An isomorphism between two deformation quantizations is a formal series

$$T(f) = f + \sum_{k=1}^{\infty} (i\hbar)^k T_k(f)$$

where $T(f) * T(g) = T(f *' g)$ and $T_k : C^\infty(M) \to C^\infty(M)$ are linear differential operators. Below we review how to classify deformation quantizations up to isomorphism using Fedosov connections.

3.3 The Bundle $\widehat{\mathbb{A}}_M$

By $\widehat{\mathbb{A}}_M$ we denote the bundle of algebras associated to the action of $\mathrm{Sp}(2n)$ on $\widehat{\mathbb{A}}$.

3.4 The Fedosov Connection

Definition 3.2 A Fedosov connection ∇ is a connection in the bundle of algebras $\mathcal{A}_M$ satisfying the following properties.

(1)
$$\nabla(fg) = \nabla(f)g + f\nabla(g)$$

for any local sections f and g of $\mathbb{A}_M$.

(2) $\nabla^2 = 0$

(3) In any local Darboux coordinates x, ξ on M and any formal Darboux coordinates $\widehat{x}, \widehat{\xi}$ of $\mathbb{A}$,

$$\nabla = d_{\mathrm{DR}} - \left(\frac{\partial}{\partial \widehat{x}} dx - \frac{\partial}{\partial \widehat{\xi}} d\xi\right) + A_{\geq 0}$$

where $A_{\geq 0}$ is a one-form with coefficients in $\mathfrak{g}_{\geq 0}$ (we use the notation of (3.1.6)).

Note that $\mathfrak{sp}(2n)$ embeds into $\widetilde{\mathfrak{g}}$ as the space of $\frac{1}{i\hbar} q(\widehat{x}, \widehat{\xi})$ where q is a quadratic polynomial.

Definition 3.3 A lifted Fedosov connection $\widetilde{\nabla}$ is a collection of $\widetilde{\mathfrak{g}}$-valued one-forms A_j on local Darboux charts U_j such that

(1)
$$A_j = -dg_{jk}g_{jk}^{-1} + \mathrm{Ad}(g_{jk})A_k$$

for any j and k.

(2) ∇^2 is central.

(3) In any local Darboux coordinates x, ξ on M and any formal Darboux coordinates $\widehat{x}, \widehat{\xi}$ of $\mathbb{A}$,

$$\nabla = d_{\mathrm{DR}} - \frac{1}{i\hbar}\widehat{\xi}dx + \frac{1}{i\hbar}\widehat{x}d\xi + A_{\geq 0}$$

where $A_{\geq 0}$ is a one-form with coefficients in $\widetilde{\mathfrak{g}}_{\geq 0}$ (we use the notation of (3.1.6)).

Any lifted Fedosov connection $\widetilde{\nabla}$ defines a Fedosov connection ∇ via the projection $\widetilde{\mathfrak{g}} \to \mathfrak{g}$. In this case we call $\widetilde{\nabla}$ a lifting of ∇.

Let
$$G = \mathrm{Sp}(2n, \mathbb{R}) \ltimes \exp(\mathfrak{g}_{\geq 1}) \tag{3.4.1}$$

This group acts on $\mathbb{A}$ by automorphisms. Let G_M be the associated bundle of groups. It acts by automorphisms on the bundle of algebras $\mathbb{A}_M$.

Definition 3.4 Two Fedosov connections are gauge equivalent if they are conjugated by a section of G_M.

Theorem 3.5 *(1) For every*

$$\theta = \frac{1}{i\hbar}\omega + \sum_{j=0}^{\infty}(i\hbar)^j\theta_j$$

where θ_j are closed two-forms on M, there exists a lifted Fedosov connection $\widetilde{\nabla}$ such that $\widetilde{\nabla}^2 = \theta$.

(2) Any Fedosov connection has a lifting. Two Fedosov connections are gauge equivalent if and only if the curvatures of their liftings are cohomologous as $\frac{1}{i\hbar}\mathbb{C}[[\hbar]]$-valued two-forms. In particular, any Fedosov connection is locally gauge equivalent to the standard one.

(3) For any Fedosov connection, the kernel of $\nabla : \Omega^0_M(\mathbb{A}_M) \to \Omega^1_M(\mathbb{A}_M)$ is isomorphic to $C^\infty_M[[\hbar]]$ as a sheaf of algebras. Therefore any Fedosov connection defines a deformation quantization of M.

(4) Any deformation quantization comes from some Fedosov connection. Two deformation quantizations are isomorphic if and only if the corresponding Fedosov connections are gauge equivalent.

This is mostly contained in [9]. The complete proof can be found in [31]. See also [4].

4 The Extended Fedosov Construction

4.1 The Algebra $\mathcal{A}$

First consider a larger completion of the Weyl algebra. Recall that the assignment

$$|\widehat{x}_j| = |\widehat{\xi}_j| = 1; \ \ |\hbar| = 2 \tag{4.1.1}$$

turns $\widehat{\mathbb{A}}$ into a complete graded algebra

$$\widehat{\mathbb{A}} = \prod_{k=0}^{\infty} \widehat{\mathbb{A}}_k \tag{4.1.2}$$

Let $\widehat{\mathbb{A}}[\hbar^{-1}]_k$ be the space of elements of degree k in $\widehat{\mathbb{A}}[\hbar^{-1}]$.

Now define

$$\widehat{\widehat{\mathbb{A}}} = \left\{ \sum_{k=-N}^{\infty} a_k | a_k \in \widehat{\mathbb{A}}[\hbar^{-1}]_k \right\} \tag{4.1.3}$$

where N runs through all integers. The product is the usual Moyal–Weyl product.

Now let $\mathrm{Sp}^4(2n)$ be the group defined in Sect. 12.3 (in the case $N = 4$). This group acts on $\widehat{\widehat{\mathbb{A}}}$ through $\mathrm{Sp}(2n)$. Consider the cross product $\mathrm{Sp}^4(2n) \ltimes \widehat{\widehat{\mathbb{A}}}$.

Remark 4.1 Here and everywhere by cross products we will mean their completed versions. In other words, elements of the cross product are infinite sums $\sum g_k a_k$ where $g_k \in \mathrm{Sp}^4$, $a_k \in \mathbb{A}[\hbar^{-1}]$, and $|a_k| \to \infty$.

Definition 4.2

$$\mathcal{A} = \left\{ \sum_{k=0}^{\infty} a_k e^{\frac{1}{i\hbar} c_k} | a_k \in \mathrm{Sp}^4(2n) \ltimes \widehat{\widehat{\mathbb{A}}}; \ c_k \in \mathbb{R}; \ c_k \to \infty \right\}$$

Let $\mathcal{A}_\Lambda$ be defined exactly as above, but with an extra condition $c_k \geq 0$. We will sometimes write $\mathcal{A}_{\mathbb{K}}$ instead of $\mathcal{A}$.

Note that we view $\mathrm{Sp}^4(2n)$ as a *discrete* group.

4.1.1 The Novikov Ring

Define

$$\Lambda = \left\{ \sum_{k=0}^{\infty} a_k e^{\frac{1}{i\hbar} c_k} | a_k \in \mathbb{C}((\hbar)); \ c_k \in \mathbb{R}; c_k \geq 0; \ \ c_k \to \infty \right\} \tag{4.1.4}$$

$$\mathbb{K} = \left\{ \sum_{k=0}^{\infty} a_k e^{\frac{1}{i\hbar} c_k} | a_k \in \mathbb{C}((\hbar));\ c_k \in \mathbb{R};\ c_k \to \infty \right\} \tag{4.1.5}$$

Clearly, $\mathcal{A}$ is an algebra over $\mathbb{K}$.

4.2 The Bundle $\mathcal{A}_M$

Since the action of $\mathrm{Sp}(2n)$ extends from $\widehat{\mathbb{A}}$ to $\mathcal{A}$, we get the associated bundle of algebras $\mathcal{A}_M$ on any symplectic manifold M.

4.3 The Extended Fedosov Connection

Note that the action of the Lie algebra $\widetilde{\mathfrak{g}}$ extends to an action on $\mathcal{A}$ and therefore any Fedosov connection $\nabla_{\mathbb{A}}$ extends canonically to a connection that we denote by $\nabla_{\mathcal{A}}$.

5 Action up to Inner Automorphisms

5.1 Groups Acting up to Inner Automorphisms

Definition 5.1 Let Γ be a group and A an associative algebra. An action of Γ on A up to inner automorphisms is the following data.

(1) Automorphisms $T_g : A \xrightarrow{\sim} A$ for all $g \in \Gamma$.

(2) Invertible elements $c(g_1, g_2)$ of A for all $g_1,\ g_2$ in Γ such that

$$T_{g_1} T_{g_2} = \mathrm{Ad}(c(g_1, g_2)) T_{g_1 g_2} \tag{5.1.1}$$

$$c(g_1, g_2) c(g_1 g_2, g_3) = T_{g_1} c(g_2, g_3) c(g_1, g_2 g_3) \tag{5.1.2}$$

An *equivalence* between (T, c) and (T', c') is a collection $\{b(g) \in A^{\times} | g \in G\}$ such that

$$T'_g = \mathrm{Ad}(b(g)) T_g;\ \ c'(g_1, g_2) = b(g_1) T_{g_1}(b_{g_2}) c(g_1, g_2) b(g_1 g_2)^{-1} \tag{5.1.3}$$

It $\{b'(g)\}$ is an equivalence between (T, c) and (T', c') and $\{b''(g)\}$ is an equivalence between (T', c') and (T'', c''), then their *composition* is defined by $b(g) = b''(g) b'(g)$ and is an equivalence between (T, c) and (T'', c'').

5.2 Derivations of Square Zero up to Inner Derivations

Definition 5.2 Let A be a graded algebra and let Γ be a group acting on A up to inner automorphisms. *A derivation of A of square zero up to inner derivations compatible with the action of* Γ is the following data.

(1) A derivation D of A of degree one;

(2) an element R of A of degree two;

(3) elements $\alpha(g)$ of A of degree one for every element g of Γ, such that

$$D^2 = \mathtt{ad}(R);\ \ DR = 0;\ \ T_g D T_g^{-1} = D + \mathtt{ad}(\alpha(g));$$

$$D\alpha(g) + \alpha(g)^2 = T_g R - R;$$

$$\alpha(g_1) + T_{g_1}\alpha(g_2) - \mathrm{Ad}(c(g_1, g_2))\alpha(g_1 g_2) + Dc(g_1, g_2) \cdot c(g_1, g_2)^{-1} = 0.$$

Now assume that we are given two sets of data: (T, c) with a compatible (D, α, R), and (T', c') with a compatible (D', α', R'). An equivalence

$$(T, c), (D, \alpha, R) \xrightarrow{\sim} (T', c'), (D', \alpha', R')$$

between them is an equivalence $\{b(g)\}$ between the actions and an element β of $\mathcal{A}$ of degree one such that

$$D' = D + \mathtt{ad}(\beta); \tag{5.2.1}$$

$$\alpha'(g) = -Db(g) \cdot b(g)^{-1} + \mathrm{Ad}_{b(g)}(\alpha(g) + T_g\beta); \tag{5.2.2}$$

$$R' = R + D\beta + \beta^2 \tag{5.2.3}$$

For two equivalences

$$(b'(g), \beta') : (T, c), (D, \alpha, R) \xrightarrow{\sim} (T', c'), (D', \alpha', R')$$

and

$$(b''(g), \beta'') : (T', c'), (D', \alpha', R') \xrightarrow{\sim} (T'', c''), (D'', \alpha'', R''),$$

their *composition* is an equivalence

$$(b(g), \beta) : (T, c), (D, \alpha, R) \xrightarrow{\sim} (T'', c''), (D'', \alpha'', R'')$$

given by

$$b(g) = b''(g)b'(g);\ \ \beta = \beta'' + \beta'. \tag{5.2.4}$$

Remark 5.3 A graded algebra with D and R as in (1) and (2) subject to the first two equations in (3) is called *a curved differential graded algebra* cf. [35]. In other words, this is an A_∞ algebra with the only nonzero operations being m_0, m_1, m_2.

Furthermore, $(T_g, \alpha(g))$ are *curved morphisms*, i.e. A_∞ morphisms with the only nonzero operations $T_0,\ T_1$.

5.2.1 Lie Algebras Acting up to Inner Derivations

The above is a partial case of the following definition (that is not used in the sequel).

Definition 5.4 Consider an action of a group Γ on an algebra A given by the data $T_g, c(g_1, g_2)$. Let $\mathcal{L}$ be a Lie algebra. An action of $\mathcal{L}$ on A up to inner derivations compatible with the action of Γ is the following data.

(1) A linear map $D : \mathcal{L} \to \mathtt{Der}(A)$, $X \mapsto D_X$;

(2) linear maps $\alpha : \mathcal{L} \to A$ for any $g \in \Gamma$, $X \mapsto \alpha_X(g)$.

(3) a bilinear skew symmetric map $R : \mathcal{L} \times \mathcal{L} \to A$, satisfying

$$[D_X, D_Y] = D_{[X,Y]} + \mathtt{ad}\, R(X, Y);$$

$$D_X(R(Y, Z)) + D_Y(R(Z, X)) + D_Z(R(X, Y)) =$$

$$= [D_X, D_{[Y,Z]}] + [D_Y, D_{[Z,X]}] + [D_Z, D_{[X,Y]}];$$

$$T_g D_X T_g^{-1} = D + \mathtt{ad}(\alpha_X(g));$$

$$D_X\alpha_Y(g) - D_Y\alpha_X(g) + [\alpha(X, g), \alpha(Y, g)] - \alpha_{[X,Y]}(g) = T_g R(X, Y) - R(X, Y);$$

$$\alpha_X(g_1) + T_{g_1}\alpha_X(g_2) - \mathrm{Ad}(c(g_1, g_2))\alpha_X(g_1g_2) + D_X c(g_1, g_2) \cdot c(g_1, g_2)^{-1} = 0.$$

More generally, let A be a graded algebra and $\mathcal{L}$ is a graded Lie algebra. The above definition makes sense with the following changes: $c(g_1, g_2)$ are of degree zero; R and α are homogeneous of degree zero; and signs are present in the formulas. Definition 5.2 describes a partial case when $\mathcal{L}$ is a one-dimensional graded Lie algebra concentrated in degree one.

5.3 *Modules with Compatible Structures*

For an algebra A with an action $(T_g,\ c(g_1, g_2))$ of a group G up to inner automorphisms and for an A-module V, a compatible action of G on V is a collection $\{T_g : V \to V | g \in G\}$ of module automorphisms such that $T_{g_1}T_{g_2} = c(g_1, g_2)T_{g_1g_2}$.

Given a graded algebra A and a graded module V as above, consider a derivation (D_A, α, R) of square zero of A up to inner derivations compatible with the action of G. A compatible derivation of V is a derivation $D_V : V^\bullet \to V^{\bullet+1}$ such that

$$D_V^2 = R;\ \ D_V(av) = D_A(a)v + (-1)^{|a|}aD_V(v);\ \ T_g D_V T_g^{-1} = D_V + \alpha(g) \tag{5.3.1}$$

for all homogeneous a in A and v in V.

Given an equivalence

$$(\{b(g)\}, \beta) : (T, c), (D, \alpha, R) \xrightarrow{\sim} (T', c'), (D', \alpha', R')$$

together with an action and a derivation on an A-module V compatible with (T, c), then

$$T'_g = b(g)T_g;\ \ D'_V = D_V + \beta \tag{5.3.2}$$

define on V an action and a derivation compatible with (T', c'). This operation is compatible with compositions of equivalences.

5.4 *Quotient Groups Acting up to Inner Automorphisms*

Assume given a surjection of groups $G \to \Gamma$ with kernel H. Assume that A is an associative algebra together with a G-equivariant morphism of groups $i : H \to A^\times$. Consider an action of G on A by automorphisms, $g \mapsto \mathbf{T}_g$. This is of course a partial case of Sect. 5.1 with $c(g_1, g_2) = 1$. We assume that $\mathbb{T}_g h = \mathrm{Ad}_{I(h)}$ for $h \in H$.

Choose a section of $G \to \Gamma$ sending $g \in \Gamma$ to $\overline{g} \in G$. Put

$$T_g = \mathbf{T}_{\overline{g}};\ \ c(g_1, g_2) = i(\overline{g}_1\overline{g}_2(\overline{g_1 g_2})^{-1}) \tag{5.4.1}$$

Furthermore, let $\mathbf{D}$, $\beta(g)$, $\mathbf{R}$ be a derivation of square zero up to inner derivations compatible with the action of G. Assume that

$$\beta(h) = -\mathbf{D}(ih)(ih)^{-1}$$

for all $h \in H$. Put

$$D = \mathbf{D};\ \ \alpha(g) = \beta(\overline{g});\ \ R = \mathbf{R} \tag{5.4.2}$$

Lemma 5.5 *(1) Formulas (5.4.2) define a derivation of square zero up to inner derivations compatible with the action of Γ given by (5.4.1). Given two different sections $s_1 : g \mapsto \overline{g}$ and $s_2 : g \mapsto \widetilde{g}$, formulas*

$$b(g) = i(\widetilde{g}\overline{g}^{-1});\ \ \beta = 0$$

define an equivalence $B(s_2, s_1)$ between corresponding derivations. One has

$$B(s_3, s_2)B(s_2, s_1) = B(s_3, s_1)$$

(2) Assume $(V, \mathbf{T}_g, \mathbf{D}_V)$ is an A-module with a compatible action of G and with a compatible derivation. Put $D_V = \mathbf{D}_V$; $T_g = \mathbf{T}_{\overline{g}}$. Then (V, T_g, D_V) is an A-module with a compatible action of Γ and a compatible derivation.

The proof is straightforward.

There is also an analog of the above Lemma for Lie algebra actions as in Sect. 5.2.1.

5.5 The Case of Groupoids

Now let G be a groupoid with the set of objects X. Let $A = \{A_x | x \in X\}$ be a family of algebras. *An action of G on A up to inner automorphisms* is the data consisting of operators $T_g : A_x \overset{\sim}{\leftarrow} A_y$ for all $g \in G_{x,y}$ and of invertible elements $c(g_1, g_2) \in A_x$ for all $g_1 \in G_{x_1,x_2}$ and $g_2 \in G_{x_2,x_3}$ such that (5.1.2) is true. We give the same definition for a family A of graded algebras where we require $c(g_1, g_2)$ to be of degree zero.

If $A = \{A_x\}$ is a family of graded algebras with an action of G up to inner derivations, *a derivation of square zero up to inner derivations compatible with the action* of A is a family of derivations $\{D_x : A_x \to A_x | x \in X\}$ and of elements $\{\alpha(g) \in A_{x_1} | x_1, x_2 \in X,\ g \in G_{x_1,x_2}\}$ such that

$$D_x^2 = \mathtt{ad}(R_x);\ \ D_x R_x = 0;\ \ T_g D_{x_2} T_g^{-1} = D_{x_1} + \mathtt{ad}(\alpha(g));$$

$$D\alpha(g) + \alpha(g)^2 = T_g R_{x_2} - R_{x_1};$$

$$\alpha(g_1) + T_{g_1}\alpha(g_2) - \mathrm{Ad}(c(g_1, g_2))\alpha(g_1 g_2) + D_{x_1} c(g_1, g_2) \cdot c(g_1, g_2)^{-1} = 0.$$

A similar definition can be given for a family of (graded) Lie algebras $\{\mathcal{L}_x | x \in X\}$.

Now consider a family of subgroups $\{H_x \in G_{x,x} | x \in X\}$, a groupoid Γ with the same set of objects X, and an epimorphism of groupoids $G \to \Gamma$ such that $H_x = \mathrm{Ker}(G_{x,x} \to \Gamma_{x,x})$. Let $\{i_x : H_x \to A_x^\times\}$ be a G-equivariant family of morphisms of groups. Choose a section $g \mapsto \overline{g}$ of $G \to \Gamma$.

Lemma 5.6 *(1) Given an action $\{\mathbf{T}_g\}$ of G on A with $c(g_1, g_2) = 1$, formulas* (5.4.1) *define an action of Γ on A up to inner automorphisms.*

(2) Given a derivation of square zero $(\mathbf{D}, \mathbf{R}, \beta)$ *up to inner derivations compatible with the action of G, assume that $\beta(h) = -\mathbf{D}i(h) \cdot i(h)^{-1}$ for all x and all $h \in H_x$ Then formulas* (5.4.2) *define a derivation of square zero up to inner derivations compatible with the action of Γ.*

(3) For two different choices of sections s_1, s_2, same formulas as in Lemma 5.5, (1), define an equivalence $B(s_2, s_1)$ between to derivations corresponding to two sections (s_1, s_2). One has

$$B(s_3, s_2)B(s_2, s_1) = B(s_3, s_1).$$

5.6 Modules with a Compatible Structure

For A and G as in Sect. 5.5, an A-module V with a compatible action of G is a collection $\{V_x | x \in X\}$ of A_x modules together with isomorphisms $\{T_g : V_x \stackrel{\sim}{\leftarrow} V_y | x, y \in X;\ g \in G_{x,y}\}$ satisfying

$$T_g(av) = T_g(a)T_g(v);\ T_{g_1}T_{g_2} = c(g_1, g_2)T_{g_1 g_2}$$

If A and V are graded and $(D_A, \alpha(g), R)$ is a compatible derivation of square zero up to inner derivations, a compatible derivation of V is a linear map $D_V : V^\bullet \to V^{\bullet+1}$ such that

$$D_V^2 = R;\ D_V(av) = D_A(a)v + (-1)^{|a|} a D_V(v);\ T_g D_V T_g^{-1} = D_V + \alpha(g)$$

for all homogeneous $a \in A_x,\ v \in V_x$.

There are analogs of Lemma 5.5 that we leave to the reader.

5.7 The Case of Lie Groupoids

5.7.1 Lie Groupoids: Notation and Conventions

Recall that a groupoid with a set of morphisms $\mathcal{G}$ and the set of objects M is a *Lie groupoid* [27] if $\mathcal{G}$ and M are (pro)manifolds and the source and target maps $s, t : \mathcal{G} \to M$ are smooth surjective submersions, and the composition, inverse, and the map $M \to \mathcal{G}$, $x \mapsto \mathtt{Id}_x$, are smooth.

For two points x_0 and x_1 of M, $\mathcal{G}_{x_0,x_1} = \{g \in \mathcal{G} | t(g) = x_0,\ s(g) = x_1\}$. This way, the composition is a map $\mathcal{G}_{x_0,x_1} \times \mathcal{G}_{x_1,x_2} \to \mathcal{G}_{x_0,x_2}$. If

$$\mathcal{G} \times_M \mathcal{G} = \{(g, g') \in \mathcal{G} \times \mathcal{G} | s(g) = t(g')\},$$

then the multiplication can be described as a map

$$m : \mathcal{G} \times_M \mathcal{G} \to \mathcal{G}.$$

We denote by $\underline{\mathcal{G}}$ the sheaf of (pro)manifolds on $M \times M$ defined by $\underline{\mathcal{G}}(W) = (s, t)^{-1}(W)$, $W \subset M \times M$.

More generally, we have the map

$$\mathrm{proj}_n : \mathcal{G} \times_M \cdots \times_M \mathcal{G} \to M \times \cdots \times M$$

where the product is n-fold on the left and $(n + 1)$-fold on the right. In particular, $\mathrm{proj}_1 = (s, t)$. Put

$$\underline{\mathcal{G}}^{(n)}(W) = \mathrm{proj}_n^{-1}(W) \tag{5.7.1}$$

This is a sheaf of pro-manifolds on M^{n+1}.

By $\mathcal{O}_M$ we denote a sheaf of (graded) algebras on M that could be C^∞_M, $\Omega^\bullet_M$, or the sheaf of Λ-valued forms or functions that we will consider later. All that we need is that $\mathcal{O}_M$ be defined for every manifold M (of given type) and that for every morphism $f : M \to N$ the inverse image $f^*\mathcal{O}_N$ be defined, together with the morphisms $f^{-1}\mathcal{O}_M \to f^*\mathcal{O}_M$ and $f^*\mathcal{O}_N \to \mathcal{O}_M$ subject to the usual identities.

By $p_j : M^{n+1} \to M$ we denote the projection onto the jth factor. Let $\mathcal{A}$ be a sheaf of $\mathcal{O}_M$-algebras.

Definition 5.7 An action of $\mathcal{G}$ on $\mathcal{A}$ up to inner derivations is a morphism of sheaves on $M \times M$

$$\underline{\mathcal{G}} \times p_2^*\mathcal{A} \to p_1^*\mathcal{A};\ \ (g, a) \mapsto T_g a$$

and a morphism of sheaves on $M \times M \times M$

$$c : \underline{\mathcal{G}}^{(2)} \to p_1^*\mathcal{A}$$

subject to

$$T_{g_1} T_{g_2}(a) = \mathrm{Ad}\, c(g_1, g_2) T_{g_1 g_2}(a)$$

in $p_1^*\mathcal{A}$, for any local section a of $p_3^*\mathcal{A}$ and any two local sections g_2 of $p_{23}^*\underline{\mathcal{G}}$ and g_1 of $p_{12}^*\underline{\mathcal{G}}$.

Remark 5.8 Given two local sections g_1, g_2 as above, by their composition we mean the following. If $g_1 = g_1(x_1, x_2, x_3) \in \mathcal{G}_{x_1,x_2}$ and $g_2 = g_2(x_1, x_2, x_3) \in \mathcal{G}_{x_2,x_3}$, then $(g_1 g_2)(x_1, x_2, x_3) = g_1(x_1, x_2, x_3) g_2(x_1, x_2, x_3)$ in $\mathcal{G}_{x_1,x_3}$. Similarly for $c(g_1, g_2)$.

5.7.2 Flat Connections up to Inner Derivations

Here we assume that the role of $\mathcal{O}_M$ as above is played by $\mathcal{O}^\bullet_M$, a differential graded algebra with a differential d. *A connection* on a sheaf of graded $\mathcal{O}^\bullet_M$-modules $\mathcal{E}$ is a morphism of sheaves $\nabla : \mathcal{E} \to \mathcal{E}$ of degree one such that $\nabla(ae) = da \cdot e + (-1)^{|a|} a \nabla e$.

We also assume that for every $f : M \to N$ and every sheaf of graded $\mathcal{O}^\bullet_N$-modules $\mathcal{E}$, a natural connection $f^*\nabla$ on $f^*\mathcal{E}$ is defined, subject to the usual properties. For us $\mathcal{O}^\bullet_M$ will be the sheaf of Λ-valued forms, and $f^*\nabla$ will be a straightforward analog of the standard inverse image of a connection that we will define in Sect. 8.1.

Definition 5.9 Let $\mathcal{A}^\bullet$ be a sheaf of graded $\mathcal{O}^\bullet_M$-algebras with an action of $\mathcal{G}$ up to inner automorphisms. A flat connection up to inner derivations compatible with the action of $\mathcal{G}$ is the following data.

(1) A connection $\nabla : \mathcal{A}^\bullet \to \mathcal{A}^{\bullet+1}$ which is a derivation.
(2) A section R of $\mathcal{A}^2$.
(3) A morphism of sheaves $\alpha : \underline{\mathcal{G}} \to p_1^*\mathcal{A}^\bullet$ of degree one, such that:

$$\nabla^2 = \mathrm{ad}(R);\ \ \nabla R = 0;\ \ T_g(p_2^*\nabla)T_g^{-1} = p_1^*\nabla + \mathrm{ad}(\alpha(g));$$

$$(p_1^*\nabla)\alpha(g) + \alpha(g)^2 = T_g(p_2^* R) - p_1^* R;$$

$$\alpha(g_1) + T_{g_1}\alpha(g_2) - \mathrm{Ad}(c(g_1, g_2))\alpha(g_1 g_2) + (p_1^*\nabla)c(g_1, g_2)\cdot c(g_1, g_2)^{-1} = 0.$$

We will often write $\alpha(g) = -\nabla g \cdot g^{-1}$.

5.8 Modules with a Compatible Structure: The Lie Groupoid Case

In the situation of Definition 5.9, let $(\mathcal{V}^\bullet, \nabla_{\mathcal{V}})$ be a differential graded $\mathcal{A}^\bullet$-module together with a morphism of sheaves $M \times M$

$$\underline{\mathcal{G}} \times p_2^*\mathcal{V} \to p_1^*\mathcal{V};\ \ (g, v) \mapsto T_g v$$

subject to:

$$T_{g_1}T_{g_2}(v) = c(g_1, g_2)T_{g_1 g_2}(v)$$

in $p_1^*\mathcal{V}^\bullet$, for any local section v of $p_3^*\mathcal{V}$ and any two local sections g_2 of $p_{23}^*\underline{\mathcal{G}}$ and g_1 of $p_{12}^*\underline{\mathcal{G}}$;

$$T_g(av) = T_g(a)T_g(v)$$

in $p_1^*\mathcal{V}$, for any local sections a of $p_2^*\mathcal{A}^\bullet$ and v of $p_2^*\mathcal{V}^\bullet$;

$$\nabla_{\mathcal{V}}^2 = R;\ \nabla_{\mathcal{V}}(av) = \nabla_{\mathcal{A}}(a)v + (-1)^{|a|}a\nabla_{\mathcal{V}}(v)$$

for any homogeneous local sections a of $\mathcal{A}^\bullet$ and v of $\mathcal{V}^\bullet$;

$$T_g(p_2^* D_V)T_g^{-1} = \pi_1^* D_V + \alpha(g)$$

5.8.1 The Action of the Quotient in the Lie Groupoid Case

Now consider two Lie groupoids $\mathcal{G}$ and Γ with the same manifold of objects M and an epimorphism of groupoids $\mathcal{G} \to \Gamma$ (over M.) Define $\mathcal{H}_x = \mathrm{Ker}(\mathcal{G}_{x,x} \to \Gamma_{x,x})$ and $\mathcal{H} = \cup_{x\in M}\mathcal{H}_x$. Consider the morphism : $\mathcal{H} \to M$. Define the sheaf of groups $\underline{\mathcal{H}}(U) = s^{-1}(U)$ for $U \subset M$. Let $i : \underline{\mathcal{H}} \to \mathcal{A}^\times$ be a $\mathcal{G}$-equivariant morphism of sheaves of groups. Choose a section $g \mapsto \overline{g}$ of $\mathcal{G} \to \Gamma$.

Lemma 5.10 *(1) Given an action* $\{\mathbf{T}_g\}$ *of* $\mathcal{G}$ *on* $\mathcal{A}$ *with* $c(g_1, g_2) = 1$, *formulas* (5.4.1) *define an action of* Γ *on* $\mathcal{A}$ *up to inner automorphisms.*

(2) Given a flat connection $(\mathbf{D}, \mathbf{R}, \beta)$ *up to inner derivations compatible with the action of* $\mathcal{G}$, *assume that* $\beta(h) = -\mathbf{D}i(h) \cdot i(h)^{-1}$ *for all local sections of* $\mathcal{H}$. *Then formulas*

$$\nabla = \mathbf{D};\ \ \alpha(g) = \beta(\overline{g});\ \ R = \mathbf{R}$$

define a flat connection up to inner derivations compatible with the action of Γ.

(3) For two different choices of sections s_1, s_2, *same formulas as in Lemma 5.5, (1), define an equivalence* $B(s_2, s_1)$ *between to derivations corresponding to two sections* (s_1, s_2). *One has*

$$B(s_3, s_2)B(s_2, s_1) = B(s_3, s_1).$$

(4) Let $\mathcal{V}$ *be a graded* $\mathcal{A}$*-module with a compatible action* $\mathbf{T}$ *of* $\mathcal{G}$ *and a compatible connection* $\mathbf{D}_{\mathcal{V}}$. *Then formulas*

$$T_g = \mathbf{T}_{\overline{g}};\ \ \nabla_{\mathcal{V}} = \mathbf{D}_{\mathcal{V}}$$

define a compatible action of Γ *and a compatible connection on* $\mathcal{V}$.

Remark 5.11 Note that the morphisms of sheaves $c : \underline{\Gamma}^{(2)} \to p_1^*\mathcal{A}$ and $\alpha : \underline{\Gamma} \to p_1^*\mathcal{A}$ are discontinuous. For us Γ will be an étale groupoid, more precisely the fundamental groupoid of M. We can only make a choice of a continuous c and α on any small coordinate chart, but that will be enough for our purposes. More precisely, this will define to a *twisted* A_∞ action as it is explained in Sect. 16.

6 From Actions up to Inner Automorphisms to A_∞ Actions

It is a well-known fact that inner isomorphisms act on the **Ext** functors trivially. Therefore, if a group acts on an algebra up to inner automorphisms, given compatible actions on two A-modules V and W, the group acts on the cohomology $\mathbf{Ext}_A^\bullet(V, W)$. In this section we prove a more precise version of this fact, namely we construct an A_∞ action of the group on the standard bar complex.

6.1 A_∞ Actions

An A_∞ action of a group G on a complex $C^\bullet$ is a collection $\{T(g_1, \ldots, g_n) \in \mathrm{Hom}^{1-n}(C^\bullet, C^\bullet) | g_1, \ldots, g_n \in G, n > 0\}$ satisfying

$$[d, T(g_1, \ldots, g_n)] + \sum_{j=1}^{n-1} (-1)^j T(g_1, \ldots, g_j) T(g_{j+1}, \ldots, g_n) - \tag{6.1.1}$$

$$\sum_{j=1}^{n-1}(-1)^j T(g_1,\ldots,g_jg_{j+1},\ldots,g_n)=0$$

We sometimes write T_g instead of $T(g)$. The operators $T(g)$ induce an action of G on the cohomology of $C^\bullet$.

An A_∞ morphism between two A_∞ actions T and T' is a collection $\{\phi(g_1,\ldots,g_n) \in \mathrm{Hom}^{-n}(C^\bullet, C^\bullet) | g_1,\ldots,g_n \in G, n\geq 0\}$ satisfying

$$[d,\phi(g_1,\ldots,g_n)]+\sum_{j=1}^{n-1}(-1)^j T'(g_1,\ldots,g_j)\phi(g_{j+1},\ldots,g_n)- \tag{6.1.2}$$

$$-\sum_{j=1}^{n-1}(-1)^j\phi(g_1,\ldots,g_j)T(g_{j+1},\ldots,g_n)-\sum_{j=1}^{n-1}(-1)^j\phi(g_1,\ldots,g_jg_{j+1},\ldots,g_n)=0$$

6.2 *The* Ext *Functors*

Let A be an associative algebra and $V,\ W$ two A-modules. By $C^\bullet(V,A,W)$, or simply $C^\bullet(V,W)$, we denote the standard complex computing $\mathbf{Ext}^\bullet{}_A(V,W)$. Namely,

$$C^m(V,W)=\prod_{p+n=m}\mathrm{Hom}(A^{\otimes n},\mathrm{Hom}^p(V,W));$$

the differential δ is defined by

$$(\delta\varphi)(a_1,\ldots,a_{n+1})=(-1)^{|\varphi||a_1|}a_1\varphi(a_2,\ldots,a_{n+1})+$$

$$\sum_{j=1}^{n}(-1)^{\sum_{i=1}^{j}(|a_i|+1)}\varphi(a_1,\ldots,a_ja_{j+1},\ldots,a_{n+1})+$$

$$(-1)^{\sum_{i=1}^{n+1}(|a_i|+1)}\varphi(a_1,\ldots,a_n)a_{n+1}$$

Lemma 6.1 *(1) Let T be an automorphism of A together with compatible automorphisms of V and W (i.e. invertible operators T such that $T(av)=T(a)T(v)$). Put*

$$(T\varphi)(a_1,\ldots,a_n)=T\varphi(T^{-1}a_1,\ldots,T^{-1}a_n)T^{-1}$$

Then $\varphi\mapsto T\varphi$ is an automorphism of $C^\bullet(V,W)$.
(2) For an invertible element c of A of degree zero define

$$(\phi(c)\varphi)(a_1, \dots, a_n) =$$

$$= -\sum_{j=0}^{n}(-1)^{\sum_{i=1}^{j}(|a_i|+1)}\varphi(a_1, \dots, a_j, c, c^{-1}a_{j+1}c, \dots, c^{-1}a_n c)c^{-1}$$

One has

$$[\delta, \phi(c)] = \mathrm{Ad}(c) - \mathtt{Id}$$

(3) More generally, for m invertible elements $c_1, \dots, c_m$ of degree zero of A, define

$$(\phi(c_1, \dots, c_m)\varphi)(a_1, \dots, a_n) =$$

$$= -\sum_{0\le j_1\le \dots j_m\le n}(-1)^{\sum_{k=1}^{m}\sum_{i=1}^{j_k}(|a_i|+1)}\varphi(a_1, \dots, a_{j_1}, c_1, c_1^{-1}a_{j_1+1}c_1, \dots, c_1^{-1}a_{j_2}c_1,$$

$$c_2, (c_1c_2)^{-1}a_{j_2+1}(c_1c_2), \dots, (c_1c_2)^{-1}a_{j_3}(c_1c_2), \dots,$$

$$c_m, (c_1\dots c_m)^{-1}a_{j_m+1}(c_1\dots c_m), \dots, (c_1\dots c_m)^{-1}a_n(c_1\dots c_m))(c_1\dots c_m)^{-1}$$

One has

$$[d, \phi(c_1, \dots, c_m)] + \mathrm{Ad}_{c_1}\phi(c_2, \dots, c_m)+$$

$$+\sum_{j=1}^{m-1}(-1)^j\phi(c_1, \dots, c_jc_{j+1}, \dots, c_m) + (-1)^m\phi(c_1, \dots, c_{m-1}) = 0$$

In other words: the group of automorphisms of (A, V, W) acts on $C^\bullet(V, A, W)$; the subgroup of inner automorphisms acts homotopically trivially, in the sense that there is an A_∞ morphism, starting with the identity, between this action and the trivial action. Note that, as in (1) above, we denote by Ad_c both the inner automorphism of A and the induced automorphism of $C^\bullet(V, A, W)$.

Lemma 6.2

$$\phi(c_1, \dots, c_m)\phi(d_1, \dots, d_n) = \sum \pm\phi(e_1, \dots, e_{n+m})$$

where the summation is over all $(e_1, \dots, e_{n+m})$ such that:

(a) as a set, $\{e_1, \dots, e_{n+m}\} = \{d_1, \dots, d_m, x_1c_1x_1^{-1}, \dots, x_nc_nx_n^{-1}\}$, with x_j defined below in (c);

(b) the order of elements d_j is preserved; the order of the elements $x_jc_jx_j^{-1}$ is the same as the order of the elements c_j;

(c) x_j is the product of all d_k^{-1} where d_k is to the left of $x_jc_jx_j^{-1}$.

For example,

$$\phi(c)\phi(d) = \phi(c, d) - \phi(d,\ d^{-1}cd)$$

6.2.1 A Lemma About A_∞ Actions

Lemma 6.3 *Let $\widetilde{G}$ be a group and H its normal subgroup. Let $G = \widetilde{G}/H$. Consider a complex $C^\bullet$ with the following data:*

(1) An action of $\widetilde{G}$, $g \mapsto \mathcal{T}_g$ for any $g \in \widetilde{G}$.

(2) Operators $\Phi(c_1, \ldots, c_m) : C^\bullet \to C^{\bullet - m}$, $m \geq 0$, for all $c_1, \ldots, c_m \in H$, satisfying

$$[d, \Phi(c_1, \ldots, c_m)] + \mathcal{T}_{c_1}\Phi(c_2, \ldots, c_m)+$$

$$+\sum_{j=1}^{m-1}(-1)^j \Phi(c_1, \ldots, c_j c_{j+1}, \ldots, c_m) + (-1)^m \Phi(c_1, \ldots, c_{m-1}) = 0$$

$$\Phi(c_1, \ldots, c_m)\Phi(d_1, \ldots, d_n) = \sum \pm \Phi(e_1, \ldots, e_{n+m})$$

as in Lemma 6.2. For any section $g \mapsto \overline{g}$ of the projection $\widetilde{G} \to G$, there is an A_∞ action of G on $C^\bullet$ such that $T_g = \mathcal{T}_{\overline{g}}$.

Proof Consider the differential graded algebra $\mathcal{B}(H, \widetilde{G})$ generated by the group algebra of $\widetilde{G}$ and by elements $\Phi(c_1, \ldots, c_m)$ of degree $-m$ for all $c_1, \ldots, c_m$ in H, such that:

(a)

$$g\Phi(c_1, \ldots, c_m)g^{-1} = \Phi(gc_1g^{-1}, \ldots, gc_mg^{-1})$$

for any $g \in \widetilde{G}$;

(b) the differential ∂ satisfies

$$\partial\Phi(c_1, \ldots, c_m) + c_1\Phi(c_2, \ldots, c_m)+$$

$$+\sum_{j=1}^{m-1}(-1)^j \Phi(c_1, \ldots, c_j c_{j+1}, \ldots, c_m) + (-1)^m \Phi(c_1, \ldots, c_{m-1}) = 0$$

(c)

$$\Phi(c_1, \ldots, c_m)\Phi(d_1, \ldots, d_n) = \sum \pm \Phi(e_1, \ldots, e_{n+m})$$

as in Lemma 6.2. This differential graded algebra is quasi-isomorphic to $k[G]$. In fact, as a complex it is the standard bar construction of H with coefficients in the right module $k[\widetilde{G}]$. The quasi-isomorphism is the morphism of algebras such that

$$\Phi(c_1, \ldots, c_m) \mapsto 0;\ g \mapsto \mathrm{proj}_G(g),\ g \in \widetilde{G}. \tag{6.2.1}$$

There is (unique up to homotopy) morphism from the standard resolution CobarBar $(k[G])$ to $\mathcal{B}(H, \widetilde{G})$ over $k[G]$. Now define $T(g_1, \ldots, g_n)$ to be the action of the image of the generator $(g_1|\ldots|g_n)$ on $C^\bullet$. □

6.2.2 The A_∞ Action on the Standard Complex

Now assume that a group G acts on an algebra A up to inner automorphisms. Assume that V and W are two A-modules with compatible actions. This means that there are linear automorphisms T_g of V and W for any $g \in G$ such that

$$T_g(av) = T_g(a)T_g(v);\ \ T_{g_1}T_{g_2} = c(g_1, g_2)T_{g_1 g_2} \tag{6.2.2}$$

($c(g_1, g_2)$ in the right hand side denotes the module action of the element of A).

Theorem 6.4 *There is an A_∞ action of G on $C^\bullet(V, A, W)$ such that $T(g)$ is equal to T_g as in Lemma 6.1.*

Proof Let $\widetilde{G} = G \ltimes_c A^\times$ be the group whose elements are expressions ag, $g \in G$ and $A^\times$, with the product

$$(a_1 g_1)(a_2 g_2) = a_1 T_{g_1}(a_2)c(g_1, g_2)(g_1 g_2) \tag{6.2.3}$$

and $H = A^\times$. The theorem follows immediately from Lemmas 6.1, 6.2, and 6.3. □

Remark 6.5 The proof of Theorem 6.4 actually leads to a rather simple recursive formula for the A_∞ action. Namely, the construction of a morphism

$$\mathrm{CobarBar}(k[G]) \to \mathcal{B}(A^\times, G \ltimes_c A^\times) \tag{6.2.4}$$

(see the proof of Lemma 6.3) is an inductive procedure in n for finding the image of $(g_1|\ldots|g_n)$ under this morphism. Let us describe this procedure. Consider the subalgebra $\mathcal{B}(A^\times, A^\times)$ of expressions $c_0\Phi(c_1, \ldots, c_m)$. This subalgebra is quasi-isomorphic to k, the homotopy being

$$s(c_0\Phi(c_1, \ldots, c_m)) = \Phi(c_0, c_1, \ldots, c_m) \tag{6.2.5}$$

Now define $\Psi(g_1, \ldots, g_n)$ in $\mathcal{B}(A^\times, A^\times)$ recursively by

$$\Psi_1(g) = g; \tag{6.2.6}$$

$$\Psi(g_1, \ldots, g_{n+1}) = \tag{6.2.7}$$

$$s\sum_{j=1}^{n}(-1)^j \Psi(g_1, \ldots, g_j)T_{g_1\ldots g_j}\Psi(g_{j+1}, \ldots, g_{n+1})c(g_1\ldots g_j, g_{j+1}\ldots g_{n+1})$$

Here the product is described in Lemma 6.2, and

$$T_g(c_0\Phi(c_1, \ldots, c_m)) = (T_g c_0\Phi(T_g c_1, \ldots, T_g c_m))$$

The elements $\Psi(g_1, \ldots, g_n)$ are some linear combinations of $\phi(c_1, \ldots, c_k)$ where c_j are algebraic expressions in $T_{h_0}c(h_1, h_2)$, h_i being some products of g_i.

Let $\psi(g_1, \ldots, g_n)$ be the image of $\Psi(g_1, \ldots, g_n)$ under the morphism of algebras

$$\mathcal{B}(A^{\times}, A^{\times}) \to \mathrm{End}(C^{\bullet}) \tag{6.2.8}$$

sending $g \in G$ to T_g, $c \in A^{\times}$ to $\mathrm{Ad}(c)$, and $\Phi(g_1, \ldots, g_n)$ to $\phi(g_1, \ldots, g_n)$. Then

$$T(g_1, \ldots, g_n) = \psi(g_1, \ldots, g_n) T_{g_1 \ldots g_n}$$

For example,

$$T(g_1, g_2) = \phi(c(g_1, g_2)) T_{g_1 g_2}$$

6.2.3 The Case of Groupoids

Let G be a groupoid with the set of objects X that acts on a family of algebras $A = \{A_x | x \in X\}$ up to inner automorphisms. Let $V = \{V_x | x \in X\}$ and $W = \{W_x | x \in X\}$ two A-modules with compatible actions of G, i.e. with families $T_g : V_x \stackrel{\sim}{\leftarrow} V_y$ and $T_g : W_x \stackrel{\sim}{\leftarrow} W_y$, satisfying (6.2.2).

Given a family of complexes $C^{\bullet} = \{C_x^{\bullet} | x \in X\}$, an A_∞ action of G on $C^{\bullet}$ is a collection of

$$T(g_1, \ldots, g_n) : C_{x_{n+1}}^{\bullet + 1 - n} \stackrel{\sim}{\leftarrow} C_{x_1}^{\bullet}$$

for any $g_j \in G_{x_j, x_{j+1}}$, $j = 1, \ldots, n$, satisfying the identities in the beginning of Sect. 6.1. Morphisms between A_∞ actions are defined similarly.

Define

$$C^{\bullet}(V, A, W)_x = C^{\bullet}(V_x, A_x, W_x)$$

Theorem 6.6 *There is an A_∞ action of G on $C^{\bullet}(V, A, W)$ such that $T(g)$ is equal to T_g as in Lemma 6.1.*

The proof is identical to the proof of Theorem 6.4.

6.2.4 A_∞ Action on the Standard Complex and Derivations

Let A be a graded algebra with an action of G up to inner automorphisms. Let D be a compatible derivation of square zero up to inner derivations. If V and W are two graded A-modules with compatible actions of G, we assume that both of them carry a compatible derivation, i.e. an operator $D : V \to V$ or $W \to W$ of degree one satisfying

$$D(av) = D(a)v + (-1)^{|a|} a D(v); \;\; D^2 = R; \;\; T_g D T_g^{-1} = D + \alpha(g) \tag{6.2.9}$$

Here R and $\alpha(g)$ stand for the action of corresponding elements of A. For any homogeneous derivation E of A that acts on V and W compatibly, put

$$(E\varphi)(a_1,\ldots,a_n)=[E,\varphi(a_1,\ldots,a_n)]- \tag{6.2.10}$$

$$-\sum_{j=1}^{n}(-1)^{\sum_{i=1}^{j-1}|E|(|a_i|+1)}\varphi(a_1,\ldots,Ea_j,\ldots,a_n)$$

Put for any homogeneous element a of A put

$$(\iota_a\varphi)(a_1,\ldots,a_n)=\sum_{j=0}^{n}(-1)^{\sum_{i=1}^{j}(|a|+1)(|a_i|+1)}\varphi(a_1,\ldots,a_j,a,a_{j+1},\ldots,a_n) \tag{6.2.11}$$

Lemma 6.7

$$[\delta,E]=0;\ \ [\delta,\iota_a]=\mathtt{ad}(a);\ \ [E,\iota_a]=(-1)^{|E|}\iota_{Ea};\ \ [\iota_a,\iota_b]=0.$$

Corollary 6.8

$$(\delta+D-\iota_R)^2=0$$

on $C^\bullet(V,A,W)$.

Remark 6.9 We will always view $C^\bullet(V,A,W)$ as the standard complex equipped with the total differential $\delta+D-\iota_R$.

We now define an A_∞ action on this standard complex. We follow the proof of Theorem 6.4. The only change is a different choice of operators T_g and $\phi(c_1,\ldots,c_n)$ (see Lemma 6.1, (3)).

$$\mathcal{T}_g=\exp(\iota_{\alpha(g)})T_g \tag{6.2.12}$$

for every $g\in G$;

$$\widetilde{\mathrm{Ad}}(c)=\exp(-\iota_{Dc\cdot c^{-1}})\,\mathrm{Ad}(c) \tag{6.2.13}$$

for every $c\in A^\times$ of degree zero.

Lemma 6.10 *(a)* $[\delta+D-\iota_R,\mathcal{T}_g]=0$;

(b) $\widetilde{\mathrm{Ad}}(c_1)\widetilde{\mathrm{Ad}}(c_2)=\widetilde{\mathrm{Ad}}(c_1c_2)$;

(c) $\mathcal{T}_g\widetilde{\mathrm{Ad}}_c\mathcal{T}_g^{-1}=\widetilde{\mathrm{Ad}}_{T_gc};\mathcal{T}_{g_1}\mathcal{T}_{g_2}=\widetilde{\mathrm{Ad}}(c(g_1,g_2))\mathcal{T}_{g_1,g_2}$

Proof (a) is straightforward. Let us prove (b).

$$\mathcal{T}_g(\delta+D-\iota_R)\mathcal{T}_g^{-1}=e^{\iota_{\alpha(g)}}T_g(\delta+D-\iota_R)T_g^{-1}e^{-\iota_{\alpha(g)}}=$$

$$= e^{\iota_{\alpha(g)}}(\delta + D + \mathtt{ad}_{\alpha(g)} - \iota_{R+D\alpha(g)+\alpha(g)^2})e^{-\iota_{\alpha(g)}}$$

(we used the equations in Definition 5.2). Now observe that

$$e^{\iota_{\alpha(g)}}De^{-\iota_{\alpha(g)}} = D + \iota_{\alpha(g)};$$

$$e^{\iota_{\alpha(g)}}\delta e^{-\iota_{\alpha(g)}} = \delta - \mathtt{ad}_{\alpha(g)} + \iota_{\alpha(g)^2}$$

which implies (a).

Now prove (b).

$$\widetilde{\mathrm{Ad}}_{c_1}\widetilde{\mathrm{Ad}}_{c_2} = \exp(-\iota_{Dc_1\cdot c_1^{-1}})\,\mathrm{Ad}_{c_1}\exp(-\iota_{Dc_2\cdot c_2^{-1}})\,\mathrm{Ad}_{c_2} =$$

$$= \exp(-\iota_{Dc_1\cdot c_1^{-1}+\mathrm{Ad}_{c_1}(Dc_2\cdot c_2^{-1})})\,\mathrm{Ad}_{c_1c_2} =$$

$$= \exp(-\iota_{D(c_1c_2)\cdot(c_1c_2)^{-1}})\,\mathrm{Ad}_{c_1c_2} = \widetilde{\mathrm{Ad}}_{c_1c_2}$$

Next, observe that, because of the third equation in Definition 5.2,

$$T_g(Dc\cdot c^{-1}) = T_g(Dc)T_g(c)^{-1} = D(T_g(c))T_g(c)^{-1} + [\alpha(g), T_g(c)]T_g(c)^{-1} =$$

$$= D(T_g(c))T_g(c)^{-1} + \alpha(g) - \mathrm{Ad}_{T_g(c)}(\alpha(g))$$

which implies

$$\mathcal{T}_g\widetilde{\mathrm{Ad}}_c\mathcal{T}_{g^{-1}} = e^{\iota_{\alpha(g)}}T_g e^{-\iota_{Dc\cdot c^{-1}}}\,\mathrm{Ad}_c\,T_g^{-1}e^{-\iota_{\alpha(g)}} =$$

$$= \exp(\iota_{\alpha(g)} - \iota_{T_g(Dc\cdot c^{-1})})\,\mathrm{Ad}_{T_g(c)}\exp(-\iota_{\alpha(g)}) =$$

$$\exp(\iota_{\alpha(g)} - \iota_{T_g(Dc\cdot c^{-1})} - \iota_{T_g(\alpha(g))})\,\mathrm{Ad}_{T_g(c)} =$$

$$\exp(-\iota_{DT_g(c)\cdot T_g(c)^{-1}})\,\mathrm{Ad}_{T_g(c)} = \widetilde{\mathrm{Ad}}_{T_g(c)}$$

which is (b). Finally,

$$\mathcal{T}_{g_1}\mathcal{T}_{g_2} = \exp(\iota_{\alpha(g_1)})T_{g_1}\exp(\iota_{\alpha(g_2)})T_{g_2} = \exp(\iota_{\alpha(g_1)+T_{g_1}\alpha(g_2)})T_{g_1g_2} =$$

$$+\exp(\iota_{\alpha(g_1)+T_{g_1}\alpha(g_2)})\,\mathrm{Ad}_{c(g_1,g_2)}\,T_{g_1}T_{g_2}$$

while

$$\mathrm{Ad}_{c(g_1,g_2)}\,\mathcal{T}_{g_1g_2} = \exp(-\iota_{Dc(g_1,g_2)c(g_1,g_2)^{-1}})\,\mathrm{Ad}_{c(g_1,g_2)}\exp(\iota_{\alpha(g_1g_2)})T_{g_1g_2} =$$

$$= \exp(-\iota_{Dc(g_1,g_2)c(g_1,g_2)^{-1}} - \iota_{\mathrm{Ad}_{c(g_1,g_2)}(\alpha(g_1g_2))})\,\mathrm{Ad}_{c(g_1,g_2)}\,T_{g_1}T_{g_2}$$

which implies (c) because of the last equation in Definition 5.2. □

Let $\mathbf{a} = (a_1, \ldots, a_n)$. Define:

$$T_g \mathbf{a} = (T_g a_1, \ldots, T_g a_n);\ \ \mathrm{Ad}_c\, \mathbf{a} = (\mathrm{Ad}_c\, a_1, \ldots \mathrm{Ad}_c\, a_n); \tag{6.2.14}$$

and

$$\iota_a \mathbf{a} = \sum_{j=0}^{n} (-1)^{\sum_{i=1}^{j}(|a|+1)(|a_i|+1)} (a_1, \ldots, a_j, a, a_{j+1}, \ldots, a_n) \tag{6.2.15}$$

(for a homogeneous element a).

By analogy with (6.2.12), (6.2.13), set

$$\mathcal{T}_g = \exp(\iota_{\alpha(g)}) T_g \tag{6.2.16}$$

for every $g \in G$;

$$\widetilde{\mathrm{Ad}}(c) = \exp(-\iota_{Dc \cdot c^{-1}})\, \mathrm{Ad}(c) \tag{6.2.17}$$

for every $c \in A^\times$ of degree zero.

Note that n could be equal to zero. In this case $\iota_a(\mathbf{a}) = 0$, $T_g \mathbf{a} = \mathbf{a}$, and $\mathrm{Ad}(c)(\mathbf{a}) = \mathbf{a}$.

For $\mathbf{a}_1 = (a_1, \ldots, a_{n_1})$, $\mathbf{a}_2 = (a_{n_1+1}, \ldots, a_{n_2})$, *etc.*, put

$$\varphi(\mathbf{a}_1, \mathbf{a}_2, \ldots) = \varphi(a_1, \ldots, a_{n_1}, a_{n_1+1}, \ldots, a_{n_2}, \ldots)$$

Every choice of $n_1, \ldots, n_{m+1} \geq 0$ such that $n_1 + \ldots + n_{m+1} = n$ defines a presentation $(a_1, \ldots, a_n) = (\mathbf{a}_1, \ldots, \mathbf{a}_{m+1})$. Define

$$|\mathbf{a}_k| = \sum_{i=n_k+1}^{n_{k+1}} |a_i|.$$

Put

$$(\phi(c_1, \ldots, c_m)\varphi)(a_1, \ldots, a_n) = \sum_{n_1, \ldots, n_{m+1}} (-1)^{N(n_1, \ldots, n_{m+1})} \tag{6.2.18}$$

$$\varphi(\mathbf{a}_1, c_1, \widetilde{\mathrm{Ad}}_{c_1}^{-1} \mathbf{a}_2, c_2, \widetilde{\mathrm{Ad}}(c_1 c_2)^{-1} \mathbf{a}_3, \ldots, c_m, \widetilde{\mathrm{Ad}}(c_1 c_2 \ldots c_m)^{-1} \mathbf{a}_{m+1})(c_1 c_2 \ldots c_m)^{-1}$$

Here

$$N(n_1, \ldots, n_{m+1}) = \sum_{j=1}^{m} \sum_{i=1}^{j} (|\mathbf{a}_i| + n_i)$$

Lemma 6.11 *The operators $\mathcal{T}_g$, $\widetilde{\mathrm{Ad}}(c)$, and $\phi(c_1, \ldots, c_m)$ satisfy all the relations of Lemmas 6.1 and 6.2.*

Proof Define for $\mathbf{a} = (a_1, \ldots, a_n)$ and for a homogenous derivation E

$$E\mathbf{a} = \sum_{j=1}^{n} (-1)^{|E| \sum_{p<j} |a_p|} (a_1, \ldots, Ea_j, \ldots, a_n) \tag{6.2.19}$$

Also put

$$\partial\mathbf{a} = \sum_{j=1}^{n-1} (-1)^{\sum_{p\leq j} |a_p|} (a_1, \ldots, a_j a_{j+1}, \ldots, a_n) \tag{6.2.20}$$

Note that Lemma 6.10 holds for $\mathcal{T}_g$ and $\widetilde{\mathrm{Ad}}_c$ as in (6.2.16), (6.2.17) and for D, ι, *etc.* as above, if one replaces δ by ∂. (In fact, a) can be easily checked, and the rest follows formally from (a)). It is easy to deduce Lemma 6.11 from this. □

We get a generalization of Theorem 6.4:

Theorem 6.12 *There is an A_∞ action of G on $C^\bullet(V, A, W)$ such that $T(g)$ is equal to $\mathcal{T}_g$ as in* (6.2.12).

6.2.5 Behavior with Respect to Equivalences

Now consider an equivalence between two actions up to inner automorphisms and compatible derivations

$$\mathbf{b} = (\{b(g)\}, \beta) : (T, c), (D, \alpha, R) \stackrel{\sim}{\to} (T', c'), (D', \alpha', R') \tag{6.2.21}$$

If V is a module with a derivation D_V and an action T_g compatible with the action on the left, let $\mathbf{b}_* V$ be V equipped with the derivation D'_V and with the action T'_g compatible with the action on the right (cf. (5.3.2)). Let

$$\mathcal{B}_c = \mathcal{B}(A^\times, G \ltimes_c A^\times);\ \mathcal{B}_{c'} = \mathcal{B}(A^\times, G \ltimes_{c'} A^\times)$$

(cf. definitions in Lemma 6.3 and in Theorem 6.4).

Lemma 6.13 *The formulas*

$$g \mapsto b(g)g,\ g \in G;\ c \mapsto c,\ c \in A^\times$$

define an isomorphism

$$G \ltimes_c A^\times \stackrel{\sim}{\leftarrow} G \ltimes_{c'} A^\times$$

of groups over G. Together with

$$\Phi(c_1, \ldots, c_m) \mapsto \Phi(c_1, \ldots, c_m),$$

they define an isomorphism of differential graded algebras

$$\mathbf{b}^\dagger : \mathcal{B}_c \xleftarrow{\sim} \mathcal{B}_{c'}$$

over $k[G]$.

Definition 6.14

$$\mathbf{b}^* = \exp(\iota_\beta) : C^\bullet(V, A, W) \xleftarrow{\sim} C^\bullet(\mathbf{b}_* V, A, \mathbf{b}_* W)$$

Proposition 6.15 *If one views* $C^\bullet(V, A, W)$ *as a differential graded* $\mathcal{B}_{c'}$*-modules* via *the morphism* $\mathbf{b}^\dagger$*, then* $\mathbf{b}^*$ *is a morphism of differential graded modules over* $\mathcal{B}_{c'}$*. For two composable equivalences* $\mathbf{b}_1$ *and* $\mathbf{b}_2$*, one has*

$$(\mathbf{b}_1\mathbf{b}_2)^\dagger = \mathbf{b}_2^\dagger \mathbf{b}_1^\dagger; \;\; (\mathbf{b}_1\mathbf{b}_2)^* = \mathbf{b}_2^* \mathbf{b}_1^*$$

Proof The statement follows from

Lemma 6.16 *(a)* $\widetilde{\mathrm{Ad}}_{b(g)}\mathcal{T}_g \exp(\iota_\beta) = \exp(\iota_\beta)\mathcal{T}'_g$

(b) $\widetilde{\mathrm{Ad}}_c \exp(\iota_\beta) = \exp(\iota_\beta)\widetilde{\mathrm{Ad}}'_c$

To prove the lemma, observe

$$\widetilde{\mathrm{Ad}}_{b(g)}\mathcal{T}_g \exp(\iota_\beta)\mathcal{T}_g'^{-1} =$$

$$\exp(-\iota_{Db(g)\cdot b(g)^{-1}})\,\mathrm{Ad}_{b(g)}\exp(\iota_{\alpha(g)})T_g \exp(\iota_\beta)T_g'^{-1}\exp(-\iota_{\alpha'(g)}) =$$

$$\exp(-\iota_{Db(g)\cdot b(g)^{-1}})\exp(\iota_{\mathrm{Ad}_{b(g)}\,\alpha(g)})\exp(\iota_{\mathrm{Ad}_{b(g)}\,T_g\beta})\exp(-\iota_{\alpha'(g)}) = \exp(\iota_\beta)$$

because of (5.2.1). This proves (a). To prove (b), note that

$$\widetilde{\mathrm{Ad}}_c \exp(\iota_\beta)\widetilde{\mathrm{Ad}}_c^{-1} = \exp(-\iota_{Dc\cdot c^{-1}})\,\mathrm{Ad}_c \exp(\iota_\beta)\,\mathrm{Ad}_c^{-1}\exp(\iota_{D'c\cdot c^{-1}}) =$$

$$\exp(-\iota_{Dc\cdot c^{-1}})\exp(\iota_{T_c\beta})\exp(\iota_{Dc\cdot c^{-1}+\beta-T_c\beta}) = \exp(\iota_\beta)$$

□

6.2.6 Behavior with Respect to Yoneda Product

Now let us describe the relation of the A_∞ action on a quotient to Yoneda product

$$\smile : C^\bullet(V_1, A, V_2) \otimes C^\bullet(V_2, A, V_3) \to C^\bullet(V_1, A, V_3) \tag{6.2.22}$$

given by

$$(\varphi \smile \psi)(a_1, \ldots, a_{m+n}) = (-1)^{(|\varphi|+m)\sigma_j(|a_j|+1)}\varphi(a_1, \ldots, a_m)\psi(a_{m+1}, \ldots, a_{m+n}) \tag{6.2.23}$$

Lemma 6.17 *The coproduct*

$$\Delta\phi(c_1, \dots, c_m) = \sum_{j=1}^{m} \phi(c_1, \dots, c_j) \otimes c_1 \dots c_j \phi(c_{j+1}, \dots, c_m)$$

turns the algebra $\mathcal{B}_c$ into a differential graded bialgebra. The morphism (6.2.1) is a bialgebra morphism. If we write $\Delta a = \sum a^{(1)} \otimes a^{(2)}$, then

$$a(\varphi \smile \psi) = \sum a^{(1)}\varphi \smile a^{(2)}\psi$$

for a in $\mathcal{B}_c$. Morphisms $b^\dagger$ from Lemma 6.13 are morphisms of bialgebras.

The proof is straightforward.

6.3 A_∞ Action on the Standard Complex: The Case of Lie Groupoids

6.3.1 A_∞ Action of a Lie Groupoid

Consider a Lie groupoid $\mathcal{G}$ with the manifold of objects M. Let $\mathcal{A}^\bullet$ be a sheaf of $\mathcal{O}_M^\bullet$-algebras with an action of $\mathcal{G}$ up to inner automorphisms and with a compatible flat connection up to inner derivations as in Sect. 5.7.2. Recall the presheaves $\underline{\mathcal{G}}^{(n)}$ on M^{n+1} (5.7.1). Let also

$$\underline{\mathcal{G}}_{jk}^{(n)} = p_{jk}^{-1}\underline{\mathcal{G}} \tag{6.3.1}$$

where $p_{jk} : M^{n+1} \to M^2$ is the projection to the jth and kth components.

Definition 6.18 An A_∞ action of $\mathcal{G}$ on a differential graded $\mathcal{O}_M^\bullet$-module $\mathcal{C}^\bullet$ is a collection of morphisms

$$T : \mathcal{G}^{(n)} \to \underline{\mathrm{Hom}}^{1-n}(p_{n+1}^*\mathcal{C}^\bullet, p_1^*\mathcal{C}^\bullet),$$

$n \geq 1$, such that (6.1.1) holds for every $g_1, \dots, g_n$ where g_j is a local section of $\underline{\mathcal{G}}_{j,j+1}^{(n)}$.

An A_∞ morphism of A_∞ actions is a collection of morphisms

$$\phi : \underline{\mathcal{G}}^{(n)} \to \underline{\mathrm{Hom}}^{-n}(p_{n+1}^*\mathcal{C}^\bullet, p_1^*\mathcal{C}^\bullet),$$

$n \geq 0$, such that (6.1.2) holds.

6.3.2 Action on the Standard Complex

Let $\mathcal{V}^\bullet$ and $\mathcal{W}^\bullet$ be two graded $\mathcal{A}^\bullet$-modules with compatible actions of $\mathcal{G}$ and with compatible connections ∇. Sometimes, to distinguish, we denote the three connections by $\nabla_{\mathcal{A}}$, $\nabla_{\mathcal{V}}$, and $\nabla_{\mathcal{W}}$ respectively. Compatibility means, as usual, that

$$\nabla(av) = \nabla(a)v + (-1)^{|a|}a\nabla(v)$$

for $a \in \mathcal{A}^\bullet$ and $v \in \mathcal{V}^\bullet$.

Definition 6.19 *The standard complex* $\mathcal{C}^\bullet(\mathcal{V}^\bullet, \mathcal{A}^\bullet, \mathcal{W}^\bullet)$ is the complex of sheaves

$$\mathcal{C}^m = \prod_{p+n=m} \underline{\mathrm{Hom}}^p_{\mathcal{O}^\bullet_M}(\otimes^n_{\mathcal{O}^\bullet_M}\mathcal{A}^\bullet, \underline{\mathrm{Hom}}_{\mathcal{O}^\bullet_M}(\mathcal{V}^\bullet, \mathcal{W}^\bullet))$$

with the differential $\delta + \nabla + \iota_R$ (cf. (6.2.10), (6.2.11), and Corollary 6.8).

Remark 6.20 In other words, $\mathcal{C}^\bullet$ is the standard complex computed over the algebra of scalars $\mathcal{O}^\bullet_M$ and sheaffied. An example arises when $\mathcal{A}$ is a bundle of algebras with a flat connection, $\mathcal{V}$ and $\mathcal{W}$ are bundles of modules with compatible flat connections, $\mathcal{O}_M$ is the differential graded algebra of forms, and $\mathcal{V}^\bullet$, resp. $\mathcal{W}^\bullet$, is the module of $\mathcal{V}$- (resp. $\mathcal{W}$)-valued forms. In this case $C^\bullet(\mathcal{V}, \mathcal{A}, \mathcal{W})$ is a bundle of complexes with an induced flat connection, and $\mathcal{C}^\bullet(\mathcal{V}^\bullet, \mathcal{W}^\bullet)$ is the complex of forms with values in this bundle. Our situation is different in only one regard. Namely, our $\mathcal{O}^\bullet$ will be mainly the algebra of Λ-valued forms. Accordingly, the exact nature of local cochains $\varphi(a_1, \ldots, a_n;\ v)$ that we allow needs to be specified. We will do this in Sect. 8.1.

Theorem 6.21 *There is an* A_∞ *action of* $\mathcal{G}$ *on* $\mathcal{C}^\bullet(\mathcal{V}^\bullet, \mathcal{A}^\bullet, \mathcal{W}^\bullet)$ *such that* $T(g)$ *is equal to* $\mathcal{T}_g$ *as in* (6.2.12).

Proof The operators $T(g_1, \ldots, g_n)$ are computed by a recursive procedure from Remark 6.5 where $\phi(c_1, \ldots, c_n)$ are as in (6.2.18). The only difference is that the morphism (6.2.8) sends c not to $\mathrm{Ad}(c)$ but to $\widetilde{\mathrm{Ad}}(c)$ (cf. (6.2.12), (6.2.13)). □

6.4 *The Cochain Complex of an* A_∞ *Action*

Given a sheaf of $\mathcal{O}^\bullet_M$-modules $\mathcal{M}^\bullet$ with an A_∞ action of a Lie groupoid $\mathcal{G}$, define

$$C^\bullet(M, \mathcal{M}^\bullet) = \prod_{n=0}^{\infty} \Gamma(M^{n+1}, \underline{\mathrm{Hom}}(\underline{\mathcal{G}}^{(n)}, p_1^*\mathcal{M}^{\bullet-n}))$$

with the differential

$$(d\Phi)(g_1, \ldots, g_{n+1}) = \nabla_{\mathcal{M}}\Phi(g_1, \ldots, g_{n+1}) + \sum_{j=1}^{n} T(g_1, \ldots, g_j)\Phi(g_{j+1}, \ldots, g_{n+1}) +$$

$$+ \sum_{j=1}^{n} (-1)^j \Phi(g_1, \ldots, g_j g_{j+1}, \ldots, g_{n+1}) + (-1)^{n+1}\Phi(g_1, \ldots, g_n)$$

Here g_j is a local section of $\underline{\mathcal{G}}^{(n)}_{j,j+1}$, cf. (6.3.1).

7 The A_∞ Action of $\pi_1(M)$ on Standard Complexes of $\mathcal{A}^\bullet_M$-Modules

7.1 *The Action of $\pi_1(M)$ up to Inner Automorphisms on $\mathcal{A}^\bullet_M$*

Assume that M is a symplectic manifold with a chosen Sp^4 structure. In this section we construct:

(1) a groupoid $\widetilde{\mathbf{G}}_M$ together with an epimorphism $\widetilde{\mathbf{G}}_M \to \pi_1(M)$ and a morphism of groups

$$\mathrm{Ker}(\widetilde{\mathbf{G}}_{x,x} \xrightarrow{p} \pi_1(M)_{x,x}) \xrightarrow{i} \mathcal{A}^\times_{M,x}; \tag{7.1.1}$$

(2) an action of $\widetilde{\mathbf{G}}_M$ on $\mathcal{A}_M$ up to inner automorphisms such that any element h of $\mathrm{Ker}(p)$ acts by conjugation with $i(h)$;

(3) a flat connection on $\mathcal{A}_M$ up to inner derivations compatible with the action of $\widetilde{\mathbf{G}}_M$, such that ∇ is a Fedosov connection $\nabla_{\mathcal{A}}$ whose lifting has curvature $\frac{1}{i\hbar}\omega$.

A more straightforward construction works in general under the assumption that M has an Sp^4 structure and yields the connection with $R = \frac{1}{i\hbar}\omega$. A construction that is a little more involved yields a connection with $R = 0$ under an additional restriction:

$$\langle \pi_2(M), [\omega] \rangle = 0 \tag{7.1.2}$$

meaning that the class of the symplectic form vanishes on the image of the Hurewicz homomorphism.

By Lemma 5.6 we will conclude that

Proposition 7.1 *The sheaf of algebras*

$$\mathcal{A}^\bullet_M = \Omega^\bullet_M(\mathcal{A}) \tag{7.1.3}$$

of $\mathcal{A}_M$-valued forms on M carries an action of $\pi_1(M)$ up to inner automorphisms and a compatible flat connection up to inner derivations such that ∇ is a Fedosov connection $\nabla_{\mathcal{A}}$ whose lifting has curvature $\frac{1}{i\hbar}\omega$.

Now Theorem 6.21 implies

Theorem 7.2 *For any two differential graded $\mathcal{A}_M^\bullet$-modules $\mathcal{V}^\bullet$, $\mathcal{W}^\bullet$ with a compatible action of $\pi_1(M)$ and a compatible connection, the standard complex $\mathcal{C}^\bullet(\mathcal{V}^\bullet, \mathcal{A}^\bullet, \mathcal{W}^\bullet)$ has a natural A_∞ action of $\pi_1(M)$.*

7.2 The Construction of the Groupoid $\widetilde{\mathbf{G}}_M$

There are two options for constructing the groupoid $\widetilde{G}_M$ and a flat connection up to inner derivations.

7.2.1 The Connection with $R = \frac{1}{i\hbar}\omega$

Assume that M is a symplectic manifold with an Sp^4 structure. Let g_{jk} be an $\mathrm{Sp}^4(2n, \mathbb{R})$-cocycle whose projection to Sp is a cocycle representing the tangent bundle. Let $\widetilde{\mathbf{G}}_M$ be the groupoid of the bundle represented by the cocycle g (viewed as a twisted bundle with $c = 1$). Here the role of G (as in Sect. 14.2) is played by the group $\widetilde{\mathbf{G}}$ as in Sect. 13.1.

Consider a lifted Fedosov connection with curvature $\frac{1}{i\hbar}\omega$ (cf. Theorem 3.5). This is a partial case of a connection defined in Sect. 14.2.2. Now we can define a flat connection up to inner derivations as in Sect. 14.2.2. (Observe that $\frac{1}{i\hbar}\mathbb{A}$ is a Lie subalgebra of the associative algebra $\mathcal{A}$ and $\mathrm{Sp}^4(2n, \mathbb{R})$ is a subgroup of $\mathcal{A}^\times$).

7.2.2 The Connection with $R = 0$

Consider the cocycle g_{jk} as above in Sect. 7.2.1. Consider $\widetilde{g}_{jk} \in \exp(\frac{1}{i\hbar}\mathbb{R})$ defined by

$$\widetilde{g}_{jk} = \exp\left(\frac{1}{i\hbar} f_{jk}\right) \tag{7.2.1}$$

where

$$\omega|U_j = d\alpha_j;\ \ \alpha_j - \alpha_k = df_{jk} \tag{7.2.2}$$

Observe that

$$c_{jkl} = \exp\left(\frac{1}{i\hbar}(f_{jk} + f_{kl} - f_{jl})\right) \tag{7.2.3}$$

takes values in $\exp(\frac{1}{i\hbar}\mathbb{R})$ and represents the class $\exp(\frac{1}{i\hbar}[\omega])$. Define $\widetilde{\mathbf{G}}_M$ to be the groupoid constructed from $\widetilde{g}_{jk}$, c_{jkl} as in Sect. 14.2. If our lifted Fedosov connection is represented by a collection of $\widetilde{\mathfrak{g}}$-valued one-forms A_j, then

$$\widetilde{A}_j = \frac{1}{i\hbar}\alpha_j + A_j \tag{7.2.4}$$

represents a *flat* connection in the twisted bundle given by $\widetilde{g}_{jk}, c_{jkl}$. Now we can define a flat connection up to inner derivations exactly as we did in Sect. 7.2.1 for which $R = 0$.

There is a short exact sequence of groups

$$1 \to \mathrm{Sp}^4(2n, \mathbb{R}) \to (\widetilde{\mathbf{G}}_M)_{x,x} \to \pi_1(M, x) \to 1 \tag{7.2.5}$$

for any point x of M.

8 Resumé of the General Procedure

We summarize the construction that we described up to this point. This includes the definition of objects and the construction of the infinity local system of morphisms between two objects. Next (in Sect. 9.1) we will present a construction of a special type of objects.

8.1 $\Omega^\bullet_{\mathbb{K},M}$-Modules and Their Inverse Images

Recall the definition of the sheaf $\Omega^\bullet_{\mathbb{K},M}$ of $\mathbb{K}$-valued forms on a manifold M (Definition 1.1). We will be considering the following class of sheaves of $\Omega^\bullet_{\mathbb{K},M}$-modules. Start with a vector bundle E (finite or profinite) and a fiber bundle $\mathfrak{X}$ on M. Local sections of the module $\mathcal{M}^\bullet_{\mathcal{E},\mathfrak{X}}$ are countable sums

$$\sum_{\varphi,\Phi} a_{\Phi,\varphi} \exp\left(\frac{1}{i\hbar}\varphi\right) e_\Phi \tag{8.1.1}$$

where $a_{\Phi,\varphi}$ are local differential forms with coefficients in E, φ are local sections of C^∞_M, e_Φ are formal symbols corresponding to local sections Φ of $\mathfrak{X}$, and $\varphi \to +\infty$. For a smooth map $M \to N$ we define

$$f^*\mathcal{M}^\bullet_{\mathcal{E},\mathfrak{X}} = \mathcal{M}^\bullet_{f^*E, f^*\mathfrak{X}} \tag{8.1.2}$$

We consider differentials of the following type on $\mathcal{M}^\bullet_{E,\mathfrak{X}}$. Let E_0 be a fiber of E and let X be a fiber of $\mathfrak{X}$. Choose any local trivialization of the bundles E and $\mathfrak{X}$ near x_0. Also choose any local coordinate systems on M near x_0 and on $\mathfrak{X}$ near $\Phi(x_0)$. Then we can identify local sections of E with local functions $M \to E_0$ and local sections of $\mathfrak{X}$ with local maps $M \to \mathbb{R}^{\dim \mathfrak{X}}$. We require the differential to be of the form

$$\nabla_{\mathcal{M}} \sum_{\varphi,\Phi} a_{\Phi,\varphi} \exp\left(\frac{1}{i\hbar}\varphi\right) e_\Phi = \sum_{\varphi,\Phi} da_{\Phi,\varphi} \exp\left(\frac{1}{i\hbar}\varphi\right) e_\Phi + \tag{8.1.3}$$

$$+ \sum_{\varphi,\Phi} \frac{1}{i\hbar}\varphi' a_{\Phi,\varphi} \exp\left(\frac{1}{i\hbar}\varphi\right) e_\Phi dx + \sum_{\varphi,\Phi} A(x, \Phi(x))\Phi'(x) a_{\Phi,\varphi} \exp\left(\frac{1}{i\hbar}\varphi\right) e_\Phi dx +$$

$$+ \sum_{\varphi,\Phi} B(x, \Phi(x)) a_{\Phi,\varphi} \exp\left(\frac{1}{i\hbar}\varphi\right) e_\Phi$$

Here A and B are local $\mathrm{End}(E_0)$-valued functions on $M \times X$. If $\nabla_{\mathcal{M}}$ is of the above form for one choice of the local trivializations then it is true for any such choice. We will use the shorthand

$$\nabla_{\mathcal{M}} e_\Phi = (A\Phi' + B)e_\Phi \tag{8.1.4}$$

Let $f : M \to N$ be a smooth map. A differential $\nabla_{\mathcal{M}}$ on $\mathcal{M}^\bullet_{E,\mathfrak{X}}$ induces a differential $f^*\nabla_{\mathcal{M}}$ on $f^*\mathcal{M}^\bullet_{E,\mathfrak{X}} = \mathcal{M}^\bullet_{f^*E, f^*\mathfrak{X}}$ as follows. Let x be local coordinates on M, y local coordinates on N, and let the map be locally of the form $y = f(x)$. If

$$\nabla_{\mathcal{M}} e_\Psi = (A(y, \Psi(y))\Psi'(y)dy + B(y, \Psi(y))dy)e_\Psi$$

for any Ψ, then

$$f^*\nabla_{\mathcal{M}} e_\Phi = (A(f(x), \Phi(x))\Phi'(x)dx + B(f(x), \Phi(x)) f'(x)dx)e_\Phi$$

In other words: let $p : \mathfrak{X} \to M$ be the projection. Locally in $\mathfrak{X}$ (near $\Phi(x)$), we require that there exist linear operators $A(z) : T_z\mathfrak{X}_{p(z)} \to \mathrm{End}\, E_{p(z)}$ and $B(z) : T_{p(z)}M \to \mathrm{End}\, E_{p(z)}$ and a linear projection $P(z) : T_z\mathfrak{X} \to T_z\mathfrak{X}_{p(z)}$, all smoothly depending on $z \in \mathfrak{X}$, such that for any point x of M and for any $\eta \in T_xM$,

$$\nabla_{\mathcal{M}} e_\Phi(x)(\eta) = (A(\Phi(x))P(d\Phi(x))\eta + B(\Phi(x))\eta)e_\Phi \tag{8.1.5}$$

Note that if $\nabla_{\mathcal{M}}$ satisfies this property for one choice of P then it satisfies it for any other choice. This is because for any two projections P_1 and P_2, $(P_1 - P_2)d\Phi(x) : T_xM \to T_{\Phi(x)}\mathfrak{X}_{\Phi(x)}$ is a linear operator depending only on the value of $\Phi(x)$.

For $f : M \to N$, if $\nabla_{\mathcal{M}}$ is locally determined by $A(z)$, $B(z)$, and $P(z)$, so is $f^*\nabla_{\mathcal{M}}$.

8.2 Oscillatory Modules

Consider the bundle $\mathcal{A}^\bullet_M$ with the action of the groupoid $\widetilde{\mathbf{G}}_M$ up to inner automorphisms and a compatible flat connection up to inner derivations as defined in

Sect. 7.2.2. By definition, an oscillatory module $\mathcal{V}^\bullet$ is a graded module over $\mathcal{A}_M^\bullet$ of the type defined in Sect. 8.1, with a compatible action of the groupoid $\widetilde{\mathbf{G}}_M$ and a compatible flat connection as in Sect. 5.8.

8.3 $\Omega^\bullet_{\mathbb{K},M}$-*Modules with* π_1-*Action*

These modules are defined in Sect. 6.3.1 (in our case here, $\mathcal{O}_M^\bullet = \Omega^\bullet_{\mathbb{K},M}$ as in Definition 1.1). More generally, *twisted* $(\Omega^\bullet_{\mathbb{K},M}, \pi_1(M))$ modules are defined in Sect. 16.3. By Theorem 6.21 and Lemma 5.6, under the assumptions $c_1(M) = 0$ and (7.1.2), the standard complex (Definition 6.19) of two oscillatory modules is a twisted $\Omega^\bullet_{\mathbb{K},M}$-module with π_1-action. We denote this complex by $\mathcal{C}^\bullet(\mathcal{V}^\bullet, \mathcal{A}^\bullet, \mathcal{W}^\bullet)$.

8.4 *Infinity Local Systems of* $\mathbb{K}$-*Modules*

An infinity local systems of $\mathbb{K}$-modules on a manifold X is a collection of complexes of $\mathbb{K}$-modules $\mathcal{C}_x^\bullet$, $x \in X$, together with linear maps

$$T(g_1, \dots, g_n) : \mathcal{C}^\bullet_{x_{n+1}} \to \mathcal{C}^{\bullet+1-n}_{x_1} \tag{8.4.1}$$

for any $g_j \in \pi_1(X)_{x_j, x_{j+1}}$, $j = 1, \dots, n$, subject to (6.1.1). In other words, this is a system of complexes with an A_∞ action of the fundamental groupoid $\pi_1(X)$, cf. Sect. 6.2.3.

8.4.1 From Twisted $(\Omega^\bullet_{\mathbb{K},M}, \pi_1(M))$ Modules to Infinity Local Systems

If $\mathcal{M}^\bullet$ is an $\Omega^\bullet_{\mathbb{K},M}$-module with a twisted π_1-action (as in Sects. 8.3, 16.3), then

$$\mathcal{C}_x^\bullet = \varinjlim_{x \in U} C^\bullet(U, \mathcal{M}^\bullet) \tag{8.4.2}$$

is an infinity local system of $\mathbb{K}$-modules. (cf. Sect. 6.4 for the definition of the cochain complex $C^\bullet(U, \mathcal{M}^\bullet)$). This is explained in detail in Sect. 16.3.2.

Definition 8.1 Given two oscillatory modules $\mathcal{V}^\bullet$ and $\mathcal{W}^\bullet$ on a symplectic manifold M that has an Sp^4 structure and satisfies (7.1.2), we denote by $\underline{\mathbb{R}\,\mathrm{HOM}}(\mathcal{V}^\bullet, \mathcal{W}^\bullet)$ the infinity local system $\mathcal{C}^\bullet$ (cf. Sect. 8.4) constructed from the complex $\mathcal{M}^\bullet = \mathcal{C}^\bullet(\mathcal{V}^\bullet, \mathcal{A}^\bullet, \mathcal{W}^\bullet)$ (cf. Sect. 8.3).

9 Objects Constructed from Lagrangian Submanifolds

9.1 Induced Modules

9.1.1 The Case of Groups Acting on Algebras

Let $i : B \to A$ be a morphism of algebras and let $j : P \to G$ be a morphism of groups. Assume that P acts on B by automorphisms and G acts on A by automorphisms. We denote these automorphisms by S_p, $p \in P$, and T_g, $g \in G$. We assume that $i(T_p b) = T_{jp}(ib)$ for any p and g. For simplicity, we consider here only true actions, i.e. those for which $c(g_1, g_2) = 1$ and $c(p_1, p_2) = 1$.

Let W be a B-module with a compatible action of P denoted by $S_p : W \xrightarrow{\sim} W$, $p \in P$. Define the induced module V as follows. First consider the A-module $A \otimes_B W$. Note that it carries a compatible action of P:

$$S_p(a \otimes w) = T_{jp}(a) S_p(w). \tag{9.1.1}$$

Now let V be the quotient of the space of formal linear combinations

$$\sum_{g \in G} T_g v_g, \;\; v_g \in A \otimes_B W, \tag{9.1.2}$$

by the linear span of $T_{gj(p)}(a \otimes w) - T_g S_p(a \otimes w), g \in G,\; p \in P,\; a \in A,\; w \in W$. Define the A-module structure on V by

$$a \sum T_g g v_g = \sum T_g (T_g^{-1} a) v_g \tag{9.1.3}$$

and a compatible group action of G

$$T_{g_0} \sum T_g v_g = \sum T_{g_0 g} v_g \tag{9.1.4}$$

This is just another way of defining the induced module

$$V = (G \ltimes A) \otimes_{P \ltimes B} W \tag{9.1.5}$$

Now assume that A and B are graded algebras. Let $\{D : A \to A;\; \alpha(g) | g \in G;\; R_A\}$ and $\{E : B \to B;\; \beta(g) | g \in B;\; R_B\}$ be derivations of square zero of A and of B up to inner derivations. We assume that these derivations are compatible with i and j, i.e.

$$i(E(b)) = D(i(b));\; i(\beta(p)) = \alpha(jp);\; i(R_B) = R_A. \tag{9.1.6}$$

Let $E_W : W \to W$ be a compatible derivation of W. Then $A \otimes_B W$ carries a derivation $E_{A \otimes_B W}$ compatible with the action of B;

$$E_{A\otimes_B W}(a\otimes w) = D_A(a)\otimes w + (-1)^{|a|} a\otimes E_W(w). \tag{9.1.7}$$

This allows to define a derivation of the induced module V compatible with the action of G:

$$D_V\left(\sum T_g v_g\right) = \sum T_g(\alpha(g^{-1})v_g) + \sum T_g E_{A\otimes_B W}(v_g) \tag{9.1.8}$$

9.1.2 The Case of Groupoids

Now generalize the situation of Sect. 9.1.1 to the case when P is a groupoid with the set of objects Y and G is a groupoid with the set of objects X. Denote by $j : Y \to X$ the action of the morphism of groupoids j on objects. In this case $A = \{A_x | x \in X\}$, $B = \{B_y | y \in Y\}$, and $W = \{W_y | y \in Y\}$. Put

$$(A\otimes_B W)_y = A_{jy}\otimes_{B_y} W_y \tag{9.1.9}$$

Formulas (9.1.1) and (9.1.7) define a compatible action of P and a compatible derivation on $A\otimes_B W$.

$$V_x = \left\{\sum_{y\in Y, g\in G_{x,jy}} T_g v_g | v_g \in (A\otimes_B W)_y\right\} / \langle T_{gj(p)}v - T_g(S_p v)\rangle \tag{9.1.10}$$

Formulas (9.1.3), (9.1.4), (9.1.7), and (9.1.8) define on V an A-module structure, a compatible action of G, and a compatible derivation.

9.1.3 The Case of Lie Groupoids

Now let $\mathcal{G}$ and $\mathcal{P}$ be Lie groupoids with the manifolds of objects X and Y respectively. Let $j : \mathcal{P} \to \mathcal{G}$ be a morphism of Lie groupoids, i.e. a smooth map $X \to Y$ and a smooth map $\mathcal{P} \to \mathcal{G}$ over $X \times X$ that preserves the composition and the unit. Let $\mathcal{B}^\bullet$ be a sheaf of $\mathcal{O}_Y^\bullet$-algebras and let $\mathcal{A}^\bullet$ be a sheaf of $\mathcal{O}_X^\bullet$-algebras, together with a morphism $i : \mathcal{B}^\bullet \to j^*\mathcal{A}^\bullet$. Consider an action S of $\mathcal{P}$ on $\mathcal{B}^\bullet$ and an action T of $\mathcal{G}$ on $\mathcal{A}^\bullet$. We assume that the morphism i preserves the action of $\mathcal{P}$. Furthermore, let $(\nabla_{\mathcal{B}}, \beta, R_{\mathcal{B}})$ be a compatible flat connection up to inner derivations on $\mathcal{B}^\bullet$ and let $(\nabla_{\mathcal{A}}, \alpha, R_{\mathcal{A}})$ be a compatible flat connection up to inner derivations on $\mathcal{A}^\bullet$. We require the following compatibility conditions generalizing (9.1.6):

$$i(\nabla_{\mathcal{B}} b) = (j^*\nabla_{\mathcal{A}})(ib) \tag{9.1.11}$$

in $j^*\mathcal{A}^\bullet$ on Y, for any local section b of $\mathcal{B}^\bullet$;

$$i(R_{\mathcal{B}}) = j^*(R_{\mathcal{A}}) \tag{9.1.12}$$

in $j^*\mathcal{A}^\bullet$; and

$$i(\beta(p)) = \alpha(jp) \tag{9.1.13}$$

in $j^*\mathcal{A}^\bullet$ for any local section p of $\mathcal{P}$.

Remark 9.1 The latter equation requires some explanation. It is not a priori clear why, for a local section g of $\mathcal{G}$, $\alpha(g)$ depends only on the restriction of g to $Y \times Y$. To ensure this, we will always assume that the form $\alpha(g)$ is obtained from a local section g by the same procedure as the factor in front of e_Φ in the right hand side of (8.1.4) is obtained from a local section Φ.

Now assume that $\mathcal{W}^\bullet$ is a $\mathcal{B}^\bullet$-module with a compatible action S of $\mathcal{P}$ and a compatible connection $\nabla_{\mathcal{W}}$. The module $j^*\mathcal{A}^\bullet \otimes_{\mathcal{B}^\bullet} \mathcal{W}$ has a compatible action S of $\mathcal{P}$ and a compatible connection $\nabla_{\mathcal{A}\otimes_{\mathcal{B}}\mathcal{W}}$ given by

$$S_p(a \otimes w) = T_{jp}(a) \otimes S_p w; \tag{9.1.14}$$

$$\nabla_{\mathcal{A}\otimes_{\mathcal{B}}\mathcal{W}}(a \otimes w) = \nabla_{\mathcal{A}}(a) \otimes w + (-1)^{|a|} a \otimes \nabla_{\mathcal{W}}(w) \tag{9.1.15}$$

Now define the induced module $\mathcal{V}$ as follows. First, for any open subsets U of X and U' of Y and any smooth map $f : U \to U'$, let $\underline{\mathcal{G}}_f$ be the inverse image of $\underline{\mathcal{G}}$ under

$$U \xrightarrow{\sim} \mathrm{graph}(f) \hookrightarrow X \times Y \hookrightarrow X \times X \tag{9.1.16}$$

The space of local sections of $\mathcal{V}^\bullet$ over U is the space of formal linear combinations

$$\sum_{U,U'} \sum_{f:U\to U'} \sum_{g\in \underline{\mathcal{G}}_f(U)} T_g v_g; \;\; v_g \in (\mathcal{A}^\bullet \otimes^\bullet_{\mathcal{B}} \mathcal{W}^\bullet)(U') \tag{9.1.17}$$

factorized by the linear span of

$$T_{gj(p)}(a \otimes w) - T_g(S_p(a \otimes w)) \tag{9.1.18}$$

for some $h : U' \to U''$, $f : U \to U'$, g a local section of $\underline{\mathcal{G}}_f(U)$, and p a local section of $\underline{\mathcal{P}}|\mathrm{graph}(h)$. We interpret $gj(p)$ as a local section of $\underline{\mathcal{G}}_{hf}$.

Formulas

$$T_{g_0} \sum T_g v_g = \sum T_{g_0 g} v_g; \;\; a \sum T_g v_g = \sum T_g(T_{g^{-1}}(a) v_g); \tag{9.1.19}$$

$$\nabla_{\mathcal{V}} \sum T_g v_g = \sum T_g \alpha(g^{-1}) v_g + T_g \nabla_{\mathcal{A}\otimes_{\mathcal{B}}\mathcal{W}}(v_g) \tag{9.1.20}$$

define an $\mathcal{A}^\bullet$-module structure, a compatible action of $\mathcal{G}$, and a compatible connection on $\mathcal{V}^\bullet$. Note that the last formula relies again on the assumption discussed in Remark 9.1. Indeed, we need to be sure that $\alpha(g^{-1})|\mathrm{graph}(f)$ depends only on $g|\mathrm{graph}(f)$.

9.1.4 General Definition of an Induced Module

Finally, let us assume, analogously to what we did in Sect. 5.8.1, that there is a Lie groupoid Γ on X and a Lie groupoid Π on Y together with a morphism $\Pi \to j^*\Gamma$, an epimorphism $\mathcal{G} \to \Gamma$, and an epimorphism $\mathcal{P} \to \Pi$ such that the diagram

$$\begin{array}{ccc} \mathcal{P} & \longrightarrow & \Pi \\ \downarrow & & \downarrow \\ j^*\mathcal{G} & \longrightarrow & j^*\Gamma \end{array}$$

commutes. Let $\mathcal{H}_x = \mathrm{Ker}(\mathcal{G}_{x,x} \to \Gamma_{x,x})$ and $\mathcal{Q}_y = \mathrm{Ker}(\mathcal{P}_{y,y} \to \Pi_{y,y})$. Denote by $\underline{\mathcal{H}}$, resp. $\underline{\mathcal{Q}}$, the sheaf of sections of the bundle of groups $\mathcal{H}$, resp. $\mathcal{Q}$. We also assume that there are morphisms of sheaves $i : \underline{\mathcal{H}} \to \mathcal{A}^\times$ and $i : \underline{\mathcal{Q}} \to \mathcal{B}^\times$ such that the diagram

$$\begin{array}{ccc} \underline{\mathcal{Q}} & \longrightarrow & \mathcal{A}^\times \\ \downarrow & & \downarrow \\ j^*\underline{\mathcal{H}} & \longrightarrow & j^*\mathcal{B}^\times \end{array}$$

commutes. We also assume that the $\mathcal{B}^\bullet$-module $\mathcal{W}^\bullet$ and the flat connection up to inner derivations $(\nabla_{\mathcal{B}}, \beta, R_{\mathcal{B}})$ satisfies

$$S_q w = i(q)w; \;\; \beta(q) = -\nabla_{\mathcal{B}} i(q) \cdot (iq)^{-1}$$

for any local sections q of $\underline{\mathcal{Q}}$ and w of $\mathcal{W}^\bullet$.

Definition 9.2 Under the assumptions above, the induced module is the quotient of the module $\mathcal{V}^\bullet$ (9.1.19), (9.1.20) by the submodule generated by elements $T_h v - i(h)v$, h being any local section of $\mathcal{H}$ and v any local section of $\mathcal{V}^\bullet$.

9.2 The Induced Oscillatory Module $\mathcal{V}_L$

9.2.1 The Algebra $\mathcal{B}$ and the Module $\widehat{\widehat{\mathbb{V}}}_{\mathbb{K}}$

Recall the grading

$$|\widehat{x}_j| = |\widehat{\xi}_j| = 1; \;\; |\hbar| = 2 \tag{9.2.1}$$

Now define

$$\widehat{\mathbb{V}} = \mathbb{C}[[\widehat{x}, \hbar]]; \;\; \widehat{\widehat{\mathbb{V}}} = \left\{ \sum_{k=-N}^{\infty} v_k | v_k \in \widehat{\mathbb{V}}[\hbar^{-1}]_k \right\} \tag{9.2.2}$$

where N runs through all integers.

Definition 9.3 Put

$$\widehat{\widehat{\mathbb{V}}}_{\mathbb{K}} = \left\{ \sum_{k=0}^{\infty} e^{\frac{1}{i\hbar}c_m} |v_m \in \widehat{\widehat{\mathbb{V}}};\ c_m \in \mathbb{R};\ c_m \to \infty \right\} \tag{9.2.3}$$

$$\widehat{\widehat{\mathbb{V}}}_{\Lambda} = \left\{ \sum_{k=0}^{\infty} e^{\frac{1}{i\hbar}c_m} |v_m \in \widehat{\widehat{\mathbb{V}}};\ c_m \geq 0;\ c_m \to \infty \right\} \tag{9.2.4}$$

Now define the subalgebra $\mathcal{B}$ of $\mathcal{A}$ (Definition 4.2) by

$$\mathcal{B} = \mathrm{MPar}(n) \ltimes \widehat{\widehat{\mathbb{A}}} \tag{9.2.5}$$

(cf. Sect. 12.1).

Lemma 9.4 *The formulas*

$$\widehat{x} \mapsto \widehat{x};\ \widehat{\xi} \mapsto i\hbar \frac{\partial}{\partial \widehat{x}};$$

$$\begin{bmatrix} b & 0 \\ 0 & b^{-1} \end{bmatrix} \mapsto T_b,\ (T_b f)(\widehat{x}) = \frac{1}{\sqrt{\det(b)}} f(b^{-1}\widehat{x});$$

$$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \mapsto \exp\left(-\frac{i\hbar}{2} a \left(\frac{\partial}{\partial \widehat{x}} \right)^2 \right)$$

define an action of $\mathrm{MPar}(n)$ *that together with the action of* $\widehat{\widehat{\mathbb{A}}}$ *turns* $\widehat{\widehat{\mathbb{V}}}_{\mathbb{K}}$ *into a* $\mathcal{B}$*-module.*

Definition 9.5

$$\widehat{\mathcal{V}} = \mathcal{A} \widehat{\otimes}_{\mathcal{B}} \widehat{\widehat{\mathbb{V}}}$$

Here by $\widehat{\otimes}$ we mean the completed tensor product. Namely,

$$\mathcal{A} \widehat{\otimes}_{\mathcal{B}} \widehat{\widehat{\mathbb{V}}} = \varprojlim_{N \to \infty} \mathcal{A} \otimes_{\mathcal{B}} \widehat{\widehat{\mathbb{V}}} / \exp\left(\frac{N}{i\hbar} \right) \mathcal{A}_{\Lambda} \otimes_{\mathcal{B}} \widehat{\widehat{\mathbb{V}}}$$

In Sect. 13 we interpret $\widehat{\mathcal{V}}$ as an algebraic version of the metaplectic representation (Proposition 13.8).

9.2.2 The Sheaf of Algebras $\mathcal{B}_L$ and the Sheaf of Modules $\widehat{\widehat{\mathbb{V}}}_L$

Let L be a Lagrangian submanifold of M. Recall that we assume the existence of an Sp^4 structure on M. Consider the restriction to L of the $\mathrm{Sp}^4(n)$-valued cocycle $\widetilde{g}_{jk}$ as in Sect. 7.2.1 or in Sect. 7.2.2 (it does not matter which one of them because $\omega|L = 0$). Consider the cohomologous $\mathrm{MPar}(n)$-valued cocycle $\widetilde{p}_{jk}$ as in (12.1.4).

The group $\mathrm{MPar}(n, \mathbb{R})$ (cf. Sect. 12.1) acts on $\mathcal{B}$ by automorphisms. It also acts on $\widehat{\widehat{\mathbb{V}}}_{\mathbb{K}}$ compatibly. Let $\mathcal{B}_L$ be the bundle of algebras and $\widehat{\widehat{\mathbb{V}}}_L$ the bundle of modules on L associated to these actions and to the principal MPar-bundle defined by p_{jk}. Note that the Lie algebra $\widetilde{\mathfrak{g}}$ (3.1.4) acts by derivations on $\mathcal{B}$ and on $\widehat{\widehat{\mathbb{V}}}_{\mathbb{K}}$. Therefore any given Fedosov connection defines a connection on $\mathcal{B}_L$ and on $\widehat{\widehat{\mathbb{V}}}_L$. If the curvature of this connection is $\frac{1}{i\hbar}\omega$ then the connection on $\widehat{\widehat{\mathbb{V}}}_L$ is flat. We denote these connections by $\nabla_{\mathcal{B}}$ and $\nabla_{\mathbb{V}}$.

Definition 9.6 By $\mathcal{B}_L^\bullet$, resp. by $\widehat{\widehat{\mathbb{V}}}_L^\bullet$, we denote the differential graded algebra of $\mathcal{B}_L$-valued $\mathbb{K}$-forms with differential $\nabla_{\mathcal{B}}$, resp. the differential graded module of $\widehat{\widehat{\mathbb{V}}}_L$-valued $\mathbb{K}$-forms with differential $\nabla_{\mathbb{V}}$.

9.2.3 The Lie Groupoid $\mathbf{P}_L$

Recall the $\widetilde{P}$-valued cocycle from Eq. (12.1.4) construct the groupoid $\mathbf{P}_L$ as the groupoid of the (twisted in general, but not in this case) bundle defined by this cocycle as in Sect. 14.2. We have a short exact sequence of groups

$$1 \to \widetilde{\mathbf{P}} \to (\mathbf{P}_L)_{x,x} \to \pi_1(L, x) \to 1 \tag{9.2.6}$$

for every point x of L. Cf. Sect. 13.1 for the definition of $\widetilde{\mathbf{P}}$.

Definition 9.7 Let $\widehat{\widehat{\mathbb{V}}}_L^\bullet$ be the $\mathcal{B}_L^\bullet$-module with the compatible action of $\mathbf{P}_L$ and the compatible connection $\nabla_{\mathbb{V}}$ as in Definition 9.6. The oscillatory module $\mathcal{V}_L^\bullet$ is the $\mathcal{A}_M^\bullet$-module with a compatible action of $\widetilde{\mathbf{G}}_M$ and a compatible connection induced from $\widehat{\widehat{\mathbb{V}}}_L^\bullet$ as in Definition 9.2.

9.3 *Filtrations*

Proposition 9.8 *Assume that L is a Lagrangian submanifold of M such that $\langle[\omega], \pi_2(M, L)\rangle = 0$. Then there is a filtration $\mathrm{Filt}^a\mathcal{V}_L^\bullet$, $a \in \mathbb{R}$, on $\mathcal{V}_L^\bullet$ such that:*

(1) $\mathrm{Filt}^a\mathcal{V}_L^\bullet \subset \mathrm{Filt}^b\mathcal{V}_L^\bullet$ *for $a \geq b$;*
(2) $\mathrm{Filt}^a\mathbb{K} \cdot \mathrm{Filt}^b\mathcal{V}_L^\bullet \subset \mathrm{Filt}^{a+b}\mathcal{V}_L^\bullet$
(3) $\mathrm{Filt}^a\mathcal{V}_L^\bullet$ *is preserved by $\nabla_{\mathcal{V}}$ and by the action of $\mathcal{A}_M^\bullet$ (but not necessarily by the action of $\widetilde{\mathbf{G}}_M$).*

Here Filt$^a\mathbb{K}$ *consists of sums as in* (4.1.5) *with the additional condition* $c_k \geq a$ *for all* k.

Proof Similarly to what we did in Sect. 14.1, for any chart T choose a one-form α_T on T such that $d\alpha_T = \omega|T$, for any two charts T and T' a function $f_{TT'}$ on $T \times_M T'$ such that $\alpha_T - \alpha_{T'} = df_{TT'}$, and for any three charts T, T', T'' put $c_{TT'T''} = \exp(\frac{1}{i\hbar}(f_{TT'} - f_{TT''} + f_{T'T''}))$ which is a locally constant function on $T \times_M T' \times_M T''$. We can choose them in such a way that they all vanish on L. For any path T from $x_0 \in L$ to x_1 in M, and for a small open U_{x_1} containing x_0, we get an open subset U_T of $\widetilde{M/L}$ (homeomorphic to the image of U_{x_1} in M/L). Consider a cover of $\widetilde{M/L}$ by such U_T. We will define Filt$^0\mathcal{V}_L^\bullet$ to be the linear span of those elements of $\mathcal{V}_L^\bullet$ that are, under the trivialization with respect to a chart T, represented as $\exp(\frac{1}{i\hbar}\varphi_T)g_T v$ where $v \in \widehat{\widehat{\mathbb{V}}}_\Lambda$ (cf. (9.2.2)), $g_T \in \mathrm{Sp}^4(2n)$, and φ_T are some functions on U_T. To make this well defined, we must have

$$\exp\left(\frac{1}{i\hbar}(\varphi_T - \varphi_{T'})\right) = \exp\left(\frac{1}{i\hbar}f_{TT'}\right)c_{TST'}$$

on $U_T \cap U_{T'}$, for any T and T' as above and for any homotopy S between them. We will find such φ_T if we show that the right hand side of the above formula (a) does not depend on S and (b) defines a one-cocycle with respect to the cover of $\widetilde{M/L}$ by U_T. But, under our assumption, (b) follows immediately from Lemma 14.2. As for (a), for two different homotopies S and S' between T and T',

$$c_{TST'}c_{SS'T'} = c_{TSS'}c_{TS'T'}$$

But $c_{SS'T'} = c_{TSS'} = 1$. Indeed, $S \times_M S' \times_M T = T$ and same is true for T', and c vanishes when restricted to L. □

9.3.1 The Microsupport of a Filtered Module

Assume $\mathcal{V}^\bullet$ has a filtration as in Proposition 9.8. Define

$$\mu\mathrm{Supp}(\mathcal{V}^\bullet) = \mathrm{supp} H^\bullet\left(s\left(\varinjlim_{a\to 0, a>0} \mathrm{Filt}^0\mathcal{V}^\bullet/\mathrm{Filt}^a\mathcal{V}^\bullet\right), \nabla_\mathcal{V}\right) \tag{9.3.1}$$

Here s denotes sheafification of a presheaf.

9.4 *The Case of* $\mathbb{R}^{2n}$

9.4.1 The Groupoid $\widetilde{\mathbf{G}}_M$

Sections of $\widetilde{\mathbf{G}}_M$ are in bijection with smooth functions $g(x_1, \xi_1; x_2, \xi_2)$ on $M \times M$ with values in $\widetilde{\mathbf{G}}$ (cf. Sect. 13.1). We will denote a section corresponding to g by a formal symbol

$$\sigma(x_1, \xi_1; x_2, \xi_2) = \exp\left(\frac{1}{i\hbar}(\xi_2 - \xi_1)\widehat{x} + (x_1 - x_2)\widehat{\xi}\right) g(x_1, \xi_1; x_2, \xi_2) \qquad (9.4.1)$$

The composition consists of formal multiplication of exponentials and multiplication of elements of $\mathrm{Sp}^4(2n)$.

9.4.2 The Flat Connection up to Inner Derivations on $\mathcal{A}_M$ Compatible with the Action of $\widetilde{\mathbf{G}}_M$

For a section σ as in (9.4.1),

$$-\alpha(\sigma) = \nabla_{\widetilde{\mathbf{G}}}\sigma \cdot \sigma^{-1} = d_{\mathrm{DR}} g \cdot g^{-1} + \frac{1}{i\hbar}(\xi_2 dx_2 - \xi_1 dx_1) +$$

$$\left(-\frac{\widehat{\xi}_1}{i\hbar}dx_1 + \frac{\widehat{x}_1}{i\hbar}d\xi_1\right) - \mathrm{Ad}_g\left(-\frac{\widehat{\xi}_2}{i\hbar}dx_2 + \frac{\widehat{x}_2}{i\hbar}d\xi_2\right)$$

(by (7.2.4)).

9.4.3 The Sheaf $\mathcal{V}_f^\bullet$

Denote by $\mathcal{V}_f^\bullet$ the oscillatory module corresponding to the Lagrangian submanifold graph(df). One has

$$\mathcal{V}_f^\bullet = \widehat{\mathcal{V}}_M^\bullet = \Omega^\bullet_{\mathbb{K},M}(\widehat{\mathcal{V}}). \qquad (9.4.2)$$

(cf. Definition 9.5). In other words, local sections of $\mathcal{V}_f^\bullet$ are $\widehat{\mathcal{V}}$-valued $\mathbb{K}$-forms on M (cf. Definition 9.5).

Remark 9.9 (a) Sections of $\widehat{\widehat{\mathbb{V}}}_L$ are identified with $\widehat{\widehat{\mathbb{V}}}$-valued functions on L as follows: if $v(x, \widehat{x})$ is a $\widehat{\widehat{\mathbb{V}}}$-valued function, then the corresponding section of $\widehat{\widehat{\mathbb{V}}}_L$ is

$$\exp\left(\frac{1}{i\hbar}(f(x + \widehat{x}) - f'(x)\widehat{x})\right) v(x, \widehat{x}) \qquad (9.4.3)$$

(b) A section $w(x, dx, \widehat{x})$ of (9.4.2) is identified with the section of $\mathcal{V}_f^\bullet$ given by

$$\underline{w} = \sigma((x, \xi); (x, f'(x))) \exp\left(\frac{1}{i\hbar}(f(x + \widehat{x}) - f'(x)\widehat{x})\right) w(x, dx, \widehat{x}) \quad (9.4.4)$$

where $\sigma(x, \xi; x, f'(x))$ is as in (9.4.1)).

9.4.4 The Connection on $\mathcal{V}_f$

$$\nabla_{\mathcal{V}} = -\frac{\xi - f'(x)}{i\hbar}dx + \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial \widehat{x}} - \frac{1}{i\hbar}f''(x)\widehat{x}\right)dx + \left(\frac{\partial}{\partial \xi} + \frac{1}{i\hbar}\widehat{x}\right)d\xi \quad (9.4.5)$$

Indeed, under the identification as in (b) in Sect. 9.4.3 above, the connection $\nabla|L$ becomes

$$\mathrm{Ad}\exp\left(-\frac{1}{i\hbar}(f(x + \widehat{x}) - f'(x)\widehat{x})\right)\left(-\frac{f'(x)}{i\hbar} + \frac{\partial}{\partial x} - \frac{\partial}{\partial \widehat{x}} + \frac{1}{i\hbar}f''(x)\widehat{x}\right)dx$$

which is equal to

$$\nabla_{\mathrm{st}} = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial \widehat{x}}\right)dx$$

Now, if we denote

$$\sigma_f = \sigma(x, \xi; x, f'(x)),$$

as well as

$$A = -\frac{1}{i\hbar}\xi dx - \frac{\partial}{\partial \widehat{x}}dx + \frac{\widehat{x}}{i\hbar}d\xi; \;\; p(x, \xi) = (x, f'(x)),$$

then

$$\nabla_{\mathcal{V}}(\sigma_f \underline{w}) = \sigma_f(A - p^*A)\sigma_f \underline{w} + \sigma_f \underline{\nabla_{\mathrm{st}} w}$$

Since

$$A - p^*A = -\frac{1}{i\hbar}\xi dx - \frac{\partial}{\partial \widehat{x}}dx + \frac{\widehat{x}}{i\hbar}d\xi + \frac{1}{i\hbar}f'(x)dx + \frac{\partial}{\partial \widehat{x}}dx - \frac{\widehat{x}}{i\hbar}f''(x)dx,$$

we conclude that (9.4.5) holds.

9.4.5 The Action of $\widehat{\widehat{\mathbb{A}}}$ on $\mathcal{V}_f$

The formal variables act as follows: $\widehat{x}$ by multiplication, and $\widehat{\xi}$ by $i\hbar\frac{\partial}{\partial \widehat{x}} + f'(x + \widehat{x}) - f'(x)$.

Indeed, under the identification (b) from Sect. 9.4.3, $\widehat{\xi}$ acts by

$$\mathrm{Ad}\exp\left(-\frac{1}{i\hbar}(f(x+\widehat{x})-f'(x)\widehat{x})\right)\left(i\hbar\frac{\partial}{\partial\widehat{x}}\right)=i\hbar\frac{\partial}{\partial\widehat{x}}+f'(x+\widehat{x})-f'(x)$$

and $\widehat{x}$ by

$$\mathrm{Ad}\exp\left(-\frac{1}{i\hbar}(f(x+\widehat{x})-f'(x)\widehat{x})\right)(\widehat{x})=\widehat{x}$$

9.4.6 The Action of $\widetilde{\mathrm{G}}_M$ on $\mathcal{V}_f$

A section σ as in (9.4.1) acts by

$$\exp\left(-\frac{1}{i\hbar}(f(x_1+\widehat{x})-f'(x_1)\widehat{x}-f(x_2+\widehat{x})+f'(x_2)\widehat{x}\right)g(x_1,\xi_1;x_2,\xi_2) \tag{9.4.6}$$

This is obvious, because of how we make the identification in (b), Sect. 9.4.3.

9.4.7 Comparison Between $\mathcal{V}_f^\bullet$ and $\mathcal{V}_0^\bullet$

Corollary 9.10 *One has an isomorphism*

$$\exp\left(\frac{1}{i\hbar}(f(x+\widehat{x})-f'(x)\widehat{x})\right):\mathcal{V}_0^\bullet\stackrel{\sim}{\to}\mathcal{V}_f^\bullet$$

This follows immediately from the constructions above.

We see that, if we disregard the filtration, all modules $\mathcal{V}_f$ are isomorphic. The filtration is what distinguishes among them.

9.5 The Filtration and Microsupport

The filtration on $\mathcal{V}_f^\bullet$ that is constructed in Sect. 9.3 is defined as follows:

$$\mathrm{Filt}^0\mathcal{V}_f^\bullet=\Omega^\bullet_{\mathbb{R}^{2n}}(\mathcal{V}_\Lambda)$$

where

$$\mathcal{V}_\Lambda=\mathrm{Sp}^4(2n)\cdot\widehat{\widehat{\mathbb{V}}}_\Lambda \tag{9.5.1}$$

The microsupport of $\mathcal{V}_f^\bullet$ is graph(df), as seen from formula for $\nabla_{\mathcal{V}}$ in Sect. 9.4.4.

10 The Complex Computing $\mathbb{R}\,\mathrm{Hom}(\mathcal{V}_0^\bullet, \mathcal{V}^\bullet)$

10.1 The Simplified Version

Let, as above, $\mathbb{A} = \mathbb{C}[\widehat{x}, \widehat{\xi}, \hbar]$ with the Moyal–Weyl product; $\widetilde{G} = \mathrm{Sp}^4(2n)$; $\widetilde{P} = \mathrm{MPar}(2n)$; $\mathcal{A}_0 = \mathbb{C}[\widetilde{G}] \ltimes \mathbb{A}[\hbar^{-1}]$; $\mathcal{B}_0 = \mathbb{C}[\widetilde{P}] \ltimes \mathbb{A}[\hbar^{-1}]$; $\mathbb{V} = \mathbb{C}[\widehat{x}, \hbar]$ on which $\widehat{\xi}$ acts by $i\hbar\frac{\partial}{\partial \widehat{x}}$ and $\widehat{x}$ by multiplication; $\mathcal{V}_0 = \mathcal{A}_0 \otimes_{\mathcal{B}_0} \mathbb{V}[\hbar^{-1}]$ (Note that, since operators $\exp(ai\hbar(\frac{\partial}{\partial \widehat{x}})^2)$ are well defined on $\mathbb{V}$, the group $\widetilde{P}$ acts on $\mathbb{V}$ compatibly with the action of $\mathbb{A}$). In this simplified version all tensor products and cross products are not completed. We start with computing $\mathrm{Ext}^\bullet_{\mathcal{A}_0}(\mathcal{V}_0, \mathcal{V}_0)$.

Proposition 10.1

$$\mathrm{Ext}^\bullet_{\mathcal{A}_0}(\mathcal{V}_0, \mathcal{V}_0) \xrightarrow{\sim} H^\bullet(\widetilde{P}, \mathbb{C}[\hbar, \hbar^{-1}]))$$

where $H^\bullet$ stands for (discrete) group cohomology.

Proof First, by a version of Shapiro's Lemma [6], we see that

$$\mathrm{Ext}^\bullet_{\mathcal{A}_0}(\mathcal{V}_0, \mathcal{V}_0) \xrightarrow{\sim} H^\bullet(\widetilde{P}, C^\bullet(\mathbb{V}, \mathbb{A}, \mathcal{V}_0))$$

where $C^\bullet$ in the right hand side is the standard complex computing $\mathrm{Ext}^\bullet_{\mathbb{A}}(\mathbb{V}, \mathcal{V}_0)$. Second, we have

$$\mathcal{V}_0 = \oplus_{\lambda \in \widetilde{G}/\widetilde{P}} \mathcal{V}_{0,\lambda}$$

An element of $\widetilde{P}$ sends $\mathcal{V}_{0,\lambda}$ to $\mathcal{V}_{0,p\lambda}$ and therefore we have a $\widetilde{P}$-module decomposition

$$\mathcal{V}_0 = \oplus_{\mathcal{O}} \mathcal{V}_{\mathcal{O}}$$

where

$$\mathcal{V}_{\mathcal{O}} = \oplus_{\lambda \in \mathcal{O}} \mathcal{V}_{0,\lambda}$$

Lemma 10.2 *For all $\mathcal{O}$ except the one-point orbit $\widetilde{P}$,*

$$H^\bullet(\widetilde{P}, C^\bullet(\mathbb{V}, \mathbb{A}, \mathcal{V}_{\mathcal{O}})) = 0$$

This follows from results of Sect. 10.4.2. Finally, $\mathbb{C}[\hbar, \hbar^{-1}] \to C^\bullet(\mathbb{V}, \mathbb{A}, \mathcal{V}[\hbar^{-1}])$ is a quasi-isomorphism and $\mathbb{V}[\hbar^{-1}] = \mathcal{V}_{\mathcal{O}}$ where $\mathcal{O}$ is the one-point orbit. □

This proves the simplified case of Theorem 10.9. The actual theorem is more complicated because our actual module consists of forms with values in completed $\mathcal{V}_0$, and we take not only the complex of derived morphisms between them but also the derived invariants of the fundamental groupoid with values in the De Rham complex. It is almost evident that taking derived invariants of the fundamental groupoid will get rid of the dependence on a point (x, ξ) of our space and reduce the problem to the above, after some completion and tensoring by the Novikov ring. The remainder of the section just makes this explicit (in addition to Sect. 10.4.2 that was mentioned and used above). The main and only point is to construct explicitly a resolution of the differential graded module $\mathcal{V}_0$ that carries an action of $\pi_1(\mathbb{R}^{2n})$.

10.2 The Statement of the Result

Here we state the general result for any $\mathcal{V}^\bullet$ (Proposition 10.4). Let $M = \mathbb{R}^{2n}$. Given an oscillatory module $\mathcal{V}^\bullet$ on M, construct the following complex. Note first that the group $\mathrm{MPar}(n, \mathbb{R})$ acts on the linear span of $d\widehat{x}_1, \ldots, d\widehat{x}_n$ through the projection $\mathrm{MPar}(n) \to \mathrm{ML}^4(n) = \{(b, z) | b \in \mathrm{GL}(n), z \in \mathbb{C}, z^4 = \det(b)^2\}$. Introduce the vector space

$$\wedge\, (d\widehat{x}_1, \ldots, d\widehat{x}_n) d^{-\frac{1}{2}}\widehat{x} \tag{10.2.1}$$

where

$$d^{\frac{1}{2}}\widehat{x} = (d\widehat{x}_1 \ldots d\widehat{x}_n)^{-\frac{1}{2}}$$

is a formal element on which a pair (A, z) in MPar (if we use notation from Definition 12.7, (b)) acts via multiplication by z. Consider the space

$$\wedge\, (d\widehat{x}_1, \ldots, d\widehat{x}_n) d^{-\frac{1}{2}}\widehat{x} \otimes \mathcal{V}^\bullet \tag{10.2.2}$$

with the following structures.

10.2.1 The Differential

Define the differential on (10.2.2) as

$$\widetilde{\nabla}_{\mathcal{V}} = \frac{1}{i\hbar}(\xi dx + \widehat{\xi} d\widehat{x} - \widehat{x} d\xi) + \nabla_{\mathcal{V}}$$

One checks that $\widetilde{\nabla}^2 = 0$. In fact,

$$\frac{1}{i\hbar}\nabla_{\mathcal{V}}(\xi dx + \widehat{\xi} d\widehat{x} - \widehat{x} d\xi) + \frac{1}{(i\hbar)^2}(\xi dx + \widehat{\xi} d\widehat{x} - \widehat{x} d\xi)^2 =$$

$$\frac{1}{i\hbar}(-d\xi d\widehat{x} + d\xi dx + dx d\xi - d\widehat{x} d\xi) = 0$$

10.2.2 The Action of MPar

Denote by $\mathrm{MPar}(n, \mathbb{R})_M$ the sheaf of smooth sections of the associated (in our case trivial) bundle of groups with fiber $\mathrm{MPar}(n)$.

There is an obvious action of $\mathrm{MPar}(n, \mathbb{R})_M$ on (10.2.2) but we have to modify it to make it commute with the differential. Put

$$\mathbf{R}_h = h + \frac{1}{i\hbar}[\iota_{d\widehat{x}} dx, h] \tag{10.2.3}$$

Here [,] stands for the commutator of operators on (10.2.2);

$$\iota_{d\widehat{x}} dx = \sum_{j=1}^{n} \iota_{d\widehat{x}_j} dx_j;$$

and $\iota_{d\widehat{x}_j}$ is the graded derivation of $\wedge(d\widehat{x}_1, \ldots, d\widehat{x}_n)$ that sends $d\widehat{x}_j$ to one and $d\widehat{x}_k$ to zero for $k \neq j$. One checks immediately that

$$\mathbf{R}_{h_1} \mathbf{R}_{h_2} = \mathbf{R}_{h_1 h_2} \tag{10.2.4}$$

Lemma 10.3

$$\widetilde{\nabla}_{\mathcal{V}} \mathbf{R}_h = \mathbf{R}_h \widetilde{\nabla}_{\mathcal{V}} \tag{10.2.5}$$

Proof For a local section h of $\mathrm{MPar}(n)_M$, define $\alpha(h) \in \Omega^1_M(\mathcal{A}_M)$ by

$$\alpha(h) = -dh \cdot h^{-1} + A_{-1} - \mathrm{Ad}_h(A_{-1}) \tag{10.2.6}$$

where $A_{-1} = \frac{1}{i\hbar}(-\widehat{\xi} dx + \widehat{x} d\xi)$. Note that

$$\nabla_{\mathcal{V}}(\mathbf{R}_h v) = -\alpha(h) \mathbf{R}_h v + \mathbf{R}_h \nabla_{\mathcal{V}} v; \tag{10.2.7}$$

$$-\frac{1}{i\hbar}(\xi dx + \widehat{\xi} d\widehat{x} - \widehat{x} d\xi)(\mathbf{R}_h v) = -\alpha(h) \mathbf{R}_h v - \mathbf{R}_h \frac{1}{i\hbar}(\xi dx + \widehat{\xi} d\widehat{x} - \widehat{x} d\xi) v \tag{10.2.8}$$

The first equation is equivalent to the fact that $\mathcal{V}^\bullet$ is a differential graded $\mathcal{A}^\bullet_M$-module. The second is checked by a direct computation:

$$\frac{1}{i\hbar}[\widehat{\xi} dx + \xi dx - \widehat{x} d\xi, h] = -\frac{1}{i\hbar}[\widehat{x} d\xi, h];$$

$$\frac{1}{i\hbar}[\widehat{\xi} dx + \xi dx - \widehat{x} d\xi, [\iota_{d\widehat{x}} dx, h]] = \frac{1}{i\hbar}[[\widehat{\xi} dx + \xi dx - \widehat{x} d\xi, \iota_{d\widehat{x}}] dx, h] +$$

$$[\iota_{d\widehat{x}} dx, \frac{1}{i\hbar}[\widehat{\xi} dx + \xi dx - \widehat{x} d\xi, h]] = \frac{1}{i\hbar}[\widehat{\xi} dx, h]$$

(the second summand vanishes). Therefore

$$\frac{1}{i\hbar}[\widehat{\xi}dx + \xi dx - \widehat{x}d\xi, \mathbf{R}_h] = \frac{1}{i\hbar}[\widehat{\xi}dx - \widehat{x}d\xi, h] = -[\nabla_{\mathcal{V}}, h].$$

Equation (10.2.5) immediately follows. □

Proposition 10.4 *The standard complex computing group cohomology*

$$C^{\bullet}(\mathrm{MPar}(n)_M, \wedge(d\widehat{x}_1, \ldots, d\widehat{x}_n)d^{-\frac{1}{2}}\widehat{x} \otimes \mathcal{V}^{\bullet})$$

is quasi-isomorphic to the complex $\mathcal{C}^{\bullet}(\mathcal{V}_0^{\bullet}, \mathcal{A}_M^{\bullet}, \mathcal{V}^{\bullet})$.

More precisely,

$$C^{\bullet} = \oplus_{m=0}^{\infty} \mathrm{Hom}((\mathrm{MPar}(n)_M)^m, \wedge(d\widehat{x}_1, \ldots, d\widehat{x}_n)d^{-\frac{1}{2}}\widehat{x} \otimes \mathcal{V}^{\bullet});$$

$$(\delta D)(h_1, \ldots, h_{m+1}) = (-1)^m \widetilde{\nabla}_{\mathcal{V}} D(h_1, \ldots, h_{m+1}) + \mathbf{R}_{h_1} D(h_2, \ldots, h_{m+1}) +$$

$$+ \sum_{j=1}^{m} (-1)^j D(h_1, \ldots, h_j h_{j+1}, \ldots, h_{m+1}) + (-1)^{m+1} D(h_1, \ldots, h_m);$$

The following Sect. 10.3 is devoted to the proof of Proposition 10.4.

10.3 The Resolution of $\mathcal{V}_0$ and the Computation of $\mathbb{R}\mathbf{Hom}(\mathcal{V}_0, \mathcal{V})$

10.3.1 A Resolution of $\mathcal{V}_0$

As above, let $M = \mathbb{R}^{2n}$. First construct a resolution $\mathbb{P}^{\bullet}$ that is only free over $\widehat{\widehat{\mathbb{A}}}_M$, not over $\mathcal{A}_M$. This resolution is a free module over

$$\widehat{\widehat{\mathbb{A}^{\bullet}}}_M = \Omega_M^{\bullet}(\widehat{\widehat{\mathbb{A}}}_M) \tag{10.3.1}$$

with the space of generators

$$\wedge(e_1, \ldots, e_n)v_0; \;\; |v_0| = 0; \;\; |e_j| = -1$$

with the differential $\nabla_{\mathbb{P}}$ defined by the following properties:

$$\nabla_{\mathbb{P}} v_0 = \frac{1}{i\hbar}(-\xi dx + \widehat{x}d\xi)v_0; \;\; \nabla e_j = \widehat{\xi}_j; \tag{10.3.2}$$

$$\nabla_{\mathbb{P}}(av) = \nabla_{\mathcal{A}} a \cdot v + (-1)^{|a|} a \nabla_{\mathbb{P}} v$$

for any a in $\widehat{\widehat{\mathbb{A}^{\bullet}}}_M$ and any v in $\mathbb{P}$; and

$$\nabla_{\mathbb{P}}(\beta v_0) = \nabla_{\mathbb{P}} \beta \cdot v + (-1)^{|\beta|} \beta \nabla_{\mathbb{P}} v_0$$

for any β in $\wedge(e_1, \ldots, e_n)$. A simple computation shows that $\nabla^2 = 0$.

Next we construct a $\mathcal{B}_M^{\bullet}$-free resolution of the $\mathcal{B}_M^{\bullet}$-module $\widehat{\mathcal{V}}_M^{\bullet}$. Here, as always, $\mathcal{B}_M^{\bullet}$ stands for forms with coefficients in the (trivial) bundle of algebras associated to $\mathcal{B}$, and $\widehat{\mathcal{V}}_M^{\bullet}$ stands for forms with coefficients in the bundle of modules associated to $\widehat{\widehat{\mathcal{V}}}$, cf. Definition 9.5. We first observe that $\mathbb{P}^{\bullet}$ is in fact a $\mathcal{A}^{\bullet}$-module, though not free. Indeed, to define an $\mathcal{B}_M^{\bullet}$-action, we have to define an MPar_M-action compatible with the action of the smaller algebra and with the differential. We are going to do this next.

10.3.2 The Action of $\mathrm{MPar}(n)_M$

The action of $\mathrm{MPar}(n)_M$ extends from $\widehat{\mathcal{V}}_M^{\bullet}$ to $\mathbb{P}^{\bullet}$ because of the following. The group MPar also acts on $\wedge(e_1, \ldots, e_n)$. The latter action is induced by the linear action on $\mathbb{R}^n$ which in our context is the easiest to describe as follows: identify e_j with $\widehat{\xi}_j$ and therefore $\mathbb{R}^n$ with the linear span of $\widehat{\xi}_j$ in $\widehat{\mathbb{A}}$. The action of MPar through the composition MPar $\to$ GL $\to$ Sp on $\widehat{\mathbb{A}}$ leaves this subspace invariant. This is the action that we mean.

Recall again that an element of $\mathrm{MPar}(n)$ may be represented by a pair

$$\left(\begin{bmatrix} b & a \\ 0 & b^{t-1} \end{bmatrix}, z \right); \ \det(b)^2 = z^4.$$

This element sends v_0 to $u^{-1} v_0$. Combined with the above, we get an action of $\mathrm{MPar}(n)$ on $\wedge(e_1, \ldots, e_n) v_0$.

Unfortunately, this action does not make $\mathbb{P}^{\bullet}$ a differential graded $\mathcal{B}_M$-module. To achieve that, we have to change the action as follows:

$$\mathbf{R}_h = h + \left[\frac{1}{i\hbar} edx, h \right] \tag{10.3.3}$$

Here $edx = \sum_j e_j dx_j$. The commutator is just the commutator of operators on $\mathbb{P}^{\bullet}$. This action, unlike the previous one, makes $\mathbb{P}^{\bullet}$ a differential graded $\mathbb{P}^{\bullet}$-module, which is equivalent to the following.

One has

$$\nabla_{\mathbb{P}}(\mathbf{R}_h v) = -\alpha(h) \mathbf{R}_h v + \mathbf{R}_h \nabla_{\mathbb{P}} v \tag{10.3.4}$$

10.3.3 The Resolution $\mathcal{P}^\bullet$

Now define

$$\mathcal{P}^\bullet = \mathcal{B}_{-\bullet}(\mathrm{MPar}(n)_M, \mathbb{P}^\bullet) = \oplus_{m=0}^{\infty}\mathbb{C}[\mathrm{MPar}(n)_M]^{\otimes m}\widehat{\otimes}\mathbb{P}^\bullet \tag{10.3.5}$$

The action of $\mathcal{B}_M$ on $\mathcal{P}^\bullet$ is given by

$$h((h_1, \ldots, h_m) \otimes v) = (hh_1, \ldots, h_m) \otimes \mathbf{R}_h v$$

(cf. (10.3.3));

$$a((h_1, \ldots, h_m) \otimes v) = (h_1, \ldots, h_m) \otimes av$$

for h in $\mathrm{MPar}(n)_M$ and a in $\widehat{\widehat{\mathbb{A}}}_M$.

This is the standard bar resolution of the MPar-module $\mathbb{P}^\bullet$. More precisely, the differential is given by

$$\nabla_{\mathcal{P}} = \nabla_{\mathcal{P}}^{(0)} + \nabla_{\mathcal{P}}^{(1)}$$

$$\nabla^{(0)}((h_1, \ldots, h_m) \otimes v) = (-1)^m (h_1, \ldots, h_m) \otimes \nabla_{\mathbb{P}} v \tag{10.3.6}$$

$$\nabla^{(1)}((h_1, \ldots, h_m) \otimes v) = \sum_{j=1}^{m-1} (-1)^j (h_1, \ldots, h_j h_{j+1}, \ldots, h_m) \otimes v \tag{10.3.7}$$

$$+(-1)^m (h_1, \ldots, h_{m-1}) \otimes v$$

Finally, put

$$\mathcal{R}^\bullet = \mathcal{A}_M^\bullet \widehat{\otimes}_{\mathcal{B}_M^\bullet} \mathcal{P}^\bullet \tag{10.3.8}$$

10.4 *The Complex* Hom$(\mathcal{R}^\bullet, \mathcal{V}^\bullet)$

The complex

$$\mathrm{Hom}_{\mathcal{A}^\bullet}(\mathcal{R}^\bullet, \mathcal{V}^\bullet) \tag{10.4.1}$$

is now straightforward to compute for any oscillatory module $\mathcal{V}$ on $\mathbb{R}^{2n}$. It is the complex of cochains of the group $\mathrm{MPar}(n)_M$ with coefficients in the module $\wedge(e_1^*, \ldots, e_n^*)v_0^* \otimes \mathcal{V}$,

$$\mathrm{Hom}_{\mathcal{A}^\bullet}(\mathcal{R}^\bullet, \mathcal{V}^\bullet) \xrightarrow{\sim} C^\bullet(\mathrm{MPar}(n)_M, \wedge(e_1^*, \ldots, e_n^*)v_0^* \otimes \mathcal{V}) \tag{10.4.2}$$

Here $|e_j^*| = 1$; $|v_0^*| = 0$; the action of MPar on $\wedge(e_1^*, \ldots, e_n^*)v_0^*$ is dual to the one from Sect. 10.3.2. It is straightforward that this complex is identical to the one in Proposition 10.4.

10.4.1 The Case $\mathcal{V} = \mathcal{V}_f$

Now we are able to compute $\mathbb{R}\,\mathrm{Hom}_{\mathcal{A}^\bullet_M}(\mathcal{V}^\bullet_0, \mathcal{V}^\bullet_f)$. Recall (9.2.3)

$$\widehat{\widehat{\mathbb{V}}}_{\mathbb{K}} = \left\{ \sum_{k=0}^{\infty} e^{\frac{1}{i\hbar}c_k} v_k | \in \widehat{\widehat{\mathbb{V}}};\ c_k \in \mathbb{R};\ c_k \to \infty \right\}$$

Here we view this space with the following action of $\mathrm{MPar}(n, \mathbb{R})$:

$$\begin{bmatrix} b & 0 \\ 0 & b^{-1} \end{bmatrix} \mapsto S_b,\ (S_b f)(x) = f(b^{-1}x);$$

$$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \mapsto \exp\left(-\frac{i\hbar}{2} a \frac{\partial}{\partial \widehat{x}}^2 \right)$$

Now define the $\mathrm{MPar}(n)$-module

$$\Omega^{\bullet;\bullet}_{\mathbb{K}} = \wedge(d\widehat{x}_1, \ldots, d\widehat{x}_n) \otimes \mathbb{C}[\mathrm{Sp}^4(n)] \otimes_{\mathrm{MPar}(n)} \widehat{\widehat{\mathbb{V}}}_{\mathbb{K}} \tag{10.4.3}$$

and the $\mathrm{MPar}(n)_M$-module of $\Omega^\bullet_{\mathbb{K}}$ forms with coefficients in (10.4.3).

Remark 10.5 Intuitively, $\Omega^{\bullet;\bullet}_{\mathbb{K}}$ is the space of expressions

$$\sum_{J,K,j} \exp\left(\frac{1}{i\hbar} \varphi_{j,J,K}(x, \xi, \widehat{x}) \right) a_{j,J,K}(x, \xi, \widehat{x}) dx_J d\widehat{x}_K \tag{10.4.4}$$

where linear term of $\varphi_{j,J,K}(x, \xi, \widehat{x})$ with respect to $\widehat{x}$ is zero, and its quadratic term may be infinite; more precisely, it is allowed to be not just a quadratic form but a point of the Lagrangian Grassmannian.

The differential on $\wedge(d\widehat{x}_1, \ldots, d\widehat{x}_n) \otimes \widehat{\mathcal{V}}^\bullet_{\mathbb{K}})$ is

$$d_f = \frac{\partial}{\partial \xi} d\xi + \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial \widehat{x}} \right) dx + \frac{\partial}{\partial \widehat{x}} d\widehat{x} + \frac{1}{i\hbar}(f'(x + \widehat{x}) - f'(x)) d\widehat{x} + \frac{1}{i\hbar}(f'(x) - f''(x)\widehat{x}) dx \tag{10.4.5}$$

One has

$$d_f = \left(\exp\left(-\frac{1}{i\hbar}(f(x + \widehat{x}) - f'(x)\widehat{x} \right) \right) d_0 \left(\exp\left(\frac{1}{i\hbar}(f(x + \widehat{x}) - f'(x)\widehat{x} \right) \right) \tag{10.4.6}$$

Proposition 10.6 *The standard complex $\mathcal{C}_{\bullet \mathcal{A}^\bullet_M}(\mathcal{V}^\bullet_0, \mathcal{V}^\bullet_f)$ is quasi-isomorphic to the complex*

$$C^\bullet(\mathrm{MPar}(n)_M, \Omega^{\bullet;\bullet}_{\mathbb{K}}). \tag{10.4.7}$$

10.4.2 A Stationary Phase Statement

Lemma 10.7 *For any positive integer p, consider* $\mathbb{R}^p$ *viewed as a discrete group. One has*

$$H^\bullet(\mathbb{R}^p, \mathbb{C}[\mathbb{R}^p]) = 0.$$

Proof One has $\mathbb{R}^p \xrightarrow{\sim} \bigoplus \mathbb{Q}$. Therefore $\mathbb{R}^p \xrightarrow{\sim} \mathbb{Q} \oplus \mathbb{R}^p$. By Künneth formula,

$$H^\bullet(\mathbb{R}^p, \mathbb{C}[\mathbb{R}^p]) \xrightarrow{\sim} H^\bullet(\mathbb{Q}, \mathbb{Q}) \otimes H^\bullet(\mathbb{R}^p, \mathbb{C}[\mathbb{R}^p]).$$

But $H^0(\mathbb{Q}, \mathbb{Q}) = 0$. If k is the minimal integer such that $H^k(\mathbb{R}^p, \mathbb{C}[\mathbb{R}^p]) \neq 0$, Künneth formula tells that $H^k = 0$, whence the contradiction. □

Corollary 10.8 *Let* Ω *be an orbit of* $\mathrm{MPar}(n, \mathbb{R})$ *in the Lagrangian Grassmannian* $\Lambda(n)$ *that consists of more than one point.Then*

$$H^\bullet(\mathrm{MPar}(n), \mathbb{C}[\Omega]) = 0.$$

Proof Let N be the subgroup of $\mathrm{MPar}(n, \mathbb{R})$ consisting of pairs

$$\left(\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}, 1\right)$$

(in other words, $N = \mathrm{Ker}(\mathrm{MPar}(n) \to \mathrm{GL}^4(n))$). Choose a point in Ω. Denote its stabilizer by Z. Then Z is a real vector subspace of N. Let W be a complementary subspace to Z. Consider the Lyndon spectral sequence

$$E_2^{pq} = H^p(N/Z, H^q(Z, \mathbb{C}[\Omega])) \implies H^{p+q}(N, \mathbb{C}[\Omega]).$$

But $\Omega \xrightarrow{\sim} Z$ as a Z-set, so $H^\bullet(N, \mathbb{C}[\Omega]) = 0$ by Lemma 10.7. Now consider the Lyndon spectral sequence

$$E_2^{pq} = H^p(\mathrm{GL}^4(n), H^q(N, \mathbb{C}[\Omega])) \implies H^{p+q}(\mathrm{MPar}(n), \mathbb{C}[\Omega]).$$

The statement follows. □

10.5 *The Computation of* $\mathbb{R}\,\underline{\mathbf{HOM}}(\mathcal{V}_0, \mathcal{V}_f)$

Let

$$\mathcal{S}^\bullet = C^\bullet(\mathrm{MPar}(n), \mathbb{K}) \tag{10.5.1}$$

Theorem 10.9

$$\underline{\mathbb{R}\,\mathrm{HOM}}^\bullet(\mathcal{V}_0^\bullet, \mathcal{V}_f^\bullet) \xrightarrow{\sim} \mathcal{S}^\bullet$$

with the action of a path from $(x_1\xi_1)$ *to* (x_2, ξ_2) *given by multiplication by* $\exp(\frac{1}{i\hbar}(f(x_1) - f(x_2))$.

Proof First one checks that all the structures for $\mathcal{V}_0^\bullet$ and $\mathcal{V}_f^\bullet$ are conjugate by multiplication by $\exp(\frac{1}{i\hbar}(f(x+\widehat{x}) - f'(x)\widehat{x}))$. So we can reduce the statement to the case $f = 0$. The cohomology in question is computed by the complex

$$C^\bullet(\pi_1(M), C^\bullet(\mathrm{MPar}(n)_M, \Omega^{\bullet,\bullet})). \tag{10.5.2}$$

First compute the cohomology of $\pi_1(M)$. An argument identical to the one in Introduction (starting before (1.1.6)) shows that this cohomology is isomorphic to

$$H^\bullet(\mathrm{MPar}(n), \mathbb{C}[\mathrm{Sp}^4]\widehat{\otimes}_{\mathbb{C}[\mathrm{MPar}(n)]}\widehat{\widehat{\mathbb{V}}}_{\mathbb{K}}) \tag{10.5.3}$$

(In other words, all dependence on x, ξ, and $dx, d\xi$ is eliminated). Now, by Corollary 10.8, all contributions from all Lagrangian submanifolds other than $L_0 = \{\xi = 0\}$ are also eliminated. Our cohomology is therefore computed by the complex

$$C^\bullet(\mathrm{MPar}(n), \wedge(d\widehat{x}_1, \ldots, d\widehat{x}_n) \otimes \widehat{\widehat{\mathbb{V}}}_{\mathbb{K}}) \tag{10.5.4}$$

of group cochains of $\mathrm{MPar}(n)$ with coefficients in the complex $\wedge(d\widehat{x}_1, \ldots, d\widehat{x}_n) \otimes \widehat{\widehat{\mathbb{V}}}_{\mathbb{K}}$ of formal forms in $\widehat{x}$ with the differential $\frac{\partial}{\partial\widehat{x}}d\widehat{x}$. □

10.6 The Case of Sheaves

Here we compare the computation above to the analogous computation for the microlocal category of sheaves as in Sect. 1.7.

Proposition 10.10 *Let f and g be two C^∞ functions on $\mathbb{R}^n$. For a bounded contractible open subset of $\mathbb{R}^n$, the module of horizontal sections of the local system $\underline{\mathbb{R}\,\mathrm{HOM}}(\mathcal{V}_g^\bullet, \mathcal{V}_f^\bullet)$ on U is a free $\mathcal{S}^\bullet$-module with one generator $J(f,g)$ lying in $\mathrm{Filt}^{-\inf_U(f-g)}$. The composition is as follows:*

$$J(f,g)J(g,h) = \exp\left(\frac{1}{i\hbar}c(f,g,h)\right)J(f,h)$$

where

$$c(f,g,h) = \inf_U(f-h) - \inf_U(f-g) - \inf_U(g-h))$$

Proof It is easy to see that

$$\underline{\mathbb{R}\mathrm{HOM}}(\mathcal{V}_g^\bullet, \mathcal{V}_f^\bullet) \xrightarrow{\sim} \underline{\mathbb{R}\mathrm{HOM}}(\mathcal{V}_0^\bullet, \mathcal{V}_{f-g}^\bullet)$$

Put

$$J(f,g) = \exp\left(\frac{1}{i\hbar}((f-g)(x+\widehat{x}) - (f-g)'(x)\widehat{x} - \inf_U(f-g)\right) \qquad (10.6.1)$$

The statement follows from Theorem 10.9. □

Compare this to the following result of Tamarkin. Recall the definitions from Sect. 1.7.1. Put

$$\mathbb{K}_{\mathbb{Z}} = \left\{\sum_{k=0}^{\infty} a_k e^{-\frac{c_k}{i\hbar}}\right\}$$

where $a_k \in \mathbb{Z}$, $c_k \in \mathbb{R}$, and $c_k \to \infty$. For any two objects $\mathcal{F}$ and $\mathcal{G}$ of $D(T^*\mathbb{R}^n)$, let $\mathrm{HOM}_{\mathbb{K}}(\mathcal{F},\mathcal{G}) = \mathbb{K}_{\mathbb{Z}} \otimes_{\Lambda_{\mathbb{Z}}} \mathrm{HOM}(\mathcal{F},\mathcal{G})$. Let $\mathrm{Filt}^c\,\mathrm{HOM}_{\mathbb{K}} = e^{\frac{c}{i\hbar}}\,\mathrm{HOM}$.

Proposition 10.11 *Let f and g be two C^∞ functions on $\mathbb{R}^n$. For a bounded contractible open subset U of $\mathbb{R}^n$, consider the objects $\mathcal{F}_f$ and $\mathcal{F}_g$ of $D(T^*U)$ as in Sect. 1.7.1. The complex* $\mathrm{HOM}_{\mathbb{K}}(\mathcal{F}_g, \mathcal{F}_f)$ *is quasi-isomorphic to a free $\mathbb{K}_{\mathbb{Z}}$-module with one generator $J(f,g)$ lying in* $\mathrm{Filt}^{-\inf_U(f-g)}$. *The composition satisfies the same formulas as in Proposition 10.10.*

Proof Recall that $\mathcal{F}_f = \mathbb{Z}_{t+f\geq 0}$. It is immediate that

$$\mathrm{HOM}_{\mathbb{K}}(\mathcal{F}_g, \mathcal{F}_f) \xrightarrow{\sim} \mathrm{HOM}_{\mathbb{K}}(\mathcal{F}_0, \mathcal{F}_{f-g}) \qquad (10.6.2)$$

Let $J(f,g)$ be the morphism $\mathbb{Z}_{t\geq 0} \to \mathbb{Z}_{t+f-g-\inf_U(f-g)\geq 0}$ which is the restriction to the subset $\{t+f-g-\inf_U(f-g)\geq 0\} \subset \{t \geq 0\}$. It is clear that the right hand side of (10.6.2) is the free $\mathbb{K}_{\mathbb{Z}}$-module generated by $J(f,g)$, that $J(f,g)$ is in $\mathrm{Filt}^{-\inf_U(f-g)}$, and that the composition is as in Proposition 10.10. □

10.6.1 Matrix Units

Now put

$$\mathbf{E}_{\mathbf{f},\mathbf{g}} = \exp\left(\frac{1}{i\hbar}\inf_U(f-g)\right) J(f,g) \in \mathrm{HOM}_{\mathbb{K}}(\mathcal{F},\mathcal{G}) \qquad (10.6.3)$$

in $D(T^*U)$. Then

$$\mathbf{E}_{\mathbf{f},\mathbf{g}}\mathbf{E}_{\mathbf{g},\mathbf{h}} = \mathbf{E}_{\mathbf{f},\mathbf{h}} \qquad (10.6.4)$$

11 $\mathbb{R}$ Hom and Theta Functions

11.1 Modules Associated to the Lagrangian Submanifold $\xi = mx$

In this section, $M = \mathbb{T}^2$ and $\widetilde{M} = \mathbb{R}^2$ with the standard symplectic form $\omega = d\xi dx$.

11.1.1 The Groupoid $\widetilde{\mathbf{G}}_M$

Local sections of $\widetilde{\mathbf{G}}_M$ are in bijection with smooth local functions $g(x_1, \xi_1; x_2, \xi_2)$ on $\widetilde{M} \times \widetilde{M}$ with values in $\widetilde{\mathbf{G}}$ for $n = 1$ (cf. Sect. 13.1). As in Sect. 9.4.1, we denote a section corresponding to g by a formal symbol

$$\sigma(x_1, \xi_1; x_2, \xi_2) = \exp\left(\frac{1}{i\hbar}(\xi_2 - \xi_1)\widehat{x} + (x_1 - x_2)\widehat{\xi}\right) g(x_1, \xi_1; x_2, \xi_2) \quad (11.1.1)$$

These sections satisfy

$$\sigma(x_1, \xi_1; x_2, \xi_2) = \exp\left(\frac{1}{i\hbar}(x_1 - x_2)\right) \sigma(x_1, \xi_1 + 1; x_2, \xi_2 + 1); \quad (11.1.2)$$

$$\sigma(x_1, \xi_1; x_2, \xi_2) = \sigma(x_1 + 1, \xi_1; x_2 + 1, \xi_2). \quad (11.1.3)$$

As in Sect. 9.4.1, the composition consists of formal multiplication of exponentials and multiplication of elements of $\mathrm{Sp}^4(2)$.

The flat connection up to inner derivations on $\widetilde{\mathbf{G}}_M$ is given exactly as in Sect. 9.4.4: for a section σ as in (9.4.1),

$$-\alpha(\sigma) = \nabla_{\widetilde{\mathbf{G}}}\sigma \cdot \sigma^{-1} = d_{\mathrm{DR}} g \cdot g^{-1} + \frac{1}{i\hbar}(\xi_2 dx_2 - \xi_1 dx_1) +$$
$$\left(-\frac{\widehat{\xi}_1}{i\hbar} dx_1 + \frac{\widehat{x}_1}{i\hbar} d\xi_1\right) - \mathrm{Ad}_g \left(-\frac{\widehat{\xi}_2}{i\hbar} dx_2 + \frac{\widehat{x}_2}{i\hbar} d\xi_2\right)$$

11.1.2 The Sheaf $\mathcal{V}^{\bullet}_{L_m}$

Denote by $\mathcal{V}^{\bullet}_{L_m}$ the oscillatory module corresponding to the Lagrangian submanifold $\xi = mx$. Local sections of $\mathcal{V}^{\bullet}_{L_m}$ are sums

$$v = \sum_{k \in \mathbb{Z}} v_k \quad (11.1.4)$$

where v_k is a local section of $\mathcal{V}^\bullet_{m\frac{x^2}{2}+kx}$ on $\widetilde{M}$. In other words, v_k is an $\Omega_{\mathbb{K}}$-form on $\widetilde{M}$ with coefficients in $\widehat{\mathcal{V}}$ (Definition 9.5). The connection $\nabla_{\mathcal{V}}$ is given by (cf. Sect. 9.4.4)

$$\nabla_{\mathcal{V}} v_k = \left(-\frac{\xi - mx - k}{i\hbar} dx + \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial \widehat{x}} - \frac{1}{i\hbar} m\widehat{x}\right) dx + \left(\frac{\partial}{\partial \xi} + \frac{1}{i\hbar}\widehat{x}\right) d\xi\right) v_k \tag{11.1.5}$$

The action of $\widehat{\widehat{\mathbb{A}}}_M$ is as follows (cf. Sect. 9.4.5): $\widehat{x}$ by multiplication, and $\widehat{\xi}$ by $i\hbar\frac{\partial}{\partial \widehat{x}} + m\widehat{x}$.

Remark 11.1 The component v_k is an element of the form $\sigma(x, \xi; x, \xi - mx - k)w_k$ where w_k is a local section of the module $\mathbb{V}_{L_m}$ (cf. Sect. 9.2). Also note that sums (11.1.4) may be infinite but we require that $v_k \in \exp(\frac{1}{i\hbar}N_k)\widehat{\mathcal{V}}_\Lambda$ where $N_k \to \infty$ as $|k| \to \infty$.

Components v_k satisfy

$$v_k(x, \xi) = v_{k+1}(x, \xi + 1) = v_{k-m}(x + 1, \xi). \tag{11.1.6}$$

The action of $\widetilde{\mathbf{G}}_M$ on $\mathcal{V}_{L_m}$ is as follows:

$$\sigma(x_1, \xi_1;\ x_2, \xi_2)v_k = \exp\left(-\frac{1}{i\hbar}\left(\frac{mx_1^2}{2} + kx_1 - \frac{mx_2^2}{2} - kx_2\right)\right) g(x_1, \xi_1; x_2, \xi_2)v_k \tag{11.1.7}$$

(cf. Sect. 9.4.6). It is easy to see directly that all the structures are compatible with each other (of course this also follows from the fact that the above construction is obtained by applying the general procedure of Sect. 15).

11.2 *The Computation of* $\underline{\mathbb{R}\,\mathbf{HOM}}(\mathcal{V}^\bullet_{L_0}, \mathcal{V}^\bullet_{L_m})$

11.2.1 Matrices with Coefficients in $\mathcal{S}^\bullet$

Let e_Λ, resp. E, be the free module over Λ, resp. $\mathbb{K}$, with generators $e_k,\ k \in \mathbb{Z}$. Recall the differential graded algebra $\mathcal{S}$ from (10.5.1). Put also

$$\mathcal{S}^\bullet_\Lambda = C^\bullet(\mathrm{MPar}(n), \Lambda) \tag{11.2.1}$$

Let

$$\mathrm{Matr}(\mathcal{S}) = \varprojlim_{N\to\infty} \mathrm{Hom}(E, \mathcal{S}^\bullet \otimes E)/\exp\left(\frac{1}{i\hbar}N\right)\mathrm{Hom}(E, \mathcal{S}^\bullet_\Lambda \otimes E) \tag{11.2.2}$$

Let $\mathbf{E}_{k\ell}$ be the matrix unit, i.e. the homomorphism sending e_k to e_ℓ and e_j to zero if $j \neq k$.

11.2.2

Theorem 11.2 *The sheaf of complexes* $\underline{\mathbb{R}\,\mathrm{HOM}}^\bullet(\mathcal{V}^\bullet_{L_0}, \mathcal{V}^\bullet_{L_m})$ *is quasi-isomorphic to the sheaf of sections of the trivial bundle with fiber* $\mathrm{Matr}(\mathcal{S}^\bullet)$*, with the action of* $\pi_1(M)$ *as follows. Let* γ_1 *and* γ_2 *be the two generators of* $\pi_1(M)$*, namely* γ_1 *the loop* $\xi = \xi_0, x = x_0 + t$ *and* γ_2 *the loop* $x = x_0, \xi = \xi_0 + t$*. Then for a matrix unit* $\mathbf{E}_{k\ell}$

$$\gamma_1^q \gamma_2^p : \mathbf{E}_{k\ell} \mapsto \exp\left(\frac{1}{i\hbar}\left(\frac{mq^2}{2} + q(\ell - k)\right)\right) \mathbf{E}_{k+p,\ell+p-mq}$$

Proof First construct the $\mathcal{A}^\bullet_M$-free resolution $\mathcal{R}^\bullet_{L_0}$ of $\mathcal{V}^\bullet_{L_0}$ as in (10.3.8). Local sections of $\mathcal{R}^\bullet_{L_0}$ are sums (11.1.4) with the same relations (11.1.6) with $m = 0$; v_k are elements of $\mathcal{R}^\bullet_k$ on $\widetilde{M}$ which is constructed exactly as $\mathcal{R}^\bullet$ in (10.3.8) with the only modification: Eq. (11.2.3) becomes

$$\nabla_{\mathbb{P}} v_{0,k} = \frac{1}{i\hbar}(-(\xi + k)dx + \widehat{x}d\xi)v_{0,k} \tag{11.2.3}$$

Now, local sections of $\mathrm{Hom}_{\mathcal{A}^\bullet_M}(\mathcal{R}^\bullet_{L_0}, \mathcal{V}^\bullet_{L_m})$ are sums $\sum_{k,\ell} b_{k\ell}$ where

$$b_{k\ell} \in \mathcal{C}^\bullet_{k\ell};$$

here $\mathcal{C}^\bullet_{k\ell}$ is the complex (10.4.7) computed for the function

$$f_{k\ell}(x, \xi) = mx^2 + (\ell - k)x \tag{11.2.4}$$

Local sections $b_{k\ell}$ satisfy the following:

$$b_{k\ell}(x, \xi) = b_{k,\ell-m}(x + 1, \xi) = b_{k+1,\ell+1}(x, \xi + 1) \tag{11.2.5}$$

(Note that all $\mathcal{C}^\bullet_{k\ell}$ are identical as graded spaces, with the differential $d_{k\ell}$ on $\mathcal{C}^\bullet_{k\ell}$ given by

$$d_{k\ell} = \mathrm{Ad}\left(\exp\left(-\frac{1}{i\hbar}\left(\frac{mx^2}{2} + (\ell - k)x + \frac{m\widehat{x}^2}{2}\right)\right)\right) d_{00}).$$

The action of the fundamental groupoid is as follows. A path $\gamma : (x_1, \xi_1) \to (x_2, \xi_2)$ in $\widetilde{M}$ preserves each $\mathcal{C}^\bullet_{k\ell}$ and acts on it by

$$(\gamma b)_{k\ell}(x_1, \xi_1) = \exp\left(\frac{1}{i\hbar}\left(\frac{mx_1^2}{2} + (\ell - k)x_1 - \frac{mx_2^2}{2} + (\ell - k)x_2\right)\right) b_{k\ell}(x_2, \xi_2) \tag{11.2.6}$$

because of (9.4.1) and because

$$(f_{k\ell}(x_1 + \widehat{x}) - f'_{k\ell}(x_2)\widehat{x}) - f_{k\ell}(x_2 + \widehat{x}) - f'_{k\ell}(x_2)\widehat{x}) =$$

$$= \frac{mx_1^2}{2} + (\ell - k)x_1 - \frac{mx_2^2}{2} - (\ell - k)x_2.$$

When $x_2 - x_1 = q$ and $\xi_2 - \xi_1 = p$, the right hand side of (11.2.6) becomes

$$(\gamma b)_{k+p,\ell+p-mq}(x,\xi) = \exp\left(\frac{1}{i\hbar}\left(\frac{mq^2}{2} + q(\ell - k)\right)\right) b_{k\ell}(x,\xi).$$

The statement now follows from Theorem 10.9. □

Corollary 11.3 *For $m > 0$, the space of horizontal sections of $\underline{\mathbb{R}\,\mathrm{HOM}}^\bullet(\mathcal{V}^\bullet_{L_0}, \mathcal{V}^\bullet_{L_m})$ is m-dimensional over $\mathbb{K}$ with the basis*

$$\theta_a = \sum_{q\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \exp\left(\frac{1}{i\hbar}(mq^2 + aq)\right) \mathbf{E}_{k,k+a-qm}$$

where $a = 0, 1, \ldots, m-1$.

11.3 The Case of Sheaves

Following Tamarkin, we define the category $D(\mathbf{T}^2)$. First define the following diffeomorphisms of $\mathbb{R} \times \mathbb{R}$:

$$S_1(x,t) = (x+1,t);\;\; S_2(x,t) = (x, t+x); \tag{11.3.1}$$

One has

$$S_2S_1 = T_1S_1S_2;\;\; T_1S_1 = S_1T_1;\, T_1S_2 = S_2T_1 \tag{11.3.2}$$

where $T_1(x,t) = (x, t+1)$. (In other words, we have an action of the Heisenberg group $\mathrm{Heis}(3,\mathbb{Z})$ on $\mathbb{R} \times \mathbb{R}$.)

Define objects of $D(\mathbf{T}^2)$ as equivariant objects of $D(\mathbb{R}^2)$, i.e. objects $\mathcal{F}$ of $D(\mathbb{R})^2$ together with isomorphisms

$$\sigma_1 : \mathcal{F} \overset{\sim}{\to} S_{1*}\mathcal{F};\;\; \sigma_2 : \mathcal{F} \overset{\sim}{\to} S_{2*}\mathcal{F} \tag{11.3.3}$$

in $\mathrm{HOM}_{\mathbb{K}}$ such that

$$\sigma_2\sigma_1\tau_1 = \sigma_1\sigma_2 \tag{11.3.4}$$

or more precisely

$$(T_1S_1)_*\sigma_2 \cdot T_{1*}\sigma_1 \cdot \tau_1 = S_{2*}\sigma_1 \cdot \sigma_2 \tag{11.3.5}$$

as morphisms $\mathcal{F} \to (S_2S_1)_*\mathcal{F} = (T_1S_1S_2)_*\mathcal{F}$.

Example 11.4 For an integer n, put

$$\mathcal{F}_m = \prod_{k \in \mathbb{Z}} \mathcal{F}_{m\frac{x^2}{2}+kx} \tag{11.3.6}$$

In fact,

$$(S_1^q S_2^p)(x, t) = (x + q, t + px);$$

$$(S_1^q S_2^p)^* \mathcal{F}_{m\frac{x^2}{2}+kx} = \mathbb{Z}_{\{t+px+m\frac{(x+q)^2}{2}+k(x+q)\geq 0\}} = T^*_{m\frac{q^2}{2}+kq} \mathcal{F}_{m\frac{x^2}{2}+(k+mq+p)x};$$

In other words, if

$$\mathcal{L}_k = \mathcal{F}_{m\frac{x^2}{2}+kx}, \tag{11.3.7}$$

then

$$\mathcal{F}_m = \prod_{k \in \mathbb{Z}} \mathcal{L}_k; \ \ (S_1^q S_2^p)^* \mathcal{L}_k = \left(T_{\frac{mq^2}{2}+kq}\right)^* \mathcal{L}_{k+mq+p} \tag{11.3.8}$$

11.4 Comparison Between the Categories

Consider the following automorphisms of the pair $(\widetilde{\mathbf{G}}_{\mathbb{R}^2}, \mathcal{A}_{\mathbb{R}^2})$. Let $\sigma(x_1, \xi_1; x_2, \xi_2)$ be as in (11.1.1). Define

$$(S_1)\sigma(x_1, \xi_1; x_2, \xi_2) = \sigma(x_1 + 1, \xi_1; x_2 + 1, \xi_2). \tag{11.4.1}$$

$$(S_2\sigma)(x_1, \xi_1; x_2, \xi_2) = \exp\left(\frac{1}{i\hbar}(x_1 - x_2)\right)\sigma(x_1, \xi_1 + 1; x_2, \xi_2 + 1); \tag{11.4.2}$$

For a section a of $\mathcal{A}_{\mathbb{R}^2}$, define

$$(S_1 a)(x, \xi, \widehat{x}, \widehat{\xi}) = a(x + 1, \xi, \widehat{x}, \widehat{\xi}); \ \ (S_2 a)(x, \xi, \widehat{x}, \widehat{\xi}) = a(x, \xi + 1, \widehat{x}, \widehat{\xi}) \tag{11.4.3}$$

It is easy to see that these maps preserve all the structures, i.e. the product on $\mathcal{A}$, the composition on $\widetilde{\mathbf{G}}$, the action of $\widetilde{\mathbf{G}}$ on $\mathcal{A}$, and the flat connection up to inner derivations. Therefore for an oscillatory module $\mathcal{V}^\bullet$ on $\mathbb{R}^2$, one can define new oscillatory modules $S_1^*\mathcal{V}^\bullet$ and $S_2^*\mathcal{V}^\bullet$ as follows. As differential graded $\Omega^\bullet_{\mathbb{K}}$-modules, they are the inverse images of $\mathcal{V}^\bullet$ under the shifts $(x, \xi) \mapsto (x + 1, \xi)$ and $(x, \xi) \mapsto (x, \xi + 1)$; the algebra $\mathcal{A}_{\mathbb{R}^2}$ and the groupoid $\widetilde{\mathbf{G}}_{\mathbb{R}^2}$ act via automorphisms S_1, S_2. One has

$$(S_2^p)^*(S_1^q)^*\mathcal{V}^\bullet_{m\frac{x^2}{2}+kx} = \mathcal{V}^\bullet_{m\frac{x^2}{2}+(k+mq-p)x} \tag{11.4.4}$$

Note that the central subgroup $\{T_c | c \in \mathbb{Z}\}$ of $\mathrm{Heis}(\mathbb{Z})$ acts on $\mathrm{HOM}(\mathcal{F}_0, \mathcal{F}_m)$. Therefore the automorphisms σ_1 and σ_2 generate an action of $\mathbb{Z}^2$.

11.4.1 Matrix Units for Tori

Put

$$f_{m,k}(x) = m\frac{x^2}{2} + kx \tag{11.4.5}$$

Let $m > 0$. Define the matrix unit $\mathbf{E}_{k\ell}$ as follows. Let

$$\mathbf{E}_{f_{m_1,k}, f_{m+m_1,\ell}} \in \mathrm{HOM}_{\mathbb{K}}(\mathcal{F}_{f m_1,k}, \mathcal{F}_{f_{m+m_1,\ell}}) \tag{11.4.6}$$

be as in (10.6.3). Let i_k, resp. pr_k, be the embedding of, resp. the projection onto, the kth component in the decomposition in (11.3.6). Define $\mathbf{E}_{k\ell}$ as the composition

$$i_\ell \circ \mathbf{E}_{f_{m_1,k}, f_{m+m_1,\ell}} \circ \mathrm{pr}_k : \mathcal{F}_{m_1} \to \mathcal{F}_{m_1\frac{x^2}{2}+kx} \to \mathcal{F}_{(m+m_1)\frac{x^2}{2}+\ell x} \to \mathcal{F}_{m+m_1}$$

One has

$$\mathrm{HOM}_{\mathbb{K}}(\mathcal{F}_{f_{m_1,k}}, \mathcal{F}_{f_{m+m_1,\ell}}) = \mathbb{K}\mathbf{E}_{kl}$$

$$\mathbf{E}_{j\ell} = \mathbf{E}_{jk}\mathbf{E}_{k\ell}$$

Proposition 11.5 *The action of the group* $\mathbb{Z}^2$ *on* $\mathrm{HOM}(\mathcal{F}_0, \mathcal{F}_m)$ *is as follows.*

$$\sigma_1^q \sigma_2^p \mathbf{E}_{k\ell} = \exp\left(\frac{1}{i\hbar}\left(m\frac{q^2}{2} + (\ell - k)q\right)\right)\mathbf{E}_{\ell+p,k+p-mq}$$

Now let $m < 0$. There is a generator

$$\mathbf{Z}(f_{m_1,k}, f_{m+m_1,\ell}) \in \mathbb{R}^1\,\mathrm{Hom}(\mathcal{F}_{f_{m_1,k}}, (T_{-\sup f_{m,\ell-k}})_* \mathcal{F}_{f_{m+m_1,\ell}}) \tag{11.4.7}$$

obtained as follows. First, to simplify notation, assume $m_1 = k = 0$, as well as $\sup(f_{m,\ell}) = 0$ (the general case follows immediately). Replace $\mathcal{F}_0 = \mathbb{Z}_{t\geq 0}$ by the complex

$$\mathbb{Z}_{t<0} \to \mathbb{Z} \tag{11.4.8}$$

The complex $\mathrm{Hom}(\mathbb{Z}_{t<0} \to \mathbb{Z}, \mathbb{Z}_{t\geq f_{m,\ell}})$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z} \xrightarrow{(1,-1)} \mathbb{Z}$ and computes $\mathbb{R}\,\mathrm{Hom}(\mathcal{F}_{f_{0,0}}, \mathcal{F}_{f_{m,\ell}})$.

Put

$$\mathbf{Z}_{f_{m_1,k}, f_{m+m_1,\ell}} = \exp(\frac{1}{i\hbar}\sup(f_{m,\ell-k}))\mathbf{Z}(f_{m_1,k}, f_{m+m_1,\ell}) \tag{11.4.9}$$

Define $\mathbf{Z}_{k\ell}$ as the composition

$$i_\ell \circ \mathbf{Z}_{f_{m_1,k}, f_{m+m_1,\ell}} \circ \mathrm{pr}_k : \mathcal{F}_{m_1} \to \mathcal{F}_{m_1\frac{x^2}{2}+kx} \to \mathcal{F}_{(m+m_1)\frac{x^2}{2}+\ell x} \to \mathcal{F}_{m+m_1}$$

We have thus defined

$$\mathbf{E}_{k\ell}(m_2, m_1) \in \mathrm{HOM}^0_{\mathbb{K}}(\mathcal{F}_{m_1}, \mathcal{F}_{m_2}),\ m_1 \le m_2; \tag{11.4.10}$$

$$\mathbf{Z}_{k\ell}(m_2, m_1) \in \mathrm{HOM}^1_{\mathbb{K}}(\mathcal{F}_{m_1}, \mathcal{F}_{m_2}),\ m_1 > m_2; \tag{11.4.11}$$

They satisfy

$$\mathbf{E}_{jk}(m_3, m_2)\mathbf{E}_{k'\ell}(m_2, m_1) = \delta_{kk'}\mathbf{E}_{k\ell}(m_3, m_1); \tag{11.4.12}$$

$$\mathbf{E}_{jk}(m_3, m_2)\mathbf{Z}_{k'\ell}(m_2, m_1) = \delta_{kk'}\mathbf{Z}_{k\ell}(m_3, m_1) \tag{11.4.13}$$

if $m_1 > m_3$ and zero otherwise;

$$\mathbf{Z}_{jk}(m_3, m_2)\mathbf{E}_{k'\ell}(m_2, m_1) = \delta_{kk'}\mathbf{Z}_{k\ell}(m_3, m_1) \tag{11.4.14}$$

if $m_1 > m_3$ and zero otherwise;

$$\mathbf{Z}_{jk}(m_3, m_2)\mathbf{Z}_{k'\ell}(m_2, m_1) = 0 \tag{11.4.15}$$

Proposition 11.6 *The action of the group* $\mathbb{Z}^2$ *on* $\mathrm{HOM}(\mathcal{F}_0, \mathcal{F}_m)$ *is as follows.*

$$\sigma_1^q\sigma_2^p\mathbf{Z}_{k\ell} = \exp\left(\frac{1}{i\hbar}\left(m\frac{q^2}{2} + (\ell - k)q\right)\right)\mathbf{Z}_{\ell+p,k+p-mq}$$

It would be interesting to compare the above to other works, for example [14].

12 Appendix. Metaplectic and Metalinear Groups

We recall the classical material that is contained, for example, in [15, 36].

12.1 Metalinear Groups and Metalinear Structures

Recall [15] that the metalinear group is by definition

$$\mathrm{ML}(n, \mathbb{R}) = \{(g,\ z)|g \in \mathrm{GL}(n, \mathbb{R}),\ z^2 = \det(g)\} \tag{12.1.1}$$

This is a twofold cover of $\mathrm{GL}(n, \mathbb{R})$. There is a morphism

$$\det{}^{\frac{1}{2}} : \mathrm{ML}(n, \mathbb{R}) \to \mathbb{C}^\times;\ \ (g, z) \mapsto z. \tag{12.1.2}$$

Denote by $\mathrm{MO}(n)$ the preimage of $\mathrm{O}(n)$ in $\mathrm{ML}(n)$. Let also

$$\mathrm{MU}(n) = \{(u,\ \zeta)|u \in \mathrm{U}(n, \mathbb{C}),\ \zeta^2 = \det(u)\} \tag{12.1.3}$$

Definition 12.1 Let $\mathrm{Mp}(2n, \mathbb{R})$ be the universal twofold cover of $\mathrm{Sp}(2n, \mathbb{R})$. We call this group *the metaplectic group*.

There is a commutative diagram

$$\begin{array}{ccc} \mathrm{MO}(n) & \longrightarrow & \mathrm{ML}(n, \mathbb{R}) \\ \downarrow & & \downarrow \\ \mathrm{MU}(n) & \longrightarrow & \mathrm{Mp}(2n, \mathbb{R}) \end{array}$$

where the horizontal embeddings are homotopy equivalences.

A metalinear structure on a real vector bundle E is a lifting of the transition automorphisms g^E_{jk} to an $\mathrm{ML}(n, \mathbb{R})$-valued cocycle $\widetilde{g}^E_{jk}$. For a real bundle E with a metalinear structure, the complex line bundle $\wedge^{\frac{1}{2}} E$ is by definition given by the transition automorphisms $\det^{\frac{1}{2}}(\widetilde{g}^E_{jk})$, cf. (12.1.2).

A metaplectic structure on a symplectic vector bundle E is a lifting of the transition automorphisms g^E_{jk} to an $\mathrm{Mp}(n, \mathbb{R})$-valued cocycle $\widetilde{g}^E_{jk}$. A metalinear structure on a manifold (resp. a metaplectic structure on a symplectic manifold) is by definition the corresponding structure on its tangent bundle.

Lemma 12.2 *A manifold X has a metalinear structure if and only if T^*X has a metaplectic structure. If a symplectic manifold has a metaplectic structure then any Lagrangian submanifold of M has a metalinear structure.*

Proof The obstruction to existence of a metalinear, resp. metaplectic, structure is as follows. Pick any transition isomorphisms g_{jk} for the tangent bundle. Lift them to a cochain $\widetilde{g}_{jk}$ with values in ML, resp. in Mp. Then compute the two-cocycle $a_{jk\ell} = \widetilde{g}_{jk}\widetilde{g}_{k\ell}\widetilde{g}_{j\ell}^{-1}$ with values in $\mathbb{Z}/2\mathbb{Z}$. The cohomology class of this cocycle is the obstruction. If $M = T^*X$, this cohomology class is determined by its restriction to X. But on X the symplectic transition functions g_{jk} for TM can be chosen as the image of $\mathrm{GL}(n)$-valued transition functions for TX under the embedding $\mathrm{GL} \to \mathrm{Sp}$. This proves the first statement of the Lemma. Now, for a Lagrangian submanifold L of M, the transition isomorphisms for $TM|L$ are cohomologous to an Mp-valued cocycle $p_{jk} : g_{jk} = h_j p_{jk} h_k^{-1}$. Lift h_j to $\mathrm{Mp}(2n)$ somehow. Put

$$\widetilde{p}_{jk} = \widetilde{h}_j^{-1}\widetilde{g}_{jk}\widetilde{h}_k. \tag{12.1.4}$$

This is a cocycle cohomologous to $\widetilde{g}_{jk}|L$. It takes values in the preimage of the subgroup of $\mathrm{Sp}(2n)$ consisting of matrices preserving the Lagrangian submanifold $L_0 = \{\widehat{\xi} = 0\}$. The image of this cocycle under the projection to GL via ML is a cocycle defining the bundle TX. □

12.2 The Maslov Class of a Lagrangian Submanifold

12.2.1 The Case $c_1(M) = 0$

Consider the cohomology class of the two-cocycle $a_{jk\ell}$ constructed as in the proof of Lemma 12.2 above but when we use the universal cover $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$ instead of $\mathrm{Mp}(n)$. This is now a class in $H^2(M, \mathbb{Z})$ that represents $c_1(M)$, the first Chern class of TM viewed as a complex bundle after we reduce the structure group Sp to the maximal compact subgroup $U(n)$. Indeed, $\widetilde{\mathrm{Sp}}$ is homotopy equivalent to

$$\widetilde{U}(n) = \{(u, x) | u \in U(n),\ x \in \mathbb{R}, \det(u) = e^{2\pi i x}\}.$$

The proof of Lemma 12.2 applied to this case establishes the following fact.

Consider the group

$$\widetilde{\mathrm{GL}}(n, \mathbb{R}) = \{(g, x) | x \in \mathrm{GL}(n, \mathbb{R}); x \in \mathbb{R}; \det(g) = e^{2\pi i x}\} \tag{12.2.1}$$

(Of course $\widetilde{\mathrm{GL}}$, unlike $\widetilde{U}$ or $\widetilde{\mathrm{Sp}}$, has nothing to do with the universal cover).

Lemma 12.3 *A trivialization of $c_1(M)$ defines a $\widetilde{GL}(n)$-structure on any Lagrangian submanifold L of M, i.e. a lifting of the transition automorphisms of TL to a $\widetilde{\mathrm{GL}}(n)$-valued cocycle.*

Assume that L is oriented. Then there is another $\widetilde{GL}(n)$-structure on L, due to the fact that $\mathrm{SL}(n)$ is a subgroup of $\widetilde{GL}(n)$. The two liftings differ by a class in $\lambda(L) \in H^1(L, \mathbb{Z})$. We will call this class *the Maslov class of an oriented Lagrangian submanifold* of a symplectic manifold M with a trivialization of $c_1(M)$.

12.2.2 The Case $2c_1(M) = 0$

Now consider the group

$$\widetilde{U}^{(2)}(n) = \{(g, x) | g \in U(n); x \in \mathbb{R}; \det(g)^2 = e^{2\pi i x}\} \tag{12.2.2}$$

Note that

$$\{(g, x) | x \in \mathrm{GL}(n, \mathbb{R}); x \in \mathbb{R}; \det(g)^2 = e^{2\pi i x}\} \xrightarrow{\sim} \mathrm{GL}(n, \mathbb{R}) \times \mathbb{Z} \tag{12.2.3}$$

Arguing exactly as before, we get

Lemma 12.4 *A trivialization of $2c_1(M)$ defines a $\mathrm{GL}(n) \times \mathbb{Z}$-structure on any Lagrangian submanifold L of M.*

Projecting to $\mathbb{Z}$, we get a class $\mu(L) \in H^1(L, \mathbb{Z})$. We call $\mu(L)$ *the Maslov class of a Lagrangian submanifold* of a symplectic manifold M with a trivialization of $2c_1(M)$.

Note that

$$\mu(L) = 2\lambda(L) \tag{12.2.4}$$

for a trivialization of c_1, the induced trivialization of $2c_1$, and an oriented L.

Remark 12.5 Let $\widetilde{\Lambda}(n)$ be the universal cover of the Lagrangian Grassmannian $\Lambda(n)$. Define the group $\widetilde{\mathrm{Sp}}^{(2)}(2n, \mathbb{R})$ by the condition that the following square be Cartesian.

$$\begin{array}{ccc} \widetilde{\mathrm{Sp}}^{(2)}(2n, \mathbb{R}) & \longrightarrow & \widetilde{\Lambda}(n) \\ \downarrow & & \downarrow \\ \mathrm{Sp}(2n, \mathbb{R}) & \longrightarrow & \Lambda(n) \end{array}$$

Then $\widetilde{U}^{(2)}$ is a homotopy equivalent subgroup of $\widetilde{\mathrm{Sp}}^{(2)}(2n, \mathbb{R})$.

Example 12.6 For $n = 1, U(1) \stackrel{\sim}{\to} S^1$; also $\Lambda(1) \stackrel{\sim}{\to} S^1$. Under these identifications, the projection $U(1) \to \Lambda(1)$ becomes the map $\zeta \mapsto \zeta^2$.

12.3 The Groups Sp^N

Here we use definitions and notation from [36]. For $N \geq 1$, let $\Lambda^N(n)$ be the universal N-fold cover of $\Lambda(n)$. Define the group $\mathrm{Sp}^N(2n, \mathbb{R})$ by requiring the following diagram to be Cartesian:

$$\begin{array}{ccc} \mathrm{Sp}^N(2n, \mathbb{R}) & \longrightarrow & \Lambda^N(n) \\ \downarrow & & \downarrow \\ \mathrm{Sp}(2n, \mathbb{R}) & \longrightarrow & \Lambda(n) \end{array}$$

In other words, $\mathrm{Sp}^N(2n) = \widetilde{\mathrm{Sp}}^{(2)}(2n)/(\mathbb{Z}/N)$. Define also

$$U^N(n) = \{(u, \zeta) | u \in U(n), \zeta \in \mathbb{C}, \det(u)^2 = \zeta^N\} = \widetilde{U}^{(2)}/(\mathbb{Z}/N)$$

This is a subgroup of $\mathrm{Sp}^N(n)$ and the embedding is a homotopy equivalence.

A $\mathrm{Sp}^N(2n)$-structure on M is the same as a trivialization of $2c_1(M)$ in $H^2(M, \mathbb{Z}/N)$.

The universal N-fold cover of $\mathrm{Sp}(2n)$ is a subgroup of $Sp^{2N}(2n)$. In particular, the metaplectic group $\mathrm{Mp}(2n)$ is a subgroup of $\mathrm{Sp}^4(2n)$. The latter is generated by $\mathrm{Mp}(2n)$ and the central subgroup $\{\pm 1, \pm i\}$. The intersection of the two is $\{\pm 1\}$, the kernel of $\mathrm{Mp} \to \mathrm{Sp}$.

The following makes sense for any N. We fix $N = 4$ just to fix the notation for the rest of the paper.

Definition 12.7 (a) Define $P(n, \mathbb{R})$ as the subgroup of $\mathrm{Sp}(2n, \mathbb{R})$ consisting of pairs (A, z) where $A = \begin{bmatrix} b & a \\ 0 & (b^{-1})^t \end{bmatrix}$ is a symplectic matrix. In other words, $P(n)$ is the subgroup of $\mathrm{Sp}(2n)$ consisting of matrices preserving the Lagrangian submanifold $L_0 = \{\widehat{\xi} = 0\}$.

(b) Define $\mathrm{MPar}(n, \mathbb{R})$ as the subgroup of $\mathrm{Sp}^4(2n, \mathbb{R})$ consisting of pairs (A, z) where $A = \begin{bmatrix} b & a \\ 0 & (b^{-1})^t \end{bmatrix}$ is a symplectic matrix, z is a complex number, and $\det(b)^2 = z^4$. In other words, this is the lifting to $\mathrm{Sp}^4(2n)$ of $P(n)$.

Lemma 12.8 *(a)* $\mathrm{MPar}(n, \mathbb{R}) \xrightarrow{\sim} P(n, \mathbb{R}) \times \{\pm 1, \pm i\}$

(b) If a symplectic manifold M has an Sp^4 *structure and L is a Lagrangian submanifold then formulas* (12.1.4) *define an* $\mathrm{MPar}(n)$*-valued cocycle cohomologous to the transition isomorphisms of $TM|L$.*

(c) If M has a real polarization then it has an $\mathrm{Sp}^4(2n)$*-structure.*

Definition 12.9 The projection of the cohomology class from Lemma 12.8, (b) to $H^1(L, \mathbb{Z}/4\mathbb{Z})$ is called *the Maslov class* of L.

When the trivialization of $2c_1(M)$ modulo 4 comes from a trivialization of $2c_1(M)$ then the Maslov class defined above is equal to $\exp(\frac{i\pi}{2}\mu(L))$ that was defined in Sect. 12.2.2.

13 Appendix. The Algebraic Metaplectic Representation

Most of the material of this section is contained in [40]. Recall the algebra $\mathcal{A}$ from Sect. 4.1 and the $\mathcal{A}$-module from Definition 9.5. In this section we give an interpretation of this module in terms of the metaplectic representation.

13.1 Symmetries of the Deformation Quantization Algebra of a Formal Neighborhood

Any continuous automorphism g of $\widehat{\mathbb{A}}$ induces a symplectic linear transformation g_0 of $\mathbb{C}^{2n}$. Denote by G the group of those g whose linear part g_0 preserves the real structure. We have

$$G = \mathrm{Sp}(2n, \mathbb{R}) \ltimes \exp(\mathfrak{g}_{\geq 1}) \tag{13.1.1}$$

Define the central extension

$$\widetilde{\mathbf{G}} = \exp\left(\frac{1}{i\hbar}\mathbb{C} \oplus \mathbb{C}\right) \times \mathrm{Sp}^4(2n, \mathbb{R}) \ltimes \exp(\widetilde{\mathfrak{g}}_{\geq 1}) \tag{13.1.2}$$

where $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$ is the universal cover of $\mathrm{Sp}(2n, \mathbb{R})$. One has an exact sequence

$$1 \to \frac{\mathbb{Z}}{4} \times \exp\left(\frac{1}{i\hbar}\mathbb{C}[[\hbar]]\right) \to \widetilde{\mathbf{G}} \to G \to 1 \tag{13.1.3}$$

Define also P to be the subgroup of G consisting of elements g whose linear part preserves the Lagrangian subspace

$$L_0 = \{\widehat{\xi_1} = \ldots = \widehat{\xi_n} = 0\} \tag{13.1.4}$$

Let $\widetilde{\mathbf{P}}$ be the preimage of P in $\widetilde{\mathbf{G}}$.

13.2 The Algebraic Fourier Transform

Let $\widehat{y} = (\widehat{y}_1, \ldots, \widehat{y}_n)$ be n formal variables. For a symmetric real $n \times n$ matrix a, put

$$\mathcal{H}_a^{\widehat{y}} = \exp\left(\frac{a\widehat{y}^2}{2i\hbar}\right)\widehat{\mathbb{C}}[[\widehat{y}, \hbar]]((e^{\frac{c}{i\hbar}}|c \in \mathbb{C})) \tag{13.2.1}$$

Here

$$\widehat{\mathbb{C}}[[\widehat{y}, \hbar]] = \left\{\sum_{k=-N}^{\infty} v_k | v_k \in \mathbb{C}[[\widehat{y}]]((\hbar))_k\right\} \tag{13.2.2}$$

with respect to the grading (3.1.3); for any vector space V, we define

$$V((e^{\frac{c}{i\hbar}}|c \in \mathbb{C})) = \left\{\sum_{k\in\mathbb{N};\mathrm{Re}(c_k)\to+\infty} e^{\frac{c_k}{i\hbar}} v_k\right\}, \tag{13.2.3}$$

$v_k \in V$. In particular, the operator of multiplication by h is automatically invertible.

For a nondegenerate a, define the Fourier transform (cf. [22])

$$F : \mathcal{H}_a^{\widehat{y}} \xrightarrow{\sim} \mathcal{H}_{-a^{-1}}^{\widehat{\eta}} \tag{13.2.4}$$

as follows. Heuristically,

$$(Ff)(\widehat{\eta}) = \frac{e^{-\frac{\pi i n}{4}}}{(2\pi i\hbar)^{n/2}} \int e^{\frac{\widehat{y}\widehat{\eta}}{i\hbar}} f(\widehat{y})d\widehat{y}; \tag{13.2.5}$$

To give the above formula a rigorous meaning, put

$$F\left(f(\widehat{y})\exp\left(\frac{a\widehat{y}^2}{2i\hbar}\right)\right)(\widehat{\eta}) = f\left(i\hbar\frac{\partial}{\partial\widehat{\eta}}\right)F\left(\exp\left(\frac{a\widehat{\eta}^2}{2i\hbar}\right)\right) =$$

$$f\left(i\hbar\frac{\partial}{\partial\widehat{\eta}}\right)\frac{e^{-\frac{\pi i n}{4}}}{\det(\sqrt{ia})}\exp\left(\frac{-a^{-1}\widehat{\eta}^2}{2i\hbar}\right) = \frac{e^{-\frac{\pi i p(a)}{2}}}{\det\sqrt{|a|}}f\left(i\hbar\frac{\partial}{\partial\widehat{\eta}}\right)\exp\left(\frac{-a^{-1}\widehat{\eta}^2}{2i\hbar}\right)$$

Here $p(a)$ is the number of positive eigenvalues of a. We used the branch of the square root for which $\sqrt{x} > 0$ if $x > 0$; it is defined on the complex plane with the line $\{x < 0,\ x \in \mathbb{R}\}$ removed. The final term in the above chain of equalities can be viewed as the definition of the first term.

Remark 13.1 The definition of the Fourier transform F extends to elements of the form

$$\mathbf{f}(\widehat{y}) = \exp\left(\frac{a\widehat{y}^2}{2i\hbar} + i\widehat{y}\widehat{z}\right)f(\widehat{y}) \tag{13.2.6}$$

where a is nondegenerate and $\widehat{z}$ is another formal parameter:

$$F(\mathbf{f})(\widehat{\eta}) = F\left(\exp\left(\frac{a\widehat{y}^2}{2i\hbar}\right)f(\widehat{y})\right)(\widehat{\eta}+\widehat{z}) \tag{13.2.7}$$

One has

$$F^2(\mathbf{f})(\widehat{y}) = i^{-n}\mathbf{f}(-\widehat{y});\ \ F\widehat{y}F^{-1} = i\hbar\frac{\partial}{\partial\widehat{\eta}};\ \ Fi\hbar\frac{\partial}{\partial\widehat{y}}F^{-1} = -\widehat{\eta} \tag{13.2.8}$$

13.3 The Two-Dimensional Case

For the readers convenience, we first present the case $n = 1$.

$$\mathcal{H} = \bigoplus_{a\in\mathbb{R}}\mathcal{H}_a^{\widehat{x}}\bigoplus\bigoplus_{a\in\mathbb{R}}F\mathcal{H}_a^{\widehat{x}}/\sim \tag{13.3.1}$$

where

$$Ff(\widehat{x})\exp\left(\frac{a\widehat{x}^2}{2i\hbar}\right) \sim \frac{e^{-\frac{\pi i}{2}p(a)}}{\sqrt{|a|}}f\left(i\hbar\frac{\partial}{\partial\widehat{x}}\right)\exp\left(-\frac{a^{-1}\widehat{x}^2}{2i\hbar}\right) \tag{13.3.2}$$

for $a \neq 0$. Here $p(a) = 1$ if $a > 0$ and $p(a) = 0$ otherwise.

13.3.1 The Action of $\widehat{\widehat{\mathbb{A}}}$ on $\mathcal{H}$

The algebra $\widehat{\widehat{\mathbb{A}}}$ acts on the space $\mathcal{H}$ as follows. If $\mathbf{f}$ is in the first summand in (13.3.1), then $\widehat{x}$ acts on it by multiplication and $\widehat{\xi}$ by $i\hbar\frac{\partial}{\partial\widehat{x}}$, the latter defined by

$$\frac{\partial}{\partial\widehat{x}}\left(\exp\left(\frac{a\widehat{x}^2}{2i\hbar}f(\widehat{x})\right)\right)=\exp\left(\frac{a\widehat{x}^2}{2i\hbar}\right)\left(\frac{\partial}{\partial\widehat{x}}+a\widehat{x}\right)f(\widehat{x}).$$

As for $F\mathbf{f}$, $\widehat{x}$ sends it to $-i\hbar F\frac{\partial}{\partial\widehat{x}}\mathbf{f}$ and $\widehat{\xi}$ sends it to $F\widehat{x}\mathbf{f}$.

13.3.2 Some Operators on $\mathcal{H}$

The operator $F:\mathcal{H}\to\mathcal{H}$. Define for $\mathbf{f}(\widehat{x})=\exp(\frac{a\widehat{x}^2}{2i\hbar})f(\widehat{x})$

$$F:\mathbf{f}\mapsto F\mathbf{f}\mapsto i^{-1}\mathbf{f}(-\widehat{x})$$

The operator $\exp(\frac{a\widehat{x}^2}{2i\hbar}):\mathcal{H}\to\mathcal{H}$. (1)

$$\exp\left(\frac{a\widehat{x}^2}{2i\hbar}\right):\exp\left(\frac{c\widehat{x}^2}{2i\hbar}\right)f(\widehat{x})\mapsto\exp\left(\frac{(a+c)\widehat{x}^2}{2i\hbar}\right)f(\widehat{x})$$

for $c\in\mathbb{R}$;

(2)

$$F\exp\left(\frac{c\widehat{x}^2}{2i\hbar}\right)f(\widehat{x})\mapsto\frac{e^{\frac{-\pi i}{2}p(c)}}{\sqrt{|c|}}f\left(-i\hbar\frac{\partial}{\partial\widehat{x}}+a\widehat{x}\right)\exp\left(\frac{(a-c^{-1})\widehat{x}^2}{2i\hbar}\right)$$

for $c\neq 0$;

(3)

$$F\exp\left(\frac{c\widehat{x}^2}{2i\hbar}\right)f(\widehat{x})\mapsto iFf\left(\widehat{x}-ai\hbar\frac{\partial}{\partial\widehat{x}}\right)\frac{e^{-\frac{\pi i}{2}(p(c)+p(\frac{-c}{1-ac}))}}{\sqrt{|1-ac|}}\exp\left(\frac{c}{1-ac}\frac{\widehat{x}^2}{2i\hbar}\right)$$

for $c\neq a^{-1}$. These maps preserve the equivalence relation and therefor define operators on $\mathcal{H}$.

13.3.3 The Action of $\mathrm{Sp}^4(2,\mathbb{R})$ on $\mathcal{H}$

The group $\mathrm{Sp}^4(2,\mathbb{R})$ acts by the algebraic version of the metaplectic representation that we are going to describe next.

13.4 The Metaplectic Projective Representations of $\mathbf{SL_2}(\mathbb{R})$

Define the action of generators of $SL_2(\mathbb{R})$ by exactly the same formula as the usual metaplectic representation

$$T: \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \mapsto \exp\left(\frac{a\widehat{x}^2}{2i\hbar}\right); \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mapsto F; \tag{13.4.1}$$

$$\begin{bmatrix} b & 0 \\ 0 & b^{-1} \end{bmatrix} \mapsto T_b;\ (T_b f)(x) = \frac{1}{\sqrt{\det(b)}} f(b^{-1}x) \tag{13.4.2}$$

The corresponding representation of $\mathfrak{sl}(2)$ is given by

$$X_- = \frac{\widehat{x}^2}{2i\hbar};\ H = -\frac{\widehat{x\xi}}{i\hbar} = -\frac{\widehat{x} * \widehat{\xi}}{i\hbar} - \frac{1}{2};\ X_+ = -\frac{\widehat{\xi}^2}{2i\hbar} \tag{13.4.3}$$

13.4.1 The Bruhat Relations

The following are well known to be the defining relations of SL_2 (together with the requirement that $a \mapsto \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}$ is a morphism from the additive group and $b \mapsto \begin{bmatrix} b & 0 \\ 0 & b^{-1} \end{bmatrix}$ is a morphism from the multiplicative group).

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -a \\ 0 & 1 \end{bmatrix} \tag{13.4.4}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} b & 0 \\ 0 & b^{-1} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} b^{-1} & 0 \\ 0 & b \end{bmatrix} \tag{13.4.5}$$

$$\begin{bmatrix} b & 0 \\ 0 & b^{-1} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}\begin{bmatrix} b & 0 \\ 0 & b^{-1} \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ b^{-2}a & 1 \end{bmatrix} \tag{13.4.6}$$

$$\begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ a^{-1} & 1 \end{bmatrix} = \begin{bmatrix} a^{-1} & 0 \\ 0 & a \end{bmatrix}\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \tag{13.4.7}$$

for $a \neq 0$.

Proposition 13.2 *Formulas* (13.4.1) *define a representation of* $\widetilde{SL}_2$ *in which an element of* $\pi_1(SL_2)$ *acts by* $e^{\frac{\pi i}{2}c}$ *where c is its image in* $\pi_1(\Lambda) \xrightarrow{\sim} \mathbb{Z}$.

Proof All the Bruhat relations except (13.4.7) are true for operators $T(g)$ defined in (13.4.1), whereas

Lemma 13.3

$$T\left(\begin{bmatrix}1 & 0\\ a & 1\end{bmatrix}\right)T\left(\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\right)T\left(\begin{bmatrix}1 & 0\\ a^{-1} & 1\end{bmatrix}\right)=$$

$$=\frac{\sqrt{|a|}}{\sqrt{a}}e^{\frac{\pi i}{2}p(a)}T\left(\begin{bmatrix}a^{-1} & 0\\ 0 & a\end{bmatrix}\right)T\left(\begin{bmatrix}1 & a\\ 0 & 1\end{bmatrix}\right)$$

□

13.5 The Case of a General n

Now define

$$\mathcal{H}=\bigoplus_{I\subset\{1,\ldots,n\}}\bigoplus_{a}F_{I,J}\mathcal{H}^{\widehat{x}}/\sim \tag{13.5.1}$$

where a runs through all symmetric $n\times n$ real matrices and the equivalence relation is defined as follows. For every subset K of $\{1,2,\ldots,n\}$, define

$$F_K:\bigoplus_{a}F_I\mathcal{H}^{\widehat{x}}\to\bigoplus_{a}F_{I\triangle K}\mathcal{H}^{\widehat{x}} \tag{13.5.2}$$

(where $\triangle$ stand for the symmetric difference) as follows: if L is the complement of $I\cap K$, then

$$(F_KF_I\mathbf{f})(\widehat{x}_{I\cap K},\widehat{x}_L)=i^{-|I\cap K|}F_{I\triangle K}\mathbf{f}(-\widehat{x}_{I\cap K},\widehat{x}_L) \tag{13.5.3}$$

Let J be the complement of I.

$$\mathbf{f}(\widehat{x}_I,\widehat{x}_J)=\exp\left(\frac{a\widehat{x}_I^2+b\widehat{x}_I\widehat{x}_J+c\widehat{x}_J^2}{2i\hbar}\right)f(\widehat{x}_I,\widehat{x}_J) \tag{13.5.4}$$

such that a_I is a nondegenerate symmetric matrix. Then

$$F_KF_I\mathbf{f}\sim F_K\frac{\exp\left(-\frac{\pi i}{2}p(a)\right)}{\sqrt{\det(|a|)}}f\left(i\hbar\frac{\partial}{\partial\widehat{x}_I}\right)\exp\left(\frac{-a^{-1}(\widehat{x}_I+b\widehat{x}_J)^2}{2i\hbar}\right) \tag{13.5.5}$$

for all K.

13.5.1 Operators on $\mathcal{H}$

Clearly, the operators F_K (13.5.2) preserve the equivalence relation and therefore descend to $\mathcal{H}$.

13.5.2 The Action of $\widehat{\widehat{\mathbb{A}}}$ on $\mathcal{H}$

The algebra $\widehat{\widehat{\mathbb{A}}}$ acts on the space $\mathcal{H}$ as follows. On the summand $F_I\mathcal{H}_a^{\widehat{x}}$,

$$\widehat{x}_j F_I \mathbf{f} = -F_I i\hbar \frac{\partial}{\partial \widehat{x}_j}\mathbf{f},\ j \in I;\ \widehat{x}_j F_I \mathbf{f} = F_I \widehat{x}_j \mathbf{f},\ j \notin I; \tag{13.5.6}$$

$$\widehat{\xi}_j F_I \mathbf{f} = F_I i\hbar \frac{\partial}{\partial \widehat{x}_j}\mathbf{f},\ j \notin I;\ \widehat{\xi}_j F_I \mathbf{f} = F_I \widehat{x}_j \mathbf{f},\ j \in I. \tag{13.5.7}$$

13.5.3 The Action of $\mathbf{Sp}^4(2n)$ on $\mathcal{H}$

This action is exactly as described in Sect. 13.3.3. In particular, $\mathrm{Sp}^4(2n, \mathbb{R})$ acts by the metaplectic representation as in Sect. 13.4:

$$T: \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \mapsto \exp\left(\frac{a\widehat{x}^2}{2i\hbar}\right);\ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mapsto F; \tag{13.5.8}$$

more generally, let $\mathbf{F}_I$ be the matrix that is the direct sum of $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ in coordinates $\widehat{x}_I, \widehat{\xi}_I$ and the identity matrix in the rest of the Darboux coordinates maps to F_I;

$$\begin{bmatrix} b & 0 \\ 0 & {}^t b^{-1} \end{bmatrix} \mapsto T_b;\ (T_b f)(x) = \frac{1}{\sqrt{\det(b)}} f(b^{-1}x) \tag{13.5.9}$$

Remark 13.4 The construction of $\mathcal{H}$ mimics very closely the construction of the orbit of 1 under the action of $\mathrm{Sp}^4(2n) \ltimes \mathbb{C}[\widehat{x}, \widehat{\xi}]$ on the space of distributions via differential operators and the standard metaplectic representation.

Lemma 13.5 *Assign to $F_I \exp(\frac{a\widehat{x}^2}{2i\hbar}) f(\widehat{x}) \in \mathcal{H}$ the Lagrangian subspace $\mathbf{F}_I(\{\widehat{\xi} = a\widehat{x}\})$ where $\mathbf{F}_I$ was defined after (13.5.8). This is a well-defined map $\mathcal{H} \to \Lambda(n)$ where $\Lambda(n)$ is the Grassmannian of Lagrangian subspaces in $\mathbb{R}^{2n}$. The space $\mathcal{H}$ is identified with the space of finitely supported sections of a $\widetilde{\mathbf{G}}$-equivariant vector bundle on $\Lambda(n)$.*

The Lagrangian Grassmannian is a homogeneous space of $\widetilde{\mathbf{G}}$ via the projection $\widetilde{\mathbf{G}} \to \mathrm{Sp}^4 \to \mathrm{Sp}$. In fact,

$$\Lambda(n) \xrightarrow{\sim} \widetilde{\mathbf{G}}/\widetilde{\mathbf{P}}.$$

Lemma 13.6 *The lines $\mathbb{C}F_I \exp(\frac{a\widehat{x}^2}{2i\hbar})$ where a runs through real symmetric $n \times n$ matrices and I through subsets of $\{1, \dots, n\}$ form a line subbundle of $\mathcal{H}$ which is isomorphic to the bundle on $\Lambda(n)$ determined by the Čech one-cohomology class $(-1)^{\mu_L}$ where μ_L is the generator of $H^1(\Lambda(n), \mathbb{Z})$ (the Maslov class).*

Lemma 13.7 *The actions described in Sects. 13.5.2 and 13.5.3 turn $\mathcal{H}$ into an $\mathcal{A}$-module.*

13.6 The Algebraic Metaplectic Representation as an Induced Module

Proposition 13.8

$$\mathcal{H} \xrightarrow{\sim} \widehat{\mathcal{V}} = \mathcal{A}\widehat{\otimes}_{\mathcal{B}}\widehat{\widehat{\mathbb{V}}}_{\mathbb{K}}$$

(cf. Sect. 9.2.1).

14 Appendix. Twisted Bundles and Groupoids

14.1 Charts and Cocycles

Suppose we have a manifold X with two sheaves of groups $\underline{C} \subset \underline{G}$ where $\underline{C}$ is constant and central in $\underline{G}$. Consider a class $c \in H^2(X, \underline{C})$. A $\underline{G}$-bundle on X twisted by c is given by an equivalence class of $g_{ij} \in \underline{G}(U_i \cap U_j)$ for an open cover $X = \cup U_i$ such that

$$g_{ij}g_{jk} = c_{ijk}g_{ik} \tag{14.1.1}$$

where c_{ijk} is a Čech cocycle representing c. Two data g_{ij} and g'_{ij} are equivalent if

$$g_{ij} = h_i g'_{ij} h_j^{-1} b_{ij} \tag{14.1.2}$$

for some common refinement of the two covers, where $h_i \in \underline{G}(U_i)$ and $b_{ij} \in \underline{C}(U_i \cap U_j)$. Note that this definition makes all $\underline{C}$-bundles equivalent.

By a *chart* we mean a map $T \to X$ where T is any topological space. *A good collection of charts* on X is a collection of charts $T \to X$, $T \in \mathcal{T}$, such that for every $T_0, \ldots, T_p$ in $\mathcal{T}$, every one-cocycle on $T_0 \times_X \ldots \times_X T_p$ with values in the pullback of $\underline{G}$, and every one- or two-cocycle with values in the pullback of $\underline{C}$, can be trivialized.

Lemma 14.1 *For any good collection of charts and any twisted bundle, one can define*

$$c_{TT'T''} \in \underline{C}(T \times_X T' \times_X T''); \;\; g_{TT'} \in \underline{G}(T \times_X T') \tag{14.1.3}$$

satisfying

$$c_{TT'T''}c_{TT''T'''} = c_{TT'T'''}c_{T'T''T'''} \tag{14.1.4}$$

and

$$g_{TT'}g_{T'T''} = c_{TT'T''}g_{TT''} \tag{14.1.5}$$

in such a way that, if T_i are a good open cover, then $c_{T_iT_jT_k}$ is cohomologous to c_{ijk} and $g_{T_iT_j}$ is equivalent to g_{ij}. The choice is unique up to equivalence in the following sense:

$$c_{TT'T''} = c'_{TT'T''}b_{TT'}b_{T'T''}b_{TT''}^{-1}; \;\; g_{TT'} = h_T g'_{TT'}h_{T'}^{-1}b_{TT'} \tag{14.1.6}$$

for some $b_{TT'} \in \underline{C}(T \times T')$ and $h_T \in \underline{G}(T)$.

Proof Consider inverse images on charts T of an open cover $\{U_i\}$ of X. Let

$$c_{ijk} = \alpha_{ij}(T)\alpha_{jk}(T)\alpha_{ik}(T)^{-1}$$

be a trivialization of c on T. Choose trivializations

$$g_{ij}\alpha_{ij}(T)^{-1} = h_i(T)h_j(T)^{-1}$$

on T and

$$\alpha_{ij}(T)\alpha_{ij}(T') = \beta_i(T, T')\beta_j(T, T')^{-1}$$

where α_{ij}, β_{ij} are sections of $\underline{C}$ and h_i are sections of $\underline{G}$. Put

$$c_{TT'T''} = \beta_i(T, T')\beta_i(T', T'')\beta_i(T, T'')^{-1} \tag{14.1.7}$$

and

$$g_{TT'} = h_i(T)^{-1}h_i(T')\beta_i(T, T') \tag{14.1.8}$$

The relations above show that these do not depend on i. □

14.2 The Groupoid of a Twisted G-Bundle

Let G be a Lie group and $\underline{G}$ the sheaf of smooth G-valued functions. Let C be a central subgroup of G and $\underline{C}$ the sheaf of locally constant C-valued functions. Consider a $\underline{C}$-valued two-cohomology class represented by a cocycle c_{ijk} and a twisted G-bundle represented by a $\underline{G}$-valued cochain g_{jk}.

Define a groupoid on X as follows.

For x_0 and x_1 in X, define the set $\widetilde{\mathbf{G}}_{x_0,x_1}$. Let $\gamma : [0, 1] \to X$ be a smooth map. View it as a chart that we denote by T. An element of $\widetilde{\mathbf{G}}_{x_0,x_1}$ is a class of an element $g_T \in G$ with respect to the following equivalence relation. Consider two charts T and T' representing two smooth maps γ, $\gamma' : [0, 1] \to X$ and a homotopy $\sigma : [0, 1]^2 \to X$ such that $\sigma(0, s) = x_0$, $\sigma(s, t) = x_1$, $\sigma(t, 0) = \gamma(t)$, and $\sigma(t, 1) = \gamma'(t)$. We will view σ as a chart S. We call S *a homotopy between S and S'*. Now generate the equivalence relation by the following.

$$g_T \sim (g_{TT'}c^{-1}_{TT'S})(x_0)g_{T'}(g_{TT'}c^{-1}_{TT'S})(x_1)^{-1} \tag{14.2.1}$$

Lemma 14.2 *Let S be a homotopy between T and T', S' a homotopy between T' and T'', and S'' a homotopy between T and T''. If we denote the right hand side of (14.2.1) by $a(S)g_T$, then $a(S)a(S') = \langle c, [\Sigma]\rangle a(S'')$ where Σ is the sphere formed by S, S', and S''.*

Proof We have

$$a(S)a(S')g_T =$$

$$g_{TT'}g_{T'T''}c^{-1}_{TT'S}c^{-1}_{T'T''S'}(x_0)g_{T''}(g_{TT'}g_{T'T''}c^{-1}_{TT'S}c^{-1}_{T'T''S'}(x_1))^{-1}$$

The right hand side is equal to

$$(g_{TT''}c_{TT'T''}c^{-1}_{TT'S}c^{-1}_{T'T''S'})(x_0)g_{T''}(g_{TT''}c_{TT'T''}c^{-1}_{TT'S}c^{-1}_{T'T''S'})(x_1)^{-1};$$

therefore

$$a(S)a(S') = \frac{c_{TT'T''}c_{TT''S''}}{c_{TT'S}c_{T'T''S'}}(x_0)(\frac{c_{TT'T''}c_{TT''S''}}{c_{TT'S}c_{T'T''S'}}(x_1))^{-1}a(S'')$$

Applying the cocyclicity condition to the quadruple of charts $TT'T''S$, we get

$$\frac{c_{TT'T''}c_{TT''S''}}{c_{TT'S}c_{T'T''S'}} = \frac{c_{TT''S''}c_{T'T''S}}{c_{TT''S}c_{T'T''S'}}$$

Applying the same condition to $TT''SS''$ and then to $SS'S''T''$, we replace the right hand side with

$$\frac{c_{TSS''}c_{T''SS'}}{c_{T''S'S''}c_{T'SS'}} = \frac{c_{SS'S''}c_{TSS''}}{c_{T'SS'}c_{T''S'S''}}.$$

But $T \times_X S \times_X S'' = T$, $T' \times_X S \times_X S' = T'$, and $T'' \times_X S' \times_X S'' = T''$. Therefore the values of $c_{TSS''}$, *etc.* at x_0 and x_1 are the same. Therefore

$$a(S)a(S') = c_{SS'S''}(x_0)c_{SS'S''}(x_1)^{-1}a(S'')$$

But

$$c_{SS'S''}(x_0)c_{SS'S''}(x_1)^{-1} = \langle c, [\Sigma]\rangle$$

for any two-cocycle c. To see this, note that the left hand side is 1 for any coboundary c. On the other hand, if we enlarge S, S', S'' a little bit to make them an open cover of Σ, take an element a of C, and define $c_{SS'S''}(x_0) = a$, $c_{SS'S''}(x_1) = 1$, the result will be $a = \langle c, [\Sigma]\rangle$. □

Corollary 14.3 *There is an epimorphism*

$$\widetilde{\mathbf{G}}_{x_0,x_1} \to \pi_1(x_0, x_1) \tag{14.2.2}$$

When $x_0 = x_1 = x$, the kernel of this epimorphism is isomorphic to $G/\langle c, \pi_2(X)\rangle$.

14.2.1 Example: The Holonomy Groupoid of a Vector Bundle

Let E be a real oriented vector bundle of rank N. Let $G = \mathrm{SO}_N(\mathbb{R})$ and $\widetilde{G} = \mathrm{Spin}_N(\mathbb{R})$ its universal cover. Reduce the structure group of E to G using a Riemannian metric. Let $\widetilde{\mathrm{Isom}}(E)_{x_0,x_1}$ be the set of equivalence classes of data (γ, u_t) where $\gamma : [0, 1] \to X$ is a smooth map, $\gamma(0) = x_0, \gamma(1) = x_1$, and $u_t : E_{\gamma(t)} \xrightarrow{\sim} E_{\gamma(0)}$ a metric-preserving linear isomorphism smoothly depending on t and satisfying $u_0 = \mathtt{Id}$. An equivalence between (γ, u_t) and (γ', u'_t) is a smooth map $\sigma : [0, 1] \times [0, 1] \to X$ such that $\sigma(0, s) = x_0$, $\sigma(1, s) = x_1$, $\sigma(t, 0) = \gamma(t)$, $\sigma(t, 1) = \gamma'(t)$, and a linear metric-preserving isomorphism $v_{t,s} : E_{\sigma(t,s)} \xrightarrow{\sim} E_{x_0}$ smooth in (t, s), such that $v_{0,s} = \mathtt{Id}$, $v_{t,0} = u_t$, $v_{t,1} = u'_t$, and $v_{1,s} = u_1 = u'_1$.

Lift the transition isomorphisms g^E_{ij} of E to some $\widetilde{g}_{ij}$. Put $c_{ijk} = \widetilde{g}_{ij}\widetilde{g}_{jk}\widetilde{g}_{ik}^{-1}$. This cocycle represents the second Stiefel–Whitney class $w_2(E)$. Note that the groupoid $\widetilde{\mathrm{Isom}}(E)$ is isomorphic to the groupoid $\widetilde{\mathbf{G}}'$ constructed from the twisted bundle defined by $\widetilde{g}_{ij}, c_{ijk}$. In fact, note that for the charts T and S defined by maps γ and σ, there is a natural lifting $\widetilde{g}_{TS}$ of g_{TS}. Namely, $\widetilde{g}_{TS}(\gamma(t))$ is the class of the path $g_{TS}(\gamma(\tau))$, $0 \le \tau \le t$. Similarly with $\widetilde{g}_{ST'}$. Identify with $\widetilde{G}$ the set of equivalence classes of (γ, u_t) with fixed γ (and with $\sigma(t, s) = \gamma(t)$ in the definition of the equivalence). Now, given an equivalence σ, v between γ, u and γ', u', $g_T \in \widetilde{G}$ gets identified with $\widetilde{g}_{TS}\widetilde{g}_{ST'} = \widetilde{g}_{TT'}c_{TST'}$.

Corollary 14.4 *There is an epimorphism*

$$\widetilde{\mathrm{Isom}}(E)_{x_0,x_1} \to \pi_1(x_0, x_1) \tag{14.2.3}$$

and every preimage is a homogeneous space $\mathrm{Spin}(N, \mathbb{R})/\langle w_2(E), \pi_2(X)\rangle$. *(We identify $\mathbb{Z}/2$ with the center of* $\mathrm{Spin}(N, \mathbb{R})$*).*

14.2.2 Connections on Twisted Bundles

As in Sect. 14.2, let G be a simply-connected (pro) Lie group and $\underline{G}$ the sheaf of smooth G-valued functions. Let C be a central subgroup of G and $\underline{C}$ the sheaf of smooth C-valued functions. In addition, fix some algebra $\mathcal{A}$ on which G acts by automorphisms. Consider a twisted bundle defined by the data (g_{ij}, c_{ijk}). A connection in this twisted bundle is a collection of $\mathcal{A}$-valued forms on U_i such that

$$\mathrm{Ad}_{g_{ij}}(d + A_j) = d + A_i$$

on every U_{ij}. Here $\mathrm{Ad}_g(d) = -dg \cdot g^{-1}$. Note that, because c_{ijk} are locally constant and central, $\mathrm{Ad}_{g_{ij}} \mathrm{Ad}_{g_{jk}}(d + A_k) = \mathrm{Ad}_{g_{ik}}(d + A_k)$, so the conditions above are consistent on U_{ijk}. The curvature $R = dA_i + A_i^2$ is a well-defined $\mathcal{A}$-valued two-form.

14.2.3 The Flat Connection up to Inner Derivations

Here we will construct a flat connection up to inner derivations on the associated bundle of algebras $\mathcal{A}$ compatible with the action of the groupoid $\widetilde{\mathbf{G}}$ of a twisted bundle (cf. Sect. 14.2). We will start from a flat connection on the twisted bundle itself.

First define special coordinate charts on $\widetilde{\mathbf{G}}$ as follows. Fix:

- two open charts U_0 and U_1 of X;
- two points $x_0^* \in U_0$ and $x_1^* \in U_1$;
- a path γ from x_0^* to x_1^* in X;
- smooth maps $\tau_0 : [0, 1] \times U_0 \to U_0$ and $\tau_1 : [0, 1] \times U_1 \to U_1$, $\tau_0(0, x_0) = x_0$, $\tau_0(1, x_0) = x_0^*$, $\tau_1(0, x_1) = x_1$, $\tau_1(1, x_1) = x_1^*$.

For every $x_0 \in U_0$ and $x_1 \in U_1$, we will denote the path $t \mapsto \tau_0(t, x_0)$ by τ_{x_0} and the path $t \mapsto \tau_1(t, x_1)$ by τ_{x_1}. For the data as above, we construct a chart T in $\widetilde{\mathbf{G}}$ as a map

$$U_0 \times U_1 \to \widetilde{\mathbf{G}};\ (x_0, x_1) \mapsto \tau_{x_0} \circ \gamma \circ \tau_{x_1} : x_0 \to x_1$$

(the composition of paths).

Now consider a flat connection in our twisted bundle. In a local trivialization, on any open chart W, we write $\nabla_{\mathcal{V}} = d + A_W$. We can identify a local section of $\widetilde{\mathbf{G}}$ on T with a $\widetilde{G}$-valued function $g_T(x_0, x_1)$ on $U_0 \times U_1$.

Definition 14.5

$$\alpha(g_T) = -dg_T \cdot g_T^{-1} - A_0 + \mathrm{Ad}_{g_T}(A_1)$$

where $A_0 = \pi_0^*(A_{U_0})$ and $A_1 = \pi_1^*(A_{U_1})$;

$$R = dA_0 + A_0^2.$$

Lemma 14.6 *The above formulas define a flat connection up to inner derivations on the associated bundle of algebras $\mathcal{A}$ compatible with the action of* $\widetilde{\mathbf{G}}$.

15 Appendix. Modules Associated to Lagrangian Submanifolds and Lagrangian Distributions

For any Lagrangian submanifold L of a symplectic manifold M with a given Sp^4 structure we constructed a bundle of modules $\widehat{\widehat{\mathbb{V}}}_L$ with a flat connection $\nabla_{\mathbb{V}}$ (cf.

Sect. 9.2.2). This is a bundle of $\widehat{\mathbb{A}}_M$-modules, and the connections $\nabla_{\mathbb{V}}$ and $\nabla_{\mathbb{A}}$ are compatible. In particular, denote by $\mathbb{A}_M$ the sheaf of algebras of horizontal sections of $\nabla_{\mathbb{A}}$ and by $\mathbb{V}_L$ the sheaf of horizontal sections of $\nabla_{\mathbb{V}}$. Then $\mathbb{V}_L$ is a sheaf of $\mathbb{A}_M$-modules.

Now apply the same construction to L but instead of M take a tubular neighborhood of L and identify it with the tubular neighborhood of L in T^*L by Darboux–Weinstein theorem. Use the Sp^4 structure provided by the Lagrangian polarization by fibers of T^*L (cf. Lemma 12.8). We get another $\mathbb{A}_M$-module that we denote by $\mathbb{V}_L^{(0)}$.

Lemma 15.1 *$\mathbb{V}_L$ is isomorphic to $\mathbb{V}_L^{(0)}$ twisted by the $\{\pm 1, \pm i\}$-valued Maslov class of L.*

We denote this class by $\exp(\frac{\pi i}{2}\mu(L))$. Note that $\mu(L)$ can be chosen as a $\mathbb{Z}$-valued cocycle only if $2c_1(M) = 0$.

15.1 The Asymptotic Construction of Hörmander and Maslov

As we have seen in Sect. 9.2.2, the oscillatory module $\mathcal{V}_L^\bullet$ is induced from the module of forms with coefficients in $\widehat{\widehat{\mathbb{V}}}$. But it is the twisted version of the latter module that serves as an asymptotic version of the classical construction of Lagrangian distributions with wave front L.

Put

$$\mathbb{V}_{L,\mathbb{K}} = \mathbb{K}\widehat{\otimes}\mathbb{V}_L = \left\{ \sum_{k=0}^{\infty} e^{\frac{1}{i\hbar}c_k} v_k | v_k \in \mathbb{V}_L;\ c_k \in \mathbb{R};\ c_k \to \infty \right\} \tag{15.1.1}$$

Definition 15.2 Assume $M = T^*X$. Let $\mathbb{V}_{L,\mathbb{K}}^{\eta}$ be the twist of the sheaf $\mathbb{V}_{L,\mathbb{K}}$ by the Čech cohomology class $\exp(-\frac{1}{i\hbar}\eta) \in H^1(L, \exp(\frac{1}{i\hbar}\mathbb{R}))$ where η is the class of the form $\xi dx|L$.

Let $X = \cup U_\alpha$ is a small open cover. Let $L = \cup W_\gamma$ be a refinement of the cover $L = \cup(T^*U_\alpha \cap L)$. In particular, a choice is made of $\gamma \mapsto \alpha = \alpha(\gamma)$ such that $W_\gamma \subset T^*U_\alpha \cap L$.

15.1.1 Quantization Procedure

First let us review our deformation quantization picture in the case $M = T^*X$. First, we have the sheaf of algebras $\mathbb{A}_{T^*X}$. It can be described by products $*_\alpha$ on $C^\infty(T^*U_\alpha)[[\hbar]]$

$$a *_\alpha b = \sum_{k=0}^{\infty} (i\hbar)^k P_{\alpha,k}(a, b) \tag{15.1.2}$$

and by transition functions

$$G_{\alpha\beta}(a) = \sum_{k=0}^{\infty} (i\hbar)^k T_{\alpha\beta,k}(a) \tag{15.1.3}$$

where $P_{\alpha,k}$ are bilinear bidifferential expressions, $T_{\alpha\beta,k}$ are differential operators, $P_{\alpha,0}(f, g) = fg$, $P_{\alpha,1}(f, g) = \frac{1}{2}\{f, g\}$, and $T_{\alpha\beta,0}(f) = f$. One has $G_{\alpha\beta}(a *_\beta b) = G_{\alpha\beta}(a) *_\alpha G_{\alpha\beta}(b)$. Actually in our C^∞ case, unlike the complex analytic or algebraic case, $G_{\alpha\beta}$ can be made the identity automorphisms, but this is not necessarily the most natural choice.

The sheaf of modules $\mathbb{V}_L^\eta$ is described by the action

$$a *_\gamma f = \sum_{k=0}^{\infty} (i\hbar)^k Q_{\gamma,k}(a, f) \tag{15.1.4}$$

where $f \in |\Omega|^{\frac{1}{2}}(W_\gamma)$ and $a \in C^\infty(U_{\alpha(\gamma)})$, and by the transition functions

$$H_{\gamma\delta}(f) = \exp\left(-\frac{1}{i\hbar}\eta_{\gamma\delta}\right) \sum_{k=0}^{\infty} (i\hbar)^k S_{\gamma\delta,k}(f) \tag{15.1.5}$$

where $Q_{\gamma,k}$ are bidifferential and $S_{\gamma\delta,k}$ are differential. Moreover, $Q_{\gamma\delta,0}(a, f) = af$ and

$$S_{\gamma\delta,0}(f) = \exp\left(\frac{\pi i}{2}\mu_{\gamma\delta}(L)\right) f. \tag{15.1.6}$$

One has

$$a *_\gamma (b *_\gamma f) = (a *_{\alpha(\gamma)} b) *_\gamma f$$

and

$$S_{\gamma\delta}(a *_\delta f) = T_{\alpha(\gamma)\alpha(\delta)}(a) *_\gamma S_{\gamma\delta}(f)$$

Again, all higher $S_{\gamma\delta,k}$ can be made zero, but this is not the most natural choice.

Let C^∞_{poly} denote functions on T^*X that are polynomial on fibers. *A quantization procedure* is the following.

(1) For any α, a map

$$\text{Op}^\alpha_\hbar : C^\infty_{\text{poly}}(T^*(U_\alpha)) \to \mathcal{D}(U_\alpha, |\Omega|^{\frac{1}{2}}_X) \tag{15.1.7}$$

such that

$$\text{Op}^\alpha_\hbar(a)\text{Op}^\alpha_\hbar(b) = \text{Op}^\alpha_\hbar(a *_\alpha b)$$

and

$$\text{Op}^\alpha_\hbar(G_{\alpha\beta}(a)) = \text{Op}^\beta(a)$$

on $U_\alpha \cap U_\beta$. (We can ask for exact equalities, not for asymptotic equalities like we use below, when a and b are polynomial).

(2) A map

$$u_\hbar^\gamma : |\Omega|_c^{\frac{1}{2}}(W_\gamma) \to |\Omega|_c^{\frac{1}{2}}(U_{\alpha(\gamma)}) \tag{15.1.8}$$

for all $\hbar > 0$, such that

$$\mathrm{Op}_\hbar^{\alpha(\gamma)}(a)u_\hbar^\gamma(f) - \sum_{k=0}^{N}(i\hbar)^k u_\hbar^\gamma(Q_{\gamma,k}(a, f)) = O(h^{N+1})$$

and

$$u^\gamma(f) - \sum_{k=0}^{N}(i\hbar)^k u_\hbar^\delta(S_{\gamma,\delta,k}(f)) = O(h^{N+1})$$

for all N.

Let us recall how a quantization procedure is carried out. For every γ choose a *phase function* for $L \cap W_\gamma$ as follows. Let $\theta = (\theta_1, \ldots, \theta_k)$ be a coordinate system on $\mathbb{R}^k$. Choose a coordinate system $x = (x_1, \ldots, x_n)$ on $U_{\alpha(\gamma)}$. Choose *a phase function* for $L \cap W_\gamma$, i.e. a function $\varphi(x, \theta)$ such that

$$L \cap W_\gamma = \{(\xi, x)|\exists\theta \text{ such that } \xi = \varphi_x(x, \theta) \text{ and } \varphi_\theta(x, \theta) = 0\} \tag{15.1.9}$$

Here φ_x and φ_θ stand for partial derivatives. We assume that the $n \times (n + k)$ matrix $(\varphi_{xx}, \varphi_{x\theta})$ is nondegenerate.

Example 15.3 Let $n = 1$. Assume that $L = \{\xi = \varphi'(x)\}$. Then we can choose $k = 0$ and $\varphi = \varphi(x)$. Now let $L = \{x = -\psi'(\xi)\}$. Then we can take $k = 1$ and $\varphi(x, \theta) = x\theta + \psi(\theta)$.

Example 15.4 More generally, one can always subdivide the coordinates into two groups and write $x = (x_1, x_2)$; $\xi = (\xi_1, \xi_2)$ so that $L \cap W_\gamma$ will be of the form

$$\xi_1 = F_{x_1}(x_1, \xi_2);\ \ x_2 = -F_{\xi_1}(x_1, \xi_2) \tag{15.1.10}$$

In this case one can take $\varphi(x_1, x_2, \theta) = x_2\theta + F(x_1, \theta)$.

Note that the condition that the matrix of second derivatives is nondegenerate means that θ in (15.1.9) is unique and therefore $L \cap W_\gamma$ can be identified with $\{(x, \theta)|\varphi_\theta(x, \theta) = 0\}$. (To do that, one may need to pass to a finer open cover). Moreover, we can choose n out of $n + k$ coordinates x, θ so that they will be coordinates on $\{\varphi_\theta = 0\}$. Namely, we can take any n coordinates such that the corresponding square submatrix of $(\varphi_{xx}, \varphi_{x\theta})$ is nondegenerate. Denote these coordinates by z and the other k coordinates by ζ. Choose a procedure for extending functions $f(z)$ to functions on $\{(x, \theta)\}$. Namely, extend $f(z)$ to $f(z)\rho(z')$ where ρ is a function with small support near zero and $\rho(z') = 0$.

Given a phase function and a compactly supported half-form $f = f(z)|dz|^{\frac{1}{2}}$, define $u_\hbar^\gamma(f)$ as follows. Denote by $f(x, \theta)$ the extension of $f(z)$ as above. Then define

$$u_\hbar(f) = \frac{e^{\frac{-\pi i k}{4}}}{(2\pi\hbar)^{\frac{k}{2}}} \int e^{-\frac{\varphi(x,\theta)}{i\hbar}} f(x, \theta) d\theta |dx|^{\frac{1}{2}} \tag{15.1.11}$$

For the sake of completeness let us outline the proof of the fact that this is indeed a quantization procedure as described above (it is contained essentially in [15, 16], as well as in [32]).

First, as proven in [16], any two local phase functions differ by a coordinate change

$$\varphi(x, \theta) \mapsto \varphi(g(x), h(x, \theta))$$

followed by iterated application of

$$\varphi(x, \theta) \mapsto \varphi(x, \theta) \pm \theta_1^2$$

to one or the other phase function. Here θ_1 is an extra variable. So we can assume that our local phase functions are as in Example 15.4, possibly with some θ_1^2 added or subtracted. We have two choices of subdivision $x = (x_1, x_2)$. Namely, for W_γ we will have

$$x_1^\gamma = (x_1, x_2);\;\; x_2^\gamma = (x_3, x_4);$$

for W_δ,

$$x_1^\delta = (x_1, x_3);\;\; x_2^\delta = (x_2, x_4).$$

Let $F_\gamma(x_1, x_2, \xi_3, \xi_4)$ and $F_\delta(x_1, x_3, \xi_2, \xi_4)$ be functions as in Example 15.4. Let us look for functions f_γ and f_δ such that (15.1.11) will give the same answer for the charts W_γ and W_δ.

$$\frac{e^{-\frac{\pi i}{4}(k_3+k_4)}}{(2\pi\hbar)^{\frac{k_3+k_4}{2}}} \int e^{-\frac{1}{i\hbar}(x_3\xi_3+x_4\xi_4+F_\gamma(x_1,x_2,\xi_3,\xi_4))} f_\gamma(x_1, x_2, \xi_3, \xi_4) d\xi_3 d\xi_4 = \tag{15.1.12}$$

$$= \frac{e^{-\frac{\pi i}{4}(k_2+k_4)}}{(2\pi\hbar)^{\frac{k_2+k_4}{2}}} \int e^{-\frac{1}{i\hbar}(x_2\xi_2+x_4\xi_4+F_\delta(x_1,x_3,\xi_2,\xi_4))} f_\delta(x_1, x_3, \xi_2, \xi_4) d\xi_2 d\xi_4$$

Applying the inverse Fourier transform we get

$$e^{-F_\gamma} f_\gamma = \frac{e^{-\frac{\pi i}{4}(k_2-k_3)}}{(2\pi\hbar)^{\frac{k_2+k_3}{2}}} \int e^{\frac{1}{i\hbar}(-x_2\xi_2+x_3\xi_3-F_\delta)} f_\delta d\xi_2 dx_3 \tag{15.1.13}$$

Compute the right hand side by the stationary phase method. The critical points satisfy

$$x_2 = -\frac{\partial F_\delta}{\partial \xi_2};\ \xi_3 = \frac{\partial F_\delta}{\partial x_3} \tag{15.1.14}$$

In other words, the critical point (ξ_2, x_3) is such that (x_1, x_2, ξ_1, ξ_2) is in L.

$$f_\gamma = \epsilon_{\gamma\delta} \exp(\frac{1}{i\hbar}((x_3\xi_3 - F_\delta) - (x_2\xi_2 - F_\gamma)))\mathrm{mod}\hbar \tag{15.1.15}$$

or

$$f_\gamma = \epsilon_{\gamma\delta} \exp\left(\frac{1}{i\hbar}(\varphi_\delta - \varphi_\gamma)\right) \mathrm{mod}\hbar \tag{15.1.16}$$

Here

$$\epsilon_{\gamma\delta} = e^{-\frac{\pi i}{4}(k_2-k_3)} e^{-\frac{\pi i}{4}(n_-(\gamma,\delta)-n_+(\gamma,\delta))} \tag{15.1.17}$$

where $n_-(\gamma, \delta)$, resp. $n_+(\gamma, \delta)$, is the number of negative, resp. positive, eigenvalues of the matrix of second derivatives of F_δ with respect to variables ξ_2 and x_3. We can re-write (15.1.17) as

$$\epsilon_{\gamma\delta} = \exp \frac{\pi i}{2}(n_+ - k_2) \tag{15.1.18}$$

where, as above, n_+ is the number of positive eigenvalues of the matrix of second derivatives of F_δ in variables x_2, ξ_3.

Example 15.5 Let $F_\gamma(x) = \varphi(x)$ and $F_\delta(x, \theta) = x\theta - \psi(\theta)$ as in Example 15.3. Let us compute ϵ_{γ_δ}. One has $k_2 = 1$. If $\varphi_{xx} > 0$ then $n_2 = 0$. If $\varphi_{xx} < 0$ then $n_2 = 1$. Therefore

$$\epsilon_{\gamma_\delta} = -1 \text{ for } \varphi_{xx} > 0;\ \epsilon_{\gamma_\delta} = 0 \text{ for } \varphi_{xx} < 0.$$

Now compute $\epsilon_{\delta\gamma}$. One has $k_2 = 0$. If $\varphi_{xx} > 0$ then $n_2 = 1$. If $\varphi_{xx} < 0$ then $n_2 = 0$. Therefore

$$\epsilon_{\delta\gamma} = 1 \text{ for } \varphi_{xx} > 0;\ \epsilon_{\delta\gamma} = 0 \text{ for } \varphi_{xx} < 0.$$

Now note that $d\varphi_\gamma = \xi dx|L$ on $L \cap W_\gamma$ and $d\varphi_\delta = \xi dx|L$ on $L \cap W_\delta$. Therefore, if $\eta_{\gamma\delta} = \varphi_\gamma - \varphi_\delta$ on $L \cap W_\gamma \cap W_\delta$, then $(\eta_{\gamma\delta})$ represents the cohomology class η corresponding to the De Rham class of $\xi dx|L$.

On the other hand, a choice of a local presentation (15.1.10) of L determines a choice of lifting of transition isomorphisms as in (12.1.4). Indeed, in a tangent space $T_{(x,\xi)}L$ to a point of $L \cap W_\gamma$, let $\widehat{x}$, $\widehat{\xi}$ be formal Darboux coordinates coming from some local coordinate system. Choose a presentation

$$\widehat{\xi_1} = A\widehat{x_1} + B\widehat{\xi_2};\ \widehat{x_2} = -C\widehat{x_1} - D\widehat{\xi_2} \tag{15.1.19}$$

Construct a symplectic matrix sending $L_0 = \{\widehat{\xi_1} = \widehat{\xi_2} = 0\}$ to $T_{(x,\xi)}L$ as follows. Let

$$p(A, B, C, D) : (\widehat{x_1}, \widehat{x_2}, \widehat{\xi_1}, \widehat{\xi_2}) \mapsto (\widehat{x_1}, \widehat{x_2}, A\widehat{x_1} + B\widehat{x_2}, C\widehat{x_1} + D\widehat{x_2}) \tag{15.1.20}$$

and

$$F_{\widehat{x}_2} : (\widehat{x}_1, \widehat{x}_2, \widehat{\xi}_1, \widehat{\xi}_2) \mapsto (\widehat{x}_1, -\widehat{\xi}_2, \widehat{\xi}_1, \widehat{x}_2) \tag{15.1.21}$$

One has

$$T_{(x,\xi)}L = F_{\widehat{x}_2} p(A, B, C, D)L_0 \tag{15.1.22}$$

Note also that both factors of the right hand side extend automatically to elements in Sp^4. Indeed, one can replace $p(A, B, C, D)$ by the homotopy class of the path $p(tA, tB, tC, tD)$, $0 \le t \le 1$, and $F_{\widehat{x}_2}$ by the homotopy class of the path

$$(\widehat{x}_1, \widehat{x}_2, \widehat{\xi}_1, \widehat{\xi}_2) \mapsto (\widehat{x}_1, \widehat{x}_2 \cos t - \widehat{\xi}_2 \sin t, \widehat{\xi}_1, \widehat{x}_2 \sin t + \widehat{\xi}_2 \cos t),\ 0 \le t \le \frac{\pi}{2}$$

It is easy to see that the Maslov class μ corresponding to the lifted transition functions thus defined is inverse to the one defined by (15.1.18).

16 Appendix. Twisted A_∞ Modules and A_∞ Functors

16.1 Differential Graded Categories of A_∞ Functors

Our references for this Section are [23] and [24] (see also [8] and the survey [41]).

Let A and B be two differential graded (DG) categories. For two maps

$$\mathbf{f}, \mathbf{g} : \mathrm{Ob}(A) \to \mathrm{Ob}(B)$$

define

$$\overline{C}^{\bullet}_{\mathbf{f},\mathbf{g}}(A, B) = \prod_{n\ge 1; x_0,\ldots,x_n} \mathrm{Hom}^{\bullet}(A(x_0, x_1) \otimes \ldots \otimes A(x_{n-1}, x_n)[n], B(f(x_0), g(x_n)))$$

where the product is taken over all $x_0, \ldots, x_n \in \mathrm{Ob}(A)$. Put

$$C^{\bullet}_{\mathbf{f},\mathbf{g}}(A, B) = \prod_{x_0 \in \mathrm{Ob}(A)} B(f(x_0), g(x_0)) \times \overline{C}^{\bullet}_{\mathbf{f},\mathbf{g}}(A, B) \tag{16.1.1}$$

Define the differential d by

$$(d_1\varphi)(a_1, \ldots, a_{n+1}) = \sum_{j=1}^{n} (-1)^{\sum_{p\le j}(|a_p|+1)} \varphi(a_1, \ldots, a_j a_{j+1}, \ldots, a_{n+1}) \tag{16.1.2}$$

($d_1 = 0$ on the first factor of (16.1.1));

$$(d_2\varphi)(a_1,\ldots,a_n)=\sum_{j=1}^{n}(-1)^{\sum_{p<j}(|a_p|+1)}\varphi(a_1,\ldots,d_A a_j,\ldots,a_{n+1})+d_B\varphi(a_1,\ldots,a_n) \tag{16.1.3}$$

Define

$$d=d_1+d_2$$

Also define the product

$$\overline{C}^{\bullet}_{\mathbf{f},\mathbf{g}}(A,B)\otimes\overline{C}^{\bullet}_{\mathbf{g},\mathbf{h}}(A,B)\to\overline{C}^{\bullet}_{\mathbf{f},\mathbf{h}}(A,B)$$

by

$$(\varphi\smile\psi)(a_1,\ldots,a_{m+n})=(-1)^{|\psi|\sum_{j=1}^{m}(|a_j|+1)}\varphi(a_1,\ldots,a_m)\psi(a_{m+1},\ldots,a_{m+n}) \tag{16.1.4}$$

(Note that here m or n can be zero, which corresponds to the case of one or both factors lying in the first factor of (16.1.1)).

Definition 16.1 An A_∞ functor $f:A\to B$ is a map $f:\mathrm{Ob}(A)\to\mathrm{Ob}(B)$ together with an element f of degree 1 in $\overline{C}^{\bullet}_{\mathbf{f},\mathbf{f}}(A,B)$ such that

$$df+f\smile f=0$$

A *curved* A_∞ functor is defined the same way but now the cochain f is allowed to be in $C^{\bullet}_{\mathbf{f},\mathbf{f}}(A,B)$.

Definition 16.2 Define the DG category $\mathbf{C}(A,B)$ as follows. Let objects be A_∞ functors $\mathbf{f}:A\to B$; set

$$\mathbf{C}^{\bullet}(A,B)(f,g)=C^{\bullet}_{\mathbf{f},\mathbf{g}}(A,B)$$

with the differential

$$\delta\varphi=d\varphi+f\smile\varphi-(-1)^{|\varphi|}\varphi\smile f$$

We define the composition to be the cup product.

Also, define the DG category $\mathbf{C}_+(A,B)$ the same way as above but with objects being curved A_∞ functors.

16.1.1 Equivalence of Objects in a DG Category

Let $\mathbf{C}_1$ be the category with two objects 0 and 1 and two mutually inverse morphisms $g:0\to 1$ and $g^{-1}:1\to 0$.

Definition 16.3 Two objects $\mathbf{x}$, $\mathbf{y}$ of a DG category C are equivalent if there is an A_∞ functor $\mathbf{C}_1\to C$ sending 0 to $\mathbf{x}$ and 1 to $\mathbf{y}$.

Lemma 16.4 *The relation defined above is an equivalence relation.*

Proof Let $\mathbf{C}_2$ be the category with three objects 0, 1, 2 and with unique morphism between any two objects. There are functors $i_{pq} : \mathbf{C}_1 \to \mathbf{C}_2$ that send 0 to p and 1 to q, $0 \leq p < q \leq 2$. If we have one equivalence between $\mathbf{x}$ and $\mathbf{y}$ and another between $\mathbf{y}$ and $\mathbf{z}$, then we have a functor (cf. Definitions and Lemma 16.7 below):

$$\text{Cobar Bar}\, k[i_{01}\mathbf{C}_1] *_{k[1]} \text{Cobar Bar}\, k[i_{12}\mathbf{C}_1] \to C \tag{16.1.5}$$

that sends 0 to $\mathbf{x}$, 1 to $\mathbf{y}$, and 2 to $\mathbf{z}$. Here $*$ stands for free product of categories; for any category $\mathbf{C}$, $k[\mathbf{C}]$ is its linearization, and $k[1]$ is the category with one object 1 whose ring of endomorphisms is k. But the left hand side of (16.1.5) is quasi-isomorphic to $k[i_{01}\mathbf{C}_1] *_{k[1]} k[i_{12}\mathbf{C}_1] \xrightarrow{\sim} \mathbf{C}_2$. By the standard transfer of structure [24, 25, 28], we get an A_∞ morphism $\mathbf{C}_2 \to C$ that sends 0 to $\mathbf{x}$, 1 to $\mathbf{y}$, and 2 to $\mathbf{z}$. Composing it with i_{02}, we get an equivalence between $\mathbf{x}$ and $\mathbf{z}$. □

Definition 16.5 Two A_∞ functors $A \to B$ are equivalent if they are equivalent as objects in $\mathbf{C}(A, B)$.

16.1.2 The Bar Construction

The bar construction of a DG category A is a DG cocategory $\text{Bar}(A)$ with the same objects where

$$\text{Bar}(A)(x, y) = \bigoplus_{n\geq 0} \bigoplus_{x_1,\ldots,x_n} A(x, x_1)[1] \otimes A(x_1, x_2)[1] \otimes \cdots \otimes A(x_n, x)[1]$$

with the differential

$$d = d_1 + d_2;$$

$$d_1(a_1|\cdots|a_{n+1}) = \sum_{i=1}^{n+1} \pm(a_1|\cdots|da_i|\cdots|a_{n+1});$$

$$d_2(a_1|\cdots|a_{n+1}) = \sum_{i=1}^{n} \pm(a_1|\cdots|a_i a_{i+1}|\cdots|a_{n+1})$$

The second sum is taken over n-tuples $x_1, \ldots, x_n$ of objects of A. The signs are $(-1)^{\sum_{j<i}(|a_i|+1)+1}$ for the first sum and $(-1)^{\sum_{j\leq i}(|a_i|+1)}$ for the second. The comultiplication is given by

$$\Delta(a_1|\cdots|a_n) = \sum_{i=1}^{n-1}(a_1|\cdots|a_i) \otimes (a_{i+1}|\cdots|a_n)$$

Dually, for a DG cocategory B one defines the DG category $\mathrm{Cobar}(B)$. The DG category $\mathrm{Cobar\,Bar}(A)$ is a cofibrant resolution of A.

It is convenient for us to work with DG (co)categories without (co)units. For example, this is the case for $\mathrm{Bar}(A)$ and $\mathrm{Cobar}(B)$ (we sum, by definition, over all tensor products with at least one factor). Let A^+ be the (co)category A with the (co)units added, i.e. $A^+(x, y) = A(x, y)$ for $x \neq y$ and $A^+(x, x) = A(x, x) \oplus k\mathtt{Id}_x$. If A is a DG category then A^+ is an augmented DG category with units, i.e. there is a DG functor $\epsilon : A^+ \to k_{\mathrm{Ob}(A)}$. (For a set I, k_I is the DG category with the set of objects I and with $k_I(x, y) = 0$ for $x \neq y$, $k_I(x, x) = k$). Dually, one defines the DG cocategory $k^{\mathrm{Ob}(B)}$ and the DG functor $\eta : k^{\mathrm{Ob}(B)} \to B^+$ for a DG cocategory B.

For DG (co)categories with (co)units, define $A \otimes B$ as follows: $\mathrm{Ob}(A \otimes B) = \mathrm{Ob}(A) \times \mathrm{Ob}(B)$; $(A \otimes B)((x_1, y_1), (x_2, y_2)) = A(x_1, y_1) \otimes B(x_2, y_2)$; the product is defined as $(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{|a_2||b_1|} a_1 a_2 \otimes b_1 b_2$, and the coproduct in the dual way. This tensor product, when applied to two (co)augmented DG (co)categories with (co)units, is again a (co)augmented DG (co)category with (co)units: the (co)augmentation is given by $\epsilon \otimes \epsilon$, resp. $\eta \otimes \eta$.

Definition 16.6 For DG categories A and B without units, put

$$A \otimes B = \mathrm{Ker}(\epsilon \otimes \epsilon : A^+ \otimes B^+ \to k_{\mathrm{Ob}(A)} \otimes k_{\mathrm{Ob}(B)}).$$

Dually, for DG cocategories A and B without counits, put

$$A \otimes B = \mathrm{Coker}(\eta \otimes \eta : k^{\mathrm{Ob}(A)} \otimes k^{\mathrm{Ob}(B)} \to A^+ \otimes B^+).$$

The following is standard (and straightforward).

Lemma 16.7 *There are natural bijections*

$$\mathrm{Ob}\,\mathbf{C}(A, B) \xrightarrow{\sim} \mathrm{Hom}(\mathrm{Cobar\,Bar}(A), B);$$

$$\mathrm{Ob}\,\mathbf{C}_+(A, B) \xrightarrow{\sim} \mathrm{Hom}(\mathrm{Cobar\,Bar}^+(A), B)$$

In other words, an A_∞ functor $A \to B$ is the same as a DG functor $\mathrm{Cobar\,Bar}(A) \to B$. A curved A_∞ functor $A \to B$ is the same as a DG functor $\mathrm{Cobar\,Bar}^+(A) \to B$.

16.1.3 The Adjunction Formula

Lemma 16.8 *There are natural bijections*

$$\mathrm{Ob}\,\mathbf{C}(A, \mathbf{C}(B, C)) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{DGcat}}(\mathrm{Cobar}(\mathrm{Bar}^+(A) \otimes \mathrm{Bar}(B)), C)$$

$$\mathrm{Ob}\,\mathbf{C}_+(A, \mathbf{C}_+(B, C)) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{DGcat}}(\mathrm{Cobar}(\mathrm{Bar}^+(A) \otimes \mathrm{Bar}^+(B)), C)$$

This (as well as Lemma 16.7) follows from Lemmas 16.10, 16.11, 16.12 below.

16.1.4 Convolution Categories

Let $\mathbb{B}$ be a DG cocategory and C a DG category. For

$$\mathbf{f}, \mathbf{g} : \mathrm{Ob}(\mathbb{B}) \to \mathrm{Ob}(C),$$

put

$$\overline{\mathrm{Conv}}_{\mathbf{f},\mathbf{g}}(\mathbb{B}, C) = \prod_{x,y\in \mathrm{Ob}(\mathcal{B})} \mathrm{Hom}^\bullet(\mathbb{B}(x, y), C(fx, gy))$$

$$\mathrm{Conv}_{\mathbf{f},\mathbf{g}}(\mathbb{B}, C) = \prod_{x,y\in \mathrm{Ob}(\mathcal{B})} \mathrm{Hom}^\bullet(\mathbb{B}^+(x, y), C(fx, gy))$$

The differential d is the usual one (induced by the differentials on $\mathbb{B}$ and C). Define the product

$$\mathrm{Conv}_{\mathbf{f},\mathbf{g}}(\mathbb{B}, C) \otimes \mathrm{Conv}_{\mathbf{g},\mathbf{h}}(\mathbb{B}, C) \to \mathrm{Conv}_{\mathbf{f},\mathbf{h}}(\mathbb{B}, C)$$

$$\overline{\mathrm{Conv}}_{\mathbf{f},\mathbf{g}}(\mathbb{B}, C) \otimes \overline{\mathrm{Conv}}_{\mathbf{g},\mathbf{h}}(\mathbb{B}, C) \to \overline{\mathrm{Conv}}_{\mathbf{f},\mathbf{h}}(\mathbb{B}, C)$$

as follows. If

$$\Delta b = \sum b^{(1)} \otimes b^{(2)}$$

then

$$(\varphi \smile \psi)(b) = \sum (-1)^{|\psi||b^{(1)}|} \varphi(b^{(1)})\psi(b^{(2)}) \qquad (16.1.6)$$

Definition 16.9 Define DG categories $\mathbf{Conv}(\mathbb{B}, C)$ and $\mathbf{Conv}_+(\mathbb{B}, C)$ as follows. Their objects are maps $\mathbf{f} : \mathrm{Ob}(\mathbb{B}) \to \mathrm{Ob}(C)$ together with elements f of degree one in $\overline{\mathrm{Conv}}_{\mathbf{f},\mathbf{f}}(\mathbb{B}, C)$ (resp. in $\mathrm{Conv}_{\mathbf{f},\mathbf{f}}(\mathbb{B}, C)$) satisfying

$$df + f \smile f = 0.$$

The complex of morphisms between f and g is $\mathrm{Conv}_{\mathbf{f},\mathbf{g}}(\mathbb{B}, C)$ with the differential

$$\delta\varphi = d\varphi + f \smile \varphi - (-1)^{|\varphi|}\varphi \smile f$$

The composition is the cup product (16.1.6).

Lemma 16.10 *There are natural isomorphisms of DG categories*

$$\mathbb{C}(A, B) \xrightarrow{\sim} \mathbf{Conv}(\mathrm{Bar}(A), B)$$

$$\mathbb{C}_+(A, B) \xrightarrow{\sim} \mathbf{Conv}_+(\mathrm{Bar}(A), B)$$

Lemma 16.11 *There is a natural bijection*

$$\mathrm{Hom}_{\mathrm{DGcat}}(\mathrm{Cobar}(\mathbb{B}), C) \xrightarrow{\sim} \mathrm{Ob}(\mathbf{Conv}(\mathbb{B}, C))$$

Lemma 16.12 *There is a natural isomorphism of DG categories*

$$\mathbf{Conv}(\mathbb{B}_1, \mathbf{Conv}(\mathbb{B}_2, C)) \xrightarrow{\sim} \mathbf{Conv}(\mathbb{B}_1 \otimes \mathbb{B}_2, C)$$

This is a reformulation of a result in [23].

16.1.5 An A_∞ Functor to A_∞ Modules

Let k be a field. By dgmod(k) we denote the differential graded category of complexes of modules over k. Let R be an associative algebra over k.

Definition 16.13 We denote the DG category $\mathbf{C}(\mathrm{Bar}(R), \mathrm{dgmod}(k))$ by $\mathrm{Mod}_\infty(R)$ and call it the DG category of A_∞ modules over R.

Let $\mathfrak{X}(R)$ be the category whose objects are pairs $(\mathcal{B} \xrightarrow{\pi} R, \mathcal{M})$ where $\mathcal{B}$ is a differential graded algebra, π a quasi-isomorphism of DGA, and $\mathcal{M}$ a DG module over $\mathcal{B}$. A morphism $(\mathcal{B} \xrightarrow{\pi} R, \mathcal{M}) \to (\mathcal{B}' \xrightarrow{\pi'} R, \mathcal{M}')$ is a morphism $\mathcal{B} \to \mathcal{B}'$ of DGA over R together with a compatible morphism $\mathcal{M} \to \mathcal{M}'$.

We will construct an A_∞ functor

$$\mathfrak{X}(R) \to \mathrm{Mod}_\infty(R) \tag{16.1.7}$$

Remark 16.14 An A_∞ functor from a category $\mathfrak{X}$ to a DG category $\mathcal{A}$ is by definition an A_∞ functor from the linearization of $\mathfrak{X}$ (viewed as a DG category with zero differential) to $\mathcal{A}$.

Define the DG category $\mathfrak{B}$ as follows. Its objects are the same as objects of $\mathfrak{X}(R)$ but repeated countably many times, i.e. an object of $\mathfrak{B}$ is a pair $(\mathbf{x}, n)$ where $\mathbf{x}$ is an object of $\mathfrak{X}(R)$ and $n \in \mathbb{Z}$. The spaces of morphisms are as follows.

$$\mathfrak{B}((\mathbf{x}, m), (\mathbf{y}, n)) = 0 \tag{16.1.8}$$

if $m < n$ or $m = n$ but $\mathbf{x} \neq \mathbf{y}$. If $m > n$ and

$$\mathbf{x} = (\mathcal{B} \xrightarrow{\pi} R, \mathcal{M}),\ \mathbf{y} = (\mathcal{B}' \xrightarrow{\pi'} R, \mathcal{M}'), \tag{16.1.9}$$

then

$$\mathfrak{B}((\mathbf{x}, m), (\mathbf{y}, n)) = \mathcal{B}' \times \mathfrak{X}(R)(\mathbf{x}, \mathbf{y}) \tag{16.1.10}$$

By $a'\mathbf{b}$ we denote the pair $(a', \mathbf{b})$ where $a' \in \mathcal{B}'$ and $\mathbf{b} : \mathbf{x} \to \mathbf{y}$ is a morphism in $\mathfrak{X}(R)$. We denote the underlying morphism $\mathcal{B} \to \mathcal{B}'$ also by $\mathbf{b}$. Put

$$\mathfrak{B}((\mathbf{x}, n), (\mathbf{x}, n)) = \mathcal{B} \tag{16.1.11}$$

We also denote the right hand side by $\mathcal{B}\mathtt{Id}_{\mathbf{x}}$. The composition is given by

$$(a''\mathbf{b}')(a'\mathbf{b}) = (a''\mathbf{b}'(a'))\mathbf{b}'\mathbf{b} \tag{16.1.12}$$

Consider the following right DG module $\mathbf{M}$ over $\mathfrak{B}$. Define

$$\mathbf{M}(\mathbf{x}, n) = \mathcal{M}$$

where $\mathbf{x} = (\mathcal{B} \to R, \mathcal{M})$. Define the action

$$\mathbf{M}(\mathbf{x}, m) \otimes \mathfrak{B}((\mathbf{x}, m), (\mathbf{y}, n)) \to \mathbf{M}(\mathbf{y}, n)$$

by

$$v \otimes (a'\mathbf{b}) = a'\mathbf{b}(v)$$

Here we denote by $\mathbf{b}$ the underlying action of the morphism $\mathbf{b} : \mathbf{x} \to \mathbf{y}$ on the module, as well as on the algebra.

Define another DG category $\mathcal{R}$ exactly like $\mathfrak{B}$ above with the only difference that we put

$$\mathcal{R}((\mathbf{x}, m), (\mathbf{y}, n)) = R \times \mathfrak{X}(R)(\mathbf{x}, \mathbf{y}) \tag{16.1.13}$$

instead of (16.1.10) and

$$\mathfrak{B}((\mathbf{x}, n), (\mathbf{x}, n)) = R \tag{16.1.14}$$

instead of (16.1.11). We also denote the right hand side by $\mathtt{Id}_{\mathbf{x}}R$. Instead of (16.1.12), the composition is given by

$$(a''\mathbf{b}')(a'\mathbf{b}) = (a''a')\mathbf{b}'\mathbf{b} \tag{16.1.15}$$

The morphisms $\pi : \mathcal{B} \to R$ induce a quasi-isomorphism of DG categories $\mathfrak{B} \xrightarrow{\pi} \mathcal{R}$. The transfer of structure argument makes $\mathbf{M}$ a right A_∞ module over $\mathcal{R}$ as follows. Fix a linear map $\mathcal{R} \xrightarrow{i} \mathcal{B}$ that is inverse to π at the level of cohomology. (This is where we use the assumption that k is a field). Fix also homotopies for $\mathtt{Id}_{\mathfrak{B}} - i\pi$ and for $\mathtt{Id}_{\mathcal{R}} - \pi i$. (By this we mean collections of maps $\mathcal{R}(\mathbf{x}, \mathbf{y}) \to \mathfrak{B}(\mathbf{x}, \mathbf{y})$, *etc.*, for any objects $\mathbf{x}$ and $\mathbf{y}$). From his data one constructs an A_∞ functor $\mathcal{R} \to \mathfrak{B}$ which is inverse to π up to equivalence (cf. [24, 25, 28]). Furthermore, the map i and the homotopies can be chosen to be invariant under the action of $\mathbb{Z}$ on $\mathfrak{B}$ and on $\mathcal{R}$. Therefore the A_∞ functor is also $\mathbb{Z}$-invariant. We denote it by $\mathbb{T}$, and the corresponding twisting cochain ρ by $\rho_{\mathbb{T}}$.

This, in turn, defines the desired A_∞ functor (16.1.7). In fact, for any object $\mathbf{x} = (\mathcal{B} \to R, \mathcal{M})$, the value of this A_∞ functor on $\mathbf{x}$ is the underlying complex $\mathcal{M}$. For $g_1, \ldots, g_p \in R$, put

$$\rho(g_1, \ldots, g_p) = \rho_{\mathbb{T}}(g_1 \mathtt{Id}_{\mathbf{x}}, \ldots, g_p \mathtt{Id}_{\mathbf{x}}) \tag{16.1.16}$$

where we view $\rho_j \mathtt{Id}_{\mathbf{x}}$ as morphisms $(\mathbf{x}, 0) \to (\mathbf{x}, 0)$ in $\mathfrak{B}$. This makes each $\mathcal{M}$ an A_∞ module over R. Now consider morphisms

$$\mathbf{x}_0 \xleftarrow{\mathbf{b}_1} \mathbf{x}_1 \xleftarrow{\mathbf{b}_2} \cdots \xleftarrow{\mathbf{b}_n} \mathbf{x}_n \tag{16.1.17}$$

in $\mathfrak{X}(R)$, as well as corresponding morphisms

$$(\mathbf{x}_0, 0) \xleftarrow{\mathbf{b}_1} (\mathbf{x}_1, 1) \xleftarrow{\mathbf{b}_2} \cdots \xleftarrow{\mathbf{b}_n} (\mathbf{x}_n, n) \tag{16.1.18}$$

in $\mathcal{R}$. Now put

$$\rho(g_1, \ldots, g_p) = \sum \pm \rho_{\mathbb{T}}(g_1 \mathtt{Id}_{\mathbf{x}_0}, \ldots, g_{p_1} \mathtt{Id}_{\mathbf{x}_0}, \mathbf{b}_1,$$

$$g_{p_1+1} \mathtt{Id}_{\mathbf{x}_1}, \ldots, g_{p_2} \mathtt{Id}_{\mathbf{x}_1}, \ldots, \mathbf{b}_n, g_{p_n+1} \mathtt{Id}_{\mathbf{x}_n}, \ldots, g_p \mathtt{Id}_{\mathbf{x}_n})$$

where the sum is taken over all $0 \le p_1 \le \ldots \le p_n \le n$. The sign rule: both $g_j \mathtt{Id}_{\mathbf{x}_k}$ and $\mathbf{b}_i$ are treated as odd (the former has degree $(-1)^{|g_j|+1}$ if R is graded).

It is straightforward to check that thus defined ρ, when viewed as a cochain

$$\rho(\mathbf{b}_1, \ldots, \mathbf{b}_n) \in \mathrm{Mod}_\infty(R)(\mathcal{M}_0, \mathcal{M}_n),$$

is an A_∞ functor $\mathfrak{X}(R) \to \mathrm{Mod}_\infty(R)$. (Here $\mathcal{M}_j$ is the underlying DG module of $\mathbf{x}_j$, viewed as a complex).

16.2 Twisted A_∞ Modules on a Space

Let $\mathcal{R}$ be a sheaf of algebras on a topological space X. Fix an open cover $\mathfrak{U}$ of X. For two collections $\mathbf{M} = \{\mathcal{M}_U | U \in \mathfrak{U}\}$ and $\mathbf{N} = \{\mathcal{N}_U | U \in \mathfrak{U}\}$ of sheaves of $\mathcal{R}_U$-modules, define the complex $C^\bullet_{\mathbf{M},\mathbf{N}}(\mathfrak{U})$ as follows. Put

$$C^{\bullet}_{\mathbf{M},\mathbf{N}}(\mathfrak{U}) = \prod_{p,q=0}^{\infty} \prod_{U_0,\ldots,U_p \in \mathfrak{U}} \underline{\mathrm{Hom}}^{\bullet - p - q}(\mathcal{R}^{\otimes q}, \underline{\mathrm{Hom}}^{\bullet}(\mathcal{N}_{U_p}, \mathcal{M}_{U_0}))(U_0 \cap \ldots \cap U_p) \tag{16.2.1}$$

Define the differentials

$$(\check{\partial}\varphi)_{U_0 \ldots U_{p+1}} = \sum_{j=1}^{p} (-1)^j \varphi_{U_0 \ldots \widehat{U_j} \ldots U_{p+1}}; \tag{16.2.2}$$

$$(\partial\varphi)(g_1, \ldots, g_{q+1}) = (-1)^{p|\varphi|} \sum_{j=1}^{q} \varphi(g_1, \ldots, g_j g_{j+1}, \ldots, g_{q+1}) \tag{16.2.3}$$

for local sections $g_1, \ldots$ of $\mathcal{R}$;

$$d\varphi = \check{\partial}\varphi + \partial\varphi + d_{\mathcal{M}}\varphi - (-1)^{|\varphi|}\varphi d_{\mathcal{N}} \tag{16.2.4}$$

Define also the product

$$C^{\bullet}_{\mathbf{M},\mathbf{N}}(\mathfrak{U}) \otimes C^{\bullet}_{\mathbf{N},\mathbf{P}}(\mathfrak{U}) \to C^{\bullet}_{\mathbf{M},\mathbf{P}}(\mathfrak{U}) \tag{16.2.5}$$

by

$$(\varphi \smile \psi)_{U_0 \ldots U_{p_1+p_2}}(g_1, \ldots, g_{q_1+q_2}) =$$

$$(-1)^{|\varphi|p_2 + (|\psi|+p_2)q_1} \varphi_{U_0 \ldots U_{p_1}}(g_1, \ldots, g_{q_1}) \psi_{U_{p_1}, \ldots, U_{p_1+p_2}}(g_{q_1+1}, \ldots, g_{q_1+q_2})$$

Set

$$C^{\bullet}_{\mathbf{M},\mathbf{N}}(X) = \varinjlim_{\mathfrak{U}} C^{\bullet}_{\mathbf{M},\mathbf{N}}(\mathfrak{U}) \tag{16.2.6}$$

The differential and the cup product are well defined on the above complexes.

Definition 16.15 A twisted A_∞ module $\mathcal{M}$ over $\mathcal{R}$ is a collection $\mathbf{M} = \{\mathcal{M}_U | U \in \mathfrak{U}\}$, of sheaves of $\mathcal{R}_U$-modules together with a cochain ρ of degree one in $C^{\bullet}_{\mathbf{M},\mathbf{M}}(X)$ such that

$$d\rho + \rho \smile \rho = 0.$$

The DG category $\mathrm{Tw\,Mod}_\infty(\mathcal{R})$ has twisted A_∞ modules as objects. The complex of morphisms between $\mathcal{M} = (\mathbf{M}, \rho)$ and $\mathcal{N} = (\mathbf{N}, \sigma)$ is the complex $C^{\bullet}_{\mathbf{M},\mathbf{N}}(X)$ with the differential $\delta\varphi = d\varphi + \rho \smile \varphi - (-1)^{|\varphi|}\varphi \smile \sigma$.

The above definition is an extension of the definition of twisted cochains from [39]. Cf. also [5, 33, 42].

Remark 16.16 The DG category of twisted A_∞ modules is obtained almost *verbatim* as a partial case of the left hand side of Lemma 16.8. Formally, one could choose B to be the category with one object whose complex of morphisms is $\mathcal{R}$, and $A = \mathrm{Op}_X$ to be the category of open subsets of X. More precisely, we perform all the computations as if A were the category whose objects are open subsets U_α, and there is one morphism $U_\alpha \to U_\beta$ for any two intersecting open subsets. This is not literally true (there may be nonempty intersections $U_\alpha \cap U_\beta$ and $U_\beta \cap U_\gamma$ but not $U_\alpha \cap U_\gamma$), but all the formulas work. The above motivation may be given rigorous meaning using the techniques of [13] or [5].

16.3 Twisted A_∞ Modules over Groupoids

For $q \geq 0$, we use notation $\mathbb{U} = (U^{(0)}, \ldots, U^{(q)})$. We denote by $\mathfrak{U}_q$ the set of all such $\mathbb{U}$ where U_j is in a given open cover $\mathfrak{U}$. For $p+1$ such q-tuples $\mathbb{U}_{j_0}, \ldots, \mathbb{U}_{j_p}$, denote

$$U^{(k)}_{j_0 \ldots j_p} = U^{(k)}_{j_0} \cap \cdots \cap U^{(k)}_{j_p} \tag{16.3.1}$$

for all $0 \leq k \leq q$. Denote also

$$\mathbb{U}_{j_0 \ldots j_p} = (U^{(0)}_{j_0 \ldots j_p}, \ldots, U^{(q)}_{j_0 \ldots j_p}). \tag{16.3.2}$$

Let Γ be an étale groupoid on a manifold X (in our applications, $\Gamma = \pi_1(X)$). For $\mathbf{M} = \{\mathcal{M}_U | U \in \mathfrak{U}\}$ and $\mathbf{N} = \{\mathcal{N}_U | U \in \mathfrak{U}\}$ as in the beginning of Sect. 16.2, put

$$C^\bullet_{\mathbf{M},\mathbf{N}}(\mathfrak{U}, \Gamma) = \prod_{p,q \geq 0} \prod_{\mathbb{U}_0, \ldots, \mathbb{U}_p \in \mathfrak{U}_q} \underline{\mathrm{Hom}}^{\bullet - p - q}(\underline{\Gamma}^{(q)}, \underline{\mathrm{Hom}}^\bullet(\mathcal{N}_{U^{(q)}_p}, \mathcal{M}_{U^{(0)}_0})(\prod_{k=0}^{q} U^{(k)}_{01\ldots p})$$

Here $\mathcal{M}_{U^{(0)}_0}$ stands for its inverse image under the map

$$\prod_k \cap_j U^{(k)}_j \to \prod_k U^{(k)}_0 \to U^{(0)}_0$$

The differential and the cup product are defined exactly as in (16.2.5), (16.2.3), (16.2.2) (with U_j replaced by $\mathbb{U}_j$). Define

$$C^\bullet_{\mathbf{M},\mathbf{N}}(X, \Gamma) = \varinjlim_{\mathfrak{U}} C^\bullet_{\mathbf{M},\mathbf{N}}(\mathfrak{U}, \Gamma) \tag{16.3.3}$$

Definition 16.17 (a) Define the DG category $\mathrm{Tw\,Mod}_\infty(\Gamma)$ exactly as in Definition 16.15 using complexes $C^\bullet_{\mathbf{M},\mathbf{N}}(X, \Gamma)$.

(b) The DG category

$$\mathrm{Tw\,Mod}_\infty(\Gamma, \Omega^\bullet_{\mathbb{K},X})$$

is defined the same way but with $\mathcal{M}_U$ being $\Omega^\bullet_{\mathbb{K},U}$-modules as in Sect. 8.1.

Remark 16.18 By

$$\mathrm{Loc}_{\infty,\mathbb{K}}(X)$$

we denote the DG category of A_∞ representations of the fundamental groupoid $\pi_1(X)$. This is the partial case of the above Definition 16.17, (a) when $\Gamma = \pi_1(X)$, the topology on X is *discrete*, and the ground ring is $\mathbb{K}$. Objects of this DG category are infinity local systems as in Sect. 8.4.

16.3.1 From $\mathcal{A}^\bullet_M$-Modules with an Action of $\pi_1(M)$ up to Inner Automorphisms to Twisted $(\Omega^\bullet_{\mathbb{K},M}, \pi_1(M))$-Modules

Given two $\mathcal{A}^\bullet_M$-modules $\mathcal{V}^\bullet$ and $\mathcal{W}^\bullet$ with an action of $\pi_1(M)$ up to inner automorphisms, consider the standard complex

$$\mathcal{M} = \mathcal{C}^\bullet(\mathcal{V}^\bullet, \mathcal{A}^\bullet, \mathcal{W}^\bullet).$$

As it is shown in Sect. 6.2.5, $\mathcal{M}$ has the following structure.

For a number of open subsets $U^{(j)}$ indexed by $j \in J$, write $\mathbb{U}_{ij} = (U^{(i)}, U^{(j)})$. We have constructed:

(a) For every $U^{(0)}$ and $U^{(1)}$, an $\Omega^\bullet_{\mathbb{K},U^{(0)}\times U^{(1)}}$-module $\mathcal{B}_{\mathbb{U}_{01}}$ together with a quasi-isomorphism

$$\mathcal{B}_{\mathbb{U}_{01}} \to \mathbb{K}\underline{\pi_1}(M)|(U^{(0)} \times U^{(1)}); \tag{16.3.4}$$

(b) a morphism

$$p_{01}^*\mathcal{B}_{\mathbb{U}_{01}} \otimes p_{12}^*\mathcal{B}_{\mathbb{U}_{12}} \to p_{02}^*\mathcal{B}_{\mathbb{U}_{02}} \tag{16.3.5}$$

which commutes with the composition on $\underline{\pi_1}(M)$ under (16.3.4);

(c) for any $U_0^{(j)}$ and $U_1^{(j)}$, an isomorphism

$$\mathbf{b}_{01} : \mathcal{B}_{\mathbb{U}_0} \xleftarrow{\sim} \mathcal{B}_{\mathbb{U}_1} \tag{16.3.6}$$

that commutes with (16.3.4) and (16.3.5) and satisfies

$$\mathbf{b}_{01}\mathbf{b}_{12} = \mathbf{b}_{02}$$

on the intersections.

Now repeat the procedure from Sect. 16.1.5, together with Remark 16.16, in the above context. First note that the constructions of Sect. 16.1.5 can be carried out in the case when R is a category (and all $\mathcal{B}$ are DG categories with the same objects). Now act as if R were the category with objects $U^{(j)}$, with

$$R(i, j) = \underline{\pi_1}(M)|(U^{(i)} \times U^{(j)})$$

and the composition being the one on $\underline{\pi_1}$. Now, let Op_M be the category whose objects are open subsets U_j, exactly as discussed in Remark 16.16. View the data (a), (b), (c) above as a DG functor $\mathrm{Op}_M \to \mathfrak{X}(R)$. Applying formulas from Sect. 16.1.5, we get an A_∞ functor $\mathrm{Op}_M \to \mathbf{C}(R, \mathrm{dgmod}(\mathbb{K}))$, which is the same as an $\Omega^\bullet_{\mathbb{K},M}$-module with a twisted action of $\pi_1(M)$.

16.3.2 From Twisted $(\Omega^\bullet_{\mathbb{K},X}, \pi_1(X))$ Modules to Infinity Local Systems

Here we extend the construction from Sect. 8.4.1. Consider all open covers of the type $\mathfrak{U} = \{U_x | x \in X\}$. For an object $\mathcal{M}$ of $\mathrm{Tw\,Mod}_\infty(\pi_1(X), \Omega^\bullet_{\mathbb{K},X})$ choose a cover $\mathfrak{U}$ as above and define

$$\mathcal{M}_x = \varinjlim_{U \subset U_x} C^\bullet(U, \mathcal{M}_{U_x}) \tag{16.3.7}$$

The A_∞ operators $T(g_1, \ldots, g_n)$ are by definition $\rho_{\mathbb{U}}(g_1, \ldots, g_n)$ where $g_j \in \pi_1(X)_{x_{j-1},x_j}$ and $\mathbb{U} = (U_{x_0}, \ldots, U_{x_n})$. Let us show that different choices of $\mathfrak{U}$ lead to equivalent infinity local systems (in the sense of Definition 16.5). Choose two covers $\mathfrak{U}'$ and $\mathfrak{U}''$. Apply (16.3.7) to all covers of the form $\mathfrak{U} = \{U_x | x \in X\}$ where for any x either $U_x = U'_x$ or $U_x = U''_x$. This data defines an A_∞ functor $\mathbb{K}\mathbf{C}_1 \otimes \mathbb{K}\pi_1(X) \to \mathrm{dgmod}(\mathbb{K})$ (cf. Sect. 16.1.1). Let $\mathbb{K}(0)$, resp. $\mathbb{K}(1)$, be the full subcategory of $\mathbf{C}_1$ with one object 0, resp. 1. When restricted to $\mathbb{K}(0)$, resp. to $\mathbb{K}(1)$, our A_∞ functor coincides with the infinity local system obtained from $\mathfrak{U}'$, resp. from $\mathfrak{U}''$. By the adjunction formula (Lemma 16.8), the two infinity local systems are equivalent.

Remark 16.19 It is easy to modify the above construction and obtain an A_∞ functor

$$\mathrm{TwMod}(\Omega^\bullet_{\mathbb{K},X}, \pi_1(X)) \to \mathrm{Loc}_{\infty,\mathbb{K}}(X).$$

Moreover, the right hand side is a monoidal category up to homotopy, and the assignment $\mathcal{M}, \mathcal{N} \mapsto \underline{\mathbb{R}\,\mathrm{HOM}}(\mathcal{M}, \mathcal{N})$ turns oscillatory modules, as well as $\Omega^\bullet_{\mathbb{K},M}$-modules with an action of $\pi_1(M)$ up to inner automorphisms, into a category enriched over it. The main reason for this is Lemma 6.17. We will provide the details in a subsequent work.

17 Appendix. Jets and Twisted Bundles

Here we will describe the deformation quantization and the twisted bundle $\mathcal{H}_M$ in terms of bundles of jets.

17.1 Jet Bundles

Let M be any manifold and let $\mathcal{E}$ be a complex vector bundle of rank N on M. Here we recall the construction of the bundle whose fiber at a point x is the space of jets of sections of $\mathcal{E}$ at x. This bundle has the canonical connection; its horizontal sections are determined by sections s of $\mathcal{E}$. The value of such a section at any x is the jet of s at x.

Let $\{U_\alpha\}$ is an open cover and $x_\alpha = (x_{\alpha,1}, \ldots, x_{\alpha,n})$ a local coordinate system on U_α. For $x \in U_\alpha \cap U_\beta$, we denote by x_α, resp. x_β, its coordinates in the corresponding coordinate system and write

$$x_\alpha = g_{\alpha\beta}(x_\beta) \tag{17.1.1}$$

Let $h_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}_N$ be the transition isomorphisms of $\mathcal{E}$. We identify a local section of $\mathcal{E}$ on $U_\alpha \cap U_\beta$ with a $\mathbb{C}^N$-valued function in the coordinates x_β.

Let $\mathbb{C}^N[[\widehat{x}]] = \mathbb{C}^N[[\widehat{x}_1, \ldots, \widehat{x}_n]]$. For $x \in U_\alpha$ define $G_{\beta\alpha}(x) : \mathbb{C}^N[[\widehat{x}]] \to \mathbb{C}^N[[\widehat{x}]]$ by $G_{\beta\alpha}(x) : f_\alpha \mapsto f_\beta$ where

$$f_\beta(\widehat{x}) = h_{\alpha\beta}(x_\beta + \widehat{x}) f_\alpha(g_{\alpha\beta}(x_\beta + \widehat{x}) - x_\alpha) \tag{17.1.2}$$

It is easy to see that different choices of covers and of local trivializations lead to isomorphic bundles. We denote the bundle defined in (17.1.2) by $\mathrm{Jets}(\Gamma(\mathcal{E}))$.

The canonical flat connection is given in any local coordinate system by

$$\nabla_{\mathrm{can}} = \left(\frac{\partial}{\partial x_\alpha} - \frac{\partial}{\partial \widehat{x}} \right) dx_\alpha \tag{17.1.3}$$

If a local section of $\mathcal{E}$ is represented by a vector-valued function $f(x_\alpha)$, it defines a horizontal section which is given in local coordinates by $f(x_\alpha + \widehat{x})$.

17.2 Real Polarization

Recall that a real polarization is an integrable distribution of Lagrangian subspaces. Let $\mathcal{P}$ be a real polarization on M. In this case, automatically $2c_1(TM) = 0$ modulo 4 (cf. [36]).

17.2.1 The Line Bundle $\mathcal{L}$

Assume that ω admits a real polarization $\mathcal{P}$ (i.e. a foliation by Lagrangian submanifolds). By $T_{\mathcal{P}}$ we denote the quotient of TM by the subbundle of vectors tangent to the leaves. Choose local Darboux coordinates ξ_α, x_α such that $x_{j,\alpha}$ are constant along the leaves. Then the transition coordinate changes are of the form

$$x_\alpha = g_{\alpha\beta}(x_\beta);\ \ \xi_\alpha = (g'_{\alpha\beta}(x_\beta)^t)^{-1}(\xi_\beta + \varphi_{\alpha\beta}(x_\beta)) \tag{17.2.1}$$

Assume that $i\omega$ is a $2\pi i\mathbb{Z}$-valued cohomology class. Construct explicitly the line bundle $\mathcal{L}$ such that $c_1(\mathcal{L}) = i\omega$. Adding some constants to $\varphi_{\alpha\beta}$, we may assume that $i\varphi_{\alpha\beta} - i\varphi_{\alpha\gamma} + i\varphi_{\beta\gamma} \in 2\pi i\mathbb{Z}$; define $\mathcal{L}$ to be the line bundle with transition isomorphisms $\exp(i\varphi_{\alpha\beta})$. Formulas

$$A_\alpha = -i\xi_\alpha dx_\alpha \tag{17.2.2}$$

define a connection in this bundle, since

$$\xi_\alpha dx_\alpha = \xi_\beta dx_\beta + d\varphi_{\alpha\beta};$$

the curvature of this connection is $-i\omega$.

17.2.2 The Jet Bundle Jets$(\Gamma_{\mathbf{hor}}(\Omega^{\frac{1}{2}} \otimes \mathcal{L}^k))$

Define for $x \in U_\alpha \cap U_\beta$

$$G_{\beta\alpha}(x) : \mathbb{C}[[\widehat{x}]] \to \mathbb{C}[[\widehat{x}, \hbar]]$$

by $(G_{\beta\alpha} f_\alpha)(\widehat{x}) = f_\beta(\widehat{x})$ where

$$f_\beta(\widehat{x}) = \det g'_{\alpha\beta}(x_\beta + \widehat{x})^{\frac{1}{2}} e^{ik\varphi_{\alpha\beta}(x_\beta+\widehat{x})} f_\alpha(g_{\alpha\beta}(x_\beta + \widehat{x}) - x_\alpha) \tag{17.2.3}$$

The square root of the determinant comes from the metalinear structure. The above formula defines the transition functions for the bundle of jets of $\mathcal{P}$-horizontal sections of the bundle $(\wedge^{\max} T^*_{\mathcal{P}})^{\frac{1}{2}} \otimes \mathcal{L}^k$.

17.2.3 The Jet Bundle Rees Jets $D(\Gamma_{\mathbf{hor}}(\Omega^{\frac{1}{2}} \otimes \mathcal{L}^{\frac{1}{\hbar}}))$

Recall the construction of the Rees ring and the Rees module [2] of a filtered ring and a filtered module. If A is a ring with an increasing filtration F_pA, $p \geq 0$, and V an A-module with a compatible filtration F_p, $p \geq 0$, we put

$$\text{Rees}\, A = \oplus_{p\geq 0}\hbar^p F_pA;\ \ \text{Rees}\, V = \oplus_{p\geq 0}\hbar^p F_pV. \tag{17.2.4}$$

$$\text{Rees}_f\, A = \prod_{p\geq 0}\hbar^p F_pA;\ \ \text{Rees}_f\, V = \prod_{p\geq 0}\hbar^p F_pV. \tag{17.2.5}$$

When applied to the ring of formal differential operators with its filtration by order, (17.2.4) produces the ring $\mathbb{C}[[\widehat{x}]][\widehat{\xi}, \hbar]$ with the usual Heisenberg relations ($\widehat{\xi_j} =$

$i\hbar\frac{\partial}{\partial\widehat{x}_j}$). When applied to the module of formal functions $V = \mathbb{C}[[\widehat{x}]]$ whose filtration is given by $F_0 V = V$, it gives $\mathbb{C}[[\widehat{x}]][\hbar]$. The completed version (17.2.5) produces the complete Weyl algebra $\mathbb{C}[[\widehat{x}, \widehat{\xi}, \hbar]]$ and the complete module $\mathbb{C}[[\widehat{x}, \hbar]]$.

Observe that in the expression $G_{\beta\alpha}(i\hbar\frac{\partial}{\partial\widehat{x}})G_{\alpha\beta}$ one can substitute $\frac{1}{i\hbar}$ for k. The result will be given (in vector/matrix notation) by the following:

$$\frac{1}{2}\left(i\hbar\frac{\partial}{\partial\widehat{x}}\right)\left(g'_{\alpha\beta}(x_\beta+\widehat{x})^t\right)+\left(g'_{\alpha\beta}(x_\beta+\widehat{x})^t\right)\left(i\hbar\frac{\partial}{\partial\widehat{x}}\right)-\varphi'_{\alpha\beta}(x_\beta+\widehat{x})$$

Define the bundle of algebras Rees Jets $D(\Gamma_{\text{hor}}(\Omega^{\frac{1}{2}}\otimes\mathcal{L}^{\frac{1}{\hbar}}))$ whose fiber is $\mathbb{C}[[\widehat{x}, \hbar]][\widehat{\xi}]$ and whose transition isomorphisms are

$$G_{\beta\alpha}(\widehat{x}) = g_{\beta\alpha}(x_\alpha+\widehat{x}) - x_\beta; \tag{17.2.6}$$

$$G_{\beta\alpha}(\widehat{\xi}) = g'_{\alpha\beta}(x_\beta+\widehat{x})^t * \widehat{\xi} - \varphi'_{\alpha\beta}(x_\beta+\widehat{x}) \tag{17.2.7}$$

(the multiplication in the left hand side is the (matrix) Moyal–Weyl multiplication). We see that our bundle is the result of formally substituting $\frac{1}{\hbar}$ for k in the bundle of jets of Rees rings of $\mathcal{P}$-horizontal differential operators on $(\wedge^{\max}T^*_{\mathcal{P}})^{\frac{1}{2}}\otimes\mathcal{L}^k$.

The above formula is the result of formally substituting k by $\frac{1}{\hbar}$ into the transition functions for the bundle

$$\text{Rees Jets } D(\Gamma_{\text{hor}}((\wedge^{\max}T^*_{\mathcal{P}})^{\frac{1}{2}}\otimes\mathcal{L}^k)).$$

17.2.4 The Bundle of Algebras $\widehat{\mathbb{A}}_M$ and the Twisted Bundle of Modules $\mathcal{H}_M$

Now apply to the bundle above the gauge transformation [32]

$$\mathrm{Ad}\exp\left(\frac{1}{i\hbar}\xi_\alpha\widehat{x}\right) \tag{17.2.8}$$

We get transition isomorphisms

$$G_{\beta\alpha}(\widehat{x}) = g_{\beta\alpha}(x_\alpha+\widehat{x}) - x_\beta; \tag{17.2.9}$$

$$G_{\beta\alpha}(\widehat{\xi}) = g'_{\alpha\beta}(x_\beta+\widehat{x})^t * (\widehat{\xi}+\xi_\alpha) - \varphi'_{\alpha\beta}(x_\beta+\widehat{x}) - \xi_\beta \tag{17.2.10}$$

Unlike in (17.2.6) and (17.2.7), these transition isomorphisms preserve the maximal ideal $\langle\widehat{x}, \widehat{\xi}, \hbar\rangle$ and therefore extend to the complete Weyl algebra $\widehat{\mathbb{A}} = \mathbb{C}[[\widehat{x}, \widehat{\xi}, \hbar]]$, cf. Sect. 2.1. We use them to construct a bundle of algebras $\mathbb{A}_M$ whose fiber is the Weyl algebra $\widehat{\mathbb{A}}$. We see immediately that the bundle of algebras $\widehat{\mathbb{A}}_M$ is a deformation of the bundle of jets of functions on M.

Moreover, after we apply the gauge transformation (17.2.8), the formula (17.2.11) allows to replace k by $\frac{1}{\hbar}$. We get new transition isomorphisms

$$f_\beta(\widehat{x}) = \det g'_{\alpha\beta}(x_\beta + \widehat{x})^{\frac{1}{2}} e^{-\frac{1}{i\hbar}(\varphi_{\alpha\beta}(x_\beta+\widehat{x})-\varphi'_{\alpha\beta}(x_\beta)\widehat{x})} f_\alpha(g_{\alpha\beta}(x_\beta + \widehat{x}) - x_\alpha) \quad (17.2.11)$$

that define a twisted bundle of modules $\mathcal{H}_M$ whose fiber is the space $\mathcal{H}$ of the formal metaplectic representation (cf. (13.5.1)). The cocycle c from the definition of a twisted module (14.1.1) is $\exp(\frac{1}{i\hbar}(\varphi_{\alpha\beta} - \varphi_{\alpha\gamma} + \varphi_{\beta\gamma}))$. (The summand $-\varphi'_{\alpha\beta}(x_\beta)\widehat{x}$ in the exponent comes from the difference of $\xi_\alpha\widehat{x}$ and $\xi_\beta\widehat{x}$ that figure in the gauge transformation).

In other words, the bundle of algebras $\widehat{\mathbb{A}}_M$ can be formally described as

$$\widehat{\mathbb{A}}_M = \mathrm{Rees}_f\ \mathrm{Jets}\ D_{\mathrm{hor}}((\wedge^{\frac{1}{2}} T^*_{\mathcal{P}})^{\frac{1}{2}} \otimes \mathcal{L}^{\frac{1}{\hbar}}) \quad (17.2.12)$$

$$\mathcal{H}_M = \mathrm{Rees}_f\ \mathrm{Jets}\ \Gamma_{\mathrm{hor}}((\wedge^{\frac{1}{2}} T^*_{\mathcal{P}})^{\frac{1}{2}} \otimes \mathcal{L}^{\frac{1}{\hbar}}) \quad (17.2.13)$$

(cf. (17.2.5) for the meaning of $\mathbf{Rees}_f$). The latter is only a twisted bundle because the transition functions of $\mathcal{L}$ stop being a one-cocycle when elevated to the power $\frac{1}{\hbar}$.

17.2.5 The Canonical Connections

The bundle of horizontal sections of $\Omega^{\frac{1}{2}} \otimes \mathcal{L}^k$ has a canonical connection that is given by the formula

$$\nabla = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial \widehat{x}}\right) dx + \frac{\partial}{\partial \xi} d\xi$$

in all local coordinate systems.

This connection induces a connection in Rees Jets $D(\Gamma_{\mathrm{hor}}(\Omega^{\frac{1}{2}} \otimes \mathcal{L}^{\frac{1}{i\hbar}}))$ that is given by the same formula. After the gauge transformation from Sect. 17.2.4 we get flat connections

$$\nabla_{\mathbb{A}} = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial \widehat{x}}\right) dx + \left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \widehat{\xi}}\right) d\xi \quad (17.2.14)$$

in $\mathcal{A}_M$ and

$$\nabla_{\mathcal{H}} = -\frac{1}{i\hbar}\xi dx + \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial \widehat{x}}\right) dx + \left(\frac{\partial}{\partial \xi} + \frac{1}{i\hbar}\widehat{x}\right) d\xi \quad (17.2.15)$$

17.3 Complex Polarization

The following is largely based on the approach to deformation quantization from [19].

17.3.1 Kähler Potentials

Let M be a Kähler manifold. We can locally choose a Kähler potential, i.e. a real-valued function Φ such that the symplectic form is given by

$$\omega = -i\partial\overline{\partial}\Phi$$

A Kähler potential is unique up to a change $\Phi \mapsto \Phi + \varphi + \overline{\varphi}$ where φ is holomorphic.

Lemma 17.1 *Put* $\zeta_j = i\frac{\partial\Phi}{\partial z_j}$. *Then*

$$\{z_j, z_k\} = 0;\ \ \{\zeta_k, z_j\} = \delta_{jk};\ \ \{\zeta_j, \zeta_k\} = 0.$$

Proof Choose local holomorphic coordinates and put

$$A_{jk} = \frac{\partial}{\partial z_j}\frac{\partial}{\partial \overline{z}_k}\Phi(z, \overline{z})$$

We have

$$\{z_j, \overline{z}_k\} = i(A^{-1})_{kj};$$

$$\{z_j, \zeta_k\} = i\sum \frac{\partial \zeta_k}{\partial \overline{z}_l}\{z_j, \overline{z}_l\} = \sum A_{kl}(A^{-1})_{lj} = \delta_{jk};$$

$$-\{\zeta_j, \zeta_k\} = \sum \left(\frac{\partial \zeta_j}{\partial z_p}\frac{\partial \zeta_k}{\partial \overline{z}_q} - \frac{\partial \zeta_k}{\partial z_p}\frac{\partial \zeta_j}{\partial \overline{z}_q}\right)\{z_p, \overline{z}_q\} =$$

$$i\sum\left(\frac{\partial^2\Phi}{\partial z_j \partial z_p}A_{kq} - \frac{\partial^2\Phi}{\partial z_k\partial z_p}A_{jq}\right)(A^{-1})_{qp} = i\left(\frac{\partial^2\Phi}{\partial z_j\partial z_k} - \frac{\partial^2\Phi}{\partial z_k\partial z_j}\right) = 0$$

□

17.3.2 The Line Bundle $\mathcal{L}$

Choose an open cover $\{U_\alpha\}$ of M and a holomorphic coordinate system $z_\alpha = (z_{\alpha,1}, \ldots, z_{\alpha,n})$ on every U_α. We write

$$z_\alpha = g_{\alpha\beta}(z_\beta). \tag{17.3.1}$$

Choose local Kähler potentials Φ_α. We have

$$i\Phi_\alpha - i\Phi_\beta = \varphi_{\alpha\beta} + \overline{\varphi_{\alpha\beta}} \tag{17.3.2}$$

where $\varphi_{\alpha\beta}$ are holomorphic.

Let us start with rewriting the transition isomorphisms in terms of the new complex Darboux coordinates z, ζ. We have

$$i\Phi_\alpha(z_\alpha) - i\Phi_\beta(z_\beta) = \varphi_{\alpha\beta} + \overline{\varphi_{\alpha\beta}(z_\beta)}$$

Applying $\frac{\partial}{\partial z_\beta}$, we get

$$\frac{\partial z_\alpha}{\partial z_\beta} i \frac{\partial \Phi}{\partial z_\alpha}(z_\alpha) - i\frac{\partial \Phi}{\partial z_\beta}(z_\beta) = \frac{\partial \varphi_{\alpha\beta}}{\partial z_\beta}(z_\beta)$$

or

$$\zeta_\alpha = (g'_{\alpha\beta}(z_\beta)^{-1})^t \left(\zeta_\beta + \frac{\partial \varphi_{\alpha\beta}}{\partial z_\beta}(z_\beta)\right) \tag{17.3.3}$$

Together with (17.3.1), this describes the rule for the change of new variables.

Assume that $i(\varphi_{\alpha\beta} + \varphi_{\beta\gamma} - \varphi_{\alpha\gamma})$ is a $2\pi i\mathbb{Z}$-valued two-cocycle. the line bundle $\mathcal{L}$ with transition functions $\exp(\varphi_{\alpha\beta})$. The curvature of this connection is $-i\omega$.

17.3.3 The Jet Bundles

Assume that the canonical sheaf has a square root $\Omega^{\frac{1}{2}}$. We call this line bundle the bundle of holomorphic half-forms on M. The transition isomorphisms of this line bundle are denoted by $\det {g'_{\alpha\beta}}^{\frac{1}{2}}$. For any integer k, consider the bundle $\mathrm{Jets}(\Gamma_{\mathrm{hol}}(\mathcal{L}^k \otimes \Omega^{\frac{1}{2}}))$ of jets of holomorphic sections of $\mathcal{L}^k \otimes \Omega^{\frac{1}{2}}$. The fiber of this bundle is $\mathbb{C}[[\widehat{z}]]$ where $\widehat{z} = (\widehat{z}_1, \ldots, \widehat{z}_n)$. The transition isomorphisms of the jet bundle take a power series $f_\alpha(\widehat{z})$ to a power series $f_\beta(\widehat{z})$ according to the following formula.

$$f_\beta(\widehat{z}) = f_\alpha(g_{\alpha\beta}(z_\beta + \widehat{z}) - z_\alpha) \det g'_{\alpha\beta}(z_\beta + \widehat{z})^{\frac{1}{2}} \exp(k\varphi_{\alpha\beta}(z_\beta + \widehat{z})) \tag{17.3.4}$$

Exactly as in Sect. 17.2.3, we can define the bundle of algebras whose fiber is $\mathbb{C}[[\widehat{z}, \hbar]][\widehat{\zeta}]$ by transition isomorphisms

$$G_{\beta\alpha}(\widehat{z}) = g_{\beta\alpha}(z_\alpha + \widehat{z}) - z_\beta; \tag{17.3.5}$$

$$G_{\beta\alpha}(\widehat{\zeta}) = g'_{\alpha\beta}(z_\beta + \widehat{z})^t * \widehat{\zeta} - \partial_{z_\beta}\varphi_{\alpha\beta}(z_\beta + \widehat{\zeta}) \tag{17.3.6}$$

We see that our bundle is the result of formally substituting $\frac{1}{\hbar}$ for k in the bundle of jets of Rees rings of holomorphic differential operators on $\Omega^{\frac{1}{2}} \otimes \mathcal{L}^k$ (if we map $\widehat{\zeta_i}$ to $i\hbar\partial_{\widehat{z}}$). On the other hand, because of (17.3.3), this bundle of algebras is a deformation of the bundle of jets of C^∞ functions on M. The gauge transformation

$$\mathrm{Ad}\exp\left(\frac{1}{i\hbar}\widehat{\zeta}_\alpha\widehat{z}\right) \tag{17.3.7}$$

produces new transition functions

$$G_{\beta\alpha}(\widehat{z}) = g_{\beta\alpha}(z_\alpha + \widehat{z}) - z_\beta; \tag{17.3.8}$$

$$G_{\beta\alpha}(\widehat{\zeta}) = g'_{\alpha\beta}(z_\beta + \widehat{z})^t * (\widehat{\zeta} + \zeta_\alpha) - \partial_{z_\beta}\varphi_{\alpha\beta}(z_\beta + \widehat{\zeta}) - \zeta_\beta \tag{17.3.9}$$

that extend to $\widehat{\mathbb{A}}_M = \mathbb{C}[[\widehat{z}, \widehat{\zeta}, \hbar]]$. The transition isomorphisms for the module of jets (17.3.4) are now as follows (when we replace k by $\frac{1}{\hbar}$) which now define only a twisted module that we denote by $\mathcal{H}_M$.

$$f_\beta(\widehat{z}) = f_\alpha(g_{\alpha\beta}(z_\beta + \widehat{z}) - z_\alpha)\det g'_{\alpha\beta}(z_\beta + \widehat{z})^{\frac{1}{2}}\exp\left(\frac{1}{i\hbar}\varphi_{\alpha\beta}(z_\beta + \widehat{z}) - \partial_{z_\beta}\varphi_{\alpha\beta}(z_\beta)\widehat{z}\right)$$

As in the case of a real polarization, the canonical connections become

$$\nabla_{\mathbb{A}} = \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial\widehat{z}}\right)dz + \left(\frac{\partial}{\partial\zeta} - \frac{\partial}{\partial\widehat{\zeta}}\right)d\zeta \tag{17.3.10}$$

on $\mathcal{A}_M$ and

$$\nabla_{\mathcal{H}} = -\frac{1}{i\hbar}\zeta dz + \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial\widehat{z}}\right)dz + \left(\frac{\partial}{\partial\zeta} + \frac{1}{i\hbar}\widehat{z}\right)d\zeta \tag{17.3.11}$$

on $\mathcal{H}_M$. We conclude that

$$\widehat{\mathbb{A}}_M \xrightarrow{\sim} \mathrm{Rees}_f\,\mathrm{Jets}\,D_{\mathrm{hol}}(\Omega^{\frac{1}{2}} \otimes \mathcal{L}^{\frac{1}{\hbar}}) \tag{17.3.12}$$

$$\mathcal{H}_M \xrightarrow{\sim} \mathrm{Rees}_f\,\mathrm{Jets}\,\Gamma_{\mathrm{hol}}(\Omega^{\frac{1}{2}} \otimes \mathcal{L}^{\frac{1}{\hbar}}) \tag{17.3.13}$$

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Part II
Analytic Microlocal Analysis

Determinantal Point Processes and Fermions on Polarized Complex Manifolds: Bulk Universality

Robert J. Berman

Abstract We consider determinantal point processes on a compact complex manifold X in the limit of many particles. The correlation kernels of the processes are the Bergman kernels associated to a high power of a given Hermitian holomorphic line bundle L over X. The empirical measure on X of the process, describing the particle locations, converges in probability towards the pluripotential equilibrium measure, expressed in term of the Monge–Ampère operator. The asymptotics of the corresponding fluctuations in the bulk are shown to be asymptotically normal and described by a Gaussian free field and applies to test functions (linear statistics) which are merely Lipschitz continuous. Moreover, a scaling limit of the correlation functions in the bulk is shown to be universal and expressed in terms of (the higher dimensional analog of) the Ginibre ensemble. This geometric setting applies in particular to normal random matrix ensembles, the two dimensional Coulomb gas, free fermions in a strong magnetic field and multivariate orthogonal polynomials.

1 Introduction

The systematic study of *determinantal point processes* was initiated by Macchi [56] in the seventies who called them *fermionic* point processes, inspired by the properties of fermion gases in statistical (quantum) mechanics. For general reviews see [47, 49, 75]. The theory concerns ensembles of "particle configurations" on a given space X which exhibit repulsion. An important class of such processes are the determinantal projection processes, which may be defined by a probability measure on the $N-$fold product X^N, the "configuration space of N particles on X", with the property that its density may be written as

$$\rho^{(N)}(x_1, ..., x_N) = \frac{1}{N!} \det(\mathcal{K}(x_i, x_j)), \tag{1.1}$$

R. J. Berman (✉)
Chalmers University of Technology, Gothenburg, Sweden
e-mail: robertb@chalmers.se

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where the kernel $\mathcal{K}$ is the integral kernel of an orthogonal projection operator onto a vector space of dimension N. As a consequence the probability distributions vanish for a configuration $(x_1, \ldots, x_N)$ of points x_i as soon as two points coincide, explaining the repulsive behavior of the ensemble. As it turns out, in many situations such ensembles are *critical* in the sense that they naturally appear in sequences with N, the number of particles, tending to infinity in such a way that a well-defined limiting ensemble may be extracted. Moreover, large classes of such sequences of ensembles often give rise to one and the same limit. This is the phenomenon of *universality* (see [31] for a nice survey). Perhaps its most famous illustration is given by ensembles of $N \times N$ *Hermitian random matrices* whose eigenvalues, in the large N limit, determine a unique determinantal point process on the real line. This latter process has also been conjectured to describe the statistics of the zeroes of the Riemann zeta function, as well as statistics of quantum systems whose classical dynamics is chaotic (references and more recent relations to random growth and tiling problems may be found in [49]).

The present paper concerns a general class of such critical ensembles, where the space X is a compact complex manifold equipped with an holomorphic line bundle L with a given Hermitian metric locally represented as $e^{-\phi}$, where ϕ is called a "weight" on L. The kernel $\mathcal{K}$ defining the ensemble may then be identified with the orthogonal projection onto the space of global holomorphic sections $H^0(X, L)$ of L (with respect to a local unitary frame of $(L, e^{-\phi})$) and the corresponding determinantal probability density on X^N may be written as the squared point-wise norm of the normalized Vandermonde type determinant $(\det S)(x_1, \ldots, x_N)$ associated to any given base $S = (s_1, \ldots, s_N)$ of sections in $H^0(X, L)$:

$$\rho^{(N)}(x_1, \ldots, x_N) = \frac{1}{\mathcal{Z}_N} |\det(S)(x_1, \ldots, x_N)|^2_\phi \tag{1.2}$$

In this setting the limit of a large number N of particles corresponds to the limit when the line bundle L is replaced by a large tensor power, written as kL in additive notation. When X is the complex projective space this setting is just a geometric formulation of the theory of (weighted) multivariate orthogonal polynomials, with the tensor power k corresponding to the degree of the polynomials (see Sect. 2). In mathematical physics terminology $H^0(X, L)$ may be identified with the quantum ground state space of a single fermion (complex spinor) on X subject to an exterior magnetic field and the density in formula (1.2) is the squared probability amplitude for the corresponding maximally filled many particle state, i.e. $(\det S)$ is the corresponding Slater determinant.

Already in the simplest case when X is the complex projective line, i.e. the Riemann sphere (viewed as the one-point compactification of $\mathbb{C}$) the corresponding ensemble is remarkably rich and admits at least three different well-known descriptions in terms of *(1) normal random matrices, (2) a free fermion gas, (3) a Coulomb gas* of repelling electric charges [77]. Compare the discussion in Sect. 2.

While there are quite recent result concerning this special case, both in mathematics and physics, there seems to be almost no previous general results in the higher dimensional situation studied in the present paper. For one reference see the recent paper [67]. As it turns out, the main new feature that appears in higher dimensions is that the role of the Laplace operator in one complex dimension (which expresses the limiting expected density of particles) is played by the *fully non-linear Monge–Ampère operator,* which is the subject of (complex) *pluripotential* theory [43, 51]. In fact, one of the motivations for the present paper and the companion paper [18] is to develop a Coulomb gas type descriptions of a gas of free fermions on complex manifolds and conversely to provide a statistical mechanical interpretation of complex pluripotential theory. An important feature of our approach is that it does not require that ϕ be positively curved, i.e. that the corresponding magnetic two-form has any definite sign properties. As will be explained below this means that the support of the limiting one-point correlation functions will only cover a proper subset D of X, which corresponds to the droplet appearing in the physical description of the Quantum Hall Effect (QHE) describing fermions in large magnetic fields [52]. We will here focus on the universality properties in the "bulk" of the droplet D leaving the case of the boundary (edge) properties as challenging open problem for the future (which from a physical point of view can be expected to be related to the properties of the edge states playing a central role in the QHE).

Yet another motivation comes from approximation theory where configurations $(x_1, \ldots, x_N)$ appear as *interpolation nodes* on X and a configuration maximizing a functional of the form (1.1) is known to have optimal interpolation properties in a certain sense [42, 74]. Sequences of such configurations, with N tending to infinity, then appear naturally in discretization schemes. Moreover, as shown very recently in [11] any such optimal sequence equidistributes asymptotically on the corresponding equilibrium measure. This fact should be compared with Theorem 1.4 in the present paper which shows that, with high probability, the same equidistribution property holds for *random* configurations of the corresponding ensemble.

One final motivation comes from the study by Shiffman, Zelditch and coworkers of random zeroes of holomorphic sections of positive line bundles, where many statistical results have been obtained and where a key role is played by Bergman kernels (cf. [22, 71, 72]).

1.1 Statement of the Main Results

Let L be a holomorphic line bundle over a compact complex manifold X. Denote by $H^0(X, L)$ the vector space of all global holomorphic sections on X with values in L and write $N := \dim H^0(X, L)$. Fixing an Hermitian metric on L (locally represented by $e^{-\phi}$(where the additive object ϕ is called a *weight* ϕ) and a suitable measure μ on X induces an inner product on $H^0(X, L)$ defined by

$$\|s\|_\phi^2 := \int_X |s|^2 e^{-\phi}\mu$$

(abusing notation slightly; see Sect. 1.4). We will denote the corresponding Hilbert space by $\mathcal{H}(X, L)$ and its *Bergman kernel* by K, which is the integral kernel of the orthogonal projection $\mathcal{C}^{\infty}(X, L) \to H^0(X, L)$:

$$K(x, y) = \sum_{i=1}^{N} s_i(x) \otimes \overline{s_i(y)}, \tag{1.3}$$

where (s_i) is an orthonormal bases in $\mathcal{H}(X, L)$.

As is essentially well-known this setup induces a probability measure γ_P on the $N-$fold product X^N whose density (w.r.t. $\mu^{\otimes N}$) is defined as the determinant of an $N \times N$ matrix:

$$\rho^{(N)}(x_1, \dots, x_N) := \frac{1}{N!} \det(K(x_i, x_j) e^{-\frac{1}{2}(\phi(x_i)+\phi(x_j))}), \tag{1.4}$$

The main object of study in the present paper is the large k asymptotics of the probability space (X^N, γ_P), when L is replaced by its kth tensor power (written as kL in our additive notation) equipped with the induced weight $k\phi$. In the following a subindex k will be used to indicate the dependence on the parameter k. We will always assume that L is *big,* i.e that

$$N_k := \dim H^0(X, kL) = V k^n + o(k^{n-1}), \quad V > 0$$

(where the constant V is usually called the *volume* of L). The main case of interest appears when L is (very) ample, so that X may be embedded as algebraic manifold in complex projective space and L is the restriction of the hyperplane line bundle. Then (X, L) is called a polarized manifold and $H^0(X, kL)$ gets identified with the restriction to X of the space of all homogeneous polynomials of degree k. Moreover, the main results in the present paper concern weighted measured (ϕ, μ) which for which we introduce the (non-standard) terminology *strongly regular*. This will mean that the weight ϕ is locally $\mathcal{C}^{1,1}$-smooth, i.e. it is differentiable and all of its first partial derivatives are locally Lipschitz continuous, and the measure $\mu = \omega_n$ is the volume form of a continuous metric ω on X. The reason that we assume that ϕ is merely $\mathcal{C}^{1,1}$-smooth, rather than $\mathcal{C}^2-$smooth (or even $\mathcal{C}^{\infty}-$smooth) is that this appears to be the essentially optimal regularity class where the results below concerning universality of the scaled correlation functions can be expected to hold. Moreover, since ϕ is not assumed to be positively curved we will anyway have to work with the corresponding equilibrium weight ϕ_e in the proofs which is almost never $\mathcal{C}^2-$smooth, even if ϕ is smooth (unless ϕ is positively curved; compare [13]). When X is the complex projective space $X := \mathbb{E}^n$ and L the hyperplane line bundle $\mathcal{O}(1)$ (so that $H^0(X, kL)$ may be identified with the space of all polynomials of total degree at most k in $\mathbb{C}^n$) we also allow ω_n to be the Lebesgue measure on the affine piece $\mathbb{C}^n$ as long as ϕ has super logarithmic growth (formula 2.5).

The notion of strongly regular weighted measures (ϕ, μ) on X that we shall focus on in the present paper should be contrasted with the considerably more general notion of weighted measures (ϕ, μ) satisfying the *Bernstein–Markov property* in the sense of [11]. From the probabilistic point of view the latter property simply means that the one-point correlation function $\rho_k^{(1)}$ of the corresponding determinantal point process has sub-exponential growth in k. For example, the Bernstein–Markov property is satisfied if ϕ is continuous and μ is a continuous volume form on a complex or real algebraic variety. In particular, the latter property applies when μ is Lebesgue measure on $\mathbb{R}^n$, as in the setting of Hermitian random matrices [30] (where $n = 1$).

As a guide line, the Bernstein–Markov property of (ϕ, μ) is enough to establish asymptotics in the "macroscopic regime", such as convergence in probability towards the corresponding equilibrium measure. In contrast, the results in the "microscopic regime", concerning length scales of the order $k^{-1/2}$ on X, only hold in the strongly regular case.

1.1.1 Correlation Functions and the Equilibrium Measure

As is well known all the $m-$*point correlation functions* $\rho_k^{(m)}$, where $1 \leq m \leq N_k$, of the ensemble above may be expressed as (weighted) determinants of $K_k(x_i, x_j)$. In particular,

$$\rho_k^{(1)}(x) = K_k(x, x)e^{-k\phi(x)}, \quad \rho^{(2).c}(x, y) = -|K_k(x, y)|^2 e^{-k\phi(x)}e^{-k\phi(y)},$$

where $\rho^{(2).c}$ is the *connected* 2-point correlation function (see Sect. 6.1). As shown in [13], in the strongly regular case,

$$\frac{1}{N_k}\rho_k^{(1)}\omega_n \to \mu_{\phi_e}, \tag{1.5}$$

weakly, when $k \to \infty$, where μ_{ϕ_e} is the pluripotential *equilibrium measure* (of (X, ϕ)), which may be written as the Monge–Ampère measure $\frac{1}{Vn!}(dd^c\phi_e)^n$ of the equilibrium weight ϕ_e and represented as

$$\frac{1}{Vn!}(dd^c\phi_e)^n = 1_S \det_\omega(dd^c\phi)(x)\frac{\omega^n}{Vn!},$$

where $S \subset X$ denotes the support of the equilibrium measure (see Sect. 3). We recall that in the case of one complex dimension (i.e. $n = 1$) the support S is referred to as the *droplet* in the physics literature on the Quantum Hall Effect (see [52, 77] and Sect. 2 below).

As later shown in [11] the convergence (1.5) holds, in the weak topology, for weighted measures (ϕ, μ) satisfying the Bernstein–Markov property. However, in the strongly regular setting that we will concentrate on here *point-wise* convergence actually holds in the sense that there is a subset of X that will be called the *weak bulk* (of (X, ϕ)) such that

$$\frac{1}{N_k}\rho_k^{(1)}(x) \to \frac{1}{V}\det_{\omega}(dd^c\phi)(x), \quad x \text{ in the weak bulk}$$

and converges to zero almost everywhere in the complement of the weak bulk. We recall that in the random matrix and Coulomb gas literature the bulk of the equilibrium measure is usually defined as the interior of the support S of the equilibrium measure. But the problem is that, for a general smooth weight ϕ, the set S may be extremely irregular and, a priori, its interior could be empty. In contrast, the weak bulk always has positive Lebesgue measure. The precise definition of the weak bulk is given in Sect. 3 and uses that, by the results in [13], the equilibrium weight ϕ_e is $C^{1,1}-$smooth and hence the second derivatives exist almost everywhere.

The following theorem gives the scaling asymptotics of the Bergman kernel, around a fixed point x in the weak bulk. It is expressed in terms of "normal" local coordinates z centered at x and a"normal" trivialization of L, i.e such that

$$\omega(z) = \frac{i}{2}\sum_{i=1}^{n} dz_i \wedge \overline{dz_i} + \cdots, \quad \phi(z) = \sum_{i=1}^{n} \lambda_i |z_i|^2 + \cdots \tag{1.6}$$

where the dots indicate "higher order terms". Hence, λ_i are the eigenvalues of the curvature form $dd^c\phi$ w.r.t the metric ω and we denote the corresponding diagonal matrix by λ.

Theorem 1.1 *Assume that the weight ϕ is in $C^{1,1}_{loc}$ and that the volume form ω_n is continuous. Let x be a fixed point in the weak bulk and take "normal" local coordinates z centered at x and a "normal" trivialization of L as above. Then*

$$k^{-n}K_k(k^{-1/2}z, k^{-1/2}w) \to \frac{\det\lambda}{\pi^n}e^{\langle\lambda z, w\rangle} \tag{1.7}$$

in the $\mathcal{C}^{\infty}-$topology on compact subsets of $\mathbb{C}^n_z \times \mathbb{C}^n_w$. In particular, the connected 2-point function has the following scaling asymptotics

$$-k^{-2n}\rho_k^{(2).c}(k^{-1/2}z, k^{-1/2}w) \to \left(\frac{\det\lambda}{\pi^n}\right)^2 e^{-\sum_{i=1}^{n}\lambda_i|z_i-w_i|^2}$$

uniformly on compacts of $\mathbb{C}^n_z \times \mathbb{C}^n_w$.

In the case when ϕ is $C^{\infty}-$smooth and strictly positively curved (and in particular the weak bulk coincides with all of X) the convergence (1.7) was shown in [22], where it was deduced from the microlocal analysis of the Boutet de Monvel–Sjöstrand parametrix for the corresponding Szegö kernel [25] following [79] (which also yields an explicit control on the remainder terms). As emphasized in [22] the previous theorem may on one hand be interpreted as a "localization" result, in the sense that the limit is expressed in terms of local data (the curvature of $dd^c\phi$ at the fixed point). On the other hand, it can be seen as a "universality" result (see [31] for a general discussion of universality in mathematics and physics). Indeed, scaling the coordinates further in order to make the Kähler metric $dd^c\phi$ at the fixed point the "yard stick" the limiting kernel becomes independent of the ensemble (and coincides with the Bergman kernel of Fock space). When $n = 1$ the corresponding limiting one-dimensional determinantal point process was studied by Ginibre, who showed that it appears from a scaling limit of random complex matrices with independent complex Gaussian entries.

As a corollary the following analog of a well-known universality result for the Hermitian random matrix model (where the limiting kernel is the sine kernel) is obtained:

Corollary 1.2 *Let ϕ be a function in $C^{1,1}_{loc}(\mathbb{C})$ with super logarithmic growth and denote by $\rho_k^{(\cdot).\cdot}$ the eigenvalue correlation functions of the associated normal random matrix model (see Sect. 2.3). Then the following convergence holds when the rank $N = k + 1$ of the matrices tends to infinity:*

$$-\frac{\rho_k^{(2).c}\left(z_0 + \frac{z}{\sqrt{\rho_k^{(1)}(z_0)}}, z_0 + \frac{w}{\sqrt{\rho_k^{(1)}(z_0)}},\right)}{\left(\rho_k^{(1)}(z_0)\right)^2} \to e^{-|z-w|^2}$$

uniformly on compacts of $\mathbb{C} \times \mathbb{C}$, when z_0 is a fixed point in the weak bulk (in the eigenvalue plane $\mathbb{C}$).

The remaining main results concern properties inside the *bulk* of (X, ϕ) which, when the weight ϕ is C^2-smooth, is defined as the interior of the support S of the equilibrium measure. In general, the bulk (which always contains the weak bulk appearing above) is defined as the largest open subset of S where

$$\omega_\phi := dd^c\phi \tag{1.8}$$

defines a continuous Kähler metric (i.e. a continuous strictly positive form). The next theorem implies that the correlations are short range on macroscopic length scales in the bulk:

Theorem 1.3 *Assume that the weight ϕ is in $C^{1,1}_{loc}$ and that the volume form ω_n is continuous. Let E be a compact subset of the bulk. Then there is a constant C (depending on E) such that the following estimate holds for all pairs (x, y) such that either x or y is in E :*

$$-k^{-2n}\rho_k^{(2).c}(x, y) \leq Ce^{-\sqrt{k}d(x,y)/C}$$

for all k, where $d(x, y)$ is the distance function with respect to a fixed smooth metric on X.

1.1.2 Fluctuations of Linear Continuous Statistics

Consider the random measure (i.e. a measure valued random variable) defined by

$$(x_1, \ldots, x_N) \mapsto \sum_{i=1}^{N} \delta_{x_i}, \tag{1.9}$$

Its expected value is the one point correlation measure $\rho^{(1)}\omega_n$. To get a real-valued random variable one fixes a function u on X and defines the random variable $\mathcal{N}[u]$ by contraction:

$$\mathcal{N}[u](x_1, \ldots, x_N) := u(x_1) + \cdots + u(x_N),$$

often called a *linear statistic* in the statistical mechanics literature. In particular, if $u = 1_E$ is the characteristic function of a subset E of X, then $\mathcal{N}[u](x_1, ..., x_N)$ counts the number of x_i contained in E. By (1.5) the expected value of the random measure (1.9) divided by N converges weakly to the *equilibrium measure* of (X, ϕ). In fact, one actually has convergence in *probability,* i.e. a (weak) "law of large numbers":

Theorem 1.4 *Assume that (ϕ, μ) has the Bernstein–Markov property and denote by μ_ϕ the corresponding equilibrium measure (supported on the support of μ). Let u be a bounded continuous function on (X, μ). Then*

$$\frac{1}{N_k}\mathcal{N}_k[u] \to \int_X \mu_\phi u \tag{1.10}$$

in probability when k tends to infinity at a rate of order $o(k^{-n})$, i.e.

$$Prob_k(\{(x_1, ..., x_{N_k}) : \left| k^{-n}(u(x_1) + \cdots + u(x_{N_k})) - \int_X \mu_\phi u \right| > \epsilon\}) \leq \frac{C}{\epsilon k^n}$$

for some constant C independent of ϵ and k.

Note that it follows from basic integration theory that the convergence also holds if u is the characteristic function of a, say smooth, domain E in X, as long as the limiting equilibrium measure μ_ϕ is absolutely continuous (w.r.t. a smooth volume form). In particular, this happens in the strongly regular case. Theorem 1.4 follows from the convergence of the expectations together with the following simple variance estimate:

$$\mathrm{Var}(\mathcal{N}_k[u]) := \mathbb{E}(\widetilde{\mathcal{N}_k}[u])^2) = O(k^n)$$

for any u as above, where $\widetilde{\mathcal{N}_k}[u]$ is the "fluctuation"

$$\widetilde{\mathcal{N}_k}[u] := \mathcal{N}_k[u] - \mathbb{E}(\mathcal{N}_k[u])$$

of the random variable $\mathcal{N}_k[u]$. Before continuing we point out that by the large deviation results in [18] the convergence in the previous theorem in fact holds at the rate $O(k^{-(n+1)})$.

Next, the fluctuations in the bulk are considered for functions u which are Lipschitz continuous, which equivalently means that differential du is point-wise defined almost everywhere on X and in L^{∞}_{loc}. In particular, given a continuous Riemannian metric g on a (measurable) subset $S \subset X$ the Dirichlet norm of u is finite and defined by

$$\|du\|^2_{(S,g)} := \int_S |du|^2_g dV_g,$$

In the present setting g mainly arises as the Kähler metric in the bulk of S defined by the Kähler form corresponding to ϕ (formula 1.8), when u is supported in the bulk of S. But in fact, the corresponding Dirichlet norm is defined on S for any Lipschitz continuous function u (see Sect. 3). The main result is the following Central Limit Theorem (CLT), which may be interpreted as saying that the (scaled) fluctuations of the random measure (1.9) converges in distribution to the Laplacian of the *Gaussian free field* in the bulk (defined with respect to the Kähler metric ω_ϕ) [70].

Theorem 1.5 *Assume that the weight ϕ is in $C^{1,1}_{loc}$ and that the volume form ω_n is continuous. Denote by S the support of the equilibrium measure of (X, ϕ).*

- *Assume that u is a Lipschitz function on X supported in a compact subset of the bulk. Then*

$$\lim_{k\to\infty} \mathbb{E}(e^{-tk^{-(n-1)/2}\widetilde{\mathcal{N}}_k[u]}) = \exp\left(\frac{t^2}{8\pi}\|du\|^2_{(S,\omega_\phi)}\right) \tag{1.11}$$

  in the $C^\infty-$topology when t is restricted to a compact subset of $\mathbb{C}$. In particular, the variance of $\mathcal{N}[u]$ has the following asymptotics

$$Var_k(\mathcal{N}[u]) = \frac{k^{n-1}}{4\pi}(\|du\|^2_{(S,\omega_\phi)}) + o(k^{n-1})$$

  and

$$k^{-(n-1)/2}\widetilde{\mathcal{N}}_k[u] := N^{(1+1/n)/2}\frac{\sum_{i=1}^N(u(x_i) - \mathbb{E}(u(x_i))}{N} \tag{1.12}$$

  (where $N = N_k \sim k^n$) converges in distribution, as $N \to \infty$, to a centered normal random variable with mean zero and variance $\frac{1}{4\pi}\|du\|^2_{\omega_\phi}$.
- *For a general continuous function u on X whose differential u exists almost everywhere the following variance estimate holds:*

$$\frac{k^{n-1}}{4\pi}(\|du\|^2_{(S,\omega_\phi)}) + o(k^{n-1}) \leq Var_k(\mathcal{N}[u]) \leq o(k^n),$$

Let us make some remarks:

- The assumptions on ϕ and u appear to be essentially sharp, in general (as discussed in Sect. 1.3).
- The scaling by $N^{(1+1/n)/2}$ in formula (1.12) gives a gain by a factor $N^{1/2n}$ compared to the classical case of the CLT for sample averages of independent random variables (appearing when the points x_i are independent and identically distributed). As explained in Sect. 7 the Large Deviation Principle established in [18] provides a simple heuristic explanation for the scaling above and for the asymptotics of the variance.
- The special case $n = 1$, i.e. when X is a Riemann surface, is singled out by the fact that the variance of $\mathcal{N}[u]$ is bounded (i.e. no scaling is required) and its leading asymptotics are independent of the weight ϕ, as follows from the conformal invariance of the Dirichlet norm when $n = 1$.
- Due to the presence of second order phase transitions (when the weight ϕ is perturbed), a central limit theorem for general smooth functions u - not supported in the bulk - is not to be expected (see the discussion in Sect. 7.2).

Applying the previous theorem gives the following normalized version of the CLT (using [76] when $n > 1$) :

Corollary 1.6 *Assume that the weight ϕ is in $C^{1,1}_{loc}$ and that the volume form ω_n is continuous. Let u be a Lipschitz function on X such that $\|du\|^2_{(S,\omega_\phi)} \neq 0$. When $n = 1$ assume moreover that u is supported in a compact subset of the bulk. Then the normalized random variable $\widetilde{\mathcal{N}_k}[u]/\sqrt{Var(\mathcal{N}_k[u])}$ converges in distribution to the standard normal variable with mean zero and unit variance.*

Just like Theorem 1.1 the previous results may be interpreted as a universality result (compare the discussion in [31]). The condition that $\|du\|^2_{(S,\omega_\phi)} \neq 0$ is natural since the CLT does not hold if u is a constant function (indeed, the variance then vanishes for any k). The validity of the normalized CLT when $n > 1$ should be contrasted with the failure of the normalized CLT in the "real setting" when $n = 1$ (see Sect. 1.2).

Remark 1.7 The previous results are actually shown to hold in a more general setting where $(kL, k\phi)$ is replaced by $(kL + F, k\phi + \phi_F)$ were (F, ϕ_F) is a Hermitian holomorphic line bundle with suitable regularity properties. In fact, this flexibility will allow us to pass directly from variance asymptotics to a central limit theorem.

1.2 Relation to Previous Results

The main point of the present paper is to apply techniques from complex geometry/pluripotential theory, in particular $\overline{\partial}$-estimates, to determinantal point processes. It should be emphasized that in the case of a smooth weight ϕ corresponding to a *smooth positively curved* metric on L the asymptotic results on the corresponding Bergman kernels are well-known and go back to the work of Tian, Bouche, Zelditch, Catlin and others. For the decay estimate in Theorem 1.3 in a $\mathbb{C}^n$-setting see [32, 55]. Note that by an example of M.Christ the rate of decay in Theorem 1.3 is essentially optimal. The extension to smooth non-positively curved metrics and the relation to equilibrium measures was initiated in [13, 16] and then developed to less regular weights and measures in [11, 17]. In the smooth positively curved case Bergman kernel asymptotics have already been applied and developed extensively by Shiffman-Zelditch and their collaborators in the different context of random zeroes of holomorphic sections (defined with respect to the Gaussian probability measure on the Hilbert space $\mathcal{H}(X, kL)$). For example, universality of the corresponding correlation functions was proved in [22] and a central limit theorem (when $n = 1$) was obtained in [72].

Let us next compare the results in the present paper with the results in the extensively studied one-dimensional "real setting" appearing when the reference measure μ is the Euclidean measure on $\mathbb{R}$. The corresponding determinantal random point process then coincides with the Hermitian random matrix model, with the points x_i representing the eigenvalues of the corresponding random matrices. In this setting the corresponding bulk universality holds at length scales of the order k^{-1} and the limiting kernel is then the sine kernel (the bulk is then usually defined as the maximal open set in $\mathbb{R}$ where the corresponding equilibrium measure has a positive continuous density; see [57] where mean-field theory methods are used and [29] for the real-analytic case, where Riemann-Hilbert methods are used). For the convergence in probability, towards the equilibrium measure (which is a special case of Theorem 1.4) see [58] and references therein. The analog in the one-dimensional real setting of the CLT in Theorem 1.5 was obtained in the seminal work [48] for a sufficiently smooth u and under the assumption that the weight ϕ be sufficiently smooth and that the support $S \subset \mathbb{R}$ of the corresponding equilibrium measure be connected (which is the case when, for example, $\phi(x)$ is strictly convex on $\mathbb{R}$). The limiting variance is then given by a Sobolev $1/2-$type norm. The proof in [48] used the method of Ward identities originating in Quantum Field Theory to compute the second order asymptotics of the corresponding Laplace transform (appearing in formula 1.11). The latter asymptotics is an analog of the classical Strong Szegö limit theorem for Toeplitz determinants (concerning the case when μ is the invariant measure on S^1). Interestingly, as shown in [59] in the case when the support $S \subset \mathbb{R}$ has several components the CLT does not hold in general (a counter-example is obtained in [59] for a non-convex real analytic ϕ with u linear on the support). More precisely, as shown in [59] the corresponding variance is bounded, but not convergent (it is asymptotically periodic in N as indicated by the formal argument in [24]) and even the normalized version of the CLT in Corollary 1.6 fails.

In the present complex setting, in the special case when $X = \mathbb{C}$ (and $\phi(z)$ has super logarithmic growth), Theorem 1.5 was obtained, independently, in [3] for real-analytic ϕ and smooth u. The proof in [3] uses the method of cumulants, which is related to the combinatorial approach for central limit theorems for general determinantal point processes used in [76] (where certain estimates on the variance are assumed, as recalled in the proof of Corollary 1.6). Just as in the present paper, the key analytic input in [3] is Bergman kernel asymptotics, obtained using the method introduced in [16] (see [2]). For the special case where $\phi = |z|^2$ in $\mathbb{C}$ a more general form of Theorem 1.5 was obtained in [63] for any u which is C^1-smooth, using combinatorics of cumulants. In particular, it is not assumed in [63] that u be supported in the bulk, which leads to a boundary contribution in the formula for the limiting variance.

1.3 Relations to Recent Developments and Outlook

The original version of the present paper appeared as a preprint on ArXiv in 2008 (which also contained some results on links to asymptotics of direct image bundles that have been removed as they appear in [20]). Since then there has been various new developments, as will be briefly recalled next. A central limit theorem allowing general (smooth and bounded) u in the one-dimensional case of the complex plane was established in [4] using the method of Ward identities (see Remark 6.8). It was assumed that ϕ be real analytic and the boundary S be a connected domain with real analytic boundary and that $\Delta\phi > 0$ in a neighborhood of S. The corresponding limiting variance can then by expressed as the Dirichlet norm of the harmonic extension of u from S to all of $\mathbb{C}$, which amounts to adding a boundary contribution to the Dirichlet norm (as in [62]). As pointed out in Sect. 7 this can - at a heuristic level - be explained in terms of the general Large Deviation Principle in [18] and related to the absence of second order phase transitions. Very recently, the results in [4] concerning $X = \mathbb{C}$ have been generalized to less regular data ϕ [7, 54] (with u assumed almost C^4-smooth; see Sect. 7.2). As for the scaling limits of the correlation function at the boundary/edge of the support they were established in [5] under suitable regularity and symmetry assumptions. It would be very interesting to consider the behavior at the boundary in higher dimensions. This appears to be a very challenging problem as it seems hard to say anything useful about the boundary regularity of the support S of the equilibrium measure, in general. In the presence of toric and circular symmetry results in this direction have been obtained recently in [61, 64, 78].

In another direction it was shown in [20] that a sharp version of the Central Limit Theorem in Theorem 1.5 holds on any Riemann surface when $dd^c\phi$ is a Kähler metric with constant curvature. The sharpness means that the convergence of the Laplace transforms of the corresponding laws (formula (1.11)) hold for any test function u with finite Dirichlet norm, $\|du\|^2 < \infty$ (in the case of the Riemann sphere the convergence in distribution of the laws was first shown in [62]). However, as pointed out in [20], the corresponding statement fails in higher dimensions (for any given ϕ). The point is that when $n > 1$, even if $\|du\|^2$ is assumed finite the local integrals of e^{-u} may, in general, diverge and hence the Laplace transform appearing in the left hand side of formula (1.11), may diverge. From this point of view the assumption that u be Lipschitz used in the present paper appears to be essentially optimal.

Let us also mention the recent work [6] where determinantal point processes defined by real multivariate orthogonal polynomials are applied to numerical integration, using a Monte Carlo type approach. In particular, a CLT (analogous to

Theorem 1.5) is established in the “real setting” of a measure μ supported on the unit-cube in $\mathbb{R}^n$ with u a C^1-smooth function (supported in the interior of the unit-cube). In the light of [6] the present results in particular provide a theoretical base for numerical integration of functions u which are periodic in $\mathbb{R}^{2n}$ (by identifying the fundamental domain with the Abelian variety $X := \mathbb{C}^n + i\mathbb{C}^n)/\Lambda$, for $\Lambda = \mathbb{Z}^n + i\mathbb{Z}^n$). But we shall not go further into this here.

It would also be interesting to study universality properties for general “beta deformations” of the determinantal point processes considered here. Such random point processes are obtained by raising the Slater determinant appearing in formula (1.2) to the βth power, for a given real number β (by [18] the empirical measure still converge in probability towards the equilibrium measure in the many particle limit). In one complex dimension such powers were introduced by Laughlin [52] to explain the experimentally observed *fractional* Quantum Hall Effect (where the fraction in question appears as $1/\beta$ when β is a suitable positive integer). For very recent field theoretical works on the Quantum Hall Effect on Riemann surfaces see the survey [50] and references therein. In another direction it was shown in [19] that letting β depend on k, yields a probabilistic construction of Kähler–Einstein metrics ω_{KE} on complex algebraic varieties X. More precisely, this happens when $\beta = \pm 1/k$, where the sign is the opposite sign of the Ricci curvature of ω_{KE}. In statistical mechanical terms this corresponds to looking at a limit of fixed non-zero temperature, which brings entropy into the picture. It would be very interesting to understand the connections between the latter probabilistic approach to Kähler–Einstein metrics, using canonical random point processes and the program of Ferrari–Klevtsov–Zelditch [38], which is based on random Bergman metrics, i.e. probability measures on the symmetric spaces $GL(N, \mathbb{C})/U(N)$ rather than on the N fold symmetric products of X.

Organization After having introduced the notation and general setup below we illustrate in Sect. 2 the general geometric setup in the special case when X is complex projective space, explaining the relations to orthogonal polynomials and Coulomb and fermion gases. Then, in Sect. 3, we recall the definition of the pluripotential equilibrium measure and define its (weak) bulk. In Sect. 4 we provide weighted L^2-estimates for $\bar{\partial}$ formulated in terms of the equilibrium potential. The latter estimates are then applied in Sect. 5 to obtain asymptotics for Bergman kernels and correlations (proving in particular Theorems 1.1 and 1.3). In Sect. 6 the main results concerning asymptotics of linear statitistics are proved, using the asymptotics in Sect. 5. An alternative proof of the CLT using second order expansions is also given, for smooth data. In the final section an outlook on the relations between the CLT in Theorem 1.5, the Large Deviation Principle (LDP) in [18] and phase transitions is given. This leads to a suggestive picture for a general CLT taking boundary contributions into account, which is consistent with the one-dimensional results in [4, 7, 54, 62].

1.4 Notation and General Setup

Weights on Line Bundles[1]

Let L be a holomorphic line bundle over a compact complex manifold X. We will represent an Hermitian metric on L by its *weight* ϕ. In practice, ϕ may be defined as certain collection of *local* functions. Namely, let s^U be a local holomorphic trivializing section of L over an open set U (i.e. $s^U(x) \neq 0$ for x in U). Then locally, $\left|s^U(z)\right|_\phi^2 =: e^{-\phi^U(z)}$. If α is a holomorphic *section* with values in L, then over U it may be locally written as $\alpha = f^U \cdot s^U$, where f^U is a local holomorphic *function.* In order to simplify the notation we will usually omit the dependence on the set U and s^U and simply say that f is a local holomorphic function representing the section α. The point-wise norm of α may then be locally expressed as

$$|\alpha|_\phi^2 = |f|^2 e^{-\phi}, \tag{1.13}$$

but it should be emphasized that it defines a *global* function on X.

The canonical curvature two-form of L is the global form on X, locally expressed as $\partial\overline{\partial}\phi$ and the normalized curvature form

$$\omega_\phi := i\partial\overline{\partial}\phi/2\pi =: dd^c\phi$$

(where $d^c := i(-\partial + \overline{\partial})/4\pi$) represents the first Chern class $c_1(L)$ of L in the second real de Rham cohomology group of X. The curvature form of a smooth weight is said to be *positive* at the point x if the local Hermitian matrix $\left(\frac{\partial^2\phi}{\partial z_i \partial \bar{z}_j}\right)$ is positive definite at the point x (i.e. $dd^c\phi_x > 0$). This means that the curvature is positive when $\phi(z)$ is strictly *plurisubharmonic (spsh)* i.e. strictly subharmonic along local complex lines. In differential geometric terms this means that the two-form ω_ϕ defines a *Kähler metric,* i.e. the corresponding symmetric two-tensor $\omega_\phi(\cdot, J\cdot)$ is a Riemannian metric compatible with the complex structure J on X. A line bundle is said to be ample (or positive) if admits a smooth metric with positive curvature. More generally, a weight ψ on L is called (possibly) *singular* if $|\psi|$ is locally integrable. Then the curvature is well-defined as a $(1, 1)-$current on X. The curvature current of a singular metric is called *positive* if ψ may be locally represented by a plurisubharmonic function and ψ will then simply be called a *psh weight.* A line bundle L is *big* if admits a psh weigh ψ whose curvature current is bounded from below by a Kähler form.

Further fixing an Hermitian metric two-form ω on X with associated volume form ω_n gives a pair (ϕ, ω_n) that will be called a weighted measure. It induces an inner product on the space $H^0(X, L)$ of holomorphic global sections of L by declaring

$$\|\alpha\|_\phi^2 := \int_X |\alpha|_\phi^2 \,\omega_n, \tag{1.14}$$

[1]General references for this section are the books [33, 60].

The corresponding Hilbert space will be denoted by $\mathcal{H}(X, L)$ and its Bergman kernel by $K(x, y)$, which is a section of the pulled back line bundle $L \boxtimes \overline{L}$ over $X \times \overline{X}$ (see Sect. 5).

The Hermitian line bundle (L, ϕ) over X induces, in functorial way, Hermitian line bundles over all products of X (and its conjugate $\overline{X}$) and we will usually keep the notation ϕ for the corresponding weights. For example, we will write

$$|K(x, y)|^2_\phi := |K(z, w)|^2 e^{-\phi(z)} e^{-\phi(w)}$$

where the right hand side is strictly speaking only defined when both x and y are contained in an open set U where L has been trivialized as above. When studying asymptotics we will replace L by its k th tensor power, written as kL in additive notation. The induced weight on kL may then be written as $k\phi$. A subindex k will indicate that the object is defined w.r.t the weight. $k\phi$ on kL for ϕ a fixed weight on L.

Regularity assumptions. A weighted measure (ϕ, μ) will be called *strongly regular* if the weight ϕ is locally $\mathcal{C}^{1,1}$-smooth (i.e. it is differentiable and all of its first partial derivatives are locally Lipschitz continuous) and $\mu = \omega_n$ is the volume form of a continuous metric ω on X. Moreover, if $(X, L) = (\mathbb{E}^n, \mathcal{O}(1))$, where is $\mathbb{E}^n$ the complex projective space, viewed as a compactification of its affine piece $\mathbb{C}^n$, then we also allow ω_n to be defined by the Lebesgue measure on $\mathbb{C}^n$ as long as the corresponding weight *function* $\phi(z)$ on $\mathbb{C}^n$ has super logarithmic growth (formula 2.5 below) with $\phi \in \mathcal{C}^{1,1}_{loc}(\mathbb{C}^n)$.

Probability notation. Given a probability space (Y, γ), i.e. a measure space where $\gamma(X) = 1$, a measurable function $\mathcal{N}$ on (Y, γ) is called a *random variable.* Its integral w.r.t to Y is denoted by $\mathbb{E}(\mathcal{N})$ and called the *expectation* of $\mathcal{N}$. Recall also that if $\mathcal{N}$ takes values in a space Z then the pushforward of γ under $\mathcal{N}$ is called the *law of $\mathcal{N}$ on Z*. A subindex k will indicate that the object is defined w.r.t. the probability measure on $Y = X^{N_k}$, defined by the density (1.4) induced by a weighted measure (ϕ, μ).

Occasionally, we will also consider the probability measures defined by the Bergman kernels $K_{k\phi+\phi_F}$ associated to a sequence of Hermitian line bundles $(kL + F, k\phi + \phi_F)$ (and a fixed reference measure μ) and we will then write $\mathbb{E} = \mathbb{E}_{k\phi+\phi_F}$ etc.

2 Examples

In this section we will illustrate our setup in the concrete case when X is the complex projective space. But it may also be worth pointing out that another concrete setting appears when $X := \mathbb{C}^n/\Lambda$ is a principally polarized torus (Abelian variety), in which case $H^0(X, kL)$ may be identified with the space of theta functions on $\mathbb{C}^n$ at level k, which are $\Lambda-$quasi periodic. In particular, the latter setting gives a geometric approach to the one-dimensional setting in [40].

2.1 From Projective Space to Orthogonal Polynomials and Vandermonde Determinants

It is a classical fact that $\mathbb{C}^n$ is compactified by the complex projective space $X := \mathbb{P}^n$. Let L be the hyperplane line bundle $\mathcal{O}(1)$ on $\mathbb{P}^n$. Then $H^0(X, kL)$ is the space of all complex homogeneous polynomials of total degree k in $\mathbb{C}^{n+1}$, which is isomorphic to the vector space $\mathcal{H}_k(\mathbb{C}^n)$ of all polynomials in $\mathbb{C}^n$ of total degree at most k. Indeed, fix a global holomorphic section s of $\mathcal{O}(1)$, whose zero-set is $\mathbb{P}^n - \mathbb{C}^n$, the "hyper plane at infinity". Then any section s_k of $L^{\otimes k}$ over the open subset $U := \mathbb{C}^n$ may be written as

$$s_k(z) = p_k s^{\otimes k}$$

where p_k is in $\mathcal{H}_k(\mathbb{C}^n)$ (concretely, this amounts to "dehomogenizing" s_k). Moreover, the point-wise norms with respect to a metric on $k\mathcal{O}(1)$ induced by a given locally bounded metric h on $\mathcal{O}(1)$ become

$$|s_k(z)|^2_{h^{\otimes k}} = |p_k(z)|^2 \, e^{-k\phi(z)} \tag{2.1}$$

for some function $\phi(z)$ on $\mathbb{C}^n$, that we will call the *weight function.* As is well-known, this gives a correspondence between locally bounded metrics h on $\mathcal{O}(1)$ and weight functions $\phi(z)$ of the form

$$\phi(z) = \phi_{FS}(z) + u(z) := \ln(1 + |z|^2) + u(z), \tag{2.2}$$

where u is a locally bounded function on $\mathbb{C}^n$. In particular, a subclass of weights corresponding to *smooth* metrics on $\mathcal{O}(1)$ are obtained by taking $u \in \mathcal{C}_c^\infty(\mathbb{C}^n)$. Note that the metric h_{FS} corresponding to $\phi_{FS}(z)$ is the Fubini-Study metric on $\mathcal{O}(1)$ which is characterized (up to a constant) by its invariance under the $SU(n)-$action. Its (normalized) curvature form $\omega_{FS} := dd^c\phi_{FS}$ is the called the Fubini-Study metric on $\mathbb{P}^n$ and a simple calculation shows that the corresponding volume form is given by

$$(\omega_{FS})_n := (dd^c\phi_{FS})^n/n! = e^{-(n+1)\phi_{FS}} \left(\frac{i}{2}\right)^n dz \wedge d\bar{z}$$

where $(\frac{i}{2})^n dz \wedge d\bar{z}$ denotes the Lebesgue measure on $\mathbb{C}^n$. The global norm of s_k induced by the weighted measure $(\phi, (\omega_{FS})_n)$ may hence be represented as

$$\|s_k\|^2_{(\phi,\omega_{FS})} := \int_{\mathbb{C}^n} |p_k(z)|^2 \, e^{-k\phi(z)} (\omega_{FS})_n. \tag{2.3}$$

Alternatively, the weight ϕ itself induces a measure $e^{-(n+1)\phi(z)}(\frac{i}{2})^n dz \wedge d\bar{z}$. The corresponding norm is hence given by

$$\|s_k\|_\phi^2 := \int_{\mathbb{C}^n} |p_k(z)|^2 e^{-(k+n+1)(\phi(z))} \left(\frac{i}{2}\right)^n dz \wedge d\bar{z}$$

Note that the contribution from the factor $e^{-(n+1)\phi}$ makes sure that the integrals are finite.

The corresponding determinantal probability density (5.6) may in this case be expressed explicitly as

$$\frac{1}{\mathcal{Z}_{k\phi}} |\Delta^{(N_k)}(z_1, \ldots, z_{N_k})|^2 e^{-k\phi(z_1)} \cdots e^{-k\phi(z_{N_k})}, \tag{2.4}$$

where $\Delta^{(N_k)}(z_1, \ldots, z_{N_k})$ is the higher dimensional *Vandermonde determinant,* i.e. the Slater determinant det S corresponding to a bases S of multinomials and where $\mathcal{Z}_{k\phi}$ is the corresponding normalizing factor (compare Lemma 5.1).

2.1.1 The Setting of Super Logarithmic Growth and Sections Vanishing Along a Hypersurface

A variant of the previous setting arises if one insists on using the Lebesgue measure as the integration measure defining the norms in (2.3). Then $\phi(z)$ has to have slightly larger growth than in formula (2.2) in order to get finite norms. More precisely, we then assume that ϕ has super logarithmic growth in the sense that

$$\phi(z) \geq (1+\epsilon) \ln |z|^2, \text{ when } |z| >> 1 \tag{2.5}$$

for some positive number ϵ. It should be emphasized that such a weight ϕ does not correspond to a locally bounded metric h on $\mathcal{O}(1)$. But as shown in [16] a slight modification of the arguments apply to this super logarithmic setting, as well. The key point is that the growth condition (2.5) forces the corresponding equilibrium measure to be compactly supported in $\mathbb{C}^n$. The model case is when $\phi(z) = |z|^2$. Then the equilibrium measure is (up to a multiplicative constant) the Lebesgue measure on the unit ball.

Remark 2.1 Another variant of the geometric setting of a line bundle $L \to X$ endowed with a, say smooth, weight ϕ is obtained by fixing a smooth complex hypersurface Z in X (of codimension one). Let $H_{k\lambda Z}$ be the subspace of $H^0(X, kL)$ consisting of all sections vanishing to order $[k\lambda]$ along Z for a fixed sufficiently small positive number λ. Then any continuous Hermitian metric $\|\cdot\|$ (with curvature form ω) and a volume form ω_n on X induce by restriction, an inner product on the subspace $H_{k\lambda Z}$. Hence, we can associate a sequence of determinantal point-processes to the corresponding sequence of Hilbert spaces $H_{k\lambda Z}$. As shown in [18, Section 5.5] the laws of the corresponding sequence of empirical measures satisfy a large deviation principle (LDP). The results in the present paper also extends with simple modifications to the determinantal point processes associated to $H_{k\lambda Z}$ (by replacing the equilibrium potential ϕ_e used in the present paper with the corresponding equilibrium potential relative to λZ, obtained by imposing that ψ in formula 3.1 has a Lelong number of at least λ along Z). In fact, the setting of super logarithmic growth in $\mathbb{C}^n$ can be fitted into this setting in the case when ϕ is of the special form

$$\phi(z) = (1+\epsilon)\log(1+|z|^2) + u(z), \tag{2.6}$$

where $u(z)$ extends smoothly from $\mathbb{C}^n$ to $\mathbb{E}^n$. Indeed, one then let Z be a hyperplane in $X := \mathbb{E}^n$ and identifies $\mathbb{C}^n$ with $X - Z$, in the usual way.

2.2 A Higher Dimensional Coulomb Type Gas

Continuing with the setting of multivariate orthogonal polynomials in $\mathbb{C}^n$ and introducing the Hamiltonian

$$E_{k\phi}(z_1,\ldots,z_N) := E_k(z_1,\ldots,z_{N_k}) + k\phi(z_1)/2 + \cdots + k\phi(z_{N_k})/2,$$

where

$$E_k(z_1,\ldots,z_{N_k}) = -\log\left|\Delta^{(N_k)}(z_1,\ldots,z_{N_k})\right|,$$

the corresponding probability density (2.4) may be written as a *Boltzmann-Gibbs density* at inverse temperature $\beta = 2$ (in suitable units):

$$\frac{e^{-\beta E_k(z_1,\ldots,z_N)}}{\mathcal{Z}_{k\phi}}, \tag{2.7}$$

describing an ensemble of N_k identical particles in thermal equilibrium interacting by the internal energy $E_k(z_1,\ldots,z_N)$ and subject to the exterior potential $k\phi/2$. In particular, in the one-dimensional case, expanding the Vandermonde determinant reveals that $E_k(z_1,\ldots,z_N)$ is precisely the Coulomb interaction for N_k unit-charge particles:

$$E_k(z_1,\ldots,z_{N_k}) = -\frac{1}{2}\sum_{1\le i,j\le N}\log|z_i - z_j|^2$$

(such a gas is also called a one component plasma in the physics literature). Using mean field theory heuristics one would expect that the corresponding random point processes satisfy a *Large Deviation Principle* (LDP) with a rate function $E(\mu) + \int\phi\mu$ defined on the space of all probability measures on $\mathbb{C}^n$ and with speed kN, i.e. that

$$\text{Prob}\left\{\frac{1}{N}\sum\delta_{z_i}\cong\mu\right\}\sim e^{-kN\left(E(\mu)+\int\phi\mu\right)}/Z$$

holds in the sense of large deviations. As shown in the companion paper [18] this is indeed the case (see Sect. 7) and, in physical terms, it can be interpreted as a higher dimensional effective fermion-boson correspondence. This LDP is also closely related to the fact that the corresponding equilibrium measure $MA(\phi_e)$ (which in the present paper is defined directly in terms of pluripotential theory in Sect. 3) may be alternatively obtained as the unique minimizer of the total "macroscopic" energy $E(\mu) + \int \phi\mu$ appearing as the rate functional above; see [18] and reference therein.

2.3 Random Normal Matrices

Consider the set of all normal matrices $\mathcal{M}_N := \{M \in gl(N, \mathbb{C}) : [M, M^*] = 0\}$ as a Riemannian subvariety of the space $gl(N, \mathbb{C})$ of all complex matrices of rank N equipped with the Euclidean metric. A given weight function ϕ of super logarithmic growth induces the following probability measure on $\mathcal{M}_N$

$$e^{-N\mathrm{Tr}(\phi(M))} dV_{\mathcal{M}_N} / \mathcal{Z}_{N\phi} \tag{2.8}$$

where $dV_{\mathcal{M}_N}$ is the Riemannian volume measure of $\mathcal{M}_N$ and $\mathcal{Z}_{N\phi}$ is a normalizing constant (usually called the partition function of the corresponding matrix model [77]). Under the map which associates the (ordered) eigenvalues $(z_1, \ldots, z_N)$ to a matrix M the probability measure (2.8) is pushed forward to a probability measure on $\mathbb{C}^N$ which turns out to coincide with the determinantal probability measure for polynomials of degree $N - 1$ weighted by ϕ (when $n = 1$). The corresponding correlation functions $\rho_k^{(m)}$ are hence usually called *eigenvalue correlation functions* in this context. It should also be pointed out that the correlation functions corresponding to the weighted set (ϕ, μ) where μ is the invariant measure supported on $\mathbb{R}$ (or the unit-circle T) coincide with eigenvalue correlation functions for random *Hermitian* (or unitary) matrices, weighted by ϕ, which have been extensively studied (cf. [30, 48, 57] and references there in).

2.4 Free Fermions in a Magnetic Field

When $n = 1$ the weighted polynomials $\Psi_{+,m} := z^m e^{-k\phi(z)/2}$ where $m = 0, \ldots, k$ each represent the quantum state of a single spin $1/2$ quantum particle (=fermion) confined to a plane subject to a magnetic field B perpendicular to the plane, where the value of B at the point z is $\frac{i}{2\pi} k \frac{\partial^2 \phi(z)}{\partial z \partial \bar{z}}$ in suitable units (and similarly in higher dimensions; see [18, 73] and references therein). Moreover, the states form a linearly independent set in the lowest possible energy level (i.e. the ground state). More precisely, this latter fact means that $\Psi_{+,m}$ is an eigenvector of finite norm with eigenvalue 0 of the Pauli operator, which in complex notation may be written as

$$(\overline{\partial}_{k\phi}+\overline{\partial}_{k\phi}^{*})^{2}\Psi_{+,m}=0,$$

where $\overline{\partial}_{k\phi}$ intertwines the space $S_+ := \Omega^{0,0}(\mathbb{C})$ of spin *up* and the space $S_- := \Omega^{0,1}(\mathbb{C})$ of spin *down* particles

$$\overline{\partial}_{k\phi}=\overline{\partial}+\frac{k}{2}\overline{\partial}\phi\wedge: \quad S_+\to S_-$$

and $\overline{\partial}_{k\phi}^{*}$ is its formal adjoint. This means that the corresponding real "vector potential" (i.e. $U(1)-$gauge field) for the magnetic two-form is given by k times

$$A:=\frac{1}{2}(\overline{\partial}\phi-\partial\phi)),$$

where $dA = iB$. Hence, the particle state $\Psi_{+,m}$ is said to have spin *up*, since it has no spin down component in $\Omega^{0,1}(\mathbb{C})$ (defined is the space of element of the form $gd\bar{z}$), where $g \in C^\infty(\mathbb{C})$). The corresponding many particle state of N free fermions, should, according to the postulates of quantum mechanics for fermions, be anti-symmetric under an exchange of two single particle states Ψ_m. Hence, it is represented by the (Slater) determinant $\Psi(z_1,\ldots,x_N) := \det(\Psi_{+,i}(z_j))$. In particular, the corresponding probability amplitude coincides (after normalization) with the corresponding determinantal probability measure (compare Lemma 5.1). The correspondence between the free fermion representation and the Coulomb bas picture above can, at a heuristic level, be explained by the process of bosonization (see [1, 18]).

Remark 2.2 The Pauli operator above is defined as the square of the Dirac operator $\mathcal{D}_{kA} := (\overline{\partial}_{k\phi}+\overline{\partial}_{k\phi}^{*})$ on the space $S := S_+\oplus S_-$of complex spinors, endowed with the L^2-norm induced by the Euclidean metric on $\mathbb{C}$ (this setup corresponds to gyromagnetic ratio $g = 2$; see for example [73] and [28, Chapter 5] for a physics reference). If one instead uses the metric induced by the curvature form B - assuming that B is positive - then the square of the corresponding Pauli operator on may be expressed as

$$\mathcal{D}_A^2=\left(\frac{1}{4}\nabla_{kA}^{*}\nabla_{kA}-k\right)\oplus\left(\frac{1}{4}\nabla_{A}\nabla_{A}^{*}+k\right), \tag{2.9}$$

where the magnetic Schrödinger operator $\nabla_{kA}^{*}\nabla_{kA}$ is the Landau Hamiltonian for a non-spinning particle subject to the magnetic vector potential kA (in our general setting this corresponds to taking the measure ω_n to be the one induced by the $dd^c\phi$). From the complex geometric point of view formula 2.9 is a special case of the Bochner-Kodaira-Nakano formula [60]. In particular, in the case of constant positive magnetic field, i.e. $\phi(z) = |z|^2$, the Pauli and the Landau operators are essentially the same (up to an additive constant depending on the spin).

3 The Pluripotential Equilibrium Measure

In this section we will give the pluripotential construction of the measure which will arise as the limiting expected distribution of the empirical measure of the point processes on X.

Let $L \to X$ be an ample line bundle over a compact complex manifold X. Given a weight ϕ on L, that we first only assume is continuous, the corresponding "equilibrium weight" ϕ_e is defined as the envelope

$$\phi_e(x) := \sup \{\psi(x) : \ \psi \leq \phi \text{ on } X\}. \tag{3.1}$$

where the sup is taken over all continuous psh weights ψ. Then ϕ_e is also a continuous psh weight on L [43] and we denote by D the corresponding coincidence set:

$$D := \{\phi_e = \phi\} \subset X$$

so that $D = X$ precisely when ϕ is a psh weight. *The equilibrium measure* (associated to the continuous weight ϕ) is in general defined as the *Monge–Ampère measure* $MA(\phi_e)$ constructed in the seminal work of Bedford-Taylor in the local setting (see [43] for the global setting). For a *smooth* psh weight ψ this measure is simply defined by

$$MA(\psi) := (dd^c\psi)^n/n! = \left(\frac{i}{2\pi}\right)^n \det\left(\frac{\partial^2\psi}{\partial z_i \partial z_j}\right) dz_1 \wedge d\bar{z}_1 \wedge ...dz_n \wedge d\bar{z}_n \tag{3.2}$$

As is well-known the equilibrium measure μ_{ϕ_e} is supported on D (see below). In the case when ϕ is smooth (and not merely continuous) it was shown in [13] that ϕ_e is $\mathcal{C}^{1,1}-$ smooth and in particular the local derivatives $\frac{\partial^2\psi}{\partial z_i \partial z_j}$ exist almost everywhere on X and are locally bounded. We may then simply *define* the equilibrium measure in this setting by the following measure which has an $L^\infty_{loc}-$density

$$\mu_{\phi_e} := \frac{1}{V} MA(\phi_e) := \frac{1}{V}(dd^c\phi_e)^n/n!$$

More precisely, the following theorem holds and is the specialization to ample line bundles of a general result in [13] concerning *big* line bundle (see Theorem 3.4 and Remark 3.6 there). It shows that if ϕ is class $\mathcal{C}^{1,1}$ on X, than ϕ_e is also in the class $\mathcal{C}^{1,1}$:

Theorem 3.1 *Suppose that L is an ample line bundle and that the given metric ϕ on L is in the class $\mathcal{C}^{1,1}$. Then*

(a) ϕ_e is in the class $\mathcal{C}^{1,1}$ on X.

(b) The Monge–Ampère measure of ϕ_e on X is absolutely continuous with respect to any given volume form and coincides with the corresponding L^∞_{loc} $(n, n)-$form obtained by a point-wise calculation:

$$(dd^c\phi_e)^n/n! = \det(dd^c\phi_e)\omega_n \tag{3.3}$$

(c) the following identity holds almost everywhere on the set $D := \{\phi_e = \phi\}$:

$$\det(dd^c\phi_e) = \det(dd^c\phi) \tag{3.4}$$

More precisely, it holds for all points where the second order jet $(\phi_e - \phi)^{(2)}$ *exists and vanishes and in particular point-wise on*

$$\{(\phi_e - \phi)^{(2)} = 0\} \cap \{\det(dd^c\phi) > 0\} \tag{3.5}$$

(d) Hence, the following identity between measures on X *holds:*

$$n!V\mu_{\phi_e} = (dd^c\phi_e)^n = 1_D(dd^c\phi)^n = 1_{D\cap X(0)}(dd^c\phi)^n, \tag{3.6}$$

where $X(0) = \{dd^c\phi > 0\}$.

We define the set

$$S := D \cap X(0)$$

that we shall call the *support* of the equilibrium measure μ_{ϕ_e}, in view of formula 3.6. Next, we are going to define the *weak bulk* (of the equilibrium measure associated to ϕ). It may seem tempting to define it as the interior of the support S of the equilibrium measure, but the problem is that there are essentially no general regularity results for S - for example it is not clear that, in general, $\overline{\text{int}(S)} = \bar{S}$. In fact, it even not clear that the interior $\text{int}(S)$ is non-empty, in general! (see [69] for the construction of examples where the coincidence set D can be extremely irregular, in the case $n = 1$).

Definition 3.2 The set in formula 3.5 above is called the *weak bulk (of* (X, ϕ)*)*. When ϕ is assumed to be in C^2_{loc} the *bulk (of* (X, ϕ)*)* is defined as the interior of the support S of the equilibrium measure. For a general ϕ in $C^{1,1}_{loc}$ the bulk is defined as the maximal open subset of the interior of S where $dd^c\phi_e$ (or equivalently, $dd^c\phi$) is represented by a continuous and strictly positive form (i.e. a continuous Kähler metric).

The definitions are made so that, in the weak bulk, the density of the equilibrium measure (w.r.t. ω_n) exists and is equal to $\det(dd^c\phi)$ and vanishes a.e. on the complement of the bulk. Moreover, the bulk is always contained in the weak bulk. We note that for a general Lipschitz continuous function the Dirichlet norm $\|du\|^2_{(S,\omega_\phi)}$ is well-defined. Indeed, by the previous regularity theorem

$$\|du\|^2_{(S,\omega_\phi)} = V\int_X |du|^2_{\omega_\phi}\,\mu_\phi$$

which is well-defined since $\omega_\phi > 0$ almost everywhere with respect to μ_ϕ.

Remark 3.3 In the general case when L is big one defines the weak bulk as above on the augmented base locus of X (also called the Kähler locus), which is a (Zariski) open subset of X. But for simplicity we will mainly stick to the case when L is ample.

3.1 Remarks on Regularity Properties of the Support S

Even in the classical one-dimensional case where $(X, L) = (\mathbb{E}^1, \mathcal{O}(1))$ and ϕ is smooth, the equilibrium weight may not have second derivatives at some points. In fact, when ϕ is radial this happens "generically" [13]. More generally, when (X, L) is a toric or abelian variety and ϕ is invariant under the corresponding torus action the envelope ϕ_e may be identified with the convexification of the function $\Phi(x)$ on $\mathbb{R}^n$ corresponding to ϕ. For a generic such Φ the corresponding support S_Φ has been classified in dimension $n \leq 3$ as a domain with piece-wise smooth boundary, with explicit algebraic singularity type. The proof uses Arnold's catastrophe theory of Lagrangian singularities (motivated by the adhesion model in cosmology where S arises in the Eulerian description of the "cosmic web"; see [23] and the appendix in [45]). However, in the general complex geometric setting there are almost no general results concerning the regularity properties of the support S. It would be interesting to find general conditions ensuring that S is a topological domain (i.e $\overline{\mathrm{int}(S)} = \bar{S}$) with some additional regularity properties. Comparing with the extensively studied Laplacian case appearing when $n = 1$ [27] suggests that a minimal requirement in order to have reasonable regularity properties is the assumption that $dd^c\phi > 0$ on the coincidence set D (which then coincides with the corresponding support set S). For example, in the setting of sections vanishing along a hypersurface described in Remark 2.1 it has recently been shown in [65] that the support S of the corresponding equilibrium measure is a domain with smooth boundary under the assumption that $dd^c\phi > 0$ on all of X and λ is sufficiently small (in fact, the complement of S is then even diffeomorphic to a tubular neighborhood of Z). In particular, this result applies in the setting of logarithmic growth in $\mathbb{C}^n$ as long as the weight $\phi(z)$ is smooth and strictly plurisubharmonic and the number ϵ appearing in formula 2.6 is sufficiently small. Anyway, it should be stressed that an important point in the present paper is to avoid making any detailed regularity assumptions on the support S.

4 Weighted L^2−Estimates for $\overline{\partial}$

In this section we will generalize, by refining the results in [13], some well-known estimates for the $\overline{\partial}$−operator concerning psh weights to more general weights. More precisely, we will assume that ϕ is a locally $\mathcal{C}^{1,1}$−smooth weight on the line bundle L over X. When $(X, L) = (\mathbb{E}^n, \mathcal{O}(1))$ we also allow weights corresponding to a weight function $\phi(z)$ in $\mathbb{C}^n$ with super logarithmic growth (see Sect. 2). But for simplicity we do not consider the latter situation in the proofs. The simple modifications needed follow precisely as in the appendix in [16].

We will denote by K_X the canonical line bundle of X, whose smooth sections are $(0, n)-$forms on X. A weight ϕ on L induces, without choosing a volume form ω_n on X, an L^2-norm on sections u of $L + K_X$ that we will write as

$$\|u\|_{\phi}^{2} := \int_X |u|^2 e^{-\phi}$$

In the statement of the following theorem, we will use the fact that $dd^c\phi$ defines a positive form with locally bounded coefficients in the bulk (by the very definition of the bulk).

Theorem 4.1 *Let L be a big line bundle and ϕ a $\mathcal{C}^{1,1}-$smooth weight. Then for any $\overline{\partial}-$closed $(0, 1)-$form g with values in $L + K_X$ and supported in the interior of the bulk, there is a smooth section u with values in $L + K_X$ such that*

$$\overline{\partial}u = g \tag{4.1}$$

and

$$\int_X |u|^2 e^{-\phi} \le \int_X |g|^2_{dd^c\phi} e^{-\phi}. \tag{4.2}$$

In particular, the previous estimate holds for any u such that u is orthogonal to $H^0(X, L + K_X)$ (w.r.t the weight ϕ).

Proof Let ψ denote a general psh weight on L. By Theorem 5.1 in [34] the theorem holds with ϕ replaced by a (possibly singular) psh weight ψ if $dd^c\phi$ is replaced with the absolutely continuous part $(dd^c\psi)_c$ of the Lebesgue decomposition of the positive form $dd^c\psi$. More precisely,

$$\int_X |u|^2 e^{-\psi} \le \int_X |g|^2_{(dd^c\psi)_c} e^{-\psi} \tag{4.3}$$

as long as the r.h.s is finite. Now set $\psi = \phi_e$, the equilibrium weight corresponding to ϕ. Since g is supposed to be supported in the bulk, the regularity Theorem 3.1, gives

$$\int_X |g|^2_{(dd^c\phi_e)_c} e^{-\phi_e} = \int_X |g|^2_{(dd^c\phi)} e^{-\phi}$$

and since g is, in fact, supposed to be supported in the *pseudo-interior* of the bulk the latter integral is finite. Finally, using that $\phi_e \le \phi$ on *all* of X finishes the proof of the estimate (4.2). The last statement of the theorem now follows since the estimate (4.2) in particular holds for the solution which minimizes the corresponding L^2-norm. □

Remark 4.2 Given a bounded function f on X it follows immediately from the inequality (4.2) that

$$\int_X |u|^2 e^{-(\phi+f)} \leq C_f \int_X |g|^2_{dd^c\phi} e^{-(\phi+f)}, \quad C_f = e^{2\|f\|_{L^\infty(X)}}$$

In particular, the previous estimate holds when u is the solution to the Eq. (4.1) which is minimal wrt the L^2-norm on L induced by the weight $\phi + f$.

The previous theorem is a generalization to non-psh weights ϕ of the fundamental result of Hörmander-Kodaira. In turn, the next theorem is a generalization to non-psh weights of a refinement of the Hörmander-Kodaira estimate which goes back to a twisting trick in the work of Donelly-Fefferman. See [32, 55] for an analogous result concerning psh weights in $\mathbb{C}^n$.

Theorem 4.3 *Let L be a big line bundle, ϕ a $\mathcal{C}^{1,1}-$smooth weight on L and v a smooth function on E such that dv is supported in the interior of the bulk of (X, ϕ) and*

$$(i) \left|\overline{\partial} v\right|^2_{dd^c\phi} \leq 1/8 \quad (ii)\, dd^c v \geq -dd^c\phi/2$$

there. Then

$$\int_X |u|^2 e^{-\phi_e + v} \leq 2 \int_X \left|\overline{\partial} u\right|^2_{dd^c\phi} e^{-\phi_e+v} \tag{4.4}$$

for any smooth section u of $L + K_X$ orthogonal to the space $H^0(L + K_X)$, w.r.t the weight ϕ, and such that $\overline{\partial} u$ is supported in the interior of the bulk of (X, ϕ). Moreover, given a bounded function f on X the function v above may be replaced by $v + f$ at the expence of multiplying the right hand side in the inequality (4.4) by $C_f := e^{2\|f\|_{L^\infty(X)}}$.

Proof By assumption

$$\langle u, h\rangle_\phi = 0, \quad \forall h \in H^0(X, L + K_X).$$

Equivalently, writing $u_v := ue^v$,

$$\langle u_v, h\rangle_{\phi+v} = 0, \quad \forall h \in H^0(X, L + K_X). \tag{4.5}$$

By Leibniz rule

$$\overline{\partial} u_v = (\overline{\partial} u + \overline{\partial} v u)e^v, \tag{4.6}$$

which by assumption is supported in the bulk of (X, ϕ). Hence, applying the estimate (4.3) in the proof of the previous theorem to $\psi = \phi_e + v$ gives, since by assumption ii $(\phi_e + v)$ is a psh weight

$$\int_X |u_v|^2 e^{-(\phi_e+v)} \leq \int_X \left|\overline{\partial} u_v\right|^2_{dd^c(\phi_e+v)} e^{-(\phi_e+v)} \leq \int_X \left|\overline{\partial} u_v\right|^2_{\frac{1}{2}dd^c\phi} e^{-(\phi+v)}$$

for *some* solution u_v of the corresponding $\overline{\partial}-$equation and hence for u_v as in formula 4.5 (we are also using that $\overline{\partial}u$ and $\overline{\partial}v$ are supported in the bulk of (X, ϕ) to replace ϕ_e with ϕ in the r.h.s). Using $\phi_e \leq \phi$, (4.6) and the "parallelogram law" then gives

$$\int_X |u|^2 e^{-\phi} e^v \leq 4 \int_X \left(\left|\overline{\partial}u\right|^2_{dd^c\phi} + \left|\overline{\partial}vu\right|^2_{dd^c\phi} \right) e^{-\phi_e} e^v$$

By assumption (i) in the theorem the term in the r.h.s involving $\overline{\partial}vu$ may be absorbed in the l.h.s. Finally, the last statement in the theorem follows from the estimate in Remark 4.2. □

Corollary 4.4 *Let L be a big line bundle and let ϕ be a $\mathcal{C}^{1,1}-$smooth weight on and ω_n a fixed volume form on X. Let E be a given compact subset of the interior of the bulk. Then there is a constant C (depending on E and F) such that the following holds. If ψ_k is a sequence of functions such that $d\psi_k$ is supported in the interior of the bulk of (X, ϕ) and*

$$(i) \ \left|\overline{\partial}\psi_k\right|^2_{dd^c\phi} \leq 1/C \quad (ii)\, dd^c\psi_k \leq \sqrt{k} dd^c\phi/C$$

Then, for any sequence f_k of smooth sections of kL such that $\overline{\partial} f_k$ is supported in the interior of the bulk of (X, ϕ)

$$\|\Pi_k(f_k) - f_k\|^2_{k\phi+\phi_F+\sqrt{k}\psi_k} \leq C\frac{1}{k} \left\|\overline{\partial} f_k\right\|^2_{k\phi+\phi_F+\sqrt{k}\psi_k},$$

where Π_k is the Bergman projection with respect to $k\phi$ (formula 5.1 and below). Moreover, the constant C can be taken to depend on ϕ_F only through an upper bound on the $L^\infty-$norm $\left\|(\phi_F - \phi_{F_0})\right\|_{L^\infty(X)}$, where ϕ_{F_0} is a fixed smooth metric on F.

Proof Replacing L with $kL + F - K_X$, ϕ with $k\phi + \phi_F$ and v with $\sqrt{k}\psi_k$ the corollary follows from the previous theorem using standard properties of orthogonal projections. □

Proposition 4.5 *The following local estimate holds for all u which are $\mathcal{C}^1-$smooth (or more generally, Lipschitz continuous):*

$$\sup_{|z|\leq Rk^{-1/2}} |u(z)|^2 e^{-k\phi(z)} \leq C_R k^n \left(\int_{|z|\leq 2Rk^{-1/2}} \left(|u|^2 + \frac{1}{k}\left|\overline{\partial}u\right|^2 \right) e^{-k\phi}\omega_n \right) \tag{4.7}$$

Proof This is a generalization of the uniformity statement in Lemma 5.3. It is proved in essentially the same way, by replacing the mean value property of holomorphic functions used to prove Lemma 5.3 by the general Cauchy formula for a smooth function u. It is also a consequence of Gårding's inequality - see (the proof of) Lemma 3.1 in [15] for a more general inequality. □

5 Asymptotics for Bergman Kernels and Correlations

5.1 Bergman Kernels

Recall that $\mathcal{H}(X, L)$ denotes the Hilbert space obtained by equipping the vector space $H^0(X, L)$ with the inner product corresponding to the norm induced by the weighted measure (ϕ, ω_n). Let (s_i) be an orthonormal base for $\mathcal{H}(X, L)$. The *Bergman kernel* of the Hilbert space $\mathcal{H}(X, L)$ may be defined as the holomorphic section

$$K_k(x, y) = \sum_i s_i(x) \otimes \overline{s_i(y)}. \tag{5.1}$$

of the pulled back line bundle $L \boxtimes \overline{L}$ over $X \times \overline{X}$. To see that is independent of the choice of base (s_I) one notes that K_k represents the integral kernel of the orthogonal projection Π_k from the space of all smooth sections with values in L onto $\mathcal{H}(X, L)$.

The restriction of K_k to the diagonal is a section of $L \otimes \overline{L}$. Hence, its point wise norm $|K_k(x, x)|_\phi$ $(= |K_k(x, x)|\, e^{-k\phi(x)})$ defines a well-defined function on X that will be denoted by $\rho^{(1)}$ (and later identified with the *one point correlation function*):

$$\rho^{(1)}(x) := \sum_i |s_i(x)|^2_{k\phi}\,. \tag{5.2}$$

It has the following well-known extremal property:

$$\rho^{(1)}(x) := \sup \left\{ |s(x)|^2_\phi : \; s \in \mathcal{H}(X, L), \; \|s\|^2_\phi \leq 1 \right\} \tag{5.3}$$

Moreover, integrating (5.2) shows that $|K_k(x, x)|_\phi$ is a "dimensional density" of the space $\mathcal{H}(X, L)$:

$$\int_X \rho^{(1)}(x) \omega_n = \dim \mathcal{H}(X, L) := N \tag{5.4}$$

In Sect. 6.1 we will consider a function on the $N-$fold product X^N that may, abusing notation slightly, be written as

$$\rho^{(N)}(x_1, \ldots, x_N) = \det_{1 \leq i, j \leq N} (K(x_i, x_j) e^{-\frac{1}{2}(\phi(x_i) + \phi(x_j))}). \tag{5.5}$$

To clarify the notation denote by $L^{\boxtimes N}$ the pulled-back line bundle on X^N with the weight induced by the weight ϕ on L. Then the base $S = (s_i)$ in $H^0(X, L)$ induces an element $\det(S)$ in $H^0(X^N, L^{\boxtimes N})$ whose value at $(x_1, \ldots, x_N)$ is defined as the (Slater) determinant

$$\det(S)(x_1, .x_N) := \det_{1 \leq i, j \leq N} (s_i(x_i))_{i,j} \in L_{x_1} \otimes \cdots \otimes L_{x_N}. \tag{5.6}$$

In particular, its point-wise norm is a *function* on X^N which according to the following lemma may be locally written in the form (5.5). The lemma also shows that after division by $N!$ this function defines the density of a probability measure on X^N. Its proof is based on the following "integrating out" property of the Bergman kernel K, which is a direct consequence of the fact that K is a projection kernel:

$$|K(x,x)|_\phi = \int_X |K(x,y)|^2_\phi \,\omega_n(y) \tag{5.7}$$

Lemma 5.1 *The following identities hold point-wise:*

$$\det_{1\leq i,j\leq N}(K(x_i,x_j)e^{-\frac{1}{2}(\phi(x_i)+\phi(x_j))}) = |\det(S)(x_1,\ldots,x_N)|^2_\phi.$$

Integrating gives

$$\int_{X^N} |\det(S)(x_1,\ldots,x_N)|^2_\phi \,\omega_n^{\otimes N} = N!.$$

Proof The identities are formal consequences of the identity (5.7), as is well-known in the random matrix literature. See for example [30]. The last identity can also be proved directly using the following general identity [17, Lemma 5.3]:

$$\int_{X^N} |\det(S)(x_1,\ldots,x_N)|^2_\phi \,\omega_n^{\otimes N} = N! \det_{1\leq i,j\leq N}(\langle s_i,s_j\rangle_{(\omega_n,\varphi)})_{i,j}, \tag{5.8}$$

given a base (s_i) in $H^0(X,L)$ and a bounded weight ϕ on L.

5.2 Scaling Asymptotics of $K_k(x,y)$ in the Weak Bulk

In this section we fix a continuous metric ω on X. Given a point x in X we can take "normal" local coordinates z centered at x and a "normal" trivialization of L, i.e such that

$$\omega_x = \frac{i}{2}\sum_{i=1}^n dz_i\wedge\overline{dz_i} + o(1)\quad \phi(0) = d\phi(0) = 0 \tag{5.9}$$

Moreover, if the second partial derivatives of ϕ exist at x then we may assume

$$(dd^c\phi)_x = \frac{i}{2\pi}\sum_{i=1}^n \lambda_i dz_i\wedge\overline{dz_i}$$

Hence, the λ_i are the eigenvalues of the curvature form $dd^c\phi$ at x w.r.t the metric ω and we denote the corresponding diagonal matrix by λ.

For proofs of the following elementary local consequences of the regularity properties of ϕ and ϕ_e see [16].

Lemma 5.2 *Given a point x in X and "normal" local coordinates z centered at x and a "normal" trivialization of L the following holds:*

$$|\phi(z)| \leq C\,|z|^2\,, \tag{5.10}$$

where C can be taken to be independent of the center x on any given compact subset of X. Moreover, if the second partial derivatives of ϕ exist at $z = 0$, then for any $\epsilon > 0$, there is a $\delta > 0$ such that

$$(|z| \leq \delta \Rightarrow \left|\phi(z) - \sum_{i=1}^{n} \lambda_i\,|z_i|^2\right| \leq \epsilon\,|z|^2 \tag{5.11}$$

and for any fixed positive number R the following uniform convergence holds when k tends to infinity

$$\sup_{|z|\leq R}\left|k\phi\left(\frac{z}{\sqrt{k}}\right) - \sum_{i=1}^{n} \lambda_i\,|z_i|^2\right| \to 0. \tag{5.12}$$

Finally, if the center x is in the weak bulk, then for any $\epsilon > 0$, there is a $\delta > 0$ such that

$$(iii)\ \ |z| \leq \delta \Rightarrow |\phi_e(z) - \phi(z)| \leq \epsilon\,|z|^2 \tag{5.13}$$

The next lemma only uses local properties of holomorphic functions and was called *local holomorphic Morse inequalities* in [15]. See [16] for the proof when the weight ϕ is merely $C^{1,1}-$smooth.

Lemma 5.3 *Fix a center x in X where the second derivatives of the weight ϕ exist and normal coordinates z centered at x. Then*

$$\limsup_k k^{-n}\rho_k^{(1)}(z/k^{1/2}) \leq \det_\omega(dd^c\phi)(x).$$

Moreover, if $|z| \leq R$ then the l.h.s. above is uniformly bounded by a constant C_R which is independent of the center x.

Now we can prove the following *lower* bound on the 1-point correlation function in the weak bulk, which is a refinement of Lemma 4.4 in [13]:

Lemma 5.4 *Fix a center x in the weak bulk and normal coordinates z centered at x. Then*

$$\liminf_k k^{-n}\rho_k^{(1)}(z/k^{1/2}) \geq \det_\omega(dd^c\phi)(x)$$

Proof Step1: construction of a smooth extremal σ_k. Fix a point x in the weak bulk. First note that there is a *smooth* section σ_k with values in $kL + F$ such that

$$(i) \lim_{k\to\infty} \frac{|\sigma_k|^2_{k\phi}(z_0/\sqrt{k})}{k^n \|\sigma_k\|^2_{k\phi+\phi_F}} = \left(\frac{1}{2\pi}\right)^n \det\lambda, \quad (ii) \left\|\overline{\partial}\sigma_k\right\|^2_{k\phi_e+\phi_F} \leq Ce^{-k/C} \tag{5.14}$$

To see this first take normal trivializations of L and F and normal coordinates z centered at x (i.e. x corresponds to $z=0$). Next, by scaling the coordinates z we can assume that

$$\omega_{x_0} = \frac{i}{2}\sum_{i=1}^n \frac{1}{\lambda_i} dz_i \wedge \overline{dz_i}, \quad (dd^c\phi)_{x_0} = \frac{i}{2\pi}\sum_{i=1}^n dz_i \wedge \overline{dz_i}$$

Fix a smooth function χ which is equal to one when $|z| \leq \delta/2$ and supported where $|z| \leq \delta$; the number δ will be assumed to be sufficiently small later on. Now $\sigma_k(z)$ is simply obtained as the local section with values in L^k represented by the function

$$\chi(z)e^{k(\bar{z}_0\cdot z - \frac{1}{2}\bar{z}_0\cdot z_0)}$$

close to $z=0$ and extended by zero to all of X. To see that (i) holds note first consider the numerator

$$|\sigma_k|^2_{k\phi}(z_0/\sqrt{k}) = e^{\bar{z}_0\cdot z_0}e^{-k\phi(z_0/\sqrt{k})} \to 1,$$

when k tends to infinity, using (5.12). Next, write the the integrand in $k^n \|\sigma_k\|^2_{k\phi+\phi_F}$, in the form

$$\chi(z)^2 k^n e^{-k(\left|z-z_0/\sqrt{k}\right|^2 + (\phi(z)-|z|^2))}((\det\lambda)^{-1} + o(1))$$

and decompose the region of integration according to the following decomposition of the radial values:

$$[0,\delta] = [0, R/\sqrt{k}]\bigsqcup[R/\sqrt{k},\delta], \tag{5.15}$$

where R is a fixed large number. In the first region, we have by (5.12),

$$\sup_{|z|\leq R/\sqrt{k}} \left|k(\phi(z) - |z|^2)\right| \to 0$$

Hence, performing the change of variables $z = z'/\sqrt{k}$ gives

$$\lim_{k\to\infty} k^n \|\sigma_k\|^2_{k\phi+\phi_F,[0,R/\sqrt{k}]} = (\det\lambda)^{-1}\int_{[0,R]} e^{-\left|z'-z_0\right|^2}\left(\frac{i}{2}\sum_{i=1}^n dz'_i \wedge \overline{dz'_i}\right)^n / n!$$

As fort the second region in (5.15) we have

$$\left|z - z_0/\sqrt{k}\right|^2 + (\phi(z) - |z|^2) \geq \frac{1}{2}|z|^2 \tag{5.16}$$

for R sufficiently large. Indeed, by (5.11)

$$|z| \leq \delta \Rightarrow \left|(\phi(z) - |z|^2)\right| \leq \frac{1}{4} |z|^2 .$$

Moreover,

$$\left|z - z_0/\sqrt{k}\right|^2 \geq \frac{1}{4} |z|^2 ,$$

for all k, if R is sufficiently large. Hence,

$$k^n \|\sigma_k\|^2_{k\phi+\phi_F,[R/\sqrt{k},\delta]} \leq \int_{[R/\sqrt{k},\delta]} k^n e^{-k\frac{1}{2}|z|^2} \to 0,$$

since it is the tail of a convergent (Gaussian) integral (using the change of variables $z = z'/\sqrt{k}$ again). Finally, letting first k and then R tend to infinity finishes the proof of (i) in (5.14).

Next, to prove (ii) in (5.14), first note that

$$\left\|\overline{\partial}\sigma_k\right\|^2_{k\phi_e+\phi_F} \leq C' \int_{\delta/2\leq|z|\leq\delta} e^{-k(\left|z-z_0/\sqrt{k}\right|^2+(\phi(z)-|z|^2)+\phi_e(z)-\phi(z)))}\omega_n(0) \tag{5.17}$$

as follows from the definition of χ. Now take δ so that, using (5.11) and (5.13) ,

$$|z| \leq \delta \Rightarrow \phi(z) + (\phi_e(z) - \phi(z)) \geq |z|^2/4 \tag{5.18}$$

for δ sufficiently small. Combining (5.16) and (5.18) shows that the exponent in (5.17) is at most$-\frac{1}{4}k|z|^2$ which proves (ii) in (5.14).

Step2: perturbation of σ_k *to a holomorphic extremal* α_k.

This step is just a repetition (word for word) of the corresponding step in the proof of Lemma 4.4 in [13]. For completeness we recall it briefly here. Equip $kL + F$ with a "strictly positively curved modification" ψ_k of the metric $k\phi_e + \phi_F$ as constructed in [13]. Let $g_k = \overline{\partial}\sigma_k$ and let α_k be the following holomorphic section

$$\alpha_k := \sigma_k - u_k,$$

where u_k is the solution of the $\overline{\partial}$-equation in the Hörmander-Kodaira Theorem 4.1 with $g_k = \overline{\partial}\sigma_k$. Using properties of ϕ_e on then obtains the estimate

$$\|u_k\|_{k\phi+\phi_F} \leq C \|g_k\|_{k\phi_e+\phi_F} \tag{5.19}$$

and then (ii) in (5.14) in the right hand side gives

$$(a)\ \|u_k\|_{k\phi+\phi_F} \leq Ce^{-k/C},\ \ (b)\ |u_k|^2_{k\phi+\phi_F}(x) \leq C'k^n e^{-k/C'},$$

where (b) is a consequence of (a) (using Proposition 4.5 at $z = 0$). Combining (a) and (b) with (i) in (5.14) then proves that (i) in (5.14) holds with σ_k replaced by the holomorphic section α_k. By the definition of $\rho_k^{(1)}$ this finishes the proof of the lemma. □

Before turning to the proof of Theorem 1.1 we also recall the following uniform estimate (which follows from Lemma 5.3 precisely as in Lemma 5.2 (i) in [14]):

Lemma 5.5 *Fix a center x in X and normal coordinates z and w centered at x with z, w contained in a fixed compact set. Then*

$$k^{-2n}\left|K_k(z/k^{1/2}, w/k^{1/2}))\right|^2_{k\phi+\phi_F} \leq C$$

for some constant independent of the center x in X.

5.2.1 Proof of Theorem 1.1

Fix a point x_0 in X and take coordinates z and w centered at x and normal trivializations of L and F as in the proof of the previous lemma, inducing corresponding trivializations around (x, x) in $X \times X$. Consider the holomorphic functions $f_k(z, w) = k^{-n}K_k(k^{-1/2}z, k^{-1/2}\bar{w})$ and $f(z, w) = \det_\omega(dd^c\phi)(x_0)e^{zw}$ on the polydisc on Δ_R of radius R centered at the origin in $\mathbb{C}^{2n}$. By Lemma 5.5:

$$\sup_{\Delta_R} |f_k| \leq C_R, \tag{5.20}$$

Moreover, combining the upper and lower bounds in Lemmas 5.3 and 5.4, respectively, shows that f_k tends to f on $M := \{(z, \bar{z}) \in \Delta_R\}$. Now, by the bound (5.20) f_k has a convergent subsequence converging uniformly on Δ_R to a holomorphic function f_∞ where necessarily $f_\infty = f$ on M. But since M is a maximally totally real submanifold it follows that $f_\infty = f$ everywhere on Δ_R. Since, the argument can be repeated for any subsequence of f_k this proves the uniform convergence in the theorem. Finally, the convergence of higher derivatives is a standard consequence of Cauchy estimates.

Remark 5.6 In fact, Theorem 1.1 also follows in a more or less formal way (using the method in [14]) from combining Lemma 5.3 with the the special case of Lemma 5.4 obtained by setting $z = 0$ (which was obtained in [13]). But the present method is more explicit and hence gives a better control on the convergence, which might be useful in other contexts.

5.3 *Off-Diagonal Decay of $K_k(x, y)$*

The next theorem is a refined version of Theorem 1.3 stated in the introduction (the dependence on the line bundle F will be important in the proof of Theorem 1.5).

Theorem 5.7 *Let L be a big line bundle and K_k the Bergman kernel of the Hilbert space $\mathcal{H}(kL+F)$. Let E be a compact subset of the interior of the bulk. Then there is a constant C (depending on E) such that the following estimate holds for all pairs (x, y) such that either x or y is in E:*

$$k^{-2n}\left|K_k(x,y)\right|^2_{k\phi+\phi_{F,t}} \leq Ce^{-\sqrt{k}d(x,y)/C}$$

for all k, where $d(x,y)$ is the distance function with respect to a fixed smooth metric ω on X. Moreover, fixing a smooth reference weight ϕ_{F_0}on L the constant C can be taken to only depend on the continuous weight ϕ_F via an upper bound on the $L^\infty-$norm $\left\|(\phi_F-\phi_{F_0})\right\|_{L^\infty(X)}$.

Proof Fix a point x in X and take an element s_k in $\mathcal{H}_k$ such that

$$|s_k|^2\, e^{-k\phi} = |K_k(x,\cdot)|^2\, e^{-k\phi(x)}e^{-k\phi(\cdot)} \tag{5.21}$$

Next, fix a point y in the set E appearing in the formulation of the theorem and "normal" local coordinates z centered at y and a "normal" trivialization of L (see the beginning of the section). In particular, $\phi(0)=\overline{\partial}\phi(0)=0$. Identify s_k with a local holomorphic function in the $z-$variable. By the mean value property of holomorphic functions

$$s_k(0)=\int \chi_k s_k,$$

where $\chi_k=c_nk^n\chi(\sqrt{k}z)$ has unit mass and is expressed in terms of a radial smooth function χ supported on the unit-ball (so that χ_k is supported on the scaled unit ball of radius $1/\sqrt{k}$). Writing $\chi_{k\phi}:=\chi_k e^{k\phi(x)}$ the relation (5.21) gives,

$$|s_k|_{k\phi}(y)=\left|\left\langle \chi_{k\phi}, s_k\right\rangle_{k\phi}\right| = \left|\Pi_k(\chi_{k\phi})(x)\right|_{k\phi}(x)$$

using the definition of s_k in the last equality. Decomposing $\Pi_k(\chi_{k\phi})=\chi_{k\phi}+(\Pi_k(\chi_{k\phi})-\chi_{k\phi})$ and applying Theorem 4.3 combined with Proposition 4.5 now yields the following estimate

$$|s_k|_{k\phi+\sqrt{k}\psi_k}(y)\leq \left|\chi_{k\phi}\right|_{k\phi+\sqrt{k}\psi_k}(x)+Ck^{(n-1)/2}\left\|\overline{\partial}\chi_{k\phi}\right\|_{k\phi+\sqrt{k}\psi_k} \tag{5.22}$$

for any function ψ_k satisfying the assumptions in Theorem 4.3. The idea now is take ψ_k to be comparable to the distance to x. In the following we will denote by R a sufficiently large (but fixed constant).

Case 1: $d(x,y)\geq 1/R$. Set $\psi_k=\psi$ for a fixed smooth function ψ on X such that $\psi(\cdot)=1/R$ when $d(x,\cdot)\geq 1/(2R)$ and $\psi(\cdot)=0$ for when $d(x,\cdot)\leq 1/(4R)$. For $R>>1$ (but fixed) the assumptions on ψ_k in Theorem 4.3 are clearly satisfied (using that y is in the interior of the bulk). Hence, the estimate (5.22) gives

$$|s_k|^2_{k\phi} e^{\sqrt{k}/C}(y) \leq 0 + Ck^n \frac{1}{k} \left\| \overline{\partial} \chi_{k\phi} \right\|^2_{k\phi+0} \leq C' k^{2n}$$

using that $\psi = 0$ on the support of $\chi_{k\phi}$ and that $\left| k\overline{\partial}\phi \right|^2$ is uniformly bounded there (since $\overline{\partial}\phi$ is assumed to be Lipschitz continuous and vanishing when $z = 0$). Since by definition s_k is related to K_k by the relation (5.21) this proves the theorem in this case.

Case 2: $d(x, y) \leq 1/R$. In this case we may assume that x is contained in the fixed coordinate neighborhood of y. By a translation of the coordinates z we now assume that they are centered at x. Set

$$\psi_k(z) = \frac{1}{R}\kappa(|z|^2 + 1/k)^{1/2}$$

where κ corresponds to a smooth function on X which is equal to one on the "ball" $\{d(, y) \leq 2/C\}$ and is supported in the set E. Accepting for for the moment that the assumptions on ψ_k in Theorem 4.3 are satisfied, the inequality (5.22) gives (with $z \leftrightarrow y$)

$$|s_k|^2_{k\phi} e^{\sqrt{k}(|z|^2+1/k)^{1/2})}(z) \leq \left|\chi_{k\phi}\right|^2_{k\phi+1}(x) + C'k^n \frac{1}{k} \left\| \overline{\partial} \chi_{k\phi} \right\|^2_{k\phi+1} \leq C'' k^{2n}$$

using that $\sqrt{k}\psi_k \geq \sqrt{k}/\sqrt{k}$ on the support of $\chi_{k\phi}$ in the first inequality. In particular,

$$|s_k|^2_{k\phi}(z) \leq C' k^{2n} e^{-\sqrt{k}|z|}$$

which proves the theorem, since the distance function $d(\cdot, y)$ is comparable, close to y, with the distance function induced by the local Euclidean metric.

Next, let us check that the assumptions on ψ_k in Theorem 4.3 are indeed satisfied. Differentiating gives

$$\overline{\partial}\psi_k = \frac{1}{R}(\overline{\partial}\kappa \cdot \left(|z|^2 + 1/k)^{1/2} - \kappa \frac{z d\bar{z}}{2(|z|^2 + 1/k)^{1/2}} \right) \tag{5.23}$$

Hence,

$$\left|\overline{\partial}\psi_k\right| \leq \frac{1}{R}(C' + C''\sqrt{k}) \tag{5.24}$$

so that (i) in Theorem 4.3 holds for $R >> 1$. Next, note that $f_k := (|z|^2 + 1/k)^{1/2}$ is a psh function. Hence, formula 5.23 combined with Leibniz rule gives

$$\partial\overline{\partial}\psi_k \geq \partial\overline{\partial}\kappa \cdot f_k + \partial\kappa \wedge \overline{\partial} f_k + \overline{\partial}\kappa \wedge \partial f_k$$

and (5.24) (which clearly also holds when ψ_k is replaced by f_k) then shows that assumption (ii) in Theorem 4.3 holds, as well (even without taking R large).

Finally, the last statement in the theorem, concerning the dependence on ϕ_F, follows immediately from writing $\phi_F = f + \phi_{F_0}$ and repeating the previous proof with $k\phi$ replaced with $k\phi + f$ and using the L^2−estimate in Remark 4.2. □

5.4 Fluctuations

Theorem 5.8 *Let L be a big line bundle and K_k the Bergman kernel of $\mathcal{H}(X, kL + F)$. Let u be a Lipschitz continuous function on X. Then*

$$\liminf_{k\to\infty} \frac{1}{2}\int_{X\times X} k^{-(n-1)} \left|K_k(x,y)\right|^2_{k\phi+\phi_F} (u(x)-u(y))^2 \geq \|du\|^2_{(S,\omega_\phi)}$$

where equality holds, with lim inf *replace by* lim, *if u is supported in a compact subset of the bulk. Moreover, if ϕ_F satisfies the assumptions in the previous theorem, then the left hand side above is uniformly bounded by a constant only depending on ϕ_F through the L^∞−norm of $\phi_F - \phi_{F_0}$.*

Proof Let us start by the first point in the theorem, i.e. the case when u is compactly supported in the bulk. Denote by E the support of u. First note that the integrand vanishes if both x and y are in $X - E$. We rewrite the integral above as follows:

$$2I_k := \int_{E\times X\cup X\times E} \left|k^{1/2}(u(y)-u(x))\right|^2 k^{-n} \left|K_k(x,y)\right|^2 e^{-k\phi(x)}e^{-k\phi(x)}\omega_n(x)\wedge\omega_n(y),$$

Decompose the integral above as $A_{k,R} + B_{k,R} + C_{k,R}$ according to the following three regions:

First region $(1 \leq d(x,y))$: By symmetry we may assume that $x \in E$. But then Theorem 5.7 shows that A_k tends to zero only using that u is bounded.

Second region $(Rk^{-1/2} \leq d(x,y) \leq 1)$: Again, by symmetry we may assume that $x \in E$. Since u is Lipschitz continuous $|u(y) - u(x)| \leq Cd(x,y)$. Hence, by Theorem 5.7

$$B_{k,R} \leq C\int_{Rk^{-1/2}\leq\{d(x,y)\leq 1} \left|\sqrt{k}d(x,y)\right|^2 k^n e^{-\sqrt{k}d(x,y)/C}\omega_n(x)\wedge\omega_n(y).$$

Performing a change of variables (with y fixed) then gives

$$I_k \leq C\int_X (\int_{2\sqrt{k}\geq|\zeta|\geq R/2} |\zeta|^2 e^{-|\zeta|}d\zeta ...)\omega_n(x) \to 0, \tag{5.25}$$

when first k and then R tends to infinity. □

Third region $(d(x,y) \leq Rk^{-1/2})$:

By the previous discussion only the third region gives a contribution to the asymptotics of the integrals I_k:

$$\lim_{k\to\infty} I_k := 0 + 0 + \lim_{R\to\infty}\lim_{k\to\infty} C_{k,R},$$

assuming that the last limits exist, as well be shown next. To this end fix $R > 0$ and note that, using a partition of unity we may as well replace the total region of integration $X \times X$ by $U \times U$, where U is a given local coordinate neighborhood. Moreover, the third region $C_{k,R}$ may as well be replaced by the region $C'_{k,R}$ defined by $|x - y| \leq Rk^{-1/2}$, expressed in terms of the Euclidean distance on U (just using that $A^{-1}|x - y| \leq d(x, y) \leq A|x - y|$ on U for some positive constant A). Upon removing a set of measure zero we may also assume that x is in the bulk (since E is a compact set in the interior of the bulk) and that the first order derivatives of u exist at x. Now take "normal coordinates" z and trivializations of L and F centered at x. Then the integral over $\{x\} \times Y$ in $C'_{k,R}$ may be written as

$$\int_{|z|\leq R} g_k(x, z)\omega_n(k^{-1/2}z), \tag{5.26}$$

where, $g_k(x, z) :=$

$$= \left|k^{1/2}(u(k^{-1/2}z) - u(0))\right|^2 k^{-2n} \left|K_k(0, k^{-1/2}z)\right|^2 e^{-k\phi(k^{-1/2}z)}e^{-k\phi(0)}\omega_n(k^{-1/2}z)$$

(using the change of variables $z \to k^{-1/2}z$). Since u is assumed to be Lipschitz continuous and differentiable at $z = 0$ we have

$$\sup_{|z|\leq R}\left|\left(k^{1/2}(u(k^{-1/2}z) - u(0))\right) - (\sum_{i=1}^n a_i z_i + \overline{a_i z_i})\right| \to 0,\ \ a_i := \frac{\partial u}{\partial z_i}(0)$$

By the scaling asymptotics in Theorem 1.1 and Lemma 5.5 and the Lipschitz assumption on u we have

$$|g_k(x, z)| \leq A_R$$

and

$$\lim_{k\to\infty} g_k(x, z) = \int_{|z|\leq R}\left|\sum_{i=1}^n a_i z_i + \overline{a_i z_i})\right|^2 \left(\frac{\det\lambda}{\pi^n}\right)^2 e^{-\langle\lambda z, z\rangle}\frac{i}{2}dz_1 \wedge d\bar{z}_1 \wedge \cdots$$

As a consequence, computing the Gaussian integrals gives

$$\lim_{R\to\infty}\lim_{k\to\infty} g_k(x, z) = \left(\frac{\det\lambda}{\pi^n}\right)\sum_i 2\left|\frac{\partial}{\partial z_i}u(0)\right|^2 \lambda_i^{-2}c_n,$$

where

$$c_n = \left(\int_0^\infty se^{-s}ds\right)^n = -\frac{d}{dt}\,|_{t=1}\int_0^\infty e^{-ts}ds = 1$$

Hence, by the dominated convergence

$$I = \frac{1}{2} \lim_{R\to\infty} \lim_{k\to\infty} C_{k,R} = \frac{1}{\pi} \int_X |\partial u|^2_{(dd^c\phi)} (dd^c\phi)^n/n!,$$

which concludes the proof of the convergence in theorem. To prove the last statement of the theorem just note that the integrand may, as above, be estimated from above by

$$C \left|\sqrt{k}d(x,y)\right|^2 k^n e^{-\sqrt{k}d(x,y)/C},$$

where C only depends on the $L^\infty-$norm of $|\phi_F - \phi_{F_0}|$, according to Theorem 5.8. Integrating over x and y then concludes the proof, as above.

Finally, for a general Lipschitz continuous u the lower bound on the second point of the theorem follows by restricting the integration to the third region above with x restricted to the weak bulk. Indeed, letting first $k \to \infty$ using the scaling asymptotics in Theorem 1.1 as above together with Fatou's lemma and then letting $R \to \infty$, using the monotone convergence theorem, gives the desired lower bound.

6 Asymptotics for Linear Statistics

Let us first recall the setup in Sect. 5. A line bundle $L \to X$ and a pair (ϕ, ω_n) induces a Hilbert space $\mathcal{H}(X, L)$ of dimension N with associated Bergman kernel $K(x, y)$. Recall also that, in general, a subindex k on an object indicates that it is defined with respect to $(kL, k\phi)$. Hence, we will set $k = 1$ in the following definitions.

We define the associated ensemble (X^N, γ) by letting γ be the probability measure with the following density:

$$\mathcal{P}(x_1, \ldots, x_N) := \frac{1}{N!} \det(K(x_i, x_j)e^{-\frac{1}{2}(\phi(x_i)+\phi(x_j))}).$$

By Lemma 5.1 this is indeed a well-defined probability measure. Note that the ensemble is symmetric in the sense that $\mathcal{P}(x_1, \ldots, x_N)$ is invariant under permutations of the components x_i.

6.1 Correlation Functions

Next, we recall the general formalism of correlation functions. But it should be pointed out that in the present paper we will mainly consider the correlation functions in formula 6.1 below, that the reader could also take as definitions.

For a general symmetric ensemble (X^N, γ) the *m−point correlation measures* on X^m may be defined as $N!/(N-m)!$ times the pushforward of γ to X^m under the projection $(x_1, \ldots, x_N) \mapsto (x_1, \ldots, x_m)$ (i.e. the $m-$dimensional *marginal* of γ). The *$m-$point correlation functions* $\rho^{(m)}$ on X^m are then defined as the corresponding densities. As is well-known [30, 75] the fact that the defining kernel K of the process represents an orthogonal projection operator leads to the following quite remarkable identities in the present context:

$$\rho^{(m)}(x_1, \ldots, x_m) = \det_{1 \leq i,j \leq m} (K(x_i, x_j)e^{-\frac{1}{2}(\phi(x_i)+\phi(x_j))})$$

A crucial role in the present paper is played by the so called *connected $2-$point correlation function* $\rho^{(2),c}$ which may be defined by

$$\rho^{(2).c}(x, y) := \rho^{(2)}(x, y) - \rho^{(1)}(x)\rho^{(1)}(y)$$

Hence, $\rho^{(1)}$ and $\rho^{(2).c}$ may be simply expressed as

$$\rho^{(1)}(x) = |K(x, x)|_{\phi}, \quad \rho^{(2).c}(x, y) = -|K(x, y)|^2_{\phi}. \tag{6.1}$$

Remark 6.1 The present setup is essentially a special case of the general formalism of determinantal random point processes [47, 49, 75]. It falls into the class of such processes where the correlation kernel is the integral kernel of an orthogonal projection operator.

6.2 Linear Statistics

A given (measurable) function u on (X, ω_n) induces the following random variable $\mathcal{N}[u]$ on $(X^N, d\mathcal{P})$:

$$\mathcal{N}[u](x_1, \ldots, x_N) := u(x_1) + \cdots + u(x_N).$$

Hence, if u is the characteristic function of a set Ω in X, then $\mathcal{N}[u](x_1, \ldots, x_n)$ simply counts the number of x_i contained in Ω. However, we will mainly focus on the case when u is continuous. For a given random variable $\mathcal{X}$ we will write its *fluctuation* as the random variable

$$\widetilde{\mathcal{X}} := \mathcal{X} - \mathbb{E}(\mathcal{X}),$$

so that $\mathbb{E}(\widetilde{\mathcal{X}}) = 0$. Recall that the variance of a random variable $\mathcal{X}$ is defined as

$$\mathrm{Var}(\mathcal{X}) := \mathbb{E}((\widetilde{\mathcal{X}})^2)$$

The following lemma is also essentially well-known.

Lemma 6.2 *The following formulas for the expectation and variance of $\mathcal{N}_k[u]$ hold:*

$$(i)\ \mathbb{E}_{\phi+tu}(\mathcal{N}[u]) = -\frac{d}{dt}\log\mathbb{E}_{\phi+tu}(e^{-t\mathcal{N}[u]}) = \int_X \left|K_{\phi+tu}(x,x)\right|_{\phi+tu} u(x)$$

and

$$(ii)\ Var_{\phi+tu}(\mathcal{N}[u])) = \frac{d^2}{d^2t}\log\mathbb{E}_{\phi+tu}(e^{-t\mathcal{N}[u]}) =$$

$$= \frac{1}{2}\int_{X\times X}\left|K_{\phi+tu}(x,y)\right|^2_{\phi+tu}(u(x)-u(y))^2\omega_n(x)\wedge\omega_n(y).$$

Proof Without loss of generality we may as well calculate the derivatives at $t=0$ (indeed, at a general $t=t_0$ one then rewrites $\phi+(t_0+\epsilon)u = (\phi+t_0u)+\epsilon u$ and applies the previous case with ϕ replaced by $\phi+t_0u$). Set $f(t) := -\log\mathbb{E}(e^{-t\mathcal{N}[u]})$ Then it follows immediately that

$$\frac{d}{dt}_{|t=0} f(t) = \int_{X^N}\sum_{i=1}^N u(x_i)\rho^{(N)}(x_1,\ldots,x_N)\omega_n^{\otimes N} = \int_X u\rho^{(1)}\omega_n,$$

which, combined with formula 6.1 proves the item (i). Similarly,

$$\frac{d^2f(t)}{d^2t}_{|t=0} = \int_{X^N}\sum_{1\leq i,j\leq N} u(x_i)u(x_j)\rho^{(N)}(x_1,\ldots,x_N)\omega_n^{\otimes N}$$

and hence splitting the sum over the indices (i,j) where $i=j$ and $i<j$ gives

$$\frac{d^2f(t)}{d^2t}_{|t=0} = \int_X u^2\rho^{(1)}\omega_n + \int_{X^2} u(x)u(y)\rho^{(2)}(x,y)\omega_n$$

Invoking formula 6.1 for $\rho^{(2)}(x,y)$ thus gives that

$$\frac{d^2f(t)}{d^2t}_{|t=0} = \int_X u^2|K(x,x)|_\phi\omega_n + \int_{X^2} u(x)u(y)\left(|K(x,x)|_\phi|K(y,y)|_\phi - |K(x,y)|^2_\phi\right)$$

Under the normalization that $\mathbb{E}_\phi(\mathcal{N}[u]) := \int u\rho^1\omega_n = 0$ this means that

$$\frac{d^2f(t)}{d^2t}_{|t=0} = \int_X u^2|K(x,x)|_\phi\omega_n - \int_{X^2} u(x)u(y)|K(x,y)|^2_\phi.$$

The proof is now concluded by first rewriting $u(x)u(y) = -(u(x)-u(y))^2/2 + u(x)^2/2 + u(y)^2/2$ and then integrating over first x and then y and using that (by the reproducing property) $|K(x,x)|_\phi = \int_X |K(x,y)|^2_\phi\omega_n(y)$. □

Remark 6.3 Let (s_i) be an orthonormal base for $H^0(X, L)$ w.r.t. (ϕ, ω_n). Then $\mathbb{E}(e^{-t\mathcal{N}[u]})$ may be alternatively expressed as a Gram determinant:

$$\mathbb{E}(e^{-t\mathcal{N}[u]}) = \det\left(\left\langle s_i, s_j\right\rangle_{\phi+tu}\right)_{i,j} \tag{6.2}$$

and hence form the point of view of Kähler geometry the functional $u \mapsto -\log \mathbb{E}(e^{-t\mathcal{N}[u]})$ can be viewed as a Donaldson $\mathcal{L}_k-$functional (see [17, 20, 37] and references therein). Formula 6.2 follows immediately from writing

$$\mathbb{E}(e^{-t\mathcal{N}[u]}) = \frac{\int_{X^N} |\det(S)(x_1, \ldots, x_N)|^2_{\phi+tu}\, \omega_n^{\otimes N}}{\int_{X^N} |\det(S)(x_1, \ldots, x_N)|^2_{\phi}\, \omega_n^{\otimes N}}.$$

and applying the identity (5.8) to the weights ϕ and $\phi + tu$.

Proposition 6.4 *Suppose that u is a bounded function on X and* (ϕ, μ) *is a general weighted measure. Then*

$$(i)\ Var_k(\mathcal{N}[u])) = O(k^n)$$

Moreover, if (ϕ, ω_n) *is strongly regular and u continuous, then*

$$(ii)\ Var_k(\mathcal{N}[u])) = o(k^n).$$

Proof By (ii) in Lemma 6.2

$$\mathrm{Var}_k(\mathcal{N}[u])) = \frac{1}{2}\int_{X\times X} |K_k(x, y)|^2_{k\phi}\, (u(x) - u(y))^2 \omega_n(x) \wedge \omega_n(y)$$

The first item of the proposition follows immediately, since u is assumed bounded, from combining (5.4) and (5.7)and using that $N_k = O(k^n)$ for *any* line bundle L. The second item follows from [13] where it is shown that

$$\int k^{-n}\, |K_k(x, y)|^2_{k\phi}\, f(x)g(y)\omega_n(x) \wedge \omega_n(y) \to \int_X fg\mu_{\phi_e},$$

for any continuous functions f, g. □

6.3 A Law of Large Numbers (Proof of Theorem 1.4)

By (i) in Lemma 6.2 and [11, Thm B]:

$$\mathbb{E}_k(k^{-n}\mathcal{N}[u]) = \int_X |K_k|_{k\phi}\, u\omega_n \to \int_X u\mu_{\phi_e}.$$

Moreover, by (i) in the previous proposition

$$\mathrm{Var}_k(k^{-n}\mathcal{N}[u])) = O(k^{-n}) \to 0.$$

Hence, the theorem follows directly from Chebishevs inequality, just like in the usual proof of the classical weak law of large numbers.

6.4 A Central Limit Theorem (Proof of Theorem 1.5)

Proof We start by taking $t \in \mathbb{R}$. Let $\mathcal{F}_k(t) := -\log \mathbb{E}_k(e^{-tk^{-(n-1)/2}\widetilde{\mathcal{N}_k}[u]})$. By (i) in Lemma 6.2

$$\frac{d\mathcal{F}_k(t)}{dt}_{t=0} = k^{-(n-1)/2}\mathbb{E}_k(\widetilde{\mathcal{N}_k}) = 0, \tag{6.3}$$

using the definition of $\widetilde{\mathcal{N}_k}$ in the last equality. Moreover, by (ii) in Lemma 6.2

$$\frac{d^2\mathcal{F}_k(t)}{d^2t} = -k^{-(n-1)}\frac{1}{2}\int_{X\times X}\left|K_{k\phi+th_k}(x,y)\right|^2_{k\phi+th_k}(h_k(x)-h_k(y))^2$$

where $h_k = u - c_k$ with $c_k = \mathbb{E}_k(\mathcal{N}_k)$. Next, note that the map $\psi \mapsto \left|K_\psi(x,y)\right|^2_\psi$ is clearly invariant under $\psi \to \psi + c$ for any constant c. Hence, we get

$$\frac{d^2\mathcal{F}_k(t)}{d^2t} = -\frac{1}{2}\int_{X\times X}\left|K_{k\phi+tu}(x,y)\right|^2_{k\phi+tu}(u(x)-u(y))^2$$

Applying Theorem 5.8 to $kL + F$ where F is the trivial holomorphic line bundle equipped with the weight $k^{-(n-1)/2}tu$ (taking for example $\phi_{F_0} \equiv 0$) gives

$$\lim_{k\to\infty}\frac{d^2\mathcal{F}_k(t)}{d^2t} = -\left\|du\right\|^2_{dd^c\phi} \tag{6.4}$$

for all t. Using that the second order derivatives of $\mathcal{F}_k(t)$ uniform bound are uniformly bounded on any fixed interval (by the uniformity in Theorems 5.8) and (6.3) the theorem now follows by integrating over t. Indeed, since $\mathcal{F}_k(t)$ and its first derivative vanish at $t = 0$ we have

$$\mathcal{F}_k(t) = \int\int\frac{d^2\mathcal{F}_k(s)}{d^2t}\chi(v,s)dvds,$$

where χ is the characteristic function of the set of all (v, s) such that $v \leq s \leq t$. Hence (6.4) gives

$$\mathcal{F}_k(t) \to a \int \int \chi(v, s) dv ds = a\frac{t^2}{2}, \ a := a := -\|du\|^2_{dd^c\phi} \tag{6.5}$$

which proves the point-wise version of the asymptotics (1.11) when $t \in \mathbb{R}$.

Next, we set $\nu_k := k^{-(n-1)/2}\widetilde{\mathcal{N}}_k[u]_*(\gamma_k)$, which gives a sequence of compactly supported probability measures on $\mathbb{R}$, obtained by pushing forward the probability measure γ_k. Then we may write

$$\mathcal{F}_k(t) = \int_{\mathbb{R}} \nu_k(s) e^{-ts}$$

which gives a well-defined holomorphic function for all t in $\mathbb{C}$ with

$$|f_k(t)| \leq C_R$$

for all $t \in \mathbb{C}$ such that $|t| \leq R$. By (6.5) we have $f_k(t) \to f(t)$, where $f(t)$ is an entire function, on the maximally totally real set $\mathbb{R}$ in $\mathbb{C}$. Hence, the same normal families argument as below formula 5.20 shows that uniform convergence actually holds on compacts of $\mathbb{C}$ (even for all derivatives). Setting $t = i\xi$ with $\xi \in \mathbb{R}$ in particular gives that the Fourier transforms $\widehat{\nu_k}$ converges uniformly om compacts in $\mathbb{R}_\xi$ towards $\widehat{\nu}$, where $\widehat{\nu}$ (and hence ν) is a centered Gaussian. As is well-known this latter convergence property is equivalent to convergence in distribution. □

Finally, the variance asymptotics then follows by evaluating the convergence of the second derivatives at $t = 0$ and using Lemma 6.2.

6.5 *Proof of Corollary 1.6 (The Normalized CLT)*

The case when u is supported in the bulk follows directly from Theorem 1.5. Next, we recall that by [76] the normalized CLT holds, for a general determinantal point processes, under the condition that $\mathrm{Var}(\mathcal{N}(u)) \to \infty$(as $N \to \infty$) and that there exists a positive numbers δ and C such that

$$\mathbb{E}(\mathcal{N}(u)) \leq C \left(\mathrm{Var}(\mathcal{N}(u))\right)^\delta .$$

Since $\mathbb{E}(\mathcal{N}(u)) \sim N \sim k^n$ the validity of these assumptions in the present setting, when $n \geq 2$, follows directly from the lower bound on the variance in Theorem 1.5 (by taking $\delta = (n-1)/n$).

6.6 An Alternative Proof of the CLT for Smooth Data Using Second Order Expansions

We start by recalling the following result in [13] generalizing the seminal asymptotic expansion of Zelditch and Catlin concerning the case when $dd^c\phi > 0$ on all of X (see [10, 79]).

Theorem 6.5 *Assume that ϕ is a smooth weight on the ample line bundle L, ω_n a smooth volume form on X and ϕ_F a smooth metric on a line bundle F. Then, on the diagonal, the point-wise norm of the Bergman kernel K_k of $H^0(X, kL + F)$ endowed with the corresponding L^2−norm admits a complete asymptotic expansion on any compact subset of bulk. More precisely, the corresponding second order expansion is given by*

$$|K_k(x,x)|_{k\phi+\phi_F}\frac{\omega^n}{n!} =$$

$$= \frac{k^n}{n!}\omega_\phi^n + \frac{k^{n-1}}{(n-1)!}\left(-\frac{1}{2}Ric\omega_\phi + Ric\omega + dd^c\phi_F\right)\wedge\omega_\phi^{n-1} + O(k^{n-2}),$$

(the form Ric $\eta := -dd^c \log \eta^n$ represents the normalized Ricci curvature of a Kähler metric η).

Remark 6.6 Strictly speaking the result in [13] was only formulated when F is trivial (which in fact will be enough for our purposes). But exactly the same proof applies for a general F. Indeed, around any point where $\omega_\phi > 0$ [10] gives the expansion for a local version of the Bergman kernel (the contribution to the coefficients coming from the line bundle F are computed in [10, Section 2.5]). Then the local Bergman kernel is shown to coincide with the global one in the bulk using Theorem 4.1 with L replaced by $kL + F - K_X$ (just as in the proof of Step 2 in Lemma 5.4).

In particular, by the previous theorem the following holds in the bulk:

$$\left(|K_k(x,x)|_{k\phi+\phi_F} - K_k(x,x)|_{k\phi}\right)\frac{\omega^n}{n!} = \frac{k^{n-1}}{(n-1)!}dd^c\phi_F\wedge(dd^c\phi)^{n-1} + O(k^{n-2}), \tag{6.6}$$

Let us now specialize to the case when $n = 1$ and apply the previous result to the trivial line bundle F endowed with the weight $\phi_F = tu$ for $t \in \mathbb{R}$ and u a smooth function supported in the interior of the bulk. Then it is not hard to see that the remainder term above is uniform in t, as long as $|t| \leq C$ (indeed, the proof in [10] shows that the remainder term only depends on an upper bound on the local sup-norm of the local derivatives of ϕ_F).

Now, combining the asymptotics in (6.6) with the first formula in Lemma 6.2 gives

$$-\frac{d}{dt}\log\mathbb{E}_{k\phi+tu}(e^{-t\tilde{\mathcal{N}}[u]}) = \int_X |K_k(x,x)|_{k\phi+tu}u\omega - \int_X |K_k(x,x)|_{k\phi}u\omega =$$

$$= t \int_X (udd^c u + o(1),$$

where the remainder term tends to zero, uniformly in k and t. Hence, integrating over t gives

$$-\log \mathbb{E}_{k\phi+tu}(e^{-t\tilde{\mathcal{N}}[u]}) = \int_0^t sds \int_X udd^c u = \frac{1}{2}\int_X udd^c u,$$

proving the asymptotics in formula 1.11 in this special case (which implies Theorem 1.5, just as before). In fact, the uniformity in t used above may be dispensed with. Indeed, by the convexity of $t \mapsto g(t) := -\log \mathbb{E}_{k\phi+tu}(e^{-t\tilde{\mathcal{N}}[u]})$ we have $g'(0) \leq g'(t) \leq g'(1)$ so that the dominated convergence theorem may be applied.

Remark 6.7 It follows immediately from Theorem 6.5 that, for u as above, the expectation of $\mathcal{N}(u)$ has a complete asymptotic expansion of the form

$$\mathbb{E}(\mathcal{N}(u)) = \int_X u \left(\frac{k^n}{n!}\omega_\phi^n + \frac{k^{n-1}}{(n-1)!}\left(-\frac{1}{2}\mathrm{Ric}\omega_\phi + \mathrm{Ric}\omega\right) \wedge \omega_\phi^{n-1} \right) + O(k^{n-2}).$$

Moreover, when $\omega_\phi > 0$ on all of X integrating the asymptotics in Theorem 6.5 yields a complete asymptotic expansion of the partition function $\log Z_{N_k}[\phi]$ corresponding to (ϕ, ω_n) (see the notation Sect. 7.2):

$$-\frac{1}{N_k k}\log Z_{N_k}[\phi] = \mathcal{F}_0[\phi] + \mathcal{F}_1[\phi]k^{-1} + \cdots,$$

where $\mathcal{F}_0$ and $\mathcal{F}_1$ are explicit functionals, well-known in Kähler geometry ($\mathcal{F}_0$ is the primitive $\mathcal{E}$ of the Monge–Ampère operator, sometimes called the Aubin-Yau energy and $\mathcal{F}_1$ is a twisted version of the K-energy functional [37]).

It seems likely that a similar argument applies when $n > 1$, using $\phi_F = k^{(n-1)/2}t$. But then one has to verify that the remainder terms are uniform in k. Alternatively, one could, at least formally, apply the *first* order asymptotics of $K_k(x,x)|_{k\tilde{\phi}}$ with the *perturbed* weight

$$\tilde{\phi} := \phi + k^{-1}k^{(n-1)/2}u \tag{6.7}$$

Indeed, setting $\phi_t := \phi + tu$, handling the limit $k \to \infty$ formally gives

$$k^{-(n-1)/2}\left(K_k(x,x)|_{k\tilde{\phi}} - K_k(x,x)_{k\phi}\right) \approx \frac{d}{dt}_{|t=0} k^{-n}K_k(x,x)_{k\phi_t} \approx$$

$$\approx \frac{d\mu_{\phi_t}}{dt}_{|t=0} = \frac{1}{(n-1)!}dd^c u \wedge (dd^c\phi)^{n-1}$$

Anyway, an important feature of the proof of Theorem 1.5 in the previous section is that it only requires that u be Lipschitz continuous. In contrast, any argument based on the second order expansion in Theorem 6.5 requires that u be, at least, C^2-smooth, ensuring that Δu is point-wise defined.

Remark 6.8 The alternative proof above is similar to the method of proof in the real setting in [48] and the second proof of the corresponding result in [3], also concerning the case $n = 1$ (the first proof in [4] uses the method of cumulants). The second proof, which was only sketched in [3], uses the formal first order argument involving the perturbed weight $\tilde{\phi}$ above which was made rigorous in [4], for real analytic ϕ, using the method of Ward identities. An important feature of the method in [4] is that it also applies on the boundary of S giving the precise "edge contribution". It would be very interesting to extend the results in [4] (and the generalizations in [7, 54]) to the case when $n > 1$, as further discussed in the following section.

7 Outlook on Relations to LDPs and Phase Transitions

7.1 From the LDP Towards a General CLT

Let us start with some general considerations. Consider an $N-$particle random point processes $(\mu^{(N)}, X^N)$ on a compact topological space X. Assume that the law of the corresponding empirical measure

$$\delta_N := \frac{1}{N}\sum_{i=1}^{N}\delta_{x_i}$$

satisfies a large deviation principle (LDP) at a speed $r_N \to \infty$ and rate functional $E(\mu)$ on $\mathcal{P}(X)$, symbolically expressed as

$$(\delta_N)_*\mu^{(N)} \sim e^{-r_N E(\mu)},\ \ N \to \infty$$

(see [36] for the precise meaning of a LDP). In particular, by the contraction principle, this implies a LDP at the same speed r_N for the real-valued random variable $\langle \delta_N, u\rangle$ on $(\mu^{(N)}, X^N)$ defined by a given continuous function $u \in C^0(X)$. It is well-known that, in general, a LDP at a speed r_N for a real-valued random variable implies, under suitable further assumptions (that are unfortunately rather strong) a CLT of the following form:

$$r_N^{1/2}(\langle \delta_N, u\rangle - \mathbb{E}(\langle \delta_N, u\rangle)) \to \mathcal{N}(0, \sigma_u), \tag{7.1}$$

in distribution, where the variance σ_u is given by

$$\sigma_u = -\frac{d^2\mathcal{F}(tu),}{d^2t}_{|t=0}, \tag{7.2}$$

expressed in terms of the concave functional $\mathcal{F}(u)$ defined by the following limit:

$$\mathcal{F}(u) := \lim_{N\to\infty} \mathcal{F}^{(N)}(u), \quad \mathcal{F}^{(N)}(u) := -\log \mathbb{E}(e^{-r_N\langle u,\delta_N\rangle}), \tag{7.3}$$

where $\frac{1}{r_N}\log\mathbb{E}(e^{-r_N t\langle u,\delta_N\rangle})$ is thus a scaling of the moment generating function $\log\mathbb{E}(e^{-\langle u,\delta_N\rangle})$ of the random variable $\langle u,\delta_N\rangle$. The existence of the limit above follows from the LDP (by Varadhan's lemma [36]) and the functional $\mathcal{F}$ on $C^0(X)$ coincides with the Legendre-Fenchel transform of the rate functional $E(\mu)$. For example, by [26], the CLT follows from the LDP under the assumption that $f(t) := \mathcal{F}(tu)$ is real-analytic and the convergence of $\mathcal{F}^{(N)}(tu)$ towards $f(t)$ can be extended to complex valued t (which, in particular, requires the absence of phase transitions at any order, as recalled below).

Conversely, we make the following simple observation:

Proposition 7.1 *If the LDP holds with a speed r_N and a CLT (as in formula 7.1) holds, then the corresponding variance σ_u is given by*

$$\sigma_u = -\lim_{N\to\infty}\frac{d^2\mathcal{F}^{(N)}(tu),}{d^2t}_{|t=0}.$$

Proof If the CLT holds then

$$g^{(N)}(t) := \log\mathbb{E}(e^{-(r_N)^{1/2}(\langle u,\delta_N\rangle-\mathbb{E}\langle u,\delta_N\rangle)}) \to a|t|^2/2$$

in the $C^\infty_{loc}-$topology, where $a\in\mathbb{R}$ is the corresponding variance (by the argument used in the end of the proof of Theorem 1.5). In particular,

$$\frac{d^2g^{(N)}(t)}{d^2t}_{|t=0} \to a.$$

But, $g^{(N)}(t) = -r_N f^{(N)}(r_N^{-1/2}t) + \mathbb{E}(\langle u,\delta_N\rangle)t$ and hence $\frac{d^2g^{(N)}(tu)}{d^2t}_{|t=0}$ coincides with $-\frac{d^2f^{(N)}(t)}{d^2t}_{|t=0}$, which concludes the proof. □

In the present setting the LDP for the laws of the empirical measure is established in [18] at a speed

$$r_N = kN_k$$

and the corresponding functional $\mathcal{F}$ (formula 7.3) may be expressed as

$$\mathcal{F}(u) = \mathcal{E}((\phi+u)_e),$$

where $\mathcal{E}$ is a primitive of complex Monge–Ampère operator, i.e. for any smooth weight ϕ and smooth function u

$$\frac{\mathcal{E}((\phi + tu))}{dt}_{|t=0} = \frac{1}{n!} \int_X (dd^c \phi)^n u$$

Moreover, by [17, Thm B], the functional $\mathcal{F}$ is Gateaux differentiable on $C^0(X)$ and its differential at ϕ is represented by the corresponding equilibrium measure, i.e. for any $u \in C^0(X)$

$$\frac{d\mathcal{F}(tu)}{dt}_{|t=0} = \frac{1}{n!} \int_X (dd^c \phi_e)^n u \tag{7.4}$$

Since the linear statistic $\mathcal{N}[u]$ is given by

$$\mathcal{N}[u] := N \langle u, \delta_N \rangle$$

and $N \sim k^n$ the general discussion above thus suggests that, under suitable assumptions, a CLT of the following form should hold:

$$N^{-(1-1/n)/2}(\mathcal{N}[u] - \mathbb{E}(\mathcal{N}[u]) \to \mathcal{N}(0, \sigma_u),$$

which is thus consistent with the CLT in Theorem 1.5 and Corollary 1.6.

Remark 7.2 As shown in [18], the LDP in the present setting follows from the asymptotics (7.3) (established in the present setting in [17, Thm A]) together with the Gärtner-Ellis theorem, using the differentiability of $\mathcal{F}$. The corresponding rate functional E on $\mathcal{P}(X)$ may then be defined as the Legendre-Fenchel transform on $\mathcal{P}(X)$ of the functional $\mathcal{F}$ and the differentiability of $\mathcal{F}$ corresponds to the strict convexity of E (on the convex subset $\{E < \infty\} \subset \mathcal{P}(X)$). In fact, the LDP in [18] holds in the very general setting where μ has the property that (ϕ, μ) satisfies the Bernstein–Markov property for any continuous weight ϕ (i.e. the corresponding one-point correlation density has sub-exponential growth). In particular, this is the case in the purely real setting where $X = \mathbb{R}^n$ and ϕ has super logarithmic growth.

7.2 Relations to Phase Transitions

In the present setting the probability measure $\mu^{(N)}$ on X^N may be represented as the Gibbs measure

$$\mu^{(N)} := \frac{e^{-\beta E^N}}{Z_N[\phi]} \mu_0^{\otimes N}, \quad Z_N[\phi] := \int_{X^N} e^{-\beta E^N} \mu^{\otimes N}$$

at inverse temperature $\beta = 2$, of the Hamiltonian

$$E^{(N)} := -\log |\det(S)(x_1, \ldots, x_N)|_{k\phi}$$

where $Z_N[\phi]$ is the corresponding partition function (see Remark 6.3). Accordingly, the scaled moment generating function may, in the terminology of statistical mechanics, be represented as a difference of scaled *free energies*:

$$\frac{1}{r_N}\log \mathbb{E}(e^{-r_N t\langle u,\delta_N\rangle}) = \frac{1}{kN_k}\log Z_N[\phi+tu] - \frac{1}{kN_k}\log Z_N[\phi].$$

The limiting functional $\mathcal{F}(u)$ can thus be viewed as the thermodynamical free energy functional, describing the leading asymptotics of the $N-$dependent free energies $\mathcal{F}^{(N)}(u)$, as $N\to\infty$. We recall that, according to Ehrenfest's classical classification of phase transitions, a system is said to exhibit a *phase transition of order m* when the m th derivative of the thermodynamical free energy has a discontinuity when considering variations of the thermodynamical variable in question (assuming that the lower order derivatives exist and are continuous). In the present setting the thermodynamical variable is the function u defining the linear statistic and we have the following

Proposition 7.3 *Given a smooth bounded function $u\in C^0(X)$ the thermodynamical free energy $t\mapsto \mathcal{F}(tu)$ has continuous first order derivatives. Moreover, the right and left second order derivatives exist at $t=0$ and are given by*

$$\frac{d^2\mathcal{F}(tu),}{d^2t}_{|t=0^\pm} = \frac{1}{(n-1)!}\int v_\pm dd^c u\wedge (dd^c\phi_e)^{n-1} \tag{7.5}$$

where the right and left derivatives

$$v_\pm := \frac{d(\phi+tu)_e}{dt}_{|t=0^\pm} \tag{7.6}$$

exist, defining bounded functions on X.

Proof As recalled above the existence of the *first* order derivatives when X is compact is the content of [17, Thm B] and the superlogarithmic setting when $X=\mathbb{C}^n$ is shown in [18]. In order to study the second order derivatives first observe that $t\mapsto(\phi+tu)_e(x)$ is concave (indeed it is defined as the sup of linear functions). In particular, it is locally Lipschitz continuous and hence the right and left derivatives $v_\pm$, at $t=0$, indeed exist and are in L^∞. Now, fixing $t\neq 0$ and setting $\psi_t := (\phi+tu)_e$ we have, by formula 7.4,

$$\frac{d\mathcal{F}(tu)}{dt} - \frac{d\mathcal{F}(0)}{dt} = \int_X u\left((dd^c\psi_t)^n - (dd^c\psi_0)^n\right)/n!.$$

Expanding the bracket and integrating by parts this means that

$$t^{-1}\left(\frac{d\mathcal{F}(tu)}{dt} - \frac{d\mathcal{F}(0)}{dt}\right) = \int_X dd^c u\wedge t^{-1}(\psi_t-\psi_0)\left((dd^c\psi_t)^{n-1}...+(dd^c\psi_0)^{n-1}\right)/n!.$$

By the regularity results in [13, 16] $dd^c\psi_t$ is a $L^\infty-$current which is uniformly bounded in t (for bounded t) and by concavity the left and right limits $v_\pm$ of $t^{-1}(\psi_t - \psi_0)$ as $t \to 0^\pm$ exist and are monotonic in t. Hence, applying the dominated convergence theorem proves formula 7.5. □

This means that there is an absence of first order phase transitions in the present setting. In the light of the discussion in the previous section it is tempting to speculate that the linear statistic corresponding to a smooth bounded function u on X satisfies a CLT, as in formula, if one assumes that $\frac{d(\phi+tu)_e}{dt}_{|t=0}$ exists, i.e.

$$v_+ = v_-$$

(perhaps with additional regularity assumptions) and that the limit σ_u of the scaled variances $N^{1/n-1}\mathrm{Var}\mathcal{N}(u)$ is then given by

$$\frac{d^2\mathcal{F}(tu)}{d^2t}_{|t=0} = -\frac{1}{(n-1)!}\int \frac{d(\phi+tu)_e}{dt}_{|t=0^\pm} dd^c u \wedge (dd^c\phi_e)^{n-1} \tag{7.7}$$

In the case when u is supported in the interior of the bulk this is consistent with Theorem 1.5. Indeed, then $v_\pm = u$ and an integration by parts thus reveals that the integral above coincides with the variance in question. The speculation above is also consistent with the results in [4, 7, 54] concerning the setting of super logarithmic growth in $\mathbb{C}$. Indeed, in the most general results appearing in [7, 54] it is, in particular, assumed that $\Delta\phi > 0$ on a neighborhood of the support S and that the boundary of the support has no singular points (cusps) in the sense of [27]. Under these assumptions it can be shown that $\frac{d(\phi+tu)_e}{dt}_{|t=0}$ exists and is given by the function $\tilde{u}$ defined as u on S and on $X - S$ as the harmonic extension of u. The point is that, assuming that the support S_{ϕ_t} varies continuously with t, the following holds in the complement of S :

$$0 = \frac{d\mu_{\phi_t}}{dt}_{|t=0} = dd^c\frac{d(\phi+tu)_e}{dt}_{|t=0}$$

In particular, one then has

$$\frac{d^2\mathcal{F}(tu),}{d^2t}_{|t=0^\pm} = -\int_X \tilde{u}dd^c\tilde{u} = \int_X d\tilde{u}\wedge d^c\tilde{u},$$

which indeed coincides with the formula for the variance in [4, 7, 54]. It would be very interesting to extend the CLTs in [4, 7, 54] to higher dimensions $n > 1$ and show that the limiting variance is given by formula 7.7. Under the regularity assumption that ϕ_e admits a Monge–Ampère foliation by Riemann surfaces in the complement S^c the role of $\tilde{u}$ is then played by the extension of u which is harmonic along the leaves $\mathcal{L}_\alpha$ of the foliation and

$$\frac{d^2\mathcal{F}(tu),}{d^2t}_{|t=0^\pm} = -\int d\alpha \int_{\mathcal{L}_\alpha} d\tilde{u} \wedge d^c\tilde{u},$$

i.e. a certain superposition of the Dirichlet norms of $\tilde{u}$ along the leaves. Even though the regularity assumption used above is rather strong (in general it holds if $\phi_e \in C^3_{loc}(S^c)$ and $(dd^c\phi_e)^{n-1}$ is of rank $n-1$ in S^c) there are certainly particular geometrically settings where it is satisfied. For example, it applies in the setting of [65] and in the equivariant settings in [61, 64, 78].

Even if the limit of the scaled variances $N^{1/n-1}\text{Var}\mathcal{N}(u)$ may not exist for a general strongly regular weighted measure (ϕ, ω_n) it seems natural to expect that the sequence is always bounded. By Lemma 6.2 this would follow from the validity of the following

Conjecture 1 *Given a strongly regular weighted measure (ϕ, μ) there exists a constant C such that*

$$\frac{1}{2}\int_{X\times X} k^{-(n-1)}\,|K_k(x,y)|^2_{k\phi}\, d(x,y)^2\mu\otimes\mu \leq C,$$

where $d(x,y)$ is the distance function corresponding to a given metric on X.

In the "real setting", i.e. case when μ is supported on a real algebraic variety (or on $X := \mathbb{R}^n$ in the super logarithmic setting) the estimate in the previous conjecture was established in [12] (in the case $X = \mathbb{R}$ with ϕ real analytic the estimate is shown in [59]). Moreover, by the second item in Proposition 6.4 a weaker form of the conjecture holds, where the constant C is replaced by $o(k)$.

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Probability Measures Associated to Geodesics in the Space of Kähler Metrics

Bo Berndtsson

Abstract We associate certain probability measures on $\mathbb{R}$ to geodesics in the space $\mathcal{H}_L$ of positively curved metrics on a line bundle L, and to geodesics in the finite dimensional symmetric space of hermitian norms on $H^0(X, kL)$. We prove that the measures associated to the finite dimensional spaces converge weakly to the measures related to geodesics in $\mathcal{H}_L$ as k goes to infinity. The convergence of second order moments implies a recent result of Chen and Sun on geodesic distances in the respective spaces, while the convergence of first order moments gives convergence of Donaldson's Z-functional to the Aubin–Yau energy. We also include a result on approximation of infinite dimensional geodesics by Bergman kernels which generalizes work of Phong and Sturm.

1 Introduction

Let X be a compact Kähler manifold and L an ample line bundle over X. If ϕ is a hermitian metric on L with positive curvature, then

$$\omega_\phi := i\partial\bar{\partial}\phi$$

is a Kähler metric on X with Kähler form in the Chern class of L, $c(L)$, and we let $\mathcal{H}_L$ denote the space of all such Kähler potentials. By the work of Mabuchi, Semmes and Donaldson (see [10, 13, 18]), $\mathcal{H}_L$ can be given the structure of an infinite dimensional, negatively curved Riemannian manifold, or even symmetric space. With this space one can associate certain finite dimensional symmetric spaces in the following way. Take a positive integer k and let V_k be the vector space of global holomorphic sections of kL,

$$V_k = H^0(X, kL).$$

B. Berndtsson (✉)
Department of Mathematics, Chalmers University of Technology, S-412 96 Göteborg, Sweden
e-mail: bob@chalmers.se

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(Later we shall consider more generally vector spaces $H^0(X, kL + F)$ where F is a fixed line bundle, but for simplicity we omit F in this introduction.) The finite dimensional symmetric spaces in question are then the spaces $\mathcal{H}_k$ of hermitian norms on V_k.

There are for any k natural maps

$$FS = FS_k : \mathcal{H}_k \mapsto \mathcal{H}_L,$$

and

$$Hilb = Hilb_k : \mathcal{H}_L \mapsto \mathcal{H}_k,$$

and a basic idea in the study of Kähler metrics on X with Kähler form in $c(L)$ is that under these maps the finite dimensional spaces $\mathcal{H}_k$ should approximate $\mathcal{H}_L$ as k goes to infinity. This will be explained a bit more closely in the next section of this paper, see also [7, 10, 14] for excellent backgrounds to these ideas.

The most basic result in this direction is the result of Bouche, [3] and Tian, [21] that for ϕ in $\mathcal{H}_L$

$$\phi_k := FS_k \circ Hilb_k(\phi)$$

tends to ϕ together with its derivatives. It is natural to ask whether geodesics between points in $\mathcal{H}_L$ also can be approximated in some sense by geodesics coming from the finite dimensional picture. This question was first raised by Arezzo and Tian in [1], and then treated by Phong and Sturm in [14], where it is proved that any geodesic in $\mathcal{H}_L$ is a limit of FS_k of geodesics in $\mathcal{H}_k$, in an 'almost uniform way' (see below for their precise statement). Later, this result has been refined in particular cases (like toric varieties) to give convergence of derivatives as well by Song-Zelditch, Rubinstein-Zelditch and Rubinstein, see [16, 17, 20]. (These works also treat more general equations than the geodesic equation.)

In a recent very interesting paper, [7], Chen and Sun have shown that moreover if ϕ_0 and ϕ_1 are two Kähler potentials in $\mathcal{H}_L$, then the geodesic distance, suitably normalized, between $Hilb_k(\phi_0)$ and $Hilb_k(\phi_1)$ in $\mathcal{H}_k$ tends to the geodesic distance between ϕ_0 and ϕ_1 in $\mathcal{H}_L$. Hence $\mathcal{H}_k$ approximates $\mathcal{H}_L$ as metric spaces in this sense.

In this paper we associate to geodesics, in $\mathcal{H}_k$ and $\mathcal{H}_L$ respectively, certain probability measures on $\mathbb{R}$ from which many quantities related to the geodesic (like length, energy) can be recovered. The main result of the paper is that the measures associated to geodesics in $\mathcal{H}_k$ converge to their counterparts in $\mathcal{H}_L$ in the weak*-topology as k goes to infinity. It follows that their moments converge, which applied to second order moments implies the result of Chen and Sun on convergence of geodesic distance.

Let H_k^0 and H_k^1 be two points in $\mathcal{H}_k$, and let H_k^t be the geodesic in $\mathcal{H}_k$ connecting them. The tangent vector to this geodesic

$$A_{t,k} := (H_k^t)^{-1} \dot{H}_k^t$$

is then an endomorphism of V_k. The geodesic condition means that it is actually independent of t so we will omit the t in the subscript. Since A_k is hermitian for the scalar products in the curve all its eigenvalues are real. Let $\nu_k = \nu_{A_k}$ be the normalized spectral measure of $k^{-1}A_k$. By this we mean that

$$\nu_k = d_k^{-1} \sum \delta_{\lambda_j},$$

where λ_j are the eigenvalues of $k^{-1}A_k$ and d_k is the dimension of V_k, so that ν_k are probability measures on $\mathbb{R}$.

The second order moment of ν_k is precisely the norm squared of the vector A_k in the tangent space of $\mathcal{H}_k$, divided by d_k. Since this is independent of t and t goes from 0 to 1, the second order moment equals the square of the normalized geodesic distance between H_k^0 and H_k^1. We shall also see in Sect. 2 that the first order moment of ν_k equals the Donaldson functional

$$Z(H_k^0, H_k^1)/d_k$$

from [11].

We next describe the corresponding objects for the infinite dimensional space $\mathcal{H}_L$. Let ϕ_0 and ϕ_1 be two points in $\mathcal{H}_L$ and let ϕ_t be the Monge–Ampère geodesic joining them. By this we mean that ϕ_t is a curve of positively curved metrics on L for t between 0 and 1. We extend the definition of ϕ_t to complex t in

$$\Omega := \{0 < \operatorname{Re} t < 1\}$$

by letting it be independent of the imaginary part of t. The geodesic equation is then

$$(i\partial\bar{\partial}\phi_t)^{n+1} = 0$$

on $\Omega \times X$.

It was proved by Chen and Błocki, in [5, 6], that such a geodesic always exists and is of class $C^{1,1}$ in the sense that all (1, 1)-derivatives are uniformly bounded. Recently it has also been proved by Lempert and Vivas that in general one can not find a classical geodesic that is smooth up to the boundary, see [12]. A 'geodesic in $\mathcal{H}_L$' is therefore not necessarily a curve in $\mathcal{H}_L$ (which consists of smooth metrics), but we will adhere to the common terminology nevertheless. For each t fixed we can now define a probability measure on $\mathbb{R}$ in the following way. Let first dV_t be the normalized volume measure on X induced by ω_{ϕ_t},

$$dV_t := (\omega_{\phi_t})^n / Vol,$$

where Vol is the volume of X

$$Vol = \int_X c(L)^n.$$

Since $\dot{\phi}_t$ is a continuous real valued function, we can consider the direct image (or 'pushforward') of dV_t

$$\mu_t = (-\dot{\phi}_t)_*(dV_t) \tag{1.1}$$

so that μ_t is a probability measure on $\mathbb{R}$. Concretely, this means that if f is a continuous function on $\mathbb{R}$, then

$$\int_{\mathbb{R}} f(x)d\mu_t(x) = \int_X f(-\dot{\phi}_t)dV_t.$$

We shall show in the next section that if ϕ_t is a Monge–Ampère geodesic, then $\mu = \mu_t$ is independent of t. This is then the measure that corresponds to the spectral measures ν_k in the infinite dimensional setting, and our main results says that ν_k converge to μ in the weak*-topology as k goes to infinity.

Theorem 1.1 *Let ϕ_0 and ϕ_1 be two points in $\mathcal{H}_L$ and let*

$$H_k^t = Hilb_k(\phi_t)$$

for $t = 0, 1$ be the corresponding norms in $\mathcal{H}_k$. Let for t between 0 and 1 H_k^t be the geodesic in $\mathcal{H}_k$ connecting these two norms and let ν_k be their normalized spectral measures as defined above. Then

$$\nu_k \longrightarrow \mu,$$

in the weak-topology, where $\mu = \mu_t$ is defined in Theorem 1.1 .*

Just like the spectral measures of the endomorphisms A_k contain part of the properties of the corresponding geodesics in $\mathcal{H}_k$, part of the properties of the Monge–Ampère geodesic can be read off from the measure μ. It is for instance immediately clear that the second order moment of μ is equal to

$$\int_X \dot{\phi}_t^{\,2} dV_t/Vol$$

which is the length square of the tangent vector to the Monge–Ampère geodesic (which is independent of t as it should be). Since the parameter interval is from 0 to 1 the length of the tangent vector is the length of the geodesic from ϕ_0 to ϕ_1. By a theorem of Chen, [6], the length of the geodesic is equal to the geodesic distance, so the convergence of second order moments implies the theorem of Chen and Sun, [7] that normalized geodesic distance in $\mathcal{H}_k$ converges to geodesic distance in $\mathcal{H}_L$. Similarly we shall see in the next section that the first order moment of μ is the Aubin–Yau energy of the pair ϕ_0 and ϕ_1, and convergence of first order moments therefore says that the Aubin–Yau energy is the limit of Donaldson's Z-functional (this is a much simpler result).

The proof of our main result is given in Sect. 3; it is based on the curvature estimates from [2]. The basic idea is as follows: The Monge–Ampère geodesic ϕ_t induces a certain curve of norms in $\mathcal{H}_k$, $H_{\phi_t,k}$. These are L^2-norms on the space of global sections, similar to the curves $Hilb_k(\phi_t)$ but defined slightly differently to fit with the results of [2]. At the end points, $t = 0, 1$,

$$H_{\phi_t,k} = H_k^t := Hilb_k(\phi_t),$$

and we define H_k^t for t between 0 and 1 to be the geodesic in $\mathcal{H}_k$ between these endpoint values. The main result of [2] immediately implies that

$$H_{\phi_t,k} \geq H_k^t$$

for t between 0 and 1, and by definition equality holds at the endpoints. Let

$$T_{t,k} := H_{\phi_t,k}^{-1} \dot{H}_{\phi_t,k}$$

Differentiating with respect to t at $t = 0, 1$ we then get that

$$\langle A_k u, u \rangle_{H_k^0} \leq \langle T_{0,k} u, u \rangle_{H_k^0}$$

and

$$\langle A_k u, u \rangle_{H_k^1} \geq \langle T_{1,k} u, u \rangle_{H_k^1}$$

This means that we get estimates for the tangent vector to the finite dimensional geodesic in terms of certain operators on V_k defined by the Monge–Ampère geodesic. These operators are Toeplitz operators on V_k with symbol $\dot{\phi}_t$, $t = 0, 1$ and their spectral measures are essentially known to converge to $\mu_t = \mu$. Since A_k is pinched between these two operators it is not hard to see that the spectral measures of A_k have the same limit, which proves the theorem.

In a final section we will give a result on the uniform convergence of FS_k of finite dimensional geodesics to Monge–Ampère geodesics, generalizing the work of Phong-Sturm mentioned earlier. This result is only a small variation of Theorem 6.1 from [2], but it has as a consequence the following theorem which is more natural than Theorem 6.1 in [2] so it seems good to state it explicitly.

Theorem 1.2 *Let ϕ_0 and ϕ_1 be two Kähler potentials in $\mathcal{H}_L$ and let ϕ_t be the Monge–Ampère geodesic joining them. Let*

$$H_k^t = Hilb_k(\phi_t)$$

for $t = 0, 1$ and let H_k^t for t between 0 and 1 be the geodesic in $\mathcal{H}_k$ between these two points. Let finally

$$B_{t,k} := FS_k(H_k^t)$$

for $0 \leq t \leq 1$. *Then*

$$\sup |k^{-1} \log B_{t,k} - \phi_t| \leq C \frac{\log k}{k}.$$

This theorem answers affirmatively a question raised by Arezzo and Tian in [1] (question 2 in that paper). It strengthens the main result of Phong and Sturm, [14], who proved that

$$\lim_{l \to \infty} (\sup_{k \geq l} k^{-1} \log B_{t,k})^* = \phi_t$$

uniformly, where u^* means the upper semicontinuous regularization of a function u.

The final parts of this work (the most important parts!) were carried out during the conference on extremal Kähler metrics at BIRS June-July 2009. I am grateful to the organizers for a very stimulating conference. I would also like to thank Jian Song for suggesting that my curvature estimates might be relevant in connection with the Chen-Sun theorem and for encouraging me to write down the details of the proof of Theorem 1.2. Finally I am grateful to Xiuxiong Chen and Song Sun for explaining me their result and to the referee for helpful comments.

2 Background and Definitions

In the first subsection we will give basic facts about the space $\mathcal{H}_L$ and its finite dimensional 'quantizations'. Since this material is well known (see e.g. [10, 14] or [7]) we will be brief and emphazise a few particularities that are relevant for this paper.

2.1 $\mathcal{H}_L$, $\mathcal{H}_k$ and Its Variants

Let L be an ample line bundle over the compact manifold X. $\mathcal{H}_L$ is the space of all smooth metrics ϕ on L with

$$\omega_\phi := i\partial\bar{\partial}\phi > 0.$$

$\mathcal{H}_L$ is an open subset of an affine space and its tangent space at each point equals the space of smooth real valued functions on X. The Riemannian norm on this tangent space at the point ϕ is the L^2-norm

$$\|\psi\|^2 = \int_X |\psi|^2 \omega_\phi^n / Vol.$$

A geodesic in $\mathcal{H}_L$ is a curve ϕ_t for $a < t < b$ that satisfies the geodesic equation

$$\frac{d^2}{dt^2}\phi_t = |\bar\partial \frac{d}{dt}\phi_t|^2_{\omega_{\phi_t}}. \tag{2.1}$$

It is useful to extend the definition of ϕ_t to complex values of t in the strip

$$\Omega = \{t; a < \mathrm{Re}\, t < b\}$$

by taking it to be independent of the imaginary part of t. Then (2.1) can be written equivalently on complex form

$$c(\phi_t) := \ddot\phi_{t\bar t} - |\bar\partial \dot\phi_t|^2_{\omega_{\phi_t}} = 0,$$

where $\dot\phi_t = \partial\phi_t/\partial t$. On the other hand the expression $c(\phi_t)$ is related to the Monge–Ampère operator through the formula

$$c(\phi_t) i dt \wedge d\bar t \wedge (\omega_{\phi_t})^n = (i\partial\bar\partial\phi_t)^{n+1}/(n+1),$$

where on the right hand side we take the $\partial\bar\partial$-operator on $\Omega \times X$. Geodesics in $\mathcal{H}_L$ are therefore given by solutions to the homogeneous Monge–Ampère equation that are independent of $\Im t$. Notice that a geodesic will automatically satisfy

$$i\partial\bar\partial\phi_t \geq 0,$$

and we shall refer to any curve with this property as a 'subgeodesic' even though this term has no meaning in Riemannian geometry in general.

A fundamental theorem of Chen, with complements of Błocki, [5, 6], says that if ϕ_0 and ϕ_1 are two points in $\mathcal{H}_L$ they can be connected by a geodesic of class $C^{1,1}$, i e such that

$$(i\partial\bar\partial\phi_t)^{n+1} = 0$$

and

$$\partial\bar\partial\phi_t$$

has bounded coefficients.

One associates with $\mathcal{H}_L$ the vector spaces

$$V_k := H^0(X, kL)$$

of global holomorphic sections of kL for k positive integer. A metric ϕ in $\mathcal{H}_L$ is mapped to a hermitian norm $Hilb_k(\phi)$ on V_k by

$$\|u\|^2_{Hilb_k(\phi)} := \int_X |u|^2 e^{-k\phi}\omega_\phi^n.$$

It will also be useful for us to consider the vector spaces

$$H^0(X, K_X + kL).$$

A metric ϕ on L also induces an hermitian norm, $H_{k\phi}$ on these spaces through

$$\|u\|^2_{H_{k\phi}} := \int_X |u|^2 e^{-k\phi}. \tag{2.2}$$

An important point is that $|u|^2 e^{-k\phi}$ is a measure on X if u lies in $H^0(X, K_X + kL)$, so the integral of this expression is naturally defined, without the introduction of any extra measure like ω^n_ϕ.

In order to treat both these types of spaces simultaneously we let F be an arbitrary line bundle over X and consider spaces

$$H^0(X, K_X + kL + F).$$

Norms on these spaces are then defined by

$$\|u\|^2_{H_{k\phi+\psi}} := \int_X |u|^2 e^{-k\phi-\psi},$$

where ψ is some metric on F. The two cases we discussed earlier then correspond to $F = -K_X$ and

$$\psi = -\log \omega^n_\phi,$$

and $F = 0$ respectively. In the first case

$$H_{k\phi+\psi} = Hilb_{k\phi}$$

as defined above, but varying ψ we get similar spaces defined by arbitrary smooth volume forms instead of ω^n_ϕ.

Let now V be the space of holomorphic sections of some line bundle, G, over X; it may be any of the choices discussed above, and denote by $\mathcal{H}_V$ the space of hermitian norms on V. For such a hermitian norm, H, let s_j be an orthonormal basis for the space of sections $H^0(X, G)$, and consider the Bergman kernel

$$B_H = \sum |s_j|^2.$$

The absolute values on the right hand side here are to be interpreted with respect to some trivialization of G. When the trivialization changes, $\log B_H$ transforms like a metric on G since

$$|u|^2/B_H$$

is a well defined function if u is a section of G. By definition $FS(H)$ is that metric

$$FS(H) = \log B_H.$$

By the well known extremal characterization of Bergman kernels we have

$$B_H(x) = \sup_{u \in H^0(X,G)} \frac{|u(x)|^2}{\|u\|_H^2}.$$

From this we can conclude that the Bergman kernel is a decreasing function of the metric; if we change the metric to a larger one, the Bergman kernel becomes smaller.

Choosing a basis for V we can represent an element in $\mathcal{H}_V$ by a matrix that we slightly abusively also call H. A curve in $\mathcal{H}_V$ then gets represented by a curve of matrices H^t. Differentiating norms we get

$$\frac{d}{dt}\|u\|_{H^t}^2 = \langle A_t u, u\rangle_{H^t},$$

with

$$A_t = (H^t)^{-1}\frac{d}{dt}H^t.$$

A_t is an endomorphism of V; the tangent vector to the curve H^t. Its norm is

$$\|A_t\|^2 = tr A_t^* A_t.$$

Here the * stands for the adjoint with respect to H, but since A is selfadjoint for this scalar product, the norm of A is the sum of the squares of its eigenvalues.

Finally, the geodesic equation is

$$\frac{d}{dt}A_t = 0.$$

It is easy to see that any two norms in $\mathcal{H}_V$ can be joined by a geodesic. Explicitly, we can find a basis s_j of V which is orthonormal w r t H^0 and diagonalizes H^1 with eigenvalues e^{λ_j}. The geodesic is then represented (in this basis) by the diagonal matrix H^t with eigenvalues $e^{t\lambda_j}$. Hence, $A = A_t$ is diagonalized by the same basis and has eigenvalues λ_j.

Just like in the case of $\mathcal{H}_L$ it is convenient to consider curves H^t defined also for complex values of t in the strip Ω, by letting it be independent of the imaginary part of t. We can then write the geodesic equation equivalently as

$$\frac{\partial}{\partial \bar{t}} H^{-1}\frac{\partial}{\partial t}H = 0.$$

This suggests that the geodesic equation can be thought of as the zero-curvature equation for a certain vector bundle. Let E be the trivial bundle over Ω with fiber V. A curve in $\mathcal{H}_V$ is then the same thing as a vector bundle metric on E, independent of the imaginary part of t, and we see that geodesics correspond to flat metrics on E. In analogy with the case of curves in $\mathcal{H}_L$, we will call curves in $\mathcal{H}_V$ that correspond to vector bundle metrics of semipositive curvature 'subgeodesics' in $\mathcal{H}_V$.

A main role in the sequel is played by Theorem 2.1 in [2]. This theorem implies that if ϕ_t is a subgeodesic in $\mathcal{H}_L$, i e satisfies

$$i\partial\bar{\partial}\phi_t \geq 0,$$

then the induced curve from formula (2.2), H_{ϕ_t}, in $\mathcal{H}_V$ for $V = H^0(X, K_X + L)$ has semipositive curvature, so it is a subgeodesic in $\mathcal{H}_V$.

Since metrics with semipositive curvature are greater than flat metrics having the same boundary values, this gives us a way of comparing L^2-norms on V induced by (sub)geodesics in $\mathcal{H}_L$ to finite dimensional geodesics in $\mathcal{H}_V$ (cf Proposition 3.1).

2.2 Measures Defined by Geodesics

Let us start with the case of a finite dimensional geodesic, H^t, in $\mathcal{H}_V$. As we have seen in the previous subsection it can be represented by a diagonal matrix with diagonal elements $e^{t\lambda_j}$ in a suitable basis, and its tangent vector A is then diagonal with diagonal elements λ_j. The measure we associate to the geodesic is then the (normalized) spectral measure of A

$$\nu_A = \frac{1}{d}\sum \delta_{\lambda_j},$$

with d the dimension of V. This is defined in terms of eigenvalues of the endomorphism A so it does not depend on the basis we have chosen.

Recall that for any pair of norms in $\mathcal{H}_V$, Donaldson [11] has defined a quantity

$$Z(H^1, H^0) = \log \frac{\det H^1}{\det H^0}$$

(the determinant is the determinant of a matrix representing the norm in some basis, but since we consider quotients of determinants, Z does not depend on which basis). Then

$$\frac{d}{dt} Z(H^t, H^0) = tr A.$$

Hence we see that, since A is constant and we have chosen our parameter interval to be [0, 1], that

$$\int_{\mathbb{R}} x d\nu_A = tr A/d = Z(H^1, H^0)/d$$

so first moments of the spectral measure gives the Donaldson Z-functional. Second order moments are

$$\int_{\mathbb{R}} x^2 d\nu_A = tr A^2/d = \|A\|^2/d$$

which in the same way equals the square of the geodesic distance from H^0 to H^1, again divided by d.

We next turn to the corresponding construction for $\mathcal{H}_L$. Let ϕ_t be a curve in $\mathcal{H}_L$ and to fix ideas we think of t as real now. We first assume that ϕ_t is smooth and denote by

$$\dot{\phi}_t = \frac{d\phi_t}{dt}$$

the tangent vector (a smooth function on X). For ease of notation we also set

$$\omega_t = \omega_{\phi_t}.$$

Lemma 2.1 *Let f be a compactly supported function on $\mathbb{R}$ of class C^1. Then*

$$\frac{d}{dt}\int_X f(\dot{\phi}_t)\omega_t^n = \int_X f'(\dot{\phi}_t)c(\phi_t)\omega_t^n.$$

Proof This is just a simple computation.

$$\frac{d}{dt}\int_X f(\dot{\phi}_t)\omega_t^n = \int f'(\dot{\phi}_t)\frac{d^2\phi_t}{dt^2}\omega_t^n + n\int_X f(\dot{\phi}_t)i\partial\bar{\partial}\dot{\phi}_t \wedge \omega_t^{n-1}.$$

By Stokes' theorem applied to the last term this equals

$$\int f'(\dot{\phi}_t)\frac{d^2\phi_t}{dt^2}\omega_t^n - n\int_X f'(\dot{\phi}_t)i\partial\dot{\phi}_t \wedge \bar{\partial}\dot{\phi}_t \wedge \omega_t^{n-1} = \int_X f'(\dot{\phi}_t)c(\phi_t)\omega_t^n.$$

□

Since for smooth geodesics $c(\phi_t) = 0$ it follows that the integrals

$$\int_X f(\dot{\phi}_t)\omega_t^n$$

do not depend on t. By approximation we can draw the same conclusion for (say) geodesics of class C^1.

Proposition 2.2 *Let ϕ_t be a curve of metrics on L with semipositive curvature which is of class C^1 and satisfies*

$$(i\partial\bar{\partial}\phi_t)^{n+1} = 0$$

in the sense of currents. Then the integrals

$$\int_X f(\dot{\phi}_t)\omega_t^n$$

do not depend on t.

Proof Let K be a compact in Ω. We can then approximate ϕ_t over $K \times X$ by smooth metrics ϕ_t^ϵ such that

$$i\partial\bar{\partial}\phi_t^\epsilon \geq 0$$

and

$$\int_{K\times X} (i\partial\bar{\partial}\phi_t^\epsilon)^{n+1}$$

tends to 0. In fact, the approximation can be carried out locally by convolution and then patched together with a partition of unity - the patching causes no problem if the initial metric is of class C^1. The proposition then follows from the lemma. □

Remark: As pointed out in [8], this proposition follows from well known facts if the geodesic is smooth and $\omega_t > 0$ for all t. In that case the geodesic defines a foliation by graphs of holomorphic functions from the parameter space to X, along which ϕ_t is harmonic. Following the leaves of the foliation we get maps F_t from X to itself that depend holomorphically on t and satisfy $F_t^*(\omega_t) = \omega_0$ (i.e. they are symplectomorphisms that carry the symplectic forms ω_t to ω_0). Moreover, $F_t^*(\dot{\phi}_t) = \dot{\phi}_0$, from which it follows directly $(\dot{\phi}_t)_*(\omega_t) = (\dot{\phi}_0)_*(\omega_0)$. In the special case when $t \to F_t$ is the flow of a holomorphic vector field V one can also interpret $\dot{\phi}_0$ as the Hamiltonian of the imaginary part of V, and $(\dot{\phi}_0)_*(\omega_0)$ is then the moment measure of the imaginary part of V. □

For a C^1-geodesic we now consider the normalized volume measures on X

$$dV_t = \omega_t^n / Vol$$

where

$$Vol = \int_X c(L)^n$$

is the volume of X, and their direct image measures under the map $-\dot{\phi}_t$

$$d\mu_t = (-\dot{\phi}_t)_*(dV_t).$$

These are probability measures on $\mathbb{R}$, supported on a compact interval $[-M, M]$, $M = \sup |\dot{\phi}_t|$ and concretely defined by

$$\int_{\mathbb{R}} f(x)d\mu_t(x) = \int_X f(-\dot{\phi}_t)\omega_t^n / Vol.$$

By the proposition, they do in fact not depend on t, so $d\mu = d\mu_t$ is a fixed probability measure on $\mathbb{R}$ associated to the given geodesic.

Recall that the Aubin–Yau energy of a pair of metrics in $\mathcal{H}_L$ is defined in the following way:

$$\frac{d}{dt}\mathcal{E}(\phi_t, \phi_0) = -\int_X \dot{\phi}_t \omega_t^n,$$

and $\mathcal{E}(\phi_0, \phi_0) = 0$. From this we see that the first order moment of $d\mu$

$$\int x d\mu(x) = -\int_X \dot{\phi}_t \omega_t^n / Vol,$$

is preciseley the derivative of the Aubin–Yau energy, which is constant for a geodesic, and hence equal to the Aubin–Yau energy itself if the parameter interval is (0, 1). This corresponds to the relation between the measures $d\nu_k$ and the Donaldson Z-functional, and Theorem 1.1 in this case is just the familiar convergence of the Z-functionals to the Aubin–Yau energy. Similarly, the second order moments

$$\int x^2 d\mu(x) = \int_X (\dot{\phi}_t)^2 \omega_t^n / Vol,$$

is the length of the tangent vector to ϕ_t squared, so second order moments give geodesic distances. Notice finally that the proposition implies that all L^p-norms of $\dot{\phi}_t$ are constant along the curve, hence also the L^∞-norm. More precisely, since $\sup(-\dot{\phi}_t)$ is the supremum of the support of μ it follows that $\inf \dot{\phi}_t$ (and $\sup \dot{\phi}_t$) are constant (where we mean *essential* sup and inf).

Remark Notice also that if we define the measures in the same way when ϕ_t is a subgeodesic, then the integrals

$$\int_{\mathbb{R}} f(x) d\mu_t(x)$$

increase with t if f is an increasing function. Intuitively, the measures μ_t move to the right as t increases.

3 The Convergence of Spectral Measures

We first state a consequence of the main result from [2]. In the statement of the proposition we shall use the notation

$$\|u\|^2_{H_\phi} = \int_X |u|^2 e^{-\phi}$$

for the hermitian norm on $H^0(X, L + K_X)$ defined by a metric ϕ on L.

Proposition 3.1 *Let L be an ample line bundle over X and let ϕ_t for $t = 0, 1$ be two elements of $\mathcal{H}_L$. Let for $t = 0, 1$ H^t be the norms H_{ϕ_t} on $H^0(X, L + K_X)$ defined by ϕ_0 and ϕ_1. Let for t between 0 and 1 H^t be the geodesic in the space of metrics on $H^0(X, L + K_X)$ joining H^0 and H^1. Let finally ϕ_t be any smooth subgeodesic in $\mathcal{H}_L$ connecting ϕ_0 and ϕ_1, i e any metric with nonnegative curvature on L over $X \times \Omega$, smooth up to the boundary. Then*

$$H^t \leq H_{\phi_t}. \tag{3.1}$$

Proof If we regard H^t and H_{ϕ_t} as vector bundle metrics on the trivial vector bundle over Ω with fiber $H^0(X, L + K_X)$, then Theorem 2.1 of [2] implies that the second of these metrics has nonnegative curvature. On the other hand the first metric has zero curvature since H^t is a geodesic . Since the two metrics agree over the boundary a comparison lemma from [15] or [18] gives inequality (3.1). □

We have been a little bit vague about what 'smoothness' means in the proposition. The proof of Theorem 2.1 in [2] requires at least C^2-regularity, but we claim that C^1 regularity is sufficient in the proposition, which can be seen from regularization of the metric (this can be done locally with the aid of a partition of unity in the case that the metric is C^1 from the start). This means that we can (and will) apply the proposition to Monge–Ampère geodesics of class $C^{1,1}$.

The next step is to differentiate the inequality (3.1) for $t = 0, 1$ (recall that equality holds at the endpoints). If u lies in $H^0(X, L + K_X)$ we get

$$\frac{d}{dt}\|u\|^2_{H^t} = \langle A_t u, u\rangle_{H^t},$$

where

$$A_t = (H^t)^{-1}\dot{H}^t.$$

Since H^t is a geodesic, $A_t = A$ is independent of t. The derivative of the right hand side of (3.1) is

$$\frac{d}{dt}\|u\|^2_{H_{\phi_t}} = \langle T_t u, u\rangle_{H_{\phi_t}},$$

where T_t is the Toeplitz operator on $H^0(X, L + K_X)$ defined by

$$\langle T_t u, u\rangle_{H_{\phi_t}} = -\int_X \dot{\phi}_t |u|^2 e^{-\phi_t}.$$

Since by Proposition 3.1

$$\|u\|_{H^t} \leq \|u\|_{H_{\phi_t}}$$

with equality for $t = 0$ it follows that

$$\langle A_0 u, u\rangle_{H^0} = \frac{d}{dt}|_{t=0}\|u\|_{H^t} \leq \frac{d}{dt}|_{t=0}\|u\|_{H_{\phi_t}} = \langle T_0 u, u\rangle_{H^0},$$

which means that

$$A = A_0 \leq T_0 \tag{3.2}$$

as operators on the space $H^0(X, L + K_X)$ equipped with the Hilbert norm H^0.

Since equality between the norms also holds for $t = 1$, we get in a similar way

$$A \geq T_1 \tag{3.3}$$

as operators on the space $H^0(X, L + K_X)$ equipped with the Hilbert norm H^1.

We are now going to apply these estimates to multiples kL of the bundle L, but in order to accommodate also the spaces $H^0(X, kL)$ and L^2-metrics of the form

$$\int_X |u|^2 e^{-k\phi} dV,$$

where dV is a smooth volume form, we need to generalize the set up first. Let therefore F be an arbitrary line bundle over X and consider line bundles of the form

$$K_X + F + kL.$$

The main examples will be $F = 0$ and $F = -K_X$, and the reader may find it convenient to focus on the case $F = 0$ first, in which case the argument below is easier, at least notationally. Put now

$$V_k = H^0(X, kL + F + K_X).$$

Fix two metrics ϕ_0 and ϕ_1 in $\mathcal{H}_L$. Let χ be some fixed metric on L considered as a bundle over $X \times \bar{\Omega}$, i e a curve of metrics χ_t for t in $\bar{\Omega}$. Assume that its curvature is bounded from below by a positive constant, so that

$$i\partial\bar{\partial}_{X,t} \geq c(\omega_{\phi_0} + idt \wedge d\bar{t}),$$

that $\chi_t = \phi_t$ for t equal to 0 and 1, and that finally χ_t depends only on Re t. Such a metric χ can be found on the form

$$t\phi_1 + (1-t)\phi_0 + \kappa(\text{Re } t)$$

where κ is a sufficiently convex function on the interval (0, 1) which equals 0 at the endpoints.

Let also ψ be an arbitrary metric on F considered as a bundle over $X \times \bar{\Omega}$, independent of $\Im t$, not necessarily with positive curvature, but smooth up to the boundary. Choose a fixed positive constant a, sufficiently large so that

$$ai\partial\bar{\partial}\chi + i\partial\bar{\partial}\psi \geq 0.$$

We next consider the vector spaces

$$H^0(X, K_X + F + kL)$$

with the induced L^2-metrics

$$\|u\|_{k,t}^2 := \int_X |u|^2 e^{-(k-a)\phi_t - a\chi_t - \psi_t}.$$

Notice that the metric on the line bundle $F + kL$ that we use here, $(k-a)\phi + a\chi + \psi$ has been chosen so that it has nonnegative curvature, meaning that we can apply the results from (3.1), (3.2) and (3.3). We denote the Toeplitz operators arising from differentiation of the norms at $t = 0$ and $t = 1$ by $T_{0,k}$ and $T_{1,k}$ now in order to keep track on how they depend on k. By immediate calculation

$$\langle T_{k,t} u, u \rangle_{k,t} = -\int_X [(k-a)\dot{\phi}_t + a\dot{\chi}_t + \dot{\psi}_t]|u|^2 e^{-(k-a)\phi_t - a\chi_t - \psi_t} \tag{3.4}$$

for $t = 0, 1$.

Let now H_k^t be the finite dimensional geodesic in the space of hermitian norms on $H^0(X, K_X + F + kL)$ that connects $\|\cdot\|_{k,t}$ for $t = 0$ and $t = 1$. Let

$$A_k = (H_k^t)^{-1} \frac{d}{dt} H_k^t$$

be the tangent vector of the finite dimensional geodesic. By (3.2) and (3.3) we have the inequalities

$$T_{0,k} \geq A_k \tag{3.5}$$

with respect to the hermitian scalar product H_k^0 and

$$T_{1,k} \leq A_k \tag{3.6}$$

with respect to the hermitian scalar product H_k^1. Let $\lambda_j(k)$ be the eigenvalues of A_k arranged in increasing order, and let $\tau_j^t(k)$ be the eigenvalues of the two Toeplitz operators, also arranged in increasing order. We then get immediately from (3.5) and (3.6) that

$$\tau_j^1(k) \leq \lambda_j(k) \leq \tau_j^0(k). \tag{3.7}$$

The final step in the argument is the following theorem on the asymptotics of Toeplitz operators; it is a variant of a theorem of Boutet de Monvel and Guillemin, [4]. Since the theorem is essentially known, we defer its proof to an appendix.

Theorem 3.2 *Let L and F be line bundles over X with smooth metrics ϕ and ψ respectively. Assume that ϕ has strictly positive curvature. Let ξ and ξ_k be continuous real valued functions on X with ξ_k tending uniformly to 0. Define Toeplitz operators with symbols $\xi + \xi_k$ on the spaces*

$$H^0(X, K_X + kL + F)$$

by

$$\langle T_k u, u\rangle_{k\phi+\psi} = \int (\xi + \xi_k)|u|^2 e^{-k\phi-\psi}.$$

Let μ_k be the normalized spectral measure of T_k.

Then the sequence μ_k converges weakly to the measure

$$\mu = \xi_*(\omega_\phi^n/Vol),$$

the direct image of the normalized volume element on X defined by ω_ϕ under the map ξ.

We apply this theorem to the Toeplitz operator $k^{-1}T_{k,t}$ for $t = 0, 1$. Its symbol is $-\dot{\phi}_t$ plus a term that goes uniformly to zero. In our operators $k^{-1}T_{k,t}$ the metric on F can be taken to be $\psi + a(\chi - \phi)$ if we take the metric on L to be ϕ. Theorem 3.2 therefore shows that the spectral measures $d\mu_{k,t}$ of $k^{-1}T_{k,t}$ converge to

$$d\mu_t = (-\dot{\phi}_t)_*(dV_t),$$

for $t = 0, 1$.

By the previous section these two measures are the same (for $t = 0$ and $t = 1$), namely the measure $d\mu$ that we associated to the geodesic in $\mathcal{H}_L$. The inequality (3.7) for the eigenvalues shows that

$$\int_{\mathbb{R}} f d\mu_{k,1} \leq \int_{\mathbb{R}} f d\nu_k \leq \int_{\mathbb{R}} f d\mu_{k,0}$$

if f is continuous and increasing (recall that ν_k is the spectral measure of A_k). It follows that

$$\lim \int_{\mathbb{R}} f d\nu_k = \int_{\mathbb{R}} f d\mu$$

for f continuous and increasing. Since any C^1-function can be written as a difference of two increasing functions, the previous limit must hold for any C^1-function too. But this implies weak convergence of the measures since all the measures involved are probability measures supported on a fixed compact interval. This finishes the proof of our main result:

Theorem 3.3 *Let ϕ_0 and ϕ_1 be two points in $\mathcal{H}_L$ and let ψ_t be two arbitrary smooth metrics on the line bundle F for t equal to 0 and 1. Let*

$$V_k = H^0(X, K_X + F + kL)$$

and let $\mathcal{H}_k$ be the space of hermitian norms on V_k. Let H_k^t be the elements in $\mathcal{H}_k$ defined by

$$\|u\|^2_{H_k^t} = \int_X |u|^2 e^{-k\phi_t - \psi_t}$$

for $t = 0, 1$. Let for t between 0 and 1 H_k^t be the geodesic in $\mathcal{H}_k$ connecting these two norms and let ν_k be its normalized spectral measures as defined above. Then

$$\nu_k \longrightarrow \mu,$$

in the weak-topology, where $\mu = \mu_t$ is defined in 1.1.*

Note that this implies Theorem 1.1 since we can take $F = -K_X$ and choose ψ_t to be equal to $-\log \omega^n_{\phi_t}$ for t equal to 0 and 1. We also see that we can replace $\omega^n_{\phi_t}$ by any other smooth volume forms. We can also take $F = 0$. Then the proof simplifies: The introduction of the auxiliary metrics is not necessary since we can work with the metrics

$$\|u\|^2_{H_{\phi_t}} = \int_X |u|^2 e^{-k\phi}$$

directly, and we get the analogue of Theorem 1.1 for these metrics.

The basic observation in the proof is that the inequality between finite dimensional geodesics and L^2-norms coming from Monge–Ampère geodesics in Proposition 3.1 also gives inequality for the first derivatives, since we have equality at the endpoint. The next proposition (cf the sup norm estimate for $\dot{\phi}_t$ from [14]) is another instance of this.

Proposition 3.4 *With the same notation as in the previous theorem, and*

$$A_k = (H_k^t)^{-1} \dot{H}_k^t,$$

let $\Lambda_{(k)}$ and $\lambda_{(k)}$ be the largest and smallest eigenvalues of $k^{-1}A_k$. Then, for all k,

$$\inf -\dot{\phi}_t \leq \lambda_{(k)} \leq \Lambda_{(k)} \leq \sup -\dot{\phi}_t.$$

Proof This follows immediately from (3.7), since the corresponding inequality for the eigenvalues of the Toeplitz operators is immediate. □

4 Approximation of Geodesics

Again we consider the spaces

$$V_k = H^0(X, K_X + F + kL)$$

equipped with metrics

$$\|u\|^2_{k\phi+\psi} := \int_X |u|^2 e^{-k\phi-\psi}$$

Let

$$B_{k\phi+\psi} = \sum |s_j|^2,$$

where s_j is an orthonormal basis for V_k . Since pointwise

$$|u|^2 / B_{k\phi+\psi}$$

is a function if u is a section of $K_X + F + kL$,

$$\log B_{k\phi+\psi}$$

can be interpreted as a metric on $K_X + F + kL$. In the proof below we will have use for the following lemma (we formulate it for $F = 0$ and $k = 1$), which is a variant on a well known theme. The basic underlying idea, to estimate Bergman kernels using the Ohsawa-Takegoshi theorem is due to Demailly, see e.g. [9].

Lemma 4.1 *Let ω^0 be a fixed Kähler form on X. Let ϕ be a metric (not necessarily smooth) on the line bundle L satisfying*

$$i\partial\bar{\partial}\phi \geq c_0\omega^0.$$

Let H_ϕ be the norm

$$\int_X |u|^2 e^{-\phi}$$

for u in $H^0(X, L + K_X)$, and let B_ϕ be its Bergman kernel. Then

$$B_\phi \geq \delta_0 e^{\phi} \omega_0^n$$

with δ_0 a universal constant, if c_0 is sufficiently large depending on X and ω^0 (only).

Proof By the extremal characterization of Bergman kernels it suffices to find a section u of $K_X + L$ with

$$|u(x)|^2 e^{-\phi(x)} \geq \delta_0 \omega_0^n \int_X |u|^2 e^{-\phi}$$

Choose a coordinate neighbourhood U with local coordinates z centered at x which is biholomorphic to the unit ball of $\mathbb{C}^n$ via the map z. By the Ohsawa-Takegoshi extension theorem we can find a section satisfying the required estimate over U. (L and K_X are trivial over the ball and the Ohsawa-Takegoshi Theorem says that we can extend the value 1 from the origin to the ball with an absolute L^2-estimate.) Let η be a cut-off function, equal to 1 in the ball of radius 1/2 and with compact support in the unit ball. We then solve, using Hörmander's L^2-estimates

$$\bar\partial v = \bar\partial \eta \wedge u =: g$$

with

$$\int_X |v|^2 e^{-\phi - 2n\eta \log|z|} \leq (C/c_0) \int_X |g|^2 e^{-\phi - 2n\eta \log|z|}$$

(z is the local coordinate). This can be done since

$$i\partial\bar\partial\phi - 2n\eta \log|z| \geq c_0\omega^0/2$$

if c_0 is large enough. Then $v(x) = 0$ since the integral in the left hand side is finite. Then

$$u - v$$

is a global holomorphic section of $K_X + L$ satisfying the required estimate. □

Let ϕ_0 and ϕ_1 be two points in $\mathcal{H}_L$, and let ψ_0 and ψ_1 be any two smooth metrics on F. We abbreviate by H_k^t the norms $\|\cdot\|_{k\phi_t+\psi_t}$ for t equal to 0 or 1, and let for t between 0 and 1 H_k^t be the geodesic in $\mathcal{H}_k$, the space of hermitian norms on V_k, joining these two endpoints.

Theorem 4.2 *For t equal to 0 and 1, let ϕ_t be two points in $\mathcal{H}_L$, and for t between 0 and 1 let ϕ_t be the geodesic in $\mathcal{H}_L$ joining them. Let $B_{t,k}$ be the Bergman kernels for the norms in the finite dimensional geodesic H_k^t. Let τ be an arbitrary smooth metric on $K_X + F$ over $\Omega \times X$. Then*

$$\sup_X |k^{-1}\log B_{t,k} - k^{-1}\tau - \phi_t| \leq Ck^{-1}\log k$$

for $0 \leq t \leq 1$

Proof Note that

$$i\partial\bar\partial \log B_{t,k} \geq 0.$$

This follows since H_k^t are geodesics. Perhaps the easiest way to see it (cf [14]) is to use the explicit description

$$B_{t,k} = \sum |e^{-t\lambda_j}||s_j|^2$$

which is immediate from the explicit formula for geodesics in $\mathcal{H}_k$ in Sect. 2. Since $\log B_{t,k}$ is a metric on $K_X + F + kL$,

$$k^{-1}(\log B_{t,k} - \tau)$$

is a metric on L. We shall now us the metric χ on L that we introduced in the previous section; it has strictly positive curvature over $\Omega \times X$ and coincides with ϕ_0 and ϕ_1 respectively when (Re)t is 0 or 1. Take a to be positive and consider

$$(k-a)k^{-1}(\log B_{t,k} - \tau) + a\chi;$$

it is a smooth metric on kL and it has positive curvature if a is sufficiently large. By standard Bergman kernel asymptotics it differs from ϕ_0 and ϕ_1 at most by $C \log k$ when (Re)t equals 0 or 1. Hence

$$(k-a)k^{-1}(\log B_{t,k} - \tau) + a\chi \leq k\phi_t + C \log k$$

since the geodesic ϕ_t is the supremum of all positively curved metrics lying below ϕ_0 and ϕ_1 on the boundary (cf [6]). Dividing by $(k-a)$ we see that

$$k^{-1} \log B_{t,k} - k^{-1}\tau - \phi_t \leq Ck^{-1} \log k$$

since χ, τ and ϕ_t are all uniformly bounded. The crux of the proof is the opposite estimate.

To estimate $B_{t,k}$ from below we first compare it to the Bergman kernel

$$B_{\phi_t,k},$$

which is defined using the hermitian norms

$$\|u\|_*^2 = \int_X |u|^2 e^{-(k-a)\phi_t - a\chi_t - \psi_t},$$

where the curve ψ_t is chosen as in the previous section. Again, the metric $(k-a)\phi_t + a\chi + \psi$ that we use here has positive curvature if a is sufficiently large. These norms coincide with H_k^t on the boundary and by Proposition 3.1 they are bigger than H_k^t in the interior. This implies (by the extremal characterization of Bergman kernels) that the respective Bergman kernels satisfy the opposite inequality, so we get

$$\log B_{t,k} \geq \log B_{\phi_t,k}.$$

To complete the proof it therefore suffices to show that

$$B_{\phi_t,k} \geq Ce^{k\phi_t + \tau},$$

or equivalently

$$B_{\phi_t,k} \geq Ce^{(k-a)\phi_t + a\chi + \tau}$$

But this follows from Lemma 4.1 since we can take a arbitrarily large so that

$$i\partial\bar{\partial}(k-a)\phi_t + a\chi + \tau$$

meets the curvature assumptions of that lemma. □

Remark: If $F = 0$ and τ is an arbitrary metric on K_X, Theorem 4.2 is exactly Theorem 6.1 in [2]. The main case is when $F = -K_X$ and we choose ($\tau = 0$ and) $\psi_t = -\log \omega_{\phi_t}$ for $t = 0$ and $t = 1$. Then

$$\|u\|^2_{k\phi_t + \psi_t} = \int_X |u|^2 e^{-k\phi_t} \omega^n_{\phi_t}$$

and we get Theorem 1.2 from the Introduction (this is the case studied in [14]). Finally, taking $F = -K_X$ and ψ_t one fixed (arbitrary) smooth metric on $-K_X$, we get the counterpart of Theorem 1.2 for the norms

$$\int_X |u|^2 e^{-k\phi_t} dV,$$

where dV is a fixed smooth volume form on X.

5 Appendix: Background on Toeplitz Operators

We consider Toeplitz operators $T_{k,\xi}$ on the spaces

$$V_k = H^0(X, K_X + F + kL)$$

with symbol ξ in $C(X)$. $T_{k,\xi}$ is defined by

$$\langle T_{k,\xi} u, u \rangle_{k\phi+\psi} = \int_X \xi |u|^2 e^{-k\phi - \psi},$$

where the inner product is

$$\langle v, u \rangle_{k\phi+\psi} = \int_X v\bar{u} e^{-k\phi-\psi}.$$

In other words

$$T_{k,\xi} u = P_k(\xi u)$$

where P_k is the Bergman projection.

Recall that if T is any hermitian endomorphism on an N-dimensional inner product space, and if we order its eigenvalues

$$\lambda_1 \leq \lambda_2 \leq \ldots \lambda_n,$$

then

$$\lambda_j = \inf_{Vj \subset V, dim V_j = j} \|T|_{V_j}\|.$$

From this it follows that if we perturb the operator T to $T + S$ where $\|S\| \leq \epsilon$, then the eigenvalues shift at most by ϵ. This means that if we consider the spectral measure of

$$T_{k,\xi+\xi_k}$$

where ξ_k goes uniformly to 0, the limit of the spectral measures is the same as the limit of the spectral measures of

$$T_{k,\xi}.$$

In other words, in the proof of Theorem 3.2 we may assume that $\xi_k = 0$. By the same token, we may assume that ξ is smooth, since continuous functions can be approximated by smooth functions. The most important part of the proof of Theorem 3.2 is the next lemma.

Lemma 5.1 *Let $d_k = dim(V_k)$. Then*

$$\lim \frac{1}{d_k} tr T_{k,\xi} = \int_X \xi \omega_\phi^n / Vol.$$

Proof Let $B_{k\phi+\psi}$ be the Bergman kernel. Then

$$\frac{1}{d_k} tr T_{k,\xi} = \frac{1}{d_k} \int_X \xi B_{k\phi+\psi} e^{-k\phi-\psi}.$$

But, by the formula for (first order) Bergman asymptotics

$$B_{k\phi+\psi} e^{-k\phi-\psi} / d_k$$

tends to ω_ϕ^n / Vol, so the lemma follows. □

Lemma 5.2 *Let ξ and η be smooth functions on X. Then*

$$\|T_{k,\xi} T_{k,\eta} - T_{k,\,\xi\eta}\|^2 \leq C k^{-1}.$$

Proof Note that if u is in V_k then

$$T_{k,\xi} u - \xi u =: v_k$$

is the L^2-minimal solution to the $\bar{\partial}$-equation

$$\bar{\partial} v_k = \bar{\partial}\xi \wedge u$$

(this is where we want ξ smooth). By Hörmander L^2-estimates

$$\|T_{k,\xi}u - \xi u\|^2_{k\phi+\psi} \leq \|\bar{\partial}\xi \wedge u\|^2_{k\phi+\psi} \leq Ck^{-1}\|u\|^2_{k\phi+\psi}$$

(the last inequality is because the pointwise norm $\|\bar{\partial}\xi\|^2_\theta \leq C/k$ when we measure with respect to the Kähler metric $\theta = i\partial\bar{\partial}(k\phi + \psi)$). Therefore, if u is of norm at most 1,

$$\|T_{k,\xi}T_{k,\eta}u - \xi T_{k,\eta}u\|^2 \leq Ck^{-1},$$

$$\|\xi T_{k,\eta}u - \xi\eta u\|^2 \leq Ck^{-1}$$

and

$$\|T_{k,\,\xi\eta}u - \xi\eta u\|^2 \leq Ck^{-1}$$

and the lemma follows. □

Let μ_k be the normalized spectral measures of $T_{k,\xi}$. In order to study their weak limits, it is enough to look at their moments

$$\int_{\mathbb{R}} x^p d\mu_k(x) = \frac{1}{d_k} tr T^p_{k,\xi}.$$

By Lemma 7.2 and induction

$$\|T^p_{k,\xi} - T_{k,\xi^p}\|^2 \leq Ck^{-1}.$$

Hence

$$\frac{1}{d_k} tr T^p_{k,\xi} = \frac{1}{d_k} tr T_{k,\xi^p} + O(k^{-1})$$

and

$$\lim \frac{1}{d_k} tr T_{k,\xi^p} = \int_X \xi^p \omega^n_\phi / Vol$$

by Lemma 7.1. Thus,

$$\lim \int_{\mathbb{R}} x^p d\mu_k(x) = \frac{1}{d_k} tr T^p_{k,\xi} = \int_X \xi^p \omega^n_\phi / Vol$$

for any power x^p. Taking linear combinations we get the same thing for any polynomial, and therefore for any continuous function. This completes the proof of Theorem 3.2.

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Intersection Bounds for Nodal Sets of Laplace Eigenfunctions

Yaiza Canzani and John A. Toth

Abstract Let (M^n, g) be a real analytic compact n-dimensional Riemannian manifold and denote by φ_λ the eigenfunctions of the Laplace operator Δ_g with eigenvalue λ^2. We prove that if $H \subset M$ is a real analytic closed curve for which there exist $\lambda_0, C > 0$ so that $\|\varphi_\lambda\|_{L^2(H)} \geq e^{-C\lambda}$ for all $\lambda > \lambda_0$, then

$$\#\{\varphi_\lambda^{-1}(0) \cap H\} = O(\lambda).$$

The purpose of this paper is to study the local geometry of the nodal sets of Laplace eigenfunctions. Let (M, g) be a compact real analytic Riemannian surface with no boundary. Denote by φ_λ the real-valued eigenfunctions of the Laplace operator Δ_g satisfying

$$-\Delta_g \varphi_\lambda = \lambda^2 \varphi_\lambda.$$

For normalization purposes we assume that $\|\varphi_\lambda\|_{L^2(M)} = 1$. Our object of study is the zero set of φ_λ as $\lambda \to \infty$, which we denote by

$$Z_{\varphi_\lambda} = \varphi_\lambda^{-1}(0).$$

From a quantum mechanics point of view, the position of a quantum particle on (M, g) of energy λ is described by the probability measure $x \mapsto |\varphi_\lambda(x)|^2 dv_g(x)$.

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Y. Canzani
Department of Mathematics, University of North Carolina at Chapel Hill, Chapel Hill, USA
e-mail: canzani@email.unc.edu

J. A. Toth (✉)
Department of Mathematics and Statistics, McGill University, Montreal, Canada
e-mail: jtoth@math.mcgill.ca

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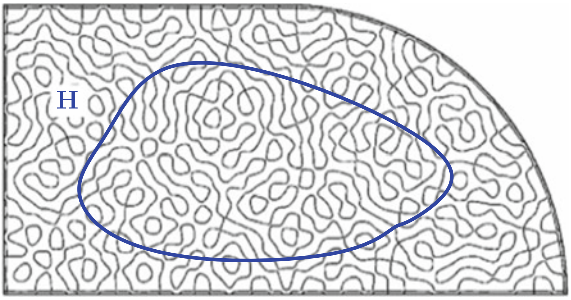

Fig. 1 Figure: Nodal lines of a high energy state, $\lambda \sim 84$, in the quarter stadium [4]

The set Z_{φ_λ} is then interpreted as the least likely place for a quantum particle in the energy state λ to be.

There are several results that aim to describe the geometric structure of Z_{φ_λ} as $\lambda \to \infty$. For example, the zero sets are rectifiable so it is possible to study their length. On a compact, real analytic, surface with no boundary it was proved by [12] that there exist two positive constants c_1 and c_2 for which

$$c_1 \lambda \le \text{length}(Z_{\varphi_\lambda}) \le c_2 \lambda \qquad \text{as } \lambda \to \infty.$$

It is also known that the nodal set Z_{φ_λ} spreads along the surface at a $1/\lambda$ scale in the sense that there exists a positive constant C so that the intersection of any geodesic ball of radius C/λ with the zero set Z_{φ_λ} is non-empty for all λ large enough. One may also try to understand the structure of Z_{φ_λ} by studying its complement. A connected component of $M \backslash Z_{\varphi_\lambda}$ is called a nodal domain, and Courant's nodal domain Theorem states that the number of nodal domains of φ_λ is bounded by λ^2 for all λ. It is also known that the inner radius of a nodal domain on a compact real analytic surface is bounded above and below by a multiple of $1/\lambda$ (see [2, 21]).

So far we have only mentioned global results on the geometry of Z_{φ_λ} as $\lambda \to \infty$. In this paper we are interested in understanding the structure of Z_{φ_λ} from a local point of view. To do this, Zelditch and the second author proposed in [23] to study the number of intersections of Z_{φ_λ} with a given fixed curve. Namely, consider a real analytic closed curve $H \subset M$. In view of the aforementioned results one should expect that

$$\#\{Z_{\varphi_\lambda} \cap H\} = O_H(\lambda) \qquad \text{as } \lambda \to \infty. \tag{1}$$

Of course there are settings in which (1) will not be satisfied in the sense that there are 'bad' curves for this problem in which the curve H is entirely contained in the nodal set of infinitely many eigenfunctions. An example of such 'bad' curve is the equator on the sphere along which we have that all odd spherical harmonics vanish. To overcome dealing with this pathological set of curves, in [23] the authors introduced the concept of a good curve (Fig. 1).

Definition 1 A curve $H \subset M$ is said to be **good** if for some $\lambda_0 > 0$ there exists $C > 0$ such that for all $\lambda \ge \lambda_0$,

$$\|\varphi_\lambda\|_{L^2(H)} \geq e^{-C\lambda}. \tag{2}$$

Using this concept it is proved in [23] that if $\Omega \subset \mathbb{R}^2$ is a real analytic bounded planar domain, and $H \subset \Omega$ is an interior good curve, then (1) holds.

The goodness condition (2) is likely generic. It was proved in [23] that $H = \partial\Omega$ is always a good curve, but in general it appears difficult to verify this condition. In [17], Jung proved that geodesic circles in compact hyperbolic surfaces are good curves, and also that the estimate (1) is satisfied for them. In [3], Bourgain and Rudnick proved that on the flat 2-torus, if H is real analytic with nowhere vanishing curvature, then H is good and (1) is also satisfied. For $\Omega \subset \mathbb{R}^2$ bounded, piecewise-smooth convex domain with ergodic billiard flow, El-Hajj and Toth [13] proved that if H is a closed real analytic interior curve with strictly positive geodesic curvature, and $(\varphi_{\lambda_j})_{j=1}^{\infty}$ is a quantum ergodic sequence of Neumann or Dirichlet eigenfunctions in Ω, then H is good and so $\#\{Z_{\varphi_{\lambda_j}} \cap H\} = \mathcal{O}(\lambda_j)$ as $j \to \infty$. The purpose of the first part of this paper is to obtain upper bounds for $\#\{Z_{\varphi_\lambda} \cap H\}$ on general compact surfaces under the assumption that the curve H is good.

Theorem 1 *Let (M, g) be a real analytic compact Riemannian surface and let $H \subset M$ be a real analytic closed* good *curve on M. Then,*

$$\#\{Z_{\varphi_\lambda} \cap H\} = O(\lambda),$$

as $\lambda \to +\infty$.

We note that it follows directly from our proof that Theorem 1 still holds if one imposes the weaker goodness condition $\|\varphi_\lambda\|_{L^\infty(H)} \geq e^{-C\lambda}$ as $\lambda \to \infty$. However, as a practical matter, such a pointwise condition is usually harder to verify than (2).

The idea of the proof of Theorem 1 uses holomorphic continuation of the heat kernel $E(t, x, y) = e^{-t\Delta}(t, x, y)$ in the outgoing x-variable at small time $t = \frac{1}{\lambda}$ to a Grauert tube complexification $M_{\mathbb{C}}$ of M. Writing $E_\lambda^{\mathbb{C}}(z, y) = E^{\mathbb{C}}(\lambda^{-1}, z, y)$ with $(z, y) \in M^{\mathbb{C}} \times M$, one has the obvious identity $E_\lambda^{\mathbb{C}}\phi_\lambda = e^{-\lambda}\phi_\lambda^{\mathbb{C}}$ where $\phi_\lambda^{\mathbb{C}}$ denotes holomorphic continuation of the eigenfunction ϕ_λ to the Grauert tube $M^{\mathbb{C}}$. Thus, to estimate $\phi_\lambda^{\mathbb{C}}$ it suffices to compute asymptotics for the complexified heat kernel $E_\lambda^{\mathbb{C}}$. The result we need here is given in Proposition 3 and is based on earlier work of Golse–Leichtnam–Stenzel [18] and Cheeger-Gromov-Taylor [10]. Next, we restrict z to a Grauert subtube $H^{\mathbb{C}}$ of $M^{\mathbb{C}}$ over a real curve H and apply a frequency function argument to estimate from above the number of complex (and hence, real) zeros of $\phi_\lambda^{\mathbb{C}}$ in the tube $H^{\mathbb{C}}$. This analysis is carried out in Sect. 1.2.

Finally, we note that it is sometimes convenient (see [13] Theorem 1) to use the notion of *weak goodness* in place of the goodness assumption in Definition 1.

Definition 2 Given $H \subset M$ be a real-analytic curve, we say that it is *weakly good* for the eigenfunction sequence ϕ_λ provided for some $\lambda_0 > 0$ there exists $C > 0$ such that for all $\lambda \geq \lambda_0$,

$$\sup_{z \in H^{\mathbb{C}}} |\phi_\lambda^{\mathbb{C}}(z)| \geq e^{-C\lambda}.$$

In Sect. 2, using a Hadamard three circles argument, we show that Definitions 1 and 2 are equivalent.

Notation. Throughout this manuscript $\alpha \in \mathbb{N} \times \mathbb{N}$ denotes a multiindex $\alpha = (\alpha_1, \alpha_2)$. We use the standard multiindex notation, $|\alpha| = \alpha_1 + \alpha_2$, $\alpha! = \alpha_1 \alpha_2$ and $\partial_x^\alpha := \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2}$.
We write $\mathrm{inj_M}$ for the injectivity radius of (M, g). For $0 < r < \mathrm{inj_M}$, we write $B_r(x)$ for the geodesic ball centred at $x \in M$ of radius r.

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1 Intersection Bounds on Real Analytic Surfaces

In this section, we begin with some background material and then prove Theorem 1. In Sect. 1.1 we give some background on the complexification of the heat kernel with the necessary bounds near and far from the diagonal. In Sect. 1.2 we explain how to obtain the bound on the number of zeros of an eigenfunction along H by complexifying the eigenfunction and reproducing it using the heat operator.

1.1 Analytic Continuation

In this section we review how to analytically continue eigenfunctions to the complexification of the real analytic manifold where they originally lived in. We refer to [18] for further details.

Throughout this section we assume (M, g) is a compact, real analytic, Riemannian surface. By a theorem of Bruhat-Whitney, M has a unique complexification $M^{\mathbb{C}}$ with $M \subset M^{\mathbb{C}}$ totally real that generalizes the complexification of $\mathbb{R}^2$ to $\mathbb{C}^2$. One defines the plurisubharmonic exhaustion function $\sqrt{\rho_g}$ on $M^{\mathbb{C}}$ as the unique solution to the complex Monge-Ampere equation

$$\begin{cases} (\partial\bar{\partial}\sqrt{\rho_g})^2 = \delta_{M,dv_g}, \\ \iota^*(i\partial\bar{\partial}\rho_g) = g, \end{cases}$$

where $\iota : M \to M^{\mathbb{C}}$ is the embedding given by the Bruhat-Whitney Theorem. For example, in the simplest model case when $M = \mathbb{R}^2$ and $M^{\mathbb{C}} = \mathbb{C}^2$, it is easy to check that $\sqrt{\rho_g}(z) = 2|\mathrm{Im}\, z|$. The open Grauert tube of radius ε is defined to be

$$M_\varepsilon^{\mathbb{C}} = \{z \in M^{\mathbb{C}} : \sqrt{\rho_g}(z) \leq \varepsilon\}.$$

There is a maximal $\varepsilon_{\max} > 0$ for which $M_\varepsilon^{\mathbb{C}}$ is defined [18, Thm 1.5], and $M_\varepsilon^{\mathbb{C}}$ is a strictly pseudoconvex domain in $M^{\mathbb{C}}$ for all $\varepsilon \leq \varepsilon_{\max}$. We denote the space of germs of holomorphic functions on an open subset $U \subset M^{\mathbb{C}}$ by $\mathcal{O}(U)$.

For all $\varepsilon \leq \varepsilon_{\max}$, we identify the radius ε ball bundle $B_\varepsilon M \subset TM$ with $B^*_\varepsilon M \subset T^*M$ using the Riemannian metric. For $x \in M$ and $0 < r < \mathrm{inj_M}$, we let $\exp_x : B_r(0) \subset T^*_x M \to M$ be the geodesic exponential map. We denote the lifted exponential map to all of $B^*_\varepsilon M$ by

$$\mathrm{Exp} : B^*_\varepsilon M \to M, \quad \mathrm{Exp}(x,\xi) = \exp_x(\xi).$$

Since (M, g) is real-analytic, for fixed $x \in M$ and $0 < r < \mathrm{inj_M}$, the geodesic exponential map $\exp_x : B_r(0) \subset T^*_x M \to M$ admits a holomorphic continuation $\exp^{\mathbb{C}}_x : (B_r(0))^{\mathbb{C}} \to M^{\mathbb{C}}$ in the fiber ξ-variables with range contained in $M^{\mathbb{C}}$. For $0 < \varepsilon < \varepsilon_{\max}$, we define the associated complexified lifted map by

$$\mathrm{Exp}^{\mathbb{C}} : B^*_\varepsilon M \to M^{\mathbb{C}}, \quad \mathrm{Exp}^{\mathbb{C}}(x,\xi) = \exp^{\mathbb{C}}_x(i\xi).$$

The complexified map, $\mathrm{Exp}^{\mathbb{C}}$, gives a diffeomorphism between $B^*_\varepsilon M$ and $M^{\mathbb{C}}_\varepsilon$ with the property that $(\mathrm{Exp}^{\mathbb{C}})^*(\rho_g) = |\cdot|_g$. Consequently, $B^*_\varepsilon M \cong M^{\mathbb{C}}_\varepsilon$ as complex manifolds via $\mathrm{Exp}^{\mathbb{C}}$. Also, the map

$$\pi_M : M^{\mathbb{C}}_\varepsilon \to M, \qquad \pi_M(\mathrm{Exp}^{\mathbb{C}}(x,\xi)) = x, \tag{3}$$

is an analytic fibration. The fibers $\pi_M^{-1}(M)$ correspond to imaginary directions over the totally real submanifold $M \subset M^{\mathbb{C}}_\varepsilon$.

1.1.1 Complexified Normal Coordinates

In this section we review the results in Lemma 1.18 of [18] regarding the existence of a holomorphic coordinate system $h(x,\xi)$ on the complex manifold $B^*_\varepsilon M$. Fix $x_0 \in M$ and $0 < r < \mathrm{inj_M}$. The map

$$\eta = r(x) \mapsto \exp_{x_0}(\eta) = x$$

is real analytic near the origin and so it can be holomorphically extended to the complex manifold $B^*_\varepsilon M$ in a neighbourhood of x_0 by

$$\eta + i\zeta = h(x,\xi) \mapsto \exp^{\mathbb{C}}_{x_0}(\eta + i\zeta) = (x,\xi).$$

According to Lemma 1.18 of [18], this coordinate system satisfies $h(x,0) = r(x)$ and $h(x_0,\xi) = i\xi$. Identifying the point $(x,\xi) \in B^*_\varepsilon M$ with $\exp^{\mathbb{C}}_x(i\xi) \in M^{\mathbb{C}}_\varepsilon$ as described above, one has $\pi_M(x,\xi) = \pi_M(\mathrm{Exp}^{\mathbb{C}}(x,\xi)) = x = \exp_{x_0}(\eta)$.

As of now we adopt the following notation: for $z = (x,\xi) \in M^{\mathbb{C}}_\varepsilon$ close to x_0 we write

$$\mathrm{Re}\, z := \mathrm{Re}\, h(x,\xi), \quad \text{and} \quad \mathrm{Im}\, z := \mathrm{Im}\, h(x,\xi). \tag{4}$$

For future purposes we remark that with this notation $\pi_M(z)$ is identified with Re z since $\pi_{_M}(z) = \pi_{_M}(x, \xi) = \exp_{x_0}(\eta) = \exp_{x_0}(\mathrm{Re}\, h(x, \xi)) = \exp_{x_0}(\mathrm{Re}\, z)$.

1.1.2 Complexified Distance

Consider the squared geodesic distance on M

$$r^2(\cdot, \cdot) : M \times M \to \mathbb{R}.$$

For $0 < \varepsilon < \varepsilon_{\max}$, there exists a connected open neighbourhood $\tilde{\Delta} \subset M_\varepsilon^{\mathbb{C}} \times M_\varepsilon^{\mathbb{C}}$ of the diagonal $\Delta \subset M \times M$ to which $r^2(\cdot, \cdot)$ can be holomorphically extended [18, Corollary 1.24]. We denote the extension by $r_{\mathbb{C}}^2(\cdot, \cdot) \in \mathcal{O}(\tilde{\Delta})$. Moreover, one can easily recover the exhaustion function $\sqrt{\rho_g}(z)$ from $r_{\mathbb{C}}$; indeed, $\rho_g(z) = -r_{\mathbb{C}}^2(z, \bar{z})$ for all $z \in M_\varepsilon^{\mathbb{C}}$.

1.1.3 Complexified Heat Operator

Consider the heat operator of (M, g) defined at time h by

$$E_h = e^{h\Delta_g} : C^\infty(M) \to C^\infty(M).$$

The Schwartz kernel of the heat operator can be written in the form

$$E_h(x, y) = \sum_{j=0}^{\infty} e^{-h\lambda_j^2} \varphi_j(x)\overline{\varphi_j(y)} \qquad \text{for } (x, y) \in M \times M,$$

where $\{\varphi_j\}_j$ is an orthonormal basis of $L^2(M)$ of eigenfunctions, $\Delta_g \varphi_j = \lambda_j^2 \varphi_j$. By a recent result of Zelditch [24, Section 11.1], the maximal geometric tube radius $\varepsilon_{\max}$ agrees with the maximal analytic tube radius in the sense that for all $0 < \varepsilon < \varepsilon_{\max}$, all the eigenfunctions φ_j extend holomorphically to $M_\varepsilon^{\mathbb{C}}$ (see also [18, Prop. 2.1]). It is also known that the kernel $E(\cdot, \cdot; h)$ admits a holomorphic extension to $M_\varepsilon^{\mathbb{C}} \times M_\varepsilon^{\mathbb{C}}$ for all $0 < \varepsilon < \varepsilon_{\max}$ and $h \in (0, 1)$, [18, Prop. 2.4]. We denote the complexification by $E_h^{\mathbb{C}}(\cdot, \cdot)$. In particular, if we write $\varphi_j^{\mathbb{C}} \in \mathcal{O}(M_\varepsilon^{\mathbb{C}})$ for the holomorphic continuation of the eigenfunctions, it is clear that

$$E_h^{\mathbb{C}}(z, y) = \sum_{j=0}^{\infty} e^{-h\lambda_j^2} \varphi_j^{\mathbb{C}}(z)\overline{\varphi_j}(y) \qquad \text{for } (z, y) \in M_\varepsilon^{\mathbb{C}} \times M,$$

and therefore

$$(E_h^{\mathbb{C}} \varphi_j)(z) = e^{-h\lambda_j^2} \varphi_j^{\mathbb{C}}(z), \quad z \in M_\varepsilon^{\mathbb{C}}. \tag{5}$$

To analyze the asymptotic behaviour of $E_h^{\mathbb{C}}(z, y)$ with $(z, y) \in M_\varepsilon^{\mathbb{C}} \times M$, we split the kernel into two pieces where

(i) the point $(\pi_M z, y) \in M \times M$ is close to the diagonal in terms of inj_M and the Grauert tube radius ε,

(ii) the point $(\pi_M z, y) \in M \times M$ is relatively far from the diagonal in terms of inj_M and ε.

In the near-diagonal case (i), one has the following result of Golse, Leichtnam and Stenzel.

Theorem 2 ([18, Theorem 0.1]) *Let (M, g) be a compact real analytic Riemannian surface. Fix $0 < \varepsilon < \varepsilon_{\max}$ and $x \in M$. Then, there exist positive constants $\beta = \beta(x, \varepsilon)$, $D = D(x, \varepsilon)$, and an open neighbourhood $W_x \subset M_\varepsilon^{\mathbb{C}}$ of x, such that*

$$E_h^{\mathbb{C}}(z, w) = e^{-\frac{r_{\mathbb{C}}^2(z,w)}{4h}} N^{\mathbb{C}}(z, w; h) + O(e^{-\beta/h}), \qquad h \to 0^+, \tag{6}$$

for $(z, w) \in W_x \times W_x$. Here,

$$N^{\mathbb{C}}(z, w; h) := \frac{1}{4\pi h} \sum_{0 \le k \le D/h} u_k^{\mathbb{C}}(z, w) h^k, \tag{7}$$

where the $u_k^{\mathbb{C}}$'s are analytic continuation of the coefficients appearing in the formal solution of the heat equation on (M, g). The asymptotic sum $\sum_{k=1}^{\infty} u_k^{\mathbb{C}}(z, w)$ is a classical symbol in the sense of Sjöstrand and the error term $O(e^{-\beta/h})$ is uniform in $(z, w) \in W_x \times W_x$.

We make use of this fact in Proposition 3 below. To control the behaviour of the complexified heat kernel for a pair of points $(\pi_M z, y) \in M \times M$ that are relatively close or far from the diagonal, we need the following result.

Proposition 3 *There exist $0 < \varepsilon_0 \le \varepsilon_{\max}$ and positive constants β, D, ε_1, δ_0 and h_0, depending only on $\varepsilon_0 > 0$, such that for $0 < \varepsilon \le \varepsilon_0$, $0 < \delta \le \delta_0$ and $(z, y) \in M_\varepsilon \times M$, the following is true:*

(i) When $r(\pi_M z, y) < \delta$ and $h \in (0, h_0]$,

$$E_h^{\mathbb{C}}(z, y) = e^{-\frac{r_{\mathbb{C}}^2(z,y)}{4h}} N^{\mathbb{C}}(z, y; h) + O(e^{-\beta/h}), \tag{8}$$

where $N^{\mathbb{C}}(z, y; h)$ is the polyhomogeneous sum in (7).

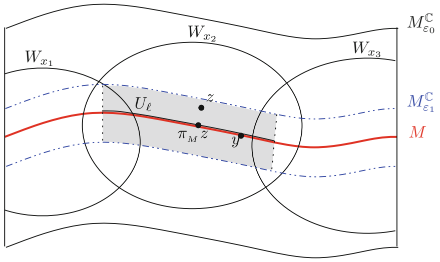
(ii) When $r(\pi_M z, y) > \frac{\delta}{2}$ and $h \in (0, 1)$,

$$\left| E_h^{\mathbb{C}}(z, y) \right| \le C\, e^{-\frac{\delta^2}{128h}}, \tag{9}$$

where C is a positive constant depending only on (M, g).

Proof For each $x \in M$ consider the open neighbourhood of x, $W_x \subset M^{\mathbb{C}}_{\varepsilon_0}$ as in (6). Since M is compact in $M^{\mathbb{C}}_{\varepsilon_0}$ there exists a finite covering $\{W_{x_j}\}_{j=1}^k$ so that $M \subset \cup_{j\le k} W_{x_j}$. For each W_{x_j} let $\beta_j = \beta(x_j, \varepsilon_0)$ and $D_j = D(x_j, \varepsilon_0)$ be as in (6) and set

$$\beta := \min_{1\le j\le k} \beta_j \qquad \text{and} \qquad D := \min_{1\le j\le k} D_j. \tag{10}$$



Next, choose $0 < \varepsilon_1 \le \varepsilon_0$ so that $M^{\mathbb{C}}_{\varepsilon_1} \subset \cup_{j\le k} W_{x_j}$. Fix $0 < \varepsilon \le \varepsilon_1$. Since

$$M \subset \bigcup \{U : \ U \subset M \text{ is open and } (\pi_M^{-1}(U) \cap M^{\mathbb{C}}_{\varepsilon}) \subset W_{x_j} \text{ for some } j\}$$

and M is compact, there is a finite covering of $M = \cup_{\ell=1}^N U_\ell$ with the property that for all $\ell = 1, \dots, N$ there exists $j_\ell \in \{1, \dots, k\}$ such that $(\pi_M^{-1}(U_\ell) \cap M^{\mathbb{C}}_{\varepsilon}) \subset W_{x_{j_\ell}}$.

Let $\delta_0 > 0$ be the Lebesgue number corresponding to the covering $\{U_\ell\}_{\ell=1}^N$. That is, if $x, y \in M$ and $r(x, y) < \delta_0$, then there exists $\ell \in \{1, \dots, N\}$ such that $x, y \in U_\ell$. Without loss of generality we assume $\delta_0 \le \frac{1}{8}\,\mathrm{inj_M}$.

Let $0 < \delta \le \delta_0$, $0 < \varepsilon \le \varepsilon_1$, and consider $(z, y) \in M^{\mathbb{C}}_{\varepsilon} \times M$. If $r(\pi_M z, y) < \delta$ then there exists $\ell \in \{1, \dots, N\}$ for which both $\pi_M z$ and y belong to U_ℓ. By the definition of U_ℓ we get $z, y \in W_{x_{j_\ell}}$ for some j_ℓ and we can use the heat kernel expansion (6) to obtain (8) for β as defined above.

We next consider the case $r(\pi_M z, y) > \frac{\delta}{2}$. Using the notation in [10, (A.2)] define $\phi_h(s) := h^{-\frac{1}{2}} e^{-\frac{s^2}{8h}}$. In [10, Theorem 3.1] it is proved that for $0 < a \le \mathrm{inj_M}$, there exists a constant $C = C(a, M) > 0$ such that for all $h \in (0, 1]$, $\alpha \in \mathbb{Z}^2_+$, and $x, y \in M$ with $r(x, y) > 2a$,

$$\sup_{B_a(y)} \left|\partial_x^\alpha E_h(x, \cdot)\right| \le C(C|\alpha|)^{|\alpha|}\ \psi_h\left(r(x, y) - 2a\right), \tag{11}$$

where $\psi_h(r) := \int_r^\infty \phi_h(s)ds$ is defined as in [10, (2.2)].

Observe that there exists $C' > 0$ for which $\psi_h(r) \le C' e^{-\frac{r^2}{8h}}$ for $h \in (0, 1)$. Therefore, by possibly adjusting C in (11), we have for $h \in (0, 1)$ and $x, y \in M$ with $r(x, y) > 2a$ that

$$\sup_{B_a(y)} \left|\partial_x^\alpha E_h(x, \cdot)\right| \le C(C|\alpha|)^{|\alpha|} e^{-\frac{(r(x,y)-2a)^2}{8h}}. \tag{12}$$

To complete the proof we make a Taylor expansion around $x = \pi_M z$ with $z \in M_\varepsilon^{\mathbb{C}}$. Assume that $r(\pi_M z, y) > \delta/2$ for some $y \in M$ and set $a = \delta/8$. Then $r(\pi_M z, y) > 4a$ and so $r(\pi_M z, y) - 2a > 2a$. From (12) it follows that

$$\left|\partial_x^\alpha E_h(\pi_M z, y)\right| \leq C(C|\alpha|)^{|\alpha|} e^{-\frac{a^2}{2h}}. \tag{13}$$

Let $0 < \varepsilon \leq \varepsilon_{\max}$ and suppose $(z, y) \in M_\varepsilon^{\mathbb{C}} \times M$. Choose $x_0 \in M$ for which $z \in (B_{\mathrm{inj_M}}(x_0))_\varepsilon$, and write $z = \operatorname{Re} z + i \operatorname{Im} z$ in the complexified normal coordinates at x_0 described in (4). By Taylor expansion at $\operatorname{Im} z = 0$ we then know

$$E_h^{\mathbb{C}}(z, y) = \sum_\alpha \frac{(i \operatorname{Im} z)^{|\alpha|}}{\alpha!} \cdot \partial_x^\alpha E_h(\operatorname{Re} z, y). \tag{14}$$

Since in complexified normal coordinates $\pi_M z$ is identified with $\operatorname{Re} z$ via $\pi_M z = \exp_{x_0}(\operatorname{Re} z)$, the proof follows from substituting the Cauchy estimates (13) in (14). Using Stirling's formula, it follows that to get convergence in (14) it suffices to work with $0 < \varepsilon \leq \varepsilon_0$ for $0 < \varepsilon_0 \leq \min\{\frac{1}{Ce}, \varepsilon_{\max}\}$. □

From now on, we always carry out our analysis in the complex Grauert tubes $M_\varepsilon^{\mathbb{C}}$ with $0 < \varepsilon \leq \varepsilon_1$, where in view of Proposition 3, we have good control of the complexified heat kernel, $E_h^{\mathbb{C}}(\cdot, y)$ for $y \in M$.

1.2 Bound for Good Curves

Since our arguments are semiclassical, by a slight abuse of notation we write φ_h for φ_{λ_j} with $\lambda_j = \frac{1}{h}$.

Without loss of generality, we assume that the length of H is $|H| = 1$ and let $q : [-\frac{1}{2}, \frac{1}{2}] \to H$ be a real analytic arc-length parametrization of H with extension $q : [-1, 1] \to H$ that is 1-periodic. In analogy with [13], we define the restricted parametrized eigenfunctions

$$u_h^H : [-1, 1] \to \mathbb{C}, \qquad u_h^H := \varphi_h \circ q.$$

For future reference, for $\epsilon > 0$ sufficiently small, we define the height ϵ level curve

$$H_\epsilon := \{q^{\mathbb{C}}(t);\ |\operatorname{Re} t| \leq \frac{1}{2},\ \ |\operatorname{Im} t| = \epsilon\}. \tag{15}$$

For $0 < \varepsilon \leq \varepsilon_1/2$, consider the complex strip around $[-1, 1]$ given by

$$[-1, 1]_\varepsilon^{\mathbb{C}} := \{\tau \in \mathbb{C} :\ \operatorname{Re}(\tau) \in [-1, 1] \quad \text{and} \quad \operatorname{Im}(\tau) \in [-\varepsilon, \varepsilon]\}.$$

Consider also the holomorphic continuation of the functions u_h^H defined as

$$u_h^{H,\mathbb{C}} : [-1,1]_\varepsilon^{\mathbb{C}} \to \mathbb{C}, \qquad u_h^{H,\mathbb{C}} := (\varphi_h \circ q)^{\mathbb{C}}. \tag{16}$$

We shall bound the number of zeros of u_h in $[-\frac{1}{2}, \frac{1}{2}]$ by the number of zeros of $u_h^{H,\mathbb{C}}$ inside a subset of $[-1,1]_\varepsilon^{\mathbb{C}}$ that contains $[-\frac{1}{2}, \frac{1}{2}]$. Let C_ε be a simply connected domain with real analytic boundary satisfying

$$C_\varepsilon \subset [-1,1]_\varepsilon^{\mathbb{C}} \qquad \text{and} \qquad [-\tfrac{1}{2}, \tfrac{1}{2}] \subset C_\varepsilon.$$

By the Riemann mapping Theorem there exists a biholomorphism $F : B_1(0) \subset \mathbb{C} \to C_\varepsilon$. The map F has a natural extension to the closure of $B_1(0)$ while being a diffeomorphism when restricted to the boundary $\partial B_1(0)$. The function $u_h^{H,\mathbb{C}} \circ F$ is holomorphic in $B_1(0)$ and so one can apply Lemma 3.2 in [19] to count the number of its zeros. Let $r \in (0,1)$ be chosen so that $[-\frac{1}{2}, \frac{1}{2}] \subset F(B_r(0))$. By a slight modification of the argument in [19, Lemma 3.2] one can show that there exists a constant $c_\varepsilon > 0$, depending only on H, r and ε, so that

$$\#\left\{\tau \in B_r(0) : \ (u_h^{H,\mathbb{C}} \circ F)(\tau) = 0\right\} \le c_\varepsilon \frac{\|\nabla(u_h^{H,\mathbb{C}} \circ F)\|^2_{L^2(B_1(0))}}{\|u_h^{H,\mathbb{C}} \circ F\|^2_{L^2(\partial B_1(0))}}.$$

Since $u_h^{H,\mathbb{C}} \circ F$ is harmonic, we may combine Green's identity together with the Cauchy-Schwartz inequality to get the bound

$$\|\nabla(u_h^{H,\mathbb{C}} \circ F)\|^2_{L^2(B_1(0))} \le \|u_h^{H,\mathbb{C}} \circ F\|_{L^2(\partial B_1(0))} \|\partial_\nu(u_h^{H,\mathbb{C}} \circ F)\|_{L^2(\partial B_1(0))},$$

where ∂_ν denotes the normal derivative along $\partial B_1(0)$. Using the Cauchy Riemann equations we may turn the normal derivative of the real (resp. imaginary) part of $u_h^{H,\mathbb{C}} \circ F$ into the tangential derivative of the imaginary (resp. real). After changing variables to work on $\partial C_\varepsilon = F(\partial B_1(0))$, and using that $[-\frac{1}{2}, \frac{1}{2}] \subset F(B_r(0))$, it follows that

$$\#\left\{t \in [-\tfrac{1}{2}, \tfrac{1}{2}] : \ u_h^H(t) = 0\right\} \le c_\varepsilon \frac{\|\partial_T u_h^{H,\mathbb{C}}\|_{L^2(\partial C_\varepsilon)}}{\|u_h^{H,\mathbb{C}}\|_{L^2(\partial C_\varepsilon)}}. \tag{17}$$

Thus, to count zeros of φ_h along H, one must bound the quotient $\|\partial_T u_h^{H,\mathbb{C}}\|_{L^2(\partial C_\varepsilon)} / \|u_h^{H,\mathbb{C}}\|_{L^2(\partial C_\varepsilon)}$. Since we are considering the case of boundaryless compact surfaces, in contrast to the planar domains case treated in [13], there are no potential layer formulas. Instead, we use the holomorphically-continued heat kernel $E^{\mathbb{C}}$ and the obvious identity (5) to represent the holomorphically-continued eigenfunctions restricted to the curve H. Indeed, from (5), we know that

$$u_h^{H,\mathbb{C}} = e^{\frac{1}{h}} (E_h^{\mathbb{C}} \varphi_h) \circ q^{\mathbb{C}}. \tag{18}$$

Applying contour deformation and an eigenfunction localization argument, we prove the following result.

Proposition 4 *Let* $0 < \varepsilon \leq \varepsilon_1$. *Identify* ∂C_ε *with* $\mathbb{R}/2\pi\mathbb{Z}$ *and define the frequency cut-off function* $\chi_R \in C_0^\infty(T^*(\partial C_\varepsilon))$, *depending only on the frequency variable, by setting*

$$\chi_R(x,\sigma) = \begin{cases} 1 & |\sigma| \leq R, \\ 0 & |\sigma| \geq R+1, \end{cases}$$

for all $(x,\sigma) \in T^*(\partial C_\varepsilon)$. *Then, there exist positive constants* $h_0 = h_0(\varepsilon)$, *and* $c_{R,\varepsilon} = c_{R,\varepsilon}(R,\varepsilon)$ *satisfying* $c_{R,\varepsilon} \gtrapprox R$ *as* $R \to \infty$, *such that for* $h \in (0, h_0]$

$$\left\| (1 - Op_h(\chi_R))(h\partial_T) u_h^{H,\mathbb{C}} \right\|_{L^2(\partial C_\varepsilon)} = O\left(Rh\, e^{-\frac{c_{R,\varepsilon}}{h}} \right).$$

Proof Let $\kappa : [-\pi,\pi] \to \partial C_\varepsilon$ be an arc-length parametrization of ∂C_ε. For $t, s \in [-\pi,\pi]$ we obtain the following formula for the Schwartz kernel of $(1 - Op_h(\chi_R)) : C^\infty(\partial C_\varepsilon) \to C^\infty(\partial C_\varepsilon)$

$$(1 - Op_h(\chi_R))(t,s) = \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} e^{\frac{i}{h}(t-s+2\pi k)\sigma} (1 - \chi_R(\sigma)) d\sigma, \tag{19}$$

where to shorten notation we write $\chi_R(\sigma)$ for $\chi_R(\kappa(s),\sigma)$ (this is possible since χ_R is a function of the fiber coordinates only). Using (18), (19), and integrating by parts we get for $t \in [-\pi,\pi]$ that

$$\begin{aligned} &(1 - Op_h(\chi_R))[h\partial_T u_h^{H,\mathbb{C}}(\kappa(t))] = \\ &\quad = \frac{i}{2\pi} \sum_{k\in\mathbb{Z}} \int_{-\pi}^{\pi} \int_{\mathbb{R}} e^{\frac{i}{h}(t-s+2\pi k)\sigma} \sigma (1 - \chi_R(\sigma)) u_h^{H,\mathbb{C}}(\kappa(s)) ds d\sigma \\ &\quad = \frac{e^{\frac{1}{h}}}{2\pi} \sum_{k\in\mathbb{Z}} \int_{-\pi}^{\pi} \int_{\mathbb{R}} \int_M i\sigma e^{\frac{i}{h}(t-s+2\pi k)\sigma} E_h^{\mathbb{C}}(q^{\mathbb{C}}(\kappa(s)), y)(1 - \chi_R(\sigma))\, \varphi_h(y)\, dv_g(y)\, ds\, d\sigma. \end{aligned}$$

For $a > 0$, $\sigma \in \mathbb{R}$ and $s \in [-\pi,\pi]$, define

$$\omega_\sigma(s) = s - ia \ \mathrm{sgn}(\sigma).$$

The curve $\kappa \circ \omega_\sigma$ is a contour deformation of κ (i.e ∂C_ε). Choose a small enough so that the image of $\kappa \circ \omega_\sigma$ is contained in $[-1,1]_\varepsilon^{\mathbb{C}}$. Since for all $y \in M$ and $\sigma \in \mathbb{R}$ the map $\tau \mapsto e^{-\frac{i}{h}\tau\cdot\sigma} E_h^{\mathbb{C}}(q^{\mathbb{C}}(\kappa^{\mathbb{C}}(\tau)), y)$ is holomorphic in $\tau \in [\pi,\pi]_\varepsilon^{\mathbb{C}}$, we apply the Cauchy Theorem to shift the contour of integration in the s-variable and get for $t \in [-\pi,\pi]$

$$(1 - Op_h(\chi_R))[h\partial_\tau u_h^{H,\mathbb{C}}(\kappa(t))] =$$
$$= \frac{e^{\frac{1}{h}}}{2\pi} \sum_{k\in\mathbb{Z}} \int_{-\pi}^{\pi} \int_{\mathbb{R}} \int_M i\sigma e^{\frac{i}{h}(t-\omega_\sigma(s)+2\pi k)\sigma}\, E_h^{\mathbb{C}}(\mu_\sigma(s), y)\, (1-\chi_R(\sigma))\, \varphi_h(y)\, dv_g(y)\, ds\, d\sigma,$$

where

$$\mu_\sigma(s) := q^{\mathbb{C}}(\kappa^{\mathbb{C}}(\omega_\sigma(s))).$$

Let $\rho \in C_0^\infty(\mathbb{R})$ be the cut-off function

$$\rho(r) = \begin{cases} 1 & r \in (-\frac{1}{2}, \frac{1}{2}), \\ 0 & r \in \mathbb{R}\backslash\{(-1,1)\}. \end{cases} \tag{20}$$

Choose $\delta = \delta(\varepsilon) > 0$ sufficiently small so that Proposition 3 applies. To simplify notation, we introduce the cutoff function

$$\rho_\delta(s, y; \sigma) := \rho\left(\delta^{-2} r^2(\pi_M \mu_\sigma(s), y)\right) \quad \text{for} \quad (s, y) \in [-\pi, \pi] \times M. \tag{21}$$

We use this function to further decompose the kernel into two pieces depending on whether $r(\pi_M \mu_\sigma(s), y)$ is relatively small (resp. large) in terms of $\mathrm{inj_M}$ and the Grauert tube radius $\varepsilon > 0$. We apply the first (resp. second) estimates for $E_h^{\mathbb{C}}$ in Proposition 3 to control the two cases. More precisely, for $t \in [-\pi, \pi]$, we write

$$2\pi e^{-\frac{1}{h}}(1 - Op_h(\chi_R))[h\partial_\tau u_h^{H,\mathbb{C}}(\kappa(t))] = A_{h,R}(t) + B_{h,R}(t),$$

for

$$A_{h,R}(t) = \sum_{k\in\mathbb{Z}} \int_{-\pi}^{\pi} \int_{\mathbb{R}} \int_M i\sigma e^{\frac{i}{h}(t-\omega_\sigma(s)+2\pi k)\sigma}\, E_h^{\mathbb{C}}(\mu_\sigma(s), y)(1-\chi_R(\sigma))\rho_\delta(s, y; \sigma)\, \varphi_h(y)\, dy\, ds\, d\sigma,$$

and

$$B_{h,R}(t) = \sum_{k\in\mathbb{Z}} \int_{-\pi}^{\pi} \int_{\mathbb{R}} \int_M i\sigma e^{\frac{i}{h}(t-\omega_\sigma(s)+2\pi k)\sigma} E_h^{\mathbb{C}}(\mu_\sigma(s), y)(1-\chi_R(\sigma))(1-\rho_\delta(s, y; \sigma))\varphi_h(y) dy\, ds\, d\sigma.$$

We estimate the terms $A_{h,R}(t)$ and $B_{h,R}(t)$ separately. To deal with the near diagonal term $A_{h,R}(t)$, we apply the asymptotic expansion (8) for the kernel of $E_h^{\mathbb{C}}$ in Proposition 3 to get

$$A_{h,R}(t) = \sum_{k\in\mathbb{Z}} \int_{-\pi}^{\pi} \int_{\mathbb{R}} \int_M e^{\frac{i}{h}\psi_k(t,s,y,\sigma)}\, i\sigma(1-\chi_R(\sigma))N^{\mathbb{C}}(\mu_\sigma(s), y; h)\rho_\delta(s, y; \sigma)\, \varphi_h(y)\, dy\, ds\, d\sigma$$
$$+ O\Big(\int_{|\sigma|>R} |\sigma| e^{-\frac{a|\sigma|+\beta}{h}} d\sigma\Big),$$

where the error term is uniform in $t \in [-\pi, \pi]$ and the phase function

$$\psi_k(t, s, y, \sigma) := (t - \omega_\sigma(s) + 2\pi k)\sigma + i\, r_{\mathbb{C}}^2(\mu_\sigma(s), y)/4, \tag{22}$$

for $(s, t, y) \in [-\pi, \pi] \times [-\pi, \pi] \times M$ and $(s, y; \sigma) \in \operatorname{supp} \rho_\delta$. Note that the imaginary part of the phase ψ_k satisfies

$$\begin{aligned} \operatorname{Im}[\psi_k(t, s, y, \sigma)] &= a|\sigma| + \operatorname{Re}(r_{\mathbb{C}}^2(\mu_\sigma(s), y))/4 \\ &\geq a|\sigma| + \alpha, \end{aligned}$$

for $\alpha := \min\{\operatorname{Re}(r_{\mathbb{C}}^2(\mu_\sigma(s), y))/4 : \ (s, y) \in [-\pi, \pi] \times M, \ r(\pi_M \mu_\sigma(s), y)) < \delta\}$.

We observe that

$$\int_{|\sigma|>R} |\sigma| e^{\frac{-a|\sigma|}{h}} d\sigma = O(Rh\, e^{-\frac{aR}{h}}). \tag{23}$$

Since for $(t, s) \in [-\pi, \pi] \times [-\pi, \pi]$, one has the lower bound $|\partial_\sigma \psi_k(t, s, y, \sigma)| \gtrapprox |k|$ as $k \to \infty$, it then follows from (23) and successive integrations by parts in σ that

$$\left|A_{h,R}(t)\right| = O\left(Rh\, e^{-\frac{aR+\alpha}{h}}\right) + O(Rhe^{-\frac{aR+\beta}{h}}). \tag{24}$$

On the other hand, when $r(\pi_M \mu_\sigma(s), y) > \delta/2$, by Proposition 3 (ii) we know $E^{\mathbb{C}}(\mu_\sigma(s), y, h) = O(\, e^{-\frac{\delta^2}{128h}})$. By an application of the Cauchy-Schwarz inequality in $y \in M$, it follows that

$$\left|B_{h,R}(t)\right| \leq C \int_{|\sigma|>R} |\sigma| e^{-\frac{\rho|\sigma|+\alpha}{h}} e^{-\frac{\delta^2}{128h}} d\sigma = O\left(Rh\, e^{-\frac{1}{h}(\frac{\delta^2}{128}+\alpha+\rho R)}\right). \tag{25}$$

Finally, since $2\pi e^{-\frac{1}{h}} |(1 - Op_h(\chi_R)) u_h^{H,\mathbb{C}}(\kappa(t))| \leq \left|A_{h,R}(t)\right| + \left|B_{h,R}(t)\right|$, the result follows from (24) and (25). □

1.2.1 Proof of Theorem 1

In view of Proposition 4, we can now complete the proof of Theorem 1. From (17),

$$\begin{aligned} &\#\{q \in H : \ \varphi_h(q) = 0\} \\ &\leq \frac{c_\varepsilon}{h} \frac{\|h\partial_T u_h^{H,\mathbb{C}}\|_{L^2(\partial C_\varepsilon)}}{\|u_h^{H,\mathbb{C}}\|_{L^2(\partial C_\varepsilon)}} \\ &\leq \frac{c_\varepsilon}{h} \left(\frac{\left\|Op_h(\chi_R)(h\partial_T)\, u_h^{H,\mathbb{C}}\right\|_{L^2(\partial C_\varepsilon)}}{\|u_h^{H,\mathbb{C}}\|_{L^2(C_\varepsilon)}} + \frac{\left\|(1 - Op_h(\chi_R))(h\partial_T)\, u_h^{H,\mathbb{C}}\right\|_{L^2(\partial C_\varepsilon)}}{\|u_h^{H,\mathbb{C}}\|_{L^2(\partial C_\varepsilon)}} \right). \end{aligned} \tag{26}$$

Since $u_h^{H,\mathbb{C}}$ is holomorphic in the strip $[-1,1]_\varepsilon^{\mathbb{C}}$, by the Cauchy Integral formula it follows that for $t \in [-\frac{1}{2}, \frac{1}{2}]$

$$u_h^H(t) = \frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{u_h^{H,\mathbb{C}}(\kappa(s))}{\kappa(s) - t}\, ds$$

where we continue to write κ for the parametrization of ∂C_ε. From the Cauchy-Schwarz inequality, it follows that there is a constant $c_1 > 0$ such that

$$\|u_h^H\|_{L^\infty([-\frac{1}{2},\frac{1}{2}])} \leq c_1 \, \|u_h^{H,\mathbb{C}}\|_{L^2(\partial C_\varepsilon)}.$$

By the goodness condition this implies that there is $c_2 > 0$ so that

$$\|u_h^{H,\mathbb{C}}\|_{L^2(\partial C_\varepsilon)} \geq c_2 \, e^{-\frac{c_0}{h}}. \tag{27}$$

Combining Proposition 4 with (27),

$$\frac{\left\|(1 - Op_h(\chi_R))(h\partial_T)\, u_h^{H,\mathbb{C}}\right\|_{L^2(\partial C_\varepsilon)}}{\|u_h^{H,\mathbb{C}}\|_{L^2(\partial C_\varepsilon)}} = O_{R,\varepsilon}(Rh\, e^{\frac{-c_{R,\varepsilon}+c_0}{h}}). \tag{28}$$

with $c_{R,\varepsilon} \gtrapprox R$ as $R \to \infty$.

Furthermore, since $(Op_h(\chi_R) \circ (h\partial_T)) \in \Psi_h^{0,-\infty}(\partial C_\varepsilon)$, by L^2-boundedness and equation (27),

$$\frac{\left\|Op_h(\chi_R)(h\partial_T)\, u_h^{H,\mathbb{C}}\right\|_{L^2(\partial C_\varepsilon)}}{\|u_h^{H,\mathbb{C}}\|_{L^2(\partial C_\varepsilon)}} = O_{R,\varepsilon}(1). \tag{29}$$

The proof follows from the estimates in (29) and (28) by choosing $c_{R,\varepsilon}$ large enough so that $-c_{R,\varepsilon} + c_0 < 0$. □

2 Goodness Versus Weak Goodness

Proposition 5 *Let $H \subset M^n$ be a real analytic curve. Then, the weak goodness assumption on H in Definition 2 is equivalent to the goodness assumption in Definition 1.*

Proof Goodness clearly implies weak goodness since there must exist a point $q \in H$ at which $|u_h(q)| \geq e^{-C/h}$.

Conversely, suppose H is weakly-good; that is,

$$\sup_{z \in H^{\mathbb{C}}} |u_h^{H,\mathbb{C}}(z)| \geq e^{-C/h}. \tag{30}$$

Let H, H_{ϵ_1} and H_{ϵ_2} with $0 < \epsilon_1 < \epsilon_2$ be three level curves in the tube $H^{\mathbb{C}}$ (see (15) for definitions). Without loss of generality, we also assume that

$$\sup_{z \in H_{\epsilon_1}} |u_h^{H,\mathbb{C}}(z)| = e^{-C/h}.$$

By Hadamard three circles theorem, with $0 < \theta < 1$,

$$\begin{aligned} \sup_{z \in H_{\epsilon_1}} |u_h^{H,\mathbb{C}}(z)| &\leq \sup_{z \in H_{\epsilon_2}} |u_h^{H,\mathbb{C}}(z)|^{1-\theta} \times \sup_{q \in H} |u_h^H(q)|^{\theta} \\ &\leq e^{2\epsilon_2(1-\theta)/h} \cdot \|u_h^H\|_{L^\infty(H)}^{\theta}. \end{aligned} \tag{31}$$

In the last line we used a sup estimate for $|u_h^{H,\mathbb{C}}|$. For this, we recall that [24]

$$\|u_h^{H,\mathbb{C}}\|_{L^\infty(H_{\epsilon_2}^{\mathbb{C}})} = O(h^{\frac{-n+1}{4}} e^{\epsilon_2/h}) = O(e^{2\epsilon_2/h}).$$

Consequently, by the weak goodness assumption (30) and (31),

$$\|u_h^H\|_{L^\infty(H)} \geq e^{-C/h}.$$

By continuity, we choose $q_0 \in H$ so that

$$|u_h^H(q_0)| = e^{-C/h}.$$

By the standard bound for Laplace eigenfunctions, one also has that

$$\|\partial_s u_h^H\|_{L^\infty(H)} = O(h^{-(n+1)/2}). \tag{32}$$

Since by (32) the tangential derivative of u_h^H along H has at most polynomial growth in h^{-1}, it follows by Taylor expansion along H centered at q_0 that there is an subinterval $I(h) \subset H$ containing q_0 of length $e^{-C'/h}$ with $C' > C > 0$ such that for $q \in I(h)$,

$$|u_h^H(q)| \geq e^{-C''/h}.$$

Consequently,

$$\|u_h^H\|_{L^2(H)} \geq e^{-C''/h}$$

and so, H is good. □

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Upper Bounds for Bergman Kernels Associated to Positive Line Bundles with Smooth Hermitian Metrics

Michael Christ

Abstract Off-diagonal upper bounds are established for Bergman kernels associated to powers L^λ of holomorphic line bundles L over compact complex manifolds, asymptotically as the power λ tends to infinity. The line bundle is assumed to be equipped with a Hermitian metric with positive curvature form, which is C^∞ but not necessarily real analytic. The bounds are of the form $\exp(-h(\lambda)\sqrt{\lambda \log \lambda})$ where h tends to infinity at a non-universal rate. This form is best possible.

1 Introduction

1.1 The Setting

Let X be a connected compact complex manifold, without boundary. Let X be equipped with a C^∞ Hermitian metric g, along with the metrics on the bundles $B^{(p,q)}(X)$ of forms of bidegree (p, q) induced by g, and the volume form on X associated to the induced Riemannian metric. Denote by $\rho(z, z')$ the Riemannian distance from $z \in X$ to $z' \in X$.

Let L be a positive holomorphic line bundle over X. Let L be equipped with a C^∞ Hermitian metric ϕ whose curvature is positive at every point. ϕ is not assumed to be real analytic.

For each positive integer λ, let the line bundle L^λ be the tensor product of λ copies of L. L^λ inherits from ϕ a Hermitian metric in a natural way; if $v \in L_z$ then the λ–fold tensor product $v \otimes v \otimes \cdots \otimes v$ satisfies $|v \otimes v \otimes \cdots \otimes v| = |v|^\lambda$.

Let $L^2_\lambda = L^2(X, L^\lambda)$ be the Hilbert space of equivalence classes of all square integrable Lebesgue measurable sections of L^λ. Likewise there are the Hilbert spaces $L^2(X, B^{(0,q)} \otimes L^\lambda)$. Let H^2_λ be the closed subspace of L^2_λ consisting of all

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M. Christ (✉)
Department of Mathematics, University of California, Berkeley, CA 94720-3840, USA
e-mail: mchrist@berkeley.edu

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holomorphic sections. The Bergman projection is defined to be the orthogonal projection B_λ from L^2_λ onto H^2_λ. The Bergman kernel $B_\lambda(z, z')$ is the associated distribution-kernel; $B_\lambda(z, z')$ is a complex linear endomorphism from the fiber $L^\lambda_{z'}$ to the fiber L^λ_z.

Much is known concerning the nature of these Bergman kernels. In particular, detailed asymptotic expansions are known near the diagonal $z = z'$, that is, when $\rho(z, z')$ is bounded by a constant multiple of $\lambda^{-1/2}$. See for instance [1, 6, 20, 23] as well as the related work [5] of Boutet de Monvel and Sjöstrand on the Bergman and Szegö kernels associated to domains in $\mathbb{C}^{n+1}$. This paper is concerned with upper bounds when z, z' are far apart, that is, behavior for large λ when $\rho(z, z')$ is bounded below by a positive quantity independent of λ. If ϕ and g are real analytic, then for large λ, $|B_\lambda(z, z')| \le C_\delta e^{-c_\delta \lambda}$ whenever $\rho(z, z') \ge \delta > 0$, where $C_\delta < \infty$ and $c_\delta > 0$ are independent of λ. This is interpreted in the theory of Bleher, Shiffman and Zelditch [3, 4, 16] of random zeroes of sections of L^λ as an exponentially small upper bound on the degree of correlation between zeros at distinct points.

1.2 Subexponential Off-Diagonal Decay

It was shown in [9] that this exponential decay fails to hold, in general, if ϕ is merely infinitely differentiable. More quantitatively, for any function h satisfying $h(t) \to \infty$ as $t \to +\infty$ there exists [9] an example for which

$$\limsup_{\lambda \to \infty} \sup_{\rho(z,z') \ge \delta} e^{h(\lambda)\sqrt{\lambda \log \lambda}} |B_\lambda(z, z')| = \infty \tag{1.1}$$

for some $\delta > 0$. In this paper we establish an upper bound which dovetails with these lower bounds.

Theorem 1 *Let L be a positive holomorphic line bundle over a connected compact complex manifold X. Let there be given a C^∞ positive metric on L with strictly positive curvature form, and a C^∞ Hermitian metric on X. For any $\delta > 0$ there exist $\Lambda < \infty$ and a function h satisfying $h(\lambda) \to \infty$ as $\lambda \to \infty$ such that for all $z, z' \in X$ satisfying $\rho(z, z') \ge \delta$,*

$$|B_\lambda(z, z')| \le e^{-h(\lambda)\sqrt{\lambda \log \lambda}} \text{ for all } \lambda \ge \Lambda. \tag{1.2}$$

The analysis below of B_λ is based on its connection with the fundamental solution of a partial differential operator, $\Box_\lambda$. Denote by $\bar\partial_\lambda$ the usual Dolbeault operator, mapping sections of $B^{(0,q)} \otimes L^\lambda$ to sections of $B^{(0,q+1)} \otimes L^\lambda$. Denote by $\bar\partial^*_\lambda$ its formal adjoint, with respect to the Hilbert space structures L^2_λ defined above. Define

$$\Box_\lambda = \begin{cases} \bar\partial^*_\lambda \bar\partial_\lambda + \bar\partial_\lambda \bar\partial^*_\lambda & \text{for } n > 1 \\ \bar\partial_\lambda \bar\partial^*_\lambda & \text{for } n = 1, \end{cases} \tag{1.3}$$

acting on sections of $B^{(0,1)} \otimes L^\lambda$. For each λ, $\Box_\lambda$ is an elliptic second-order linear system of partial differential operators with C^∞ coefficients. When it is expressed in local coordinates, its coefficients are $O(\lambda^2)$ in any C^N norm.

Because the metric ϕ is positive, there exists a constant $c > 0$ such that for all sufficiently large $\lambda \in \mathbb{N}$,

$$\langle \Box_\lambda u, u \rangle \geq c\lambda \|u\|_{L^2}^2 \tag{1.4}$$

for all twice continuously differentiable sections u of $B^{(0,1)} \otimes L^\lambda$. This bound is deduced from a well-known integration by parts calculation [13]. Because of this lower bound and because $\Box_\lambda$ is formally self-adjoint and elliptic, there exists a unique self-adjoint bounded linear operator G_λ on $L^2(X, B^{(0,1)} \otimes L^\lambda)$ satisfying $\Box_\lambda \circ G_\lambda = I$, the identity operator.

The operator B_λ is related to $\Box_\lambda$ by

$$B_\lambda = I - \bar\partial_\lambda^* \circ G_\lambda \circ \bar\partial_\lambda. \tag{1.5}$$

Thus the Bergman kernel is expressed in terms of certain derivatives of the distribution-kernel for the operator G_λ. We denote this distribution-kernel by $G_\lambda(z, z')$. Because $G_\lambda(z, z')$ is a solution of $\Box_\lambda G_\lambda = 0$ with respect to the variable z and its complex conjugate is a solution of the same equation with respect to z', elliptic regularity theory guarantees that $G_\lambda(z, z')$ is a C^∞ function of (z, z') on the complement of the diagonal.

We will show that $G_\lambda(z, z') = O(e^{-h(\lambda)\sqrt{\lambda \log \lambda}})$ for (z, z') at any positive distance from the diagonal. The corresponding bound holds for those partial derivatives that express the distribution-kernel for $\bar\partial_\lambda^* \circ G_\lambda \circ \bar\partial_\lambda$ at (z, z') will be an easy consequence.

For real analytic metrics, the Bergman kernel is $O(e^{-c\lambda})$ away from the diagonal. Combining the result established here with that of [9], one knows that for C^∞ metrics, decay can in some instances be essentially as slow as $e^{-h(\lambda)\sqrt{\lambda \log \lambda}}$, but is never slower. Zelditch has raised the question of which, or what, behavior is typical, and which properties of a metric can be inferred from the off-diagonal decay rate of the associated Bergman kernels. This issue is examined in [10, 24].

1.3 Orientation

A weaker upper bound $|B_\lambda(z, z')| \leq e^{-c\sqrt{\lambda}}$, valid whenever $\rho(z, z') \geq \delta$, is a simple consequence of (1.4), and requires only C^2 or even $C^{1,1}$ regularity of ϕ. In the context of global analysis on $\mathbb{C}^1$, this was shown in [9]. For positive line bundles over complex manifolds, it was noted by Berndtsson [2]. Closely related results are found in works of Delin [11] and Lindholm [15]. The novelty in Theorem 1 is a double improvement of the exponent, from $c\sqrt{\lambda}$ to $h(\lambda)\sqrt{\lambda \log \lambda}$.

To establish the weaker bound, consider any real-valued auxiliary weight $\psi \in C^2(X)$. For any $\varepsilon > 0$ and all sufficiently large λ,

$$
\begin{aligned}
\mathrm{Re}\ & \left(\langle e^{\varepsilon\sqrt{\lambda}\psi}\Box_\lambda e^{-\varepsilon\sqrt{\lambda}\psi}u, u\rangle\right) \\
& \geq \langle \Box_\lambda u, u\rangle - C\lambda^{1/2}\varepsilon\|\bar\partial_\lambda u\|\cdot\|u\| - C\lambda^{1/2}\varepsilon\|\bar\partial_\lambda^* u\|\cdot\|u\| - C\lambda\varepsilon^2\|u\|^2 \\
& \geq (c - C\varepsilon)\lambda\|u\|_{L^2}^2
\end{aligned}
\tag{1.6}
$$

for all sections $u \in C^2(X, B^{(0,1)})$, where C depends on the C^2 norm of ψ. This is $\geq \|u\|^2$ for all sufficiently large λ, provided that ε is chosen to be sufficiently small as a function of $\|\psi\|_{C^2}$. The inequality (1.6) can alternatively be interpreted as a weighted inequality for the inverse operator $\Box_\lambda^{-1}$, with weight $e^{2\varepsilon\sqrt{\lambda}\psi}$. Whenever U, U' are disjoint sets satisfying distance $(U, U') \geq \delta > 0$, by choosing ψ so that $\psi \geq 1$ on U' and $\psi \leq 0$ on U we conclude that $\Box_\lambda^{-1}$ maps $L^2(U)$ to $L^2(U')$, where these norms are defined without reference to the auxiliary weight ψ, with operator norm $O(e^{-c\sqrt{\lambda}})$ where $c > 0$ depends on δ. The pointwise bound for $B_\lambda(z, z')$ for $(z, z') \in U \times U'$ is a simple consequence, by a routine elliptic regularity bootstrapping argument which will be used below in the main body of the proof. See the proof of Lemma 5.

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2 Unweighted Bounds and Twisted Operators

It will be convenient to work in an equivalent framework, in a coordinate patch $U \subset X$ over which L is holomorphically trivial, and in which norms are defined by integrals without λ–dependent weights, but the underlying operators $\bar\partial_\lambda$, $\Box_\lambda$ are twisted. This framework is more natural for discussion of regularity.

Let U be a small coordinate patch on X, over which L may be identified with $U \times \mathbb{C}$. Functions and differential forms may be regarded as scalar–valued. For each degree q, there is an operator $\bar\partial_\lambda$, which maps sections of $B^{(0,q)} \otimes L^\lambda$ over U to sections of $B^{(0,q+1)} \otimes L^\lambda$ over U. $\bar\partial_\lambda$ is naturally identified with the standard Cauchy-Riemann operator $\bar\partial$, which maps sections of $B^{(0,q)}$ to sections of $B^{(0,q+1)}$.

$\phi \in C^\infty$ is $\mathbb{R}$-valued, and the positive curvature assumption means precisely that its complex Hessian matrix $\left(\partial^2\phi/\partial z_j\partial\bar z_k\right)$ is strictly positive definite at each point of U. The C^∞ Hermitian metric g given for X is interpreted as a C^∞ Hermitian metric on U, and gives rise to a volume form, expressed as a measure μ on U, which is a smooth nonvanishing multiple of Lebesgue measure on $\mathbb{C}^n$. It also gives rise, for each q, to a C^∞ metric on $B^{(0,q)}$ over U. The L^2 norm squared of a section of $B^{(0,q)}$ over U, regarded as a scalar-valued function f, is expressed as $\int_U |f(z)|^2 e^{-2\lambda\phi(z)}\,d\mu(z)$, where $|f(z)|$ is measured according to g.

Substituting $fe^{-\lambda\phi} = u$, the norm squared of f with respect to the weight ϕ becomes $\|f\|_{L^2}^2 = \int_U |u(z)|^2\,d\mu(z)$; there is no weight in this integral. Moreover

$$
e^{-\lambda\phi}\bar\partial f = e^{-\lambda\phi}\bar\partial(ue^{\lambda\phi}) = \bar\partial u + \lambda a \wedge u \tag{2.1}
$$

where $a = \bar{\partial}\phi \in C^\infty$. For each q define

$$\bar{D}_\lambda = e^{-\lambda\phi} \circ \bar{\partial} \circ e^{\lambda\phi} = \bar{\partial} + \lambda a \wedge \cdot. \tag{2.2}$$

This is a first-order linear partial differential operator with smooth coefficients, but with a zero-th order term proportional to the large parameter λ. The formal adjoint(s) $\bar{D}^*_\lambda$ are defined with respect to the given metric g and associated volume form. These data are assumed to be only C^∞, rather than C^ω, but their potential lack of analyticity is less significant than that of ϕ because they are not multiplied by the large parameter λ.

Define

$$\Delta_\lambda = \begin{cases} \overline{D}_\lambda \overline{D}^*_\lambda + \overline{D}^*_\lambda \overline{D}_\lambda & \text{for } n > 1, \\ \overline{D}_\lambda \overline{D}^*_\lambda & \text{for } n = 1, \end{cases} \tag{2.3}$$

acting on (0, 1) forms over U. Under these identifications,

$$\Delta_\lambda = e^{-\lambda\phi} \circ \Box_\lambda \circ e^{\lambda\phi}. \tag{2.4}$$

The function

$$\mathcal{G}_\lambda(z, w) = e^{-\lambda\phi(z)+\lambda\phi(w)} G_\lambda(z, w) \tag{2.5}$$

represents a fundamental solution for Δ_λ with singularity at $z = w$, in the usual sense. This is a section of the complex endomorphism bundle of $B^{(0,1)}$ over $U \times U$ minus the diagonal; in this local coordinate system, it is a matrix-valued function. Its size $|\mathcal{G}_\lambda(z, w)|$ is defined with respect to given smooth metrics which do not depend on λ, so upper bounds with respect to these metrics are uniformly equivalent to upper bounds with respect to the standard metrics on these bundles.

Theorem 1 is therefore equivalent to an upper bound for all (z, w) in $U \times U$ minus the diagonal of the form

$$|\mathcal{G}_\lambda(z, w)| \le e^{-A\sqrt{\lambda \log \lambda}} \quad \text{for all } \lambda \ge \Lambda(\delta, A), \text{ whenever } |z - w| \ge \delta \tag{2.6}$$

with corresponding upper bounds for all first and second–order derivatives of $\mathcal{G}_\lambda$ with respect to z, w in this same region.

3 A Near-Diagonal Upper Bound

Theorem 1, which is concerned with the nature of G_λ far from the diagonal, will be derived from a description of G_λ much nearer the diagonal. The main point is the manner in which the bounds depend on λ, A; these bounds are completely independent of the exponent A, provided only that λ exceeds a certain threshold, which does depend on A. The reasoning below will require bounds for derivatives of G_λ, as well as for G_λ itself. These bounds are more naturally expressed in terms of the twisted kernels $\mathcal{G}_\lambda$ introduced above. ∇ will denote the gradient in $\mathbb{C}^n \times \mathbb{C}^n$, with respect to both coordinates z, z'.

Proposition 2 *There exist $c_0, A_0 \in \mathbb{R}^+$ such that for any $A \in [A_0, \infty)$ there exists $\Lambda = \Lambda(A) < \infty$ such that for any $\lambda \geq \Lambda$ and any $z, z' \in U$ satisfying*

$$A_0 \lambda^{-1/2} \sqrt{\log \lambda} \leq |z - z'| \leq A \lambda^{-1/2} \sqrt{\log \lambda}, \tag{3.1}$$

$\mathcal{G}_\lambda(z, z')$ satisfies

$$|\mathcal{G}_\lambda(z, z')| + |\nabla_{z,z'} \mathcal{G}_\lambda(z, z')| \leq e^{-c_0 \lambda |z-z'|^2}. \tag{3.2}$$

As is well understood, there is a natural scale $\asymp \lambda^{-1/2}$ inherent in this situation. In the model situation in which $X = \mathbb{C}^n$ and $\phi(z) \equiv \frac{1}{2}|z|^2$, $|\mathcal{G}_\lambda(z, z')| \asymp e^{-c_0 \lambda |z-z'|^2} |z - z'|^{2-2n}$ for $n > 1$, with the power of $|z - z'|$ replaced by $\log(1/|z - z'|)$ for $n = 1$. Proposition 2 asserts essentially that this model upper bound persists up to a distance which is greater by a multiplicative factor of $A\sqrt{\log \lambda}$ than the natural scaled distance, for arbitrarily large A. The lower bound $|z - z'| \geq A_0 \lambda^{-1/2} \sqrt{\log \lambda}$ is an inessential technicality introduced in order to simplify the statement and proof of the lemma; otherwise the upper bound would have to be modified in order to take the unbounded near-diagonal factor $|z - z'|^{2-n}$ into account.

In the next section we will show how Theorem 1 is an essentially formal consequence of Proposition 2. We will then review and establish foundational results, none of which involve significant novelty, before proving Proposition 2.

4 The Near-Diagonal Bound Implies the Far-From-Diagonal Bound

$\|T\|_{\text{op}}$ will denote the operator norm of T, as an operator on $L^2(X, B^{(0,1)} \otimes L^\lambda)$. Recall that ρ denotes the Riemannian distance function on X^2. The following obvious statement is at the heart of the construction.

Lemma 3 *Let T_1, T_2 be bounded linear operators on $L^2(X, B^{(0,q)} \otimes L^\lambda)$. Let $r_i > 0$ and suppose that for $i = 1, 2$, the distribution-kernel associated to T_i is supported in $\{(z, z') \in X^2 : \rho(z, z') \leq r_i\}$. Then the distribution-kernel associated to $T_1 \circ T_2$ is supported in $\{(z, z') \in X^2 : \rho(z, z') \leq r_1 + r_2\}$.*

This will be used to prove:

Lemma 4 *Let $A < \infty$ and $\delta > 0$. There exist $C < \infty$ and $\Lambda < \infty$ such that for every $\lambda \geq \Lambda$ there exists a bounded linear map T from the space of L^2 sections of $B^{(0,1)} \otimes L^\lambda$ to itself with these two properties: Firstly, the distribution-kernel for T is supported in $\{(z, z') : \rho(z, z') \leq \delta\}$. Secondly,*

$$\|T \circ \Box_\lambda - I\|_{op} \le e^{-A\lambda^{1/2}\sqrt{\log\lambda}}. \tag{4.1}$$

Proof Choose an auxiliary function $\eta \in C^\infty([0,\infty))$ that satisfies $\eta(x) \equiv 1$ for $x \le \frac{1}{2}$, and $\eta(x) \equiv 0$ for all $x \ge 1$. Let $A < \infty$. Let P be the operator with distribution-kernel

$$K(z,w) = G_\lambda(z,w)\eta(A^{-2}\lambda(\log\lambda)^{-1}\rho^2(z,w)).$$

Letting $\Box_\lambda$ act with respect to the z variable, and applying Leibniz's rule and the chain rule,

$$|\Box_\lambda(K(z,w) - G_\lambda(z,w))| \le C\lambda^2|G_\lambda(z,w)| + C\lambda^2|\nabla G_\lambda(z,w)|.$$

On the complement of the diagonal, $\Box_\lambda K(z,w)$ is supported where $\rho(z,w) \asymp A\lambda^{-1/2}(\log\lambda)^{1/2}$. In this region, according to Proposition 2,

$$|G_\lambda(z,w)| + |\nabla G_\lambda(z,w)| \le C\lambda^C e^{-c\lambda A^2\lambda^{-1}\log\lambda} \le C\lambda^{C-cA^2}.$$

So in all,

$$|\Box_\lambda(K(z,w) - G_\lambda(z,w))| \le \lambda^{C-cA^2}$$

for all sufficiently large λ, uniformly for all pairs (z,w) in X^2 minus the diagonal. Since $\Box_\lambda \circ G_\lambda = I$, this is an upper bound for the operator norm of $\Box_\lambda \circ P - I$. Since both $\Box_\lambda$ and P are formally self-adjoint, the same bound holds for $P \circ \Box_\lambda - I$.

Given $\delta > 0$, choose N to be the largest integer such that $NA\lambda^{-1/2}(\log\lambda)^{1/2} \le \delta$. Thus

$$N \asymp A^{-1}\lambda^{1/2}(\log\lambda)^{-1/2}\delta.$$

Set

$$E = I - \Box_\lambda \circ P \qquad \text{and} \qquad T = P \circ \sum_{j=0}^{N-1} E^j$$

so that

$$\Box_\lambda \circ T = I - E^N.$$

Because the distribution-kernel for P is supported where $\rho(z,w) \le A\lambda^{-1/2}\sqrt{\log\lambda}$, the distribution-kernel for T is supported where

$$\rho(z,w) \le NA\lambda^{-1/2}\sqrt{\log\lambda} \le \delta,$$

according to Lemma 3.

Since $\|E\|_{\text{op}} = \|\Box_\lambda \circ P - I\|_{\text{op}} \le \lambda^{C-cA^2}$,

$$\|E^N\|_{\text{op}} \le \lambda^{(C-cA^2)N} \le \lambda^{(C-cA^2)c(A^{-1}\lambda^{1/2}(\log\lambda)^{-1/2}\delta)} \le e^{-c'A\lambda^{1/2}\sqrt{\log\lambda}}$$

for all sufficiently large A. $\Box$

Proof of Theorem 1 Consider any $z' \neq z'' \in X$. To prove the upper bound for $B_\lambda(z', z'')$, consider any L^2 section f of $B^{(0,1)} \otimes L^\lambda$ that is supported in $B'' = B(z'', \frac{1}{4}\rho(z', z''))$ and satisfies $\|f\|_{L^2} \le 1$. Choose T as in Lemma 4, with distribution-kernel supported within distance $\frac{1}{2}\rho(z', z'')$ of the diagonal. Then in $B' = B(z', \frac{1}{4}\rho(z', z''))$,

$$\begin{aligned} G_\lambda f &= T\Box_\lambda G_\lambda f + O\big(e^{-A\sqrt{\lambda\log\lambda}}\|G_\lambda f\|\big) \\ &= Tf + O\big(e^{-A\sqrt{\lambda\log\lambda}}\|f\|\big). \end{aligned}$$

Since T has distribution-kernel supported within distance $\frac{1}{2}\rho(z, z')$ of the diagonal, $Tf \equiv 0$ in B'. Therefore

$$G_\lambda f = O(e^{-A\sqrt{\lambda\log\lambda}}\|f\|) \text{ in } L^2(B') \text{ norm}.$$

Thus as an operator from $L^2(B'')$ to $L^2(B')$, G_λ has operator norm $O(e^{-A\sqrt{\lambda\log\lambda}})$. Because $G_\lambda(z, w)$ is a solution of elliptic linear partial differential equations with C^∞ coefficients with respect to both variables z, w, and because the coefficients of those equations are $O(\lambda^2)$ in every C^N norm, it follows from standard bootstrapping arguments that for any N, $G_\lambda \in C^N(B' \times B'')$, with norm $O(e^{-A\sqrt{\lambda\log\lambda}})$. Since the Bergman kernel is the distribution-kernel for $I - \bar\partial_\lambda^* G_\lambda \bar\partial_\lambda$, this result with $N = 2$ includes the desired upper bound. □

5 Off-the-Shelf Upper Bounds

Thus far the argument has been purely formal. We now state two quantitative estimates on which the proof of Proposition 2 will rely. One concerns metrics with nearly minimal regularity; the other, real analytic metrics. The C^∞ case is intermediate between these two.

5.1 Low Regularity Upper Bounds

Lemma 5 *For each $n \ge 1$ there exists $N < \infty$ with the following property. Let L be a positive holomorphic line bundle over a compact complex manifold X of dimension n, equipped with a Hermitian metric ϕ of class C^N. Let X likewise be equipped with a Hermitian metric g of class C^N. Let $U, \mathcal{G}_\lambda$ be as defined above. Then there exists $C < \infty$ such that for all sufficiently large positive integers λ,*

$$|\mathcal{G}_\lambda(z, z')| + |\nabla\mathcal{G}_\lambda(z, z')| \le (\lambda + |z - z'|^{-1})^C \tag{5.1}$$

for all $z \neq z' \in U$.

Here ∇ denotes the gradient with respect to both variables z, z'.

Considerably sharper upper bounds can be established, but they will not be needed in the proof of Theorem 1.

Proof Let μ be any fixed smooth multiple of Lebesgue measure. Let $r > 0$ be small, and consider two balls $B', B \subset U$ of radii r satisfying $|z - z'| \geq r$ for all $(z', z) \in B' \times B$. For any square integrable $(0, q)$ form f supported in B, consider the $(0, q)$ form F with domain B' defined by

$$F(z) = \int \mathcal{G}_\lambda(z, w) f(w)\, d\mu(w).$$

Then $\|F\|_{L^2} = O(\lambda^{-1}\|f\|_{L^2})$. Since B' is at positive distance from B, F is annihilated by Δ_λ.

Now Δ_λ is a second order elliptic differential operator, whose coefficients are majorized by $C_N\lambda^2$ in any C^N norm. Therefore a routine bootstrapping argument, exploiting elliptic regularity, gives

$$\|F\|_{C^N(B'')} \leq C'_N(r^{-1} + \lambda)^{2N}\|f\|_{L^2}$$

for any $N < \infty$, for any ball $B'' \subset B'$ of radius $r/2$ whose distance to the boundary of B' is comparable to r. Here C'_N is a constant that may depend on X, L, φ, g and on the choice of coordinate patch U, but is independent of r, λ. Factors with order of magnitude equal to powers of r^{-1} arise from Leibniz's formula when Δ_λ is applied to products of F with auxiliary cutoff functions, chosen to satisfy natural bounds dictated by scaling.

Because f was arbitrary and the upper bounds for F are proportional to the L^2 norm of f, this conclusion can be equivalently restated as

$$\|\partial^\alpha_{z'}\mathcal{G}_\lambda(z', z)\|_{L^2_z(B)} \leq C_\alpha(r^{-1} + \lambda)^{C_\alpha}$$

for every multi-index α, where ∂^α denotes an arbitrary partial derivative of order $|\alpha|$ in the coordinates of U, and where the notation $L^2_z(B)$ indicates that the $L^2(B)$ norm is taken with respect to the variable z. For an arbitrary point $z' \in B'$, consider the $(0, q)$ form $B \ni z \mapsto \partial^\alpha_{z'}\mathcal{G}_\lambda(z', z)$, with domain B. It is annihilated by the transpose of Δ_λ, which is another second order elliptic differential operator, whose coefficients are likewise $O(\lambda^2)$ in any C_N norm. □

This type of analysis is employed for instance in [8].

5.2 High Regularity Upper Bounds

We work now in the unweighted twisted framework introduced above. Let $B \subset \mathbb{C}^n$ be any fixed open ball of positive radius, and let $\tilde{B} \Subset B$ be any relatively compact subball.

Lemma 6 *Let the ball $B \subset \mathbb{C}^n$ be equipped with a C^ω Hermitian metric g. Let L be any holomorphic line bundle over B, equipped with a positive C^ω Hermitian metric ϕ. There exist $\Lambda < \infty$ and $c > 0$ such that for any $\lambda \geq \Lambda$ and any solution u of $\Delta_\lambda u \equiv 0$ on B*

$$|u(z)| \leq e^{-c\lambda}\|u\|_{L^2(B)} \text{ for all } z \in \tilde{B}. \tag{5.2}$$

Moreover, given a family of such metrics g, ϕ, c may be taken to be independent of g, ϕ, provided that g, ϕ are uniformly C^ω and that the metrics ϕ are uniformly positive.

Positivity of ϕ means that in local coordinates, $\sum_{i,j=1}^n \frac{\partial^2 \phi(z)}{\partial z_i \partial \bar{z}_j} \zeta_i \bar{\zeta}_j \geq a|\zeta|^2$ for all $\zeta \in \mathbb{C}^n$ and all z, for some $a > 0$. We say that a family of metrics ϕ is uniformly positive if a is bounded below by some positive constant uniformly for all elements of the family in question. Likewise, we say that such a family is uniformly C^ω if there exists $C < \infty$ for which

$$\Big|\frac{\partial^\alpha \phi}{\partial(z,\bar{z})^\alpha}\Big| \leq C^{1+|\alpha|}|\alpha|! \text{ uniformly on } B. \tag{5.3}$$

for every multi-index α and all metrics ϕ. The same applies to g.

Naturally associated to the pair (L, ϕ) are the dual bundle L^*, a strictly pseudoconvex domain $D \subset L^*$ defined in terms of ϕ, and the unit circle bundle Y in L^* which is the boundary of D, a Cauchy-Riemann manifold. Temporarily denote by $\pi_D : D \to X$ and $\pi_Y : Y \to X$ the associated projections. Sections of L^λ over an open subset $W \subset X$ are in natural one-to-one correspondence with scalar-valued holomorphic functions defined on $\pi_D^{-1}(W)$ whose restrictions to fibers are of monomial form $\mathbb{C} \ni \zeta \mapsto \zeta^\lambda$. Such sections can equivalently be identified with CR functions defined on $\pi_Y^{-1}(W)$, satisfying the corresponding identity. See for instance [6, 23] for this correspondence. In the next proof, this same construction is employed, but in a more local formulation. For a coordinate patch $W \subset X$, $Y|_W$ can be identified with a trivial bundle $W \times \mathbb{T}$. The CR structure on $Y|_W$ then induces a CR structure on $W \times \mathbb{R}^1$ via the mapping $(z, t) \mapsto (z, e^{it})$. Sections of L^λ over W are thus identified with CR functions on $W \times \mathbb{R}$ of the form $(z, t) \mapsto u(z)e^{i\lambda t}$.

Proof of Lemma 6 This is a consequence of a fundamental result on analytic hypoellipticity of related subelliptic partial differential equations. Consider first the case $n > 1$. Identify a coordinate patch in X with a ball $B \subset \mathbb{C}^n$, and work in $B \times \mathbb{R}^1$ with coordinates (z, t), and set $U(z, t) = u(z)e^{i\lambda t}$. Then

$$e^{i\lambda t}\overline{D}_\lambda u(z) = \bar{\partial}_b U(z, t),$$

where $\bar{\partial}_b$, acting on forms f, is defined by

$$\bar{\partial}_b f(z,t) = \frac{\partial f}{\partial \bar{z}} + i\bar{\partial}\phi(z) \wedge \frac{\partial f}{\partial t}.$$

$\bar{\partial}_b$ is a Cauchy-Riemann operator associated to a strictly pseudoconvex CR structure on $B \times \mathbb{R}$. See for instance Sect. 2 of [23], where essentially the same construction appears, with $B \times S^1$ in place of $B \times \mathbb{R}$, and with the CR structure on $B \times S^1$ that lifts to $B \times \mathbb{R}^1$ under the inverse of the mapping $t \mapsto e^{it}$ from $\mathbb{R}^1$ to S^1.

Δ_λ is related to the Kohn Laplacian $\Box_b$ for this CR structure by the corresponding equation

$$e^{i\lambda t}\Delta_\lambda u(z) = \Box_b U(z,t).$$

For $n > 1, \Box_b$ is analytic hypoelliptic on $(0, 1)$ forms, for any real analytic strictly pseudoconvex CR structure. This and/or closely related results are proved in [7, 12, 18, 19, 22]. Identifying $B \subset \mathbb{C}^n$ with a ball in $\mathbb{R}^{2n}$, we regard $B \times \mathbb{R}$ as a subset of $\mathbb{R}^{2n+1}$, hence as a totally real submanifold of $\mathbb{C}^{2n+1}$. Any real analytic function of $(z,t) \in B \times \mathbb{R}$ thus extends holomorphically to a neighborhood in $\mathbb{C}^{2n+1}$.

Analytic hypoellipticity of $\Box_b$ implies such extension, in a quantitative sense: there exist a complex neighborhood Ω of $\tilde{B} \times [-1, 1]$ and a constant $C < \infty$ such that any bounded solution U of $\Box_b U = 0$ in $B \times (-2, 2)$ extends to a bounded holomorphic function in Ω, and moreover,

$$\sup_{\Omega} |U| \le C \sup_{B\times(-2,2)} |U|.$$

By analytic continuation, any holomorphic extension of $u(z)e^{i\lambda t}$ must take the product form $\tilde{u}(z)e^{i\lambda t}$. For positive λ we then set $t = -i$ to deduce that

$$\sup_{\tilde{B}} |u|e^{\lambda} \le C \sup_{B} |u|.$$

An examination of any of the proofs [18, 19, 22] of analytic hypoellipticity of $\Box_b$ confirms that these provide *uniform* upper bounds, given uniform upper bounds on the coefficients of $\bar{\partial}_b$ in some fixed coordinate patch, and on the Hermitian metric used to define $\bar{\partial}_b^*$, and given that the hypothesis of strict pseudoconvexity holds in a uniform way. In our setting, the latter amounts to uniform strict positivity of the metric ϕ.

The case $n = 1$ requires an alternative treatment, because $\Box_b = \bar{\partial}_b\bar{\partial}_b^*$ fails to be analytic hypoelliptic for three-dimensional CR manifolds. Instead, a variant of analytic hypoellipticity holds in two alternative (but equivalent) forms. One of these[1]

[1]The other alternative asserts that u is C^ω, microlocally outside a conic neighborhood of one of the two ray bundles whose union is the characteristic variety of $\bar{\partial}_b$. This implies holomorphic extendibility to an appropriate wedge, and the above reasoning may then be repeated to gain the factor $\exp(-c\lambda)$.

asserts that if $\bar{\partial}\bar{\partial}^* U = 0$ then

$$\sup_{\Omega} |\bar{\partial}^* U| \leq C \sup_{B\times(-2,2)} (|U| + |\bar{\partial}^* U|), \tag{5.4}$$

with the same type of uniform dependence of the constant C on the data as for $n > 1$. Together with the reasoning above, this yields the conclusion

$$\sup_{\tilde{B}} |\overline{D}_\lambda^* u| \leq e^{-c\lambda} \sup_B (|u| + |\overline{D}_\lambda^* u|). \tag{5.5}$$

The bound for u itself now follows from Lemma 7 below. □

The justification of the above form of analytic hypoellipticity rests on several facts and results, combined according to an outline introduced by Kohn [14] for the analysis of related questions concerning (weakly) pseudoconvex three-dimensional CR manifolds. Denote by $\square = \bar{\partial}_b \bar{\partial}_b^*$ the Kohn Laplacian over a strictly pseudoconvex three (real) dimensional CR manifold M. Assume that $\square u \in C^\omega$ in an open set.
(i) The analytic wave front set of u is contained in the characteristic variety of $\square$.
(ii) This characteristic variety is a real line bundle over M, thus a union of two ray bundles.
(iii) In a conic neighborhood of one of these two ray bundles, $\bar{\partial}_b$ is of principal type and satisfies (microlocally) the Poisson bracket hypothesis which ensures analytic hypoellipticity [21], and therefore is microlocally analytic hypoelliptic. The microlocal version of this theorem of Treves follows for instance by the techniques in [17]. Consequently since $\bar{\partial}_b(\bar{\partial}_b^* u) \in C^\omega$, the analytic wave front set of $\bar{\partial}_b^* u$ is disjoint from this ray bundle.
(iv) In a conic neighborhood of the complementary ray bundle, $\square$ has double characteristics and satisfies the hypotheses of the theorem of Sjöstrand [18]; see also [12] where more degenerate operators are analyzed by the same techniques. Therefore the analytic wave front set of u, and hence also the analytic wave front set of $\bar{\partial}_b^* u$, are disjoint from this ray bundle.
(v) If a distribution has empty analytic wave front set, then it is analytic.
(vi) These steps can be made quantitative, where appropriate, to justify the stated uniformity.

5.3 Exponential Localization for a First-Order Equation

Lemma 7 *Let $n = 1$. Let U, U' be open subsets of X with $U \Subset U'$. There exists $c > 0$ such that for all sufficiently large $\lambda \geq 0$, and all $(0, 1)$–forms $u \in C^1(U')$,*

$$\|u\|_{L^2(U)} \leq C \|D_\lambda^* u\|_{L^2(U')} + C e^{-c\lambda} \|u\|_{L^2(U')}.$$

Proof It suffices to show that for each $z_0 \in U$, there exists a neighborhood V of z_0 such that $\|u\|_{L^2(V)}$ satisfies the required upper bound. In a small open set, represent $\bar{D}_\lambda^*$ as $-e^{\lambda\phi}(\partial + a)e^{-\lambda\phi}$ where $a \in C^\infty$. In a sufficiently small neighborhood it is possible to solve $\partial\alpha = a$ and thus to write $\bar{D}_\lambda^* = -e^{-\alpha}e^{\lambda\phi}\partial e^{-\lambda\phi}e^{\alpha}$. Since multiplication by $e^{\pm\alpha}$ preserves L^2 norms up to a bounded factor, it suffices to prove the inequality with $\alpha \equiv 0$.

It is possible to write, for all z, w in a sufficiently small neighborhood of z_0,

$$\phi(w) = \psi(z, w) + \varphi(z, w)$$

where ψ, φ are C^∞ functions, $\varphi(z, w)$ is an antiholomorphic function of w for each z, and

$$\mathrm{Re}\,(\psi(z, w)) \geq \mathrm{Re}\,(\psi(z, z)) + c|z - w|^2 \tag{5.6}$$

for a certain constant $c > 0$. Indeed, the Taylor series of order 2 for ϕ at z provides a unique expansion

$$\phi(z, w) = \mathrm{Re}\,(Q(z, w)) + R(z, w)$$

where $w \mapsto Q(z, w)$ is a quadratic holomorphic polynomial in w for each z, and $R(z, w)$ is real-valued and takes the form

$$R(z, w) = \sum_{j,k} a_{j,k}(z)(w_j - z_j)(\bar{w}_k - \bar{z}_k) + O(|z - w|)^3$$

with $a_{j,k}$ real-valued and C^∞. Moreover, $\sum_{j,k}(z)a_{j,k}(z)\zeta_j\bar{\zeta}_k \geq c|\zeta|^2$ uniformly for all z in a neighborhood of z_0 and $\zeta \in \mathbb{C}^d$. Set $\varphi(z, w) = \overline{Q(z, w)}$. Then φ and $\psi(z, w) = \phi(w) - \varphi(z, w)$ have the required properties.

For each z, when acting on functions of w,

$$\bar{D}_\lambda^* u(w) = -e^{\lambda\psi(z,w)}\big(\partial e^{-\lambda\psi(z,\cdot)}\big)u(w).$$

Let $\eta \in C^\infty(X)$ be a function supported in a neighborhood of z_0 which is contained in a coordinate patch contained in a relatively compact subset of U', within which the above expression for ϕ is valid; and η is identically equal to one in a smaller neighborhood. Then ηu can be regarded as a function defined on $\mathbb{C}^1$. Let

$$v = \bar{D}_\lambda^*(\eta u) = \eta\bar{D}_\lambda^* u - u\partial\eta.$$

Since

$$\partial_w e^{-\lambda\psi(z,w)}(\eta u)(w) = -e^{-\lambda\psi(z,w)}v(w)$$

is a compactly supported continuous function defined on $\mathbb{C}^1$, for each z sufficiently close to z_0 one may recover $\eta(z)u(z) = u(z)$ by

$$u(z) = -c_0 \int_{\mathbb{C}^1} v(w)(\bar{z} - \bar{w})^{-1} e^{\lambda(\psi(z,z)-\psi(z,w))}\, dm(w) \tag{5.7}$$

where m denotes Lebesgue measure on $\mathbb{C}^1$ and c_0 is a certain constant. Now

$$\left| e^{\lambda(\psi(z,z)-\psi(z,w))} \right| = e^{\lambda(\mathrm{Re}\,(\psi(z,z)-\psi(z,w)))} \le e^{-c\lambda|w-z|^2}.$$

Therefore

$$\begin{aligned} |u(z)| &\le C \int_{\mathbb{C}^1} |z-w|^{-1} |v(w)| e^{-c\lambda|z-w|^2}\, dm(w) \\ &\le C \int_{\mathbb{C}^1} \big(|\eta(w) \bar{D}_\lambda^* u(w)| + |u(w) \partial\eta(w)| \big) |z-w|^{-1} e^{-c\lambda|z-w|^2}\, dm(w)\,. \end{aligned}$$

Since $|z-w|$ is bounded below by a positive quantity uniformly for all z in U and w in the support of $\nabla\eta$, the required bound follows. □

6 Proof of Proposition 2

6.1 Globalization

We introduce a variant situation in which X is replaced by $\mathbb{C}^n$ and sections of $B^{(0,1)} \otimes L^\lambda$ over X are replaced by sections of $B^{(0,1)}(\mathbb{C}^n)$ over $\mathbb{C}^n$. This variant will facilitate λ–dependent coordinate changes to be made below.

Let $\varepsilon > 0$ be given. Let U be a relatively compact open subset of a coordinate patch in X. Fix a holomorphic coordinate system on that coordinate patch, and express $\overline{D}_\lambda = e^{-\lambda\phi} \bar{\partial} e^{\lambda\phi}$ where $\phi \in C^\infty$ is $\mathbb{R}$-valued, and satisfies the positivity hypothesis

$$\left(\frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j} \right)_{i,j} \ge c(\delta_{i,j})_{i,j} \tag{6.1}$$

in the sense of Hermitian forms.

Sections of L^λ over U are thus identified with $\mathbb{C}$–valued functions in such a way that the L^2 norm squared, over U, of such a section can be expressed as $\int_U |f(z)|^2 a(z)\, d\mu(z)$ where μ is Lebesgue measure on $\mathbb{C}^n$, $a \in C^\infty(\mathbb{C}^n)$ is bounded above in C^N norm for all N by constants independent of λ, z', and $a(z)$ is positive and bounded below by a positive constant independent of λ, z, z'. Extend a to a strictly positive C^∞ function $\tilde{a}$ on $\mathbb{C}^n$, still with uniform upper and lower bounds. Likewise extend g to a C^∞ Hermitian metric on $\mathbb{C}^n$, independent of λ. Assign to $(0, k)$ forms f defined on $\mathbb{C}^n$ the L^2 norm squared $\int_{\mathbb{C}^n} |f(z)|^2 \tilde{a}(z)\, d\mu(z)$ where $|f(z)|$ is measured using this extension of g.

Fix an auxiliary function $\eta \in C_0^\infty(\mathbb{C}^n)$, supported in $\{z : |z| < 4\}$ and satisfying $\eta(z) \equiv 1$ for $|z| \le 2$. For each z' in a fixed relatively compact subset $U \Subset U'$, make the affine coordinate change

$$B \times U \ni (\zeta, z') \mapsto (z, z') = (z' + \zeta, z') \in U \times U,$$

where B is the ball of radius ε_0 centered at $0 \in \mathbb{C}^n$. In these coordinates, z' is the origin, $\zeta = 0$. We will work in the variable $z \in B$, suppressing z' in the notation; all estimates will be uniform in $z' \in U$, as the proof will show.

Let Q_2 be the Taylor polynomial of degree 2 for ϕ at $\zeta = 0$. Define

$$\tilde{\phi}(\zeta) = Q_2(\zeta) + \eta(\varepsilon_0^{-1}\zeta)(\phi(\zeta) - Q_2(\zeta)).$$

Consider the modified operator $e^{-\lambda\tilde{\phi}}\bar{\partial}_\zeta e^{\lambda\tilde{\phi}}$, which agrees with $e^{-\lambda\phi}\bar{\partial}_\zeta e^{\lambda\phi}$ for all sufficiently small ζ, but has the advantage of being defined globally for $\zeta \in \mathbb{C}^n$. For sufficiently large λ,

$$\nabla^2\tilde{\phi}(z) - \nabla^2\phi(0) = O(\varepsilon_0)$$

uniformly for all $z \in \mathbb{C}^n$. Therefore it is possible to choose $\varepsilon_0 > 0$ sufficiently small that for all sufficiently large λ, the quadratic form defined by $(\partial^2\tilde{\phi}(z)/\partial z_i \partial\bar{z}_j)_{i,j=1}^n$ is bounded below by a strictly positive constant, independent of z and λ. This holds uniformly in $z' \in U$. Choose and fix such a value of ε_0.

Consider the associated operator defined for $n > 1$ by

$$\tilde{\Delta}_\lambda = \big(e^{-\lambda\tilde{\phi}}\bar{\partial}e^{\lambda\tilde{\phi}}\big)\big(e^{-\lambda\tilde{\phi}}\bar{\partial}e^{\lambda\tilde{\phi}}\big)^* + \big(e^{-\lambda\tilde{\phi}}\bar{\partial}e^{\lambda\tilde{\phi}}\big)^*\big(e^{-\lambda\tilde{\phi}}\bar{\partial}e^{\lambda\tilde{\phi}}\big),$$

and for $n = 1$ by

$$\tilde{\Delta}_\lambda = \big(e^{-\lambda\tilde{\phi}}\bar{\partial}e^{\lambda\tilde{\phi}}\big)\big(e^{-\lambda\tilde{\phi}}\bar{\partial}e^{\lambda\tilde{\phi}}\big)^*,$$

where adjoints are interpreted with respect to the Hilbert space structure on $L^2(\mathbb{C}^n)$ introduced above.

For $n > 1$, for all sufficiently large λ, a well-known computation based on integration by parts [13] gives

$$\langle\tilde{\Delta}_\lambda u, u\rangle \ge c\lambda\|u\|_{L^2}^2 \tag{6.2}$$

for all twice continuously differentiable and compactly supported $(0, 1)$ forms u, where $c > 0$ is a positive constant.

For $n = 1$, for all sufficiently large λ,

$$\big[e^{-\lambda\tilde{\phi}}\bar{\partial}e^{\lambda\tilde{\phi}}, \big(e^{-\lambda\tilde{\phi}}\bar{\partial}e^{\lambda\tilde{\phi}}\big)^*\big] \ge c\lambda I, \tag{6.3}$$

in the sense of operators on $L^2(\mathbb{C}^n)$ with respect to the same Hilbert space structure. Consequently (6.2) also holds for $n = 1$.

Since $\tilde{\Delta}_\lambda$ is a formally self-adjoint operator, it follows that there exists a bounded linear operator $\tilde{\mathcal{G}}_\lambda$ from $L^2(\mathbb{C}^n, B^{(0,1)})$ to itself such that $\tilde{\Delta}_\lambda \circ \tilde{\mathcal{G}}_\lambda$ is the identity operator on $L^2(\mathbb{C}^n, B^{(0,1)})$, and the operator norm of $\tilde{\mathcal{G}}_\lambda$ is $O(\lambda^{-1})$ for all sufficiently large λ.

This inverse is bounded in L^2 operator norm, uniformly for all sufficiently large λ, provided that ε_0 is kept fixed. Lemma 5 also applies to this situation, so the distribution-kernel $\tilde{\mathcal{G}}_\lambda(z, 0)$ for $\tilde{\mathcal{G}}_\lambda$ with singularity at $z = 0$ satisfies

$$|\tilde{\mathcal{G}}_\lambda(z, 0)| \le (\lambda + |z|^{-1})^C \tag{6.4}$$

for all sufficiently large λ, and the same holds for all of its partial derivatives. These bounds are uniform in λ provided that λ is sufficiently large.

6.2 Gauge Change

Denote by p the harmonic part of the Taylor polynomial of $\tilde{\phi}$ of degree 2 at $w = 0$. That is, expand

$$\tilde{\phi}(z) = \tilde{\phi}(0) + \operatorname{Re}\left(\sum_{k=1}^{n} \alpha_k z_k + \sum_{i,j=1}^{n} \beta_{i,j} z_i z_j\right) + \sum_{i,j=1}^{n} \gamma_{i,j} z_i \bar{z}_j + O(|z|^3),$$

and set

$$p(z) = \tilde{\phi}(0) + \operatorname{Re}\left(\sum_{k=1}^{n} \alpha_k z_k + \sum_{i,j=1}^{n} \beta_{i,j} z_i z_j\right).$$

Define

$$\tilde{p}(z) = \operatorname{Im}\left(\sum_{k=1}^{n} \alpha_k z_k + \sum_{i,j=1}^{n} \beta_{i,j} z_i z_j\right),$$

so that $p + i\tilde{p}$ is analytic and has real part p. Then $[\bar{\partial}, e^{\lambda(p+i\tilde{p})}] = \bar{\partial}(p + i\tilde{p}) \equiv 0$ and consequently

$$e^{-\lambda\tilde{\phi}}\bar{\partial}e^{\lambda\tilde{\phi}} = e^{i\lambda\tilde{p}}e^{-\lambda(\tilde{\phi}-p)}\bar{\partial}e^{\lambda(\tilde{\phi}-p)}e^{-i\lambda\tilde{p}}. \tag{6.5}$$

Likewise

$$\left(e^{-\lambda\tilde{\phi}}\bar{\partial}e^{\lambda\tilde{\phi}}\right)^* = \left(e^{i\lambda\tilde{p}}e^{-\lambda(\tilde{\phi}-p)}\bar{\partial}e^{\lambda(\tilde{\phi}-p)}e^{-i\lambda\tilde{p}}\right)^* = e^{i\lambda\tilde{p}}\left(e^{-\lambda(\tilde{\phi}-p)}\bar{\partial}e^{\lambda(\tilde{\phi}-p)}\right)^* e^{-i\lambda\tilde{p}}$$

and consequently

$$
\begin{aligned}
e^{-\lambda\tilde{\phi}}\bar{\partial}e^{\lambda\tilde{\phi}}\big(e^{-\lambda\tilde{\phi}}\bar{\partial}e^{\lambda\tilde{\phi}}\big)^* + \big(e^{-\lambda\tilde{\phi}}\bar{\partial}e^{\lambda\tilde{\phi}}\big)^* e^{-\lambda\tilde{\phi}}\bar{\partial}e^{\lambda\tilde{\phi}} \\
= e^{i\lambda\tilde{p}}\Big(e^{-\lambda(\tilde{\phi}-p)}\bar{\partial}e^{\lambda(\tilde{\phi}-p)}\big(e^{-\lambda(\tilde{\phi}-p)}\bar{\partial}e^{\lambda(\tilde{\phi}-p)}\big)^* \\
+ \big(e^{-\lambda(\tilde{\phi}-p)}\bar{\partial}e^{\lambda(\tilde{\phi}-p)}\big)^* e^{-\lambda(\tilde{\phi}-p)}\bar{\partial}e^{\lambda(\tilde{\phi}-p)}\Big)e^{-i\lambda\tilde{p}}.
\end{aligned}
$$

Hence upon replacement of $\tilde{\phi}$ by $\tilde{\phi} - p$ in the definition of $\tilde{\Box}_\lambda$, a unitarily equivalent operator on $L^2(\mathbb{C}^n, B^{(0,1)})$ is obtained. Moreover, the absolute value of the distribution-kernel for the inverse of this unitarily equivalent operator is identically equal to $|\tilde{G}_\lambda|$.

In deriving upper bounds for $|G_\lambda(z, w)|$, where G_λ is the distribution-kernel for $\Box_\lambda^{-1}$ on X, we may therefore assume without loss of generality that the pluriharmonic part of the Taylor polynomial of degree 2 for ϕ at w vanishes identically. Likewise, because $\bar{\partial}_\lambda$ and $\bar{\partial}_\lambda^*$ have been conjugated by the unitary multiplicative factor $e^{i\tilde{p}}$, the same assumption can be made when deriving upper bounds for $|\bar{\partial}_\lambda G_\lambda(z, w)|$ and $|\bar{\partial}_\lambda^* G_\lambda(z, w)|$.

6.3 *Taylor Expansion and Dilation*

Let $\tilde{\phi}$ be as above, and suppose, as we may achieve through a gauge change, that the pluriharmonic portion of the Taylor polynomial of degree 2 for $\tilde{\phi}$ at 0 vanishes identically, while the complex Hessian matrix of $\tilde{\phi}$ is bounded below by a strictly positive constant, and all partial derivatives of $\tilde{\phi}$ are bounded above, uniformly in λ.

Let N be a large positive integer, independent of λ, to be chosen below. Define P_N to be the Taylor polynomial of degree N for $\tilde{\varphi}$, at $\zeta = 0$. For any $r > 0$ satisfying $\lambda^{-1/2} \le r \le \lambda^{-1/4}$ define

$$\psi(z) = r^{-2}P_N(rz) + r^{-2}(1 - \eta(z))(P_2(rz) - P_N(rz)). \tag{6.6}$$

For all sufficiently large λ, the complex Hessian of ψ evaluated at an arbitrary point $z \in \mathbb{C}^n$, equals the complex Hessian of $\tilde{\phi}$ evaluated at 0, plus $O(r) = O(\lambda^{-1/4})$.

Moreover on $\{z : |z| < 3\}$, where $1 - \eta \equiv 0$, ψ is real analytic, *uniformly in* λ *and in* N provided that $\lambda \ge \Lambda(N)$ where $\Lambda(N)$ is some appropriately large quantity depending only on N and the data X, L, ϕ, g. This uniformity, which is crucial to our analysis, is a consequence of the normalizations $\tilde{\phi}(0) = 0$, $\nabla\phi(0) = 0$ achieved by subtracting the degree one Taylor polynomial[2] of $\tilde{\phi}$; indeed, for z in any bounded set and $N \ge 2$, $P_N(rz) = P_2(rz) + O(r^3|z|)$ so that

$$r^{-2}P_N(rz) = P_2(z) + O_{M,N}(r)$$

[2]Subtraction of the pluriharmonic second degree terms is natural, but is inessential here.

in any C^M norm on any bounded set. Once M, N are chosen, the term $O_{M,N}(r)$ becomes arbitrarily small as λ becomes arbitrarily large.

In the same spirit, define a globalized locally analytic approximation $g^\dagger$ to the Hermitian metric g by

$$g^\dagger(z) = \tilde{P}_N(rz) + (1 - \eta(z))(g(0) - \tilde{P}_N(rz))$$

where $\tilde{P}_N$ is the Taylor polynomial of degree N for g at 0, in the natural sense. The alternative expression

$$g^\dagger(z) = \eta(z)\tilde{P}_N(rz) + (1 - \eta(z))g(0) = g(0) - \eta(z)(g(0) - P_N(rz)) = g(0) + O(r)$$

demonstrates that $g^\dagger$ is a globally well-defined Riemannian metric on $\mathbb{C}^n$.

Define

$$\kappa = r^2\lambda \tag{6.7}$$

and

$$\bar{\mathcal{D}} = e^{-\kappa\psi}\bar{\partial}e^{\kappa\psi}, \tag{6.8}$$

that is, $\bar{\mathcal{D}}u = e^{-\kappa\psi}\bar{\partial}(e^{\kappa\psi}u)$, for $(0, q)$ forms u defined on $\mathbb{C}^n$. Define $\bar{\mathcal{D}}^*$ to be the adjoint of $\bar{\mathcal{D}}$ with respect to the Hilbert space structures on L^2 sections of $B^{(0,q)}(\mathbb{C}^n)$ specified by $g^\dagger(z)$. Define

$$\Box^\dagger = \begin{cases} \bar{\mathcal{D}}\bar{\mathcal{D}}^* + \bar{\mathcal{D}}^*\bar{\mathcal{D}} & \text{for } n > 1, \\ \bar{\mathcal{D}}\bar{\mathcal{D}}^* & \text{for } n = 1. \end{cases}$$

These are differential operators. On the region $|z| < 3$, in which $\eta(z) \equiv 1$, $\Box^\dagger$ is related to $\Box_\lambda$ as follows: If $u(z) = v(rz)$ then

$$\Box^\dagger u(z) = r^2\Box_\lambda v(rz) + O(\lambda^{-cN})\, O(v, \bar{\partial}_\lambda v, \bar{\partial}_\lambda^* v, \bar{\partial}_\lambda(bv), \bar{\partial}_\lambda^*(bv)) \tag{6.9}$$

where the error term denoted $O(v, \bar{\partial}_\lambda v, \bar{\partial}_\lambda^* v, \bar{\partial}_\lambda(bv), \bar{\partial}_\lambda^*(bv))$ is a linear combination of v, $\bar{\partial}_\lambda(v)$, $\bar{\partial}_\lambda^*(v)$, $\bar{\partial}_\lambda(bv)$ and $\bar{\partial}_\lambda^*(bv)$ where all coefficients are bounded uniformly in λ, z, and bv denotes either the wedge product or the interior product of v with a real analytic $(0, 1)$ form b. The factor of r^2 is a consequence of the chain rule and the substitution $z \mapsto rz$. The terms that are $O(\lambda^{-cN})$ result from approximating $g(rz)$ by its Taylor polynomial $\tilde{P}_N(rz)$, and likewise from approximating $r^{-2}\tilde{\varphi}(rz)$ by $r^{-2}P_N(rz)$, each time incurring an error that is $O(|rz|^N) = O(\lambda^{-N/4})$ in any C^K norm for $|z| \le 3$. Moreover, in this region, these forms b are uniformly analytic as $\lambda \to \infty$.

Applying (6.9) with

$$u(z) = \mathcal{G}_\lambda(rz, 0),$$

using the upper bounds $|\mathcal{G}_\lambda(z,0)| \le C\lambda^C$ and $|\bar\partial_\lambda \mathcal{G}_\lambda(z,0)| + |\bar\partial_\lambda^* \mathcal{G}_\lambda(z,0)| \le C\lambda^C$ for $|z| \ge \lambda^{-1/2}$, and using the assumption $\lambda^{-1/2} \le r \le \lambda^{-1/4}$, we conclude that

$$|\Box^\dagger u(z)| \le \lambda^{-cN}$$

for $\frac{1}{2} \le |z| \le 2$, where $c > 0$ is independent of λ, z and of N, provided that $\lambda \ge \Lambda(N)$.

Provided that $\kappa = r^2\lambda$ is sufficiently large, a routine integration by parts calculation, together with the uniform lower bound for the complex Hessian of ψ, give the lower bound

$$\langle \Box^\dagger u, u\rangle \ge c\kappa \|u\|_{L^2}^2 \tag{6.10}$$

for all C^2 forms u of bidegree (0, 1) with compact support. The effect of the localization and rescaling has been to replace λ by κ.

6.4 Conclusion of Proof of Proposition 2

Let N be a large positive integer. Suppose that λ is large, that $\lambda^{-1/2} \le r \le \lambda^{-1/4}$, and that $\kappa = r^2\lambda$ is large. Consider $u(z) = \mathcal{G}_\lambda(rz, 0)$, defined as above using Taylor polynomials of order N. In the annular region $\frac{1}{2} < |z| < 2$, $|u| \le \lambda^C$ and $|\Box^\dagger u| \le \lambda^{-cN}$, provided that $\lambda \ge \Lambda(N)$.

Let $\tilde\eta$ be a C^∞ function which is identically equal to 1 in $\left\{z : \frac{1}{3} \le |z| \le 3\right\}$ and supported in $\left\{z : \frac{1}{4} < |z| < 4\right\}$. Provided that κ is sufficiently large, the global lower bound (6.10) ensures that the equation $\Box^\dagger v = \tilde\eta \Box^\dagger u$ is solvable in $L^2(\mathbb{C}^n)$, and that there exists a solution satisfying

$$\|v\|_{L^2} \le C\kappa^{-1} \|\tilde\eta \Box^\dagger u\|_{L^2} \le \lambda^{-cN}, \tag{6.11}$$

provided that $\lambda \ge \Lambda(N)$.

Now $\Box^\dagger(u - v) \equiv 0$ where $\frac{1}{2} < |z| < 2$, so Lemma 6 can be applied to conclude that

$$|(u - v)(z)| \le Ce^{-c\kappa} = Ce^{-cr^2\lambda} \text{ for } \tfrac{3}{4} \le |z| \le \tfrac{4}{3}. \tag{6.12}$$

Therefore in this same region,

$$|\mathcal{G}_\lambda(rz, 0)| \le Ce^{-cr^2\lambda} + C\lambda^{-cN} \tag{6.13}$$

for all $\lambda \ge \Lambda(N)$.

Equivalently, by choosing $r = |z|^{-1}$, we find that there exists a constant $B < \infty$ such that for all $\lambda \ge \Lambda(N)$ and all $|\zeta| \ge B\lambda^{-1/2}$,

$$|\mathcal{G}_\lambda(\zeta, 0)| \le Ce^{-c\lambda|\zeta|^2} + C\lambda^{-cN} = Ce^{-c\lambda|\zeta|^2} + Ce^{-cN\log\lambda}. \tag{6.14}$$

If A_0 is sufficiently large, if $A < \infty$ is fixed, and if $A_0\lambda^{-1/2}\sqrt{\log\lambda} \le |\zeta| \le A\lambda^{-1/2}\sqrt{\log\lambda}$, choose $N = A^2$ to obtain

$$|\mathcal{G}_\lambda(\zeta, 0)| \le Ce^{-c\lambda|\zeta|^2}. \tag{6.15}$$

After reversing the change of variables made above, this is the desired bound $|\mathcal{G}_\lambda(z, z')| \le Ce^{-c\lambda\rho(z,z')^2}$.

This analysis cannot be extended to a larger range of $|\zeta|$, because bounds only hold for $\lambda \ge \Lambda(N)$ and a larger range would require that N depend on $|\zeta|$, hence that N depend on λ, introducing circularity into the reasoning.

Since $\mathcal{G}_\lambda(z, z')$ is a solution on the complement of the diagonal $z = z'$ of homogeneous elliptic linear partial differential equations, separately with respect to each of the two variables z, z', and since the coefficients of these equations are $O(\lambda^2)$ in any C^M norm, it follows from routine bootstrapping arguments that each derivative of $\mathcal{G}_\lambda$ satisfies the same upper bound with a possibly smaller value of the constant $c > 0$. Each of the finitely many steps in the bootstrapping process loses at most a factor of $C\lambda^2$. Since

$$\lambda^C e^{-A\sqrt{\lambda\log\lambda}} \le e^{-(A-1)\sqrt{\lambda\log\lambda}}$$

for all sufficiently large λ, the loss of finitely many such factors is of no importance here. □

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Off-Diagonal Decay of Bergman Kernels: On a Question of Zelditch

Michael Christ

Abstract We study the orthogonal projection from $L^2(\mathbb{C}^d, e^{-2\lambda\phi})$ to its subspace of entire holomorphic functions, as $\lambda \to \infty$, for weights ϕ that depend only on $\text{Re}(z)$ and are uniformly strictly plurisubharmonic. We show that the associated Bergman kernels are $O(e^{-c\lambda})$ away from the diagonal, if and only if ϕ is real analytic.

1 Introduction

This paper investigates an inverse problem concerning asymptotic behavior of Bergman kernels. Let X be a connected compact complex manifold, without boundary. Let X be equipped with a C^∞ Hermitian metric g, along with the metrics on the bundles $B^{(p,q)}(X)$ of forms of bidegree (p, q) induced by g, and the volume form on X associated to the induced Riemannian metric. Denote by $\rho(z, z')$ the Riemannian distance from $z \in X$ to $z' \in X$.

Let L be a positive holomorphic line bundle over X. Let L be equipped with a C^∞ Hermitian metric ϕ whose curvature form is positive at every point.

For each positive integer λ, let the line bundle L^λ be the tensor product of λ copies of L. L^λ inherits from ϕ a Hermitian metric so that the λ–fold tensor product of any $v \in L_z$ satisfies $|v \otimes v \otimes \cdots \otimes v| = |v|^\lambda$.

Let $L^2(X, L^\lambda)$ be the Hilbert space of equivalence classes of all square integrable Lebesgue measurable sections of L^λ. Let $H^2(X, L^\lambda)$ be the closed subspace of $L^2(X, L^\lambda)$ consisting of all holomorphic sections. The Bergman projection operator B_λ is by definition the orthogonal projection from $L^2(X, L^\lambda)$ onto $H^2(X, L^\lambda)$. The Bergman kernel $B_\lambda(z, z')$ is the associated distribution-kernel; $B_\lambda(z, z')$ is a complex linear endomorphism from the fiber $L^\lambda_{z'}$ to the fiber L^λ_z.

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M. Christ (✉)
Department of Mathematics, University of California, Berkeley, CA 94720-3840, USA
e-mail: mchrist@berkeley.edu

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The asymptotic behavior of the Bergman kernels as $\lambda \to \infty$, on and within distance $O(\lambda^{-1/2})$ of the diagonal, has been intensively studied. This paper is concerned instead with the large λ behavior at a positive distance from the diagonal. It is well known that B_λ tends rapidly to zero as $\lambda \to \infty$, away from the diagonal. For real analytic g, ϕ there is decay at an exponential rate: provided that $\rho(z, z') \ge \delta > 0$,

$$B_\lambda(z, z') = O(e^{-c\lambda}) \tag{1.1}$$

for some $c = c(\delta) > 0$. For C^∞ metrics g, ϕ,

$$B_\lambda(z, z') = O(e^{-A\sqrt{\lambda \log \lambda}}) \tag{1.2}$$

for all $A < \infty$ [3], provided again that $\rho(z, z') \ge \delta$. This rate of decay is optimal [2]; if $h(\lambda) \to \infty$ as $\lambda \to \infty$ then there exist X, L, ϕ, g with $\phi, g \in C^\infty$ and points $z \ne z'$ such that

$$\limsup_{\lambda \to \infty} \sup_{\rho(z,z') \ge \delta} e^{h(\lambda)\sqrt{\lambda \log \lambda}} |B_\lambda(z, z')| = \infty. \tag{1.3}$$

Zelditch [5] has asked to what extent exponential decay (1.1) is tied to real analyticity of ϕ, and moreover whether exponential decay for even an arbitrarily sparse sequence of values of λ tending to infinity implies analyticity. Sjöstrand [7] has pointed out that exponential decay does hold for any structure that is real analytic on the complement of a finite set. Nonetheless, this note answers Zelditch's question in the affirmative for a special class of structures that enjoy a real d-dimensional symmetry, but are otherwise essentially arbitrary. This is the same framework in which examples of subexponential decay (1.3) were constructed [2]. An affirmative answer within this limited framework is therefore of some interest.

This framework involves spaces of entire functions on $\mathbb{C}^d$ rather than sections of positive line bundles. The two settings are closely related. It seems likely that our results could be adapted to a certain class of compact toric manifolds, perhaps by a simple transplantation of the auxiliary functions constructed here, but we have not examined this question in detail.

2 The Framework

We work in $\mathbb{C}^d$, with coordinates $z = (z_1, \ldots, z_d)$. Write $z_j = x_j + iy_j$ and $x = (x_1, \ldots, x_d)$, $y = (y_1, \ldots, y_d) \in \mathbb{R}^d$. Write $z = x + iy \in \mathbb{R}^d + i\mathbb{R}^d$.

Let X be the noncompact complex manifold $X = \mathbb{C}^d$, and let L be the trivial line bundle $L = \mathbb{C}^d \times \mathbb{C}^1$. $B^{(0,1)}$ denotes the bundle of forms of bidegree $(0, 1)$ over $\mathbb{C}^d$. X is equipped with its usual flat metric as a complex Euclidean space. Integration over $\mathbb{C}^d$ or $\mathbb{R}^d$ will be performed with respect to Lebesgue measure. $B^{(0,1)}$ is equipped with the usual metric under which $|\bar\omega_J| = 1$ where $\bar\omega_J = d\bar z_{j_1} \wedge \cdots \wedge d\bar z_{j_q}$, whenever $j_1 < j_2 < \cdots < j_q$.

The metric ϕ on L is represented by a C^∞ real-valued function $\mathbb{C}^d \ni z \mapsto \phi(z)$. The norm of an element $(z, t) \in \mathbb{C}^d \times \mathbb{C} = L$ is $e^{-\phi(z)}|t|$; the norm of an element $(z, t) \in \mathbb{C}^d \times \mathbb{C} = L^\lambda$ is $e^{-\lambda\phi(z)}|t|$; $L^2(X, L^\lambda)$ is the Hilbert space of all Lebesgue measurable functions $f : \mathbb{C}^d \to \mathbb{C}$ that satisfy

$$\|f\|^2_{L^2(X,L^\lambda)} = \int_{\mathbb{C}^d} |f(z)|^2 e^{-2\lambda\phi(z)}\, dm(z) < \infty.$$

The essential feature of the framework under discussion here is that $\phi(x + iy)$ is a function of x alone. We therefore write $\phi \equiv \phi(x)$. We assume that the curvature form of ϕ is strictly positive, and uniformly bounded above and below. Thus there exists $C \in (0, \infty)$ such that for all $z \in \mathbb{C}^d$ and all $v \in \mathbb{C}^d$,

$$C^{-1}|v|^2 \le \sum_{j,k=1}^{d} \frac{\partial^2\phi}{\partial z_j \partial \bar{z}_k}(z)\, v_j\, \bar{v}_k \le C|v|^2 \tag{2.1}$$

Because $\phi(x + iy)$ depends only on x, this simplifies to

$$C^{-1}|v|^2 \le \sum_{j,k=1}^{d} \frac{\partial^2\phi}{\partial x_j \partial x_k}(x)\, v_j\, v_k \le C|v|^2 \ \text{ for all } \ x \in \mathbb{R}^d \text{ and } v \in \mathbb{R}^d. \tag{2.2}$$

Under this positivity assumption, the space $H^2(X, L^\lambda)$ of all entire holomorphic functions satisfying $\iint_{\mathbb{C}^d} |f(x + iy)|^2 e^{-2\lambda\phi(x)}\, dx\, dy < \infty$ is a closed subspace of the space $L^2(X, L^\lambda)$ of all equivalence classes of Lebesgue measurable functions for which the same integral is finite. The Bergman kernel B_λ represents the orthogonal projection of $L^2(X, L^\lambda)$ onto $H^2(X, L^\lambda)$. $B_\lambda(z, z')$ is a C^∞ function off of the diagonal for all $\lambda > 0$. These objects are well-defined for all $\lambda \in (0, \infty)$; one need not restrict to integer values.

Theorem 2.1 *Let $X = \mathbb{C}^d$. Let L be the trivial line bundle $X \times \mathbb{C}$. Let ϕ take the form $x + iy \mapsto \phi(x)$, and let the real Hessian of ϕ be uniformly positive in the sense (2.2). Let $U \subset \mathbb{C}^d$ be an open set, and suppose that for each $\delta > 0$ there exist a sequence λ_ν tending to ∞ and $c > 0$ such that for all $(z, z') \in U \times U$ satisfying $|z - z'| \ge \delta$ and for all sufficiently large ν,*

$$|B_{\lambda_\nu}(z, z')| \le e^{-c\lambda_\nu}. \tag{2.3}$$

Then ϕ is real analytic in U.

That is, the function $x \mapsto \phi(x)$ is real analytic on the projection of U onto $\mathbb{R}^d$.

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3 Outline

Two constructions mediate between the Bergman projections and the metric ϕ. The first is a family of holomorphic functions $z \mapsto \psi_{\lambda,\xi}(z)$ which depend on an external parameter $\xi \in \mathbb{C}^d$, as well as on λ. Secondly, for each λ we consider a scalar-valued holomorphic function $\xi \mapsto \mathcal{F}_\lambda(\xi)$, which is a Fourier-Laplace transform of $e^{-\lambda\phi}$.

We link $\mathcal{F}_\lambda$ to ϕ, showing that if $\xi \mapsto |\mathcal{F}_{\lambda_\nu}(\xi)|$ satisfies suitable lower bounds in a suitable region for some sequence λ_ν tending to infinity, then ϕ is real analytic. We link $\mathcal{F}_\lambda$ to the Bergman projections by reasoning by contradiction. If $|\mathcal{F}_{\lambda_\nu}(\xi_\nu)|$ is anomalously small for a sequence $\lambda_\nu \to \infty$, and if the Bergman kernels do have exponential off-diagonal decay, then it is shown that $\psi_{\lambda_\nu,\xi_\nu}$ nearly lies in the range of the adjoint $\bar\partial^*$ with respect to a suitable weighted L^2 structure. Since it belongs to the nullspace of $\bar\partial$, this leads to a contradiction.

Complex zeroes of $\mathcal{F}_\lambda$ having suitably small imaginary parts were the key to the construction in [2] of metrics ϕ for which the Bergman kernels decay slowly as $\lambda \to \infty$. Here we show that conversely, exponential decay not only precludes such zeros, but also precludes exponentially small values of $\mathcal{F}_\lambda$.

From a more technical perspective, the construction of [2] was executed only in the lowest-dimensional case $d = 1$, but here matters are investigated in arbitrary dimensions. Two new issues thereby arise. Firstly, while the formula defining $\mathcal{F}_\lambda$ extends straightforwardly, its interpretation and relevance are not immediately clear. Secondly, in order to obtain auxiliary functions with suitable growth properties needed to conclude that $\mathcal{F}_\lambda$ cannot take on any exponentially small values, we are led to solve the divergence equation $\mathrm{div}(u) = f$ in $\mathbb{R}^d$, for an unknown one-form u, in Hilbert spaces defined by weighted L^2 norms. The necessary condition for solvability of $\mathrm{div}(u) = f$ with the specific f that arises, related to the functions $\psi_{\lambda,\xi}$, turns out to be the vanishing of $\mathcal{F}_\lambda$ — so that the two new issues are intimately intertwined.

The analysis requires bounds with respect to weights $e^{-\Phi}$ with Φ concave. Concavity is the real analogue of pluri*super*harmonicity, rather than of the standard plurisubharmonicity of $\bar\partial$ theory. This is at variance with the usual situation; indeed the equation cannot be solved with satisfactory bounds for arbitrary (closed) data. We expend some effort to establish solvability with the desired bounds.

4 Notations and Framework

Variables in $\mathbb{C}^d$ will often be denoted by $z = x + iy$ where $x, y \in \mathbb{R}^d$. For $z, w \in \mathbb{C}^d$ we will write

$$z \cdot w = \sum_{j=1}^{d} z_j w_j, \tag{4.1}$$

with no complex conjugation. $x \asymp y$ means that x, y are positive quantities whose ratios x/y and y/x are bounded above by uniform constants.

Lebesgue measure on either $\mathbb{C}^d$ or $\mathbb{R}^d$ will be denoted by m. $w : \mathbb{C}^d \to (0, \infty)$ denotes a positive continuous function. $L^2(\mathbb{C}^d, w)$ is the Hilbert space of all equivalence classes of Lebesgue measurable scalar-valued functions with norm squared $\int_{\mathbb{C}^d} |f(z)|^2 w(z)\, dm(z)$. The same notation $L^2(\mathbb{C}^d, w)$ is also used to denote the Hilbert space of all equivalence classes of Lebesgue measurable (0, 1) forms with norm squared $\int_{\mathbb{C}^d} |f(z)|^2 w(z)\, dm(z)$, where $|\sum_{j=1}^d f_j(z)\, d\bar{z}_j|^2 = \sum_{j=1}^d |f_j(z)|^2$. The Bergman projections B_λ associated to the weights $e^{-2\lambda\phi}$ are the orthogonal projections from $L^2(\mathbb{C}^d, e^{-2\lambda\phi})$ to its closed subspace of entire holomorphic functions.

The following hypotheses concerning $\phi : \mathbb{C}^d \to \mathbb{R}$ will be in force throughout the paper.

$$\phi \in C^\infty. \tag{4.2}$$

$$\phi(x + iy) \text{ depends on } x \text{ alone.} \tag{4.3}$$

$$\phi \text{ is strictly convex.} \tag{4.4}$$

$$C^{-1}|t|^2 \le \sum_{i,j=1}^d \frac{\partial^2\phi}{\partial x_i \partial x_j} t_i t_j \le C|t|^2 \tag{4.5}$$

uniformly for all $x, t \in \mathbb{R}^d$, for some positive constant C. We will abuse notation by using the symbol ϕ to denote two functions, one with domain $\mathbb{C}^d$ and one with domain $\mathbb{R}^d$, related by $\phi(x + iy) = \phi(x)$. It will be clear from the context which of the two is intended.

The Cauchy–Riemann operator $\bar\partial$, mapping scalar-valued functions to (0, 1)–forms, is defined by

$$\bar\partial f = \sum_{k=1}^d \frac{\partial f}{\partial \bar{z}_k}\, d\bar{z}_k. \tag{4.6}$$

where

$$\frac{\partial}{\partial \bar{z}_k} = \frac{1}{2}\Big(\frac{\partial}{\partial x_k} + i\frac{\partial}{\partial y_k}\Big). \tag{4.7}$$

We also write

$$\partial_{z_k} = \frac{\partial}{\partial z_k} = \frac{1}{2}\Big(\frac{\partial}{\partial x_k} - i\frac{\partial}{\partial y_k}\Big). \tag{4.8}$$

For $w = e^{-\lambda\phi}$, the formal adjoint $\bar\partial^*_{2\lambda\phi}$ of $\bar\partial$ is

$$\bar\partial^*_{2\lambda\phi}\Big(\sum_j f_j\, d\bar{z}_j\Big) = -\sum_j \big(e^{2\lambda\phi}\partial_{z_j} e^{-2\lambda\phi}\big) f_j = -\sum_j \big(\partial_{z_j} - \lambda\partial_{x_j}\phi\big) f_j \tag{4.9}$$

since ϕ depends only on x.

Most of the analysis focuses on $\mathbb{C}$–valued functions, and (0, 1) forms, of the special type $z = x + iy \mapsto e^{i\lambda\xi\cdot y} f(x)$ where $\xi \in \mathbb{C}^d$ is a parameter. Forms and functions not of this special form will not appear until Sect. 6.4. $z \mapsto e^{i\lambda\xi\cdot y} f(x)$ is holomorphic if and only if

$$0 = \bar{\partial}(e^{i\lambda\xi\cdot y} f(x)) = \tfrac{1}{2} e^{i\lambda\xi\cdot y} \sum_{j=1}^{d} (\partial_{x_j} - \lambda\xi_j) f \, d\bar{z}_j. \tag{4.10}$$

The operator $\bar{\partial}^*_{2\lambda\phi}$ can be applied to $e^{i\lambda\xi\cdot y} f(x)$ even though such a function rarely lies in $L^2(\mathbb{C}^d, e^{-2\lambda\phi})$, by using the expression (4.9).

A pivotal question for our analysis is for which pairs (ξ, λ) the function $e^{\lambda\xi\cdot z}$ is close to the range of $\bar{\partial}^*_{2\lambda\phi}$, in a suitable sense. Formulation of this closeness must take into account the infinite $L^2(\mathbb{C}^d, e^{-2\lambda\phi})$ norm of the function $e^{\lambda z\cdot\xi}$.

Denote by div the divergence operator, which maps 1–forms with domain $\mathbb{R}^d$ to scalar-valued functions with the same domain:

$$\operatorname{div}(\sum_j u_j \, dx_j) = \sum_j \frac{\partial u_j}{\partial x_j} = \sum_j \partial_{x_j} u_j. \tag{4.11}$$

A form u and function f satisfy

$$\bar{\partial}^*_{2\lambda\phi}\big(e^{i\lambda\xi\cdot y} u(x)\big) = e^{i\lambda\xi\cdot y} f(x) \tag{4.12}$$

if and only if

$$-\tfrac{1}{2} \sum_j (\partial_{x_j} + \lambda\xi_j - 2\lambda\partial_{x_j}\phi) u_j = f. \tag{4.13}$$

For $f(x) = \frac{1}{2} e^{\lambda\xi\cdot x}$, this relation can be equivalently written as

$$-\operatorname{div}(e^{\lambda\xi\cdot x - 2\lambda\phi(x)} u) = 2e^{\lambda\xi\cdot x - 2\lambda\phi(x)} f = e^{2\lambda(\xi\cdot x - \phi(x))}. \tag{4.14}$$

At issue will be the possible existence of pairs (ξ, λ) for which Eq. (4.12) with right-hand side $f(x) = \frac{1}{2} e^{\lambda\xi\cdot x}$, or equivalently $e^{\lambda\xi\cdot x}$, admits an exact or approximate solution u which enjoys suitable upper bounds. The range of the divergence operator consists, formally, of all functions satisfying $\int_{\mathbb{R}^d} g(x)\, dm(x) = 0$. Therefore the discussion will turn on the approximate vanishing of $\int_{\mathbb{R}^d} e^{2\lambda(\xi\cdot x - \phi(x))}\, dm(x)$.

This integral has an alternative interpretation as the analytic continuation to the complex domain, with respect to ξ, of the function $\mathbb{R}^d \ni \xi \mapsto \|e^{\lambda x\cdot\xi}\|^2_{L^2(\mathbb{R}^d, e^{-2\lambda\phi})}$. That interpretation does not seem to be directly useful for our purpose.

5 Preparations

Define

$$\Phi(x) = \Phi_{\xi,\lambda}(x) = \lambda(\operatorname{Re}\xi \cdot x - \phi(x)). \tag{5.1}$$

Φ depends only on the real part of ξ, and is real-valued. The Hessian matrix of Φ is comparable to $-\lambda$ times the identity matrix, uniformly in λ, ξ, x. Consequently there exist a unique point $x^\dagger \in \mathbb{R}^d$, depending on ξ, λ, satisfying

$$\Phi(x^\dagger) = \max_{x \in \mathbb{R}^d} \Phi(x), \tag{5.2}$$

and constants $c_1, c_2 \in \mathbb{R}^+$ such that

$$e^{-c_1\lambda|x-x^\dagger|^2} e^{2\Phi(x^\dagger)} \le e^{2\Phi(x)} \le e^{-c_2\lambda|x-x^\dagger|^2} e^{2\Phi(x^\dagger)} \tag{5.3}$$

uniformly for all ξ, λ, x.

Let $a > 0$ be an exponent, depending only on the dimension d, which is to be chosen below to be sufficiently large that certain properties hold. The value of a is otherwise of no importance. Define the auxiliary weight

$$\gamma(x) = a \ln(1 + |x - x^\dagger|^2). \tag{5.4}$$

γ also depends on the parameter ξ, through $x^\dagger$. We require that $a > d/2$, which ensures that

$$\int_{\mathbb{R}^d} e^{-\gamma(x)}\, dm = \int_{\mathbb{R}^d} (1 + |x - x^\dagger|^2)^{-a}\, dm(x) < \infty.$$

Therefore the function $e^{2\lambda(\xi \cdot x - \phi(x))}$ satisfies

$$\int_{\mathbb{R}^d} |e^{2\lambda(\xi \cdot x - \phi(x))}|^2 e^{-4\Phi(x)-\gamma(x)}\, dm(x) = \int_{\mathbb{R}^d} e^{-\gamma(x)}\, dm(x) < \infty, \tag{5.5}$$

and this quantity is independent of ξ, λ even though γ depends through $x^\dagger$ on the real part of ξ.

We will solve the equation

$$-\operatorname{div}(u) = e^{2\lambda(\xi \cdot x - \phi(x))} \tag{5.6}$$

approximately, with u in the space of one-forms satisfying $\int_{\mathbb{R}^d} |u(x)|^2 e^{-4\Phi(x)-2\gamma(x)}\, dm(x) < \infty$. This is the reverse of the usual situation; the weight Φ is concave rather than convex, so the standard weighted theory [4], adapted from the complex case to the real case, does not apply.

Let $\mathcal{H}_1$ be the Hilbert space of all equivalence classes of Lebesgue measurable complex–valued (0, 1) forms defined on $\mathbb{R}^d$, with norm

$$\|u\|^2_{\mathcal{H}_1} = \int_{\mathbb{R}^d} |u(x)|^2 e^{-4\Phi(x)-2\gamma(x)}\, dm(x). \tag{5.7}$$

Let $\mathcal{H}_2$ be the Hilbert space of all equivalence classes of Lebesgue measurable functions $f : \mathbb{R}^d \to \mathbb{C}$, with norm

$$\|f\|_{\mathcal{H}_2}^2 = \int_{\mathbb{R}^d} |f(x)|^2 e^{-4\Phi(x)-\gamma(x)}\, dm(x). \tag{5.8}$$

If a is sufficiently large then $f(x) = e^{2\lambda(x\cdot\xi-\phi(x))}$ satisfies

$$\|f\|_{\mathcal{H}_2}^2 = \int_{\mathbb{R}^d} e^{4\lambda(x\cdot\operatorname{Re}\xi-\phi(x))} e^{-4\lambda(x\cdot\operatorname{Re}\xi-\phi(x))} e^{-\gamma(x)}\, dm(x) = \int_{\mathbb{R}^d} e^{-\gamma(x)}\, dm(x) < \infty;$$

this norm is independent of ξ, λ. In particular, $e^{2\lambda(x\cdot\xi-\phi(x))} \in \mathcal{H}_2$.

Regard div as an unbounded linear operator from $\mathcal{H}_1$ to $\mathcal{H}_2$, whose domain is the closure of the space of continuously differentiable compactly supported one-forms with respect to the graph norm. The formal adjoint div* of div in this Hilbert space setting is

$$\operatorname{div}^*(f) = e^{\gamma(x)} \sum_{j=1}^{d} (-\partial_{x_j} + 4\partial_{x_j}\Phi + \partial_{x_j}\gamma) f\, dx_j. \tag{5.9}$$

$\mathcal{H}_2 \subset L^1(\mathbb{R}^d)$ by virtue of the Cauchy–Schwarz inequality and the rapid decay of $e^{4\Phi}$, and thus $\int_{\mathbb{R}^d} f\, dm$ is well-defined for all $f \in \mathcal{H}_2$. Define $\mathfrak{F} \subset \mathcal{H}_2$ to be the set of all $f \in \mathcal{H}_2$ that satisfy

$$\int_{\mathbb{R}^d} f(x) = 0. \tag{5.10}$$

$\mathfrak{F}$ is a closed subspace of $\mathcal{H}_2$, of codimension one, which contains the image under div of the set of all compactly supported continuously differentiable forms, and hence by closedness contains the range of div.

Lemma 5.1 *Let $d \geq 1$. Let ϕ satisfy the hypotheses* (4.2), (4.3), (4.4), (4.5). *There exist constants $a, C < \infty$ with the following properties. Let λ be sufficiently large, let $\xi \in \mathbb{C}$, and suppose that $f \in \mathcal{H}_2$ satisfies $\int_{\mathbb{R}^d} f\, dm = 0$. There exists a 1-form $u \in \mathcal{H}_1$ satisfying*

$$\operatorname{div}(u) = f \quad on\ \mathbb{R}^d, \tag{5.11}$$

$$\int_{\mathbb{R}^d} |u(x)|^2 e^{-4\Phi(x)-2\gamma(x)}\, dm(x) \leq C \int_{\mathbb{R}^d} |f(x)|^2 e^{-4\Phi(x)-\gamma(x)}\, dm(x). \tag{5.12}$$

Recall that a is the parameter that appears in the definition (5.4) of γ. It will be essential for the ensuing argument that a, C may be chosen to be independent of λ, ξ.

Only the real part of ξ enters into the formulation of Lemma 5.1, so throughout its proof we will assume that $\xi \in \mathbb{R}^d$. The main step in that proof will be the following lemma, whose justification is deferred until Sect. 8.

Lemma 5.2 *Let ϕ satisfy the hypotheses (4.2), (4.3), (4.4), (4.5). If the exponent a is chosen to be sufficiently large then there exists $C < \infty$ such that for all sufficiently large λ and all $\xi \in \mathbb{R}^d$, for any function f in the intersection of $\mathfrak{F}$ with the domain of* div^*,

$$\|f\|_{\mathcal{H}_2} \le C\|\operatorname{div}^* f\|_{\mathcal{H}_1}. \tag{5.13}$$

According to Lemmas 4.1.1 and 4.1.2 of [4], it follows that the range of div, as a closed unbounded linear operator from $\mathcal{H}_1$ to $\mathcal{H}_2$, equals $\mathfrak{F}$, and moreover that for any $f \in \mathcal{F}$ there exists u in the intersection of $\mathcal{H}_1$ with the domain of div satisfying $\operatorname{div}(u) = f$ with $\|u\|_{\mathcal{H}_1} \le C\|f\|_{\mathcal{H}_2}$, with C independent of $\xi \in \mathbb{R}^d$ and $\lambda \in \mathbb{R}^+$ provided that λ is sufficiently large. Lemma 5.1 is thus a corollary of Lemma 5.2.

A solution of the divergence equation with additional desirable properties can be obtained, and will be needed in the application below. Because the divergence equation is underdetermined, one cannot hope that arbitrary solutions will have favorable properties; it is necessary to select an appropriate solution. To this end, consider the operator $T = \operatorname{div} \circ \operatorname{div}^*$.

For f in the intersection of $\mathfrak{F}$ with the domain of T,

$$\|Tf\|_{\mathcal{H}_2}\|f\|_{\mathcal{H}_2} \ge \langle Tf, f\rangle_{\mathcal{H}_2} = \|\operatorname{div}^* f\|^2_{\mathcal{H}_1} \ge C\|f\|^2_{\mathcal{H}_2}.$$

Therefore $\|f\|_{\mathcal{H}_1} \le C\|Tf\|_{\mathcal{H}_1}$. T is symmetric, and because of this inequality, maps its domain in $\mathcal{H}_2$ onto $\mathfrak{F}$.

Lemma 5.3 *Let ϕ satisfy the hypotheses (4.2), (4.3), (4.4), (4.5). Let the parameter a be sufficiently large. Then for all sufficiently large $\lambda \in \mathbb{R}^+$, all $\xi \in \mathbb{C}^d$, and any ϕ, f satisfying the hypotheses of Lemma 5.1, there exists a solution of* $\operatorname{div}(u) = f$ *satisfying*

$$\int_{\mathbb{R}^d} \big(|u(x)|^2 + |\nabla u(x)|^2\big)e^{-4\Phi(x)-3\gamma(x)}\,dm(x) \le C\lambda^C \int_{\mathbb{R}^d} |f(x)|^2 e^{-4\Phi(x)-\gamma(x)}\,dm(x). \tag{5.14}$$

Proof We have shown that there exists a solution $h \in \mathcal{H}_2$ of the equation $\operatorname{div}\operatorname{div}^*(h) = f$. Define $u = \operatorname{div}^*(h)$. $\operatorname{div}\operatorname{div}^*$ is a second order elliptic differential operator, whose coefficients are $O(\lambda^2 + |x|^2)$, together with all of their derivatives. Cover $\mathbb{R}^d$ by a union of suitable balls, so that if x belongs to a ball B in this cover then the radius of B is comparable to $\lambda^{-1}(1 + |x - x^\dagger|)^{-1}$. This ensures that

$$\max_B e^{-4\Phi} \le C \min_B e^{-4\Phi}$$

for a finite constant C independent of λ, ξ, B. Apply standard elliptic regularity estimates in each ball to obtain upper bounds for the second partial derivatives of h, and sum the results. If the parameter a is chosen to be sufficiently large then the extra factor $e^{-\gamma(x)}$ on the left-hand side of the inequality, together with the factor λ^C on the right-hand side, compensate for the resulting losses due to growth in the coefficients as $\lambda \cdot (1 + |x|) \to \infty$. $\square$

6 Absence of Near-Resonances

6.1 Exponential Decay Implies Absence of Near-Resonances

For $\xi \in \mathbb{C}^d$ and $\lambda \in \mathbb{R}^+$ define

$$\mathcal{F}(\xi, \lambda) = \int_{\mathbb{R}^d} e^{2\lambda(\xi \cdot x - \phi(x))} \, dm(x). \tag{6.1}$$

The goal of this section is to establish the following lemma, which links the off-diagonal behavior of Bergman kernels with the function $\mathcal{F}$. In Sect. 7 we will establish a link between $\mathcal{F}$ and ϕ.

Proposition 6.1 *Suppose that there exist sequences $\lambda_\nu, A_\nu \in \mathbb{R}^+$ and $\xi_\nu \in \mathbb{C}^d$ such that*

$$\lambda_\nu \to \infty, \tag{6.2}$$

$$A_\nu \to +\infty, \tag{6.3}$$

$$\mathrm{Re}(\xi_\nu) \to \xi^* \in \mathbb{R}^d, \tag{6.4}$$

$$\mathrm{Im}(\xi_\nu) \to 0, \tag{6.5}$$

$$|\mathcal{F}(\xi_\nu, \lambda_\nu)| \le e^{-A_\nu \lambda_\nu}. \tag{6.6}$$

Define $x^* \in \mathbb{R}^d$ by

$$\nabla \phi(x^*) = \xi^*. \tag{6.7}$$

Then there is no neighborhood of x^* in $\mathbb{C}^d$ in which $(B_{\lambda_\nu} : \nu \in \mathbb{N})$ decays exponentially fast away from the diagonal.

6.2 Beginning of the Proof of Proposition 6.1

Proof Let (λ_ν), (A_ν), (ξ_ν) be sequences with the stated properties. Suppose to the contrary that there does exist a neighborhood W of x^* in $\mathbb{C}^d$ in which $(B_{\lambda_\nu} : \nu \in \mathbb{N})$ decays exponentially fast away from the diagonal. That is, for any $\eta > 0$ there exist $C, c \in (0, \infty)$ such that for all $(z, z') \in W \times W$ satisfying $|z - z'| \ge \eta$, $|B_{\lambda_\nu}(z, z')| \le Ce^{-c\lambda_\nu}$.

Writing $z = x + iy \in \mathbb{R}^d + i\mathbb{R}^d$, define

$$\psi_\nu(z) = e^{\lambda_\nu z\cdot\xi_\nu}, \tag{6.8}$$

$$f_\nu(x) = e^{2\lambda_\nu(x\cdot\xi_\nu - \phi(x))} \tag{6.9}$$

$$\Phi_\nu(x) = \lambda_\nu(x\cdot \operatorname{Re}\xi_\nu - \phi(x)) \tag{6.10}$$

where $z\cdot\xi = \sum_{j=1}^d z_j\xi_j$. Define $x_\nu \in \mathbb{R}^d$ to be the unique solution of

$$\nabla\phi(x_\nu) = \operatorname{Re}\xi_\nu; \tag{6.11}$$

this is the quantity denoted by $x^\dagger$ in (5.2). Thus the auxiliary function $\gamma = \gamma_\nu$ introduced in (5.4) takes the form

$$\gamma_\nu(x) = a\ln(1 + |x - x_\nu|^2).$$

Since $\operatorname{Re}\xi_\nu \to \xi^* \in \mathbb{R}^d$, $x_\nu \to x^*$.

Introduce the scalar

$$b_\nu = \left(\int_{\mathbb{R}^d} e^{2\Phi_\nu}\right)^{-1}\int_{\mathbb{R}^d} f_\nu = \left(\int_{\mathbb{R}^d} e^{2\Phi_\nu}\right)^{-1}\mathcal{F}(\xi_\nu,\lambda_\nu). \tag{6.12}$$

This quantity is asymptotically very small:

$$|b_\nu| = O(\lambda_\nu^d)\cdot|\mathcal{F}(\xi_\nu,\lambda_\nu)| = e^{-A_\nu\lambda_\nu/2}. \tag{6.13}$$

Now $\int_{\mathbb{R}^d}(f_\nu - b_\nu e^{2\Phi_\nu})\,dm = 0$, and $f_\nu - b_\nu e^{2\Phi_\nu} \in \mathcal{H}_2$, so this function belongs to the range of the divergence operator. Let u_ν be a complex-valued one-form with domain $\mathbb{R}^d$ that satisfies the equation

$$\operatorname{div}(u_\nu) = f_\nu - b_\nu e^{2\Phi_\nu} \tag{6.14}$$

and the upper bounds for such a solution provided by Lemma 5.3:

$$\begin{aligned}
\int_{\mathbb{R}^d}&\big(|u_\nu(x)|^2 + |\nabla u_\nu(x)|^2\big)e^{-4\Phi_\nu(x)-3\gamma_\nu(x)}\,dm(x)\\
&\le C\lambda_\nu^C\int_{\mathbb{R}^d}|f_\nu(x) - b_\nu e^{2\Phi_\nu(x)}|^2 e^{-4\Phi(x)-\gamma(x)}\,dm(x)\\
&\le C\lambda_\nu^C\int_{\mathbb{R}^d}|f_\nu(x)|^2 e^{-4\Phi_\nu(x)-\gamma_\nu(x)}\,dm(x) + C\lambda_\nu^C|\int_{\mathbb{R}^d} f_\nu\,dm|^2\\
&\le C\lambda_\nu^C
\end{aligned}$$

with $C < \infty$ independent of ν.

Define

$$v_\nu(z) = e^{i\lambda_\nu y\cdot\xi_\nu} e^{-\lambda_\nu(x\cdot\xi_\nu - 2\phi(x))} u_\nu(z). \tag{6.15}$$

v_ν satisfies the equation

$$\bar\partial^*_{2\lambda_\nu\phi} v_\nu(x+iy) = \psi_\nu(z) - b_\nu e^{i\lambda_\nu y\cdot\xi_\nu} e^{\lambda_\nu x\cdot(2\operatorname{Re}\xi_\nu - \xi_\nu)} \tag{6.16}$$

with upper bounds

$$\int_{\mathbb{R}^d} \big(|v_\nu(x+iy)|^2 + |\nabla v_\nu(x+iy)|^2\big) e^{-2\lambda_\nu x\cdot\operatorname{Re}\xi_\nu} e^{-2\gamma_\nu(x)}\, dm(x) \\ \le C\lambda_\nu^C e^{2\lambda_\nu|\operatorname{Im}(\xi_\nu)|\cdot|y|} \tag{6.17}$$

uniformly for all $y \in \mathbb{R}^d$.

Let $V \subset \mathbb{C}^d$ be a ball centered at x^*, independent of ν, to be chosen below. Then

$$\|\psi_\nu\|^2_{L^2(V,e^{-2\lambda\phi})} \ge \lambda_\nu^{-d} e^{2\lambda_\nu(x_\nu\cdot\operatorname{Re}\xi_\nu - \phi(x_\nu))} e^{-C\lambda_\nu|\operatorname{Im}(\xi_\nu)|} \tag{6.18}$$

for all sufficiently large ν, since $x_\nu \to x^* \in V$. There is a corresponding upper bound with the sign of the exponent reversed in the final exponential factor. This allows us to express bounds for v_ν in terms of ψ_ν: (6.18) and (6.17) together give

$$\int_{\mathbb{R}^d} |v_\nu(x+iy)|^2 e^{-2\lambda_\nu((x-x_\nu)\cdot\operatorname{Re}\xi_\nu - (\phi(x)-\phi(x_\nu)) - 3\gamma_\nu(x)} e^{-2\lambda_\nu\phi(x)}\, dm(x) \\ \le C\lambda_\nu^C e^{C\lambda_\nu|\operatorname{Im}(\xi_\nu)|} e^{2\lambda_\nu(x_\nu\cdot\operatorname{Re}\xi_\nu - \phi(x_\nu))} \\ \le C\lambda_\nu^C e^{C\lambda_\nu|\operatorname{Im}(\xi_\nu)|} \|\psi_\nu\|^2_{L^2(V,e^{-2\lambda_\nu\phi})}. \tag{6.19}$$

The left-hand side is the squared norm $\int_{\mathbb{R}^d} |v_\nu(x+iy)|^2 e^{-2\lambda_\nu\phi(x)}\, dm(x)$, modified by incorporation into the integrand of an advantageous supplementary factor

$$e^{-2\lambda_\nu((x-x_\nu)\cdot\operatorname{Re}\xi_\nu - (\phi(x)-\phi(x_\nu)) - 3\gamma_\nu(x)} \ge e^{c\lambda_\nu|x-x_\nu|^2}.$$

This weight is of no help in overcoming the disadvantageous factor $e^{2\lambda_\nu|\operatorname{Im}(\xi_\nu)|\cdot|y|}$ on the right-hand side when $x \approx x_\nu$ but $y \ne 0$; overcoming that factor will be a crucial issue. This supplementary factor will consequently be of no further use, and will now be dropped, so (6.19) simplifies to

$$\int_{\mathbb{R}^d} \big(|v_\nu(x+iy)|^2 + |\nabla v_\nu(x+iy)|^2\big) e^{-2\lambda_\nu\phi(x)}\, dm(x) \\ \le C\lambda_\nu^C e^{C\lambda_\nu|\operatorname{Im}(\xi_\nu)|} \|\psi_\nu\|^2_{L^2(V,e^{-2\lambda_\nu\phi})}. \tag{6.20}$$

6.3 Localized Solutions F_ν

With no loss of generality, change coordinates so that $x^* = 0$, that is, $x_\nu \to 0$. Let V be a small open neighborhood of 0 with the property that in $V \times V$, the Bergman kernels B_{λ_ν} are $O(e^{-c_\varepsilon \lambda_\nu})$ as ν (and hence λ_ν) tends to infinity, for all pairs of points at distance from the diagonal greater than ε, for any $\varepsilon > 0$.

Let $\eta \in C_0^\infty(\mathbb{C}^d)$ be identically equal to 1 in a neighborhood of 0, and be supported in V. Consider the functions $F_\nu : \mathbb{C}^d \to \mathbb{C}$ defined by

$$F_\nu = \bar\partial^*_{2\lambda_\nu\phi}(\eta v_\nu)\,. \tag{6.21}$$

These are supported in V, a bounded set independent of ν. The relation (6.16) for $\bar\partial^*_{2\lambda_\nu\phi} v_\nu$ gives

$$F_\nu = \eta\,\bar\partial^*_{2\lambda_\nu\phi} v_\nu + v_\nu\,\partial\eta = \eta\psi_\nu - b_\nu \eta e^{i\lambda_\nu y\cdot\xi_\nu} e^{\lambda_\nu x\cdot(2\,\mathrm{Re}\,\xi_\nu - \xi_\nu)} + v_\nu\,\partial\eta \tag{6.22}$$

with both sides evaluated at $z = x + iy$.

Let $W \Subset W'$ and V' be bounded open subsets of $\mathbb{C}^d$ such that $0 \in W$, the closure of W is contained in W', $\eta \equiv 1$ in a neighborhood of the closure of W', the closure of V' is disjoint from the closure of W', and the support of $\nabla\eta$ is contained in V'. For all sufficiently large indices ν,

$$\begin{aligned}\|F_\nu - \eta\psi_\nu\|_{L^2(\mathbb{C}^d, e^{-2\lambda_\nu\phi})} &+ \|\nabla(F_\nu - \eta\psi_\nu)\|_{L^2(\mathbb{C}^d, e^{-2\lambda_\nu\phi})} \\ &\le \lambda_\nu^C e^{C\lambda_\nu|\,\mathrm{Im}(\xi_\nu)|}\|\psi_\nu\|_{L^2(V, e^{-2\lambda_\nu\phi})},\end{aligned} \tag{6.23}$$

and

$$\begin{aligned}\|F_\nu - \eta\psi_\nu\|_{L^2(W', e^{-2\lambda_\nu\phi})} &+ \|\nabla(F_\nu - \eta\psi_\nu)\|_{L^2(W', e^{-2\lambda_\nu\phi})} \\ &\le \lambda_\nu^C e^{\lambda_\nu(-cA_\nu + C|\,\mathrm{Im}(\xi_\nu)|)}\|\psi_\nu\|_{L^2(V, e^{-2\lambda_\nu\phi})}.\end{aligned} \tag{6.24}$$

The second inequality holds because in W', $\nabla\eta$ vanishes identically and therefore $F - \eta\psi$ is equal to the constant b_ν multiplied by $\bar\partial e^{i\lambda_\nu y\cdot\xi_\nu} e^{\lambda_\nu x\cdot(2\,\mathrm{Re}\,\xi_\nu - \xi_\nu)}$. Growth of the second factor is amply compensated for by the factor $b_\nu = O(e^{-A_\nu\lambda_\nu/2})$.

In particular, since $\bar\partial\psi_\nu = 0$,

$$\|\bar\partial F_\nu\|_{L^2(\mathbb{C}^d, e^{-2\lambda_\nu\phi})} \le \lambda_\nu^C e^{C\lambda_\nu|\,\mathrm{Im}(\xi_\nu)|}\|\psi_\nu\|_{L^2(V, e^{-2\lambda_\nu\phi})} \tag{6.25}$$

$$\|\bar\partial F_\nu\|_{L^2(W', e^{-2\lambda_\nu\phi})} \le e^{\lambda_\nu(-cA_\nu + C|\,\mathrm{Im}(\xi_\nu)|)}\|\psi_\nu\|_{L^2(V, e^{-2\lambda_\nu\phi})}. \tag{6.26}$$

6.4 Solution of a Final $\bar\partial$ Equation

The hypothesis that the Bergman kernels decay exponentially away from the diagonal will be applied not to the functions ψ_ν themselves, but rather, to functions G_ν that satisfy the approximate relation $\psi_\nu \approx -B_{\lambda_\nu} G_\nu$. These functions G_ν will not be of the product form $e^{i\lambda_\nu y\cdot\xi_\nu} f_\nu(x)$, and will be constructed by solving a final $\bar\partial$ equation. To prepare for their construction, choose a C^∞ function $\tilde\phi : \mathbb{C}^d \to \mathbb{R}$ and a constant $\varepsilon > 0$ with the following properties:

1. $\tilde\phi$ is plurisubharmonic.
2. $\tilde\phi \le \phi$.
3. $\tilde\phi \equiv \phi$ in a neighborhood of the support of $\nabla\eta$.
3. There exists $\varepsilon > 0$ such that $\tilde\phi(z) \le \phi(z) - \varepsilon$ for all $z \in \mathbb{C}^d \setminus V'$.

These exist, because ϕ is strictly plurisubharmonic. In particular, $\tilde\phi < \phi$ in a neighborhood of $x^* = 0$, with strict inequality.

The right-hand side in our $\bar\partial$ equation will be $\bar\partial F_\nu$. The norm of $\bar\partial F_\nu$ is still under satisfactory control with respect to the modified weight $\tilde\phi$:

$$\|\bar\partial F_\nu\|_{L^2(\mathbb{C}^d, e^{-2\lambda_\nu\tilde\phi})} \le \lambda_\nu^C e^{C\lambda_\nu|\operatorname{Im}(\xi_\nu)|} \|\psi_\nu\|_{L^2(V, e^{-2\lambda_\nu\phi})} \tag{6.27}$$

for all sufficiently large ν. Note that the norm on the left-hand side involves the weight function $\tilde\phi$, while ϕ appears on the right-hand side. This relies on (6.25) and (6.26), the fact that $\tilde\phi \equiv \phi$ on the support of $\nabla\eta$, the Cauchy–Riemann relation $\bar\partial\psi_\nu \equiv 0$, and the crucial assumption that $A_\nu \to \infty$ as $\nu \to \infty$, which guarantees that the contribution of the term involving b_ν is exponentially small.

Lemma 6.2 *Let $\tilde\phi$ have the properties listed. There exists a constant $C < \infty$ such that for each sufficiently large ν there exists a solution G_ν of the equation*

$$\bar\partial G_\nu = \bar\partial F_\nu \tag{6.28}$$

satisfying

$$\int_{\mathbb{C}^d} |G_\nu|^2 e^{-2\lambda_\nu\tilde\phi} e^{-\gamma}\, dm \le C \int_{\mathbb{C}^d} |\bar\partial F_\nu|^2 e^{-2\lambda_\nu\tilde\phi}\, dm. \tag{6.29}$$

Proof A direct application of the well-known weighted theory for the $\bar\partial$ equation [4] suffices. □

For each sufficiently large ν, choose a solution G_ν of (6.28) satisfying (6.29), with C independent of ν. Concerning these functions, two consequences of Lemma 6.2 together with (6.27) will be useful. Firstly, in the whole space $\mathbb{C}^d$,

$$\int_{\mathbb{C}^d} |G_\nu|^2 e^{-2\lambda_\nu\phi}\, dm \le C\lambda_\nu^C e^{C\lambda_\nu|\operatorname{Im}(\xi_\nu)|} \|\psi_\nu\|^2_{L^2(V, e^{-2\lambda_\nu\phi})}. \tag{6.30}$$

Secondly, in the complement of V' there is an improved upper bound

$$\int_{\mathbb{C}^d \setminus V'} |G_\nu|^2 e^{-2\lambda_\nu \phi}\, dm \le e^{-\varepsilon\lambda_\nu} e^{C\lambda_\nu |\operatorname{Im}(\xi_\nu)|} \|\psi_\nu\|^2_{L^2(V, e^{-2\lambda_\nu \phi})} \tag{6.31}$$

with the extra factor $e^{-\varepsilon\lambda_\nu}$ on the right-hand side.

6.5 Arrival at a Contradiction

We now complete the proof of Proposition 6.1, modulo the deferred proof of Lemma 5.2, by showing how its hypotheses, together with the assumption that the Bergman kernels B_{λ_ν} decay exponentially fast away from the diagonal, lead to a contradiction. Throughout the discussion, it is assumed that ν is sufficiently large. An upper bound of the form "$O(M)$ in W" indicates a function whose norm in $L^2(W, e^{-2\lambda_\nu \phi})$ is $O(M)$, uniformly in ν.

Recalling that $F_\nu = \bar\partial^*_{2\lambda_\nu \phi}(\eta v_\nu)$ and that $\bar\partial G_\nu = \bar\partial F_\nu$, the equation

$$G_\nu = (G_\nu - \bar\partial^*_{2\lambda_\nu \phi}(\eta v_\nu)) \;+\; \bar\partial^*_{2\lambda_\nu \phi}(\eta v_\nu) \tag{6.32}$$

expresses G_ν as the sum of an element of the nullspace of $\bar\partial$ plus a function orthogonal to that nullspace. Therefore

$$(I - B_{\lambda_\nu})G_\nu = \bar\partial^*_{2\lambda_\nu \phi}(\eta v_\nu). \tag{6.33}$$

Consequently

$$(I - B_{\lambda_\nu})G_\nu = \bar\partial^*_{2\lambda_\nu \phi} v_\nu \quad \text{in } W \tag{6.34}$$

since $\eta \equiv 1$ in W and $\bar\partial^*_{2\lambda_\nu \phi}$ is a local operator.

v_ν satisfies (6.16) $\bar\partial^*_{2\lambda_\nu \phi} v_\nu = \psi_\nu - b_\nu e^{i\lambda_\nu y\cdot\xi_\nu} e^{\lambda_\nu x\cdot(2\operatorname{Re}\xi_\nu - \xi_\nu)}$, and b_ν is small in the sense that $|b_\nu| \le e^{-cA_\nu\lambda_\nu}$ (6.13). Therefore

$$(I - B_{\lambda_\nu})G_\nu = \psi_\nu \;+ O(e^{-c\lambda_\nu A_\nu} \|\psi_\nu\|_{L^2(V, e^{-2\lambda_\nu \phi})}) \;\text{ in } W. \tag{6.35}$$

Using the strong bound provided by inequality (6.31) in the complement of V', and in particular in W, this can be rewritten as

$$\psi_\nu = -B_{\lambda_\nu} G_\nu + O\big(e^{-c\lambda_\nu} e^{C\lambda_\nu |\operatorname{Im}(\xi_\nu)|} \|\psi_\nu\|_{L^2(V, e^{-2\lambda_\nu \phi})}\big) \text{ in } W. \tag{6.36}$$

In this rewriting, we have expanded $(I - B_{\lambda_\nu})G_\nu = G_\nu - B_{\lambda_\nu} G_\nu$ and have incorporated the term G_ν into the $O(\cdot)$ term, exploiting the factor $e^{-\varepsilon\lambda_\nu}$ in (6.31) and replacing ε by c, a positive constant independent of ν.

Let $\mathbf{1}_{V'}$ denote the indicator function of V'. Because B_{λ_ν} is a contraction on $L^2(\mathbb{C}^d, e^{-2\lambda_\nu\phi})$,

$$\begin{aligned}\|B_{\lambda_\nu}G_\nu\|_{L^2(W,e^{-2\lambda_\nu\phi})} &\le \|B_{\lambda_\nu}(\mathbf{1}_{V'}G_\nu)\|_{L^2(W,e^{-2\lambda_\nu\phi})} + \|B_{\lambda_\nu}(\mathbf{1}_{\mathbb{C}^d\setminus V'}G_\nu)\|_{L^2(\mathbb{C}^d,e^{-2\lambda_\nu\phi})}\\ &\le \|B_{\lambda_\nu}(\mathbf{1}_{V'}G_\nu)\|_{L^2(W,e^{-2\lambda_\nu\phi})} + \|G_\nu\|_{L^2(\mathbb{C}^d\setminus V',e^{-2\lambda_\nu\phi})}\\ &\le \|B_{\lambda_\nu}(\mathbf{1}_{V'}G_\nu)\|_{L^2(W,e^{-2\lambda_\nu\phi})} + e^{-c\lambda_\nu}e^{C\lambda_\nu|\operatorname{Im}(\xi_\nu)|}\|\psi_\nu\|_{L^2(V,e^{-2\lambda_\nu\phi})}.\end{aligned}$$

To obtain the final line we have again invoked (6.31) to control G_ν in the complement of V'.

The sets W, V' were constructed to have disjoint closures, and so that both are contained in a region in which the Bergman kernels B_{λ_ν} decay exponentially fast away from the diagonal. Thus there exists $c > 0$ such that for all sufficiently large ν,

$$\begin{aligned}\|B_{\lambda_\nu}(\mathbf{1}_{V'}G_\nu)\|_{L^2(W,e^{-2\lambda_\nu\phi})} &\le e^{-c\lambda_\nu}\|G_\nu\|_{L^2(V',e^{-2\lambda_\nu\phi})}\\ &\le e^{-c\lambda_\nu}e^{C\lambda_\nu|\operatorname{Im}(\xi_\nu)|}\|\psi_\nu\|_{L^2(V,e^{-2\lambda_\nu\phi})}.\end{aligned}$$

Inserting these bounds into (6.36) gives

$$\|\psi_\nu\|_{L^2(W,e^{-2\lambda_\nu\phi})} \le e^{-c\lambda_\nu}e^{C\lambda_\nu|\operatorname{Im}(\xi_\nu)|}\|\psi_\nu\|_{L^2(V,e^{-2\lambda_\nu\phi})} \tag{6.37}$$

with $c > 0$ independent of ν.

We have normalized ϕ so that $\phi(0) = 0$ and $\nabla\phi(0) = 0$, so $\phi(x) \asymp |x|^2$. It is thus apparent from the explicit formula $\psi_\nu(z) = e^{\lambda_\nu z\cdot\xi_\nu}$ and the assumption that $x_\nu \to 0$ that the functions ψ_ν peak near 0 in the sense that

$$\|\psi_\nu\|_{L^2(W,e^{-2\lambda_\nu\phi})} \ge e^{-C\lambda_\nu|\operatorname{Im}(\xi_\nu)|}\|\psi_\nu\|_{L^2(V,e^{-2\lambda_\nu\phi})}. \tag{6.38}$$

Therefore (6.37) implies that

$$\|\psi_\nu\|_{L^2(V,e^{-2\lambda_\nu\phi})} \le e^{-c\lambda_\nu}e^{C\lambda_\nu|\operatorname{Im}(\xi_\nu)|}\|\psi_\nu\|_{L^2(V,e^{-2\lambda_\nu\phi})} \tag{6.39}$$

with $c > 0$ independent of ν. Since $|\operatorname{Im}(\xi_\nu)| \to 0$ as $\nu \to \infty$, and since none of the functions ψ_ν vanish identically, this is a contradiction for all sufficiently large ν. □

This completes the proof of Proposition 6.1, modulo the deferred proof of Lemma 5.2 concerning solvability of the divergence equation. In Sect. 7 we complete the proof of Theorem 2.1. In Sect. 8 we take up the proof of Lemma 5.2.

7 Conclusion of the Proof

The second of the two main steps of the proof links properties of $\mathcal{F}$ with analyticity of the metric ϕ. The strong convexity of ϕ implies that the mapping $x \mapsto \nabla\phi(x)$ from $\mathbb{R}^d$ to $\mathbb{R}^d$ is a bijection. Define the function $\tau : \mathbb{R}^d \to \mathbb{R}^d$ to be the inverse of $\nabla\phi$, that is,

$$\nabla\phi(\tau(\xi)) = \xi. \tag{7.1}$$

Section 7 is devoted to the proof of the following result.

Proposition 7.1 *Let* $(\lambda_\nu : \nu \in \mathbb{N})$ *be a sequence of positive real numbers tending to infinity. Suppose that* $(B_{\lambda_\nu} : \nu \in \mathbb{N})$ *decays exponentially fast away from the diagonal, in some neighborhood of* $a \in \mathbb{C}^d$*. Then the function* τ *is real analytic in some neighborhood of* $\xi = \nabla\phi(\mathrm{Re}(a))$.

Consequently under the hypotheses of Proposition 7.1, the inverse function $\nabla\phi$, and hence ϕ itself, are real analytic in a corresponding neighborhood of $\mathrm{Re}(a)$.

By the hypothesis of exponentially fast decay in some neighborhood of a, we mean that there exists $\varrho > 0$ such that for each $\delta > 0$ there exists $c < \infty$ such that

$$|B_{\lambda_\nu}(z, z')| \le e^{-c\lambda_\nu} \tag{7.2}$$

for all ordered pairs of elements of $\mathbb{C}^d$ satisfying $|a - z| < \varrho$, $|a - z'| < \varrho$, and $\delta \le |z - z'|$.

Let $a \in \mathbb{C}^d$. By making the change of variables $z \mapsto z - a$ and subtracting from ϕ a real-valued affine function, we may assume without loss of generality that $a = 0$ and that $\phi : \mathbb{R}^d \to \mathbb{R}$ satisfies $\phi(0) = \nabla\phi(0) = 0$.

Lemma 7.2 *Under the hypotheses of Proposition 7.1 with* $a = 0$ *and* $\nabla\phi(0) = 0$*, there exist an open ball* $\mathcal{B} \subset \mathbb{C}^d$ *centered at* 0*, a sequence of indices* ν_k *tending to* ∞*, and a real analytic function* $u : \mathcal{B} \to \mathbb{R}$ *such that*

$$\tfrac{1}{2}\lambda_{\nu_k}^{-1} \log|\mathcal{F}(\xi, \lambda_{\nu_k})| \to u(\xi) \tag{7.3}$$

uniformly as a function of $\xi \in \mathcal{B}$ *as* $k \to \infty$.

The functions

$$F_\nu(\xi) = \mathcal{F}(\xi, \lambda_\nu) \tag{7.4}$$

are all holomorphic in some common neighborhood of $\xi = 0$, independent of ν. Moreover, straightforward estimation gives

$$|F_\nu(\xi)| \le e^{C\lambda_\nu} \tag{7.5}$$

for all ξ in that neighborhood and for all ν, with $C < \infty$ independent of ξ, ν.

Proof of Lemma 7.2 According to Proposition 6.1, there exists an open ball $\mathcal{B}$ centered at 0 such that for every sufficiently large ν, $F_\nu)$ has no zeros in $\mathcal{B}$, and moreover there exists $C < \infty$ such that

$$\lambda_\nu^{-1} \log|F_\nu(\xi)| \ge -C \quad \text{for all } \xi \in \mathcal{B}, \tag{7.6}$$

uniformly in ν. Combining this with the upper bound (7.5) gives

$$\left| \lambda_\nu^{-1} \log |F_\nu(\xi)| \right| \le C \tag{7.7}$$

uniformly for all $\xi \in \mathcal{B}$, for all sufficiently large indices ν.

Since F_ν is holomorphic and zero-free in $\mathcal{B}$, $u_\nu = \frac{1}{2}\lambda_\nu^{-1} \log |F_\nu|$ is pluriharmonic there. Because these functions are uniformly bounded, they form a normal family. Therefore after replacing $\mathcal{B}$ by a concentric ball of strictly smaller radius, there exist a pluriharmonic function u in $\mathcal{B}$ and a sequence $\nu_k \to \infty$ such that $u_{\nu_k} \to u$ uniformly on all compact subsets of $\mathcal{B}$.

Being pluriharmonic, u is real analytic. □

Lemma 7.3 *The function u in the conclusion of Lemma 7.2 is*

$$u(\xi) = \xi \cdot \tau(\xi) - \phi \circ \tau(\xi). \tag{7.8}$$

Proof Consider any $\xi \in \mathbb{R}^d$. For large λ, $\mathcal{F}(\xi, \lambda) = \int_{\mathbb{R}^d} e^{2\lambda(\xi\cdot t - \phi(t))}\,dt$ can be calculated via the method of real stationary phase: Set $\tau = \tau(\xi)$. As $\lambda \to +\infty$,

$$\int_{\mathbb{R}^d} e^{2\lambda(\xi\cdot t - \phi(t))}\,dt = c_d e^{2\lambda(\xi\cdot\tau - \phi(\tau))}\lambda^{-d/2}\big(\det \nabla^2\phi(\tau)\big)^{-1/2} + O\big(e^{2\lambda(\xi\cdot\tau-\phi(\tau))}\lambda^{-(d+2)/2}\big). \tag{7.9}$$

Thus

$$\lambda_\nu^{d/2}|\mathcal{F}(\xi, \lambda_\nu)| = e^{2\lambda_\nu(\xi\cdot\tau - \phi(\tau))}\big(\alpha(\xi) + O(\lambda_\nu^{-1})\big) \tag{7.10}$$

for a certain strictly positive $\alpha(\xi)$. Taking logarithms of both sides and dividing by λ_ν gives

$$u_\nu(\xi) = \xi \cdot \tau(\xi) - \phi \circ \tau(\xi) + O(\lambda_\nu^{-1} \log \lambda_\nu) \tag{7.11}$$

as $\nu \to \infty$. Restricting attention to the subsequence ν_k obtained above and letting $k \to \infty$ gives (7.8). □

Lemma 7.4 *The function $u(\xi) = \xi \cdot \tau(\xi) - \phi(\tau(\xi))$ satisfies*

$$\nabla u \circ \nabla \phi(x) \equiv x. \tag{7.12}$$

Proof Substitute $\xi = \nabla\phi(x)$ to the equation for u as

$$u(\nabla\phi(x)) = x \cdot \nabla\phi(x) - \phi(x). \tag{7.13}$$

Apply $\nabla = \nabla_x$ to both sides to obtain

$$(\nabla u \circ \nabla\phi(x)) \star \nabla^2\phi(x) = \nabla\phi(x) + x \star \nabla^2\phi(x) - \nabla\phi(x) = x \star \nabla^2\phi(x) \tag{7.14}$$

where $\star$ denotes the product of a vector with a matrix. The Hessian $\nabla^2\phi(x)$ is by the positivity hypothesis an invertible matrix for each x, so the conclusion of the lemma follows. □

Lemma 7.5 *Let τ be the inverse of the mapping $\mathbb{R}^d \ni x \mapsto \nabla\phi(x)$. If the function $u(\xi) = \xi \cdot \tau(\xi) - \phi \circ \tau(\xi)$ is real analytic in a neighborhood of ξ_0 then ϕ is real analytic in a neighborhood of $\tau(\xi_0)$.*

Proof By (7.12), $x \mapsto \nabla u(x)$ is a locally invertible function. This function is the gradient of a real analytic function, so is analytic. Therefore its inverse, $x \mapsto \nabla\phi$, is also real analytic. Therefore ϕ itself is analytic. □

This completes the proof of Proposition 7.1, and with it the proof of the main theorem, except for the deferred proof of Lemma 5.2.

8 Proof of Lemma 5.2

Recall from (5.1) and (5.4) the definitions $\Phi(x) = \lambda(\mathrm{Re}\,\xi \cdot x - \phi(x))$ and $\gamma(x) = a\ln(1 + |x - x^\dagger|^2)$, where $x^\dagger$ denotes the unique point of $\mathbb{R}^d$ at which Φ attains its maximum value. We seek to prove that

$$\int_{\mathbb{R}^d} |f(x)|^2 e^{-4\Phi(x)-\gamma(x)}\,dm(x) \le C \int_{\mathbb{R}^d} |\operatorname{div}^* f(x)|^2 e^{-4\Phi(x)-2\gamma(x)}\,dm(x), \quad (8.1)$$

under the assumptions that f is continuously differentiable, compactly supported, and satisfies $\int_{\mathbb{R}^d} f\,dm = 0$. Substituting $f(x)e^{-2\Phi(x)} = g(x)$, one has $\int_{\mathbb{R}^d} e^{2\Phi} g\,dm = 0$. Using the expression (5.9) for div* gives

$$\begin{aligned}\int_{\mathbb{R}^d} |\operatorname{div}^* f(x)|^2 e^{-4\Phi(x)-2\gamma(x)}\,dm(x) &= \int_{\mathbb{R}^d} |\operatorname{div}^* e^{2\Phi} g|^2 e^{-4\Phi-2\gamma}\,dm \\ &= \int_{\mathbb{R}^d} |(-\nabla + 2\nabla\Phi + \nabla\gamma)g|^2\,dm \\ &= \int_{\mathbb{R}^d} |e^{2\Phi+\gamma}\nabla e^{-2\Phi-\gamma} g|^2\,dm.\end{aligned}$$

Thus Lemma 5.2 is equivalent to

Lemma 8.1 *There exists $C < \infty$ such that for every sufficiently large $\lambda \in \mathbb{R}^+$, every $\xi \in \mathbb{C}^d$, and every continuously differentiable compactly supported function $g : \mathbb{C}^d \to \mathbb{C}$ satisfying $\int_{\mathbb{R}^d} e^{2\Phi} g\,dm = 0$,*

$$\int_{\mathbb{R}^d} |g(x)|^2 e^{-\gamma(x)}\,dm(x) \le C \int_{\mathbb{R}^d} |e^{2\Phi+\gamma}\nabla e^{-2\Phi-\gamma} g|^2\,dm. \quad (8.2)$$

The rest of Sect. 8 is devoted to a proof of Lemma 8.1. Define the conjugated gradient

$$Sg = e^{2\Phi+\gamma}\nabla(e^{-2\Phi-\gamma} g). \quad (8.3)$$

Introduce

$$\begin{cases} M_\Phi = \max_{x\in\mathbb{R}^d} \Phi(x) = \Phi(x^\dagger) \\ \Phi^* = 2\Phi + \gamma - 2M_\Phi. \end{cases} \tag{8.4}$$

Since ϕ is uniformly strictly convex, the Hessian matrix of $\Phi(x) = \lambda(\mathrm{Re}\,\xi \cdot x - \phi(x))$ is uniformly comparable to $-\lambda$. Therefore for all sufficiently large $\lambda \in \mathbb{R}^+$,

$$e^{4M_\Phi} \le C\lambda^{d/2} \int_{\mathbb{R}^d} e^{4\Phi}\, dm \le C\lambda^{d/2} \int_{\mathbb{R}^d} e^{4\Phi+\gamma}\, dm. \tag{8.5}$$

The second inequality holds because $\gamma \ge 0$. Also define

$$I(x,u) = \int_1^\infty e^{\Phi^*(x+su)} s^{d-1}\, ds. \tag{8.6}$$

Lemma 8.2 *There exists $C < \infty$ such that for any sufficiently large $\lambda \in \mathbb{R}^+$, any $\xi \in \mathbb{R}^d$, and any compactly supported continuously differentiable function $g : \mathbb{R}^d \to \mathbb{C}$ satisfying $\int_{\mathbb{R}^d} g e^{2\Phi}\, dm = 0$, for any $x \in \mathbb{R}^d$*

$$|g(x)| \le C\lambda^{d/2} \int_{\mathbb{R}^d} |u|\, I(x,u)\, |Sg(x+u)|\, du. \tag{8.7}$$

Proof For any $x, y \in \mathbb{R}^d$,

$$e^{-2\Phi(x)-\gamma(x)} g(x) = e^{-2\Phi(y)-\gamma(y)} g(y) + \int_0^1 (x-y)\cdot\nabla(e^{-2\Phi-\gamma} g)(y+t(x-y))\, dt \tag{8.8}$$

and therefore

$$e^{4\Phi(y)+\gamma(y)} e^{-2\Phi(x)-\gamma(x)} g(x) = e^{2\Phi(y)} g(y) + e^{4\Phi(y)+\gamma(y)} \int_0^1 (x-y)\cdot\nabla(e^{-2\Phi-\gamma} g)(y+t(x-y))\, dt. \tag{8.9}$$

Integrating over $\mathbb{R}^d$ with respect to $dm(y)$ and invoking the condition $\int g e^{2\Phi}\, dm = 0$ gives

$$e^{-2\Phi(x)-\gamma(x)} g(x) \int_{\mathbb{R}^d} e^{4\Phi+\gamma}\, dm = \int_{\mathbb{R}^d} e^{4\Phi(y)+\gamma(y)} \int_0^1 (x-y)\cdot\nabla(e^{-2\Phi-\gamma} g)(y+t(x-y))\, dt\, dm(y). \tag{8.10}$$

Using (8.5) to estimate the factor $\int_{\mathbb{R}^d} e^{4\Phi+\gamma}$ on the left-hand side of this identity gives the pointwise upper bound

$$\begin{aligned}|g(x)| &\le C\lambda^{d/2}e^{-4M_\Phi}\int_{\mathbb{R}^d}\int_0^1 e^{2\Phi(x)+\gamma(x)}e^{4\Phi(y)+\gamma(y)} \\ &\qquad |x-y|\,|\nabla(e^{-2\Phi-\gamma}g)(y+t(x-y))|\,dt\,dm(y) \\ &= C\lambda^{d/2}\int_{\mathbb{R}^d}\int_0^1 e^{\Phi^*(x)+2\Phi^*(y)-\gamma(y)-\Phi^*(tx+(1-t)y)}\,|x-y|\,|Sg(y+t(x-y))|\,dt\,dm(y) \\ &\le C\lambda^{d/2}\int_{\mathbb{R}^d}\int_0^1 e^{\Phi^*(x)+2\Phi^*(y)-\Phi^*(tx+(1-t)y)}\,|x-y|\,|Sg(y+t(x-y))|\,dt\,dm(y).\end{aligned}$$

A factor of $e^{-\gamma(y)}$ was dropped to obtain the final inequality; this is valid since $\gamma \ge 0$. Substitute $(t, y) \leftrightarrow (t, u)$ where $tx + (1-t)y = x + u$, so that $y = x + (1-t)^{-1}u$, and then substitute $s = (1-t)^{-1}$ to deduce that

$$\begin{aligned}|g(x)| &\le C\lambda^{d/2}\int_{\mathbb{R}^d}\Big(\int_0^1 e^{\Phi^*(x)+2\Phi^*(x+(1-t)^{-1}u)-\Phi^*(x+u)}(1-t)^{-d-1}\,dt\Big)|u|\,|Sg(x+u)|\,dm(u) \\ &= C\lambda^{d/2}\int_{\mathbb{R}^d}\Big(\int_1^\infty e^{\Phi^*(x)+2\Phi^*(x+su)-\Phi^*(x+u)}s^{d-1}\,ds\Big)\,|u|\,|Sg(x+u)|\,dm(u).\end{aligned}$$

$\Phi^* = 2\Phi + \gamma - 2M_\Phi$ is a concave function for any sufficiently large λ since the Hessian matrix of Φ is uniformly comparable to $-\lambda$ while γ is independent of λ and has bounded Hessian. Since

$$x + u = s^{-1}(x + su) + (1 - s^{-1})x$$

is a convex linear combination of $x + su$ and x, the concavity of Φ^* implies that

$$\Phi^*(x+u) \ge s^{-1}\Phi^*(x+su) + (1-s^{-1})\Phi^*(x) \tag{8.11}$$

for every $s \in [1, \infty)$ and $x, u \in \mathbb{R}^d$. Using (8.11) to majorize $-\Phi^*(x+u)$ gives

$$\begin{aligned}&\Phi^*(x) + 2\Phi^*(x+su) - \Phi^*(x+u) \\ &\qquad\le \Phi^*(x+su) + s^{-1}\Phi^*(x) + (1-s^{-1})\Phi^*(x+su).\end{aligned} \tag{8.12}$$

Since $\Phi^* = 2(\Phi - M_\Phi) + \gamma$ is nonnegative, one concludes that

$$\Phi^*(x) + 2\Phi^*(x+su) - \Phi^*(x+u) \le \Phi^*(x+su). \tag{8.13}$$

Insertion of this bound into the inner integral in the last bound for $|g(x)|$ above gives the conclusion of Lemma 8.2. □

The factor $|u|I(x,u)$ that appears on the right-hand side of the inequality in Lemma 8.2 satisfies a useful upper bound.

Lemma 8.3

$$|u|I(x,u) \le \begin{cases} C(1+|x-x^\dagger|)^{d-1}|u|^{1-d} & \text{for all } u, x \\ Ce^{-c|u|^2} & \text{if } |u| \ge 2|x-x^\dagger|. \end{cases} \tag{8.14}$$

Proof Recall that for all sufficiently large parameters λ, $\Phi^* = \Phi - M_\Phi + \gamma$ is real-valued, nonpositive, and concave, and vanishes at $x^\dagger$. Φ has a negative definite Hessian which is uniformly comparable to $-\lambda$, while γ is independent of λ and has a Hessian which is bounded above and below. Therefore

$$\Phi^*(x) \le -c\lambda|x-x^\dagger|^2, \tag{8.15}$$

uniformly in x, λ, ξ for all sufficiently large λ.

Define $\bar{s} \in \mathbb{R}$ to be the point at which $|(x-x^\dagger) - su|$ is minimized, and let h be its minimum value. Then

$$|I(x,u)| \le \int_{-\infty}^{\infty} e^{-c\lambda|h|^2} e^{-c\lambda|u|^2|s-\bar{s}|^2} |s|^{d-1}\,ds \le |u|^{-d} \int_{-\infty}^{\infty} e^{-ct^2}(|t|+|x-x^\dagger|)^{d-1}\,dt$$

for $\lambda \ge 1$. The first bound stated in (8.14) follows directly.

If $|u| \ge 2|x-x^\dagger|$ then for all $s \ge 1$,

$$\Phi^*(x+su) \le -c\lambda|x+su-x^\dagger|^2 \le -c\lambda s^2|u|^2/4 \tag{8.16}$$

and consequently by (8.13),

$$|I(x,u)| \le \int_1^{\infty} e^{\Phi^*(x+su)} s^{d-1}\,ds \le e^{-c|u|^2}$$

for $\lambda \ge 1$. □

Inserting the bound of Lemma 8.3 for $|u|I(x,u)$ into (8.7), we conclude that

$$|g(x)| \le C(1+|x-x^\dagger|)^{d-1}\lambda^{d/2} \int_{|u|\le 2|x-x^\dagger|} |u|^{1-d}|Sg(x+u)|\,dm(u) + C\lambda^{d/2}\int_{\mathbb{R}^d} e^{-c|u|^2}|Sg(x+u)|\,dm(u) \tag{8.17}$$

for certain constants $C, c \in \mathbb{R}^+$. The second term on the right-hand side represents the action on $|Sg|$ of a bounded linear operator from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$, whose operator norm is proportional to $\lambda^{d/2}$. Since the function $u \mapsto |u|^{1-d}$ is a positive decreasing function of $|u|$ and satisfies

$$\int_{|u|\leq 2|x-x^{\dagger}|} |u|^{1-d} \leq C|x-x^{\dagger}|,$$

one has

$$(1+|x-x^{\dagger}|)^{d-1}\lambda^{d/2}\int_{|u|\leq 2|x-x^{\dagger}|} |u|^{1-d}|Sg(x+u)|\,dm(u) \leq C\lambda^{d/2}(1+|x-x^{\dagger}|)^{d}M(Sg)(x), \tag{8.18}$$

where M is the Hardy–Littlewood maximal function. Now M is bounded on $L^2(\mathbb{R}^d)$, while multiplication by $(1+|x-x^{\dagger}|)^d$ defines a bounded operator from $L^2(\mathbb{R}^d)$ to the weighted space $L^2(\mathbb{R}^d, w)$ with $w(x) = (1+|x-x^{\dagger}|)^{-2d}$. This completes the proof of Lemma 8.2, and hence the proof of Lemma 8.1.

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Two Minicourses on Analytic Microlocal Analysis

Michael Hitrik and Johannes Sjöstrand

In memory of Lars Gårding and Lars Hörmander

Abstract These notes correspond roughly to the two minicourses prepared by the authors for the workshop on Analytic Microlocal Analysis, held at Northwestern University in May 2013. The first part of the text gives an elementary introduction to some global aspects of the theory of metaplectic FBI transforms, while the second part develops the general techniques of the analytic microlocal analysis in exponentially weighted spaces of holomorphic functions.

1 Introduction to Metaplectic FBI Transforms

1.1 Introduction

The metaplectic Fourier–Bros–Iagolnitzer (FBI) transform allows one to pass from the standard Hilbert space $L^2(\mathbf{R}^n)$ to an exponentially weighted space of holomorphic functions on $\mathbf{C}^n$. Such transforms occur under various other names in the literature, such as the Bargmann, Segal, Gabor, and wave packet transforms, and from the general point of view of microlocal analysis, these can all be viewed as Fourier integral operators with complex phase. In this part of the text, the connection to ana-

M. Hitrik (✉)
Department of Mathematics, University of California, Los Angeles, CA 90095-1555, USA
e-mail: hitrik@math.ucla.edu

J. Sjöstrand
IMB, Université de Bourgogne, UMR 5584, CNRS, 9 avenue Alain Savary,
BP 47870, 21078 Dijon Cedex, France
e-mail: johannes.sjostrand@u-bourgogne.fr

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lytic microlocal analysis will be emphasized, and we shall therefore refer to these transforms as FBI transforms, as they were used by J. Bros and D. Iagolnitzer to give a definition of the analytic wave front set. Pseudodifferential operators can be transported to the FBI transform side, and in this way, one obtains some flexible and powerful techniques for their analysis, particularly in the analytic case. In this chapter, we give an elementary introduction to the theory of metaplectic FBI transforms. In Sect. 1.2 we discuss aspects of the geometry of positive complex Lagrangian planes and some closely related complex canonical transformations, following Appendix A of [5] and Chap. 11 of [65]. In Sect. 1.3, following [70, 73], we introduce metaplectic FBI transforms, derive a representation for the Bergman projection and establish the unitarity of the FBI transform between $L^2(\mathbf{R}^n)$ and a suitable weighted space of holomorphic functions on $\mathbf{C}^n$. See also [36, 78]. Section 1.4 is concerned with pseudodifferential operators on the FBI transform side. We discuss their mapping properties and prove the metaplectic Egorov theorem, finishing with a brief discussion of the case of pseudodifferential operators with holomorphic symbols. Our presentation here follows [70, 73] closely.

1.2 Complex Symplectic Linear Algebra. Positivity

We shall work in the complex space $\mathbf{C}^{2n} = \mathbf{C}^n_x \times \mathbf{C}^n_\xi$, which is equipped with the complex symplectic (2,0)-form

$$\sigma = \sum_{j=1}^{n} d\xi_j \wedge dx_j, \quad (x,\xi) \in \mathbf{C}^{2n}. \tag{1.2.1}$$

The form σ is non-degenerate and closed, and we can write

$$\sigma(X,Y) = JX\cdot Y, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad X, Y \in \mathbf{C}^{2n}. \tag{1.2.2}$$

Here and in what follows we shall use the complex bilinear scalar product on $\mathbf{C}^k$, given by $X\cdot Y = \sum_{j=1}^k X_j Y_j$.

The corresponding real 2-forms

$$\mathrm{Re}\, \sigma = \frac{\sigma + \bar{\sigma}}{2}, \quad \mathrm{Im}\, \sigma = \frac{\sigma - \bar{\sigma}}{2i}. \tag{1.2.3}$$

are closed and non-degenerate, and hence give rise to real symplectic structures on $\mathbf{C}^{2n}$.

Definition 1.2.1 A complex linear map $\kappa : \mathbf{C}^{2n} \to \mathbf{C}^{2n}$ is called a complex canonical transformation if

$$\sigma(\kappa(X), \kappa(Y)) = \sigma(X, Y), \quad X, Y \in \mathbf{C}^{2n}. \tag{1.2.4}$$

If $\kappa : \mathbf{C}^{2n} \to \mathbf{C}^{2n}$ is a complex canonical transformation, then κ preserves the complex volume form $\sigma^n/n!$ on $\mathbf{C}^{2n}$, and therefore $\det \kappa = 1$. If $n = 1$, the converse is also true.

Let us consider the following configuration: Let $\Sigma \subseteq \mathbf{C}^{2n}$ be a real subspace which is *I-Lagrangian* in the sense that $\dim_{\mathbf{R}} \Sigma = 2n$ and $\mathrm{Im}\ \sigma|_\Sigma = 0$. Assume also that Σ is *R-symplectic*: $\mathrm{Re}\ \sigma|_\Sigma$ is non-degenerate. Such a subspace is automatically maximally totally real, $\Sigma \cap i\Sigma = \{0\}$, and we can write

$$\mathbf{C}^{2n} = \Sigma \oplus i\Sigma.$$

Let $\Gamma = \Gamma_\Sigma : \mathbf{C}^{2n} \to \mathbf{C}^{2n}$ be the unique antilinear map such that $\Gamma|_\Sigma = 1$. Clearly, we have

$$\sigma(\Gamma X, \Gamma Y) = \overline{\sigma(X, Y)}, \quad X, Y \in \mathbf{C}^{2n}. \tag{1.2.5}$$

Examples.

1. $\Sigma = \mathbf{R}^{2n}$, $\Gamma X = \bar{X}$, the complex conjugation.
2. Let Φ be a real valued quadratic form on $\mathbf{C}^n_x$, such that the Levi matrix, $\partial_{\bar{x}}\partial_x \Phi = (\partial_{\bar{x}_j}\partial_{x_k}\Phi)^n_{j,k=1}$, is non-degenerate.

   Let us set

$$\Sigma = \Lambda_\Phi := \left\{ \left(x, \frac{2}{i}\frac{\partial \Phi}{\partial x}(x) \right);\ x \in \mathbf{C}^n \right\}. \tag{1.2.6}$$

We claim that the linear subspace Σ is I-Lagrangian and R-symplectic. Indeed, using $x \in \mathbf{C}^n$ to parametrize Λ_Φ, we get

$$\sigma|_{\Lambda_\Phi} = \sum_{k=1}^n d\left(\frac{2}{i}\frac{\partial \Phi}{\partial x_k} \right) \wedge dx_k = \sum_{j,k=1}^n \frac{2}{i}\frac{\partial^2 \Phi}{\partial \bar{x}_j \partial x_k} d\bar{x}_j \wedge dx_k. \tag{1.2.7}$$

Using only the fact that Φ is real, we see that $\sigma|_{\Lambda_\Phi}$ is real, so that Λ_Φ is I-Lagrangian. Since the Levi form of Φ is non-degenerate, (1.2.7) also shows that $\sigma|_{\Lambda_\Phi}$ is non-degenerate.

Let us now describe the involution $\Gamma|_{\Lambda_\Phi}$ explicitly. We have

$$\Phi(x) = \frac{1}{2}\Phi''_{xx}x \cdot x + \Phi''_{\bar{x}x}x \cdot \bar{x} + \frac{1}{2}\Phi''_{\bar{x}\bar{x}}\bar{x} \cdot \bar{x}, \tag{1.2.8}$$

and therefore,

$$\Lambda_\Phi = \left\{ \left(x, \frac{2}{i}\left(\Phi''_{xx}x + \Phi''_{x\bar{x}}\bar{x} \right) \right);\ x \in \mathbf{C}^n \right\}. \tag{1.2.9}$$

Using that $\Gamma_{\Lambda_\Phi}(X + iY) = X - iY$, $X, Y \in \Lambda_\Phi$, we see that $\Gamma = \Gamma_{\Lambda_\Phi}$ is given by

$$\left(y, \frac{2}{i}\left(\Phi''_{xx}y + \Phi''_{x\bar{x}}\bar{x}\right)\right) \mapsto \left(x, \frac{2}{i}\left(\Phi''_{xx}x + \Phi''_{x\bar{x}}\bar{y}\right)\right) \tag{1.2.10}$$

Notice that the map (1.2.10) is well-defined since det $\left(\Phi''_{\bar{x}x}\right) \neq 0$.

Now let $\Lambda \subseteq \mathbf{C}^{2n}$ be a $\mathbf{C}$-Lagrangian subspace, i.e. a complex linear subspace such that $\dim_{\mathbf{C}} \Lambda = n$ and $\sigma|_{\Lambda} = 0$. If $\Sigma \subseteq \mathbf{C}^{2n}$ is I-Lagrangian, R-symplectic as above, with the associated involution Γ, we can introduce the Hermitian form

$$b(X, Y) = \frac{1}{i}\sigma(X, \Gamma Y), \quad (X, Y) \in \Lambda \times \Lambda. \tag{1.2.11}$$

Here the Hermitian property, $\overline{b(X, Y)} = b(Y, X)$, follows from (1.2.5).

Remark. When $\Sigma = \mathbf{R}^{2n}$, the Hermitian form (1.2.11) was introduced in [31]. The general case was considered in [65].

Proposition 1.2.2 *The form b is non-degenerate if and only if the subspaces Λ and Σ are transversal, i.e. $\Lambda \cap \Sigma = \{0\}$.*

Proof Consider the radical of b,

$$\mathrm{Rad}\,(b) = \{X \in \Lambda; b(X, Y) = 0 \text{ for all } Y \in \Lambda\}.$$

If $0 \neq X \in \mathrm{Rad}\,(b)$, then $\sigma(\Gamma X, Y) = 0$ for all $Y \in \Lambda$, and therefore, $\Gamma X \in \Lambda$, since Λ is Lagrangian. We see, using the fact that Γ is an antilinear involution, that the vectors $(1/2)\,(X + \Gamma X)$ and $(1/2i)\,(X - \Gamma X)$ both belong to $\Lambda \cap \Sigma$, and at least one of them is $\neq 0$, so that $\Lambda \cap \Sigma \neq \{0\}$. Conversely, $\Lambda \cap \Sigma \subseteq \mathrm{Rad}\,(b)$, and the result follows. □

Example 1.2.3 Let $\Sigma = \mathbf{R}^{2n}$ and assume that Λ is transversal to the fiber $F = \{(0, \xi);\ \ \xi \in \mathbf{C}^n\}$, $\Lambda \cap F = \{0\}$. Then necessarily, $\Lambda = \Lambda_{\varphi}$ is of the form $\xi = \varphi'(x) = \varphi'' x$, where φ is a holomorphic quadratic form on $\mathbf{C}^n_x$. We can compute the form b explicitly using this representation of Λ. When $X = (x, \varphi'' x) \in \Lambda$, we get, using (1.2.11),

$$\frac{1}{2}b(X, X) = \left(\mathrm{Im}\ \varphi''\right) x \cdot \bar{x}. \tag{1.2.12}$$

Here

$$\mathrm{Im}\,\varphi'' = \frac{1}{2i}\left(\varphi'' - \left(\varphi''\right)^*\right).$$

Definition 1.2.4 Let $\Lambda \subseteq \mathbf{C}^{2n}$ be $\mathbf{C}$-Lagrangian and let $\Sigma \subseteq \mathbf{C}^{2n}$ be I-Lagrangian, R-symplectic, with the involution Γ. We say that Λ is Σ-*positive (negative)* if the Hermitian form b is positive definite (negative definite) on Λ.

Proposition 1.2.5 *Let $\Sigma = \mathbf{R}^{2n}$. Then Λ is Σ–positive if and only if $\Lambda = \Lambda_{\varphi}$, where* $\mathrm{Im}\ \varphi'' > 0$.

Proof If $\Lambda = \Lambda_\varphi$ with $\operatorname{Im} \varphi'' > 0$, then in view of (1.2.12), we see that Λ is Σ–positive. Conversely, if Λ is Σ-positive, then Λ is transversal to the fiber F, so that $\Lambda = \Lambda_\varphi$, and Example 1.2.3 applies again. □

Proposition 1.2.6 *The set* $\{\Lambda \subseteq \mathbf{C}^{2n};\ \Lambda \text{ is } \mathbf{C} - \text{Lagrangian and } \Lambda \text{ is } \Sigma - \text{positive}\}$ *is a connected component in the set of all* **C**-*Lagrangian spaces that are transversal to* Σ.

Proof After applying a suitable linear complex canonical transformation, we may assume that $\Sigma = \mathbf{R}^{2n}$. Proposition 1.2.5 shows then that the set of all Σ-positive **C**-Lagrangian spaces is a connected (even convex) and open subset of the set of all **C**-Lagrangian spaces that are transversal to Σ. It is also closed, for if Λ is a **C**-Lagrangian space transversal to Σ, such that the form b is positive semi-definite on Λ, then b is necessarily positive definite on Λ, in view of Proposition 1.2.2. We conclude that the set of all Σ-positive **C**-Lagrangian spaces is a component in the set of all **C**-Lagrangian spaces that are transversal to Σ. □

Let us return to the situation where $\Sigma = \Lambda_\Phi$, with Φ being a real quadratic form on $\mathbf{C}^n_x$. Assume that the Levi form of Φ is positive definite,

$$\sum_{j,k=1}^{n} \frac{\partial^2 \Phi}{\partial \bar{x}_j \partial x_k} \bar{\xi}_j \xi_k > 0, \quad \forall 0 \neq \xi \in \mathbf{C}^n, \tag{1.2.13}$$

i.e. the quadratic form Φ is strictly pluri-subharmonic.

Proposition 1.2.7 *The fiber* $F = \{(0, \eta);\quad \eta \in \mathbf{C}^n\}$ *is* Λ_Φ-*negative.*

Proof Using (1.2.10) we see that $\Gamma(0, \eta) = (x, \xi)$, where $\xi = \frac{2}{i} \Phi''_{xx} x$, $\eta = \frac{2}{i} \Phi''_{x\bar{x}} \bar{x}$, which implies that

$$\frac{1}{i} \sigma((0, \eta), (x, \xi)) = \frac{1}{i} \eta \cdot x = -2\Phi''_{x\bar{x}} \bar{x} \cdot x \leq \frac{-1}{C} |x|^2 \leq -\frac{1}{\widetilde{C}} |\eta|^2 .$$

□

Now the space $\Gamma(F) : \xi = \frac{2}{i} \Phi''_{xx} x = \frac{1}{i} \partial_x \left(\Phi''_{xx} x \cdot x \right)$ is **C**-Lagrangian and Λ_Φ-positive. Let us write

$$\Phi(x) = \Phi_{\text{plh}}(x) + \Phi_{\text{herm}}(x),$$

where

$$\Phi_{\text{plh}}(x) = \operatorname{Re} \left(\Phi''_{xx} x \cdot x \right)$$

is the pluri-harmonic part, and

$$\Phi_{\text{herm}}(x) = \Phi''_{\bar{x}x} x \cdot \bar{x}$$

is the positive definite Hermitian part. Using that

$$\partial_x \left(\Phi''_{xx} x \cdot x \right) = 2\partial_x \Phi_{\rm plh}(x),$$

we conclude that $\Gamma(F)$ is of the form $\Lambda_{\Phi_{\rm plh}}$, where $\Phi(x) - \Phi_{\rm plh}(x) \sim |x|^2$.

Proposition 1.2.8 *Assume that $\partial_{\bar{x}} \partial_x \Phi > 0$. A $\mathbf{C}$-Lagrangian space Λ is Λ_Φ-positive if and only if $\Lambda = \Lambda_{\widetilde{\Phi}}$, where $\widetilde{\Phi}$ is pluri-harmonic quadratic and $\Phi - \widetilde{\Phi} \sim |x|^2$.*

Proof If $\widetilde{\Phi}$ is pluri-harmonic quadratic and $\Phi - \widetilde{\Phi} > 0$ then clearly, $\Lambda_{\widetilde{\Phi}}$ is $\mathbf{C}$-Lagrangian and transversal to Λ_Φ. It follows that the set

$$\{\Lambda_{\widetilde{\Phi}};\ \ \widetilde{\Phi} \text{ pluri-harmonic },\ \ \Phi - \widetilde{\Phi} > 0\}$$

is an open connected subset of the set of all $\mathbf{C}$–Lagrangian spaces that are transversal to Λ_Φ. It is also closed, for if $\widetilde{\Phi}$ is pluri-harmonic, $\Phi - \widetilde{\Phi} \geq 0$, and $\Lambda_{\widetilde{\Phi}}$ is transversal to Λ_Φ, then the quadratic form $\Phi - \widetilde{\Phi}$ is necessarily positive definite. (The transversality forces a non-strict inequality to become strict.) It follows that the set $\{\Lambda_{\widetilde{\Phi}};\ \widetilde{\Phi}$ pluri-harmonic , $\Phi - \widetilde{\Phi} > 0\}$ is a connected component of the set of all $\mathbf{C}$-Lagrangian spaces that are transversal to Λ_Φ. It contains $\Lambda_{\Phi_{\rm plh}}$, as we saw above, which is Λ_Φ-positive. An application of Proposition 1.2.6 allows us to conclude the proof. □

Example. Let $\Sigma = \mathbf{R}^{2n}$, and let $\Lambda_\pm \subseteq \mathbf{C}^{2n}$ be $\mathbf{C}$-Lagrangian spaces such that Λ_+ is positive and Λ_- is negative, with respect to Σ. Let us verify that there exists a holomorphic quadratic form $\varphi(x, y)$ on $\mathbf{C}^n_x \times \mathbf{C}^n_y$ such that

$$\det \varphi''_{xy} \neq 0, \quad \operatorname{Im} \varphi''_{yy} > 0, \tag{1.2.14}$$

and such that the complex linear canonical transformation

$$\kappa_\varphi : \mathbf{C}^{2n} \ni (y, -\varphi'_y(x, y)) \mapsto (x, \varphi'_x(x, y)) \in \mathbf{C}^{2n}$$

satisfies

$$\kappa_\varphi(\Lambda_+) = \{(x, 0);\ x \in \mathbf{C}^n\}, \tag{1.2.15}$$

and

$$\kappa_\varphi(\Lambda_-) = \{(0, \xi);\ \xi \in \mathbf{C}^n\}. \tag{1.2.16}$$

When showing the existence of the quadratic form $\varphi(x, y)$, let us recall from Proposition 1.2.5 that $\Lambda_\pm$ has the form $\eta = F_\pm y$, where $F_\pm$ is a complex symmetric matrix such that $\pm\operatorname{Im} F_\pm > 0$. Looking for φ in the form

$$\varphi(x, y) = \frac{1}{2} Ax \cdot x + Bx \cdot y + \frac{1}{2} Cy \cdot y,$$

where the matrices A and C are symmetric and B is bijective, we observe first that (1.2.16) is equivalent to the fact that

$$\kappa_\varphi^{-1}(\{(0,\xi);\ \xi\in\mathbf{C}^n\}) = \{(y,-Cy);\ y\in\mathbf{C}^n\} = \Lambda_-,$$

so we must have

$$C = -F_-. \tag{1.2.17}$$

The second condition in (1.2.14) is then satisfied, and we also see that

$$\begin{aligned}\kappa_\varphi^{-1}(\{(x,0);\ x\in\mathbf{C}^n\}) &= \{(y,-Bx-Cy);\ Ax+B^t y=0\}\\ &= \{(-(B^t)^{-1}Ax, -Bx + C(B^t)^{-1}Ax)\}.\end{aligned} \tag{1.2.18}$$

In order to have (1.2.15), the matrix A should necessarily be bijective, and we assume that this is the case. Writing $y=-(B^t)^{-1}Ax$, $x=-A^{-1}B^t y$, we then get from (1.2.18),

$$\begin{aligned}\kappa_\varphi^{-1}(\{(x,0);\ x\in\mathbf{C}^n\}) &= \{(y, BA^{-1}B^t y - C(B^t)^{-1}AA^{-1}B^t y)\}\\ &= \{(y, \left(BA^{-1}B^t - C\right) y)\}.\end{aligned}$$

The condition (1.2.15) therefore holds precisely when

$$BA^{-1}B^t - C = F_+. \tag{1.2.19}$$

Using (1.2.17), we may rewrite (1.2.19) in the form

$$BA^{-1}B^t = F_+ - F_-,$$

and observe that the matrix $F_+ - F_-$ is invertible, since $\operatorname{Im}(F_+ - F_-) > 0$. It follows that $A^{-1} = B^{-1}(F_+ - F_-)(B^t)^{-1}$, and choosing the invertible symmetric matrix A in the form

$$A = B^t\,(F_+ - F_-)^{-1}\,B,$$

we achieve (1.2.15). The general solution to (1.2.15), (1.2.16), satisfying (1.2.14), is therefore of the form

$$\varphi(x,y) = \frac{1}{2}B^t\,(F_+ - F_-)^{-1}\,Bx\cdot x + Bx\cdot y - \frac{1}{2}F_- y\cdot y.$$

Here B is an arbitrary invertible matrix.

1.3 Metaplectic FBI Transforms and Bergman Kernels

Last time we discussed the geometry of complex Lagrangian planes in the complexified phase space and that motivated us to look at complex canonical transformations of the form

$$\kappa_\varphi : \mathbf{C}^{2n} \ni (y, -\varphi'_y(x,y)) \mapsto (x, \varphi'_x(x,y)) \in \mathbf{C}^{2n}.$$

Here φ is a holomorphic quadratic form on $\mathbf{C}^n_x \times \mathbf{C}^n_y$ such that

$$\det \varphi''_{xy} \neq 0, \quad \text{Im } \varphi''_{yy} > 0. \tag{1.3.1}$$

Definition 1.3.1 The metaplectic Fourier–Bros–Iagolnitzer (FBI) transform associated to the quadratic form φ satisfying (1.3.1) is the operator

$$T : \mathcal{S}'(\mathbf{R}^n) \to \mathrm{Hol}(\mathbf{C}^n), \tag{1.3.2}$$

given by

$$Tu(x;h) = Ch^{-\frac{3n}{4}} \int e^{i\varphi(x,y)/h} u(y)\, dy, \quad 0 < h \leq 1. \tag{1.3.3}$$

To understand the growth properties of the entire function Tu in the complex domain, let us set

$$\Phi(x) = \sup_{y \in \mathbf{R}^n} (-\text{Im } \varphi(x,y)). \tag{1.3.4}$$

Since $\text{Im } \varphi''_{yy} > 0$, we see that the supremum in (1.3.4) is achieved at a unique point $y(x) \in \mathbf{R}^n$, which is the unique critical point of the function

$$\mathbf{R}^n \ni y \mapsto -\text{Im}\, \varphi(x,y).$$

Letting vc_y stand for the critical value, we get

$$\Phi(x) = \text{vc}_{y \in \mathbf{R}^n} (-\text{Im}\, \varphi(x,y)) = -\text{Im}\, \varphi(x, y(x)), \tag{1.3.5}$$

and by Taylor's formula, we can write, for $y \in \mathbf{R}^n$,

$$-\text{Im}\, \varphi(x,y) = \Phi(x) - \frac{1}{2} \text{Im}\, \varphi''_{yy}(y - y(x)) \cdot (y - y(x)) \leq \Phi(x) - \frac{1}{C} |y - y(x)|^2 .$$

It is therefore clear that for some $M > 0$ depending on the order of the distribution u, we have

$$|Tu(x;h)| \leq Ch^{-M} \langle x \rangle^M e^{\Phi(x)/h}, \quad x \in \mathbf{C}^n. \tag{1.3.6}$$

We also observe that the quadratic form $\Phi(x) = \sup_{y \in \mathbf{R}^n} (-\text{Im } \varphi(x,y))$ is pluri-subharmonic, being the supremum of a family of pluri-harmonic quadratic forms.

Example. Let $\varphi(x,y) = \frac{i}{2}(x-y)^2$. Then $\Phi(x) = \frac{1}{2}(\text{Im } x)^2$, and the canonical transformation κ_φ is given by

$$\kappa_\varphi(y, \eta) = (y - i\eta, \eta).$$

Remark. In microlocal analysis, microlocal properties of $u \in \mathcal{S}'(\mathbf{R}^n)$ near $(y,\eta) \in T^*\mathbf{R}^n \setminus \{0\}$ can be characterized using local properties of the holomorphic function Tu near $\pi_x\left(\kappa_\varphi(y,\eta)\right) \in \mathbf{C}^n$. Here $\pi_x : \mathbf{C}^{2n}_{x,\xi} \ni (x,\xi) \to x \in \mathbf{C}^n$ is the natural projection map. We refer to [65] and to Sect. 2.6 of this text for further details. In this elementary discussion, we shall only be concerned with global aspects of the metaplectic FBI transforms.

The following proposition indicates that there is a dictionary between the real side and the FBI transform side, where $\mathbf{R}^{2n}$ corresponds to the linear manifold

$$\Lambda_\Phi = \left\{\left(x, \frac{2}{i}\frac{\partial \Phi}{\partial x}(x)\right);\ x \in \mathbf{C}^n\right\} \subseteq \mathbf{C}^{2n}. \tag{1.3.7}$$

Proposition 1.3.2 *The complex canonical transformation*

$$\kappa_\varphi : \mathbf{C}^{2n} \ni (y, -\varphi'_y(x,y)) \mapsto (x, \varphi'_x(x,y)) \in \mathbf{C}^{2n} \tag{1.3.8}$$

maps $\mathbf{R}^{2n}$ bijectively onto Λ_Φ. The quadratic form Φ introduced in (1.3.4) is strictly pluri-subharmonic.

Proof We claim that for any $x \in \mathbf{C}^n$ there is a unique $(y(x), \eta(x)) \in \mathbf{R}^{2n}$ such that $\pi_x \circ \kappa_\varphi(y(x), \eta(x)) = x$. Indeed, if $y \in \mathbf{R}^n$, then $\varphi'_y(x,y)$ is real if and only if $\nabla_y(-\operatorname{Im}\varphi(x,y)) = 0$, in other words, if and only if $y = y(x)$, the critical point in (1.3.5). The claim follows with $\eta(x) = -\varphi'_y(x, y(x))$. We let next $\xi(x) \in \mathbf{C}^n$ be such that $\kappa_\varphi(y(x), \eta(x)) = (x, \xi(x))$, i.e. $\xi(x) = \varphi'_x(x, y(x))$. Writing

$$\Phi(x) = -\operatorname{Im}\varphi(x, y(x)) = \frac{i}{2}\left(\varphi(x,y(x)) - \overline{\varphi(x,y(x))}\right),$$

we check, using the fact that $\varphi'_y(x, y(x))$ and $y(x)$ are real that

$$\xi(x) = \frac{2}{i}\frac{\partial \Phi}{\partial x}(x). \tag{1.3.9}$$

It follows that $\kappa_\varphi(\mathbf{R}^{2n}) = \Lambda_\Phi$, and since $\sigma|_{\mathbf{R}^{2n}}$ is non-degenerate, we obtain that $\sigma|_{\Lambda_\Phi}$ is non-degenerate, or equivalently, the Levi form $\partial_{\bar{x}}\partial_x\Phi$ is non-degenerate. Since we already know that Φ is pluri-subharmonic, we conclude that Φ is strictly pluri-subharmonic. □

We shall now establish the following basic result, concerning the mapping properties of the FBI transform on $L^2(\mathbf{R}^n)$.

Theorem 1.3.3 *If $C > 0$ is suitably chosen in (1.3.3), then T is unitary,*

$$T : L^2(\mathbf{R}^n) \to H_\Phi(\mathbf{C}^n) := L^2(\mathbf{C}^n, e^{-2\Phi/h}\, L(dx)) \cap \operatorname{Hol}(\mathbf{C}^n).$$

Here $L(dx)$ is the Lebesgue measure on $\mathbf{C}^n$.

As a preparation for the proof, let us first derive an expression for the orthogonal (Bergman) projection:

$$\Pi : L^2_\Phi(\mathbf{C}^n) \to H_\Phi(\mathbf{C}^n),$$

where $L^2_\Phi(\mathbf{C}^n) = L^2(\mathbf{C}^n, e^{-2\Phi/h} L(dx))$ and $H_\Phi(\mathbf{C}^n) \subseteq L^2_\Phi(\mathbf{C}^n)$ is the closed subspace of holomorphic functions. Let $\psi(x, y)$ be the unique holomorphic quadratic form on $\mathbf{C}^n_x \times \mathbf{C}^n_y$ such that $\psi(x, \bar{x}) = \Phi(x)$. Here we may notice that the anti-diagonal $\{(x, \bar{x}); x \in \mathbf{C}^n\}$ is maximally totally real $\subseteq \mathbf{C}^n_x \times \mathbf{C}^n_y$. Explicitly, we have

$$\psi(x, y) = \frac{1}{2}\Phi''_{xx} x \cdot x + \Phi_{\bar{x}x} x \cdot y + \frac{1}{2}\Phi''_{\bar{x}\bar{x}} y \cdot y,$$

so that in particular, $\psi''_{xy} = \Phi''_{x\bar{x}}$ is non-degenerate. It also follows that when $y = \bar{x}$, we have

$$\partial_y \psi = \partial_{\bar{x}} \Phi, \quad \partial_x \psi = \partial_x \Phi. \tag{1.3.10}$$

These observations have the following useful consequence:

$$2\mathrm{Re}\, \psi(x, \overline{y}) - \Phi(x) - \Phi(y) = -\Phi''_{\bar{x}x}(y - x) \cdot (\overline{y - x}) \sim -|y - x|^2, \tag{1.3.11}$$

on $\mathbf{C}^n_x \times \mathbf{C}^n_y$. Here the last conclusion follows since Φ is strictly pluri-subharmonic, and to verify the first equality in (1.3.11) it suffices to Taylor expand the quadratic functions $y \mapsto \Phi(y)$ and $y \mapsto \psi(x, \bar{y})$ at the point $y = x$, and exploit (1.3.10) to obtain some cancellations.

Proposition 1.3.4 *The orthogonal projection* $\Pi : L^2_\Phi(\mathbf{C}^n) \to H_\Phi(\mathbf{C}^n)$ *is given by*

$$\Pi u(x) = \frac{2^n \det \psi''_{xy}}{(\pi h)^n} \int_{\mathbf{C}^n} e^{2\psi(x,\bar{y})/h} u(y) e^{-2\Phi(y)/h}\, L(dy). \tag{1.3.12}$$

Proof Let Π be the operator given in (1.3.12). To see that

$$\Pi = \mathcal{O}(1) : L^2_\Phi(\mathbf{C}^n) \to H_\Phi(\mathbf{C}^n), \tag{1.3.13}$$

we consider the reduced kernel

$$\widetilde{\Pi}(x, y) = e^{-\Phi(x)/h} \Pi(x, y) e^{\Phi(y)/h}, \tag{1.3.14}$$

and observe that thanks to (1.3.11), we have

$$\left|\widetilde{\Pi}(x, y)\right| \leq \frac{C}{h^n} e^{-|x-y|^2/Ch}.$$

The uniform boundedness of Π on L^2_Φ is therefore a consequence of Schur's lemma, and since the range of Π consists of holomorphic functions, the property (1.3.13) follows. The selfadjointness of Π on L^2_Φ follows since $\overline{\psi(x, \bar{y})} = \psi(y, \bar{x})$. We finally need to show the reproducing property of Π,

$$\Pi u = u, \quad u \in H_\Phi(\mathbf{C}^n). \tag{1.3.15}$$

To see (1.3.15), we start by establishing the Fourier inversion formula in the complex domain,

$$u(x) = \frac{1}{(2\pi h)^n} \iint_{\Gamma(x)} e^{\frac{i}{h}(x-y)\cdot\theta} u(y)\, dy \wedge d\theta, \quad u \in H_\Phi(\mathbf{C}^n). \tag{1.3.16}$$

Here $dy \wedge d\theta$ is a $(2n, 0)$–form in $\mathbf{C}^n_y \times \mathbf{C}^n_\theta$, and the integration in (1.3.16) is carried out over the $2n$-dimensional contour (chain) $\Gamma(x)$, parametrized by $y \in \mathbf{C}^n$ and given by

$$\Gamma(x) : \mathbf{C}^n \ni y \mapsto (y, \theta) \in \mathbf{C}^n \times \mathbf{C}^n, \quad \theta = \frac{2}{i}\frac{\partial \Phi}{\partial x}(x) + iC\overline{(x-y)}. \tag{1.3.17}$$

Here $C \gg 1$ is large enough. We have

$$dy \wedge d\theta|_{\Gamma(x)} = \left(\frac{C}{i}\right)^n dy \wedge d\bar{y} \tag{1.3.18}$$

is real and non-vanishing, and it what follows we shall tacitly assume that the orientation on $\Gamma(x)$ has been chosen so that the form in (1.3.18) is a positive multiple of the Lebesgue measure on $\mathbf{C}^n_y$. Let us also notice that the unique critical point of the function $\mathbf{C}^n \times \mathbf{C}^n \ni (y, \theta) \mapsto -\mathrm{Im}\,(x-y)\cdot\theta + \Phi(y)$ is given by $y = x$, $\theta = \frac{2}{i}\frac{\partial \Phi}{\partial x}(x)$, with the critical value $\Phi(x)$, and the contour $\Gamma(x)$ passes through the critical point for all C. To see (1.3.16), we first observe that the contour $\Gamma(x)$ is good [65], in the sense that along $\Gamma(x)$, we have in view of Taylor's formula,

$$\mathrm{Re}\,(i(x-y)\cdot\theta) + \Phi(y) - \Phi(x) \leq -|x-y|^2,$$

provided that $C > 1$ is large enough. The integral in (1.3.16) therefore converges absolutely for all $u \in \mathrm{Hol}(\mathbf{C}^n)$ such that $|u(x)| \leq \mathcal{O}_h(1)\langle x\rangle^{N_0} e^{\Phi(x)/h}$, for some $N_0 > 0$, and in particular, for all $u \in H_\Phi$. We also notice that it is independent of $C \gg 1$, in view of Stokes' formula.

Using (1.3.17), we see that the right hand side in (1.3.16) is given by

$$\frac{2^n C^n}{(2\pi h)^n} \int e^{-C|x-y|^2/h} e^{\frac{2}{h}\frac{\partial \Phi}{\partial x}(x)\cdot(x-y)} u(y)\, L(dy). \tag{1.3.19}$$

Here the Gaussian

$$\mathbf{C}^n \ni y \mapsto \frac{C^n}{(\pi h)^n} e^{-C|y|^2/h}$$

is spherically symmetric of integral one, and therefore, by the mean value theorem for holomorphic functions, here applied to the function

$$y \mapsto e^{\frac{2}{h}\frac{\partial \Phi}{\partial x}(x)\cdot(x-y)}u(y),$$

we conclude that the expression (1.3.19) is equal to $u(x)$ — see also Lemma 7.3.11 in [35]. This establishes the validity of (1.3.16), and we may observe that the argument given above is in some sense simpler than the usual proof of Fourier's inversion formula in the real domain, since all the integrals involved converge absolutely, thanks to the choice of a family of good contours, such as $\Gamma(x)$ above.

We shall now finish the proof of Proposition 1.3.4 by passing from (1.3.16) to (1.3.12). To this end, we make a linear complex change of variables $\theta \mapsto w$, given by

$$\theta = \frac{2}{i}\frac{\partial \psi}{\partial x}\left(\frac{x+y}{2}, w\right) = \frac{2}{i}\left(\Phi''_{xx}\left(\frac{x+y}{2}\right) + \Phi''_{x\bar{x}}w\right).$$

It follows, since ψ is quadratic, that

$$2\left(\psi(x,w) - \psi(y,w)\right) = i(x-y)\cdot\theta,$$

and we get therefore from (1.3.16),

$$u(x) = \frac{1}{(2\pi h)^n}\iint_{\widetilde{\Gamma}(x)} e^{\frac{2}{h}(\psi(x,w)-\psi(y,w))}\left(\frac{2}{i}\right)^n (\det \Phi_{x\bar{x}})\, u(y)\, dy \wedge dw. \quad (1.3.20)$$

Here $\widetilde{\Gamma}(x)$ is the natural image of $\Gamma(x)$, so that $(y, w) \in \widetilde{\Gamma}(x)$ precisely when $(y, \theta) \in \Gamma(x)$. The contour $\widetilde{\Gamma}(x)$ is good in the sense that along $\widetilde{\Gamma}(x)$, we have

$$2\,\mathrm{Re}\left(\psi(x,w) - \psi(y,w)\right) + \Phi(y) - \Phi(x) \le -\left|x-y\right|^2,$$

and another good contour $\widehat{\Gamma}(x)$ is given by $w = \bar{y}$. Indeed, we have in view of (1.3.11),

$$2\,\mathrm{Re}\left(\psi(x,\bar{y}) - \psi(y,\bar{y})\right) + \Phi(y) - \Phi(x) \le -\frac{1}{C}\left|x-y\right|^2.$$

The good contour $\widehat{\Gamma}(x)$ is homotopic to $\widetilde{\Gamma}(x)$, with the homotopy being within the set of good contours, and we conclude, in view of Stokes' formula, that

$$u(x) = \frac{\det \Phi_{x\bar{x}}}{i^n(\pi h)^n}\iint_{\widehat{\Gamma}(x)} e^{\frac{2}{h}(\psi(x,w)-\psi(y,w))}u(y)\, dy \wedge dw = \Pi u. \quad (1.3.21)$$

This completes the proof of Proposition 1.3.4. □

We shall return to the proof of Theorem 1.3.3, where, without loss of generality, we may assume that

$$\varphi''_{xx} = \mathrm{Re}\,\varphi''_{yy} = 0,$$

so that we can write

$$\varphi(x, y) = Ax \cdot y + \frac{i}{2} By \cdot y, \quad B > 0, \quad \det A \neq 0. \tag{1.3.22}$$

We shall first show that $T : L^2(\mathbf{R}^n) \to H_\Phi(\mathbf{C}^n)$ is an isometry. To this end, we observe that $Tu(A^{-1}x; h)$ is equal to $Ch^{-3n/4}$ times the semiclassical Fourier–Laplace transform of $u(y)e^{-By\cdot y/2h}$, and therefore, by Parseval's formula,

$$\int \left|Tu(A^{-1}x; h)\right|^2 d\mathrm{Re}\, x = (2\pi h)^n C^2 h^{-3n/2} \int e^{-By\cdot y/h} e^{-2\mathrm{Im}\, x\cdot y/h} \left|u(y)\right|^2 dy.$$

Next, a computation using (1.3.22) shows that

$$\Phi(x) = \frac{1}{2} B^{-1} \mathrm{Im}\,(Ax) \cdot \mathrm{Im}\,(Ax), \tag{1.3.23}$$

and therefore

$$\iint \left|Tu(A^{-1}x; h)\right|^2 e^{-2\Phi(A^{-1}x)/h} L(dx)$$
$$= (2\pi)^n C^2 h^{-n/2} \iint e^{-(By\cdot y + 2\xi\cdot y + B^{-1}\xi\cdot\xi)/h} \left|u(y)\right|^2 dy\, d\xi.$$

We have $By \cdot y + 2\xi \cdot y + B^{-1}\xi \cdot \xi = B^{-1}(\xi + By)\cdot(\xi + By)$, and therefore the integral with respect to ξ in the right hand side is equal to $(\pi h)^{n/2}\,(\det B)^{1/2}$. On the other hand, the left hand side is given by $|\det A|^2\, ||\, Tu\, ||^2_{H_\Phi}$, so that we get

$$|\det A|^2\, ||\, Tu\, ||^2_{H_\Phi} = 2^n \pi^{3n/2} C^2\, (\det B)^{1/2}\, ||\, u\, ||^2_{L^2}.$$

Choosing

$$C = 2^{-n/2} \pi^{-3n/4}\, (\det B)^{-1/4}\, |\det A| > 0, \tag{1.3.24}$$

we conclude that $T : L^2(\mathbf{R}^n) \to H_\Phi(\mathbf{C}^n)$ is an isometry.

We shall finally show that $TT^* = 1$ on $H_\Phi(\mathbf{C}^n)$. Here the Hilbert space adjoint T^* of $T : L^2(\mathbf{R}^n) \to L^2_\Phi(\mathbf{C}^n)$ is given by

$$T^* v(y) = Ch^{-3n/4} \int e^{-i\varphi^*(\bar{x}, y)/h} v(x) e^{-2\Phi(x)/h}\, L(dx), \tag{1.3.25}$$

where $\varphi^*(x,y) = \overline{\varphi(\bar{x},\bar{y})}$ is the holomorphic extension of $\mathbf{R}^n_x \times \mathbf{R}^n_y \ni (x,y) \mapsto \overline{\varphi(x,y)}$. We get, for $v \in \mathrm{Hol}(\mathbf{C}^n)$, such that $|v(x)| \le \mathcal{O}_{N,h}(1)\langle x\rangle^{-N} e^{\Phi(x)/h}$, for all N,

$$(TT^*v)(x) = C^2 h^{-3n/2} \iint e^{i(\varphi(x,y)-\varphi^*(\bar{w},y))/h} v(w) e^{-2\Phi(w)/h}\, L(dw)\, dy. \quad (1.3.26)$$

The integral with respect to y can be computed by exact stationary phase and we get, writing $q(x,\bar{w},y) = \varphi(x,y) - \varphi^*(\bar{w},y)$,

$$\int e^{iq(x,\bar{w},y)/h}\, dy = h^{n/2}\left(\det \frac{q''_{yy}}{2\pi i}\right)^{-1/2} e^{i\,\mathrm{vc}_y q(x,\bar{w},y)/h}. \quad (1.3.27)$$

Here

$$\frac{i}{2}\mathrm{vc}_y(q(x,z,y)) = \frac{i}{2}\mathrm{vc}_y\left(\varphi(x,y) - \varphi^*(z,y)\right) \quad (1.3.28)$$

is a holomorphic quadratic form on $\mathbf{C}^n_x \times \mathbf{C}^n_z$, and when $z = \bar{x}$, we see using (1.3.22) that the unique critical point y in (1.3.28) is real and that (1.3.28) is equal to $\Phi(x)$. It follows that

$$\frac{i}{2}\mathrm{vc}_y\left(\varphi(x,y) - \varphi^*(z,y)\right) = \psi(x,z),$$

and using also that $q''_{yy} = 2iB$, we obtain from (1.3.27) that

$$\int e^{iq(x,\bar{w},y)/h}\, dy = h^{n/2}\pi^{n/2}(\det B)^{-1/2} e^{2\psi(x,\bar{w})/h}.$$

Returning to (1.3.26) and recalling the explicit expression for the constant C in (1.3.24), we see that

$$\begin{aligned}(TT^*v)(x) &= C^2 h^{-3n/2} h^{n/2}\pi^{n/2}(\det B)^{-1/2} \int e^{2\psi(x,\bar{w})/h} v(w) e^{-2\Phi(w)/h}\, L(dw) \\ &= \frac{2^{-n}(\det B)^{-1}\,|\det A|^2}{(\pi h)^n}\int e^{2\psi(x,\bar{w})/h} v(w) e^{-2\Phi(w)/h}\, L(dw) = (\Pi v)(x) = v(x),\end{aligned}$$

where the penultimate equality follows from Proposition 1.3.4. Here we have also used that

$$\det \Phi''_{x\bar{x}} = 4^{-n}\,|\det A|^2\,(\det B)^{-1},$$

in view of (1.3.23). The proof of Theorem 1.3.3 is complete.

1.4 Pseudodifferential Operators on FBI Transform Side

Let Φ be a strictly pluri-subharmonic quadratic form on $\mathbf{C}^n$, and let us recall the linear IR-manifold $\Lambda_\Phi \subset \mathbf{C}^n_x \times \mathbf{C}^n_\xi$, defined in (1.3.7). Introduce

$$S(\Lambda_\Phi) = \{a \in C^\infty(\Lambda_\Phi);\ \partial^\alpha a = \mathcal{O}_\alpha(1),\ \ \forall \alpha\} \tag{1.4.1}$$

Here we identify Λ_Φ linearly with $\mathbf{C}^n$ via the projection map $\Lambda_\Phi \ni (x,\xi) \mapsto x \in \mathbf{C}^n$. If $a \in S(\Lambda_\Phi)$ and $u \in \mathrm{Hol}(\mathbf{C}^n)$ is such that $u = \mathcal{O}_{h,N}(1)\langle x\rangle^{-N} e^{\Phi(x)/h}$, for all $N \geq 0$, we put

$$\mathrm{Op}_h^w(a)u(x) = \frac{1}{(2\pi h)^n} \iint_{\Gamma(x)} e^{\frac{i}{h}(x-y)\cdot\theta} a\left(\frac{x+y}{2}, \theta\right) u(y)\, dy \wedge d\theta. \tag{1.4.2}$$

Here $\Gamma(x)$ is the only possible integration contour given by

$$\theta = \frac{2}{i}\frac{\partial \Phi}{\partial x}\left(\frac{x+y}{2}\right).$$

Along $\Gamma(x)$, we get, by Taylor's formula,

$$\mathrm{Re}\,(i(x-y)\cdot\theta) - \Phi(x) + \Phi(y) = \left\langle x-y, \nabla\Phi\left(\frac{x+y}{2}\right)\right\rangle_{\mathbf{R}^{2n}} - \Phi(x) + \Phi(y) = 0,$$

and let us notice also that

$$dy \wedge d\theta|_{\Gamma(x)} = \frac{1}{i^n}\det\,(\Phi''_{x\bar{x}})dy \wedge d\bar{y}.$$

It follows that the integral in (1.4.2) converges absolutely, and for a suitable constant $C \neq 0$, we may write,

$$\mathrm{Op}_h^w(a)u(x) = \frac{C}{h^n}\int K(x,y)u(y)\,L(dy), \tag{1.4.3}$$

where

$$K(x,y) = e^{\frac{2}{h}(x-y)\cdot\frac{\partial\Phi}{\partial x}\left(\frac{x+y}{2}\right)} a\left(\frac{x+y}{2}, \frac{2}{i}\frac{\partial\Phi}{\partial x}\left(\frac{x+y}{2}\right)\right).$$

It follows that $\partial_{\bar{x}}K(x,y) = \partial_{\bar{y}}K(x,y)$, and using an integration by parts we conclude that the function $\mathrm{Op}_h^w(a)u(x)$ is holomorphic, since u is.

Theorem 1.4.1 *Let $a \in S(\Lambda_\Phi)$. The operator $\mathrm{Op}_h^w(a)$ extends to a bounded operator: $H_\Phi(\mathbf{C}^n) \to H_\Phi(\mathbf{C}^n)$, whose norm is $\mathcal{O}(1)$, as $h \to 0^+$.*

Proof Following [73], we shall prove this result by means of a contour deformation argument. When $0 \leq t \leq 1$, let $\Gamma_t(x)$ be the $2n$-dimensional contour, given by

$$\theta = \frac{2}{i}\frac{\partial \Phi}{\partial x}\left(\frac{x+y}{2}\right) + it\frac{\overline{x-y}}{\langle x-y\rangle}. \tag{1.4.4}$$

We also introduce the $(2n+1)$-dimensional contour $G(x) \subset \mathbf{C}^n_y \times \mathbf{C}^n_\theta$, given by

$$G(x) = \bigcup_{0\le t\le 1} \Gamma_t(x).$$

We would like to replace the contour $\Gamma(x) = \Gamma_0(x)$ by $\Gamma_1(x)$ in (1.4.2), and to that end, we let $\widetilde{a} \in C^\infty(\mathbf{C}^{2n}_{x,\xi})$ be an almost holomorphic extension of $a \in S(\Lambda_\Phi)$, so that $\mathrm{supp}\,(\widetilde{a}) \subseteq \Lambda_\Phi + \mathrm{neigh}(0, \mathbf{C}^{2n})$, all derivatives of $\widetilde{a}$ are bounded, $\widetilde{a}|_{\Lambda_\Phi} = a$, and

$$\left|\partial_{\bar{x},\bar{\xi}}\widetilde{a}(x,\xi)\right| \le \mathcal{O}_N(1)\left|\xi - \frac{2}{i}\frac{\partial\Phi}{\partial x}(x)\right|^N, \tag{1.4.5}$$

for all $N \ge 0$. Let us recall that to construct $\widetilde{a}$, we may first make a complex linear change of coordinates to replace Λ_Φ by $\mathbf{R}^{2n}$ and consider the problem of constructing an almost holomorphic extension of $a \in C^\infty(\mathbf{R}^{2n})$, with $\partial^\alpha a \in L^\infty(\mathbf{R}^{2n})$ for all α. To this end, following the classical construction by Hörmander, explained in [8], we set

$$\widetilde{a}(X+iY) = \sum_{|\alpha|\ge 0} \frac{\partial^\alpha a(X)}{\alpha!}(iY)^\alpha \chi(t_{|\alpha|}Y), \tag{1.4.6}$$

where $\chi \in C_0^\infty(\mathbf{R}^{2n})$, $\chi = 1$ near 0, and $t_j \to \infty$ sufficiently rapidly, so that

$$\left|\partial_X^\beta \partial_Y^\gamma c_\alpha(X,Y)\right| \le 2^{-|\alpha|}, \quad |\beta|+|\gamma| \le |\alpha|-1.$$

Here $c_\alpha(X,Y) = (\partial^\alpha a(X)/\alpha!)(iY)^\alpha\chi(t_{|\alpha|}Y)$.

Returning to (1.4.2), we get by Stokes' formula, assuming that $u \in \mathrm{Hol}(\mathbf{C}^n)$, with $u(x) = \mathcal{O}_{h,N}(1)\langle x\rangle^{-N}e^{\Phi(x)/h}$, for all $N \ge 0$,

$$\mathrm{Op}_h^w(a)u = I_1u + I_2u, \tag{1.4.7}$$

where

$$I_1u(x) = \frac{1}{(2\pi h)^n}\iint_{\Gamma_1(x)} e^{\frac{i}{h}(x-y)\cdot\theta}\widetilde{a}\left(\frac{x+y}{2},\theta\right)u(y)\,dy\wedge d\theta, \tag{1.4.8}$$

and

$$I_2u(x) = \frac{1}{(2\pi h)^n}\int_{G(x)} d_{y,\theta}\left(e^{\frac{i}{h}(x-y)\cdot\theta}\widetilde{a}\left(\frac{x+y}{2},\theta\right)u(y)\right)\wedge dy\wedge d\theta. \tag{1.4.9}$$

We have $dy \wedge d\theta|_{\Gamma_1(x)} = \mathcal{O}(1)L(dy)$, and it follows from (1.4.4) that the reduced kernel of I_1 satisfies

$$\left|e^{-\Phi(x)/h} I_1(x, y) e^{\Phi(y)/h}\right| \leq \frac{C}{h^n} e^{-\frac{|x-y|^2}{h\langle x-y\rangle}}.$$

In order to conclude that $I_1 = \mathcal{O}(1) : L^2_\Phi(\mathbf{C}^n) \to L^2_\Phi(\mathbf{C}^n)$, in view of Schur's lemma, it suffices to check that

$$\frac{1}{h^n} \int e^{-\frac{|x|^2}{h\langle x\rangle}}\, L(dx) = \mathcal{O}(1),$$

which is easily seen by considering the integrals over the regions where $|x| \leq 1$ and $|x| \geq 1$. When estimating the contribution of I_2, we write

$$d_{y,\theta}\left(e^{\frac{i}{h}(x-y)\cdot\theta}\widetilde{a}\left(\frac{x+y}{2}, \theta\right) u(y)\right) \wedge dy \wedge d\theta = e^{\frac{i}{h}(x-y)\cdot\theta} u(y) \partial_{\bar{y},\bar{\theta}}\left(\widetilde{a}\left(\frac{x+y}{2}, \theta\right)\right) \wedge dy \wedge d\theta,$$

and notice that in view of (1.4.5), we have along $G(x)$,

$$\partial_{\bar{y},\bar{\theta}}\left(\widetilde{a}\left(\frac{x+y}{2}, \theta\right)\right) \wedge dy \wedge d\theta = \mathcal{O}_N(1) t^N \frac{|x-y|^N}{\langle x-y\rangle^N} dt\, L(dy), \quad N \geq 0.$$

It follows that the reduced kernel of I_2 satisfies

$$\left|e^{-\Phi(x)/h} I_2(t, x, y) e^{\Phi(y)/h}\right| \leq \frac{C}{h^n} e^{-\frac{t|x-y|^2}{h\langle x-y\rangle}} t^N \frac{|x-y|^N}{\langle x-y\rangle^N},$$

and by an application of Schur's lemma, we see that in order to control the norm of the operator

$$I_2 : L^2_\Phi(\mathbf{C}^n) \to L^2_\Phi(\mathbf{C}^n),$$

it suffices to estimate

$$\frac{1}{h^n} \int e^{-\frac{t|x|^2}{h\langle x\rangle}} t^N \frac{|x|^N}{\langle x\rangle^N}\, L(dx),$$

uniformly in $t \in [0, 1]$. In doing so, we consider first the contribution of the region where $|x| \leq 1$. We get

$$\frac{1}{h^n} \int_{|x|\leq 1} e^{-\frac{t|x|^2}{h\langle x\rangle}} t^N \frac{|x|^N}{\langle x\rangle^N}\, L(dx) = \mathcal{O}(1) h^{-n} \int_0^1 e^{-\frac{tr^2}{2h}} t^N r^{N+2n-1} dr$$
$$\leq \mathcal{O}(1) h^{-n} \int_0^\infty e^{-s^2} t^N \left(\frac{2h}{t}\right)^{N/2+n} s^{N+2n-1}\, ds = \mathcal{O}(1) h^{N/2} t^{N/2-n} = \mathcal{O}(h^{N/2}),$$

uniformly in $t \in [0, 1]$, for N large enough. Next, the contribution of the integral over the region $|x| \geq 1$ does not exceed a constant times

$$
\begin{aligned}
h^{-n} \int_{|x|\geq 1} e^{-\frac{t|x|}{2h}} t^N L(dx) &= \mathcal{O}(1) h^{-n} \int_1^\infty e^{-\frac{tr}{2h}} t^N r^{2n-1}\, dr \\
&= \mathcal{O}(1) h^n t^{N-2n} \int_{t/h}^\infty e^{-\rho/2} \rho^{2n-1}\, d\rho = \mathcal{O}(1) h^n t^{N-2n} \mathcal{O}\left(\left(1+\frac{t}{h}\right)^{-M}\right),
\end{aligned}
$$

for all $M \geq 0$. If $t \leq h^{1/2}$, we use the factor t^{N-2n} to get the bound $\mathcal{O}(h^{N/2})$, while for $t \geq h^{1/2}$, we use the factor

$$
\mathcal{O}\left(\left(1+\frac{t}{h}\right)^{-M}\right) = \mathcal{O}(h^{M/2}),
$$

to get the bound $\mathcal{O}(h^{n+M/2})$. We conclude, in view of (1.4.7) that

$$
\mathrm{Op}_h^w(a) u(x) = \frac{1}{(2\pi h)^n} \iint_{\Gamma_1(x)} e^{\frac{i}{h}(x-y)\cdot\theta} \widetilde{a}\left(\frac{x+y}{2}, \theta\right) u(y)\, dy \wedge d\theta + Ru, \tag{1.4.10}
$$

where

$$
R = \mathcal{O}(h^\infty) : L^2_\Phi(\mathbf{C}^n) \to L^2_\Phi(\mathbf{C}^n).
$$

This completes the proof. □

We shall next discuss the link between the h-pseudodifferential operators on the FBI transform side and the semiclassical Weyl quantization on $\mathbf{R}^n$. We have the following metaplectic Egorov theorem.

Theorem 1.4.2 *Let $T : L^2(\mathbf{R}^n) \to H_\Phi(\mathbf{C}^n)$ be a metaplectic FBI transform with the associated canonical transformation*

$$
\kappa_T : \mathbf{R}^{2n} \to \Lambda_\Phi.
$$

If $a \in S(\Lambda_\Phi)$ then we have

$$
T^* \mathrm{Op}_h^w(a) T = \mathrm{Op}_h^w(a \circ \kappa_T).
$$

Here the operator in the right hand side is the h-Weyl quantization of the symbol $a \circ \kappa_T \in S(1)$ on $\mathbf{R}^n$.

Proof The starting point is the following fact that can be verified by means of an explicit computation: let ℓ be a real linear form on $\mathbf{R}^{2n}$ and let k be the linear form on Λ_Φ such that $k \circ \kappa_T = \ell$. Then we have on $\mathcal{S}(\mathbf{R}^n)$,

$$
\mathrm{Op}_h^w(k) \circ T = T \circ \mathrm{Op}_h^w(l). \tag{1.4.11}
$$

In the computation, it is convenient to use that if $k(x,\xi) = x^* \cdot x + \xi^* \cdot \xi$, $x, \xi \in \mathbf{C}^n$, then

$$\mathrm{Op}_h^w(k) = k(x, hD_x) = x^* \cdot x + \xi^* \cdot hD_x,$$

and there is a similar formula for $\mathrm{Op}_h^w(\ell)$. Now let us recall from [8] that the first order operator $\ell(x, hD_x) = \mathrm{Op}_h^w(\ell)$ is essentially selfadjoint on $L^2(\mathbf{R}^n)$ from $\mathcal{S}(\mathbf{R}^n)$, and

$$e^{i\ell(x,hD_x)/h} = \mathrm{Op}_h^w\left(e^{i\ell(x,\xi)/h}\right). \tag{1.4.12}$$

It follows from (1.4.11) and the unitarity of T that $k(x, hD_x)$ is essentially selfadjoint on $H_\Phi(\mathbf{C}^n)$ from $T\mathcal{S}(\mathbf{R}^n)$, and therefore, the corresponding unitary groups are intertwined by T,

$$e^{ik(x,hD_x)/h} \circ T = T \circ e^{il(x,hD_x)/h}.$$

Here we claim that in analogy with (1.4.12), we have

$$e^{ik(x,hD_x)/h} = \mathrm{Op}_h^w(e^{ik(x,\xi)/h}), \tag{1.4.13}$$

where the right hand side is still given by the contour integral in (1.4.2). Indeed, let us write, for $u \in T\mathcal{S}(\mathbf{R}^n)$,

$$\mathrm{Op}_h^w\left(e^{ik(x,\xi)/h}\right) u(x) = \frac{1}{(2\pi h)^n} \iint_{\Gamma(x)} e^{\frac{i}{h}\left((x-y+\xi^*)\cdot\theta + x^*\cdot\left(\frac{x+y}{2}\right)\right)} u(y)\, dy \wedge d\theta. \tag{1.4.14}$$

Here by Stokes' theorem, the integration contour can be deformed to the following,

$$\theta = \frac{2}{i}\frac{\partial\Phi}{\partial x}(x) + iC\overline{(x - y + \xi^*)},$$

for $C \gg 1$ large enough, and the expression (1.4.14) becomes

$$\frac{2^n C^n}{(2\pi h)^n} \int e^{-C|x-y+\xi^*|^2/h} e^{\frac{2}{h}(x-y+\xi^*)\cdot\frac{\partial\Phi}{\partial x}(x) + \frac{i}{h}x^*\cdot\left(\frac{x+y}{2}\right)} u(y)\, L(dy),$$

which, by the mean value theorem for holomorphic functions, is equal to

$$x \mapsto e^{\frac{i}{h}x^*\cdot x} e^{\frac{i}{2h}x^*\cdot\xi^*} u(x + \xi^*) = e^{ik(x,hD_x)/h} u(x).$$

This establishes (1.4.13) and therefore, we get

$$\mathrm{Op}_h^w\left(e^{\frac{i}{h}k(x,\xi)}\right) \circ T = T \circ \mathrm{Op}_h^w\left(e^{\frac{i}{h}\ell(x,\xi)}\right). \tag{1.4.15}$$

If $a \in \mathcal{S}(\Lambda_\Phi)$ and $b \in \mathcal{S}(\mathbf{R}^{2n})$ are related by $b = a \circ \kappa_T$, then by Fourier's inversion formula, we can represent a and b as superpositions of bounded exponentials of the form $e^{ik(x,\xi)/h}$ and $e^{il(x,\xi)/h}$, respectively. Here the linear forms k and ℓ are related by $\ell = k \circ \kappa_T$, and passing to the h–Weyl quantizations, we get, in view of (1.4.15),

$$\mathrm{Op}_h^w(a) \circ T = T \circ \mathrm{Op}_h^w(b). \tag{1.4.16}$$

A density argument allows us to complete the proof. □

We shall finally make some remarks concerning pseudodifferential operators with holomorphic symbols, referring to [65], as well as to the second part of this text, for a much more extensive discussion. Let us assume that $a(x,\xi)$ is a holomorphic bounded function in a region of the form $\Lambda_\Phi + W \subset \mathbf{C}_x^n \times \mathbf{C}_\xi^n$. Here W is a bounded open neighborhood of $0 \in \mathbf{C}^{2n}$. It follows from the proof of Theorem 1.4.1 that in this case we have, for $u \in H_\Phi(\mathbf{C}^n)$,

$$\mathrm{Op}_h^w(a)u(x) = \frac{1}{(2\pi h)^n} \iint_{\Gamma_C(x)} e^{\frac{i}{h}(x-y)\cdot\theta} a\left(\frac{x+y}{2}, \theta\right) u(y)\, dy \wedge d\theta, \tag{1.4.17}$$

where the contour $\Gamma_C(x)$ is given by

$$\theta = \frac{2}{i}\frac{\partial \Phi}{\partial x}\left(\frac{x+y}{2}\right) + \frac{i}{C}\frac{\overline{(x-y)}}{\langle x-y\rangle},$$

and $C > 0$ is large enough fixed, so that $\Gamma_C(x) \subset \Lambda_\Phi + W$. The holomorphy of the symbol allows us to consider weight functions different from Φ as well, and study boundedness properties of $\mathrm{Op}_h^w(a)$ in the corresponding exponentially weighted spaces.

Following [73], we have the following result.

Theorem 1.4.3 *Let $\widetilde{\Phi} \in C^{1,1}(\mathbf{C}^n)$ be such that $\widetilde{\Phi}(x) = \Phi(x) + f(x)$, where $f \in C_0^{1,1}(\mathbf{C}^n)$ is such that $||\, \nabla f\, ||_{L^\infty}$, $||\, \nabla^2 f\, ||_{L^\infty}$ are sufficiently small. We then have a uniformly bounded operator*

$$\mathrm{Op}_h^w(a) = \mathcal{O}(1) : H_{\widetilde{\Phi}}(\mathbf{C}^n) \to H_{\widetilde{\Phi}}(\mathbf{C}^n). \tag{1.4.18}$$

Here we set $H_{\widetilde{\Phi}}(\mathbf{C}^n) = \mathrm{Hol}(\mathbf{C}^n) \cap L^2(\mathbf{C}^n, e^{-2\widetilde{\Phi}/h} L(dx))$.

Proof We make a deformation to the new contour and set

$$\mathrm{Op}_h^w(a)u(x) = \frac{1}{(2\pi h)^n} \iint_{\widetilde{\Gamma}_C(x)} e^{\frac{i}{h}(x-y)\cdot\theta} a\left(\frac{x+y}{2}, \theta\right) u(y)\, dy \wedge d\theta, \tag{1.4.19}$$

where

$$\widetilde{\Gamma}_C(x) = \frac{2}{i}\frac{\partial \widetilde{\Phi}}{\partial x}\left(\frac{x+y}{2}\right) + \frac{i}{C}\frac{\overline{x-y}}{\langle x-y\rangle}. \tag{1.4.20}$$

Along the contour $\widetilde{\Gamma}_C(x)$, we have

$$
\begin{aligned}
&-\widetilde{\Phi}(x)+\operatorname{Re}\,(i(x-y)\cdot\theta)+\widetilde{\Phi}(y)\\
&\qquad=-\widetilde{\Phi}(x)+\left\langle x-y,\nabla\widetilde{\Phi}\left(\frac{x+y}{2}\right)\right\rangle_{\mathbf{R}^{2n}}+\widetilde{\Phi}(y)-\frac{1}{C}\frac{|x-y|^2}{\langle x-y\rangle}\\
&\qquad\qquad=-f(x)+\left\langle x-y,\nabla f\left(\frac{x+y}{2}\right)\right\rangle_{\mathbf{R}^{2n}}+f(y)-\frac{1}{C}\frac{|x-y|^2}{\langle x-y\rangle},
\end{aligned}
$$

and applying Taylor's formula we see that this expression does not exceed

$$
\mathcal{O}(1)||\, f''\,||_{L^\infty}\frac{|x-y|^2}{\langle x-y\rangle}-\frac{1}{C}\frac{|x-y|^2}{\langle x-y\rangle}\leq-\frac{1}{2C}\frac{|x-y|^2}{\langle x-y\rangle},
$$

provided that $||\, f''\,||_{L^\infty}$ is small enough. The proof can therefore be concluded as before, by an application of Schur's lemma. □

Remark. Let us notice that $H_{\widetilde{\Phi}}(\mathbf{C}^n)=H_\Phi(\mathbf{C}^n)$ as linear spaces, with the norms being equivalent, but not uniformly as $h\to 0^+$. We observe also that the Lipschitz IR-manifold $\Lambda_{\widetilde{\Phi}}$ is close to Λ_Φ, in the sense of Lipschitz graphs.

It turns out that the natural symbol associated to the operator in (1.4.18) is $a|_{\Lambda_{\widetilde{\Phi}}}$. Indeed, we have the following fundamental quantization-multiplication formula, due to [6, 69].

Proposition 1.4.4 *We have*

$$
\left(\mathrm{Op}_h^w(a)u,v\right)_{H_{\widetilde{\Phi}}}=\int a\left(x,\frac{2}{i}\frac{\partial\widetilde{\Phi}}{\partial x}(x)\right)u(x)\overline{v(x)}e^{-\frac{2}{h}\widetilde{\Phi}(x)}\,L(dx)+\mathcal{O}(h)||\,u\,||_{H_{\widetilde{\Phi}}}\,||\,v\,||_{H_{\widetilde{\Phi}}},
$$

for $u,v\in H_{\widetilde{\Phi}}(\mathbf{C}^n)$.

Proof We represent the operator $\mathrm{Op}_h^w(a)$ as in (1.4.19) with the contour (1.4.20), and Taylor expand a, writing $\xi(x)=\frac{2}{i}\frac{\partial\widetilde{\Phi}}{\partial x}(x)$,

$$
\begin{aligned}
a\left(\frac{x+y}{2},\theta\right)&=a(x,\xi(x))+(\partial_\xi a)(x,\xi(x))(\theta-\xi(x))\\
&\qquad+(\partial_x a)(x,\xi(x))\left(\frac{y-x}{2}\right)+\mathcal{O}(|y-x|^2)+\mathcal{O}(|\theta-\xi(x)|^2).
\end{aligned}
$$

Here the remainder terms are both $\mathcal{O}(|x-y|^2)$ along the contour $\widetilde{\Gamma}_C(x)$, and therefore, in view of Schur's lemma, their contribution gives rise to an operator of the norm $\mathcal{O}(h):H_{\widetilde{\Phi}}(\mathbf{C}^n)\to L^2_{\widetilde{\Phi}}(\mathbf{C}^n)$. Next, observing that the term $(\partial_x a)(x,\xi(x))\left(\frac{y-x}{2}\right)$ drops out, when passing to the quantizations, we conclude that

$$
\mathrm{Op}_h^w(a)=a(x,\xi(x))+(\partial_\xi a)(x,\xi(x))\cdot(hD_x-\xi(x))+R,
$$

where

$$
R=\mathcal{O}(h):H_{\widetilde{\Phi}}(\mathbf{C}^n)\to L^2_{\widetilde{\Phi}}(\mathbf{C}^n).
$$

It remains to estimate the integral

$$\int \left(\partial_{\xi_j} a\right)(x, \xi(x)) \left(\left(hD_{x_j} - \xi_j(x)\right) u(x)\right) \overline{v(x)} e^{-2\widetilde{\Phi}(x)/h} L(dx), \quad 1 \leq j \leq n, \tag{1.4.21}$$

and since the function $\left(\partial_{\xi_j} a\right)(x, \xi(x))$ is Lipschitz, we can integrate by parts in (1.4.21), getting $\mathcal{O}(h) || u ||_{H_{\widetilde{\Phi}}} || v ||_{H_{\widetilde{\Phi}}}$ plus the term

$$\int \left(\partial_{\xi_j} a\right)(x, \xi(x)) u(x) \overline{v(x)} \left(-hD_{x_j} - \xi_j(x)\right) e^{-2\widetilde{\Phi}(x)/h} L(dx) = 0.$$

This completes the proof. □

We shall finish with the following general idea suggested by the discussion above: given an h–pseudodifferential operator of the form $\mathrm{Op}_h^w(a)$, with a holomorphic in a tubular neighborhood of Λ_Φ, try to find an IR-manifold $\Lambda_{\widetilde{\Phi}}$ close to Λ_Φ so that the operator

$$\mathrm{Op}_h^w(a) : H_{\widetilde{\Phi}}(\mathbf{C}^n) \to H_{\widetilde{\Phi}}(\mathbf{C}^n)$$

acquires some improved properties, such as the invertibility, ellipticity, normality, etc. We refer to the works [7, 20, 21, 26, 27, 49, 50], where implementations of this idea have led to some precise results in the spectral theory of semiclassical non-selfadjoint operators. It may also be interesting to compare this idea with the recent developments around Carleman estimates with limiting Carleman weights for second order elliptic differential operators, see [39].

2 Analytic Microlocal Analysis Using Holomorphic Functions with Exponential Weights

2.1 Introduction

There are several approaches to analytic microlocal analysis:

- One very natural approach consists in adapting the classical theory of pseudodifferential operators on the real domain to the analytic category. The basic calculus was developed by L. Boutet de Monvel and P. Krée [3]. K.G. Andersson [1] and L. Hörmander [33] studied propagation of analytic singularities. The work [33] also introduced the analytic wave front set of distributions, a corresponding notion in the framework of hyperfunctions had previously been introduced by M. Sato (see [58]). The two works [1, 33] use special sequences of cutoff functions, remedying for the lack of analytic functions with compact support. Such special sequences have an earlier history, see L. Ehrenpreis [9], S. Mandelbrojt [44, 45]. The book [77] of F. Treves gives the theory of analytic pseudodifferential operators, with the help of such cutoffs.

- A second approach is based on the representation of distributions and more generally hyperfunctions as sums of boundary values of holomorphic functions. The main work in this direction is the one of M. Sato, T. Kawai and M. Kashiwara [58].
- A third approach is to work with Fourier transforms that have been modified by the introduction of a Gaussian (avoiding the use of the special cutoffs mentioned above). Such transforms come under different names: FBI, Bargmann-Segal, Gabor, wavepacket .... transforms. Microlocal properties are now described in terms of exponential growth/decay of the transformed functions. In the context of analytic microlocal analysis they were introduced and used by D. Iagolnitzer, H. Stapp [37], J. Bros, Iagolnitzer [4]. This is the method we follow here. See [46, 65].

The aim of this part of the text is to explain the basic ingredients in the approach of [65], that was preceded by some work on propagation of analytic singularities for boundary value problems, see [63]. The main observation is that an FBI-transform produces holomorphic functions whose exponential growth rate reflect the regularity and that such transforms are Fourier integral operators with complex phase functions. This leads to a calculus of Fourier integral operators and pseudodifferential operators in the complex domain via a Egorov theorem. In this calculus oscillatory integrals are systematically replaced by contour integrals, leading to "Cauchy integral operators".

This chapter splits roughly into 4 unequal parts:

- In Sects. 2.2–2.5 we discuss pseudodifferential operators and Fourier integral operators acting on exponentially weighted spaces of holomorphic functions.
- In Sects. 2.6, 2.7 we introduce FBI (generalized Bargmann-) transforms and the analytic wave front set of a distribution.
- The Sects. 2.8, 2.9 are devoted to some applications: propagation of singularities, construction of exponentially accurate quasi-modes for non-self-adjoint differential operators.
- In Sect. 2.10, we discuss the possibility of going from local to global results and in Sect. 2.11 we give a very quick review of related developments.

The theory that we develop is designed to analyze *existing* distributions (and operators), their singularities and sometimes their asymptotic behaviour. Thus for instance, if we consider an elliptic equation $Pu = v$, we do not try to construct the solution u, by constructing an inverse or parametrix of P directly. But if we assume that the solution u exists, we can analyze it by applying an FBI-transform T to get a conjugated equation $\widetilde{P}Tu = Tv$ near some point in $\mathbf{C}^n$, where $\widetilde{P}$ is an elliptic pseudodifferential operator in the complex domain to be described below, which we can invert. Hence we get Tu from the knowledge of Tv, and this allows us to analyze u up to analytic functions, (and sometimes up to exponentially decaying functions when we have an asymptotic problem).

2.2 Classical Analytic Symbols and Pseudodifferential Operators

Let $\Omega \subset \mathbf{C}^n$ be open, $\phi \in C(\Omega; \mathbf{R})$. By definition, the function $u = u(z; h)$ on $\Omega\times]0, h_0[$ belongs to $H_\phi^{\mathrm{loc}}(\Omega)$ if

- $u(\cdot; h) \in \mathrm{Hol}(\Omega)$, for all h, where Hol (Ω) denotes the space of holomorphic functions on Ω.
- $\forall K \Subset \Omega$, $\varepsilon > 0$, $\exists C > 0$ such that $|u(z; h)| \leq Ce^{(\phi(z)+\varepsilon)/h}$, $z \in K$.

When $u \in H_0(\Omega)$, i.e. $u \in H_\phi(\Omega)$ with $\phi = 0$, we say that u is an analytic symbol. When $u = \mathcal{O}(h^{-m})$ locally uniformly on Ω, we say that u is of finite order $m \in \mathbf{R}$. Finite order symbols are useful for symbolic calculus, like inversion of elliptic operators, while general symbols (of subexponential growth in $1/h$) are sometimes more convenient for general discussions.

We frequently identify equivalent elements of $H_\phi^{\mathrm{loc}}(\Omega)$, where the equivalence $u \sim v$ of $u, v \in H_\phi^{\mathrm{loc}}(\Omega)$ means that there exists $C^0(\Omega) \ni \phi_0 < \phi$, such that $u - v \in H_{\phi_0}^{\mathrm{loc}}(\Omega)$. When ϕ is pluri-subharmonic, the $\overline{\partial}$-method of Hörmander [34] allows us in principle (and without going into any details) to patch together local "representatives" $u_j \in H_\phi^{\mathrm{loc}}(\Omega_j)$ with $u_j \sim u_k$ in $H_\phi^{\mathrm{loc}}(\Omega_j \cap \Omega_k)$ into a holomorphic function u of class H_ϕ^{loc} on the union of the Ω_j such that $u \sim u_j$ in $H_\phi^{\mathrm{loc}}(\Omega_j)$ for every j.

By H_{ϕ,x_0} we denote the intersection of all spaces $H_\phi(\Omega)$ where Ω is a small neighborhood of $x_0 \in \mathbf{C}^n$ and ϕ is defined in some fixed neighborhood of x_0. We have a corresponding equivalence relation.

Classical analytic symbols (Boutet de Monvel, Krée [3]). We restrict the attention to symbols of order 0. Let $a_k \in \mathrm{Hol}\,(\Omega)$, $k = 0, 1, \dots$ and assume that for every $\widetilde{\Omega} \Subset \Omega$, $\exists C = C_{\widetilde{\Omega}} > 0$ such that

$$|a_k(z)| \leq C^{k+1}k^k,\ z \in \widetilde{\Omega}. \tag{2.2.1}$$

$a = \sum_0^\infty a_k(z)h^k$ is called a (formal) classical analytic symbol.

This series may very well be divergent for every $h > 0$, but for $z \in \widetilde{\Omega}$, $0 \leq k \leq (eC_{\widetilde{\Omega}}h)^{-1}$ we have

$$|a_k(z)|h^k \leq C_{\widetilde{\Omega}}(C_{\widetilde{\Omega}}hk)^k \leq C_{\widetilde{\Omega}}e^{-k},$$

so in this range the terms of the series behave like those of a geometrically convergent one. We can define a *realization* of a on $\widetilde{\Omega}$ by

$$a_{\widetilde{\Omega}}(z; h) = \sum_{0\leq k\leq (eC_{\widetilde{\Omega}}h)^{-1}} a_k(z)h^k.$$

and we get $|a_{\widetilde{\Omega}}(z; h)| \leq C_{\widetilde{\Omega}}e/(e-1)$.

If $\widehat{\Omega} \supset \widetilde{\Omega}$ is another relatively compact subset of Ω, and assuming, as we may, that $C_{\widehat{\Omega}} > C_{\widetilde{\Omega}}$, then $a_{\widehat{\Omega}}$ and $a_{\widetilde{\Omega}}$ are equivalent in $H_0(\widetilde{\Omega})$. It is sometimes convenient to consider (formal) classical symbols of the form

$$a=\sum_0^\infty a_k(z)h^k,\ \ a_k\in \mathrm{Hol}\,(\Omega)$$

without the growth condition (2.2.1).

Let

$$p(x,\xi;h)=\sum_0^\infty h^k p_k(x,\xi),\ \ q(x,\xi;h)=\sum_0^\infty h^k q_k(x,\xi)$$

be classical symbols defined near $(x_0,\xi_0)\in \mathbf{C}^{2n}$. Denote by $p(x,hD;h), q(x,hD;h)$ the corresponding formal pseudodifferential operators. The formal composition of p and q is defined by

$$p\#q=\sum_{\alpha\in\mathbf{N}^n}\frac{h^{|\alpha|}}{\alpha!}\partial_\xi^\alpha p(x,\xi;h)D_x^\alpha q(x,\xi;h),$$

which is a finite sum for each power of h. Here, we use standard PDE-notation, $D_x=i^{-1}\partial_x$,

$$\partial_x^\alpha=\partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n},\ \ |\alpha|=|\alpha|_{\ell^1}=\alpha_1+\cdots+\alpha_n,\ \text{ for } \alpha=(\alpha_1,\ldots,\alpha_n)\in\mathbf{N}^n.$$

When p,q are polynomials in ξ, the differential operators $p(x,hD;h), q(x,hD;h)$ are well defined and

$$p(x,hD_x;h)\circ q(x,hD_x;h)=(p\#q)(x,hD;h).$$

If r is a third symbol, also polynomial in ξ, it follows that

$$(p\#q)\#r=p\#(q\#r). \tag{2.2.2}$$

In general, we can approximate p,q,r with finite Taylor polynomials at any given point and see that we still have (2.2.2).

To p, we associate

$$A(x,\xi,hD_x;h)=p(x,\xi+hD_x;h)=$$
$$\sum_\alpha\frac{h^\alpha}{\alpha!}\partial_\xi^\alpha p(x,\xi)D_x^\alpha=\sum_{k=0}^\infty h^k A_k(x,\xi,D_x),$$

where

$$A_k=\sum_{\nu+|\alpha|=k}\frac{1}{\alpha!}(\partial_\xi^\alpha p_\nu)(x,\xi)D_x^\alpha \tag{2.2.3}$$

is a differential operator of order $\leq k$.

Formally, $A = e^{-ix\cdot\xi/h} \circ p(x, hD_x; h) \circ e^{ix\cdot\xi/h}$ which is exact and well defined, when p is a polynomial in ξ. Let $B = q(x, \xi + hD_x; h) = \sum_0^\infty h^\ell B_\ell$. Then $C = A \circ B$ is well defined by $C = \sum_0^\infty h^m C_m$, $C_m = \sum_{k+\ell=m} A_k \circ B_\ell$. By Taylor approximation with polynomials in ξ, we see that

$$C = r(x, \xi + hD_x; h), \text{ if } r = p\#q.$$

Quasi-norms Such norms are needed to control the symbolic calculus in the analytic setting. Boutet de Monvel, Krée introduced quasinorms by means of a method of majorant series. The present version is more conceptual, using norms of associated operators.

Let $\Omega_t \Subset \mathbf{C}^{2n}$, $0 \le t \le t_0$, $t_0 > 0$ be a family of open neighborhoods of a point (x_0, ξ_0) such that

$$(y, \xi) \in \Omega_s \text{ and } |x - y|_{\ell^\infty} < t - s \Longrightarrow (x, \xi) \in \Omega_t,$$

whenever $0 \le s \le t \le t_0$. Here,

$$|x|_{\ell^\infty} = \sup |x_j|, \; x = (x_1, \dots, x_n) \in \mathbf{C}^n.$$

Then D_x^α is a bounded operator: $B(\Omega_t) \to B(\Omega_s)$ where $B(\Omega)$ denotes the space of bounded holomorphic functions on Ω. Moreover, by the Cauchy inequalities,

$$\|D_x^\alpha\|_{t,s} := \|D_x^\alpha\|_{\mathcal{L}(B(\Omega_t), B(\Omega_s))} \le \frac{\alpha!}{(t-s)^{|\alpha|}} \le \frac{C_0^{|\alpha|}|\alpha|^{|\alpha|}}{(t-s)^{|\alpha|}},$$

for some constant $C_0 > 0$.

If Ω_{t_0} is a relatively compact subset of the domain of definition of p, then on Ω_{t_0},

$$|\partial_\xi^\alpha p_\nu| \le C^{1+\nu+|\alpha|}\nu^\nu \alpha!.$$

Hence, with a new constant

$$\|\frac{1}{\alpha!}\partial_\xi^\alpha p D_x^\alpha\|_{t,s} \le C^{1+\nu+|\alpha|}\nu^\nu \frac{|\alpha|^{|\alpha|}}{(t-s)^{|\alpha|}}.$$

The number of terms in (2.2.3) is $\le (1+k)^{n+1}$, so with a new constant $C > 0$, we have

$$\|A_k\|_{t,s} \le \frac{C^{k+1}k^k}{(t-s)^k}, \; 0 \le s < t \le t_0. \tag{2.2.4}$$

Conversely, if p is a classical symbol such that (2.2.4) holds for some $C > 0$, then p is a classical analytic symbol near (x_0, ξ_0). In fact, since $p_k = A_k(1)$, we get for some new $C > 0$ that

$$\sup_{\Omega_{t_0/2}} |p_k| \leq C^{k+1}k^k. \tag{2.2.5}$$

Put $f(A) = (f_k(A))_{k=0}^{\infty}$, where $f_k(A)$ is the smallest constant ≥ 0 such that

$$\|A_k\|_{t,s} \leq f_k(A)k^k(t-s)^{-k},\ 0 \leq s < t \leq t_0.$$

When (2.2.4) holds, $f_k(A)$ is of at most exponential growth.

Let $B = \sum_0^\infty h^k B_k$ be an operator of the same type, so that B_k is a differential operator of order $\leq k$.

Lemma 2.2.1 *If $C = A \circ B$, then $f_k(C) \leq \sum_{\nu+\mu=k} f_\nu(A) f_\mu(B)$ or in other terms, $f(C) \leq f(A) * f(B)$.*

Proof We have $C_k = \sum_{\nu+\mu=k} A_\nu \circ B_\mu$ and for $0 \leq s < r < t \leq t_0$:

$$\|A_\nu \circ B_\mu\|_{t,s} \leq f_\nu(A) f_\mu(B) \frac{\nu^\nu \mu^\mu}{(r-s)^\nu (t-r)^\mu}.$$

Choose r such that

$$r - s = \frac{\nu}{\nu+\mu}(t-s),\ t - r = \frac{\mu}{\nu+\mu}(t-s).$$

Then

$$\|A_\nu \circ B_\mu\|_{t,s} \leq f_\nu(A) f_\mu(B) \frac{(\nu+\mu)^{\nu+\mu}}{(t-s)^{\nu+\mu}},$$

$$\|C_k\|_{t,s} \leq \left(\sum_{\nu+\mu=k} f_\nu(A) f_\mu(B) \right) \frac{k^k}{(t-s)^k}.$$

□

For $\rho > 0$, put

$$\|A\|_\rho = \sum_0^\infty \rho^k f_k(A).$$

Then (2.2.4) holds iff $\|A\|_\rho < \infty$ for $\rho > 0$ small enough.

Lemma 2.2.2 *Let $C = A \circ B$. If $\|A\|_\rho,\ \|B\|_\rho < \infty$, then $\|C\|_\rho < \infty$ and we have $\|C\|_\rho \leq \|A\|_\rho \|B\|_\rho$.*

Proof By Lemma 2.2.1, we have pointwise with respect to k:

$$(\rho^k f_k(C))_0^\infty \leq (\rho^k f_k(A))_0^\infty * (\rho^k f_k(B))_0^\infty$$

and we have the corresponding inequality for the ℓ^1-norms. □

If $p(x, \xi; h)$ is a classical symbol on a neighborhood of $\overline{\Omega}_{t_0}$, we put $\|p\|_\rho = \|A\|_\rho$. If p is a classical analytic symbol then there exists $\rho > 0$ such that $\|p\|_\rho < \infty$ and similarly for q corresponding to B. Since $p\#q$ corresponds to $A \circ B$, we obtain $\|p\#q\|_\rho \leq \|p\|_\rho \|q\|_\rho$ and we conclude that $p\#q$ is a classical analytic symbol in Ω_{t_0}. Next we give a semi-classical formulation of a fundamental result of L. Boutet de Monvel, P. Krée [3]:

Theorem 2.2.3 *Let p be an elliptic classical analytic symbol ($p_0 \neq 0$) on a neighborhood of $\overline{\Omega}_{t_0}$ and let q be the classical symbol given by $p\#q = 1$. Then q is a classical analytic symbol in Ω_{t_0}.*

Proof Let $q_0 = 1/p_0$, so that q_0 is a classical analytic symbol. Then $p\#q_0 = 1 - r$ where r is a classical analytic symbol of order -1 in the sense that its asymptotic expansion starts with a term in h. Consequently $\|r\|_\rho < 1/2$ if $\rho > 0$ is small enough. We have

$$q = q_0\#(1 + r + r\#r + \cdots),$$

so that

$$\|q\|_\rho \leq \|q_0\|_\rho (1 + \|r\|_\rho + \|r\|_\rho^2 + \cdots) \leq 2\|q_0\|_\rho < \infty.$$

□

2.3 Stationary Phase – Steepest Descent

Let $B = B_{\mathbf{R}^n}(0, 1)$ be the open unit ball in $\mathbf{R}^n$ and put

$$\widetilde{B} = \{\lambda x;\ x \in \overline{B},\ \lambda \in \mathbf{C},\ |\lambda| \leq 1\}.$$

Theorem 2.3.1 *There exist a constant $C > 0$ depending only on the dimension, such that for all $N \in \mathbf{N}$, $0 < h \leq 1$, $u \in \mathrm{Hol}\,(\mathrm{neigh}\,(\widetilde{B}))$,*

$$\int_B e^{-x^2/(2h)} u(x)dx = \sum_{\nu=0}^{N-1} (2\pi)^{\frac{n}{2}} h^{\frac{n}{2}+\nu} \frac{1}{\nu!} \left(\frac{1}{2}\Delta\right)^\nu u(0) + R_N(h),$$

where

$$|R_N(h)| \leq C h^{\frac{n}{2}+N} (N+1)^{\frac{n}{2}} N! 2^N \sup_{\widetilde{B}} |u(z)|.$$

We omit the proof and refer to [65], Chap. 2.

Example 2.3.2 Consider

$$J(h) = \left(\frac{1}{2\pi h}\right)^n \iint_{\substack{|x|\le C_1 \\ \xi = -C_2 i\overline{x}}} e^{-ix\cdot\xi/h} u(x,\xi) dx d\xi.$$

Then,

$$\begin{aligned} J(h) &= \sum_0^{N-1} \frac{1}{k!}\left(\frac{h}{i}\sum_1^n \frac{\partial}{\partial x_j}\frac{\partial}{\partial \xi_j}\right)^k u(0,0) + R_N(h) \\ &= \sum_{|\alpha|\le N-1} \frac{1}{\alpha!}\left(\frac{h}{i}\right)^{|\alpha|}\left(\partial_x^\alpha \partial_\xi^\alpha u\right)(0,0) + R_N(h), \end{aligned}$$

$$|R_N(h)| \le C(n)(N+1)^n N! \left(\frac{h}{C_1^2 C_2}\right)^N \sup_{\substack{|x|\le C_1 \\ |\xi|\le C_1 C_2}} |u(x,\xi)|.$$

This follows from Theorem 2.3.1, some calculations and the following three observations:

- $\Gamma : \xi = (C_2/i)\overline{x}$ is a maximally totally real subspace of $\mathbf{C}^{2n}$, hence $\simeq \mathbf{R}^{2n}$ after a complex linear change of coordinates.
- The restriction of $e^{-ix\cdot\xi/h}$ to Γ is equal to $e^{-C_2|x|^2/h}$.
- The corresponding restriction of $i^{-1}\partial_x \cdot \partial_\xi$ is equal to

$$\frac{1}{i}\partial_x \cdot \frac{i}{C_2}\partial_{\overline{x}} = \frac{1}{4C_2}\Delta_{\mathrm{Re}\, x, \mathrm{Im}\, x}.$$

Non-quadratic case. The holomorphic version of the Morse lemma is the following:

Lemma 2.3.3 *Let* $\phi \in \mathrm{Hol}\,(\mathrm{neigh}\,(z_0, \mathbf{C}^n))$, $\phi'(z_0) = 0$, $\det \phi''(z_0) \neq 0$. *Then there exist local holomorphic coordinates* $\widetilde{z}_1, \ldots, \widetilde{z}_n$, *centered at* z_0 *such that*

$$\phi(z) = \phi(z_0) + \frac{1}{2}(\widetilde{z}_1^2 + \cdots + \widetilde{z}_n^2).$$

The main ingredient in the standard proof of the Morse lemma in the real smooth category is the implicit function theorem in the same category. To get the proof of the holomorphic Morse lemma it suffices to use the holomorphic implicit function theorem.

Theorem 2.3.4 *Let* $0 \in V \Subset U \subset \mathbf{C}^n$, V, U *open*, $\phi \in \mathrm{Hol}\,(U)$, $\phi(0) = 0$, $\phi'(0) = 0$, $\phi''(0)$ *non degenerate. Assume that* $\mathrm{Re}\,\phi \ge 0$ *on* $V_{\mathbf{R}} := V \cap \mathbf{R}^n$, $\mathrm{Re}\,\phi > 0$ *on* $\partial V_{\mathbf{R}}$, $\phi'(x) \neq 0$ *on* $V_{\mathbf{R}} \setminus \{0\}$. *Then, for every* $C > 0$ *large enough, there exists a constant* $\varepsilon > 0$ *such that for every* $u \in \mathrm{Hol}\,(U)$,

$$\int_{V_{\mathbf{R}}} e^{-\phi(x)/h} u(x) dx = \sum_{0\le k\le 1/(Ch)} (2\pi h)^{\frac{n}{2}} \frac{h^k}{k!} \left(\frac{1}{2}\widetilde{\Delta}\right)^k \left(\frac{u}{J}\right)(0) + R(\lambda),$$

where

$$|R(h)| \le \frac{1}{\varepsilon} e^{-\frac{\varepsilon}{h}} \sup_U |u(z)|,\ 0 < h \le 1.$$

Here, $\widetilde{\Delta}$ *denotes the Laplacian in the Morse coordinates,* $J = \det \frac{d\tilde{z}}{dz}$, $J(0) = (\det \phi''(0))^{\frac{1}{2}}$*, with the choice of the branch of the square root that tends to 1, when we deform* $\phi''(0)$ *to* 1 *in the space of invertible symmetric matrices with real part* ≥ 0.

Proof Up to an exponentially small modification, we may replace the integral by

$$I_\chi = \int_{\mathbf{R}^n} e^{-\phi(x)/h} u(x)\chi(x) dx,\ \chi \in C_0^\infty(V_{\mathbf{R}}),$$
$$\mathrm{supp}\,(1-\chi) \subset \text{ small neighborhood of } \partial V_{\mathbf{R}}.$$

Make a first contour deformation $\Gamma_\delta :\ V_{\mathbf{R}} \ni x \mapsto x + \delta\overline{\phi'}(x),\ 0 \le \delta \le \delta_0 \ll 1$. Along Γ_δ we have

$$\phi(z) = \phi(x) + \delta|\phi'(x)|^2 + \mathcal{O}(\delta^2|\phi'(x)|^2) \ge \frac{\delta}{C}|z|^2,$$

when δ_0 is small enough.

Let G be the $(n+1)$-dimensional contour formed by the union of the Γ_δ for $0 \le \delta \le \delta_0$. Then Stokes' formula gives (with χ denoting also a suitable smooth extension to the complex domain),

$$I_\chi = \int_{\Gamma_{\delta_0}} e^{-\phi(z)/h} u(z)\chi(z) dz - \int_G d(e^{-\phi/h} u(z)\chi(z) dz).$$

The last integral is equal to

$$\int_{G\cap \mathrm{neigh}\,(\partial V_{\mathbf{R}})} e^{-\phi(z)/h} u(z)\overline{\partial}\chi(z) \wedge dz.$$

When estimating the integral over Γ_{δ_0}, we can restrict the attention to a small neighborhood of 0 and then use Morse coordinates for which $\phi = \frac{1}{2}\tilde{z}^2$. Since $\mathrm{Re}\phi \asymp |\tilde{z}|^2$ along Γ_{δ_0}, we see that Γ_{δ_0} must be of the form $\tilde{y} = k(\tilde{x})$ $(\tilde{z} = \tilde{x} + i\tilde{y})$, where $|k'| \le \theta < 1, k(0) = 0$. (Use the implicit function theorem, to see that the projection $\Gamma_{\delta_0} \ni \tilde{z} \mapsto \tilde{x}$ is a diffeomorphism near 0.) The last step is then to deform the contour $\tilde{y} = k(\tilde{x})$ to $\tilde{y} = 0$ in the simplest possible way and to apply Theorem 2.3.1. □

2.4 Contour Integrals and Fourier Transforms

a. Remarks about real quadratic forms on $\mathbf{C}^n$. Let q be a real quadratic form on $\mathbf{C}^n \simeq \mathbf{R}^{2n}$. Let sign $(q) = (m_+, m_-)$ where $m_\pm = m_\pm(q)$ are given by

$$q = \sum_1^{m_+} \xi_j^2 - \sum_{m_++1}^{m_++m_-} \xi_j^2,$$

for suitable real-linear coordinates on $\mathbf{C}^n$. We know that m_+ (m_-) is the largest possible dimension of a real-linear subspace on which q is positive (negative) definite.

Using the complex structure, put $Jq(x) = q(ix)$, so that $J^2 q = q$ (since q is even).

Notice that q is pluriharmonic iff $Jq = -q$.

We say that q is Levi if $Jq = q$.

In general we have the decomposition

$$q = h + \ell = 2\mathrm{Re}\left(\sum a_{j,k} z_j z_k\right) + \sum b_{j,k} \bar{z}_j z_k,$$

where $h = (1 - J)q/2$ is pluri-harmonic and $\ell = (1 + J)q/2$ is Levi.

Proposition 2.4.1 *Let q be a pluri-subharmonic quadratic form on $\mathbf{C}^n$. Then*

(a) $m_+(q) \geq m_-(q)$
(b) If q is non-degenerate of signature (n, n), then the same fact holds for every pluri-subharmonic quadratic form $\widetilde{q} \leq q$.

Proof The pluri-subharmonicity of q means that $\ell \geq 0$. (a) Let $L \subset \mathbf{C}^n$ be a real-linear subspace of dimension $m_- = m_-(q)$ such that $q_{|L} < 0$. Use the decomposition $q = h + \ell$. Then $h(x) = q(x) - \ell(x) < 0$ for $0 \neq x \in L$. Consequently, $h(ix) > 0$, so $q(ix) = h(ix) + \ell(ix) > 0$. Thus q is positive definite on the m_--dimensional space iL, so $m_+ \geq m_-$. (b) Now assume that $m_+ = m_- = n$. Let $\widetilde{q} \leq q$ be pluri-subharmonic and choose the subspace L as in (a). Then $\widetilde{q}$ is negative definite on L so $m_-(\widetilde{q}) \geq m_-(q) = n$ and from the part (a) of the proposition we conclude that $\widetilde{q}$ has signature (n, n). □

b. Fundamental lemma.

Lemma 2.4.2 *Let $\phi \in C^\infty(\mathrm{neigh}\,((0, 0), \mathbf{C}^{n+k}); \mathbf{R})$ be pluri-subharmonic. Assume that $\nabla_y \phi(0, 0) = 0$ and that $\nabla_y^2 \phi(0, 0)$ is nondegenerate of signature (k, k). For $x \in \mathrm{neigh}\,(0, \mathbf{C}^n)$, let $y(x) \in \mathrm{neigh}\,(0, \mathbf{C}^k)$ be the unique critical point of $\phi(x, \cdot)$, so that $y(x)$ is a smooth function of x by the implicit function theorem. Then the critical value of $y \mapsto \phi(x, y)$,*

$$\Phi(x) = \phi(x, y(x)) = \mathrm{vc}_y \phi(x, y)$$

is pluri-subharmonic. If $\widetilde{\phi} \le \phi$ *is pluri-subharmonic with* $\widetilde{\phi}(0,0) = \phi(0,0)$*, then* $\nabla_y^2 \widetilde{\phi}(0,0)$ *is also non-degenerate of signature* (k,k) *and*

$$\mathrm{vc}_y \widetilde{\phi}(x,y) \le \mathrm{vc}_y \phi(x,y), \ \textit{for } x \in \mathrm{neigh}\,(0,\mathbf{C}^n).$$

Proof Let $L \subset \mathbf{C}^k$ be a subspace of real dimension k such that $\nabla_y^2 \phi(0,0)_{|_L} < 0$. Then $\nabla_y^2 \phi(0,0)_{|_{iL}} > 0$. For $t \in \mathrm{neigh}\,(0, iL)$, put $L_t = t + L$, so that the Γ_t form a foliation of a neighborhood of $0 \in \mathbf{C}^k$. It is easy to check (and closely related to the circles of ideas around the Mountain Pass Theorem, see Theorem 2, Sect. 8.5 in [10]) that

$$\phi(x, y(x)) = \inf_t \sup_{y \in \Gamma_t} \phi(x,y), \ \ x \in \mathrm{neigh}\,(0, \mathbf{C}^n).$$

If $\widetilde{\phi} \le \phi$ is as in the statement of the lemma, we have $\nabla_y^2 \widetilde{\phi}(0,0)_{|_L} < 0$, so $\nabla_y^2 \widetilde{\phi}(0,0)$ is non-degenerate of signature 0. Then $y \mapsto \widetilde{\phi}(x,y)$ has a non-degenerate critical point $\widetilde{y}(x)$ and we have the same mini-max formula as for ϕ:

$$\widetilde{\phi}(x, y(x)) = \inf_t \sup_{y \in \Gamma_t} \widetilde{\phi}(x,y), \ \ x \in \mathrm{neigh}\,(0, \mathbf{C}^n).$$

It is then clear that $\widetilde{\phi}(x, \widetilde{y}(x)) \le \phi(x, y(x))$. Replacing $\phi, \widetilde{\phi}$ by their quadratic Taylor polynomials $\phi^{(2)}(x,y)$, $\widetilde{\phi}^{(2)}(x,y)$ at $(0,0)$, and the critical points by their linear Taylor polynomials $y^{(1)}(x)$ and $\widetilde{y}^{(1)}(x)$, we see that $\phi^{(2)}(x, y^{(1)}(x))$, $\widetilde{\phi}^{(2)}(x, \widetilde{y}^{(1)}(x))$ are the quadratic Taylor polynomials of $\phi(x, y(x))$, $\widetilde{\phi}(x, \widetilde{y}(x))$. Taking $\widetilde{\phi}^{(2)}$ pluri-harmonic it is clear that $\widetilde{\phi}^{(2)}(x, \widetilde{y}^{(1)}(x))$ is pluri-harmonic and $\le \phi^{(2)}(x, y^{(1)}(x))$, so the latter is pluri-subharmonic. This shows that $\mathrm{vc}_y \phi(x,y)$ has a positive semi-definite Levi form at 0. The same argument now works with 0 replaced by any other point in $\mathrm{neigh}\,(0, \mathbf{C}^n)$ and we get the desired plurisubharmonicity. □

c. Contour integration. Let $\phi(y) \in C^\infty(\mathrm{neigh}\,(0, \mathbf{C}^k); \mathbf{R})$. Assume that 0 is a “saddle point” for ϕ in the sense that $\nabla_y \phi(0) = 0$ and $\nabla_y^2 \phi(0)$ is non-degenerate of signature (k,k). Consider a smooth contour $\Gamma : \mathrm{neigh}\,(0, \mathbf{R}^k) \to \mathrm{neigh}\,(0, \mathbf{C}^k)$ with $\Gamma(0) = 0$, $d\Gamma$ injective. We say that Γ is a good contour if

$$\phi(y) - \phi(0) \le -\frac{1}{C}|y|^2, \ \ y \in \Gamma. \tag{2.4.1}$$

In practice, we find such good contours, by studying the set of “good points” y that satisfy (2.4.1) for some $C > 0$.

If $u \in H_{\phi,0}$ i.e. an element of $H_\phi(\mathrm{neigh}\,(0, \mathbf{C}^k))$, then

$$I_\Gamma(h) = e^{-\phi(0)/h} \int_\Gamma u(y; h)\, dy$$

is well-defined up to an exponentially small ambiguity (and also up to a factor ± 1 depending on a choice of orientation, that we shall simply forget). As we have seen, a second good contour passing through 0 can be deformed to Γ within the set of such good contours.

Now take $\phi(x, y) \in C^\infty(\mathrm{neigh}((0,0), \mathbf{C}^{n+k}); \mathbf{R})$ with $\phi(0, y)$ as above. If Γ is a good contour for the latter function and $u \in H_{\phi,(0,0)}$, then by deforming Γ into an x-dependent good contour for $\phi(x, \cdot)$, we see that

$$U(x; h) = \int_\Gamma u(x, y; h) dy$$

is a well defined element of $H_{\Phi,0}$, where $\Phi(x) = \mathrm{vc}_y \phi(x, y)$.

When working with differential forms of other degrees, we may be interested in other signatures than (k, k). Also, for instance when composing Fourier integral operators, one is frequently in the situation of integrating along a good contour with respect to one group of variables and then for the resulting integral we want a good contour for the last group of variables. The following discussion (that we state only for quadratic forms) shows that this will always work as well as one can possibly hope for.

This has nothing to do with the complex structure, so we consider a decomposition $x = (x', x'') \in \mathbf{R}^n$, $x' \in \mathbf{R}^{n-d}$, $x'' \in \mathbf{R}^d$. Let q be a quadratic form on $\mathbf{R}^n$ such that $q''(x'') := q(0, x'')$ is a non-degenerate quadratic form on $\mathbf{R}^d$. Then $x'' \mapsto q(x', x'')$ has a unique critical point $x'' = x''(x')$ depending linearly on x'. Consequently, the corresponding critical value $q'(x') = q(x', x''(x'))$ is a quadratic form on $\mathbf{R}^{n-d}$. Let $(m_+(q), m_-(q))$ be the signature of q and denote the signatures of q' and q'' similarly. Then by assumption, $m_+(q'') + m_-(q'') = d$.

Proposition 2.4.3 *Under the above assumptions we have*

$$m_+(q) = m_+(q') + m_+(q''), \;\; m_-(q) = m_-(q') + m_-(q''). \tag{2.4.2}$$

If L'_-, L''_- are subspaces of $\mathbf{R}^n$ *of dimension $m_-(q')$ and $m_-(q'')$ respectively such that $q'_{|L'_-}$, $q''_{|L''_-}$ are negative definite, and we put $L_- = \{(x', x''(x') + x''; \, x' \in L'_-, \, x'' \in L''_-\}$, then $q_{|L_-}$ is negative definite.*

Proof After the change of variables $x' = \widetilde{x}'$, $x'' = x''(\widetilde{x}') + \widetilde{x}''$, we are reduced to the case when $x''(x') \equiv 0$. This means (after dropping the tildes on the new variables) that

$$q(x) = q'(x') + q''(x'')$$

and the conclusion follows. □

d. Application to Fourier transforms. Let $\phi \in C^\infty(\mathrm{neigh}(x_0, \mathbf{C}^n); \mathbf{R})$ be plurisubharmonic with $\phi''(x_0)$ non-degenerate of signature (n, n). Let $\xi_0 = \frac{2}{i} \frac{\partial \phi}{\partial x}(x_0)$. For $\xi \in \mathrm{neigh}(\xi_0, \mathbf{C}^n)$, we put

$$\phi^*(\xi) = \mathrm{vc}_x(\phi(x) + \mathrm{Im}(x \cdot \xi)),$$

where the critical point $x = x(\xi)$ is given by

$$\xi = \frac{2}{i}\partial_x \phi(x),\ x(\xi_0) = x_0.$$

Guided by the Fourier inversion formula (that we shall study below), we look at

$$(y, \xi) \mapsto -\mathrm{Im}(x \cdot \xi) + \mathrm{Im}(y \cdot \xi) + \phi(y)$$

which is pluri-subharmonic with the critical point $y = x$, $\xi = \frac{2}{i}\partial_x \phi(x)$ and the corresponding critical value $\phi(x)$. The critical point is non-degenerate of signature $(2n, 2n)$ since we have the good contour

$$\Gamma_R(x) :\ \xi = \frac{2}{i}\partial_x \phi(x) + iR\overline{(x-y)},\ |x-y| < r,$$

parametrized by $y \in B_{\mathbf{C}^n}(x, r)$. Indeed by Taylor expanding, we get:

$$-\mathrm{Im}((x-y)\cdot\xi) + \phi(y) = \phi(x) - (R - \mathcal{O}(1))|x-y|^2,\ (y, \xi) \in \Gamma_R(x).$$

with the "$\mathcal{O}(1)$" uniform in R. Hence Γ_R is a good contour for R large enough and $r > 0$ small enough.

Applying Proposition 2.4.3, we now see that

$$\xi \mapsto -\mathrm{Im}(x\cdot\xi) + \phi^*(\xi)$$

has a non-degenerate critical point $\xi = \xi(x)$ of signature (n, n) at $\xi(x) = \frac{2}{i}\partial_x \phi(x)$ and

$$\phi(x) = \mathrm{vc}_\xi(-\mathrm{Im}(x\cdot\xi) + \phi^*(\xi)).$$

This is a standard inversion formula for Legendre transforms when viewing ϕ^* as the Legendre transform of ϕ.

Using a good contour, we can define the Fourier transform

$$\mathcal{F}u(\xi; h) = \int_{\Gamma_\xi} \underbrace{e^{-ix\cdot\xi/h}u(x; h)}_{\in H_{\phi(\cdot)+\mathrm{Im}\,((\cdot)\cdot\xi)}} dx \in H_{\phi^*, \xi_0^*}.$$

For $v \in H_{\phi^*, \xi_0^*}$, we put

$$\mathcal{G}v(x; h) = \frac{1}{(2\pi h)^n}\int_{\Gamma_x^*} e^{ix\cdot\xi/h} v(\xi) d\xi,$$

where Γ_x^* is a good contour such that

$$\phi^*(\xi) - \mathrm{Im}(x \cdot \xi) - \phi(x) \le -\frac{1}{C}|\xi - \xi(x)|^2,\ \xi(x) = \frac{2}{i}\partial_x \phi(x).$$

Proposition 2.4.4 *For $u \in H_{\phi,x_0}$, we have $u = \mathcal{G}\mathcal{F}u$ in H_{ϕ_0,x_0} (up to equivalence).*

Proof We have

$$\mathcal{G}\mathcal{F}u(x) = \frac{1}{(2\pi h)^n}\int_{\Gamma_x^*}\int_{\Gamma_\xi} e^{i(x-y)\cdot\xi/h}u(y)dyd\xi \text{ (iterated integral).}$$

Along the composed contour we have (cf. Proposition 2.4.3)

$$\begin{aligned} -\mathrm{Im}(x\cdot\xi) + \phi^*(\xi) &\le \phi(x) - \frac{1}{C}|\xi - \xi(x)|^2,\ \xi \in \Gamma_x^*, \\ \mathrm{Im}(y\cdot\xi) + \phi(y) &\le \phi^*(\xi) - \frac{1}{C}|y - x(\xi)|^2,\ y \in \Gamma_\xi, \end{aligned}$$

so

$$-\mathrm{Im}((x-y)\cdot\xi) + \phi(y) \le \phi(x) - \frac{1}{C}(|\xi - \xi(x)|^2 + |y - x(\xi)|^2).$$

The composed contour is a good contour like Γ_R.

Thus, up an exponentially small error, we can replace the composed contour by Γ_R for R large enough and get

$$\begin{aligned} &\frac{1}{(2\pi h)^n}\iint_{\Gamma_R(x)} e^{i(x-y)\cdot\xi/h}u(y)dyd\xi = \\ &\left(\frac{R}{i2\pi h}\right)^n \iint_{|x-y|<r} e^{\frac{2}{h}(x-y)\cdot\partial_x\phi(x) - \frac{R}{h}|x-y|^2}u(y)dy\wedge d\overline{y} \\ &= (1 + \mathcal{O}(e^{-Rr^2/h}))u(x) \end{aligned}$$

by the spherical mean-value property for holomorphic functions. □

2.5 Pseudodifferential Operators and Fourier Integral Operators

Let $a(x, y, \theta; h)$ be an analytic symbol defined near $(x_0, x_0, \xi_0) \in \mathbf{C}^{3n}$, so that $a \in H_{0,(x_0,x_0,\xi_0)}$. Let $\phi \in C^\infty(\text{neigh}(x_0, \mathbf{C}^n); \mathbf{R})$ with $(2/i)\partial_x\phi(x_0) = \xi_0$. For $u \in H_{\phi,x_0}$, we define $Au \in H_{\phi,x_0}$ by

$$Au(x;h) = \frac{1}{(2\pi h)^n}\iint_{\Gamma(x)} e^{i(x-y)\cdot\theta/h}a(x,y,\theta;h)u(y;h)dyd\theta,$$

where $\Gamma(x) = \Gamma_R(x)$ is the good contour introduced at the end of the preceding section so that (for R large enough)

$$e^{-\phi(x)/h} \left|e^{i(x-y)\cdot\theta/h}\right| e^{\phi(y)/h} \le e^{-\frac{1}{h}(R-\mathcal{O}(1))|x-y|^2}$$

along $\Gamma(x)$. It follows that

$$Au(x;h) = A_\Gamma u(x;h) = \int k_\Gamma(x,y;h)u(y)L(dy),$$

where

$$|k_\Gamma(x,y;h)|e^{(-\phi(x)+\phi(y))/h} \le C_\Gamma h^{-n} e^{-\frac{1}{h}(R-\mathcal{O}(1))|x-y|^2}.$$

A_Γ is uniformly bounded $L^2_{\phi,x_0} \to L^2_{\phi,x_0}$. Here, we assume for simplicity that $|a(x,y,\theta;h)| \le \mathcal{O}(1)$. Without that assumption we would need to insert a factor $C_\epsilon e^{\epsilon/h}$ to the right in the last estimate and the boundedness statement about A_Γ has to be modified accordingly.

We define the symbol of A by

$$\sigma_A(x,\xi;h) = e^{-ix\cdot\xi/h} A(e^{i(\cdot)\cdot\xi/h}),\ \ (x,\xi) \in \mathrm{neigh}\,((x_0,\xi_0), \mathbf{C}^{2n}).$$

The method of stationary phase gives

$$\sigma_A(x,\xi;h) \equiv \sum_{|\alpha|\le 1/(Ch)} \frac{h^{|\alpha|}}{\alpha!}(\partial_\xi^\alpha D_x^\alpha a)(x,x,\xi;h)$$

and this is (a realization of) a classical analytic symbol when a is a classical analytic symbol. Clearly $\sigma_A \equiv a$ when a does not depend on y.

Lemma 2.5.1 *Assume that $\sigma_A = 0$ in $H_{0,(x_0,\xi_0)}$. Then $\exists b \in H_{0,(x_0,x_0,\xi_0)}$ with values in the $(n-1)$-forms in θ such that*

$$e^{i(x-y)\cdot\theta/h}a(x,y,\theta)d\theta \equiv ihd_\theta\left(e^{i(x-y)\cdot\theta/h}b\right),\ \ in\ H_{-\mathrm{Im}\,((x-y)\cdot\theta),(x_0,x_0,\xi_0)}.$$

Applying the Stokes formula along the good contour, it then follows that $A = 0$ as an operator in H_{ϕ,x_0}.

Proof By a simple change of variables,

$$\begin{aligned}(2\pi h)^n\sigma_A(x,\eta) &= \iint e^{-iy\cdot\eta/h}\underbrace{a(x,x-y,\theta;h)e^{iy\cdot\theta/h}}_{u(x,y,\theta;h)}dyd\theta \\ &= \mathcal{F}_{(y,\theta)\to(\eta,\theta^*)}(u)(\eta,0;h) = v(x,\eta,0;h),\end{aligned}$$

where x is treated as a parameter and $v := \mathcal{F}_{(y,\theta)\to(\eta,\theta^*)}(u)$.

We have $u \in H_\phi$, $v \in H_{\phi^*}$, $\phi = -\mathrm{Im}(y \cdot \theta)$, $\phi^* = \mathrm{Im}(\eta \cdot \theta^*)$ and we observe that ϕ and ϕ^* are pluri-harmonic. Now $v(x, \eta, 0; h) = 0$ and Taylor's formula gives

$$v(x, \eta, \theta^*; h) = \sum_1^n \widehat{v}_j(x, \eta, \theta^*; h)\theta_j^*, \ \widehat{v}_j \in H_{\phi^*},$$

and $\widehat{v}_j$ depend holomorphically on x. By Fourier inversion

$$u(x, y, \theta; h) = \sum_1^n hD_{\theta_j} v_j \text{ in } H_\phi, \ \ v_j \in H_\phi,$$

so $v_j = b_j(x, y, \theta; h)e^{iy\cdot\theta/h}$, $b_j \in H_0$. Going back to the original variables, we get the identity in the lemma. □

General remarks about Fourier integral operators. Let

$$\phi(z, y, \theta) \in C^2(\text{neigh}\,((z_0, y_0, \theta_0), \mathbf{C}^{n_z+n_y+n_\theta}); \mathbf{R}), \quad f \in C^2(\text{neigh}\,(y_0, \mathbf{C}^{n_y}); \mathbf{R})$$

be pluri-subharmonic and assume that $(y, \theta) \mapsto \phi(z, y, \theta) + f(y)$ has a saddle point at (y_0, θ_0). If $a \in H_{\phi,(z_0,y_0,\theta_0)}$, we can define $A : H_{f,y_0} \to H_{g,z_0}$ by

$$Au(z; h) = \int_{\Gamma_1(z)} a(z, y, \theta; h)u(y)dyd\theta,$$

where $g(z) = \mathrm{vc}_{y,\theta}(\phi(z, y, \theta) + f(y)$ and $\Gamma_1(z)$ is a good contour.

Let $b(x, z, w; h) \in H_{\psi,(x_0,z_0,w_0)}$, $x \in \mathbf{C}^{n_x}$ and assume that ψ, g fulfill the same assumptions as ϕ, f. Then for $v \in H_{g,z_0}$, we define $Bv \in H_{k,x_0}$ by

$$Bv(x; h) = \int_{\Gamma_2(x)} b(x, z, w; h)v(z)dzdw,$$

where $\Gamma_2(x)$ and $k(x)$ denote a good contour and the critical value respectively, for $(z, w) \mapsto \psi(x, z, w) + g(z)$.

We can then define $B \circ A : H_{f,y_0} \to H_{k,x_0}$ by

$$B \circ Au(x; h) = \iiiint_{\Gamma(x)} b(x, z, w)a(z, y, \theta)u(y)dyd\theta dzdw,$$

where $\Gamma(x)$ is the composed contour given by $(z, w) \in \Gamma_2(x)$, $(y, \theta) \in \Gamma_1(z)$. It is a good contour for

$$(z, w, y, \theta) \mapsto \psi(x, z, w) + \phi(z, y, \theta) + f(y).$$

Now assume that

$$(z,w)\mapsto \psi(x_0,z,w)+\phi(z,y_0,\theta_0) \tag{2.5.1}$$

has a saddle point at (z_0,w_0). Let $F(x,y,\theta)$ be the critical value when (z,y,θ) varies near (x_0,y_0,θ_0). Then F is pluri-subharmonic, and knowing that $(z,w,y,\theta)\mapsto \psi+\phi+f$ has saddle point, we see that

$$(y,\theta)\mapsto F(x,y,\theta)+f(y) \tag{2.5.2}$$

has a saddle point. Hence, if $\Gamma_3(x,y,\theta)$ is a good contour for (2.5.1) and $\Gamma_4(x)$ a good contour for (2.5.2), the composed contour

$$\widetilde{\Gamma}(x):\ (y,\theta)\in\Gamma_4(x),\ (z,w)\in\Gamma_3(x,y,\theta)$$

is good for

$$(z,w,y,\theta)\mapsto \psi(x,z,w)+\phi(z,y,\theta)+f(y).$$

By Stokes, we can replace $\Gamma(x)$ in the formula for $B\circ Au(x)$ by $\widetilde{\Gamma}(x)$ and write

$$\begin{aligned} B\circ Au(x;h) &= \iiiint_{\widetilde{\Gamma}(x)} b(x,z,w)a(z,y,\theta)u(y)dyd\theta dzdw \\ &= \iint_{\Gamma_4(x)} \underbrace{\left(\iint_{\Gamma_3(x,y,\theta)} b(x,z,w)a(z,y,\theta)dzdw\right)}_{=:c(x,y,\theta)\in H_{F,(x_0,y_0,\theta_0)}} u(y)dyd\theta \end{aligned}$$

This remark can be applied to the case when A, B are pseudodifferential operators and when combining it with the stationary phase, we get

Theorem 2.5.2 *Let $A, B : H_{\phi,x_0}\to H_{\phi,x_0}$ be two pseudodifferential operators. Then $B\circ A$ is a pseudodifferential operator with symbol*

$$\sigma_{B\circ A}(x,\xi;h)=\sum_{|\alpha|\le \frac{1}{Ch}}\frac{1}{\alpha!}h^{|\alpha|}\partial_\xi^\alpha\sigma_B(x,\xi;h)D_x^\alpha\sigma_A(x,\xi;h).$$

2.6 FBI-Transforms and Analytic Wavefront Sets

Let $\phi\in \mathrm{Hol}\,(\mathrm{neigh}\,((x_0,y_0),\mathbf{C}^{2n}))$, $y_0\in\mathbf{R}^n$ and assume that

$$\begin{gathered}\phi'_y(x_0,y_0)=-\eta_0\in\mathbf{R}^n,\ \ \mathrm{Im}\phi''_{yy}(x_0,y_0)>0,\\ \det\phi''_{xy}(x_0,y_0)\neq 0.\end{gathered} \tag{2.6.1}$$

Let $a(x, y; h)$ be an elliptic classical analytic symbol defined near (x_0, y_0) and let $\chi \in C_0^\infty(\text{neigh}\,(y_0, \mathbf{R}^n))$ be equal to one near y_0. If $u \in \mathcal{D}'(\mathbf{R}^n)$ (or just defined in a neighborhood of the support of χ), we put

$$Tu(x; h) = \int e^{i\phi(x,y)/h} a(x, y; h)\chi(y)u(y)dy, \; x \in \text{neigh}\,(x_0, \mathbf{C}^n). \tag{2.6.2}$$

Proposition 2.6.1 $Tu \in H_\Phi(\text{neigh}\,(x_0))$, *where*

$$\Phi = \sup_{y \in \text{neigh}\,(y_0, \mathbf{R}^n)} -\text{Im}\phi(x, y) \in C^\infty(\text{neigh}\,(x_0, \mathbf{C}^n); \mathbf{R}).$$

This is evident since $\mathbf{R}^n \ni y \mapsto -\text{Im}\phi(x, y)$ has a non-degenerate maximum at $y = y(x) \in \text{neigh}\,(y_0, \mathbf{R}^n)$.

Introduce

$$\Lambda_\Phi = \left\{ \left(x, \frac{2}{i}\partial_x \Phi(x) \right); \; x \in \text{neigh}\,(x_0, \mathbf{C}^n) \right\}$$

Then (and here we only use that Φ is real and smooth), the restriction to Λ_Φ of the complex symplectic 2-form $\sigma = \sum d\xi_j \wedge dx_j$ is real, so Λ_Φ is an I-Lagrangian manifold, i.e. a Lagrangian manifold for the real symplectic form $\text{Im}\sigma$.

Proposition 2.6.2 $\Lambda_\Phi = \kappa_T(\mathbf{R}^{2n})$, *where*

$$\kappa_T : \text{neigh}\,((y_0, \eta_0)) \ni (y, -\phi'_y(x, y)) \mapsto (x, \phi'_x(x, y)) \in \text{neigh}\,((x_0, \xi_0))$$

is the complex canonical transformation associated to T, *when viewed as a Fourier integral operator. Here* $(x_0, \xi_0) = \kappa_T(y_0, \eta_0) = (x_0, (2/i)\partial_x \Phi(x_0))$. *In particular* $\sigma_{|\Lambda_\Phi}$ *is real and non-degenerate. (*Λ_Φ *is I-Lagrangian and R-symplectic.) Further,* Φ *is strictly pluri-subharmonic.*

Proof The real critical point of $-\text{Im}\phi(x, \cdot)$ is characterized by the property that $\eta(x) := -\phi'_y(x, y(x))$ is real. Further,

$$\frac{2}{i}\partial_x \Phi(x) = \frac{2}{i}(\partial_x(-\text{Im}\phi))(x, y(x)) = \phi'_x(x, y(x)).$$

Hence Λ_Φ is contained in $\kappa_T(\mathbf{R}^{2n})$ and the two manifolds have the same dimension so they have to coincide (near (x_0, ξ_0)).

We then know that

$$\sigma_{|\Lambda_\Phi} = \sum_1^n d\left(\frac{2}{i}\partial_{x_j}\Phi(x)\right) \wedge dx_j = \frac{2}{i}\sum_k \sum_j \partial_{\overline{x}_k}\partial_{x_j}\Phi \, d\overline{x}_k \wedge dx_j$$

is non-degenerate, so the Levi-form of Φ is non-degenerate. Since Φ by definition is the supremum of the family of pluri-harmonic functions $x \mapsto -\mathrm{Im}\phi(x, y)$ we know that Φ is pluri-subharmonic and hence strictly pluri-subharmonic. $\square$

For $y \in \mathbf{R}^n$ (close to y_0) let

$$\Gamma_y = \{x \in \mathbf{C}^n;\, y(x) = y\} = \pi_x \kappa_T (T_y^* \mathbf{R}^n),$$

where $\pi_x : \mathbf{C}^{2n}_{x,\xi} \to \mathbf{C}^n_x$ is the natural projection, so that Γ_y is of real dimension n and the Γ_y form a foliation of neigh $(x_0, \mathbf{C}^n)$. Γ_y is totally real: $T_x\Gamma_y \cap iT_x\Gamma_y = 0$, $\forall x \in \Gamma_y$. In fact, $T_x\Gamma_y = \{t_x \in \mathbf{C}^n;\ \phi''_{yx} t_x \in \mathbf{R}^n\}$.

For every fixed real y:

$$\Phi(x) + \mathrm{Im}\phi(x, y) = -\mathrm{Im}\phi(x, y(x)) + \mathrm{Im}\phi(x, y) \asymp \mathrm{dist}\,(x, \Gamma_y)^2. \tag{2.6.3}$$

Since $x \mapsto -\mathrm{Im}\phi(x, y)$ is pluri-harmonic, this gives another proof of the fact that $\Phi(x)$ is strictly pluri-subharmonic.

Exercise Explore the standard case of Bargmann transforms with $\phi(x, y) = i(x - y)^2/2$.

Exercise Let $f(y)$ be analytic near y_0, real valued on the real domain and with $f'(y_0) = \eta_0$. Show that

$$T(e^{if/h}) = h^{n/2} c(x; h) e^{ig(x)/h}, \tag{2.6.4}$$

where $c(x; h)$ is a classical analytic symbol of order 0 and

$$g(x) = \mathrm{vc}_{y \in \mathrm{neigh}\,(y_0, \mathbf{C}^n)} (\phi(x, y) + f(y)) \tag{2.6.5}$$

is holomorphic, $\Lambda_g := \{(x, g'(x))\} = \kappa_T(\Lambda_f)$ where Λ_f is defined as Λ_g.

Let $(\Lambda_f)_{\mathbf{R}} = \Lambda_f \cap \mathbf{R}^{2n}$. Show that $-\mathrm{Im}g \le \Phi$ and that more precisely,

$$\Phi(x) + \mathrm{Im}g(x) \asymp \mathrm{dist}\,(x, \pi_x(\kappa_T((\Lambda_f)_{\mathbf{R}})))^2. \tag{2.6.6}$$

Observe also that $\pi_x(\kappa_T((\Lambda_f)_{\mathbf{R}}))$ is transversal to Γ_y. See the end of this section for a solution of the exercise.

Assume that $\eta_0 \neq 0$. For $x \in \mathrm{neigh}\,(x_0)$, write

$$(y(x), \eta(x)) = (y(x), -\partial_y \phi(x, y(x))) \in T^*\mathbf{R}^n \setminus 0,$$

where $y(x)$ is the local real maximum of $-\mathrm{Im}\phi(x, \cdot)$. Also, we have

$$(y(x), \eta(x)) = \kappa_T^{-1} \left(x, \frac{2}{i} \partial_x \Phi(x) \right).$$

Definition 2.6.3 Let u be a distribution defined near y_0, independent of h. We say that $(y(x), \eta(x)) \notin \mathrm{WF}_a(u)$ if $Tu = 0$ in $H_{\Phi,x}$.

We shall see that this defines a closed conic subset $\mathrm{WF}_a(u)$ of $T^*(\mathrm{neigh}\,(y_0, \mathbf{R}^n)) \setminus 0$, *independent* of the choice of T.

In order to prove that the definition does not depend on the choice of T we would like to construct "the inverse T^{-1}". However, this can never succeed completely since Tu only carries microlocal information about u near (y_0, η_0). We can however give meaning to this inverse on certain smaller spaces and that will suffice to be able to describe a second FBI-transform $\widetilde{T}u$ in terms of Tu.

Put

$$Sv(x; h) = h^{-n} \int e^{-i\phi(z,x)/h} b(z, x; h) v(z) dz, \tag{2.6.7}$$

where b is an elliptic classical analytic symbol of order 0, defined near (x_0, y_0). Formally,

$$STu(x; h) = h^{-n} \iint e^{i(-\phi(z,x)+\phi(z,y))/h} b(z, x; h) a(z, y; h) u(y) dy dz \tag{2.6.8}$$

and we can apply the Kuranishi trick[1] to see that formally

$$STu(x; h) = \frac{1}{(2\pi h)^n} \iint e^{\frac{i}{h}(x-y)\cdot\theta} c(x, y, \theta; h) u(y) dy d\theta, \tag{2.6.9}$$

where c is an elliptic classical analytic symbol of order 0, defined near (y_0, y_0, η_0). According to Lemma 2.5.1 and the previously given definition of the symbol of a pseudodifferential operator, we can replace c by $\widetilde{c}(x, \theta; h)$, independent of y and still elliptic to get a new pseudodifferential operator which has the same action on expressions as in the last exercise above.

Let $\widetilde{d}$ satisfy $\widetilde{d}\#\widetilde{c} = 1$. Then

$$\widetilde{d}(x, hD_x; h) \circ ST = 1$$

when acting on functions as in the exercise. On the other hand we can apply stationary phase to get formally

$$\widetilde{d}(x, hD; h) Sv = h^{-n} \int e^{-i\phi/h} \widetilde{b} v(z) dz =: \widetilde{S}v(x; h)$$

---

[1]The usual and no doubt original application of this trick is to changes of variables for pseudodifferential operators in the standard setting, see for instance [14], pp. 34–35, 40. In the present situation we use Taylor's formula to write $-\phi(z, x) + \phi(z, y) = (x - y) \cdot \theta(x, y, z)$, where θ is holomorphic near (y_0, y_0, x_0) and $\theta(x, x, z) = -\partial_x \phi(z, x)$. We observe that $z \mapsto \theta(x, y, z)$ is a local holomorphic diffeomorphism, and use this to replace the integration variables z by θ.

Our compositions are well defined and hence associative when restricted to expressions as in the exercise and we therefore get

$$\widetilde{S}T = 1.$$

Dropping the tildes, we have shown that we can find S of the form (2.6.7) such that

$$ST = 1$$

when acting on expressions as in the exercise.

When trying to define $Sv(x;h)$ for $v \in H_\Phi$, we would like to have a contour Γ in z space such that

$$\mathrm{Im}\phi(z,x) + \Phi(z) \le 0, \ \ z \in \Gamma,$$

with strict inequality near the boundary. In view of (2.6.3) the best possible choice in general is $\Gamma = \Gamma_x$ and we then just achieve equality.

If however $v \in H_\Psi$, where $\Psi - \Phi \asymp -\mathrm{dist}\,(z, \widetilde{\Gamma})^2$ and $\widetilde{\Gamma}$ is a real manifold of dimension n transversal to Γ_x, then Sv is well-defined. In particular if u is as in the exercise, $v = Tu$, this is the case with $\Psi = -\mathrm{Im} g$, so Sv is well-defined up to an exponentially small ambiguity, and we get $Sv \equiv u$ in $H_{-\mathrm{Im}\, f}$.

Let

$$\widetilde{T}u(x;h) = \int e^{i\widetilde{\phi}(x,y)/h}\widetilde{a}(x,y;h)u(y)dy$$

be a second FBI-transform with $\widetilde{\phi}, \widetilde{a}$ defined near $(\widetilde{x}_0, y_0)$ and with $-\widetilde{\phi}'_y(\widetilde{\xi}_0, y_0) = \eta_0$. Then formally

$$\widetilde{T}Sv(x;h) = h^{-n} \iint e^{\frac{i}{h}(\widetilde{\phi}(x,y)-\phi(z,y))}\widetilde{a}(x,y;h)a(z,y;h)v(z)dydz. \tag{2.6.10}$$

This is a Fourier integral operator[2] with associated canonical transformation $\kappa_{\widetilde{T}} \circ \kappa_T^{-1}$, mapping Λ_Φ to $\Lambda_{\widetilde{\Phi}}$ and it follows from this observation, or by direct verification, that

$$(y,z) \mapsto -\mathrm{Im}\widetilde{\phi}(x,y) + \mathrm{Im}\phi(z,y) + \Phi(z) =: F$$

has a non-degenerate critical point $(y,z) = (y_c(x), z_c(x))$, given by the conditions

$$\left(z, \frac{2}{i}\partial_z\Phi(z)\right) = \kappa_T(y,\eta), \ \left(x, \frac{2}{i}\partial_x\widetilde{\Phi}(x)\right) = \kappa_{\widetilde{T}}(y,\eta),$$

where (y,η) is real $(y = y(z) = \widetilde{y}(x), \eta = \eta(z) = \widetilde{\eta}(x))$.

Next, we show that there is a good contour for (2.6.10): As a first attempt, we take

[2] A general local theory for Fourier integral operators can be developed in the spirit of Sect. 2.5. See [65], Chap. 11.

$$\gamma_0 = \gamma_0(x) : \; y \in \mathbf{R}^n, \; z \in \Gamma_y,$$

which passes through $(y_c(x), z_c(x))$. Along that contour we have

$$F(y, z) - \widetilde{\Phi}(x) = -(\widetilde{\Phi}(x) + \mathrm{Im}\widetilde{\phi}(x, y)) \asymp -|y - \widetilde{y}(x)|^2.$$

Thus the contour is "almost good". Since our critical point is non-degenerate, we can make a small deformation and find a good contour. In fact, it suffices to take $\gamma = \gamma_\epsilon(x)$, $0 < \epsilon \ll 1$, given by

$$\gamma_0 \ni (y, z) \mapsto (y, z) - \epsilon \nabla_{y,z} F(x, y, z),$$

where $\nabla_{y,z}$ denotes the gradient when $\mathbf{C}^n_y \times \mathbf{C}^n_z$ has been identified with $\mathbf{R}^{4n}$. In conclusion

$$\widetilde{T} S \text{ is a well-defined Fourier integral operator } H_{\Phi,x_0} \to H_{\widetilde{\Phi},\widetilde{x}_0}.$$

Proposition 2.6.4 *For* $x \in \mathrm{neigh}\,(x_0)$*,* $\widetilde{x} \in \mathrm{neigh}\,(\widetilde{x}_0)$ *related by*

$$\widetilde{\kappa}^{-1}_{\widetilde{T}}(\widetilde{x}, (2/i)\partial_{\widetilde{x}}\widetilde{\Phi}(\widetilde{x})) = \kappa_T^{-1}(x, (2/i)\partial_x \Phi(x)),$$

the following two statements are equivalent:

(1) $\widetilde{T} u = 0$ *in* $H_{\widetilde{\Phi},\widetilde{x}}$.
(2) $Tu = 0$ *in* $H_{\Phi,x}$.

Proof Take $x = x_0$, $\widetilde{x} = \widetilde{x}_0$ for simplicity. Let $\chi \in C_0^\infty(\mathrm{neigh}\,(\eta_0, \mathbf{R}^n))$ be equal to one near η_0. Without loss of generality, we may assume that the distribution u has compact support in a neighborhood of y_0. Then from the (classical!) Fourier inversion formula,

$$u(x) = \frac{1}{(2\pi h)^n} \int e^{ix\cdot\eta/h} \mathcal{F}u(\eta) d\eta,$$

and contour deformations, we see that

$$Tu = T\chi(hD_y)u \text{ in } H_{\Phi,x_0}, \quad \widetilde{T}u = \widetilde{T}\chi(hD_y)u \text{ in } H_{\widetilde{\Phi},\widetilde{x}_0}.$$

On the other hand $v = \chi(hD_y)u$ is a superposition of plane waves (special cases of states as in the last exercise), so

$$\chi(hD_y)u = ST\chi(hD_y)u + \mathcal{O}(e^{-1/Ch}),$$

where now

$$Sv(y) = \int_{\Gamma_y} e^{-i\phi(x,y)/h} b(x, y; h) v(x) dx.$$

Consequently,

$$\widetilde{T}\chi(hD_y)u = \widetilde{T} \circ ST\chi(hD_y)u \text{ in } H_{\widetilde{\Phi},\widetilde{x}_0}.$$

Here, for each plane wave in $\chi(hD_y)u$, we can make a contour deformation to the good contour discussed above for the Fourier integral operator $\widetilde{T}S$ and putting everything together, we get

$$\widetilde{T}u = (\widetilde{T}S)(Tu) \text{ in } H_{\widetilde{\Phi},\widetilde{x}_0}.$$

Since the Fourier integral operator $\widetilde{T}S$ maps $H_{\Phi,x_0} \to H_{\widetilde{\Phi},\widetilde{x}_0}$, we see that $\widetilde{T}u = 0$ in $H_{\widetilde{\Phi},\widetilde{x}_0}$ if $Tu = 0$ in $H_{\widetilde{\Phi},\widetilde{x}_0}$. The converse implication also holds. □

This shows that the definition of $\mathrm{WF}_a(u)$ does not depend on the choice of T. By a simple dilation in h we then see that it is a conic subset of $T^*X \setminus 0$ (if $X \subset \mathbf{R}^n$ is the open set where u is defined). Another basic property of the analytic wavefront set is given by

Proposition 2.6.5 *We have*

$$\pi_y(\mathrm{WF}_a(u)) = \mathrm{Sing\,Supp}_a(u),$$

where the right hand side denotes the analytic singular support, i.e. the complement in X of the largest open subset where u is real analytic.

Idea of the proof. We start by using a resolution of the identity of the form $1 = \int_{T^*\mathbf{R}^n} \pi_\alpha d\alpha$ where π_α is a Gaussian Fourier integral operator "concentrated at α". If $y_0 \notin \pi_y(\mathrm{WF}_a(u))$, then a simple adaptation of the proof above shows that $\pi_\alpha u$ decays exponentially when α_η tends to infinity while α_y is confined to a small neighborhood of y_0. (Here we write $\alpha = (\alpha_y, \alpha_\eta)$.)

Solution to the second exercise in this section. Ignoring the cutoff to a neighborhood of $y = y_0$, we write

$$T(e^{if/h})(x) = \int_{\mathrm{neigh}\,(y_0,\mathbf{R}^n)} a(x, y; h)e^{\frac{i}{h}(\phi(x,y)+f(y))}dy, \tag{2.6.11}$$

where

$$-\operatorname{Im}(\phi(x, y) + f(y)) - \Phi(x) \asymp -|y - y(x)|^2, \ \ y \in \mathrm{neigh}\,(y_0, \mathbf{R}^n). \tag{2.6.12}$$

When $x = x_0$, the function

$$y \mapsto \phi(x, y) + f(y) \tag{2.6.13}$$

has a critical point at $y = y_0$ which is nondegenerate by (2.6.12) and by the same relation we know that neigh $(y_0, \mathbf{R}^n)$ is a good contour in (2.6.11). For $x \in$ neigh $(x_0, \mathbf{C}^n)$, the function (2.6.13) has a nondegenerate critical point $y_c(x) \in$ neigh $(y_0, \mathbf{C}^n)$, depending holomorphically on x with $y_c(x_0) = y_0$ and in (2.6.11) we can shift the contour neigh $(y_0, \mathbf{R}^n)$ to $y_c(x) + \text{neigh}\,(0, \mathbf{R}^n)$, then apply stationary phase to get (2.6.4), (2.6.5).

From (2.6.5) we get $\Lambda_g = \kappa_T(\Lambda_f)$, moreover, since $e^{if/h}$ is bounded on the real domain, we must have $|e^{ig/h}| \leq e^{\Phi/h}$, i.e. $-\mathrm{Im} g \leq \Phi$. We also see that $-\mathrm{Im} g = \Phi$ on $\pi_x\kappa_T((\Lambda_f)_{\mathbf{R}})$, since this set is the set of points x for which $y_c(x)$ is real.

One way to get (2.6.6) is to notice that $\Gamma := \pi_x\kappa_T((\Lambda_f)_{\mathbf{R}})$ is a maximally totally real manifold ($\mathbf{C}^n = T_x\Gamma \oplus i(T_x\Gamma)$ for all $x \in \Gamma$), that $-\mathrm{Im} g$ is pluriharmonic so that $\Phi(x) + \mathrm{Im} g(x)$ is strictly plusrisubharmonic, ≥ 0 with equality on Γ. It then follows that $\Phi(x) + \mathrm{Im} g \asymp \mathrm{dist}\,(x, \Gamma)^2$.

A more direct way is to observe that $|\mathrm{Im} y_c(x)| \asymp \mathrm{dist}\,(x, \Gamma)$ and the saddle point method (for instance via the minimax formula) shows that for the critical value

$$
\begin{aligned}
&- \mathrm{Im} g(x) = -(\mathrm{Im}\phi(x, y_c) + f(y_c)) \\
&\leq \inf_{y \in \mathrm{neigh}\,(y_0, \mathbf{R}^n)} (-\mathrm{Im}\phi(x, y) + f(y)) - \frac{1}{C}|\mathrm{Im} y_c|^2 \\
&= \Phi(x) - \frac{1}{C}|\mathrm{Im} y_c|^2.
\end{aligned}
$$

2.7 Egorov's Theorem and Elliptic Regularity

Let $\widetilde{P}(y, D_y) = \sum_{|\alpha| \leq m} a_\alpha(y) D_y^\alpha$ be a differential operator with analytic coefficients, defined on an open set $X \subset \mathbf{R}^n$. Let T be an FBI-transform as above. Then we have the Egorov theorem which states that there exists a pseudodifferential operator with classical analytic symbol, $P(x, hD_h; h) : H_{\Phi,x_0} \to H_{\Phi,x_0}$ such that

$$
PTu = Th^m \widetilde{P}u \text{ in } H_{\Phi,x_0}
$$

when $u \in \mathcal{D}'(X)$ is independent of h. Indeed, we can take $P = Th^m \widetilde{P} S$. For the leading symbols, we have the relation

$$
p \circ \kappa_T = \widetilde{p}. \tag{2.7.1}
$$

Theorem 2.7.1 *In the above situation, let $u \in \mathcal{D}'(X)$ be independent of h and assume that $\widetilde{P}u$ is analytic on X. Then* $\mathrm{WF}_a(u) \subset \widetilde{p}^{-1}(0)$.

Proof Let $(y_0, \eta_0) \in T^*X \setminus 0$ be a point where $\widetilde{p}(y_0, \eta_0) \neq 0$ and assume that $(y_0, \eta_0) \notin \mathrm{WF}_a(\widetilde{P}u)$ (which is a weaker assumption than in the theorem). We choose T adapted to the point (y_0, η_0). Then

$$
PTu = 0 \text{ in } H_{\Phi,x_0} \text{ and } p\left(x_0, \frac{2}{i}\partial_x\Phi(x_0)\right) \neq 0.
$$

Let $Q(x,\xi;h)$ be a classical analytic symbol $Q\sim\sum_0^\infty h^k q_k(x,\xi)$ such that

$$Q\#P=1 \text{ near } (x_0,\xi_0).$$

Correspondingly, we have $Q(x,hD;h): H_{\Phi,x_0}\to H_{\Phi,x_0}$ so that

$$Q(x,hD;h)\circ P(x,hD;h)=1: H_{\Phi,x_0}\to H_{\Phi,x_0}.$$

Apply this to Tu:

$$Tu=QPTu=0 \text{ in } H_{\Phi,x_0}.$$

Hence $(y_0,\eta_0)\notin \mathrm{WF}_a(u)$. We have thus shown that $\mathrm{WF}_a(u)\subset \mathrm{WF}_a(\widetilde{P}u)\cup\widetilde{p}^{-1}(0)$ which is a stronger statement than in the theorem. □

For the notes of a course of more than 3 hours, it would here be the natural place to discuss the method of non-characteristic deformations and the Kawai-Kashiwara theorem about propagation of analytic regularity for micro-hyperbolic operators. See [65], Chap. 10.

2.8 Analytic WKB and Quasi-modes

Let $P(x,hD;h)$ be a classical analytic pseudodifferential operator of order 0, defined near $(0,\xi_0)\in\mathbf{C}^{2n}$, such that the leading symbol satisfies

$$p(0,\xi_0)=0,\ \partial_{\xi_n}p(0,\xi_0)\neq 0.$$

Let $\phi\in \mathrm{Hol}\,(\mathrm{neigh}\,(0,\mathbf{C}^n))$ solve the eikonal problem

$$p(x,\phi'(x))=0,\ \phi'(0)=\xi_0. \tag{2.8.1}$$

Let H be the hypersurface $x_n=0$. We use the standard notation $x=(x',x_n)\in\mathbf{C}^n$.

Theorem 2.8.1 *Let $v(x;h)$, $w(x';h)$ be classical analytic symbols of order 0 defined near 0 in $\mathbf{C}^n$ and $\mathbf{C}^{n-1}$ respectively. Then there exists a classical analytic symbol $u(x;h)$ defined near $0\in\mathbf{C}^n$ such that*

$$e^{-i\phi(x)/h}\circ P\circ e^{i\phi/h}u=hv,\ u_{|H}=w. \tag{2.8.2}$$

Proof We may assume that $w=0$. Also $e^{-i\phi(x)/h}\circ P\circ e^{i\phi/h}$ is a classical analytic pseudodifferential operator of order 0 with leading symbol $p(x,\phi'_x(x)+\xi)$, so we may assume that $\phi=0$, $p(x,0)=0$. After a change of variables, which does not modify H, we may also assume that $\partial_{\xi'}p(x,0)=0$, $\partial_{\xi_n}p=i$, or in other words, $p(x,\xi)=i\xi_n+\mathcal{O}(\xi^2)$.

Writing $P=\sum_0^\infty h^k p_k(x,\xi)$, $p_0=p$, the first equation in (2.8.2) becomes

$$\partial_{x_n} u + p_1(x,0)u(x;h) + \frac{1}{h}Au = v$$
$$A = \sum_{k+|\alpha|\geq 2} \frac{h^k}{\alpha!}(\partial_\xi^\alpha p_k)(x,0)(hD_x)^\alpha = \sum_{k=2}^{\infty} h^k A_k, \tag{2.8.3}$$

where A has the same general properties as in Sect. 2.2. Assume for simplicity that $p_1(x,0) = 0$ (which otherwise can be achieved by conjugation).

Let $\Omega = \{x \in \mathbf{C}^n;\ \frac{|x'|}{R} + \frac{|x_n|}{r} < 1\}$, where $R, r > 0$ are small enough so that we stay in the domains of definition of the various symbols and operators. For $0 \leq t \leq r$, we define $\Omega_t \subset \mathbf{C}^n$ by

$$\frac{|x'|}{R - \frac{Rt}{r}} + \frac{|x_n|}{r-t} < 1.$$

Let $a \in \mathrm{Hol}\,(\Omega_0)$ have the property that for some $k > 1$:

$$\sup_{\Omega_t} |a| \leq C(a,k)t^{-k},\ \ 0 < t \leq r.$$

Put

$$\partial_{x_n}^{-1} a(x) = \int_0^{x_n} a(x', y_n)dy_n.$$

Then

$$\sup_{\Omega_t} |\partial_{x_n}^{-1} a| \leq C(a,k)\int_t^{+\infty} s^{-k}ds = \frac{C(a,k)}{(k-1)t^{k-1}}.$$

Let $a = \sum_2^\infty a_k h^k$ be a classical analytic symbol of order -2 such that

$$\sup_{\Omega_t} |a_k| \leq \frac{f(a,k)k^k}{t^k},\ \ 0 < t \leq r, \tag{2.8.4}$$

where $k \mapsto f(a,k)$ grows at most exponentially. Then,

$$b := (h\partial_{x_n})^{-1} a = \sum_1^\infty b_k h^k,\ \ b_k = \partial_{x_n}^{-1} a_{k+1},$$

$$\sup_{\Omega_t} |b_k| \leq \frac{f(a,k+1)(k+1)^{k+1}}{kt^k} \leq 2ef(a,k+1)\frac{k^k}{t^k}.$$

Hence, $f(b,k) \leq 2ef(a,k+1)$, when defining $f(b,k)$ as in (2.8.4).

Put

$$\|a\|_\rho = \sum_2^\infty f(a,k)\rho^k,\ \ \|b\|_\rho = \sum_1^\infty f(b,k)\rho^k.$$

Then

$$\|b\|_\rho \le \frac{2e}{\rho}\|a\|_\rho. \tag{2.8.5}$$

The problem (2.8.2), (2.8.3), with $w = 0$ and $p_1(x, 0) = 0$, can be written

$$u + (h\partial_{x_n})^{-1}Au = h(h\partial_{x_n})^{-1}v =: \widetilde{v}, \tag{2.8.6}$$

where $\widetilde{v}$ is a classical analytic symbol of order 0. Defining $\|A\|_\rho$ as in Sect. 2.2 with respect to the family Ω_t, we have

$$\|Au\|_\rho \le \|A\|_\rho\|u\|_\rho \le \mathcal{O}(\rho^2)\|u\|_\rho,$$

when ρ is small enough. Hence by (2.8.5),

$$\|(h\partial_{x_n})^{-1}Au\|_\rho \le \mathcal{O}(1)\rho\|u\|_\rho.$$

We then see from (2.8.6) that $\|u\|_\rho < \infty$ when $\rho > 0$ is small enough and we conclude that u is an analytic symbol in Ω_0. □

We next discuss *quasimodes for non-self-adjoint differential operators* in the semi-classical limit. Let

$$P = P(x, hD_x; h) = \sum_{|\alpha|\le m} a_\alpha(x; h)(hD_x)^\alpha$$

be a semi-classical differential operator defined on an open set $\Omega \subset \mathbf{R}^n$. Assume that

$$a_\alpha(x; h) \sim \sum_0^\infty a_\alpha^k(x)h^k \tag{2.8.7}$$

are (realizations of) classical analytic symbols. The semi-classical principal symbol of P is then

$$p(x, \xi) = \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha. \tag{2.8.8}$$

Let $(x_0, \xi_0) \in T^*\Omega$ be a point where

$$p(x_0, \xi_0) = 0, \quad \frac{1}{2i}\{p, \overline{p}\}(x_0, \xi_0) > 0. \tag{2.8.9}$$

Here, $\{a, b\} = a'_\xi \cdot b'_x - a'_x \cdot b'_\xi$ denotes the Poisson bracket of two sufficiently smooth functions $a(x, \xi)$, $b(x, \xi)$. The following result, in a different non-semi-classical formulation is due to Hörmander [29, 30] in the smooth setting and goes back to Sato-Kawai-Kashiwara [58] in the analytic case. See [7] for references and direct proofs in the semi-classical formalism.

Theorem 2.8.2 *There exist an analytic function $\phi(x)$ and a classical analytic symbol $b(x;h)$ of order 0, defined in a neighborhood of x_0 such that*

$$\phi(x_0)=0,\ \ \phi'(x_0)=\xi_0, \tag{2.8.10}$$

$$p(x,\phi'(x))=0,\ x\in \mathrm{neigh}\,(x_0,\Omega), \tag{2.8.11}$$

$$\mathrm{Im}\phi''(x_0)>0, \tag{2.8.12}$$

$$P(\chi(x)b(x;h)e^{i\phi(x)/h})=\mathcal{O}(1)e^{-\frac{1}{Ch}},\ C=C_\chi>0, \tag{2.8.13}$$

if $\chi\in C_0^\infty(\mathrm{neigh}\,(x_0,\Omega))$ is equal to 1 near x_0 and has its support sufficiently close to x_0,

$$\|\chi b e^{i\phi/h}\|_{L^2}=h^{n/4}(1+\mathcal{O}(e^{-1/(Ch)})). \tag{2.8.14}$$

As usual, it follows from the proof that the conclusion remains uniformly valid if we replace P by $P-z$ for $z\in \mathrm{neigh}\,(0,\mathbf{C})$. More generally the conclusion is valid for $P-z$ for $z\in\mathrm{neigh}\,(z_0,\mathbf{C})$, if we replace the condition $p(x_0,\xi_0)=0$ by $p(x_0,\xi_0)=z_0$ in (2.8.9).

When P can be realized as a closed operator on $L^2(\Omega)$ or on $L^2(M)$ for some manifold containing Ω, then we conclude that $\|(P-z)^{-1}\|\ge e^{1/(Ch)}/C$ for some $C>0$ and for $z\in\mathrm{neigh}\,(z_0,\mathbf{C})\setminus\sigma(P)$, where $\sigma(P)$ denotes the spectrum of P. Notice that $i^{-1}\{p,\overline{p}\}$ is the semi-classical principal symbol of the commutator $h^{-1}[P,P^*]$, so P is non-normal.

When P is a fixed elliptic operator in the classical sense, with analytic h-independent coefficients, the result with some obvious modifications applies to $P-z$ when z tends to infinity in a narrow sector.

We refer to [7] for a fuller discussion of the spectral aspects.

Proof of Theorem 2.8.2. The assumption (2.8.9) implies that $p'_\xi(x_0,\xi_0)\neq 0$. The existence of analytic solutions to (2.8.10), (2.8.11) then follows from complex Hamilton-Jacobi theory or simply from the Cauchy–Kowalevska theorem. More precisely, if H is a complex hypersurface in x-space that passes through x_0 transversally to $p'_\xi(x_0,\xi_0)\cdot\partial_x$ and ψ is holomorphic on neigh (x_0,H) with $d\psi=\xi_0\cdot dx_{|H}$ at x_0, then (2.8.10), (2.8.11) has a solution ϕ such that $\phi_{|H}=\psi$, unique near x_0.

For (2.8.12) we recall a geometric characterization by Hörmander [32]. Let Λ_ϕ be the complex Lagrangian manifold defined near (x_0,ξ_0) by $\xi=\phi'(x)$ where $\phi(x)$ is holomorphic near x_0 and $\phi'(x_0)=\xi_0$. Then,

- (2.8.12) $\Longrightarrow$
$$\frac{1}{i}\sigma(t,\overline{t})>0,\ \forall t\in T_{x_0,\xi_0}(\Lambda_\phi)\setminus\{0\}, \tag{2.8.15}$$

  where we view the symplectic form σ as an alternate bilinear form.

- If Λ is a complex Lagrangian manifold containing (x_0, ξ_0) such that (2.8.15) holds, then after restricting Λ to a small neighborhood of (x_0, ξ_0), we get $\Lambda = \Lambda_\phi$, where ϕ is holomorphic near x_0 and satisfies (2.8.10), (2.8.12).

The geometric formulation of the problem (2.8.10)–(2.8.12) is then to find a complex Lagrangian manifold $\Lambda \subset \Gamma := p^{-1}(0)$ which contains (x_0, ξ_0) and is strictly positive in the sense of (2.8.15). Notice that the strict positivity of Λ at (x_0, ξ_0) implies that Λ intersects $T^*\Omega$ transversally at (x_0, ξ_0).

Here $\Gamma = p^{-1}(0)$ denotes the *complex* hypersurface and we recall that H_p is tangent to Γ. We also know by elementary symplectic geometry that H_p is everywhere tangent to Λ.

Let $\Sigma = p^{-1}(0) \cap \mathrm{neigh}\,((x_0, \xi_0), T^*\Omega)$ be the real characteristic manifold. It is symplectic and of codimension 2. Let $\Sigma^{\mathbf{C}} \subset \mathrm{neigh}\,((x_0, \xi_0), \mathbf{C}^{2n})$ denote its complexification. It is a complex symplectic manifold of codimension 2 in $\mathbf{C}^{2n}$, given by the equations $p(\rho) = 0$, $p^*(\rho) = 0$, where $p^*(\rho) = \overline{p(\overline{\rho})}$. The assumption (2.8.9) implies that $\Sigma^{\mathbf{C}}$ is a complex hypersurface in Γ, given there by the equation $p^*(\rho) = 0$, transversal to H_p since $H_p p^* = \{p, \overline{p}\} \neq 0$.

It is now clear that the complex Lagrangian manifolds Λ with $(x_0, \xi_0) \in \Lambda \subset \mathrm{neigh}\,((x_0, \xi_0), \Gamma)$ coincide near that point with the ones of the form

$$\{\exp(zH_p)(\rho');\ \rho' \in \Lambda',\ z \in D(0, \varepsilon)\},$$

where $\varepsilon > 0$ is small and Λ' is a complex Lagrangian submanifold of $\Sigma^{\mathbf{C}}$ containing (x_0, ξ_0). By the Darboux theorem, Σ, $\Sigma^{\mathbf{C}}$ can locally be identified with $\mathbf{R}^{2(n-1)}$, $\mathbf{C}^{2(n-1)}$, and we see that Λ is strictly positive at (x_0, ξ_0) iff Λ' is. Indeed, a general $t \in T_{(x_0,\xi_0)}\Lambda$ is of the form $t = t' + zH_p(x_0, \xi_0)$, for $t' \in T_{(x_0,\xi_0)}\Lambda'$, $z \in \mathbf{C}$ and since $\sigma(t', H_p) = \sigma(t', \overline{H_p}) = 0$, we get

$$\begin{aligned}\frac{1}{2i}\sigma(t, \overline{t}) &= \frac{1}{2i}\sigma(t', \overline{t'}) + \frac{|z|^2}{2i}\sigma(H_p, \overline{H_p}) \\ &= \frac{1}{2i}\sigma(t', \overline{t'}) + \frac{|z|^2}{2i}\{p, \overline{p}\} \asymp |t'|^2 + |z|^2 \asymp |t|^2.\end{aligned}$$

Now there are plenty of strictly positive Lagrange manifolds $\Lambda' \subset \Sigma^{\mathbf{C}}$ passing through (x_0, ξ_0) and hence there are plenty of strictly positive Lagrange manifolds $\Lambda \subset \Gamma$ containing that point. This means that we have plenty of solutions to the problem (2.8.10)–(2.8.12).

We choose one such solution $\phi(x)$ and apply Theorem 2.8.1 to conclude that there exists an elliptic classical analytic symbol $b(x; h) \sim \sum_0^\infty b_k(x)h^k$ such that formally,

$$P(x, hD; h)(b(x; h)e^{i\phi(x)/h}) = 0,\ x \in \mathrm{neigh}\,(x_0, \Omega).$$

This means that (if b also denotes a realization as in Theorem 2.8.2)

$$P(x, hD_x; h)(be^{i\phi/h}) = \mathcal{O}(e^{-1/(Ch)})e^{i\phi/h}.$$

From (2.8.12) we see that $e^{i\phi(x)/h}$ is exponentially decaying on the real domain away from any fixed neighborhood of x_0. Thus, if χ is a cutoff as in the statement of the theorem,

$$P(\chi b e^{i\phi/h}) = \mathcal{O}(e^{-1/(Ch)}).$$

By analytic stationary phase,

$$\|\chi b e^{i\phi/h}\|^2_{L^2} = h^{\frac{n}{2}} c(h),$$

where $c(h) \sim c_0 + c_1 h + \cdots$ is a positive elliptic analytic symbol. Applying the quasinorms of Sect. 2.2 (that simplify a lot since the family Ω_t is absent), we see that $c^{-1/2}$ is a classical analytic symbol. Replacing b with $c^{-1/2}b$, we get (2.8.13), (2.8.14).

2.9 Propagation of Regularity Along a Real Bicharacteristic Strip

Let P be a differential operator with analytic coefficients on an open set $X \subset \mathbf{R}^n$. Let p be the principal symbol. The following theorem is due to N. Hanges [16]. It improves the classical propagation theorem of L. Hörmander [33] and Sato, Kawai and Kashiwara [58] for operators of real principal type in that it only requires one real bicharacteristic strip. See also [17].

Theorem 2.9.1 *Assume that* $H_p = p'_\xi \cdot \partial_x - p'_x \cdot \partial_\xi$ *has a real integral curve* $\gamma : [a, b] \to p^{-1}(0) \cap T^*X \setminus 0$, $a < b$. *If* $u \in \mathcal{D}'(X)$, $\mathrm{WF}_a(Pu) \cap \gamma([a, b]) = \emptyset$, *then* $\gamma([a, b])$ *is either contained in, or disjoint from* $\mathrm{WF}_a(u)$.

The proof uses a WKB-construction and the variant we give here is slightly different from the one in Chap. 9 in [65].

If dp vanishes at some point of γ, then γ is reduced to a point and the statement in the theorem becomes trivial. Hence, we may assume that $dp \neq 0$ along γ.

Theorem 2.9.2 *Assume that* $p(y_0, \eta_0) = 0$, $dp(y_0, \eta_0) \neq 0$. *Then we can find an FBI-transform* T *defined near* (y_0, η_0) *such that* $hD_{x_n} Tu = Th^m Pu$ *in* H_{Φ, x_0}, *for* $u \in \mathcal{D}'(X)$ *independent of* h.

Proof We start with the phase. For $(x_0, y_0) \in \mathbf{C}^n \times \mathbf{R}^n$ we call

$$\phi \in \mathrm{Hol}\,(\mathrm{neigh}\,((x_0, y_0), \mathbf{C}^{2n}))$$

an FBI-phase if it fulfills (2.6.1).

Lemma 2.9.3 *There exists an FBI-phase* $\phi(x, y)$, *defined near* (x_0, y_0) *such that*

$$\partial_{x_n} \phi = p(y, -\partial_y \phi(y)). \tag{2.9.1}$$

Proof We put

$$\phi(x', 0, y) = \frac{i}{2}(x' - y')^2 - \eta_{0,n} y_n + iC(y_n - y_0)^2,$$

and choose $x_0 = (y_0' - i\eta_0', 0)$. Here C will be chosen with $\mathrm{Re}C > 0$. Then $\phi'_y((x_0', 0), y_0) = -\eta_0$ and we let $\phi(x, y)$ be the corresponding local solution of (2.9.1). Then ϕ fulfills the first two conditions in (2.6.1). In order to have $\det \phi''_{xy}(x_0, y_0) \neq 0$, we may assume, after a change of coordinates in y, that

$$\partial_{\eta_n} p(y_0, \eta_0) \neq 0, \text{ or } [\partial_\eta p(y_0, \eta_0) = 0 \text{ and } \partial_{y_n} p(y_0, \eta_0) \neq 0.]$$

Then we can find C with $\mathrm{Re}C > 0$ such that

$$\partial_{y_n}(p(y, -\partial_y \phi)) \neq 0 \text{ at } (x_0, y_0). \tag{2.9.2}$$

Now the following statements are equivalent:

- $\det \phi''_{xy}(x_0, y_0) \neq 0$,
- $y \mapsto \partial_x \phi$ has bijective differential at $x = x_0$, $y = y_0$,
- $y \mapsto (\partial_{x'}\phi, p(y, -\partial_\eta \phi))$ has bijective differential at $x = x_0$, $y = y_0$,
- Equation (2.9.2).

The last equivalence follows from

$$\det \phi''_{x',y'} \neq 0, \ \phi''_{y_n,x'} = 0 \text{ at } (x_0, y_0).$$

Thus ϕ is an FBI-phase. □

We can now finish the proof of the last theorem. Take ϕ as in the lemma. It suffices to choose a in (2.6.2) such that

$$(hD_{x_n} - h^m P^{\mathrm{t}}(y, D_y)) \left(e^{i\phi(x,y)/h} a(x, y; h)\right) = 0,$$

which we can solve locally as in the preceding section with a prescribed $a(x', 0, y; h)$. □

Proof of Hanges' theorem: We may decompose $[a, b]$ into finitely many short intervals, each being covered by one FBI transform. Thus we may assume that $\gamma([a, b])$ is contained in a small neighborhood of (y_0, η_0). Let T be a corresponding FBI transform as in the last theorem. Then $\kappa_T \circ \gamma$ is an integral curve in Λ_Φ of $H_{\xi_n} = \partial_{x_n}$ on which ξ_n vanishes. Assume for simplicity that $x_0 = 0$. Then we know that

$$\frac{2}{i}\partial_x \Phi(0, t) = \xi_0 = (\xi_0', 0)$$

and consequently $\Phi(x) = -\mathrm{Im}(x' \cdot \xi_0') + \mathcal{O}(x'^2)$.

By the intertwining property and the fact that $\gamma([a,b])$ is disjoint from $\mathrm{WF}_a(Pu)$, we know that

$$hD_{x_n}Tu = 0 \text{ in } H_\Phi(\mathrm{neigh}(\{0\}\times[a,b], \mathbf{C}^n)),$$

so by integration,

$$Tu = v(x') + \mathcal{O}(e^{-\mathrm{Im}\,(x'\cdot\xi_0')/h-\epsilon/h}) \text{ near } \{0\}\times[a,b].$$

Consequently, if $Tu = 0$ in $H_{\Phi,\gamma(t)}$ for some $t\in[a,b]$ we have the same fact for all $t\in[a,b]$. In other words, if $\gamma(t)\notin \mathrm{WF}_a(u)$ for some $t\in[a,b]$, the same must hold for all $t\in[a,b]$.

2.10 Some Further Comments

This section is motivated by questions and comments by the referee. We have described some elements of the *microlocal* theory in book [65], which does not try to develop any global, constructive operator theory, but as mentioned at the end of Sect. 2.1 it can be used to describe singularities and asymptotics of globally defined operators and functions.

In particular it could be of interest to apply analytic microlocal analysis to spectral theory when the operators and the underlying manifolds are analytic (though this was not a major motivation 36 years ago). For instance, if Δ is the Laplace–Beltrami operator on a compact real analytic Riemannian manifold, we could consider $\sqrt{-\Delta}$ and the associated unitary group $t\mapsto \exp -it\sqrt{-\Delta}$. There is no doubt that $\sqrt{-\Delta}$ has (or can) be constructed with the methods of [3, 77]. However, we think that the methods above would lead to at least equally sharp information about these operators:

If T is an FBI-transform, that we can choose locally unitary (up to exponentially small errors) $L^2\to H_{\Phi_0}$, let P be the "image of $-\Delta$", defined by

$$PT = T(-\Delta)$$

(up to negligible errors). Then P is a pseudodifferential operator in H_{Φ_0} and we can define $P^{1/2}$ by using the standard functional formula,

$$P^{1/2} = \frac{1}{2\pi i}\int_\gamma (z-P)^{-1}z^{1/2}dz, \tag{2.10.1}$$

where γ is the positively oriented boundary of the sector $-1+e^{i[-\pi/4,\pi/4]}[0,+\infty[$ and we would obtain that $P^{1/2}$ is a classical analytic pseudodifferential operator in H_{Φ_0}, which is self-adjoint and satisfies $P^{1/2}T = T(-\Delta)^{1/2}$. (To show this we could need to study the resolvent also for large z in the spirit of R. Seeley, or more radically, let γ be a closed contour since we work microlocally always in a region where the leading symbol of P is bounded.) After that, the methods of Sects. 2.5, 2.8 should permit us to study $\exp -itP^{1/2}$ with no limitation on the size of t, since crucial caustics cannot appear in this setting. This group will be the conjugation of $\exp -it(-\Delta)^{1/2}$ under T, so we get access to the FBI transforms of the latter operator, and we could in principle study its trace in the analytic framework.

2.11 Related Results and Developments

The work [65] was the natural continuation of a series of works on the propagation of singularities for solutions of boundary value problems of order 2 and higher in the analytic category, [54, 60–64]. In the case of second order operators, the main result here is that the analytic wavefront set for solutions to homogenous problems is a union of maximally extended analytic rays (and a more general microhyperbolic propagation theorem for operators of higher order). This is analogous to the corresponding result in the C^∞ by M. Taylor, R. Melrose, G. Eskin, V. Ivrii, culminating in [51, 52], stating that the ordinary C^∞ wavefront set of solutions to the homogeneous problem is a union of maximally extended C^∞-rays. Such rays have (with the exception of some slightly pathological cases) unique extensions while analytic rays have non-unique extensions from points where they are tangential to the boundary and the domain is concave in the ray direction so that the complement, that we may call "the obstacle", is convex in the same direction. Roughly, analytic rays may glide along the boundary into the C^∞ shadow region.

The methods used another kind of FBI-transforms, closely related to Gaussian resolutions of the identity. In [65] such resolutions still play a role, while in the present text, we have eliminated them completely. It would have been nice if there had been time and energy to revisit the boundary propagation in [65] with the improved methods there.

G. Lebeau [43] explored the propagation of singularities for the wave equation outside a strictly convex obstacle in the whole scale of Gevrey spaces G^s that interpolate between the smooth and the analytic functions and found that the essential difference between the two types of propagations appears at the value $s = 3$. See also [42].

A related area is that of analytic hypoellipticity for non-elliptic operators. Here F. Treves [76] and later D. Tartakoff [75] established analytic hypoellipticity for operators of the type $\Box_b$ that degenerate to order 2 on a symplectic submanifold of the real cotangent space. The approach of Treves is based on a full fledged machinery of analytic pseudodifferential operators and reductions to model-like cases while the one of Tartakoff is restricted to a more special class of operators and uses very sophisticated iterated a priori-estimates to gain control of high order derivatives directly. G. Métivier [53] in a still very long paper generalized the results to operators with multiple characteristics following the general approach of Treves.

In [66] the second author gave a short proof of Métivier's result as well as some generalizations. We refer to [13, 15] for some related results. The method of [66] is that of subelliptic deformations: After an FBI-transform we work in a space $H^{\mathrm{loc}}_{\Phi_0}$ for some strictly plurisubharmonic weight Φ_0 and the given subelliptic operator satisfies an a priori-estimate in that space. We then look for a small deformation $\Phi \approx \Phi_0$ such that P satisfies a nice a priori estimate also in H^{loc}_{Φ} and such that $\Phi < \Phi_0$ where we want to obtain analytic regularity and $\Phi \geq \Phi_0$ near the boundary of a neighborhood of those points. A variant of the method used when we have micro-hyperbolicity, is to make deformations such that the operator on the FBI-side is elliptic on Λ_Φ, $\Phi < \Phi_0$ in a region where we want to gain analytic regularity and such that on the boundary of a slightly larger region we have that $\Phi > \Phi_0$ only at points where we already have analytic regularity by assumption. The deformation of weights on the FBI-side corresponds to a local deformation $\kappa_T^{-1}(\Lambda_\Phi)$ of the real phase space $T^*\Omega$ (locally equal to $\kappa_T^{-1}(\Lambda_{\Phi_0})$). See [62, 65].

In the theory of scattering poles (resonances) and other branches of spectral theory for non-self-adjoint (pseudo-)differential operators, many works rely on phase space deformations which are now global. Since this activity started later we simply refer to some of the works which also include some of those devoted to other global questions: [2, 7, 11, 12, 18, 19, 21–25, 27, 28, 38, 40, 41, 47–50, 55–57, 59, 67–72, 74].

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A Proof of a Result of L. Boutet de Monvel

Gilles Lebeau

Abstract We give a detailed proof of a theorem of L. Boutet de Monvel formulated in 1978 in (C.R.A.S. Paris, t.287, série A, 855–856, 1978) [2] about the convergence in the complex domain of sums of eigenfunctions of the Laplace operator on a compact analytic manifold.

1 Introduction

These notes are a written version of a 3 h course given at Northwestern university in may 2013. The main purpose is to give a detailed proof of a theorem of L. Boutet de Monvel formulated in 1978 [2]. There is no need to have any knowledge about analytic microlocal analysis to read these notes. The only "analytic" things that we will use are: Cauchy–Kowalewski theorem, Zerner-lemma, and the analytic regularity for solutions of elliptic linear differential operator with analytic coefficients. Moreover, we will only use basic facts on classical pseudodifferential calculus and wave front sets, for which we refer to [5].

Let (M, g) be a compact, connected, analytic Riemannian manifold of dimension m. Let us recall that the metric g on the tangent bundle TM gives a canonical identification of TM with the cotangent bundle T^*M. Let d_gx be the volume form on M associated to the metric g. The Laplace operator $\triangle_g$ on M is defined by the formula

$$\int_M \triangle_g(u)\overline{v}d_gx = -\int_M (du|dv)d_gx \tag{1.1}$$

Here df denotes the differential of the function f so one has by definition $\triangle_g = -d^*d$ where d^* is the adjoint of d for the natural Hilbert structure induced by g on

G. Lebeau (✉)
Département de Mathématiques, Université de Nice Sophia-Antipolis, Parc Valrose, 06108 Nice Cedex 02, France
e-mail: lebeau@unice.fr

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sections of T^*M. The unbounded operator $-\triangle_g$ with domain $H^2(M)$ is self-adjoint on $L^2(M, d_g x)$, non negative, with compact resolvant. We will denote by $(e_j)_{j\geq 0}$ an orthonormal basis of $L^2(M, d_g x)$ of *real* eigenfunctions of $-\triangle_g$ associated to the eigenvalues ω_j^2, with $\omega_0 = 0 < \omega_1 \leq \omega_2 \leq ...$, $\lim_{j\to\infty} \omega_j = +\infty$, so that one has

$$- \triangle_g(e_j) = \omega_j^2 e_j, \quad \int_M e_j e_k d_g x = \delta_{j,k} \tag{1.2}$$

Since $\triangle_g$ is a second order elliptic operator with analytic coefficients, the eigenfunctions e_j are real analytic functions on M.

Let X be a complexification of M. This means that X is a complex analytic manifold of complex dimension m, and $M \subset X$ is a totally real submanifold of X (this means $TM \cap iTM = M$ where $M \subset TM$ is view as the zero section). Let $d(x, y)$ be the distance function on $M \times M$. Then $d^2(x, y)$ is an analytic function near the diagonal $Diag_M = \{(x, x), x \in M\} \subset M \times M$, and therefore extends as an holomorphic function in a complex neighborhood of $Diag_M$ in $X \times X$. Let us define $\Phi(z)$ by the formula

$$\Phi(z) = \frac{1}{2} \sup_{y\in M} Re(-d^2(z, y)) \tag{1.3}$$

We will see in Sect. 3, Lemma 3.3, that this function is well defined for $z \in X$ close to M, and is real analytic and strictly pluri-subharmonic. Moreover, one has $\Phi|_M = 0$, $d\Phi|_M = 0$ and the signature of the Hessian of Φ is equal to $(m, 0)$ at any point of M; in particular, one has $\Phi(z) \geq 0$ and $\Phi(z) = 0$ if and only if $z \in M$.[1]

This function allows to define, for $\epsilon > 0$ small enough, the tubular neighborhood B_ϵ of M in X

$$B_\epsilon = \left\{ z \in X, \ \Phi(z) < \frac{\epsilon^2}{2} \right\} \tag{1.4}$$

Let us denote by $\mathcal{O}(B_\epsilon)$ the space of holomorphic functions defined on B_ϵ. For $f \in \mathcal{O}(B_\epsilon)$, its boundary value $f|_{\partial B_\epsilon}$ on ∂B_ϵ is well defined as an hyperfunction on ∂B_ϵ which is an analytic compact real manifold of dimension $2m - 1$. This boundary value is a distribution on ∂B_ϵ if and only if the function f satisfies a polynomial growth condition at the boundary of the form $|f(z)| \leq C dist(z, \partial B_\epsilon)^{-N}$. Let us recall that the Hardy space $H(B_\epsilon)$ is the Hilbert space defined by

$$H(B_\epsilon) = \{f \in \mathcal{O}(B_\epsilon), \ f|_{\partial B_\epsilon} \in L^2(\partial B_\epsilon)\} \tag{1.5}$$

We can now state the Boutet theorem formulated in [2] (in a slightly different but equivalent form). Let us recall that a family $(u_j)_{j\geq 0}$ is a Riesz basis of an Hilbert space H if and only if any $x \in H$ can be written in a unique way as the sum of a

[1]The function $\Phi(z)$ is one half of the square of the Grauert tube function introduced by Guillemin-Stenzel [3] and Lempert-Szoke [7], namely $\Phi(z) = -\frac{1}{8} d^2(z, \bar{z})$.

convergent series in H, $x = \sum c_j(x)u_j$ and $\sum |c_j(x)|^2$ is equivalent to $\|x\|_H^2$. We use the classical notation $< x >= (1 + x^2)^{1/2}$.

Theorem 1.1 *For $\epsilon > 0$ small enough the following holds true. The eigenfunctions e_j extends holomorphically to B_ϵ and the family $(e^{-\epsilon\omega_j} < \omega_j >^{(m-1)/4} e_j(z))_{j\geq 0}$ is a Riesz basis of $H(B_\epsilon)$. For $f \in H(B_\epsilon)$ and $a_j = \int_M f e_j d_g x$, one has*

$$f(z) = \sum a_j e_j(z) \tag{1.6}$$

where the sum is uniformly convergent on any compact subset of B_ϵ and convergent in $H(B_\epsilon)$. There exists a constant C_ϵ such that one has the equivalence of norms

$$\frac{1}{C_\epsilon}\|f\|^2_{H(B_\epsilon)} \leq \sum_j |e^{\epsilon\omega_j} < \omega_j >^{-(m-1)/4} a_j|^2 \leq C_\epsilon \|f\|^2_{H(B_\epsilon)} \tag{1.7}$$

A detailed proof of this theorem has been given recently by S. Zelditch in [13], following the lines indicate in [2] and using the Hadamard parametrix for the wave equation, and also by M. Stenzel in [12] which uses the asymptotic expansions of the heat kernel. Here, we will give a proof based on non-characteristic deformation techniques and a direct calculus of the Hadamard type parametrix for the Poisson Kernel.

The paper is organized as follows:

In Sect. 2, we just recall explicit formulas in the euclidian space $\mathbb{R}^m$ and we give a proof of the Boutet theorem in the special case of the flat torus $(\mathbb{R}/2\pi\mathbb{Z})^m$.

In Sect. 3, we recall basic facts on symplectic geometry. We introduce the fundamental function Φ and we give some of his properties. We refer to [9] for a detailed study of the relationships between real and complex symplectic geometry.

Section 4 is devoted to the proof of the "analytic version" of the Boutet theorem, see Theorem 4.1, which describes the space $\mathcal{O}(B_\epsilon)$ as the space of sums of the form $\sum b_j e^{-\epsilon\omega_j} e_j(z)$ with coefficients b_j with sub-exponential growth, i.e $\forall\delta > 0,\ \exists C_\delta,\ \forall j,\ |b_j| \leq C_\delta e^{\delta\omega_j}$. The proof of this result is purely geometric: it uses only non-characteristic deformation techniques and the Zerner lemma.

Section 5 is devoted to the proof of the Boutet theorem. The main ingredient is the construction of the Hadamard parametrix for the Poisson kernel.

The appendix contains two proofs of classical technical results.

Finally, let us recall that the representation of the analytic wave front set as the analytic singular support of boundary values of holomorphic functions defined inside a strictly pseudoconvex domain, which is one of the most fundamental results in microlocal analysis, (and which is closely related to the Boutet theorem) is due to M. Sato, T. Kawai and M. Kashiwara and is explicit in their foundation article of 1971 [10].

2 Explicit Formulas in the Flat Case

In this section, we just recall what are the explicit formulas for the Poisson kernel, heat kernel, and FBI transform on the euclidean space $\mathbb{R}^m$. Replacing $\mathbb{R}^m$ by the standard m-dimensional torus $\mathbb{T}^m = (\mathbb{R}/2\pi\mathbb{Z})^m$, this will give a straightforward proof of the Boutet theorem in this special case.

First observe that on $\mathbb{R}^m$ one has $d^2(x, y) = (x - y)^2$, and therefore the function $\Phi(z)$ given by (1.3) is defined on all $\mathbb{C}^m$ by

$$\Phi(z) = Im(z)^2/2 \tag{2.1}$$

The heat kernel in $\mathbb{R}^m$ is equal to $p_t(x, y) = (2\pi t)^{-m/2} e^{-(x-y)^2/2t}$. The solution of the heat equation

$$\partial_t f - \frac{1}{2}\triangle f = 0 \text{ (in } t > 0), \quad f|_{t=0} = g \in \mathcal{S}'(\mathbb{R}^m) \tag{2.2}$$

is given by the formula

$$f(t, x) = \int_{\mathbb{R}^m} p_t(x, y) g(y) dy \tag{2.3}$$

On the Fourier side, one has the obvious identity

$$\hat{f}(t, \xi) = e^{-t\xi^2/2}\hat{g}(\xi) \tag{2.4}$$

Observe that if we replace $x \in \mathbb{R}^m$ by $z \in \mathbb{C}^m$, and if we set $\lambda = 1/t > 0$, we get

$$f(t, z) = \left(\frac{\lambda}{2\pi}\right)^{m/2} \int_{\mathbb{R}^m} e^{-\lambda(z-y)^2/2} g(y) dy = T_\lambda(g)(z) \tag{2.5}$$

where T_λ is exactly the most usual FBI transform introduced by J. Sjöstrand in [11] (up to the factor $(\frac{\lambda}{2\pi})^{m/2}$ in front of it). Therefore, we get that this FBI transform is just a complexification of the usual heat kernel. One has the obvious bound

$$|f(t, z)| \leq \left(\frac{\lambda}{2\pi}\right)^{m/2} e^{\lambda\Phi(z)} \|g\|_{L^1} \tag{2.6}$$

Now we recall the formula for the Poisson kernel $\mathbb{P}_s(x, y)$. The solution of the elliptic boundary value problem, with $f(s, .)$ bounded in $s \geq 0$ with values in $L^2(\mathbb{R}^m)$

$$\partial_s^2 f + \triangle f = 0 \text{ (in } s > 0), \quad f|_{s=0} = g \in L^2(\mathbb{R}^m) \tag{2.7}$$

is given by the formula

$$f(s,x) = \mathbb{P}_s(g)(x) = \int_{\mathbb{R}^m} \mathbb{P}_s(x,y)g(y)dy \tag{2.8}$$

One has the obvious identity

$$\mathbb{P}_s(g)(x) = (2\pi)^{-m} \int e^{ix\xi - s|\xi|}\hat{g}(\xi)d\xi \tag{2.9}$$

Fix now $s > 0$. Then (2.9) clearly implies that $\mathbb{P}_s(g)$, (with g in any Sobolev space $H^\mu(\mathbb{R}^m)$) extends holomorphically for $s > 0$ in the domain

$$B_s = \{|Im(z)| < s\} = \{\Phi(z) < s^2/2\}$$

For $z \in B_s$, set $z = a + ib$. Then the map $g \mapsto T_s(g) = \mathbb{P}_s(g)|_{\partial B_s}$ is given by

$$T_s(g)(a,b) = (2\pi)^{-m} \int e^{i(a-x)\xi - b.\xi - s|\xi|} g(x)dxd\xi \tag{2.10}$$

Clearly, T_s extends for all real μ to a map defined on the Sobolev space $H^\mu(\mathbb{R}^m)$ with values in $\mathcal{D}'(\partial B_s)$. Let $d\sigma_s$ be the standard measure on the sphere of radius s in $\mathbb{R}^m$, and let c_m be the volume of the unit sphere S^{m-1} in $\mathbb{R}^m$. Let $d\mu_s$ be the volume form on ∂B_s

$$d\mu_s = c_m^{-1} s^{-(m-1)} da d\sigma_s(b) \tag{2.11}$$

Let T_s^* be adjoint of T_s with respect to $L^2(\partial B_s, d\mu_s)$. One has

$$T_s^*(f)(x) = (2\pi)^{-m} \int e^{i(x-a)\xi - b.\xi - s|\xi|} f(a,b)d\mu_s d\xi \tag{2.12}$$

and therefore we get

$$\begin{aligned} T_s^* T_s(g)(x) &= (2\pi)^{-m} \int e^{ix\xi}\Gamma_m(s\xi)\hat{g}(\xi)d\xi \\ \Gamma_m(\eta) &= c_m^{-1} \int_{S^{m-1}} e^{-2(|\eta| + u.\eta)} d\sigma(u) \end{aligned} \tag{2.13}$$

It is clear that Γ_m is a real strictly positive function and $\Gamma_m(0) = 1$. The function $\Gamma_m(\eta)$ depends only on $|\eta|$ and $e^{2|\eta|}\Gamma_m(\eta)$ is an entire function of $|\eta|^2$. Moreover, by stationary phase, we get that $\Gamma_m(\eta)$ is an elliptic symbol of degree $-(m-1)/2$ in η (and even an analytic symbol). Therefore, with $<\eta>= (1 + |\eta|^2)^{1/2}$ there exists $c > 1$ such that

$$\frac{1}{c} < \eta >^{-(m-1)/2} \le \Gamma_m(\eta) \le c < \eta >^{-(m-1)/2}, \quad \forall \eta \in \mathbb{R}^m$$

Since $T_s^* T_s$ is the Fourier multiplier by $\Gamma_m(s\xi)$, this shows that $T_s^* T_s$ is a self adjoint, non negative, elliptic pseudodifferential operator of degree $-(m-1)/2$. Thus $T_s^* T_s$ is an isomorphism of the Sobolev space $H^{\mu-(m-1)/2}(\mathbb{R}^m)$ onto $H^{\mu}(\mathbb{R}^m)$ for any real μ. From the identity

$$(T_s^* T_s(g)|g)_{L^2(\mathbb{R}^m,dx)} = \|T_s(g)\|^2_{L^2(\partial B_s,d\mu_s)}$$

we get

$$T_s(g) \in L^2(\partial B_s) \quad \text{if and only if} \quad g \in H^{-(m-1)/4}(\mathbb{R}^m)$$

From the above formulas, it is easy to get the Boutet theorem for $M = \mathbb{T}^m = (\mathbb{R}/2\pi\mathbb{Z})^m$. The standard L^2 orthonormal basis is in that case $e_k(x) = (2\pi)^{-m/2} e^{ik.x}$, with $k \in \mathbb{Z}^m$, and associated eigenvalue $|k|^2$. The Poisson operator is given by

$$\mathbb{P}_s(\sum c_k e_k)(x) = \sum c_k e^{-s|k|} e_k(x)$$

which clearly extends to $B_s = \{z = a + ib \in (\mathbb{C}/2\pi\mathbb{Z})^m, |b| < s\}$. If T_s still denotes the map $g \mapsto T_s(g) = \mathbb{P}_s(g)|_{\partial B_s}$, one has ($T_s^*$ is the adjoint for the volume form (2.11) on ∂B_s)

$$T_s^* T_s(\sum c_k e_k) = \sum c_k \Gamma_m(sk) e_k$$

thus $T_s(g) \in L^2(\partial B_s)$ if and only if $g \in H^{-(m-1)/4}(\mathbb{T}^m)$. One has

$$T_s(\sum c_k e_k)(a + ib) = (2\pi)^{-m/2} \sum c_k e^{-s|k|} e^{ik.a-k.b}$$

The functions $(2\pi)^{-m/2} e^{ik.a-k.b} = E_k(a, b)$ are trivially orthogonal in $L^2(\partial B_s, d\mu_s)$, and the computation we have done to get (2.13) shows that one has

$$\|E_k\|^2_{L^2(\partial B_s)} = e^{2s|k|} \Gamma_m(sk)$$

It will be proven in Sect. 4 that the family $(e_k(z))_k$ is dense in the Hardy space $H(B_s)$ (we leave this as an exercise in the special case of the flat torus). Thus, in the flat case, we get the more precise statement that the family

$$e^{-s|k|} \Gamma_m^{-1/2}(sk) e_k(z), \quad k \in \mathbb{Z}^m$$

is *an orthonormal basis* of the Hardy space $H(B_s)$. Thus the Boutet theorem holds true in the special case of the flat torus.

Remark 2.1 As one can see, in the flat case, $T_s^* T_s$ is in fact a function of the Laplace operator, and the eigenfunctions $e_k(z)|_{\partial B_s}$ remains orthogonal for any s for a natural choice of the volume form on ∂B_s. There is no reason for this statement to be true in the general case. Also, one has to notice that with respect to s, viewed as a small semi-classical parameter and not viewed as a fixed constant, formula (2.13) shows that $T_s^* T_s$ is a semi-classical pseudodifferential operator with s as small parameter, and not at all an usual pseudo-differential operator uniformly in $s \in]0, 1]$. This is related to the geometric fact that the boundary of the s-Grauert tube blows down to M when s goes to 0.

Let us now recall how one can recover the Poisson kernel from the heat kernel. We start from the formula, valid for all $x \in [0, \infty[$.

$$e^{-x} = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-x^2/4u} e^{-u} \frac{du}{\sqrt{u}} \tag{2.14}$$

This formula is easy to prove, since both side are continuous functions of $x \geq 0$, and satisfy the equation $f'' - f = 0$ in $x > 0$ and $f(0) = 1$, $\lim_{x \to \infty} f(x) = 0$. From (2.14), we get for $s > 0$, $\omega \geq 0$ (change of variable $u = s^2/2t$)

$$e^{-s\omega} = \frac{s}{\sqrt{2\pi}} \int_0^\infty e^{-s^2/2t} e^{-t\omega^2/2} \frac{dt}{t^{3/2}} \tag{2.15}$$

Therefore, one has the following identity which allows to recover the Poisson kernel from the heat kernel, (and which remains obviously valid on any Riemmannian compact manifold (M, g) by decomposition on the orthonormal basis of the Laplace operator):

$$\mathbb{P}_s(x, y) = \frac{s}{\sqrt{2\pi}} \int_0^\infty e^{-s^2/2t} p_t(x, y) \frac{dt}{t^{3/2}} \tag{2.16}$$

This identity is used by M. Stenzel in [12] in his proof of the Boutet theorem. If we express this in term of the FBI transform defined in (2.5), we get (recall $T_\lambda(x, y) = p_{1/\lambda}(x, y)$)

$$\mathbb{P}_s(z, y) = \frac{s}{\sqrt{2\pi}} \int_0^\infty e^{-\lambda s^2/2}\, T_\lambda(z, y)\, \lambda^{-1/2} d\lambda \tag{2.17}$$

From (2.6), we recover from (2.17) that in the flat case, $\mathbb{P}_s(z, y)$ extends holomorphically in the domain $|Im(z)| < s$. Therefore, the FBI transform (i.e the complexification of the heat kernel) contains at least as much information than the Poisson Kernel. In fact, the two points of view are essentially equivalent if the FBI transform acts on functions independent of λ. The use of the FBI transform is of course more relevant in semi-classical analysis, with small parameter $h = 1/\lambda = t$. We refer to the article by F.Golse, E.Leichtnam and M. Stenzel, [6] for a study of the FBI transform as a complexification of the heat kernel on compact Riemannian analytic manifolds.

3 Symplectic Geometry

Let T^*X be the complex cotangent bundle to the complex manifold X. Let us recall that for $(z,\zeta) \in T^*X$, ζ is a $\mathbb{C}$-linear form on the complex vector space T_zX with values in $\mathbb{C}$, i.e $\zeta(iu) = i\zeta(u)$ for all $u \in TX$. As usual, if f is a function defined on X with values in $\mathbb{C}$, we denote by ∂f (resp $\overline{\partial} f$) its holomorphic (resp. antiholomorphic) derivative, that is

$$\partial f(u) = \frac{1}{2}(df(u) - idf(iu)), \quad \overline{\partial} f(u) = \frac{1}{2}(df(u) + idf(iu))$$

Then ∂f is a section of T^*X and one has $d = \partial + \overline{\partial}$.

Let us denote by $X^{\mathbb{R}}$ the real analytic manifold X without its complex structure. In these notes, we shall identify the real cotangent bundle $T^*(X^{\mathbb{R}})$ with the complex cotangent bundle T^*X by the following rule

$$(z,\zeta) \in T^*X \text{ is identified with } (z,\xi) \in T^*X^{\mathbb{R}} : \quad \xi(u) = Re(\zeta(u)) \tag{3.1}$$

With this identification, for any smooth function $\varphi : X \to \mathbb{R}$,

$$d\varphi(z) \in T_z^*X^{\mathbb{R}} \text{ is identified with } 2\partial\varphi(z) \in T_z^*X \tag{3.2}$$

Let $\omega = d\zeta \wedge dz$ be the canonical complex symplectic 2-form on T^*X. Then $Re(\omega)$ and $Im(\omega)$ are real symplectic 2-forms on $T^*X^{\mathbb{R}}$, and moreover, $Re(\omega) = \omega^{\mathbb{R}}$ is the canonical symplectic 2-form on $T^*X^{\mathbb{R}}$. This facts are easy to verify in local coordinates. We shall say that a real submanifold Λ of T^*X is R-symplectic (resp I-lagrangian) iff Λ is symplectic for $Re(\omega) = \omega^{\mathbb{R}}$ (resp lagrangian for $Im(\omega)$). In other words, Λ is R-symplectic iff $dim_{\mathbb{R}}\Lambda = 2m$ and $Re(\omega)|_\Lambda$ is non degenerate, and Λ is I-lagrangian iff $dim_{\mathbb{R}}\Lambda = 2m$ and $Im(\omega)|_\Lambda = 0$.

Lemma 3.1 *Let $z \mapsto \zeta(z)$ be a smooth section of $T^*X \simeq T^*X^{\mathbb{R}}$ defined on an open contractible subset Ω of X and let $\Lambda = \{(z,\zeta(z)),\ z \in \Omega\}$. Then Λ is I-lagrangian iff there exists a smooth function $\varphi : \Omega \to \mathbb{R}$ such that $\zeta(z) = 2i\partial\varphi(z)$. Moreover, Λ is also R-symplectic iff the 2-form of type $(1,1)$ $2i\overline{\partial}\partial\varphi$ on $TX|_\Omega$ is non degenerate.*

Proof If Λ is I-lagrangian, then $-i\Lambda = \{(z,-i\zeta(z)),\ z \in \Omega\}$ is R-lagrangian, $\omega^{\mathbb{R}}|_{-i\Lambda} = 0$. Since Ω is contractible, there exists a function $\varphi : \Omega \to \mathbb{R}$ such that $-i\Lambda$ view as a subset of $T^*X^{\mathbb{R}}$ is of the form $\{(z,d\varphi(z))\}$. With the identification $T^*X \simeq T^*X^{\mathbb{R}}$, and by (3.2), we get $-i\zeta(z) = 2\partial\varphi(z)$, i.e

$$\zeta(z) = 2i\partial\varphi(z)$$

Let $j : \Omega \to T^*X$ be defined by $j(z) = (z, 2i\partial\varphi(z))$. One has $j^*(Im(\omega)) = 0$. Moreover Λ is R-symplectic iff $j^*(\omega^{\mathbb{R}})$ is non degenerate and the result follows from

$$j^*(\omega^{\mathbb{R}}) = j^*(\omega) = j^*(d(\zeta dz)) = d(j^*(\zeta dz)) = d(2i\partial\varphi) = 2i\overline{\partial}\partial\varphi$$

□

The Levi form on $TX|_\Omega$, $\mathcal{L}_\varphi(u, v) = 2i\overline{\partial}\partial\varphi(u, v)$ is given in local complex coordinates $(z_1, ..., z_m)$ by the formula

$$\mathcal{L}_\varphi(u, v) = 2i \sum_{j,k} \frac{\partial^2\varphi}{\partial\overline{z_j}\partial z_k}(z)(\overline{u}_j v_k - \overline{v}_j u_k)$$

One has obviously $\mathcal{L}_\varphi(u, v) \in \mathbb{R}$, and $\mathcal{L}_\varphi$ is entirely determinate by the associated hermitian form $q_\varphi(u) = \mathcal{L}_\varphi(iu, u)$. In local coordinates, one has

$$q_\varphi(u) = 4 \sum_{j,k} \frac{\partial^2\varphi}{\partial\overline{z_j}\partial z_k}(z)\overline{u}_j u_k \tag{3.3}$$

Therefore, Λ is I-lagrangian and R-symplectic iff the hermitian form q_φ is non degenerate, hence of signature (p, q) with $p + q = m$.

The real cotangent bundle T^*M is a subset of T^*X: for $x \in M$, any $u \in T_x X$ can be written in a unique way $u = a + ib,\ a, b \in T_x M$, and $(x, \xi) \in T^*M$ defines $(x, \zeta) \in T^*X,\ \zeta(u) = \xi(a) + i\xi(b)$. Then it is obvious that T^*M is both R-symplectic and I-lagrangian. Moreover, T^*M is a totally real submanifold of T^*X and the complex symplectic manifold T^*X is a complexification of the real symplectic manifold T^*M.

Let $p(z, \zeta)$ be the holomorphic extension of $p(x, \xi) = \frac{1}{2}|\xi|_x^2$. In local coordinates, one has

$$p(z, \zeta) = \frac{1}{2} \sum_{j,k} g^{j,k}(z)\zeta_j\zeta_k$$

and $p(z, \zeta)$ is well defined on $T^*X|_W$ if W is a small neighborhood of M in X. For $t \in \mathbb{C}$, let us denote by $exp(tH_p)(z, \zeta) = (Z(t, z, \zeta), \Xi(t, z, \zeta))$ the complex integral curve of the hamiltonian vector field of p starting at (z, ζ). One has the Hamilton–Jacobi equations

$$\begin{aligned} \partial_t Z &= (\partial_\zeta p)(Z, \Xi), \quad Z(0, z, \zeta) = z \\ \partial_t \Xi &= -(\partial_z p)(Z, \Xi), \quad \Xi(0, z, \zeta) = \zeta \end{aligned} \tag{3.4}$$

Since $p(z, \zeta)$ is homogeneous of degree 2 in ζ, one has for $\lambda \neq 0$

$$Z(\lambda t, z, \zeta/\lambda) = Z(t, z, \zeta), \quad \Xi(\lambda t, z, \zeta/\lambda) = \lambda^{-1}\Xi(t, z, \zeta) \tag{3.5}$$

Therefore, $exp(tH_p)(z, \zeta)$ is well defined for $|t\zeta|$ small and $(z, \zeta) \in T^*X|_W$ if W is small enough, and one has the Taylor expansion

$$\begin{aligned} Z(t,z,\zeta) &= z + t(\partial_\zeta p)(z,\zeta) + 0(|t\zeta|^2) \\ \Xi(t,z,\zeta) &= \zeta - t(\partial_z p)(z,\zeta) + 0(|\zeta||t\zeta|^2) \end{aligned} \tag{3.6}$$

Let $\epsilon_0 > 0$ given and small. For $s \in]0,1]$, set

$$\Lambda_s = \{(z,\zeta) = exp(isH_p)(x,\xi) \in T^*X,\ (x,\xi) \in T^*M,\ |\xi|_x < \epsilon_0/s\} \tag{3.7}$$

Then for ϵ_0 small enough and all $s \in]0,1]$, Λ_s is well defined and from (3.5), one has $\Lambda_s = s^{-1}\Lambda_1$. Moreover, since the map $exp(tH_p)$ preserves the complex symplectic structure of T^*X for any $t \in \mathbb{C}$, Λ_s is both R-symplectic and I-lagrangian. By (3.6), the map $(x,\xi) \mapsto Z(is,x,\xi)$ is given in local coordinates by

$$(x,\xi) \mapsto Z(is,x,\xi), \quad Z_k(is,x,\xi) = x_k + is\sum_j g^{j,k}(x)\xi_j + 0(|s\xi|^2) \tag{3.8}$$

hence is an isomorphism near $\xi = 0$. By Lemma 3.1, near any point $x \in M$ there exists a unique function $\Phi_s(z) = s^{-1}\Phi(z)$ define in a neighborhood of x, with $\Phi_s(x) = 0$ such that one has

$$\Lambda_s = \{(z,\zeta),\ \zeta = 2i\partial\Phi_s(z) = 2is^{-1}\partial\Phi(z)\}$$

From (3.6) and (3.8) , one has $\partial\Phi|_M = 0$, and therefore the function Φ is globally defined in a neighborhood of M in X and one has

$$\Phi|_M = 0, \quad d\Phi|_M = 0 \tag{3.9}$$

Lemma 3.2 *The following identity holds true*

$$\Phi(Z(i,x,\xi)) = |\xi|_x^2/2 \tag{3.10}$$

Proof For $s \in [0,1]$, set $(\gamma(s),\eta(s)) = (Z(is,x,\xi),\Xi(is,x,\xi))$ and $\zeta(s) = 2i\partial\Phi(\gamma(s))$. One has, for $s > 0$, $(\gamma(s),\eta(s)) \in \Lambda_s = s^{-1}\Lambda_1$, and therefore $\eta(s) = s^{-1}2i\partial\Phi(\gamma(s)) = s^{-1}\zeta(s)$. Let

$$g(s) = \Phi(Z(is,x,\xi)) = \Phi(\gamma(s))$$

Then we get

$$\begin{aligned} g'(s) &= d\Phi(\gamma(s))(\gamma'(s)) = Re\Big(2\partial\Phi(\gamma(s))(i\partial_t Z(is,x,\xi))\Big) \\ &= Re\Big(2i\partial\Phi(\gamma(s))(\partial_t Z(is,x,\xi)) = Re(\zeta(s)\partial_\zeta p(\gamma(s),\eta(s))) \\ &= sRe(2p(\gamma(s),\eta(s))) = sRe(2p(\gamma(0),\eta(0)) = s|\xi|_x^2 \end{aligned} \tag{3.11}$$

Here we have used that $\partial\Phi$ is $\mathbb{C}$-linear, the Hamilton–Jacobi equations (3.4), $\zeta(s) = s\eta(s)$, and the fact that $p(z,\zeta)$ is homogeneous of degree 2 in ζ and invariant by the flow of the hamiltonian vector field H_p. Since $g(0) = 0$, we thus get $g(s) = s^2|\xi|_x^2/2$. The proof of Lemma 3.2 is complete. □

As a byproduct of Lemma 3.2 and formula (3.8), the function Φ is strictly pluri-subharmonic, i.e the hermitian form q_Φ defined in (3.3) is strictly positive. Moreover the map

$$(x,\xi) \mapsto Z(i,x,\xi) \tag{3.12}$$

gives a real analytic identification between the neighborhood $\{|\xi|_x < \epsilon_0\}$ of the zero section in the symplectic manifold T^*M, and the neighborhood $B_{\epsilon_0} = \{\Phi(z) < \epsilon_0^2/2\}$ of M in the complex manifold X. With this identification, the symplectic structure on B_{ϵ_0} is defined by the *real and closed* 2-form $2i\overline{\partial}\partial\Phi$, and the associated hermitian metric q_Φ defines a Kahlerian structure on B_{ϵ_0}. Since Φ is an exhaustion strictly pluri-subharmonic function on B_{ϵ_0}, B_{ϵ_0} is a Stein manifold.

Moreover, this identification induces a complex structure J on $\{|\xi|_x < \epsilon_0\}$. We refer to the article of Lempert and Szöke [7] for more details on this complex structure J on T^*M, which is canonically defined by the metric g on M. In particular, it is shown in [7], Theorem 4.3, that if this complex structure can be extended to $\{|\xi|_x < R\}$, then the sectional curvatures of g are bounded from below by $-\pi^2/(4R^2)$.

We denote by β_z (resp ζ_z) the *real* (resp *complex*) 1-form on the real (resp complex) analytic manifold B_{ϵ_0} defined by

$$\beta_z = Re(\zeta_z), \quad \zeta_z = \Xi(i,x,\xi),\ z = Z(i,x,\xi),\ (x,\xi) \in T^*M \tag{3.13}$$

By construction, one has

$$\zeta_z = 2i\partial\Phi(z) \tag{3.14}$$

Let $q(x,\xi) = |\xi|_x$. Then the hamiltonian $exp(tH_q)(z,\zeta) = (\tilde{Z}(t,z,\zeta), \tilde{\Xi}(t,s,\zeta))$ is well defined for $t \in \mathbb{C}$ close to 0 and $(z,\zeta) \in T^*X$ in a conic neighborhood of $T^*M \setminus M$. Since $p = q^2/2$, one has by homogeneity, with the notation $|\zeta|_z = (g^{-1}(z)(\zeta))^{1/2}$, which is preserved by the flow of H_q,

$$\tilde{Z}(t,z,\zeta) = Z(t,z,\zeta/|\zeta|_z), \quad \tilde{\Xi}(t,s,\zeta)) = |\zeta|_z\Xi(t,z,\zeta/|\zeta|_z) \tag{3.15}$$

For $s \in]0,\epsilon_0[$ let $\kappa(is) = exp(isH_q)$. Then $\kappa(is)$ is an homogeneous canonical complex transformation of T^*X, defined in a conic neighborhood U of $T^*M \setminus M$. From (3.5), one has

$$\kappa(is)(z,\zeta) = (Z(i,z,s\zeta/|\zeta|_z), |\zeta|_z\Xi(i,z,s\zeta/|\zeta|_z)) \tag{3.16}$$

Since $\kappa(is)$ preserves the canonical 1-form ζdz on T^*X, one has

$$|\zeta|_z \Xi(i, z, s\zeta/|\zeta|_z) d_{z,\zeta}(Z(i, z, s\zeta/|\zeta|_z)) = \zeta dz \tag{3.17}$$

For $y \in M$, let $\Lambda_{s,y} = \kappa(is)(T^*_y M \setminus 0)$, and let $\Lambda^{\mathbb{C}}_{s,y} = \kappa(is)(U \cap T^*_y X \setminus 0)$ be its complexification. Then $\Lambda^{\mathbb{C}}_{s,y} \subset T^*X$ is a $\mathbb{C}$-lagrangian homogeneous submanifold of T^*X. One has by (3.5), (3.13), and (3.15):

$$\Lambda_{s,y} = \{(z = Z(i, y, \eta), \zeta = t\zeta_z),\ (y, \eta) \in T^*_y M,\ |\eta|_y = s,\ t > 0\} \tag{3.18}$$

Since for real t one has $d^2(Z(t, y, \eta), y) = t^2|\eta|^2_y$, and these functions are analytic in t, we get

$$d^2(Z(i, y, \eta), y) = -|\eta|^2_y = -2\Phi(Z(i, y, \eta)), \quad \forall \eta \in T^*_y M \tag{3.19}$$

and therefore the function $s^2 + d^2(z, y)$ vanishes on $\pi(\Lambda_{s,y})$, where π is the projection $T^*X \to X$. Since $\pi(\Lambda^{\mathbb{C}}_{s,y})$ is a complexification of $\pi(\Lambda_{s,y})$ (a real analytic manifold of real dimension $m - 1$) , we get that $\Lambda^{\mathbb{C}}_{s,y}$ is the conormal bundle to the complex hypersurface $s^2 + d^2(z, y) = 0$ near the points $z = Z(i, y, \eta)$, $|\eta|_y = s$:

$$\Lambda^{\mathbb{C}}_{s,y} = T^*_{\Sigma_{s,y}} X \setminus 0, \quad \Sigma_{s,y} = \{z,\ s^2 + d^2(z, y) = 0\} \tag{3.20}$$

The following lemma (and (3.10)) gives in particular a proof for the properties of the function Φ stated in the introduction (see formula (1.3)).

Lemma 3.3 *There exists $c > 0$ and a neighborhood U of $Diag(M)$ in $M \times M$ such that for all $s \in]0, \epsilon_0]$, all $(x, y) \in U$ and all $z = Z(i, x, \xi) \in \partial B_s$ (i.e $|\xi|_x = s$), one has*

$$\partial_z d^2(z, y)|_{z=Z(i,y,\xi)} = 2i\zeta_z \tag{3.21}$$

and

$$Re(d^2(z, y) + s^2) \geq c d^2(x, y) \tag{3.22}$$

Proof From (3.19) one has $d^2(Z(i, y, \eta), y) = -|\eta|^2_y$ and from (3.18) and (3.20), one has $\partial_z d^2(z, y)|_{z=Z(i,y,\eta)} = \lambda\zeta_z$ for some $\lambda \in \mathbb{C} \setminus 0$. Let $z(t) = Z(i, y, e^t\eta) = Z(ie^t, y, \eta)$. One has $z(0) = Z(i, y, \eta) = z$ and $d^2(z(t), y) = -e^{2t}|\eta|^2_y$. By evaluation of the derivative at $t = 0$, we find:

$$-2|\eta|^2_y = d_t(d^2(z(t), y))|_{t=0} = \lambda\zeta_z(d_t z(t)|_{t=0}) = i\lambda\zeta_z \frac{\partial p}{\partial \zeta}(z, \zeta_z) = 2i\lambda p(z, \zeta_z) = i\lambda|\eta|^2_y$$

This implies $\lambda = 2i$. Let us now verify (3.22). In geodesic coordinates $exp_x(a)$ centered at x, set $d^2(a, b) = (a - b)^2 + R_x(a, b)$. The function $R_x(a, b)$ is symmetric in a, b. From $d^2(0, b) = b^2$, we get $R_x(0, b) = 0$, thus $R_x(a, 0) = 0$, and $R_x(a, b) = \sum_{j,l} a_j b_l Q^{j,l}_x(a, b)$. From $(\nabla_a d^2)(0, b) = -2b$, one gets $\sum_l b_l Q^{j,l}_x(0, b) = 0$, hence $d^2(a, b) = (a - b)^2 + O(a^2 b)$, and since $R_x(a, b)$ is symmetric

$$d^2(a,b) = (a-b)^2 + O(a^2b^2) \tag{3.23}$$

Set $y = exp_x(a)$ and $z = Z(i, x, \xi)$. In geodesic coordinates centered at x, one has $g(x) = Id$, $Z(t, x, \xi) = t\xi$, thus $z = i\xi$, and from $|\xi|_x = s$ and $d^2(x, y) = a^2$, we get

$$d^2(z, y) = d^2(x, y) - s^2 - 2ia\xi + O(s^2 d^2(x, y)) \tag{3.24}$$

Since s is small, (3.22) holds true. The proof of Lemma 3.3 is complete. □

4 The Analytic Version of the Boutet de Monvel Theorem

Recall that for $\epsilon \in]0, \epsilon_0]$, B_ϵ is the tubular neighborhood of M in X

$$B_\epsilon = \{z,\ \Phi(z) < \epsilon^2/2\} = \{Z(i, x, \xi),\ (x, \xi) \in T^*M,\ |\xi|_x < \epsilon\} \tag{4.1}$$

The Poisson kernel $\mathbb{P}_s(x, y)$ on (M, g) is the smooth function on $]0, \infty[\times M \times M$ given by the formula

$$\mathbb{P}_s(x, y) = \sum_j e^{-s\omega_j} e_j(x) e_j(y) \tag{4.2}$$

For any $v \in L^2(M)$, the smooth function on $]0, \infty[\times M$ defined by
$u(s, x) = \int_M \mathbb{P}_s(x, y) v(y) d_g y$ satisfies the elliptic boundary problem

$$(\partial_s^2 + \triangle_g)u = 0, \quad \lim_{s \to 0} u(s, x) = v(x) \text{ in } L^2(M) \tag{4.3}$$

We start with purely geometric lemmas about the holomorphic extension of the e_j, and more generally of solutions to the elliptic operator $\partial_s^2 + \triangle_g$.

Lemma 4.1 *Let $u(s, x)$ be a solution of the elliptic equation $(\partial_s^2 + \triangle_g)u = 0$ on $]0, \infty[\times M$. Then u extends holomorphically in the open set*

$$\mathbb{D} = \{(s, z) \in \mathbb{C} \times X,\ Re(s) > 0\ z \in B_{\min(\epsilon_0, Re(s))}\} \tag{4.4}$$

Proof By translation invariance in s, it is sufficient to prove the following property: Let $a \in]0, \epsilon_0[$, and $u(s, x)$ a solution of the equation $(\partial_s^2 + \triangle_g)u = 0$ on $]-a, a[\times M$. Then u extends holomorphically in the open set

$$\mathbb{G}_a = \{(s, z) \in \mathbb{C} \times X,\ Re(s) \in]-a, a[,\ z \in B_{a-|Re(s)|}\} \tag{4.5}$$

The proof of this fact uses a classical non-characteristic deformation argument based on the following Zerner lemma (see [14]). This lemma is a consequence of the precise form of the Cauchy–Kowalewski theorem given by J. Leray (see [5], Theorem 9.4.7 for a proof).

Lemma 4.2 (Zerner) *Let $Q(z,\partial_z) = \sum_{\alpha,|\alpha|\le m} q_\alpha(z)\partial_z^\alpha$ be a linear differential operator with holomorphic coefficients defined near* 0 *in* $\mathbb{C}^N$ *and let* $q(z,\zeta) = \sum_{|\alpha|=m} q_\alpha(z)\zeta^\alpha$ *be its principal symbol. Let* $f : \mathbb{C}^N \to \mathbb{R}$ *be a* C^1 *function such that* $f(0) = 0$ *and such that, with* $\zeta_0 = 2i\partial f(0)$*, one has* $q(0,\zeta_0) \neq 0$*. Then, if* $u(z)$ *is an holomorphic function defined in a half-neighborhood of* 0 *in* $f < 0$*, such that* $Q(u)$ *extends holomorphically near* 0*, then* u *extends holomorphically near* 0*.*

Let us recall that S. Zelditch [13] uses the Zerner lemma in the space variable to prove that eigenfunctions extend to the maximal tube in which the coefficients of $\triangle$ extend. Here, we have to take care of the fact that the open set $\mathbb{G}_a$ is unbounded with respect to $Im(s)$ and also of the effect of the boundary $Re(s) = \pm a$. Thus, we will introduce a parameter $\tau > 0$ to handle the large values of $Im(s)$, and a family of compact sets $K_{0,\tau}$ such that $\mathbb{G}_a = \cup_{\tau>0} Int(K_{0,\tau})$. Then, for each fixed τ, we will define an explicit decreasing family of compact sets $K_{\mu,\tau}$, $\mu \in [0,a]$ which interpolate nicely between $K_{0,\tau}$ and $K_{a,\tau} = \{s = 0\} \times M$ and we will use Zerner lemma and the hypothesis u is analytic on $]-a,a[\times M$ to prove that u extends holomorphically to $Int(K_{0,\tau})$.

For $\mu \in [0,a]$ let $\psi_\mu(t)$, $t \in \mathbb{R}$, be the non negative Lipschitz function

$$\psi_\mu(t) = \max(a - (\mu^2 + t^2)^{1/2}, 0) \tag{4.6}$$

This function interpolate between the zero function $\psi_a(t) = 0$ and the triangle function $\psi_0(t) = \max(a - |t|, 0)$. Let $\tau > 0$ be given. For $\mu \in [0,a]$, let $K_{\mu,\tau}$ be the set

$$K_{\mu,\tau} = \{(s,z) \in \mathbb{C} \times B_{\epsilon_0},\ \Phi(z) + \tau Im(s)^2 \le \psi_\mu(Re(s))^2/2,\ |Re(s)| \le (a^2 - \mu^2)^{1/2}\} \tag{4.7}$$

From $0 \le \psi_\mu \le a < \epsilon_0$, we get that $K_{\mu,\tau}$ is a compact set, and its interior, $Int(K_{\mu,\tau})$ is the subset of $\mathbb{G}_a$ defined by the equation

$$Int(K_{\mu,\tau}) = \{(s,z),\ \Phi(z) + \tau Im(s)^2 < \psi_\mu(Re(s))^2/2,\ |Re(s)| < (a^2 - \mu^2)^{1/2}\} \tag{4.8}$$

One has $K_{\mu,\tau} \subset K_{\mu',\tau}$ for $\mu' \le \mu$ and the closure of $Int(K_{\mu,\tau})$ is equal to $K_{\mu,\tau}$ for $\mu < a$. Since one has

$$\mathbb{G}_a = \cup_{\tau>0} Int(K_{0,\tau})$$

we have just to prove that u extends holomorphically to $Int(K_{0,\tau})$. Set

$$J = \{\mu, u \text{ extends holomorphically to } Int(K_{\mu,\tau})\}$$

Since $K_{a,\tau} = \{s = 0\} \times M$, J contains a neighborhood of a, and it remains to show that for $\mu > 0$ in J, u extends holomorphically to a neighborhood of $K_{\mu,\tau}$. Let $\mu > 0$ in J.

Let $(s_0, z_0) \in \partial K_{\mu,\tau} = K_{\mu,\tau} \setminus Int(K_{\mu,\tau})$. Set $s_0 = \alpha + i\beta$. If $\psi_\mu(\alpha) = 0$, then one has $z_0 \in M$, $\beta = 0$, and therefore u is holomorphic near (s_0, z_0) since u is analytic on $]-a,a[\times M$. We may thus assume $\psi_\mu(\alpha) \neq 0$, i.e $|\alpha| < (a^2 - \mu^2)^{1/2}$. The function

$$f(s, z) = \Phi(z) + \tau Im(s)^2 - \psi_\mu(Re(s))^2/2$$

is smooth for $|Re(s)| < (a^2 - \mu^2)^{1/2}$, one has $f(s_0, z_0) = 0$, and $2i\partial f$ is equal to

$$2i\partial f = (\zeta_s, \zeta_z) = i(-\psi_\mu \psi'_\mu(Re(s)) - 2i\tau Im(s), 2\partial_z \Phi(z))$$

The differential of f at (s_0, z_0), $(\zeta_s(s_0), \zeta_z(z_0))$ does not vanish.(Otherwise, we will have $z \in M$ and $Im(s_0) = 0$ and this contradict $f(s_0, z_0) = 0$ and $\psi_\mu(\alpha) \neq 0$) Moreover, u satisfies the equation $Qu = (\partial_s^2 + \Delta_g)u = 0$ in a half-neighborhood of (s_0, z_0) in $f < 0$. The principal symbol of Q is $q(s, z; \zeta_s, \zeta_z) = \zeta_s^2 + 2p(z, \zeta_z)$. Therefore, by the Zerner lemma, it remains to show

$$(\psi_\mu \psi'_\mu(\alpha) + 2i\tau\beta)^2 \neq 2p(z_0, \zeta_{z_0}) \tag{4.9}$$

Let $(x_0, \xi_0) \in T^*M$ such that $Z(i, x_0, \xi_0) = z_0$. Then one has $(z_0, \zeta_{z_0}) = exp(iH_p)$ (x_0, ξ_0), and since the function p is invariant by the hamiltonian flow H_p, one has by (3.10)

$$2p(z_0, \zeta_{z_0}) = |\xi_0|^2_{x_0} = 2\Phi(z_0) = \psi_\mu(\alpha)^2 - 2\tau\beta^2 \in \mathbb{R}$$

We first verify that (4.9) holds true for $\beta \neq 0$. For $\beta \neq 0$, equality in (4.9) implies (take imaginary part) $\psi'_\mu(\alpha) = 0$, and equality of the real part gives $-4\tau^2\beta^2 = 2\Phi(z_0) \geq 0$ which is impossible. It remains to verify $\psi'_\mu(\alpha) \neq \pm 1$ for $\mu > 0$ and $|\alpha| < (a^2 - \mu^2)^{1/2}$, which is obvious since one has

$$\psi'_\mu(\alpha) = \frac{-\alpha}{\sqrt{\mu^2 + \alpha^2}}$$

The proof of Lemma 3.1 is complete. □

If one apply the above lemma to the function $u(s, x) = e^{-s\omega_j} e_j(x)$, we get that all the eigenfunctions $e_j(x)$ extends holomorphically to the neighborhood B_{ϵ_0} of M in X, which is independent of j. In fact, we can deduce easily from Lemma 3.1 a more precise statement.

Lemma 4.3 *Let* $a \in]0, \epsilon_0[$. *For all* $\delta > 0$ *small, there exists* C_δ *such that*

$$\forall j, \quad \sup_{z \in B_a} |e_j(z)| \leq C_\delta e^{(a+\delta)\omega_j} \tag{4.10}$$

Proof Set $E = L^2(M, d_g x)$ and $F = \{f \in \mathcal{O}(B_a), \sup_{z \in B_a} |f(z)| < \infty\}$. These are Banach spaces, and the canonical injection $i : F \to E$, $i(f) = f|_M$ is continuous. Let $\delta > 0$ such that $a + \delta < \epsilon_0$ and let A_δ be the linear continuous map from E to E defined by

$$A_\delta(\sum_j c_j e_j(x)) = \sum_j e^{-(a+\delta)\omega_j} c_j e_j(x)$$

The function $u(s,x) = \sum_j e^{s\omega_j} e^{-(a+\delta)\omega_j} c_j e_j(x)$ is a solution of $(\partial_s^2 + \triangle_g)u = 0$ on the interval $]-a-\delta, a+\delta[\times M$. By the proof of Lemma 4.1, see formula (4.5), $u(0,x)$ extends holomorphically in $B_{a+\delta}$. Therefore, one has $Im(A_\delta) \subset \mathcal{O}(B_{a+\delta}) \subset F$. By the closed graph theorem, the map A_δ from E to F is continuous, and therefore, there exists a constant C_δ such that

$$\|A_\delta(f)\|_F \leq C_\delta \|f\|_E, \quad \forall f \in E \tag{4.11}$$

If one applies (4.11) to $f = e_j$, we get that (4.10) holds true. The proof of Lemma 4.3 is complete. □

Remark 4.1 The estimate (4.10) on the sup-norm of the eigenfunctions in B_a is of course very weak. The exponential factor $e^{a\omega_j}$ is the correct one, but the sub-exponential factor $C_\delta e^{\delta\omega_j}$ (for any $\delta > 0$) is far to be optimal. To my knowledge, the best estimate is proven by S.Zelditch in [13], corollary 3: $\sup_{z\in B_a} |e_j(z)| \leq C\omega_j^{(m+1)/4} e^{a\omega_j}$.

Another interesting by-product of Zerner-lemma is the following characterization of the space $\mathcal{O}(B_a)$ of holomorphic functions on B_a. This gives the "analytic" version of the Boutet theorem (i.e without any precise information on Sobolev spaces and polynomial growth of the Fourier coefficients). It implies in particular that the Poisson operator $\mathbb{P}_a(\sum c_j e_j(x)) = \sum c_j e^{-a\omega_j} e_j(z)$ is an isomorphism from the space $\mathcal{A}'(M)$ of Sato-hyperfunctions on M, onto the space $\mathcal{O}(B_a)$ of holomorphic functions in B_a.

Theorem 4.1 (Analytic version of the Boutet theorem)
Let $a \in]0, \epsilon_0[$ and let $f(x) = \sum c_j e_j(x)$ an analytic function on M. Then f extends holomorphically to B_a iff

$$\forall \delta > 0,\ \exists C_\delta,\ \textit{such that for all } j \textit{ one has} \quad |c_j| \leq C_\delta e^{-(a-\delta)\omega_j} \tag{4.12}$$

Moreover, for any function $f(z) \in \mathcal{O}(B_a)$, the Fourier coefficients $c_j = \int_M f(x) e_j(x) d_g x$ satisfy (4.12), and one has $f(z) = \sum_j c_j e_j(z)$ for all $z \in B_a$, where the sum is uniformly convergent on compact subsets of B_a.

Proof If (4.12)is satisfied, then by Lemma 4.3, formula (4.10), the sum $\sum c_j e_j(z)$ is uniformly convergent on $B_{a'}$ for all $a' < a$,(since by Weyl formula, $\sharp\{j, \omega_j \leq R\} \leq CR^m$) hence f extends holomorphically to B_a. It remains to show that for a function $f(z) \in \mathcal{O}(B_a)$, its Fourier coefficients $c_j = \int_M f(x) e_j(x) d_g x$ satisfy (4.12): with $g(z) = \sum c_j e_j(z)$, we will have $g \in \mathcal{O}(B_a)$ by the first part of the lemma, and since $(f-g)|_M = 0$, we will get $f = g$ by analytic continuation. The proof of the estimate (4.12) on the Fourier coefficients c_j of a function $f \in \mathcal{O}(B_a)$ uses the Zerner Lemma. Let $F(s,z)$ be the Cauchy–Kowalewski solution of the analytic Cauchy problem:

$$(\partial_s^2 + \triangle_z)F = 0,\ F(a,z) = f(z) \in \mathcal{O}(B_a),\ \partial_s F(a,z) = 0 \tag{4.13}$$

Let us first show that Zerner lemma implies that F extends holomorphically to the set

$$\mathbb{F}_a = \{(s,z) \in \mathbb{C} \times X,\ |Re(s) - a| < a,\ z \in B_{a-|Re(s)-a|}\} \tag{4.14}$$

The proof of this point follows the same line as the proof of Lemma 4.1. We first change s in $s+a$ so that the Cauchy data for (4.13) are now on the set $\{s=0\} \times B_a$, and we have to prove that F extends to the open set $\mathbb{G}_a$ defined in (4.5). We use the non-characteristic deformation associated to the function, with $\tau > 0$,

$$\tilde{f}_\tau(s,z) = \frac{1}{2}Re(s)^2 + \tau Im(s)^2 - \frac{1}{2}\Big(max(a - \sqrt{\mu^2 + 2\Phi(z)}, 0)\Big)^2 \tag{4.15}$$

Observe that in comparison with the proof of Lemma 4.1, we just exchange the role of $Re(s)^2/2$ and $\Phi(z)$. For $\mu \in [0,a]$, we define $\tilde{K}_{\mu,\tau}$ by

$$\tilde{K}_{\mu,\tau} = \{(s,z) \in \mathbb{C} \times X,\ \tilde{f}_\tau(s,z) \le 0,\ 2\Phi(z) \le a^2 - \mu^2\} \tag{4.16}$$

The function F is holomorphic in a neighborhood of $\tilde{K}_{a,\tau} = \{s=0\} \times M$, and as in the proof of Lemma 4.1, we just have to verify that for $\mu \in]0,a[$, if F extends to $Int(\tilde{K}_{\mu,\tau})$, then F extends to a neighborhood of $\tilde{K}_{\mu,\tau}$. Let $(s_0,z_0) \in \partial K_{\mu,\tau} = K_{\mu,\tau} \setminus Int(K_{\mu,\tau})$. Set $s_0 = \alpha + i\beta$. If $2\Phi(z_0) = a^2 - \mu^2 < a$, then one has $z_0 \in B_a$ and $s_0 = 0$, and therefore F is holomorphic near (s_0,z_0) by Cauchy–Kowalewski theorem. We may thus assume $2\Phi(z_0) < a^2 - \mu^2$. Then the function $\tilde{f}_\tau$ is smooth near (s_0,z_0) and $2i\partial\tilde{f}_\tau$ is equal to

$$2i\partial\tilde{f}_\tau = (\eta_{s_0}, \eta_{z_0}) = 2i(\alpha/2 - i\tau\beta, \frac{a - \sqrt{\mu^2 + 2\Phi(z_0)}}{\sqrt{\mu^2 + 2\Phi(z_0)}}\partial\Phi(z_0)) \tag{4.17}$$

By the Zerner lemma, it remains to show $\eta_s^2 + 2p(z_0,\eta_{z_0}) \neq 0$. Since $2p(z_0,\zeta_{z_0}) = 2\Phi(z_0)$, and $\zeta_{z_0} = 2i\partial\Phi(z_0)$, this is equivalent to verify

$$(\alpha - 2i\tau\beta)^2 \neq 2\Phi(z_0)\frac{(\sqrt{\mu^2 + 2\Phi(z_0)} - a)^2}{\mu^2 + 2\Phi(z_0)} \in [0,\infty[\tag{4.18}$$

We first verify that (4.18) holds true for $\beta \neq 0$. For $\beta \neq 0$, equality in (4.18) implies (take imaginary part) $\alpha\beta = 0$, hence $\alpha = 0$ and $-4\tau^2\beta^2 \ge 0$ which is impossible. For $\beta = 0$, from $\tilde{f}_\tau(s_0,z_0) = 0$ and $2\Phi(z_0) < a^2 - \mu^2$, we get $|\alpha| = a - \sqrt{\mu^2 + 2\Phi(z_0)} > 0$. It remains to verify

$$\alpha^2 \neq \frac{2\Phi(z_0)}{\mu^2 + 2\Phi(z_0)}\alpha^2$$

for $\mu \in]0, a[$ and $\alpha \neq 0$ which is obvious. Thus F extends holomorphically to $Int(\tilde{K}_{0,\tau})$ for all $\tau > 0$, and since one has $\cup_{\tau>0} Int(\tilde{K}_{0,\tau}) = \mathbb{G}_a$, we get the desired result.

For $s \in]0, 2a[$, and F solution of (4.13), set now $F_j(s) = \int_M F(s,x)e_j(x)dx$. Then $F_j(s)$ is analytic on $]0, 2a[$ and satisfies the equation

$$\partial_s^2 F_j - \omega_j^2 F_j = 0,\ \ F_j(a) = c_j,\ \ \partial_s F_j(a) = 0$$

This gives $F_j(s) = c_j ch((a-s)\omega_j)$. Since for all $s \in]0, a]$, the function $x \mapsto F(s,x)$ is analytic on M, its Fourier coefficients are bounded, i.e

$$\forall s \in]0, a],\ \exists C_s \text{ such that } \sup_j |c_j ch((a-s)\omega_j)| \leq C_s$$

By taking $s = \delta$ small, this implies (4.12). The proof of Theorem 4.1 is complete. □

5 A Proof of the Boutet de Monvel Theorem

Recall that the Hardy space $H(B_\epsilon)$ is defined as the Hilbert space:

$$H(B_\epsilon) = \{f \in \mathcal{O}(B_\epsilon),\ f|_{\partial B_\epsilon} \in L^2(\partial B_\epsilon)\},\quad \|f\|_{H(B_\epsilon)} = \|f|_{\partial B_\epsilon}\|_{L^2(\partial B_\epsilon)} \tag{5.1}$$

For $f \in \mathcal{O}(B_\epsilon)$, f satisfies the elliptic system of Cauchy Riemann equations $\overline{\partial} f = 0$. Hence the trace $f|_{\partial B_\epsilon}$ is well defined as an hyperfunction on ∂B_ϵ, and if this trace is analytic, then f is analytic up to the boundary. In particular, if the trace is equal to 0, the extension $\tilde{f}$ of f by 0 outside B_ϵ still satisfy $\overline{\partial}\tilde{f} = 0$; therefore $\tilde{f}$ is holomorphic, and since $\tilde{f}$ vanishes outside B_ϵ, one gets $\tilde{f} = 0$. This shows that $\|f|_{\partial B_\epsilon}\|_{L^2(\partial B_\epsilon)}$ is a norm, and thus $H(B_\epsilon)$ is an Hilbert space.

Recall that the Poisson kernel is defined by

$$\mathbb{P}_s(x,y) = \sum_j e^{-s\omega_j} e_j(x) e_j(y)$$

In his famous book [4] Hadamard gives a parametrix construction for the wave kernels $\cos t\sqrt{\Delta}$ and $\frac{\sin t\sqrt{-\Delta}}{\sqrt{-\Delta}}$ (see also [1, 8]). We will first recall this classical construction of the Hadamard type parametrix for the Poisson kernel $e^{-s\sqrt{-\Delta}}$ near $s = 0$ and $x = y$. Observe that in formulas (5.4), if one set $s = it$, we will get the Hadamard parametrix for the half wave propagator $e^{-it\sqrt{-\Delta}}$; in fact, the Poisson kernel is holomorphic in s for $Re(s) > 0$ and the half wave propagator $e^{-it\sqrt{-\Delta}}$ is its boundary value on $Re(s) = 0$.

Let $\delta(s, x, y)$ be defined by the formula

$$\delta(s, x, y) = s^2 + d^2(x, y) \tag{5.2}$$

The function δ is holomorphic in a small neighborhood W of $\{s = 0\} \times Diag_M$ in $\mathbb{C} \times X \times X$. Let $c_W = \sup_W |\delta|$. Clearly, we may assume c_W as small as we want by choosing W small enough. Set $\mu = -(m+1)/2$.

Proposition 5.1 *For W small enough, the following holds true.*

For all $j \in \mathbb{N}$, there exists holomorphic functions $a_j(s, x, y)$ defined on W, such that

$$\sum_j \sup_W |a_j| c_W^j < \infty \tag{5.3}$$

and such that if one defines $G(s, x, y)$ by the formula

$$\begin{aligned} G &= s\delta^\mu \sum_{j \geq 0} \delta^j a_j \quad \textit{if } m \textit{ is even} \\ G &= s\delta^\mu \sum_{j=0}^{|\mu|-1} \delta^j a_j + s\log(\delta) \sum_{j \geq |\mu|} \delta^{j+\mu} a_j \quad \textit{if } m \textit{ is odd} \end{aligned} \tag{5.4}$$

then the function $\mathbb{P}_s(x, y) - G(s, x, y)$ which is defined a priori for $s > 0$ small and $(x, y) \in M \times M$ close to $Diag_M$, extends holomorphically to W. Moreover, the functions a_j are even in s and one has

$$a_0(0, y, y) = d_m^{-1}, \quad d_m = \int_{\mathbb{R}^m} (1 + x^2)^{-(m+1)/2} dx \tag{5.5}$$

Proof Let us denote by ∇f the gradient of a function f, i.e the vector fields on M which is associated to the differential df via the identification of TM and T^*M. An easy computation shows that the following formula holds true:

$$\begin{aligned} &(\partial_s^2 + \triangle)(f^l b) = l(l-1) f^{l-2}((\partial_s f)^2 + |\nabla f|_g^2) b \\ &+ l f^{l-1}\Big(2\partial_s f \partial_s b + 2(\nabla f|\nabla b)_g + (\partial_s^2 f + \triangle f) b\Big) + f^l(\partial_s^2 b + \triangle b) \end{aligned} \tag{5.6}$$

For a given y, the function $f(s, x) = \delta(s, x, y)$ satisfies the identity $(\partial_s \delta)^2 + |\nabla_x \delta|_g^2 = 4\delta$ (the analog of the eiconal equation). Thus we get from (5.6)

$$\begin{aligned} (\partial_s^2 + \triangle)(\delta^l b) = l\delta^{l-1}\Big(4s\partial_s b + 2(\nabla_x d^2|\nabla b)_g + (\triangle_x(d^2) + 4l - 2) b\Big) \\ + \delta^l(\partial_s^2 b + \triangle b) \end{aligned} \tag{5.7}$$

If we set $b = sa$, with a even in s, we thus get

$$(\partial_s^2 + \triangle)(s\delta^l a) = sl\delta^{l-1}\Big(4s\partial_s a + 2(\nabla_x d^2|\nabla a)_g + (\triangle_x(d^2) + 4l + 2)a\Big) \\ + s\delta^l(\partial_s^2 a + 2s^{-1}\partial_s a + \triangle a) \tag{5.8}$$

Let us first assume that m is even. We will apply the identity (5.8) with $l = \mu + j$, $j \in \mathbb{N}$. Then for all $j \in \mathbb{N}$, one has $l \neq 0$. Let us denote by Z_l the first order operator

$$Z_l(a) = 4s\partial_s a + 2(\nabla_x d^2|\nabla a)_g + (\triangle_x(d^2) + 4l + 2)a \tag{5.9}$$

Then the function G defines by the first line of (5.4) will be formally a solution of the equation $(\partial_s^2 + \triangle)G = 0$ if one choose the functions a_j solutions of the transport equations:

$$\begin{aligned} &Z_\mu(a_0) = 0 \\ &Z_{\mu+j}(a_j) = -\frac{1}{\mu + j}(\partial_s^2 + 2s^{-1}\partial_s a + \triangle_x)a_{j-1} \quad \forall j \geq 1 \end{aligned} \tag{5.10}$$

The key point here is that the equation $Z_\mu(a_0) = 0$ admits a unique even in s holomorphic solution in W for any given data $a(0, y, y)$, and the equation $Z_{\mu+j}(a) = b$ with $j \geq 1$ and $b(s, x, y)$ even in s and holomorphic in W, admits a unique solution $a(s, x, y)$, even in s and holomorphic in W. Therefore, the system of transport equations (5.10) admits a unique solution such that formula (5.5) holds true. We refer to the appendix for a proof of these affirmations, and also for a proof of the estimate (5.3) for small enough W. From the estimate (5.3), the function $\sum_{j\geq 0} \delta^j a_j$ is a holomorphic function on W, and therefore

$$G = s\delta^\mu \sum_{j\geq 0} \delta^j a_j$$

is an holomorphic function on the set $W \cap \{Re(\delta) > 0\}$. In this set, which clearly contains $W \cap \{s > 0, x, y \in M\}$, G satisfies by construction the equation $(\partial_s^2 + \triangle_x)G = 0$, and extends as a holomorphic function on the two sheets covering of the set $W \setminus \{\delta = 0\}$. Now we claim that with the choice (5.5) of the initial data for the solution a_0 of the transport equation $Z_\mu(a_0) = 0$, one has

$$\lim_{s\to 0} G(s, x, y) = \delta_{x=y} \tag{5.11}$$

Here, we identify a measure on M with a distribution by factorization of the volume form $d_g x$. In other words, (5.11) means

$$\lim_{s\to 0} \int_M G(s, x, y)\varphi(x)d_g x = \varphi(y) \tag{5.12}$$

for any smooth test function φ with support close to y. The verification of (5.12) is easy: take near y, the geodesic coordinate system $v \mapsto exp_y(v)$, $v \in T_y M$. Then one has $d^2(x, y) = v^2$ and $d_g x = (1 + O(v^2))dv$. For f smooth with support near 0 one has

$$\begin{aligned}&\lim_{s\to 0}\int_{\mathbb{R}^m} s(s^2+v^2)^{-(m+1)/2} a_0(s, exp_y v, y) f(v)(1+O(v^2))dv \\ &= a_0(0, y, y) f(0)\int_{\mathbb{R}^m}(1+w^2)^{-(m+1)/2} dw = f(0)\end{aligned} \tag{5.13}$$

by the choice (5.5) of $a_0(0, y, y)$ (use the change of variables $v = sw$ and Lebesgue dominated convergence theorem). The same argument shows that the other terms in the development of G in powers of δ do not contribute to the limit in (5.11).

Therefore, $H(s, x, y) = \mathbb{P}_s(x, y) - G(s, x, y)$ satisfies the elliptic boundary value problem in variables (s, x) close to $(0, y)$

$$(\partial_s^2 + \triangle_x)H = 0 \text{ in } s > 0, \quad \lim_{s\to 0} H = 0 \tag{5.14}$$

Hence $H(s, x, y)$ is analytic in (s, x) near $(0, y)$. This is a classical result for this kind of elliptic boundary problem with analytic coefficients, but here, one can use a most elementary reflection argument: near $(0, y)$ in $\mathbb{R} \times M$, the function $u(s, x) = sign(s)H(|s|, x, y)$ satisfies the elliptic equation $(\partial_s^2 + \triangle_x)u = 0$, hence is analytic. The proof of the fact that $H(s, x, y)$ is analytic in (s, x, y) near $\{s = 0\} \times Diag_M$ is of the same kind: One has the symmetry $\mathbb{P}_s(x, y) = \mathbb{P}_s(y, x)$ and from the uniqueness in the construction of the coefficients a_j, one has also $G(s, x, y) = G(s, y, x)$. Hence, $H(s, x, y)$ satisfies the elliptic boundary value problem in variables (s, x, y) close to $\{s = 0\} \times Diag_M$

$$(2\partial_s^2 + \triangle_x + \triangle_y)H = 0 \text{ in } s > 0, \quad \lim_{s\to 0} H = 0 \tag{5.15}$$

Therefore, we conclude that $H(s, x, y)$ is analytic near $\{s = 0\} \times Diag_M$.

In the case m odd, the proof follows the same lines . In addition to formulas (5.6) and (5.8), one also use the formulas with $n \in \mathbb{N}$

$$\begin{aligned}&(\partial_s^2 + \triangle)(f^n \log(f)b) = nf^{n-2}(2 + (n-1)\log(f))((\partial_s f)^2 + |\nabla f|_g^2)b \\ &+ f^{n-1}(1 + n\log(f))\Big(2\partial_s f\partial_s b + 2(\nabla f|\nabla b)_g + (\partial_s^2 f + \triangle f)b\Big) + f^n\log(f)(\partial_s^2 b + \triangle b)\end{aligned} \tag{5.16}$$

which gives since $(\partial_s\delta)^2 + |\nabla\delta|_g^2 = 4\delta$

$$\begin{aligned}&(\partial_s^2 + \triangle)(\delta^n \log(\delta)b) = n\delta^{n-1}\log(\delta)\Big(4s\partial_s b + 2(\nabla_x d^2|\nabla b)_g + (\triangle_x(d^2) + 4n - 2)b\Big) \\ &+ \delta^n\log(\delta)(\partial_s^2 b + \triangle b) + \delta^{n-1}\Big(4s\partial_s b + 2(\nabla_x d^2|\nabla b)_g + (\triangle_x(d^2) + 8n + 2)b\Big)\end{aligned} \tag{5.17}$$

If we set $b = sa$, with a even in s, we thus get

$$\begin{aligned}(\partial_s^2+\triangle)(s\delta^n\log(\delta)a) &= sn\delta^{n-1}\log(\delta)\Big(4s\partial_s a+2(\nabla_x d^2|\nabla a)_g+(\triangle_x(d^2)+4n+2)a\Big)\\ &\quad + s\delta^n\log(\delta)(\partial_s^2 a+2s^{-1}\partial_s a+\triangle a)\\ &\quad + s\delta^{n-1}\Big(4s\partial_s a+2(\nabla_x d^2|\nabla a)_g+(\triangle_x(d^2)+8n+6)a\Big)\end{aligned} \tag{5.18}$$

Then one find that the second line of (5.4) holds true with an additional term of the form $sh(s,x,y)$ with h holomorphic near $s=0, x=y$, and this term plays no role in the verification of the boundary condition at $s=0$ nor in the fact that $\mathbb{P}_s(x,y)-G(s,x,y)$ is analytic near $\{s=0\}\times Diag_M$. The proof of Proposition 5.1 is complete. □

Lemma 5.1 *There exists $\epsilon_0>0$, such that for all $s\in]0,\epsilon_0[$ the following holds true.*

(i) The function $\mathbb{P}_s(z,y)$ is holomorphic in (z,y) near any point $(z,y)\in B_s\times M$.

(ii) The function $\mathbb{P}_s(z,y)$ extends holomorphically near any point $(z,y)\in\partial B_s\times M$ such that $z\notin\{Z(i,y,\eta),|\eta|_y=s\}$.

Proof Point (i) follows directly from the identity (4.2) and the bound (4.10) of Lemma 4.3. Point (ii) is also easy to prove: the function $(s,x)\in]0,\infty[\times M\mapsto\mathbb{P}_s(x,y)$ satisfies the elliptic boundary value problem

$$(\partial_s^2+\triangle_x)\mathbb{P}_s(x,y)=0 \text{ in } s>0,\quad \mathbb{P}_0(x,y)=\delta_{x=y}$$

Therefore, as in the proof of Proposition 5.1, we get that $\mathbb{P}_s(x,y)$ is analytic in (s,x) near any point $(0,x)$ with $x\neq y$. By choosing $\epsilon_0>0$ small enough, we may thus assume that $z=Z(i,x,\xi)$, $|\xi|_x=s$ and x close to $y\in M$. Then by Proposition 5.1, the singularities of $\mathbb{P}_s(z,y)$ near such points are on the subcomplex manifold $\{(z,y), s^2+d^2(z,y)=0\}$, and the result follows from the formula (3.22) of Lemma 3.3. The proof of Lemma 5.1 is complete. □

Recall that we use the identification of $\{(x,\xi)\in T^*M, |\xi|_x=s\}$ with ∂B_s given by the map $(x,\xi)\mapsto Z(i,x,\xi)$, and that c_m is the volume of the unit sphere in $\mathbb{R}^m$, so c_m/m is the volume of the unit ball in $\mathbb{R}^m$. Let $dxd\xi$ be the canonical Liouville measure on T^*M. We define the measure $d\mu_s$ on ∂B_s by the formula

$$\int_{\partial B_s} f d\mu_s=\frac{m}{c_m}\int_{|\xi|_x\le 1} f(x,\frac{s\xi}{|\xi|_x})dxd\xi=\int_M\Big(\int_{S^{m-1}} f(x,sg_x^{1/2}(u))\frac{d\sigma(u)}{c_m}\Big)d_g x \tag{5.19}$$

This is compatible with the definition of $d\mu_s$ that we have used in the flat case in Sect. 2, and if $f(z)$ is a smooth function on X defined near M, one has

$$\lim_{s\to 0}\int_{\partial B_s} f d\mu_s=\int_M f(x)d_g x \tag{5.20}$$

The real 1-form β_z introduced in (3.13) defines by restriction to ∂B_s a 1-form that we still denote by β_z. This defines a canonical half line bundle $L^-\subset T^*(\partial B_s)$

$$L^- = \{(z,\zeta) \in T^*(\partial B_s),\ \zeta = t\beta_z, t < 0\} \tag{5.21}$$

For $s \in]0, \epsilon_0[$, we denote by T_s the map from $\mathcal{D}'(M)$ into $\mathcal{D}'(\partial B_s)$

$$\sum c_j e_j = f \mapsto T_s(f) = \mathbb{P}_s(f)|_{\partial B_s} = \sum_j e^{-s\omega_j} c_j e_j|_{\partial B_s} \tag{5.22}$$

Lemma 5.3 *For all $s \in]0, \epsilon_0[$, T_s is a well defined and injective map. The Hörmander wave front set of its distribution kernel $T_s(z, y)$ is given by*

$$WF(T_s) = \{(z,\zeta; y,\eta) \in T^*(\partial B_s) \times T^*(M) \setminus M, z = Z(i, y, s\eta/|\eta|), \zeta = -\beta_z|\eta|/s\} \tag{5.23}$$

In particular, $WF(T_s)$ is parametrized by $(y, \eta) \in T^*(M) \setminus M$.

Moreover, for any $f \in \mathcal{D}'(M)$, one has $WF(T_s(f)) \subset L^-$ and

$$T_s(f) = \lim_{\mathcal{D}', r\to 0^+} \int_M \mathbb{P}_{s+r}(z, y) f(y) d_g y = \lim_{\mathcal{D}', r\to 0^+} \int_M T_s(e^{-r|\triangle_g|^{1/2}} f) \tag{5.24}$$

Proof One has $\mathbb{P}_s(f) = \int_M \mathbb{P}_s(z, y) f(y) d_g y \in \mathcal{O}(B_s)$, thus the injectivity of T_s is obvious. The fact that $T_s(f) \in \mathcal{D}'(\partial B_s)$ for any $f \in \mathcal{D}'(M)$ follows easily from Proposition 5.1 and point ii) of Lemma 5.1. By Lemma 5.1, the singular support of the Kernel $T_s(z, y)$ is contained in $\{(z, y), \exists\eta \in T^*_y M, |\eta|_y = s, \text{ and } z = Z(i, y, \eta)\}$. Then to compute $WF(T_s)$, we may use Proposition 5.1, and this reduce to the computation of $WF(s^2 + d^2(z, y))^\mu$, which is easy if one uses Lemma 3.3, and gives formula (5.23). Finally, the assertion (5.24) is obvious. The proof of Lemma 5.3 is complete. □

In the following proposition, T_s^* is the adjoint of T_s for the measures $d_g x$ on M and $d\mu_s$ on ∂B_s.

Proposition 5.4 *Let $I = [c, d] \subset]0, \epsilon_0[$. Then $T_s^* T_s$ is a smooth family in $s \in I$ of elliptic pseudodifferential operators of degree $-(m-1)/2$. Moreover, there exists a constant $C(I) > 1$ such that one has the equivalence of norms*

$$\frac{1}{C(I)} \|T_s g\|_{L^2(\partial B_s, d\mu_s)} \le \|g\|_{H^{-(m-1)/4}(M)} \le C(I) \|T_s g\|_{L^2(\partial B_s, d\mu_s)} \tag{5.25}$$

The proof of this proposition is suggested in [2]: essentially, it uses the fact that T_s is a "Fourier Integral Operator with complex phase", which is a direct consequence of Proposition 5.1 and Lemma 5.1 and then it remains to apply the general machinery. (this is the proof given in [13]). Since the reader of these notes may not be familiar with the theory of FIO's with complex phases, we shall directly verify below that $T_s^* T_s$ is an elliptic pseudodifferential operator of degree $-(m-1)/2$, by computing its distribution kernel. This will just involve the knowledge of the stationary phase theorem in the case of complex phase, but with phase and symbol analytic in the parameters, which is not so difficult. We postponed the proof of Proposition 5.4 to the end of this section.

End of Proof of the Boutet Theorem.
Take $s \in]0, \epsilon_0[$. From Proposition 5.4, the map

$$g \in H^{-(m-1)/4}(M) \mapsto \mathbb{P}_s(g)(z) = \int_M \mathbb{P}_s(z, y) d_g y \in H(B_s) \tag{5.26}$$

is well defined, continuous, injective, and has closed range. Let us prove that $\mathbb{P}_s$ is surjective, hence an isomorphism of Hilbert space. Let $f \in H(B_s) \subset \mathcal{O}(B_s)$. From Theorem 4.1, one has

$$f(z) = \sum c_j e_j(z), \quad c_j = \int_M f(x) e_j(x) d_g x \tag{5.27}$$

where the sum is uniformly convergent on compact subset of B_s and the Fourier coefficients c_j satisfy the bounds $|c_j| \leq C_\delta e^{-(s-\delta)\omega_j}$ for all $\delta > 0$. For $0 < s' < s$, one has

$$f(z)|_{B_{s'}} = \sum c_j e_j(z) = \mathbb{P}_{s'}(g_{s'}), \quad g_{s'} = \sum e^{s'\omega_j} c_j e_j \tag{5.28}$$

From the bounds on the c_j, the function $g_{s'}$ is smooth (and in fact analytic) on M, and from (5.25), we get with a constant C independent of $s' \in [s/2, s[$

$$\begin{aligned} &\Big(\sum < \omega_j >^{-(m-1)/2} e^{2s'\omega_j} |c_j|^2\Big)^{1/2} = \|g_{s'}\|_{H^{-(m-1)/4}(M)} \\ &\leq C \|T_{s'} g_{s'}\|_{L^2(\partial B_{s'}, d\mu_{s'})} = C \|f\|_{L^2(\partial B_{s'}, d\mu_{s'})} \end{aligned} \tag{5.29}$$

Since one has

$$\lim_{s' \to s} \|f\|_{L^2(\partial B_{s'}, d\mu_{s'})} = \|f\|_{L^2(\partial B_s, d\mu_s)} = \|f\|_{H(B_s)}$$

we get the “optimal” bound on the c_j:

$$\sum < \omega_j >^{-(m-1)/2} e^{2s\omega_j} |c_j|^2 < \infty$$

and therefore,

$$f(z) = \mathbb{P}_s(g_s), \quad g_s = \sum e^{s\omega_j} c_j e_j \in H^{-(m-1)/4}(M)$$

Finally, the family $< \omega_j >^{(m-1)/4} e_j$ is an orthonormal basis of $H^{-(m-1)/4}(M)$, and $\mathbb{P}_s$ is an isomorphism of Hilbert spaces. Therefore, the family

$$\mathbb{P}_s(< \omega_j >^{(m-1)/4} e_j) = e^{-s\omega_j} < \omega_j >^{(m-1)/4} e_j$$

is a Riesz basis of $H(B_s)$. The proof of the Boutet theorem 1.1 is complete.

Let us now give a proof of Proposition 5.4. We start with the following lemma. For the definition of "analytic symbol", we refer to [11].

Lemma 5.5 *There exists a classical analytic symbol of degree* 0*,* $\sigma(\lambda; s, x, y)$*, defined for* $\lambda \geq 0$*,* $\sigma \simeq \sum_{n\geq 0} \lambda^{-n}\sigma_n$*, with holomorphic dependance on* $(s, x, y) \in W$*, such that the function defined for* $s > 0$ *and* (x, y) *close to* $Diag_M$

$$G(s, x, y) - s \int_1^{\infty} e^{-\lambda(s^2+d^2(x,y))} \lambda^{(m+1)/2} \sigma(\lambda; s, x, y) \frac{d\lambda}{\lambda} \tag{5.30}$$

extends holomorphically in W*. One has for some constants* A, B *and every* $N \geq 1$

$$\sup_W |\sigma(\lambda; s, x, y) - \sum_{j=0}^{N-1} \lambda^{-j} \sigma_j(s, x, y)| \leq AB^N (N)! \lambda^{-N}, \quad \forall \lambda \geq 1 \tag{5.31}$$

and

$$\sigma_0(0, y, y) = \pi^{-(m+1)/2} \tag{5.32}$$

Proof The proof of this lemma is classical, and is an easy by-product of Proposition 5.1. The functions $\sigma_j(s, x, y)$ are given explicitly in terms of the functions $a_j(s, x, y)$. For the convenience of the reader, we have include the explicit construction of the symbol σ in the appendix, where we recall the Borel summation technique which allows to associate to a given formal analytic asymptotic expansion a function σ such that (5.31) holds true. □

Let us verify that $T_s^* T_s$ is an elliptic pseudodifferential operator of degree $-(m-1)/2$. From Lemma 5.3 formula (5.23), and general results on wave front set of tensor product, non characteristic trace, and proper direct image (see [5]), the distribution product $\overline{\mathbb{P}_s(z, x)}\mathbb{P}_s(z, y) \in \mathcal{D}'(M \times M \times \partial B_s)$ is well defined. Moreover, the distribution $K_s \in \mathcal{D}'(M \times M)$ defined by

$$K_s(x, y) = \int_{\partial B_s} \overline{\mathbb{P}_s(z, x)}\mathbb{P}_s(z, y) d\mu_s(z) \tag{5.33}$$

satisfies

$$WF(K_s) \subset \{(x, y, \xi, \eta), x = y, \xi + \eta = 0\} = T^*_{Diag(M)} M \tag{5.34}$$

Since for $f \in C^\infty(M)$, one has $T_s^* T_s(f)(x) = \int_M K_s(x, y) f(y) d_g y$, it remains to verify that K_s is an elliptic pseudodifferential operator of degree $-(m-1)/2$. In order to compute the kernel $K_s(x, y)$ modulo a smooth function, by (5.34), we may assume that (x, y) is close to $(p, p) \in Diag(M)$. For (x, y) near $Diag(M)$, we will choose the coordinate system $(p, w) \in TM$, w small and $x = exp_p(w/2)$, $y = exp_p(-w/2)$ so that p is the middle point of the geodesic connecting y to x, and in these geodesic coordinates centered at p, one has $w = x - y$. By Lemma 5.2 we may also localize the integral in (5.33) for $z = Z(i, u, \xi)$, $|\xi|_u = s$, with u close to

p. Moreover, from Proposition 4.2, and Lemmas 5.3 and 5.5. we may replace in the definition (5.33) the kernel $\overline{\mathbb{P}_s(z,x)}\mathbb{P}_s(z,y)$ by the kernel

$$s^2\int_1^\infty\int_1^\infty e^{-\lambda(s^2+d^2(z,y))-\mu(s^2+\overline{d}^2(z,x))}\lambda^{(m+1)/2}\sigma(\lambda;s,z,y)\mu^{(m+1)/2}\overline{\sigma}(\mu;s,z,x)\frac{d\lambda}{\lambda}\frac{d\mu}{\mu} \tag{5.35}$$

Let $n_{j,p}$, $1\leq j\leq m$ be an orthonormal basis of T_pM. In geodesic coordinates centered at p, we write $u=exp_p(\sum a_jn_{j,p})$, and we denote by $\xi=(\xi_1,...,\xi_m)$ the dual coordinates of the (a_j). Recall that in geodesic coordinates, one has $g(a)=Id+O(a^2)$ and we define new coordinates b by the formula

$$b=b(a,\xi)=(g^{-1}(a))^{1/2}(\xi)=\xi+0(a^2\xi) \tag{5.36}$$

Then one has $b^2=|\xi|_a^2$, and we shall parametrize the set of points $z=Z(i,u,\xi)$, u close to p and $|\xi|_u=s$ by the coordinates $a\in\mathbb{R}^m$ close to 0 and $b=sv$, $v\in S^{m-1}$. We set also

$$\lambda=\rho\cos(\theta),\ \ \mu=\rho\sin(\theta),\ \ I_\rho(\theta)=1_{min(\cos(\theta),\sin(\theta))\geq 1/\rho}$$

Then, from formulas (5.19), (5.24) and (5.35), one find that near $Diag(M)$, the kernel $K_s(x,y)$ is equal to (modulo a smooth function)

$$\begin{aligned}
&\lim_{\mathcal{D}',r\to 0^+}\int_0^\infty\int_{S^{m-1}}E_{s+r}(s,x,y;\rho,v)s^2\rho^m d\rho\frac{d\sigma(v)}{c_m},\quad (\rho\in]0,\infty[,\ u\in S^{m-1})\\
&E_{s+r}(s,x,y;\rho,v)=\int_0^{\pi/2}I_\rho(\theta)\int_{\mathbb{R}^m}e^{-\rho\Psi_{s+r}}\Sigma_{s+r}(\sin\theta\cos\theta)^{(m-1)/2}\chi(a)\sqrt{det(g(a))}d\theta da\\
&\Psi_{s+r}(s,x,y,v;a,\theta)=\sin\theta((s+r)^2+\overline{d}^2(z,x))+\cos\theta((s+r)^2+d^2(z,y))\\
&\Sigma_{s+r}(s,x,y,v,\rho;a,\theta)=\sigma(\rho\cos\theta,s+r,z,y)\overline{\sigma}(\rho\sin\theta,s+r,z,x)\\
&z=Z(i,exp_p(\textstyle\sum a_jn_{j,p}),sg^{1/2}(a)(v))
\end{aligned} \tag{5.37}$$

Here, $\chi\in C_0^\infty(|a|\leq 2c_0)$ is a smooth cutoff function, equal to 1 in the ball $|a|\leq c_0$, with c_0 such that one has $|w|<<c_0<<inf(s\in I)$ (recall $x=exp_p(w/2)$, $y=exp_p(-w/2)$). By Lemma 3.3, one has

$$Re(\Psi_{s+r})\geq(\sin\theta+\cos\theta)((s+r)^2-s^2)+c_I(\sin\theta\, d^2(a,x)+\cos\theta\, d^2(a,y))$$

and in particular, for $r>0$, the integral in (5.37) is absolutely convergent. The key technical point is to verify that the analytic function

$$(a,\theta)\mapsto\Psi_{s+r}(s,x,y,v;a,\theta)$$

admits a unique non degenerate critical point $(a_c(r,s,x,y,v),\theta_c(r,s,x,y,v))$ close to $(0,\pi/4)$ for $s\in I$, r close to 0, x,y close to p and any $v\in S^{m-1}$, and that the

Hessian of Ψ_{s+r} at the critical point is non degenerate. To this end, we have just to verify that it is true for $r = 0$, $x = y = p$, and since s is small, we may even assume that the metric is flat. But in that case, we get easily

$$\Psi_s(s, p, p, v; a, \theta) = a^2(\sin\theta + \cos\theta) - 2is(\sin\theta - \cos\theta)a.v \tag{5.38}$$

which admits a unique critical point $(a_c, \theta_c) = (0, \pi/4)$. From the Taylor expansion

$$\Psi_s(s, p, p, v; a, \pi/4 + \varphi) = \sqrt{2}(a^2 - 2is\varphi\, a.v) \tag{5.39}$$

we get that this critical point is non degenerate. Observe also that the Hessian of $Re(\Psi_s)$ is strictly positive in the a directions. Therefore, for $s \in I$, r close to 0, x, y close to p and any $v \in S^{m-1}$, Ψ_{s+r} has a unique non degenerate critical point, and the Hessian of Ψ_{s+r} is strictly positive in the a directions. Let

$$\psi_s(r, x, y, v) = \Psi_{s+r}(s, x, y, v; a_c(r, s, x, y, v), \theta_c(r, s, x, y, v)))$$

be the critical value, which depends analytically on all parameters. In the flat case, one verifies easily that one has $(a_c, \theta_c) = (0, \pi/4)$ independently of $(x, y) = (w/2, -w/2)$. By Lemma 3.3 and Taylor expansion in $w = x - y$, one gets $(a_c, \theta_c) = (0, \pi/4) + O(w^2)$, and

$$\psi_s(r, x, y, v) = \sqrt{2}\Big((s + r)^2 - s^2 + is(x - y).v + Q(p, s, v; r, x - y)\Big) \tag{5.40}$$

where $Q(p, s, v; r, w)$ is analytic in $(p, s, v; r, w)$ and satisfies

$$Q(p, s, v; r, 0) = 0,\ \nabla_w Q(p, s, v; r, 0) = 0,\ Re(\partial_w^2 Q(p, s, v; 0, 0)) >> 0 \tag{5.41}$$

To compute the integral in (5.37), one has also to take care of the contribution of the end points near $\theta = 0$ and $\theta = \pi/2$, which comes from the truncation by $I_\rho(\theta)$. Let $1 = \chi_0(\theta) + \chi_c(\theta) + \chi_{\pi/2}(\theta)$ with $\chi_0(\theta)$ supported near 0, $\chi_{\pi/2}(\theta)$ supported near $\pi/2$ and $\chi_c(\theta) \in C_0^\infty(]0, \pi/2[)$ equal to 1 near $\pi/4$. Then the contribution of χ_0 (and the contribution of $\chi_{\pi/2}$) to the kernel $K_s(x, y)$ is a smooth function near $Diag(M)$: in fact, by integration by parts in (a, θ), we find that the contribution of χ_0 gives a kernel defined by an integral on the set $\lambda = \rho\cos(\theta) = 1$, which means that we are reduced to a kernel of the form $F(x, y) = \int_{\partial B_s} f(z, y)\overline{\mathbb{P}}_s(z, x)d\mu_s(z)$ with f smooth, and by (5.23) and the classical result on the wave front set of an integral, we get that F is smooth. (observe that we have already used this argument in formula (5.35)), since we have replaced $\mathbb{P}_s$ by $\mathbb{P}_s + f$ with f smooth).

Now, we can apply the phase stationary theorem to the contribution of χ_c, and we get

$$E_{c,s+r}(s, x, y; \rho, v) = e^{-\rho\psi_s(r,x,y,v)}\rho^{-(m+1)/2}\tilde{\sigma}_s(r, x, y, v; \rho) \tag{5.42}$$

where $\tilde{\sigma}_s(r, x, y, v; \rho)$ is a classical symbol of degree 0 in ρ, $\tilde{\sigma}_s \simeq \sum_{j\geq 0} \tilde{\sigma}_{s,j}$ $(r, x, y, v)\rho^{-j}$ with $\tilde{\sigma}_{s,j}$ analytic in (r, x, y, v).Then it is easy to pass to the limit $r \to 0^+$, and we get for (x, y) near $Diag(M)$, the equality, modulo a smooth function near $Diag(M)$:

$$K_s(x, y) = \int_1^\infty \int_{S^{m-1}} e^{i((x-y)s\sqrt{2}\rho v + i\rho Q(p,s,v;0,x-y))}\rho^{-(m-1)/2}\tilde{\sigma}_s(0, x, y, v; \rho)\frac{\rho^{m-1}d\rho d\sigma(v)}{c_m} \tag{5.43}$$

Then from (5.41) and (5.43), we get that $T_s^* T_s$ is a pseudodifferential operator of degree $-(m-1)/2$ (set $\xi = s\sqrt{2}\rho v$). The ellipticity follows easily from the definition of Σ_s given in (5.37) and formula (5.32).

Finally, from the identity

$$(T_s^* T_s(g)|g)_{L^2(M,d_g x)} = \|T_s(g)\|^2_{L^2(\partial B_s, d\mu_s)}$$

and the injectivity of T_s, we get that (5.25) holds true. The proof of Proposition 5.4 is complete.

Let us end these section by some results about the principal symbol of $T_s^* T_s$. The calculus we have done gives the principal symbol A of $T_s^* T_s$ equal to

$$\begin{aligned} A(s, x, \xi) &= C^{-1/2}(s, x, \xi/|\xi|_x)\Gamma_m(s|\xi|_x), \quad (\text{mod } |\xi|_x^{-(m+1)/2}) \\ C(s, x, u) &= s^{-2}(2\sqrt{2})^{-(m+1)} det(Hess(\Psi_s(s, x, x, u; ., .)))_{a_c=0,\theta_c=\pi/4} \end{aligned} \tag{5.44}$$

where the function Γ_m is defined in formula (2.13). To prove this point, we use formula (5.43) which gives

$$A(s, x, \xi) = (2\pi)^m(|\xi|_x/s\sqrt{2})^{-(m-1)/2}\tilde{\sigma}_{s,0}(0, x, x, \xi/|\xi|_x)(s\sqrt{2})^{-m}c_m^{-1}$$

Now we use stationary phase expansion to compute $\tilde{\sigma}_{s,0}(0, x, x, \xi/|\xi|_x)$. One has

$$\Psi_s(s, x, x, u; a, \theta) = \sin\theta(\overline{d}^2(z, x) + s^2) + \cos\theta(d^2(z, x) + s^2)$$

$$z = Z(i, exp_x(a), sg^{1/2}(a)(u))$$

From Lemma 3.3, we get that the critical point is $(a_c, \theta_c) = (0, \pi/4)$. Thus the function $A(s, x, \xi)$ is equal to (here we use (5.32) and the formula (5.37) for Σ_s)

$$A(s, x, \xi) = (2\pi)^m(|\xi|_x/s\sqrt{2})^{-(m-1)/2}s^2\pi^{-(m+1)}(\frac{1}{2})^{(m-1)/2}(det^{-1/2}(2\pi)^{(m+1)/2})(s\sqrt{2})^{-m}c_m^{-1}$$

where det is the value of the Hessian determinant of $\Psi_s(s, x, x, u; a, \theta)$ at the critical point $(a_c, \theta_c) = (0, \pi/4)$ which is equal to $s^2(2\sqrt{2})^{m+1}C(s, x, \xi/|\xi|_x)$. Hence we get

$$A(s,x,\xi) = C(s,x,\xi/|\xi|_x)\frac{\pi^{(m-1)/2}(s|\xi|)^{-(m-1)/2}}{c_m}$$

and the result follows from the fact that the principal symbol of $\Gamma_m(\eta)$ is equal to $(\pi/|\eta|)^{(m-1)/2}c_m^{-1}$.

The function C involves the second derivative in z of $d^2(z,x)$ at $z = Z(i,x,su)$, hence the curvature tensor of M. In the special case where $M = S_R^m = \{x \in \mathbb{R}^{m+1}, x^2 = R^2\}$ is the sphere of radius R in $\mathbb{R}^{m+1}$, one has

$$d^2(x,y) = R^2\psi(\frac{x.y}{R^2}), \quad \psi(u) = \theta^2 \Leftrightarrow \cos\theta = u$$

and

$$Z(i,x,su) = x\cosh(s/R) + iRu\sinh(s/R), \quad x \in S_R^m,\ u \in S_1^m,\ x.u = 0$$

which gives

$$d^2(Z(i,x,su),y) = R^2\psi\left(\frac{x.y\cosh(s/R) + iRu.y\sinh(s/R)}{R^2}\right)$$

These formulas allows to find the Taylor expansion at order 2 of Ψ_s at the critical point $(a_c,\theta_c) = (0,\pi/4)$, $(\theta = \pi/4 + \varphi)$:

$$\Psi_s \simeq \sqrt{2}\Big(|a|^2L(s/R) + (1 - L(s/R))(a.u)^2 - 2is\varphi a.u\Big), \quad L(u) = u\frac{\cosh(u)}{\sinh(u)}$$

Observe that $L(0) = 1$, thus when $R \to \infty$, this is compatible with the formula (5.39) of the flat case. Therefore, in the case of S_R^m, we get

$$C(s,x,u) = C(s) = (L(s/R))^{m-1}$$

which depends effectively on the parameter s.

6 Appendix

(1) **Analysis of the transport equations** (5.10) **and proof of the estimate** (5.3).

In the geodesic system of coordinates centered at y, $v \mapsto exp_y(v)$, the first order operator Z_l defined in (5.9) is of the form:

$$Z_l = 4(s\partial_s + \sum v_j\partial_{v_j} + l - \mu + g_y(v))$$

with $g_y(v)$ holomorphic in v near $v = 0$, analytic in y, and $g_y(0) = 0$. All the constructions below will depend analytically on y. Let us denote by z the coordinates $(s, v_1, ..., v_m)$. One has to study an equation of the form

$$\left(\sum_{j=1}^{m+1} z_j \partial_{z_j} + A(z)\right) f = g \tag{6.1}$$

where $A(z)$ is a holomorphic function defined near $z = 0$. The behavior of this equation depends on the value of $A(0) = l - \mu$.

When $A(0) = 0$, which is the case when $l = \mu$, the equation (6.1) with g=0, admits for any given $f(0)$, a unique holomorphic solution f defined near $z = 0$; this is easy to see, since with $A = \sum_\alpha A_\alpha z^\alpha$, and $f = \sum_\alpha f_\alpha z^\alpha$, the equation (6.1) with $g = 0$ is equivalent to

$$\forall \alpha, \quad |\alpha| f_\alpha + \sum_{\beta+\gamma=\alpha} A_\beta f_\gamma = 0.$$

Next assume that $Re(A(z)) \geq \nu > 0$ in the ball $\{z; |z| < r_0\}$. This will be the case when $l = \mu + j$, $j \geq 1$. Then for any given g holomorphic in this ball, the equation (6.1) admits a unique solution f holomorphic in this ball, and one has

$$f(z) = \int_0^1 exp\left(-\int_u^1 A(vz)\frac{dv}{v}\right) g(uz)\frac{du}{u} \tag{6.2}$$

From $Re(A(z)) \geq \nu > 0$ and (6.2), we get for any $\rho < r_0$

$$\sup_{|z|\leq\rho} |f(z)| \leq \frac{1}{\nu} \sup_{|z|\leq\rho} |g(z)| \tag{6.3}$$

Using the Cauchy inequalities to estimate the derivatives of an holomorphic function, we thus get that there exists a constant C such that the functions $a_j(z)$ defined by the transports equations (5.10) satisfies for any $j \geq 1$ and any $\rho_1 < \rho_2 < r_0$ the estimates

$$\sup_{|z|\leq\rho_1} |a_j(z)| \leq \frac{C}{j^2(\rho_2 - \rho_1)^2} \sup_{|z|\leq\rho_2} |a_{j-1}(z)| \tag{6.4}$$

Let $r < r_0/2$. For a given $j \geq 1$, set $\rho_{j,l} = r + lr/j$. Then we get from (6.4) by induction on $l \in \{1, ..., j\}$

$$\sup_{|z|\leq r} |a_j(z)| \leq \left(\frac{C}{r^2}\right)^j \sup_{|z|\leq 2r} |a_0(z)| \tag{6.5}$$

This proves the estimate (5.3) on the growth of the functions a_j in the complex domain.

2. Proof of Lemma 5.5
We will assume m even (the case m odd requires some minor modifications due to the logarithmic terms in formulas (5.4)). Recall $\mu = -(m+1)/2 \notin -\mathbb{N}$. Let $(a_j)_{j\geq 0}$ be a sequence of complex numbers satisfying bounds

$$|a_j| \leq A_1 B_1^j$$

Let us define the sequences σ_j and b_j by the formulas

$$\sigma_j \Gamma(-\mu - j) = a_j, \quad b_j = \sigma_j / j! \tag{6.6}$$

Here, $\Gamma(z) = \int_0^\infty e^{-x} x^{z-1} dx$ is the usual Gamma function. These sequences satisfy bounds of the form (with A_2, B_2 depending only on μ, A_1, B_1)

$$|\sigma_j| \leq A_2 B_2^j \ j!, \quad |b_j| \leq A_2 B_2^j$$

Let $\rho_0 \leq \frac{1}{4B_2}$ and let $\sigma(\lambda)$ be the holomorphic function of $\lambda \in \mathbb{C}$

$$\sigma(\lambda) = \lambda \int_0^{\rho_0} e^{-\lambda x} (\sum_j b_j x^j) dx \tag{6.7}$$

Then $\sigma(\lambda)$ is a classical analytic symbol of degree zero, with asymptotic expansion when $\lambda \to \infty$, $\sigma(\lambda) \simeq \sum_{j\geq 0} \lambda^{-j} \sigma_j$. In order to prove Lemma 5.5, we have just to verify that the function $H(\delta)$ defined for $\delta \in]0, B_1^{-1}[$ by the formula

$$H(\delta) = \delta^\mu \sum a_j \delta^j - \int_1^\infty e^{-\lambda\delta} \lambda^{-\mu} \sigma(\lambda) \frac{d\lambda}{\lambda} \tag{6.8}$$

extends holomorphically in the complex disc $|\delta| < r$ with r depending only on the constants μ, A_1, B_1. From (6.6), we will get the value of $\sigma_0(0, y, y)$ given in (5.32), since with d_m defined in (5.5), one has

$$d_m \Gamma((m+1)/2) = \int_{\mathbb{R}^m} \int_0^\infty e^{-t(1+x^2)} t^{(m+1)/2} \frac{dt dx}{t} = \pi^{m/2} \int_0^\infty e^{-t} \frac{dt}{\sqrt{t}} = \pi^{(m+1)/2}$$

Let $D \geq min(1, \rho_0^{-1})$ and define a function $\sigma_0(\lambda)$ of $\lambda \geq 1$ by the formula

$$\sigma_0(\lambda) = \sum_{j \leq \lambda/D} \sigma_j \lambda^{-j} \tag{6.9}$$

One has for $\lambda \geq 1$

$$\sigma(\lambda) - \sigma_0(\lambda) = \lambda \int_0^{\rho_0} e^{-\lambda x} \left(\sum_{j > \lambda/D} b_j x^j \right) dx - \lambda \sum_{j \leq \lambda/D} b_j \int_{\rho_0}^{\infty} e^{-\lambda x} x^j dx \tag{6.10}$$

From (6.6) and $\rho_0 B_2 \leq 1/2$, we get

$$|\lambda \int_0^{\rho_0} e^{-\lambda x} \left(\sum_{j > \lambda/D} b_j x^j \right) dx| \leq 2A_2 \left(\frac{1}{2}\right)^{\lambda/D}$$

One has

$$\int_R^{\infty} e^{-y} y^j dy = e^{-R} \int_0^{\infty} e^{-z} (R+z)^j dz \leq 2^j e^{-R} (R^j + j!)$$

From $B_2\rho_0 \leq 1/4$ and $1/D \leq \rho_0$ we thus get

$$|\lambda \sum_{j \leq \lambda/D} b_j \int_{\rho_0}^{\infty} e^{-\lambda x} x^j dx| \leq 4A_2 e^{-\lambda\rho_0}$$

Therefore one has

$$|\sigma(\lambda) - \sigma_0(\lambda)| \leq 6A_2 e^{-\lambda r_0}, \quad r_0 = min(\rho_0, \log(2)/D) \tag{6.11}$$

This implies that the function

$$\int_1^{\infty} e^{-\lambda\delta} \lambda^{-\mu} (\sigma(\lambda) - \sigma_0(\lambda)) \frac{d\lambda}{\lambda}$$

is holomorphic in the complex disc $|\delta| < r_0$, and it remains to analyze the function $H_0(\delta)$:

$$H_0(\delta) = \delta^{\mu} \sum_{j \geq 0} a_j \delta^j - \int_1^{\infty} e^{-\lambda\delta} \lambda^{-\mu} \sigma_0(\lambda) \frac{d\lambda}{\lambda} \tag{6.12}$$

Let us now verify the holomorphy of $H_0(\delta)$ in a complex disc $|\delta| < r$. For $\delta > 0$ and $z \in \mathbb{C}$, set

$$F(z, \delta) = \int_1^{\infty} e^{-\lambda\delta} \lambda^{z-1} d\lambda = \delta^{-\mu} \int_{\delta}^{\infty} e^{-x} x^{z-1} dx \tag{6.13}$$

The function $z \to F(z, \delta)$ is holomorphic in $z \in \mathbb{C}$ and one has the identity

$$\frac{\partial}{\partial\delta} F(z, \delta) = -F(z+1, \delta) \tag{6.14}$$

For $Re(z) > 0$, one has

$$\int_{\delta}^{\infty} e^{-x} x^{z-1} dx = \Gamma(z) - \int_{0}^{\delta} e^{-x} x^{z-1} dx$$

Thus, for $Re(z) > 0$, we get

$$F(z, \delta) = \Gamma(z)\delta^{-z} - \sum_{0}^{\infty} \frac{(-1)^l}{l!} \frac{\delta^l}{z+l} \tag{6.15}$$

and since $F(z, \delta)$ is holomorphic in $z \in \mathbb{C}$, this formula remains valid for any $z \in \mathbb{C} \setminus (-\mathbb{N})$.

Remark 6.1 To get a formula for $z = -k$, $k \in \mathbb{N}$, recall

$$\Gamma(-k+\varepsilon) = \frac{(-1)^k}{k!}(\varepsilon^{-1} + d_k + O(\varepsilon)), \quad d_k = \Gamma'(1) + 1 + \ldots + 1/k$$

Inserting this formula in (6.15) and passing to the limit $\varepsilon \to 0$, we get

$$F(-k, \delta) = \frac{(-1)^k}{k!}\delta^k(-\log(\delta) + d_k) - \sum_{l \neq k} \frac{(-1)^l}{l!} \frac{\delta^l}{l-k} \tag{6.16}$$

This formula is used to treat the case m odd. We leave the details to the reader.

By (6.6), (6.15), (6.9), and $D \geq 1$, one has

$$\begin{aligned} H_0(\delta) &= a_0\delta^{\mu} - \sigma_0 F(-\mu, \delta) + \delta^{\mu} \sum_{j \geq 1} a_j \delta^j - \sum_{j \geq 1} \sigma_j (jD)^{-(\mu+j)} F(-\mu - j, jD\delta) \\ &= \sum_{l=0}^{\infty} (-1)^l \frac{\delta^l}{l!} \Big(\frac{\sigma_0}{l-\mu} + \sum_{j \geq 1} \frac{\sigma_j (jD)^{l-\mu-j}}{l-\mu-j} \Big) \end{aligned} \tag{6.17}$$

Since one has $D \geq 4B_2$, the result follows from the estimates:

$$\sum_{j \geq 1} \frac{\sigma_j (jD)^{l-\mu-j}}{l-\mu-j} \leq A_2 C(\mu) D^{l-\mu} \sum_{j \geq 1} j^{l-\mu} (\frac{B_2}{D})^j, \quad \text{and} \quad \sum_{j \geq 1} j^{l-\mu} 4^{-j} \leq C(\mu) C_0^l l!$$

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Propagation of Analytic Singularities for Short and Long Range Perturbations of the Free Schrödinger Equation

André Martinez, Shu Nakamura and Vania Sordoni

Abstract We study the propagation of the analytic wave front set for solutions to the Schrödinger equation associated with perturbations of the free Laplacian.

1 Introduction

We are interested in the analytic singularities of the distributions $u = u(t, x)$ that are solutions in $\mathbb{R} \times \mathbb{R}^n$ to the Schrödinger equation,

$$\text{(Sch)} : \qquad \begin{cases} i\frac{\partial u}{\partial t} = Pu; \\ u_{|t=0} = u_0, \end{cases}$$

where $P = P(x, D_x)$ is a second-order symmetric differential operator on $\mathbb{R}^n$ with analytic coefficients (typically a perturbation of the Laplace operator $P_0 := -\frac{1}{2}\Delta$), and u_0 is in $L^2(\mathbb{R}^n)$ or, more generally, in some Sobolev space.

For such a problem, it is quite natural to wonder if the analyticity of u_0 implies that of $u(t)$ at time $t \neq 0$. But actually this is not true, as it can be seen from the example where $P = P_0$ and $u_0 = (-2i\pi)^{-\frac{n}{2}} e^{-i|x|^2/2}$. In this case, using that the distributional kernel of e^{-itP_0} is $(2i\pi t)^{-\frac{n}{2}} e^{i|x-y|^2/2t}$, one can see that $u(t)$ just coincides with $v(t-1)$, where v solves the same Schrödinger equation with initial date $v(0) = \delta$ (the Dirac measure at $x = 0$). In particular, $u(1) = \delta$ is singular, while $u(0)$ is analytic. Such a phenomenon is called "infinite propagation speed of singularities", and a question one may ask is: Is there any way to read the singularities of $u(t)$ easily on u_0?

A. Martinez (✉) · V. Sordoni
Dipartimento di Matematica, Università di Bologna, Piazza di Porta San Donato 5, 40127 Bologna, Italy
e-mail: andre.martinez@unibo.it

S. Nakamura
Graduate School of Mathematical Science, University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan

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As we shall see, the answer is essentially yes, in the sense that (under some non-trapping conditions) the analytic wave front set of $e^{itP_0}u(t)$ propagates in a very precise way (while that of $u(t)$ does not at all!).

As an example, in the particular case $P = P_0 + V$ where $V = V(x)$ is an analytic function tending to 0 at infinity (and thus, in that case, $u(t) = e^{-itP}u_0$), we will prove that, for all $t \in \mathbb{R}$, one has,

$$WF_a(e^{itP_0}u(t)) = WF_a(u_0)$$

or, equivalently,

$$WF_a(u(t)) = WF_a(e^{-itP_0}u_0).$$

Here, WF_a stands for the analytic wave front set, and the details of the proofs of the results we present here can be found in [7, 8] (see also [6] for related results).

2 Assumptions and Results

Let

$$P = \frac{1}{2}\sum_{j,k=1}^{n} D_j a_{j,k}(x) D_k + \frac{1}{2}\sum_{j=1}^{n}(a_j(x)D_j + D_j a_j(x)) + a_0(x)$$

on $\mathcal{H} = L^2(\mathbb{R}^n)$, where $D_j = -i\partial_{x_j}$, and assume that the coefficients $\{a_\alpha(x)\}$ satisfy to the following hypothesis. For $\nu > 0$ we denote

$$\Gamma_\nu = \left\{ z \in \mathbb{C}^n \;\middle|\; |\mathrm{Im}\, z| < \nu\langle \mathrm{Re}\, z\rangle \right\}.$$

Assumption A For each α, $a_\alpha(x) \in C^\infty(\mathbb{R}^n)$ is real-valued and can be extended to a holomorphic function on Γ_ν with some $\nu > 0$. Moreover, for $x \in \mathbb{R}^n$, the matrix $(a_{j,k}(x))_{1\le j,k\le n}$ is symmetric and positive definite, and there exists $\sigma > 0$ such that,

$$\begin{aligned} &\left|a_{j,k}(x) - \delta_{j,k}\right| \le C_0\langle x\rangle^{-\sigma}, \quad j,k = 1,\dots,n,\\ &\left|a_j(x)\right| \le C_0\langle x\rangle^{1-\sigma}, \qquad j = 1,\dots,n,\\ &\left|a_0(x)\right| \le C_0\langle x\rangle^{2-\sigma}, \end{aligned}$$

for $x \in \Gamma_\nu$ and with some constant $C_0 > 0$.

The case $\sigma > 1$ will be referred to as the *short range case*, while the case $\sigma \in (0, 1]$ as the *long range case*.

We denote by $p(x,\xi) := \frac{1}{2}\sum_{j,k=1}^{n} a_{j,k}(x)\xi_j\xi_k$ the principal symbol of P, and by $P_0 := -\frac{1}{2}\Delta$ the free Laplace operator. For any $(x,\xi) \in \mathbb{R}^{2n}$, we also denote by $(y(t;x,\xi), \eta(t;x,\xi)) = \exp tH_p(x,\xi)$ the solution to the Hamilton system,

$$\frac{dy}{dt} = \frac{\partial p}{\partial \xi}(y, \eta), \ \frac{d\eta}{dt} = -\frac{\partial p}{\partial x}(y, \eta), \tag{2.1}$$

with initial condition $(y(0), \eta(0)) = (x, \xi)$.

We say that a point $(x, \xi) \in T^*\mathbb{R}^n \backslash 0$ is forward non-trapping (respectively backward non-trapping) when $|y(t, x, \xi)| \to \infty$ as $t \to +\infty$ (resp. as $t \to -\infty$).

In that case, one can prove the existence of $\eta_+(x, \xi) \in \mathbb{R}^n$ (resp. $\eta_-(x, \xi)$) such that $\eta(t, x, \xi) \to \eta_+(x, \xi)$ as $t \to +\infty$ (resp. $\eta(t, x, \xi) \to \eta_-(x, \xi)$ as $t \to -\infty$).

If in addition $\sigma > 1$ (short range case), then one can also prove the existence of $y_\pm(x, \xi) \in \mathbb{R}^n$ such that,

$$|y_+(x, \xi) + t\eta_+(x, \xi) - y(t, x, \xi)| \to 0 \text{ as } t \to +\infty,$$

(resp. $|y_-(x, \xi) + t\xi_-(x, \xi) - y(t, x, \xi)| \to 0$ as $t \to -\infty$).

A proof of these two facts can be found, e.g., in [1], Lemma 2.2 (indeed, though only the short range case is treated, the proof given for the existence of $\eta_\pm(x, \xi)$ still works in the long range case).

Denoting by NT^+ (resp. NT^-) the set of forward (resp. backward) non-trapping points, we define the applications,

$$S_\pm \, : NT^\pm \to \mathbb{R}^{2n}$$

by

$$S_\pm(x, \xi) := (y_\pm(x, \xi), \eta_\pm(x, \xi)).$$

They respectively correspond to the forward and backward classical wave maps. For any distribution $u \in \mathcal{D}'(\mathbb{R}^n)$, we denote by $WF_a(u)$ the analytic wave front set of u (see, e.g., [13]), that can be described by introducing the FBI transform T defined by,

$$Tu(z, h) = \int e^{-(z-y)^2/2h} u(y) dy,$$

where $z \in \mathbb{C}^n$ and $h > 0$ is a small extra-parameter. Then, Tv belongs to the Sjöstrand space $H^{loc}_{\Phi_0}$ with $\Phi_0(z) := |\mathrm{Im}\, z|^2/2$ (see [13]), and a point (x, ξ) is not in $WF_a(u)$ if and only if there exists some $\delta > 0$ such that $Tu = \mathcal{O}(e^{(\Phi_0(z)-\delta)/h})$ uniformly for z close enough to $x - i\xi$ and $h > 0$ small enough (in this case, we also use the notation: $Tu \sim 0$ in $H_{\Phi_0, x-i\xi}$). By Cauchy-formula, this is also equivalent to the existence of some $\delta' > 0$ such that $\|e^{-\Phi_0/h} Tu\|_{L^2(\Omega)} = \mathcal{O}(e^{-\delta'/h})$ for some complex neighborhood Ω of $x - i\xi$.

In the **short range case**, our main result is,

Theorem 2.1 *Suppose Assumption A with $\sigma > 1$, and let $u_0 \in L^2(\mathbb{R}^n)$. Then,*

(i) For any $t < 0$, one has,

$$WF_a(e^{-itP}u_0) \cap NT^+ = S_+^{-1}(WF_a(e^{-itP_0}u_0)); \tag{2.2}$$

(ii) For any $t > 0$, one has,

$$WF_a(e^{-itP}u_0) \cap NT^- = S_-^{-1}(WF_a(e^{-itP_0}u_0)). \tag{2.3}$$

Remark 2.2 In the particular case where the metric is globally non-trapping, this result gives a complete characterization of the analytic wave front set of $u(t)$ in terms of that of $e^{-itP_0}u_0$.

Remark 2.3 By substituting $e^{itP}u_0$ to u_0, and $-t$ to t, this result implies that one has,

$$\forall t > 0, \quad WF_a(e^{itP_0}u(t)) = S_+(WF_a(u_0) \cap NT^+);$$
$$\forall t < 0, \quad WF_a(e^{itP_0}u(t)) = S_-(WF_a(u_0) \cap NT^-).$$

In particular, this set does not depend on $t > 0$ (resp. $t < 0$).

In the important case where $a_{j,k} = \delta_{j,k}$, then one has $NT^\pm = \mathbb{R}^{2n}\backslash 0$ and $S_\pm = Id$, and we obtain the following immediate corollary:

Corollary 2.4 *Suppose Assumption A with $\sigma > 1$ and $a_{j,k} = \delta_{j,k}$ for all pair (j,k). Then, for all $t \in \mathbb{R}$ and all $u_0 \in L^2(\mathbb{R}^n)$, one has,*

$$WF_a(e^{-itP}u_0) = WF_a(e^{-itP_0}u_0).$$

Remark 2.5 In the C^∞ setting, analogous results have been obtained Hassell and Wunsch in [2]. They involve a notion of "scattering wave front set" in a more general context of manifolds. In the case of $\mathbb{R}^n$, this notion mainly coincides with that of $WF(e^{itP_0}u)$ (see also [3, 4, 9–12, 14] for related questions).

Remark 2.6 Using the FBI transform (see, e.g., [5, 13]) and the expression of the distributional kernel of e^{-itP_0}, one can see that a point $(x_0,\xi_0) \in \mathbb{R}^{2n}\backslash 0$ is not in $WF_a(e^{-itP_0}u_0)$ if and only if there exists some $\delta > 0$ such that the quantity,

$$\mathcal{T}u_0(x,\xi : h) := \int e^{i(x-hy)\xi/h-(x-hy)^2/2h} e^{iy^2/2t} u_0(y)dy,$$

is $\mathcal{O}(e^{-\delta/h})$, uniformly for $h > 0$ small enough and (x,ξ) in a neighborhood of $(-\frac{1}{t}\xi_0, \frac{1}{t}x_0)$.

In the **long range case** ($0 < \sigma \leq 1$), the maps $S_\pm$ are not defined anymore, and one need to modify the free evolution near infinity in order to be able to define similar maps.

For $h > 0$ sufficiently small and $(x,\xi) \in \mathbb{R}^{2n}$, we denote by $\widetilde{p}(x,\xi;h)$ the quantity,

$$\widetilde{p}(x,\xi) := \frac{1}{2}\sum_{j,k} a_{j,k}(x)\xi_j\xi_k + h\sum_j a_j(x)\xi_j + h^2 a_0(x),$$

and by $(\widetilde{y}(t, x, \xi; h), \widetilde{\eta}(t, x, \xi; h)) := \exp tH_{\widetilde{p}}(x, \xi)$ the corresponding Hamilton flow. Then, we have the preliminary result,

Lemma 2.7 *For any $\delta_0 > 0$, there exist two h-dependent smooth functions,*

$$W_\pm \,:\, \mathbb{R}_\pm \times \{\xi \in \mathbb{R}^n\,;\, |\xi| > \delta_0\} \to \mathbb{R},$$

that are solutions to,

$$\frac{\partial W_\pm}{\partial t}(t, \xi) = \widetilde{p}(\nabla_\xi W_\pm(t, \xi), \xi; h), \tag{2.4}$$

and such that, for any $\pm t > 0$ and $(x, \xi) \in NT^\pm$, the quantity,

$$\widetilde{y}(t/h, x, \xi) - \nabla_\xi W_\pm(t/h, \widetilde{\eta}(t/h, x, \xi)) + \nabla_\xi W_\pm(0, \eta_\pm(x, \xi)) \tag{2.5}$$

admits a limit $\widetilde{y}_\pm(x, \xi) \in \mathbb{R}^n$ independent of t as $h \to 0_+$.

Remark 2.8 Actually, Eq. (2.4) must be satisfied up to short range terms only, in order to have (2.5). For instance, in the previous short range case, one can take $W_\pm(t, \xi) = t\xi^2/2$, that gives $\widetilde{y}_\pm(x, \xi) = y_\pm(x, \xi)$.

Using the notations of the previous lemma, we set,

$$\begin{aligned} \widetilde{S}_\pm(x, \xi) &:= (\widetilde{y}_\pm(x, \xi), \eta_\pm(x, \xi)), \quad ((x, \xi) \in NT^\pm); \\ z_\pm(x, \xi) &:= \widetilde{y}_\pm(x, \xi) - i\eta_\pm(x, \xi); \\ \widetilde{W}_\pm(t, \xi) &:= W_\pm(t, \xi) - W_\pm(0, \xi). \end{aligned} \tag{2.6}$$

Then, the result for the long range case is,

Theorem 2.9 *Suppose Assumption A with $0 < \sigma \le 1$, and let $u_0 \in L^2(\mathbb{R}^n)$. Then, with the notations (2.6), one has,*

(i) For any $t < 0$ and $(x, \xi) \in NT^+$, one has the equivalence,

$$(x, \xi) \notin WF_a(e^{-itP}u_0) \iff e^{i\widetilde{W}_+(-t/h, hD_z)/h}Tu_0 \sim 0 \text{ in } H_{\Phi_0, z_+(x,\xi)};$$

(ii) For any $t > 0$ and $(x, \xi) \in NT^-$, one has the equivalence,

$$(x, \xi) \notin WF_a(e^{-itP}u_0) \iff e^{i\widetilde{W}_-(-t/h, hD_z)/h}Tu_0 \sim 0 \text{ in } H_{\Phi_0, z_-(x,\xi)};$$

Remark 2.10 Here, the operator $e^{i\widetilde{W}_\pm(-t/h, hD_z)/h}$ appearing in the statement is not defined by the Spectral Theorem, but rather as a Fourier integral operator acting on Sjöstrand's spaces (see [8]).

Remark 2.11 Actually, $W_\pm$ can be constructed in such a way that the quantity $W_1^\pm(t,\xi) := \widetilde{W}_\pm(-t/h, hD_x)/h$ does not depend on h, and in principle, the fact that $e^{i\widetilde{W}_\pm(-t/h,hD_z)/h}Tu_0 \sim 0$ in $H_{\Phi_0, z_\pm(x,\xi)}$ essentially means that $\widetilde{S}_\pm(x,\xi) \notin WF_a(e^{iW_1^\pm(-t,D_x)}u_0)$ (and in this sense, the result is very similar to that of the C^∞ setting appearing in [11]). However, in order to define $e^{iW_1^\pm(-t,D_x)}$ properly one needs to extend $\widetilde{W}_\pm$ to all values of $\xi \in \mathbb{R}^n$, and this requires the use of cut-off functions. In the analytic setting, this introduces technical difficulties that can probably be overcome by the use of analytic pseudodifferential operators on the real domain (see [13]).

3 Sketch of Proof

We explain the proof for the forward non-trapping case only (the backward non-trapping case being similar), and we start by considering the short range case with a flat metric (that is, $a_{j,k} = \delta_{j,k}$ for all j, k, and thus $S_\pm(x,\xi) = (x,\xi)$).

Replacing u_0 by $e^{itP}u_0$, and then changing t to $-t$, we see that we have to prove that for any $t > 0$, one has

$$WF_a(u_0) = WF_a(e^{itP_0}e^{-itP}u_0).$$

Following [10], we set $v(t) := e^{itP_0}e^{-itP}u_0$, that solves the system,

$$i\frac{\partial v}{\partial t} = L(t)v \quad ; \quad v(0) = u_0. \tag{3.1}$$

Here,

$$L(t) = e^{itP_0}(P - P_0)e^{-itP_0} = L_2(t) + L_1(t) + L_0(t), \tag{3.2}$$

with,

$$\begin{aligned}
L_2(t) &:= \frac{1}{2}\sum_{j,k=1}^{n} D_j(a_{j,k}^W(x+tD_x) - \delta_{j,k})D_k \\
L_1(t) &:= \frac{1}{2}\sum_{\ell=1}^{n}(a_\ell^W(x+tD_x)D_\ell + D_\ell a_\ell^W(x+tD_x)) \\
L_0(t) &:= a_0^W(x+tD_x),
\end{aligned}$$

where we have denoted by $a^W(x,D_x)$ the usual Weyl-quantization of a symbol $a(x,\xi)$, defined by,

$$a^W(x,D_x)u(x) = \frac{1}{(2\pi)^n}\int e^{i(x-y)\xi}a((x+y)/2,\xi)u(y)dyd\xi.$$

Observe that, in the flat case, one has $L_2(t) = 0$. The expressions for $L_j(t)$, $0 \le j \le 2$ can be proved directly (using the fact that $e^{\pm iP_0}$ is just the multiplication by $e^{\pm i\xi^2/2}$ in the Fourier variables), but they also result from the standard Egorov theorem (that becomes exact in this case).

Since the FBI transform T is a convolution operator, we immediately observe that $TD_{x_j} = D_{z_j}T$. However, in order to study the action of $L(t)$ after transformation by T, we need the following key-lemma that will allow us to enter the framework of Sjöstrand's microlocal analytic theory. Mainly, this lemma tells us that, if f is holomorphic near Γ_ν, then, the operator $\tilde{T} := T \circ f^W(x + thD_x)$ is a FBI transform with the same phase as T, but with some symbol $\tilde{f}(t, z, x; h)$.

Lemma 3.1 ([7], Lemma 3.1) *Let f be a holomorphic function on Γ_ν, verifying $f(x) = \mathcal{O}(\langle x\rangle^\rho)$ for some $\rho \in \mathbb{R}$, uniformly on Γ_ν. Let also K_1 and K_2 be two compact subsets of $\mathbb{R}^n$, with $0 \notin K_2$. Then, there exists a function $\tilde{f}(t, z, x; h)$ of the form,*

$$\tilde{f}(t, z, x; h) = \sum_{k=0}^{1/Ch} h^k f_k(t, z, x), \tag{3.3}$$

where f_k is defined, smooth with respect to t and holomorphic with respect to (z, x) near $\Sigma := \mathbb{R}_t \times \{(z, x)\,;\ \mathrm{Re}\, z \in K_1,\ |\mathrm{Re}\,(z - x)| + |\mathrm{Im}\, x| \le \delta_0,\ \mathrm{Im}\, z \in K_2\}$ with $\delta_0 > 0$ small enough, and such that, for any $u \in L^2(\mathbb{R}^n)$, one has,

$$Tf^W(x + thD_x)u(z, h) = \int_{|x-\mathrm{Re}\, z|<\delta_0} e^{-(z-x)^2/2h}\tilde{f}(t, z, x, h)u(x)dx + \mathcal{O}(\langle t\rangle^{\rho_+} e^{(\Phi_0(z)-\varepsilon)/h}),$$

for some $\varepsilon = \varepsilon(u) > 0$ and uniformly with respect to $h > 0$ small enough, z in a small enough neighborhood of $K := K_1 + iK_2$, and $t \in \mathbb{R}$. (Here, we have set $\rho_+ = \max(\rho, 0)$.)

Moreover, the f_k's verify,

$$f_0(t, z, x) = f(x + it(z - x))\,;$$
$$|\partial_{z,x}^\alpha f_k(t, z, x)| \le C^{k+|\alpha|+1}(k + |\alpha|)!\langle t\rangle^\rho,$$

for some constant $C > 0$, and uniformly with respect to $k \in \mathbb{Z}_+$, $\alpha \in \mathbb{Z}_+^{2n}$, and $(t, z, x) \in \Sigma$.

Thanks to this lemma, and using again Sjöstrand's theory of microlocal analytic singularities [13], we deduce the existence of an analytic second-order (that is, with a symbol $\mathcal{O}(h^{-2})$) pseudodifferential operator $Q(t, h)$ on $H^{loc}_{\Phi_0}(\mathbb{C}^n \backslash \{\mathrm{Im}\, z = 0\})$, such that,

$$TL(t) = Q(t, h)T.$$

Moreover, in the flat case, $Q(t, h)$ becomes of the first order, and its symbol is mainly given by,

$$q(t,h;z,\zeta) \sim h^{-1}\sum_{\ell=1}^{n} a_\ell(z+i\zeta+th^{-1}\zeta)\zeta_\ell + a_0(z+i\zeta+th^{-1}\zeta).$$

Actually, using Lemma 3.1, an exact formula can be obtained for the symbol of $Q(t,h)$, that coincides with the previous expression up to $\mathcal{O}(1)$-terms as $h\to 0_+$. We refer to [7], Sect. 4, for more details.

Then, applying T to (3.1), multiplying it by h^2, and changing the time-scale by setting $s := t/h$, we obtain the new evolution equation,

$$ih\frac{\partial Tv}{\partial s} = B(s,h)Tv \quad ; \quad Tv(0) = Tu_0, \tag{3.4}$$

where $B(s,h)$ is an analytic pseudodifferential operator of order -1 (still in the sense of [13]), acting on $H^{loc}_{\Phi_0}(\mathbb{C}^n\backslash\{\mathrm{Im}\, z=0\})$, with symbol $b(s,h)$ verifying,

$$b(s,h) \sim \sum_{k\geq 1} h^k b_k(s)$$

(in the sense of analytic symbols), with

$$\begin{aligned} &b_1(s;z,\zeta) = \mathcal{O}(\langle s\rangle^{1-\sigma});\\ &b_k(s;z,\zeta) = \mathcal{O}(\langle s\rangle^{2-\sigma}) \text{ for } k\geq 2, \end{aligned} \tag{3.5}$$

uniformly with respect to $s>0$, and locally uniformly with respect to $z\in\mathbb{C}^n\backslash\{\mathrm{Im}\, z=0\}$ and ζ close enough to $-\mathrm{Im}\, z$ (note that, in particular, for $k\geq 2$ and $s=\mathcal{O}(h^{-1})$, one also has: $hb_k = \mathcal{O}(\langle s\rangle^{1-\sigma})$.)

Let us recall from [13] that the quantization of such a symbol $b(s,h;z,\zeta)$ on $H^{loc}_{\Phi_0}$ is given by,

$$B(s,h)w(z;h) = \frac{1}{(2\pi h)^n}\int_{\gamma(z)} e^{i(z-y)\zeta/h} b(s,h;z,\zeta)w(y)dyd\zeta,$$

where $\gamma(z)$ is a complex contour of the form,

$$\gamma(z)\ :\ \zeta = -\mathrm{Im}\, z + iR(\overline{z-y})\ ;\ |y-z|<r,$$

with $R>0$ is fixed large enough, and $r>0$ can be taken arbitrarily small. In particular, we deduce from (3.5) that $B(s,h)$ can be written as,

$$B(s,h) = hB_1(s,h),$$

where $B_1(s,h)$ admit a symbol uniformly $\mathcal{O}(\langle s\rangle^{1-\sigma}+h\langle s\rangle^{2-\sigma})$, for $s>0$, z in a compact subset of $\mathbb{C}^n\backslash\{\mathrm{Im}\, z=0\}$, and $(y,\zeta)\in\gamma(z)$.

Then, for $z_0 \in \mathbb{C}^n \backslash \{\mathrm{Im}\, z = 0\}$ and $\varepsilon_0 > 0$, if we set,

$$L^2_{\Phi_0}(z_0, \varepsilon_0) := L^2(\{|z - z_0| < \varepsilon_0\}; e^{-2\Phi_0/h} d\mathrm{Re}\, z\, d\mathrm{Im}\, z) \cap H_{\Phi_0}(|z - z_0| < \varepsilon_0),$$

we see that $B_1(s, h)$ is a bounded operator from $L^2_{\Phi_0}(z_0, \varepsilon_0)$ to $L^2_{\tilde{\Phi}_0}(z_0, \varepsilon_0/2)$, and its norm can be easily estimated in terms of the supremum of its symbol. Thus, here we obtain,

$$\|B_1(s)\|_{\mathcal{L}\left(L^2_{\Phi_0}(z_0,\varepsilon_0); L^2_{\Phi_0}(z_0,\varepsilon_0/2)\right)} = \mathcal{O}(\langle s\rangle^{1-\sigma} + h\langle s\rangle^{2-\sigma}) = \mathcal{O}(\langle s\rangle^{1-\sigma}), \tag{3.6}$$

uniformly with respect to $h > 0$ small enough and $|s| \le T_0/h$ ($T_0 > 0$ fixed arbitrarily).

Now, let us denote by $\tilde{\Phi}_0 = \tilde{\Phi}_0(z, \bar{z})$ a smooth real-valued function defined near $z = z_0$, such that $|\tilde{\Phi}_0 - \Phi_0|$ and $|\nabla_{(z,\bar{z})}(\tilde{\Phi}_0 - \Phi_0)|$ are small enough, and verifying,

$$\tilde{\Phi}_0 \ge \Phi_0 \text{ in } \{|z - z_0| \le \varepsilon_0\}; \tag{3.7}$$

$$\tilde{\Phi}_0 = \Phi_0 \text{ in } \{|z - z_0| \le \varepsilon_0/4\}; \tag{3.8}$$

$$\tilde{\Phi}_0 > \Phi_0 + \varepsilon_1 \text{ in } \{|z - z_0| \ge \varepsilon_0/2\}, \tag{3.9}$$

for some $\varepsilon_1 > 0$. By modifying the contour defining $B_1(s)$ (see [13], Remarque 4.4), we know that $B_1(s)$ is also bounded from $L^2_{\tilde{\Phi}_0}(z_0, \varepsilon_0)$ to $L^2_{\tilde{\Phi}_0}(z_0, \varepsilon_0/2)$, and its norm on these space verifies the same estimate (3.6) as on $L^2_{\Phi_0}$.

Setting $w = Tv$, Eq. (3.4) gives,

$$i\partial_s w(s) = B_1(s, h)w(s) \text{ in } H_{\Phi_0}(|z - z_0| < \varepsilon_0), \tag{3.10}$$

with $\varepsilon_0 > 0$ fixed small enough, and thus,

$$\partial_s \|w(s)\|^2_{L^2_{\tilde{\Phi}_0}(z_0,\varepsilon_0/2)} = 2\mathrm{Im}\, \langle B_1(s)w(s), w(s)\rangle_{L^2_{\tilde{\Phi}_0}(z_0,\varepsilon_0/2)}.$$

Using Cauchy–Schwarz inequality and (3.6), we obtain,

$$\left|\partial_s \|w(s)\|^2_{L^2_{\tilde{\Phi}_0}(z_0,\varepsilon_0/2)}\right| = \mathcal{O}(\langle s\rangle^{1-\sigma})\|w(s)\|^2_{L^2_{\tilde{\Phi}_0}(z_0,\varepsilon_0)}. \tag{3.11}$$

On the other hand, using (3.9) and the fact that $\|v(t)\|_{L^2} = \|u_0\|_{L^2}$ does not depend on t, we also have the estimate,

$$\|w(s)\|^2_{L^2_{\tilde{\Phi}_0}(z_0,\varepsilon_0)} = \|w(s)\|^2_{L^2_{\tilde{\Phi}_0}(z_0,\varepsilon_0/2)} + \mathcal{O}(e^{-\varepsilon_1/h}),$$

that, inserted into (3.11), gives,

$$\left|\partial_s \|w(s)\|^2_{L^2_{\tilde{\Phi}_0}(z_0,\varepsilon_0/2)}\right| \leq C\langle s\rangle^{1-\sigma}\|w(s)\|^2_{L^2_{\tilde{\Phi}_0}(z_0,\varepsilon_0/2)} + Ce^{-\varepsilon_1/h},$$

with some constant $C > 0$. Setting $g(s) := C\int_0^s \langle s'\rangle^{1-\sigma} ds'$, and using Gronwall's lemma, we finally obtain,

$$\|w(s)\|^2_{L^2_{\tilde{\Phi}_0}(z_0,\varepsilon_0/2)} \leq e^{g(s)}\|w(0)\|^2_{L^2_{\tilde{\Phi}_0}(z_0,\varepsilon_0/2)} + C\int_0^s e^{g(s)-g(s')-\varepsilon_1/h} ds';$$
$$\|w(0)\|^2_{L^2_{\tilde{\Phi}_0}(z_0,\varepsilon_0/2)} \leq e^{g(s)}\|w(s)\|^2_{L^2_{\tilde{\Phi}_0}(z_0,\varepsilon_0/2)} + C\int_0^s e^{g(s')-\varepsilon_1/h} ds'.$$

Then, replacing s by t/h and observing that $g(s) = \mathcal{O}(\langle s\rangle^{2-\sigma}) = \mathcal{O}(h^{\sigma-2}) = o(h^{-1})$, the equivalence $(x_0,\xi_0) \notin WF_a(u_0) \Longleftrightarrow (x_0,\xi_0) \notin WF_a(u(t))$ follows immediately, and the result is proved in this case.

Now, let us still consider the case where the perturbation is short range, but the metric is not necessarily flat anymore. Then, the result we have to prove is the following: for any $t > 0$ and $(x_0,\xi_0) \in NT^+$, one has the equivalence,

$$(x_0,\xi_0) \in WF_a(u_0) \Longleftrightarrow S_+(x_0,\xi_0) \in WF_a(e^{itP_0}e^{-itP}u_0).$$

Proceeding as in the flat case, we arrive again at Eq. (3.4), but this time $B(s,h)$ is of order 0, and can be written as,

$$B(s,h) = B_0(s,h) + hB_1(s,h),$$

where B_1 is as before, and the symbol of B_0 is,

$$b_0(s;z,\zeta) = \frac{1}{2}\sum_{j,k=1}^n (a_{j,k}(z+i\zeta+s\zeta) - \delta_{j,k})\zeta_j\zeta_k.$$

Then, in order to get rid of $B_0(s)$, we construct a Fourier integral operator $F(s,h)$ on H_{Φ_0,z_0}, verifying,

$$\begin{cases} ih\partial_s F(s,h) - B_0(s,h)F(s,h) \sim \mathcal{O}(h); \\ F\,|_{s=0} = I. \end{cases}$$

More precisely, we look for $F(s,h)$ of the form,

$$F(s)v(z) = \frac{1}{(2\pi h)^n}\int_{\gamma_s(z)} e^{i(\psi(s,z,\eta)-y\eta)/h} v(y)\,dy\,d\eta, \tag{3.12}$$

where $\gamma_s(z)$ is a convenient contour and ψ is a holomorphic function that must solve the system (eikonal equation),

$$\begin{cases} \partial_s\psi + b_0(s, z, \nabla_z\psi) = 0; \\ \psi\,|_{s=0} = z.\eta. \end{cases} \tag{3.13}$$

The construction of $\psi(s)$ for small s just follows from standard Hamilton-Jacobi theory, and the extension to larger values of s can be made by using the classical flow R_s of $b_0(s)$, that is related to the Hamilton flow of p through the formula,

$$R_s = \kappa \circ \exp(-sH_{p_0}) \circ \exp sH_p \circ \kappa^{-1}, \tag{3.14}$$

where $\kappa(x, \xi) = (x - i\xi, \xi)$ is the complex canonical transformation associated with T. We refer to [7], Sect. 6, for the detailed construction.

In that way, we find a solution $\psi(s, \zeta, \eta)$ of (3.13), defined for $s \in \mathbb{R}$, z close to $z_0 := x_0 - i\xi_0$ (where $(x_0, \xi_0) \in NT^+$ is fixed arbitrarily), and η close to ξ_0. One also has the relation,

$$(z, \nabla_z\psi(s, z, \eta)) = R_s(\nabla_\eta\psi(s, z, \eta), \eta), \tag{3.15}$$

which means that ψ is a generating function of the complex canonical transformation R_s. In other words, the operator $F(s, h)$ defined by (3.12) quantizes the canonical relation R_s, and, setting $z_s := \pi_z R_s(z_0, \xi_0)$ (where $\pi_z : (z, \zeta) \mapsto z$), one can show that for any $\varepsilon_0 > 0$ small, $F(s, h)$ acts as,

$$F(s) \;:\; H_{\Phi_0}(|z - z_0| < \varepsilon_0) \to H_{\Phi_0}(|z - z_s| < \varepsilon_1), \tag{3.16}$$

for some $\varepsilon_1 = \varepsilon_1(\varepsilon_0) > 0$. A priori, ε_1 also depends on s, but as a matter of fact, since R_s tends to $R_\infty := \kappa \circ S_+ \circ \kappa^{-1}$ on a neighborhood of (z_0, ξ_0) as $s \to +\infty$, one can prove that $F(s; h)$ admits a limit $F_\infty(h)$ that is a FIO quantizing R_∞. Then, the action (3.16) remains valid for $0 \le s \le +\infty$ (with $z_\infty := \pi_z R_\infty(z_0, \xi_0)$), ε_1 can be taken independent of s, and the norm of $F(s)$ is uniformly bounded both with respect to h and $s \ge 0$.

Now, by construction, for $s \in \mathbb{R}$, $F(s)$ verifies,

$$ih\partial_s F(s) - B_0(s)F(s) = hF_1(s),$$

where $F_1(s) \;:\; H_{\Phi_0}(|z - z_0| < \varepsilon_0) \to H_{\Phi_0}(|z - z_s| < \varepsilon_1)$ is of the form,

$$F_1(s)v(z) = \frac{1}{(2\pi h)^n} \int_{\gamma_s(z)} e^{i(\psi(s,z,\eta) - y\eta)/h} f_1(s, z, \eta; h)v(y)dyd\eta,$$

with f_1 is an analytic symbol that is $\mathcal{O}(\langle s\rangle^{-1-\sigma})$ as $s \to \infty$.

In the same way, for any y close enough to z_0, we can define a Fourier integral operator $\tilde{F}(s)$ of the form,

$$\tilde{F}(s)v(y) := \frac{1}{(2\pi h)^n} \int_{\tilde{\gamma}_s(y)} e^{i(y\eta - \psi(s,z,\eta))/h} v(z)dzd\eta,$$

(where $\tilde{\gamma}_s(y)$ is again a convenient contour), such that $\tilde{F}(s)$ maps $H_{\Phi_0}(|z - z_s| < \varepsilon_0)$ into $H_{\Phi_0}(|z - z_0| < \varepsilon_1)$, and verifies,

$$ih\partial_s \tilde{F}(s) + \tilde{F}(s)B = h\tilde{F}_1(s), \tag{3.17}$$

where $\tilde{F}_1(s) : H_{\Phi_0}(|z - z_s| < \varepsilon_0) \to H_{\Phi_0}(|z - z_0| < \varepsilon_1)$ is a FIO with same phase as $\tilde{F}(s)$ and symbol $\tilde{f}_1 = \mathcal{O}(\langle s\rangle^{-1-\sigma})$.

Now, setting,

$$\tilde{w}(s) = \tilde{F}(s)Tu(hs) \in H_{\Phi_0}(|z - z_0| < \varepsilon_1),$$

by (3.4) and (3.17), we see that $\tilde{w}$ verifies,

$$i\partial_s \tilde{w}(s) = \left[\tilde{F}(s)B_1(s) + \tilde{F}_1(s)\right] Tu(hs).$$

Moreover, since $A(s) := F(s)\tilde{F}(s)$ is an elliptic pseudodifferential operator on H_{Φ_0,z_s}, by taking a parametrix $\tilde{A}(s)$, we have,

$$Tu(hs) = \tilde{A}(s)F(s)w(s) \text{ in } H_{\Phi_0}(|z - z_s| < \varepsilon), \tag{3.18}$$

(for some $\varepsilon > 0$ independent of s), and thus, we obtain,

$$i\partial_s \tilde{w}(s) = \tilde{B}_1(s)\tilde{w}(s). \tag{3.19}$$

in $H_{\Phi_0}(|z - z_0| < \varepsilon')$, where $\tilde{B}_1(s) := \left[\tilde{F}(s)B_1(s) + \tilde{F}_1(s)\right]\tilde{A}(s)F(s)$ is a pseudodifferential operator on $H_{\Phi_0}(|z - z_0| < \varepsilon')$ with the same properties as $B_1(s)$ when $s \to +\infty$.

Thus, we are reduced to a situation completely similar to that of the flat case, and, if for instance $(x_0, \xi_0) \notin WF_a(u_0)$, the same arguments show that,

$$\|w(s)\|_{L^2_{\Phi_0}(z_0,\delta)} \le Ce^{-\delta/h},$$

for some positive constant δ independent of $h > 0$ small enough and $s \in [0, T/h]$. As a consequence, using (3.18) and the fact that $\tilde{A}(s)F(s)$ is uniformly bounded from $L^2_{\Phi_0}(z_0, \delta)$ to $L^2_{\Phi_0}(z_s, \delta')$ for some $\delta' > 0$, we obtain (with some new constant $C > 0$),

$$\|Tu(hs)\|_{L^2_{\Phi_0}(z_s,\delta')} \le Ce^{-\delta/h}.$$

Replacing s by t/h with $t > 0$ fixed, and observing that $z_{t/h}$ tends to $\kappa \circ S_+(x_0, \xi_0)$ as $h \to 0_+$, we conclude that $S_+(x_0, \xi_0) \notin WF_a(u(t))$. The converse can be seen in the same way, and thus Theorem 2.1 is proved.

In the **long range** case, the construction of $W_\pm$ results from standard Hamilton-Jacobi theory, and the proof is very similar, except that we now have to handle expressions like

$$e^{i\widetilde{W}_\pm(s,hD_z)/h}v(z;h) := \int_{\gamma(s,z)} e^{i(z-y)\zeta/h+i\widetilde{W}_\pm(s,\zeta)/h}v(y)dydz,$$

where $\gamma(s,z)$ is a good contour in the sense of [13], with some uniformity as $s \to \infty$. Then, one can show that $e^{i\widetilde{W}_\pm(s,hD_z)/h}$ is a Fourier integral operator acting of Sjöstrand's spaces H_{Φ_0}, in the sense that one has,

$$e^{i\widetilde{W}_\pm(s,hD_z)/h} : H_{\Phi_0}(\Omega_s(z_0,\varepsilon_1)) \to H_{\Phi_0}(\Omega_s(Z_s(z_0),\varepsilon_2)),$$

with $\varepsilon_1, \varepsilon_2 > 0$ small enough, and where we have set,

$$\Omega_s(Z,\varepsilon) := \{z \in \mathbb{C}^n\,;\ \langle s\rangle^{-1}|\text{Re }(z-Z)| + |\text{Im }(z-Z)| < \varepsilon\}.$$

We refer to [8] for more details.

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Pointwise Weyl Law for Partial Bergman Kernels

Steve Zelditch and Peng Zhou

Abstract This article is a continuation of a series by the authors on partial Bergman kernels and their asymptotic expasions. We prove a 2-term pointwise Weyl law for semi-classical spectral projections onto sums of eigenspaces of spectral width $\hbar = k^{-1}$ of Toeplitz quantizations $\hat{H}_k$ of Hamiltonians on powers L^k of a positive Hermitian holomorphic line bundle $L \to M$ over a Kähler manifold. The first result is a complete asymptotic expansion for smoothed spectral projections in terms of periodic orbit data. When the orbit is 'strongly hyperbolic' the leading coefficient defines a uniformly continuous measure on $\mathbb{R}$ and a semi-classical Tauberian theorem implies the 2-term expansion. As in previous works in the series, we use scaling asymptotics of the Boutet-de-Monvel–Sjostrand parametrix and Taylor expansions to reduce the proof to the Bargmann–Fock case.

This article is part of a series [18, 19] devoted to partial Bergman kernels on polarized (mainly compact) Kähler manifolds $(L, h) \to (M^m, \omega, J)$, i.e. Kähler manifolds of (complex) dimension m equipped with a Hermitian holomorphic line bundle whose curvature form is $\omega_h = \omega$. Partial Bergman kernels

$$\Pi_{k,<E} : H^0(M, L^k) \to \mathcal{H}_{k,<E} \tag{1}$$

are orthogonal projections onto proper subspaces $\mathcal{H}_{k,<E} \subset H^0(M, L^k)$ of the space of holomorphic sections of L^k. Let $H \in C^\infty(M, \mathbb{R})$ denote a classical Hamiltonian, let $\xi = \xi_H$ denote the Hamilton vector field of H, let ∇ be the Chern connection. The quantization of H is the Toeplitz Hamiltonian

$$\hat{H}_k := \Pi_{h^k} \left(\frac{i}{k} \nabla_\xi + H \right) \Pi_{h^k} : H^0(M, L^k) \to H^0(M, L^k). \tag{2}$$

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S. Zelditch (✉) · P. Zhou
Department of Mathematics, Northwestern University, Evanston, IL 60208, USA
e-mail: zelditch@math.northwestern.edu

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Here, $\Pi_{h^k} : L^2(M, L^k) \to H^0(M, L^k)$ is the orthogonal (Szegö or Bergman) projection. Let $\{\mu_{k,j}\}_{j=1}^{d_k}$ denote the eigenvalues of $\hat{H}_k$ on the d_k-dimensional space $H^0(M, L^k)$ and denote the eigenspaces by

$$V_k(\mu_{k,j}) := \{s \in H^0(M, L^k) : \hat{H}_k s = \mu_{k,j} s\}.$$

Also, denote the eigenspace projections by

$$\Pi_{k,j} := \Pi_{\mu_{k,j}} : H^0(M, L^k) \to V_k(\mu_{k,j}).$$

Then the partial Bergman kernels (1) are the projections onto the spectral subspaces

$$\mathcal{H}_{k,<E} := \{\hat{H}_k < E\} := \{s \in H^0(M, L^k) : \langle \hat{H}_k s, s\rangle < E\langle s, s\rangle\}$$

of (2).

In this article, we study the pointwise semi-classical Weyl asymptotics of $\Pi_{k,<E}(z)$ (1) in the conventional semi-classical scaling by $h = \frac{1}{k}$. The main results give asymptotics for the scaled pointwise Weyl sums,

$$\Pi^E_{k,f}(z) = \sum_j f(k(\mu_{k,j} - E))\Pi_{k,j}(z, z)$$

for various types of test functions f. Equivalently, we consider a sequence of measures on $\mathbb{R}$,

$$d\mu_k^{z,1,E}(\lambda) = \sum_j \Pi_{k,j}(z)\delta_{k(\mu_{k,j-E})}(\lambda). \tag{3}$$

then $\Pi^E_{k,f}(z) = \int_{\mathbb{R}} f(\lambda)d\mu_k^{z,1,E}(\lambda)$. When $f \in \mathcal{S}(\mathbb{R})$ with $\hat{f} \in C_c^\infty(\mathbb{R})$, Theorem 2.2 gives a complete asymptotic expansion. When $f = \mathbf{1}_{[a,b]}$ (the indicator function) one has sharp Weyl sums, and Theorem 1.7 gives a pointwise Weyl formula with 2 term asymptotics.

The $\frac{1}{k}$ scaling originates in the Gutzwiller trace formula and has been studied in numerous articles in diverse settings. Two-term pointwise Weyl laws is a standard topic in spectral asymptotics. The pointwise asymptotics in the Kähler setting are quite analogous to Safarov's asymptotic results for spectral projections of the Laplacian of a compact Riemannian manifold [14, 15] and we use Safarov's notations to emphasize the similarity. For general Kähler manifolds, integrated Weyl laws and dual Gutzwiller trace expansions were studied in [17] using the Toeplitz calculus of [3]. Pointwise Weyl laws of the type studied in this article are given in Borthwick-Paul-Uribe [2], based on the Boutet-de-Monvel–Guillemin Hermite Toeplitz calculus [3].

The main purpose of this paper is to prove pointwise Weyl asymptotics using the techniques developed in [18, 19]. Existence of an asymptotic expansion for smoothed Weyl sums is a straightforward consequence of a parametrix construction and of the method of stationary phase, replacing the elaborate symplectic spinor symbol calculus of [3]. However, the coefficients are complicated to compute. In the Toeplitz theory of [2, 3] they are calculated using the symplectic spinor symbol calculus of Toeplitz operators, while we use scaling asymptotics of the quantized flow in the sense of [13, 16, 19]. It is shown that the leading coefficients depend only on the quadratic part of the Taylor expansions. Hence, the coefficients are the same as in the linear model of [5] once the flow is linearized at a period. Our approach gives a somewhat simpler formula for the leading term than in [2] and it is not completely obvious that the formulae agree; in Sect. 8 we show that the formulae do agree with those of [2]. Related calculations using the scaling approach of this article are also given in articles of Paoletti [10, 11].

In the previous articles, we studied the scaling asymptotics of $\Pi_{k,<E}(z) := \Pi_{k,<E}(z,z)$ in a $\frac{1}{\sqrt{k}}$-tube around the interface $\partial\mathcal{A}$ between the allowed and forbidden regions,

$$\mathcal{A} := \{z : H(z) < E\}, \quad \mathcal{F} = \{z : H(z) > E\}.$$

This $\frac{1}{\sqrt{k}}$ scaling was the new feature of the Weyl asymptotics of [19] and is reminiscent of the scaling of the central limit theorem. The $\frac{1}{k}$-scaling was also studied in [19], but it was sufficient for the purposes of that article to obtain the crude asymptotics corresponding to the singularity of the Fourier transform $\widehat{d\mu_k^{z,1,E}}(t)$ at $t = 0$. Technically speaking, the main difference with respect to [19] is that the asymptotics of the $\frac{1}{\sqrt{k}}$ scaling only involve 'Heisenberg translations' while those of $d\mu_k^{z,1,E}$ involve the metaplectic representation. Although the notation and approach of this article have considerable overlap with [19] we give a rather detailed exposition for the sake of completeness.

1 Statement of Results

To state the results, we need some further notation. Given a Hermitian metric h on L, we denote by $X_h = \partial D_h^* \subset L^*$ the unit S^1 bundle $\pi : X_h \to M$ over M defined as the boundary of the unit co-disc bundle in the dual line bundle L^* to L. As reviewed in Sect. 3.5, X_h is a strictly pseudo-convex CR-manifold, and we denote the CR sub-bundle by $HX \subset TX_h$. As reviewed in Sect. 3.8, the Hamilton flow $g^t : M \to M$ lifts to a contact flow $\hat{g}^t : X_h \to X_h$ (Lemma 3.5) with respect to the contact structure α associated to the Kähler potential of ω. Then $HX = \ker\alpha$ and therefore $D\hat{g}^t : HX \to HX$. Moreover, HX inherits a complex structure J from that of M under the identification $\pi_* : H_xX \to T_{\pi(x)}M$, for all $x \in X$. Its complexification has a splitting $H_xX_{\mathbb{C}} = H_xX \otimes \mathbb{C} = H_x^{1,0}X \oplus H_x^{0,1}X$ into subspaces of types $(1,0)$ resp. $(0,1)$. In the generic case where $\hat{g}^t$ is non-holomorphic, it does not preserve this splitting.

At each point $x \in X$, the complexified CR subspace $HX_{\mathbb{C}}$ equipped with J_x together with the Hermitian metric h_x determines an *osculating Bargmann–Fock* *space* $\mathcal{H}_{J_x}$ (see Sects. 3.5 and 4.6 for background). Thus, $\mathcal{H}_{J_x}$ is the space of entire holomorphic functions on $H_x^{1,0}X$ which are square integrable with respect to the *ground state* Ω_{J_x} (defined in (19)). Symplectic transformations $T : H_x X \to H_x X$ resp. $T : T_z M \to T_z M$ may be quantized by the metaplectic representation as complex linear symplectic maps (see (29) and Sect. 4.4) on the osculating Bargmann–Fock space,

$$W_{J_x}(T) : \mathcal{H}_{J_x} \to \mathcal{H}_{J_x}. \tag{4}$$

The asymptotics of $\mu_k^{z,1,E}(f)$ depend on whether or not $z \in M$ is a periodic point for g^t.

Definition 1.1 Define periodic points of g^t, as follows:

$$\mathcal{P}_E := \{z \in H^{-1}(E) : \exists T > 0 : g^T z = z\}.$$

For $z \in \mathcal{P}_E$, let T_z denote the minimal period $T > 0$ of z.

It may occur that $z \in \mathcal{P}_E$ but the orbit $g_h^t(x)$ with $\pi(x) = z$ is not periodic, where g_h^t is the flow generated by the horizontal lift ξ_H^h of the Hamiltonian vector field ξ_H. This is due to holonomy effects: parallel translation of sections of L^k around the closed curve $t \mapsto g^t(z)$ may have non-trivial holonomy. We denote the holonomy by

$$e^{in\theta_z^h} := \text{the unique element } e^{i\theta} \in S^1 : g_h^{nT_z} x = r_\theta x.$$

Let $z \in \mathcal{P}_E$, $T = nT_z$ be a period for $n \in \mathbb{Z}$. Then Dg_z^T induces linear symplectic map

$$S := Dg_z^T : T_z M \to T_z M, \tag{5}$$

When working in the Kähler context it is better to conjugate to the complexifications,

$$T_z M \otimes \mathbb{C} = T_z^{1,0} M \oplus T_z^{0,1} M.$$

We denote the projection to the 'holomorphic component' by

$$\pi^{1,0} : T_z M \otimes \mathbb{C} \to T^{1,0} M.$$

The spaces $T^{1,0}M$, $T^{0,1}M$ are paired complex Lagrangian subspaces.

Relative to a symplectic basis $\{e_j, Je_k\}$ of $T_z M$ in which J assumes the standard form J_0, the matrix of Dg^{nT_z} has the form,

$$Dg_z^{nT_z} := S^n := \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix} \in Sp(m, \mathbb{R}). \tag{6}$$

If we conjugate to the complexification $T_z M \otimes \mathbb{C}$ by the natural map $\mathcal{W}$ defined in (27), then (6) conjugates to

$$\begin{pmatrix} P_n & Q_n \\ \bar{Q}_n & \bar{P}_n \end{pmatrix} \in Sp_c(m).$$

The holomorphic block

$$P_n = (A_n + D_n + i(-B_n + C_n)) = \pi^{1,0} \, \mathcal{W} S^n \mathcal{W}^{-1} \, \pi^{1,0} : T_z^{1,0} M \to T_z^{1,0} M \quad (7)$$

plays a particularly important role.

The symplectic map (5) is quantized by the metaplectic representation W_{J_z} (4) (see Sect. 4.4) on the osculating Bargmann–Fock space $\mathcal{H}_{J_z}$ of square integrable holomorphic functions on $T_z^{1,0}M$, that is, the metaplectic representation defines a unitary operator

$$W_{J_z} \, (Dg_z^{nT_z}) : \mathcal{H}_{J_z}(T_z^{1,0}M) \to \mathcal{H}_{J_z}(T_z^{1,0}M). \quad (8)$$

The two-term Weyl law is stated in terms of certain data associated to Dg^{nT_z} and $W_{J_z}(Dg^{nT_z})$ (8). First, we let $\mathcal{W}\xi_H$ be the image of the Hamilton vector field ξ_H in $T_z M \otimes \mathbb{C}$. Let $\alpha = \pi^{1,0}\mathcal{W}\xi_H$, let $\bar{\alpha} \in \pi^{0,1}\mathcal{W}\xi_H$, and let P_n be as in (7). Set,

$$\mathcal{G}_n(z) := (\det P_n)^{-\frac{1}{2}} \cdot (\bar{\alpha} \cdot P_n^{-1} \alpha)^{-\frac{1}{2}}. \quad (9)$$

The factor $(\det P)^{-\frac{1}{2}}$ has an interpretation,

$$(\det P_n)^{-1/2} = \langle W_{J_x} \, (Dg_x^{nT(x)}) \, \Omega_{J_z}, \Omega_{J_z} \rangle \quad (10)$$

as the matrix element of (8) relative to the ground state Ω_{J_z} in $\mathcal{H}_{J_z}$. This relation is essentially proved by Bargmann and by Daubechies [5]. It can be proved by comparing the Bargmann–Fock metaplectic representation of Sect. 4.4 with Daubechies' Toeplitz construction of metaplectic representation in Sect. 4.5. Daubechies did not explicitly use the conjugation $\mathcal{W}$ to the complexification, and therefore did not record the identity (10).

Also let $e^{in\theta_x^h}$ denote the holonomy of the horizontal lift of the orbit $t \to g^t(z)$ at $t = nT_z$. We define the function $Q_{z,k}^E(s)$ by:

Definition 1.2

$$Q_{z,k}^E(s) = \begin{cases} \mathcal{G}_0(z) & z \notin \mathcal{P}_E \\ \sum_{n\in\mathbb{Z}} (2\pi)^{-1} e^{-inT_z s} e^{-ink\theta_z^h} \, \mathcal{G}_n(z) & z \in \mathcal{P}_E. \end{cases} \quad (11)$$

Definition 1.3 For $z \in \mathcal{P}_E$, define the distributions $d\nu_k^z$ on $f \in \mathcal{S}(\mathbb{R})$ by

$$\int_{\mathbb{R}} f(\lambda) d\nu_k^z(\lambda) = \sum_{n\in\mathbb{Z}} \hat{f}(nT_z) \, \mathcal{G}_n(z) e^{-ink\theta_z^h} = \int_{\mathbb{R}} f(s) Q_{z,k}^E(s) ds$$

The nature of $Q_{z,k}(s)$ and ν_k^z depends on the type of periodic orbit of $z \in \mathcal{P}_E$. In this article we confine ourselves to the case where the orbit of z is 'real positive definite symmetric' in the following sense:

Definition 1.4 Let $z \in \mathcal{P}_E$, with $T_z = T$, and let $(T_z M, J_z, \omega_z)$ be the tangent space equipped with its complex structure and symplectic structure. Let $\{e_j, f_k\}_{j,k=1}^m$ be a symplectic basis of $T_z M$ in which $J = J_0$ and $\omega = \omega_0$ take the standard forms. We say that DG_z^T is *positive definite symmetric symplectic* if its matrix $S \in Sp(m, \mathbb{R})$ in the basis $\{e_j, f_k\}_{j,k=1}^m$ is a symmetric positive definite symplectic matrix.

Positive definite symplectic matrices are discussed in Sects. 3.1 and 3.2 and in Sect. 6.1. They are diagonalizable by orthogonal matrices in $O(2n)$ and by unitary matrices in $U(n)$. In invariant terms, $O(2n)$ is the orthogonal group of $(T_z M, g_{J_z})$ where $g_{J_z}(X, Y) = \omega_z(X, J_z Y)$. Unitary matrices commute with J_z. The eigenvalues of DG_z^T are real and to come in inverse pairs. The eigenvalue 1 corresponds to the Hamilton vector field ξ_H of H and there is a second eigenvector of eigenvalue 1 coming from the fact that periodic orbits come in 1-parameter families (symplectic cylinders) as the energy level E is varied (see [1]). The eigenvalues in the symplectic orthogonal complement of the eigenspace $V(1)$ of eigenvalue 1 come in unequal real inverse pairs λ, λ^{-1}. For expository simplicity, we omit the case where eigenvalues are complex of modulus $\neq 1$ and arise in 4-tuples $\lambda, \lambda^{-1}, \bar{\lambda}, \bar{\lambda}^{-1}$ (sometimes called loxodromic). We do discuss the elliptic case where $S \in U(n)$, and thus all of the eigenvalues have modulus 1 and come in complex conjugate pairs.

We refer to [6] for background on positive definite symmetric symplectic matrices and to [8] for types of periodic orbits of Hamiltonian flows.

Definition 1.5 We say that z satisfies the *strong hyperbolicity hypothesis* if $Dg_z^T : (T_z M, J_z) \to (T_z M, J_z)$ is a positive symplectic map, with a 2-dimensional symplectic eigenspace $V(1)$ for the eigenvalue 1.

The main motivation for this hypothesis is that we can explicitly compute (9) in this case (see Proposition 6.1). Almost the same computation works if Dg_z^T is unitary (the elliptic case) However, in the strong hyperbolic case, we can prove that the infinite series defining (11) converges absolutely and uniformly, and therefore:

Proposition 1.6 *If z satisfies the strong hyperbolicity hypothesis, then ν_k^z is an absolutely continuous measure.*

The main result is a sharp 2-term Weyl law in this case:

Theorem 1.7 *Assume that $z \in H^{-1}(E)$ and that z satisfies the strong hyperbolicity hypothesis. Then,*

$$\int_a^b d\mu_k^{z,1,E} = \begin{cases} \left(\frac{k}{2\pi}\right)^{m-1/2} \mathcal{G}_0(z)(b-a)(1+o(1)).,\ z \in H^{-1}(E),\ z \notin \mathcal{P}_E \\ \left(\frac{k}{2\pi}\right)^{m-1/2} \nu_k^z(a,b)(1+o(1)), \qquad z \in H^{-1}(E), z \in \mathcal{P}_E, \end{cases}.$$

Theorem 1.7 is a Kähler Toeplitz analogue of [15, Theorem 1.8.14] (originally proved in [14]). The difference between $z \notin \mathcal{P}_E$ and $z \in \mathcal{P}_E$ is that in the former case, there is a contribution only from the $t = 0$ times of g^t (the identity map) and in the latter case there are contributions from all iterates of g^{T_z}.

It may be expected that Theorem 1.7 extends in some suitable way to any type of periodic orbit. In the somewhat analogous Riemannian setting studied in [15], the pointwise Weyl law involves first return maps on the set of geodesic loop directions $\xi \in S^*_x M$ at a point $x \in M$ rather than closed orbits. In some cases (such as where x is a focus of an ellipsoid), the corresponding measures or Q-functions are calculated in [15, Example 1.8.20]. Otherwise, the authors say simply that it is difficult to determine when the "Q" function of [15, (1.8.11)] is uniformly continuous. It is likely that Theorem 1.7 can be extended to any orbit for which none of the eigenvalues on the symplectic orthogonal complement of the $V(1)$-eigenspace of S have modulus one. This is certainly the case, by the same proof as in Proposition 1.6, if S is diagonalizable by a unitary matrix.

2 Outline of the Proof

The proof is a continuation of that in [19], adding information on the remainder term and its relation to periodic orbits of periods $T > 0$. Given a function $f \in \mathcal{S}(\mathbb{R})$ (Schwartz space) one defines

$$f(k\hat{H}_k) = \int_{\mathbb{R}} \hat{f}(\tau) e^{ik\tau \hat{H}_k} d\tau = \int_{\mathbb{R}} \hat{f}(t) U_k(t) dt, \tag{12}$$

where

$$U_k(t) = \exp itk\hat{H}_k. \tag{13}$$

is the unitary group on $H^0(M, L^k)$ generated by $k\hat{H}_k$. Note that $f(k\hat{H}_k)$ is the operator on $H^0(M, L^k)$ with the same eigensections as $\hat{H}_k$ and with eigenvalues $f(k\mu_{k,j})$. The metric contraction of the Schwarz kernel on the diagonal is given by,

$$\Pi^E_{k,f}(z) = \int_{\mathbb{R}} \hat{f}(t) e^{-iktE} e^{ikt\hat{H}_k}(z, z) dt = \int_{\mathbb{R}} \hat{f}(t) e^{-iktE} U_k(t, z) dt. \tag{14}$$

Here, and henceforth, the metric contraction of a kernel $K_k(z, w)$ is denoted by $K(z)$.

Definition 2.1 The metric contraction of a kernel $M_k(z, w) := \sum_{j=1}^{d_k} \mu_{k,j} s_{k,j}(z) \overline{s_{k,j}(w)}$ expressed in an orthonormal basis $\{s_{k,j}\}_{j=1}^{d_k}$ of $H^0(M, L^k)$ is defined by

$$M_k(z) := \sum_{j=1}^{d_k} \mu_{k,j} |s_{k,j}(z)|^2_{h^k}, \quad (d_k = \dim H^0(M, L^k))$$

In Sect. 3.8 below, we lift sections and kernels to the associated $U(1)$ frame bundle of L^*; then metric contractions are the same as values of the lifts along the diagonal.

In [19] it is shown that $U_k(t)$ is a semi-classical Toeplitz Fourier integral operator of a type defined in [17]. As in [19] we construct a parametrix of the form,

$$\hat{\Pi}_{h^k}\sigma_{k,t}(\hat{g}^{-t})^*\hat{\Pi}_{h^k} \tag{15}$$

where $(\hat{g}^{-t})^*$ is the pullback of functions on X_h by $\hat{g}^t$ and where $\sigma_{k,t}$ is a semi-classical symbol originally calculated in [17, Unitarization Lemma 1 (2b.5) and (3.10)]. In fact, to leading order in k, and up to a phase factor,

$$\sigma_{kt}(z) = \langle \Omega_{Dg_z^T J_z}, \Omega_{J_{g^t z}} \rangle^{-\frac{1}{2}}. \tag{16}$$

Here, $Dg^T J_z$ is the image of the complex structure at z and $J_{g^t z}$ is the complex structure of $T_{g^t z}M$ and Ω_J denotes the ground state in the Bargmann–Fock Hilbert space with complex structure J. It was proved in [5, 17] that (16) equals $(\det P)^{-\frac{1}{2}}$ by calculating the inner product of the two Gaussians.

Combining (3) and (14) shows that

$$\mu_k^{z,1,E}(f) := \int_{\mathbb{R}} f(x) d\mu_k^{z,1,E} = \int_{\mathbb{R}} \hat{f}(t) e^{-iEkt} \hat{\Pi}_{h^k}\sigma_{kt}(\hat{g}^t)^*\hat{\Pi}_{h^k}(z) dt, \tag{17}$$

or equivalently

$$\widehat{\mu_k^{z,1,E}}(t) = e^{-iEkt} U_k(t, z, z). \tag{18}$$

Using a semi-classical Tauberian theorem, it is proved in Sect. 7 that the singularities of (18) determine the 2-term asymptotics of $\mu_k^{z,1,E}[a, b]$ for any interval. Proposition 1.6 follows because the singularities are of a different type depending on the convergence of $Q_z(k)$.

To prove the two-term Weyl law, we begin by obtaining asymptotics for the smoothed partial density of states (17). In the first case where $z \notin \mathcal{P}_E$, the only singularity occurs at $t = 0$ and so the expansion is the same as in [19, Theorem 3] (recalled here as Theorem 7.1). The time interval $[-\epsilon, \epsilon]$ is assumed to be so short that it contains no non-zero periods of periodic orbits. When $z \notin H^{-1}(E)$ the expansion is rapidly decaying. Thus, the new aspect is the second case where $z \in \mathcal{P}_E$.

Theorem 2.2 *For $f \in \mathcal{S}(\mathbb{R})$ with $\hat{f} \in C_c^\infty(\mathbb{R})$, we have (see Definitions 1.4 and 2.1)*

$$\Pi_{k,f}(z) := \int_{\mathbb{R}} f d\mu_k^{z,1,E} = \begin{cases} \left(\frac{k}{2\pi}\right)^{m-1/2} \hat{f}(0)\mathcal{G}_0(z)(1 + O(k^{-1}))., & z \in H^{-1}(E),\ z \notin \mathcal{P}_E \\ \left(\frac{k}{2\pi}\right)^{m-1/2} \sum_{n\in\mathbb{Z}} \hat{f}(nT_z)\ \mathcal{G}_n(z) e^{-ikn\theta_z^h} + O(k^{m-3/2}), & z \in H^{-1}(E), z \in \mathcal{P}_E, \\ O(k^{-\infty}), & z \notin H^{-1}(E) \end{cases}$$

To prove Theorem 2.2 we use the Boutet de Monvel–Sjöstrand parametrix for $\hat{\Pi}_{h^k}$. This gives a parametrix for (12) and (17) as semi-classical oscillatory integrals with complex phases. The phase has no critical points when the orbit does not lie in $H^{-1}(E)$ and no critical points for $t \neq 0$ when $z \notin \mathcal{P}_E$. The main difficulty is to evaluate or interpret the phases and the Hessian determinant (and other invariants that arise) dynamically, and to determine whether or not they are invariants of $D\hat{g}^T$ or invariants of the full orbit. One phase factor is a holonomy integral around the periodic orbit $\hat{g}^t(x)$. In Proposition 5.6 it is shown that although the holonomy is apriori a 'global invariant' of the orbit rather than an invariant of the first return map, in fact the Hessian of the holonomy can be expressed as an invariant of the first return map.

To evaluate the Hessian determinants, we first do so in the linear Bargmann–Fock setting, where H is a quadratic Hamiltonian on the Kähler manifold $\mathbb{C}^m$, equipped with a general complex structure J and a Hermitian metric h.

Proposition 2.3 *Let H be a quadratic Hamiltonian in the Bargmann–Fock setting. Assume that H has compact level sets and non-degenerate periodic orbits on level E. Then, in the notation of Definition 2.1,*

$$\int_{\mathbb{R}} \hat{f}(t) U_k(t, z) e^{-itEk} dt \simeq \left(\frac{k}{2\pi}\right)^{m-\frac{1}{2}} \sum_{n \in \mathbb{Z}} \hat{f}(nT_z) e^{-ik\theta_{z nT_z}} (\bar{\alpha} P_n^{-1} \alpha)^{-1/2} (\det P_n)^{-1/2},$$

where P_n is the holomorphic block of Dg^{nT_z} (7) and $\pi^{1,0}\mathcal{W}\xi_H = \alpha$.

We give a detailed proof in Sect. 5.4 because the general case is reduced to the Bargmann–Fock case. It is shown in this article that the linearized calculation is the principal symbol of non-linear problem (17), hence that Theorem 1.7 can be reduced to Proposition 2.3. The proof consists of nothing more than Taylor expansions of the phase in suitable Kähler normal coordinates and stationary phase.

3 Background

The background to this article is largely the same as in [19], and we refer there for many details. Here we give a quick review to setup the notation. First we introduce co-circle bundle $X \subset L^*$ for a positive Hermitian line bundle (L, h), so that holomorphic sections of L^k for different k can all be represented in the same space of CR-holomorphic functions on X, $\mathcal{H}(X) = \oplus_k \mathcal{H}_k(X)$. The Hamiltonian flow g^t generated by ξ_H on (M, ω) will be lifted to a contact flow $\hat{g}^t$ generated by $\hat{\xi}_H$ on X. Then we review the Toeplitz quantization for a contact flow on X following [13, 17].

3.1 Symplectic Linear Algebra

Let (V, σ) be a real symplectic vector space of dimension $2n$ and let J be a compatible complex structure on V. There exists a symplectic basis in which $V \simeq \mathbb{R}^{2m}$, σ takes the standard form $\omega = 2\sum_{j=1}^{m} dx_j \wedge dy_j$ and J has the standard form, $J_0 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$. Let $H_J^{1,0}$ resp. $H_J^{0,1}$, denote the $\pm i$ eigenspaces of J in $V \otimes \mathbb{C}$. The projections onto these subspaces are denoted by

$$P_J = \frac{1}{2}(I - iJ) : V \otimes \mathbb{C} \to H_J^{1,0}, \quad \bar{P}_J = \frac{1}{2}(I + iJ) : V \otimes \mathbb{C} \to H_J^{0,1}.$$

Let $S \in Sp(m, \mathbb{R})$ be a real symplectic matrix. Then its transpose $S^t = JS^{-1}J^{-1}$ also lies in $Sp(m, \mathbb{R})$ and $SJ = J(S^t)^{-1}$.

3.2 Symmetric Symplectic Matrices

A matrix S is called a symmetric symplectic matrix if $S \in Sp(n, \mathbb{R})$ and $S^t = S$. For such S it follows that $SJ = JS^{-1}$. A good reference for positive definite symplectic matrices is [8, p. 6] and [8, p. 52]. For the following see [6, Proposition 22]. Let $U(n) = Sp(n) \cap O(2n, \mathbb{R})$. Then $UJ = JU$ and

$$U = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}, \quad AB^t = B^tA, \quad AA^t + BB^t = I, \quad U^{-1} = \begin{pmatrix} A^t & B^t \\ -B^t & A^t \end{pmatrix} = U^t.$$

Proposition 3.1 *If S is a positive definite symmetric symplectic matrix and $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n; \lambda_1^{-1}, \ldots, \lambda_n^{-1})$ is the given diagonal matrix, then there exists $U \in U(n)$ so that $S = U^t \Lambda U$.*

The following is [6, Proposition 26].

Proposition 3.2 *A symplectic matrix S is symmetric positive definite if and only if $S = e^X$ with $X \in \mathfrak{sp}(n)$ and $X = X^t$. The map $\exp : \mathfrak{sp}(n) \cap Sym(2n, \mathbb{R}) \to Sp(n) \cap Sym_+(2n, \mathbb{R})$ is a diffeomorphism.*

If $e_1, \ldots, e_n$ are orthonormal eigenvectors of S corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_n$ then since $SJ = JS^{-1}$,

$$SJe_k = JS^{-1}e_k = \frac{1}{\lambda_j} Je_k.$$

Hence $\pm Je_1, \ldots, \pm Je_n$ are orthonormal eigenvectors of U corresponding to eigenvalues $\lambda_1^{-1}, \ldots, \lambda_n^{-1}$ and $\begin{pmatrix} A \\ B \end{pmatrix} = [e_1, \ldots, e_n]$.

3.3 The Bargmann–Fock Space of a Complex Hermitian Vector Space

The Bargmann–Fock spaces can be defined more generally for any complex structure J on $\mathbb{R}^{2n}$ and any Hermitian metric on $\mathbb{C}^n$.

Let (V, ω) be a real symplectic vector space. Define

$$\mathcal{J} = \{J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}, \quad J^2 = -I, \quad \omega(JX, JY) = \omega(X, Y), \quad \omega(X, JX) >> 0\}$$

to be the space of complex structures on $\mathbb{R}^n$ compatible with ω. The Bargmann–Fock space of a symplectic vector space (V, σ) with compatible complex structure $J \in \mathcal{J}$ is the Hilbert space,

$$\mathcal{H}_J = \{fe^{-\frac{1}{2}\sigma(v,Jv)} \in L^2(V, dL), \ f \text{ is entire J-holomorphic}\}.$$

Here,

$$\Omega_J(v) := e^{-\frac{1}{2}\sigma(v,Jv)} \tag{19}$$

is the 'vacuum state' and dL is normalized Lebesgue measure (normalized so that square of the symplectic Fourier transform is the identity). The orthogonal projection onto $\mathcal{H}_J$ is denoted by P_J in [5] but we denote it by Π_J in this article. Its Schwartz kernel relative to $dL(w)$ is denoted by $\Pi_J(z, w)$.

Remark: The Bargmann–Fock space with $J = i$ the standard complex structure is often defined instead as the weighted Hilbert space of entire holomorphic functions with Gaussian weight $C_n e^{-|z|^2} dL(z)$ where C_n is a dimensional constant. In this definition the vacuum state is 1. There is a natural isometric 'ground states' isomorphism to $\mathcal{H}_J$ defined by multiplying by $\sqrt{\Omega_J}$. With the Gaussian measure, the Bergman kernel is $B(z, w) = e^{z\cdot\bar{w}}$. When $V = \mathbb{C}^n$ we write $v = Z$, $JZ = iZ$, and $\sigma(Z, W) = \text{Im}\overline{Z} \cdot W$. Then $\Omega_J(Z) = e^{-\frac{1}{2}|Z|^2}$.

3.4 Bargmann–Fock Bergman Kernels

For BF model, we have $\Pi_k : L^2(M, L^k) \to H^0(M, L^k)$ the Bergman projection operator. And $\hat{\Pi}_k : L^2(X) \to \mathcal{H}_k(X)$, the Szego projection operator on X to Hardy space's Fourier component. Let H also denote its pull back on X.

The semi-classical Bargmann–Fock Bergman kernels (23) on $\mathbb{C}^n$ are given by

$$\Pi^{\mathbb{C}^m}_{k,h_0,i}(z,w) = \left(\frac{k}{2\pi}\right)^m e^{k(z\bar{w}-|z|^2/2-|w|^2/2)}.$$

Their lifts to X are given by

$$\hat{\Pi}^{\mathbb{C}^m}_{k,h_0,i}(\hat{z},\hat{w}) = \left(\frac{k}{2\pi}\right)^m e^{k\psi(\hat{z},\hat{w})}$$

where

$$\hat{\psi}(\hat{z},\hat{w}) = i(\theta_z-\theta_w)+\psi(z,w) = i(\theta_z-\theta_w)+z\cdot\bar{w}-|z|^2/2-|w|^2/2.$$

where $\hat{z} = (\theta_z, z) \in S^1 \times M \cong X$ denotes a lift of z.[1]

In the general case, by (3.1) of [5], one has

$$\Pi_J\psi(z) = \langle \Omega^z_J, \psi\rangle = \int_{\mathbb{C}^n} \psi(v)\overline{\Omega^z_J}(v)dv,$$

i.e.

$$\Pi_J(z,w) = \overline{\Omega^z_J}(w) = e^{i\sigma(z,w)}e^{-\frac{1}{2}\sigma(z-w,J(z-w))} \tag{20}$$

which reduces to $e^{i\mathrm{Im} z\bar{w}}e^{-\frac{1}{2}(|z-w|^2)} = e^{z\bar{w}}e^{-\frac{1}{2}(|z|^2+|w|^2)}$ in the case $J = i$, $h = h_0$.

3.5 Holomorphic Sections in L^k and CR-Holomorphic Functions on X

Let $(L,h) \to (M,\omega)$ be a positive Hermitian line bundle, L^* the dual line bundle. Let

$$X := \{p \in L^* \mid \|p\|_h = 1\}, \quad \pi : X \to M$$

be the unit circle bundle over M.

Let $e_L \in \Gamma(U,L)$ be a non-vanishing holomorphic section of L over U, $\varphi = -\log\|e_L\|^2$ and $\omega = i\partial\bar{\partial}\varphi$. We also have the following trivialization of X:

$$U \times S^1 \cong X|_U,\ (z;\theta) \mapsto e^{i\theta}\frac{e^*_L|_z}{\|e^*_L|_z\|}. \tag{21}$$

[1]We also use the notation $x = (z,\theta_z)$, $y = (w,\theta_w)$.

X has a structure of a contact manifold. Let ρ be a smooth function in a neighborhood of X in L^*, such that $\rho > 0$ in the open unit disk bundle, $\rho|_X = 0$ and $d\rho|_X \neq 0$. Then we have a contact one-form on X

$$\alpha = -\mathrm{Re}(i\bar{\partial}\rho)|_X,$$

well defined up to multiplication by a positive smooth function. We fix a choice of ρ by

$$\rho(x) = -\log \|x\|_h^2, \quad x \in L^*,$$

then in local trivialization of X (21), we have

$$\alpha = d\theta - \frac{1}{2} d^c \varphi(z). \tag{22}$$

X is also a strictly pseudoconvex CR manifold. The *CR structure* on X is defined as follows: The kernel of α defines a horizontal hyperplane bundle

$$HX := \ker \alpha \subset TX,$$

invariant under J since $\ker \alpha = \ker d\rho \cap \ker d^c \rho$. Thus we have a splitting

$$TX \otimes \mathbb{C} \cong H^{1,0}X \oplus H^{0,1}X \oplus \mathbb{C}R.$$

A function $f : X \to \mathbb{C}$ is CR-holomorphic, if $df|_{H^{0,1}X} = 0$.

A holomorphic section s_k of L^k determines a CR-function $\hat{s}_k$ on X by

$$\hat{s}_k(x) := \langle x^{\otimes k}, s_k \rangle, \quad x \in X \subset L^*.$$

Furthermore $\hat{s}_k$ is of degree k under the canonical S^1 action r_θ on X, $\hat{s}_k(r_\theta x) = e^{ik\theta}\hat{s}_k(x)$. The inner product on $L^2(M, L^k)$ is given by

$$\langle s_1, s_2 \rangle := \int_M h^k(s_1(z), s_2(z)) d\,\mathrm{Vol}_M(z), \quad d\,\mathrm{Vol}_M = \frac{\omega^m}{m!},$$

and inner product on $L^2(X)$ is given by

$$\langle f_1, f_2 \rangle := \int_X f_1(x)\overline{f_2(x)} d\,\mathrm{Vol}_X(x), \quad d\,\mathrm{Vol}_X = \frac{\alpha}{2\pi} \wedge \frac{(d\alpha)^m}{m!}.$$

Thus, sending $s_k \mapsto \hat{s}_k$ is an isometry.

3.6 Szegö Kernel on X

On the circle bundle X over M, we define the orthogonal projection from $L^2(X)$ to the CR-holomorphic subspace $\mathcal{H}(X) = \hat{\bigoplus}_{k\geq 0}\mathcal{H}_k(X)$, and degree-$k$ subspace $\mathcal{H}_k(X)$:

$$\hat{\Pi} : L^2(X) \to \mathcal{H}(X), \quad \hat{\Pi}_k : L^2(X) \to \mathcal{H}_k(X), \quad \hat{\Pi} = \sum_{k\geq 0} \hat{\Pi}_k.$$

The Schwarz kernels $\hat{\Pi}_k(x, y)$ of $\hat{\Pi}_k$ is called the degree-k Szegö kernel, i.e.

$$(\hat{\Pi}_k F)(x) = \int_X \hat{\Pi}_k(x, y) F(y) d\,\mathrm{Vol}_X(y), \quad \forall F \in L^2(X).$$

If we have an orthonormal basis $\{\hat{s}_{k,j}\}_j$ of $\mathcal{H}_k(X)$, then

$$\hat{\Pi}_k(x, y) = \sum_j \hat{s}_{k,j}(x)\overline{\hat{s}_{k,j}(y)}.$$

The degree-k kernel can be extracted as the Fourier coefficient of $\hat{\Pi}(x, y)$

$$\hat{\Pi}_k(x, y) = \frac{1}{2\pi} \int_0^{2\pi} \hat{\Pi}(r_\theta x, y) e^{-ik\theta} d\theta. \tag{23}$$

We refer to (23) as the *semi-classical Bergman kernels*.

3.7 Boutet de Monvel–Sjöstrand Parametrix for the Szegö Kernel

Near the diagonal in $X \times X$, there exists a parametrix due to Boutet de Monvel–Sjöstrand [4] for the Szegö kernel of the form,

$$\hat{\Pi}(x, y) = \int_{\mathbb{R}^+} e^{\sigma\hat{\psi}(x,y)} s(x, y, \sigma) d\sigma + \hat{R}(x, y). \tag{24}$$

where $\hat{\psi}(x, y)$ is the almost-CR-analytic extension of $\hat{\psi}(x, x) = -\rho(x) = \log \|x\|^2$, and $s(x, y, \sigma) = \sigma^m s_m(x, y) + \sigma^{m-1} s_{m-1}(x, y) + \cdots$ has a complete asymptotic expansion. In local trivialization (21),

$$\hat{\psi}(x, y) = i(\theta_x - \theta_y) + \psi(z, w) - \frac{1}{2}\varphi(z) - \frac{1}{2}\varphi(w),$$

where $\psi(z, w)$ is the almost analytic extension of $\varphi(z)$.

3.8 Lifting the Hamiltonian Flow to a Contact Flow on X_h

In this section we review the definition of the lifting of a Hamiltonian flow to a contact flow, following [19, Section 3.1]. Let $H : M \to \mathbb{R}$ be a Hamiltonian function on (M, ω). Let ξ_H be the Hamiltonian vector field associated to H, such that $dH = \iota_{\xi_H}\omega$. The purpose of this section is to lift ξ_H to a contact vector field $\hat{\xi}_H$ on X. Let α denote the contact 1-form (22) on X, and R the corresponding Reeb vector field determined by $\langle \alpha, R\rangle = 1$ and $\iota_R d\alpha = 0$. One can check that $R = \partial_\theta$.

Definition 3.3 (1) The horizontal lift of ξ_H is a vector field on X denoted by ξ_H^h. It is determined by

$$\pi_* \xi_H^h = \xi_H, \quad \langle \alpha, \xi_H^h \rangle = 0.$$

(2) The contact lift of ξ_H is a vector field on X denoted by $\hat{\xi}_H$. It is determined by

$$\pi_* \hat{\xi}_H = \xi_H, \quad \mathcal{L}_{\hat{\xi}_H}\alpha = 0.$$

Lemma 3.4 *The contact lift $\hat{\xi}_H$ is given by*

$$\hat{\xi}_H = \xi_H^h - HR.$$

The Hamiltonian flow on M generated by ξ_H is denoted by g^t

$$g^t : M \to M, \quad g^t = \exp(t\xi_H).$$

The contact flow on X generated by $\hat{\xi}_H$ is denoted by $\hat{g}^t$

$$\hat{g}^t : X \to X, \quad \hat{g}^t = \exp(t\hat{\xi}_H).$$

Lemma 3.5 *In local trivialization* (21)*, we have a useful formula for the flow, $\hat{g}^t$ has the form (see [19, Lemma 3.2]):*

$$\hat{g}^t(z, \theta) = \left(g^t(z), \quad \theta + \int_0^t \frac{1}{2}\langle d^c\varphi, \xi_H\rangle (g^s(z))ds - tH(z) \right).$$

Since $\hat{g}^t$ preserves α it preserves the horizontal distribution $H(X_h) = \ker \alpha$, i.e.

$$D\hat{g}^t : H(X)_x \to H(X)_{\hat{g}^t(x)}.$$

It also preserves the vertical (fiber) direction and therefore preserves the splitting $V \oplus H$ of TX. Its action in the vertical direction is determined by Lemma 3.5. When g^t is non-holomorphic, $\hat{g}^t$ is not CR holomorphic, i.e. does not preserve the horizontal complex structure J or the splitting of $H(X) \otimes \mathbb{C}$ into its $\pm i$ eigenspaces.

3.9 Toeplitz Quantum Dynamics

Here we consider quantization for the Hamiltonian flow g^t on holomorphic sections of L^k, or CR-functions of degree k on X. An operator $T : C^\infty(X) \to C^\infty(X)$ is called a *Toeplitz operator of order* k, denoted as $T \in \mathcal{T}^k$, if it can be written as $T = \hat{\Pi} \circ Q \circ \hat{\Pi}$, where Q is a pseudo-differential operator on X. Its principal symbol $\sigma(T)$ is the restriction of the principal symbol of Q to the symplectic cone

$$\Sigma = \{(x, r\alpha(x)) \mid r > 0\} \cong X \times \mathbb{R}_+ \subset T^*X.$$

The symbol satisfies the following properties

$$\begin{cases} \sigma(T_1 T_2) = \sigma(T_1)\sigma(T_2); \\ \sigma([T_1, T_2]) = \{\sigma(T_1), \sigma(T_2)\}; \\ \text{If } T \in \mathcal{T}^k, \text{ and } \sigma(T) = 0, \text{ then } T \in \mathcal{T}^{k-1}. \end{cases}$$

The choice of the pseudodifferential operator Q in the definition of $T = \hat{\Pi}\, Q\, \hat{\Pi}$ is not unique. However, there exists some particularly nice choices.

Lemma 3.6 ([3] Proposition 2.13) *Let T be a Toeplitz operator on Σ of order p, then there exists a pseudodifferential operator Q of order p on X, such that* $[Q, \hat{\Pi}] = 0$ *and* $T = \hat{\Pi}\, Q\, \hat{\Pi}$.

Now we specialize to the setup here, following closely [13]. Consider an order one self-adjoint Toeplitz operator

$$T = \hat{\Pi} \circ (H \cdot \mathbf{D}) \circ \hat{\Pi},$$

where $\mathbf{D} = (-i\partial_\theta)$ and ∂_θ is the fiberwise rotation vector field on X, and H is multiplication by $\pi^{-1}(H)$, where we abuse notation and identify H downstairs with its pullback upstairs $\pi^{-1}(H)$. We note that $\mathbf{D}$ decompose $L^2(X)$ into eigenspaces $\oplus_{k\in\mathbb{Z}} L^2(X)_k$ with eigenvalue $k \in \mathbb{Z}$. The symbol of T is a function on $\Sigma \cong X \times \mathbb{R}_+$, given by

$$\sigma(T)(x, r) = (\sigma(H)\sigma(\mathbf{D})|_\Sigma)(x, r) = H(x)r, \quad \forall (x, r) \in \Sigma.$$

Definition 3.7 ([13], Definition 5.1) Let $\hat{U}(t)$ denote the one-parameter subgroup of unitary operators on $L^2(X)$, given by

$$\hat{U}(t) := \hat{\Pi}\, e^{it\hat{\Pi}(\mathbf{D}H)\hat{\Pi}}\, \hat{\Pi} : \mathcal{H}(X) \to \mathcal{H}(X),$$

and let $\hat{U}_k(t)$ (13) denote the Fourier component acting on $L^2(X)_k$:

$$\hat{U}_k(t) := \hat{\Pi}_k \, e^{it\hat{\Pi}(kH)\hat{\Pi}} \, \hat{\Pi}_k : \mathcal{H}_k(X) \to \mathcal{H}_k(X) \tag{25}$$

We use $U_k(t)$ to denote the corresponding operator on $H^0(M, L^k)$.

Proposition 3.8 ([13], Proposition 5.2) $\hat{U}(t)$ *is a group of Toeplitz Fourier integral operators on* $L^2(X)$*, whose underlying canonical relation is the graph of the time* t *Hamiltonian flow of* rH *on the symplectic cone* Σ *of the contact manifold* (X, α)*.*

Proposition 3.9 ([17]) *There exists a semi-classical symbol* $\sigma_k(t)$ *so that the unitary group* (25) *has the form*

$$\hat{U}_k(t) = \hat{\Pi}_k (\hat{g}^{-t})^* \sigma_k(t) \hat{\Pi}_k$$

modulo smooth kernels of order $k^{-\infty}$*.*

It follows from the above proposition and the Boutet de Monvel–Sjöstand parametrix construction that $\hat{U}_k(t, x, x)$ admits an oscillatory integral representation of the form,

$$\hat{U}_k(t, x, x) \simeq \int_X \int_0^\infty \int_0^\infty \int_{S^1} \int_{S^1} e^{\sigma_1 \hat{\psi}(r_{\theta_1} x, \hat{g}^t y) + \sigma_2 \hat{\psi}(r_{\theta_2} y, x) - ik\theta_1 - ik\theta_2} S_k d\theta_1 d\theta_2 d\sigma_1 d\sigma_2 dy$$

where S_k is a semi-classical symbol, and the asymptotic symbol $\simeq$ means that the difference of the two sides is rapidly decaying in k.

4 Bargmann–Fock Space

In this section, we illustrate the various definition of the background section using the example of Bargmann–Fock (BF) space. We also define the osculating BF space for at the tangent space $T_z M$ for a general Kähler manifold, and show that in the semi-classical limit as $k \to \infty$ the Bergman kernel near the diagonal reduces to the BF model at leading order.

4.1 Set-Up

Let $M = \mathbb{C}^m$ with coordinate $z_i = x_i + \sqrt{-1} y_i$, $L \to M$ be the trivial line bundle. We fix a trivialization and identify $L \cong \mathbb{C}^m \times \mathbb{C}$. We use Kähler form $\omega = i \sum_i dz_i \wedge d\bar{z}_i$ and Kähler potential $\varphi(z) = |z|^2 := \sum_i |z_i|^2$.[2] The Bargmann–Fock space of degree k on $\mathbb{C}^m$ is defined by

$$\mathcal{H}_k = \left\{ f(z) e^{-k|z|^2/2} \mid f(z) \text{ holomorphic function on } \mathbb{C}^m, \quad \int_{\mathbb{C}^m} |f|^2 e^{-k|z|^2} < \infty \right\}.$$

The volume form on $\mathbb{C}^m$ is $d\,\mathrm{Vol}_{\mathbb{C}^m} = \omega^m / m!$.

[2] Our choice of ω may differ from other conventions by factors of 2 or π.

More generally, fix (V, ω) be a real $2m$ dimensional symplectic vector space. Let $J : V \to V$ be a ω compatible linear complex structure, that is $g(v, w) := \omega(v, Jw)$ is a positive-definite bilinear form and $\omega(v, w) = \omega(Jv, Jw)$. There exists a canonical identification of $V \cong \mathbb{C}^m$ up to $U(m)$ action, identifying ω and J. We denote the BF space for (V, ω, J) by $\mathcal{H}_{k,J}$.

The circle bundle $\pi : X \to M$ can be trivialized as $X \cong \mathbb{C}^m \times S^1$. The contact form on X is

$$\alpha = d\theta + (i/2)\sum_j (z_j d\bar{z}_j - \bar{z}_j dz_j).$$

If $s(z)$ is a holomorphic function (section of L^k) on $\mathbb{C}^m$, then its CR-holomorphic lift to X is

$$\hat{s}(z, \theta) = e^{k\left(i\theta - \frac{1}{2}|z|^2\right)} s(z).$$

Indeed, the horizontal lift of $\partial_{\bar{z}_j}$ is $\partial^h_{\bar{z}_j} = \partial_{\bar{z}_j} - \frac{i}{2} z_j \partial_\theta$, and $\partial^h_{\bar{z}_j} \hat{s}(z, \theta) = 0$. The volume form on $X = \mathbb{C}^m \times S^1$ is $d\,\mathrm{Vol}_X = (d\theta/2\pi) \wedge \omega^m/m!$.

4.2 Bergman Kernel on Bargmann–Fock Space

The degree k Bergman kernel downstairs on $\mathbb{C}^m$ is given by

$$\Pi_k(z, w) = \left(\frac{k}{2\pi}\right)^m e^{z\bar{w}}.$$

Given any function $f \in L^2(\mathbb{C}^m, e^{-k|z|^2/2} dVol_{\mathbb{C}^m})$, its orthogonal projection to holomorphic function is given by

$$(\Pi_k f)(z) = \int_{\mathbb{C}^m} \Pi_k(z, w) f(w) e^{-k|w|^2} d\,\mathrm{Vol}_{\mathbb{C}^m}(w).$$

The degree k Bergman (Szegö) kernel $\hat{\Pi}_k(\hat{z}, \hat{w})$ upstairs for $X = \mathbb{C}^m \times S^1$ is given by

$$\hat{\Pi}_k(\hat{z}, \hat{w}) = \left(\frac{k}{2\pi}\right)^m e^{k\hat{\psi}(\hat{z}, \hat{w})},$$

where $\hat{z} = (z, \theta_z)$, $\hat{w} = (w, \theta_w)$ and the phase function is

$$\psi(\hat{z}, \hat{w}) = i(\theta_z - \theta_w) + z\bar{w} - \frac{1}{2}|z|^2 - \frac{1}{2}|w|^2. \tag{26}$$

4.3 Heisenberg Representation

The space $\mathbb{C}^m \times S^1$ can be identified with the reduced Heisenberg group $\mathbb{H}^m_{red}$, where the group multiplication is given by

$$(z, \theta) \circ (z', \theta') = (z + z', \theta + \theta' + \mathrm{Im}(z\bar{z}')).$$

Lemma 4.1 *The contact form* $\alpha = d\theta + \frac{i}{2}\sum_j (z_j d\bar{z}_j - \bar{z}_j dz_j)$ *on* $\mathbf{H}^m_{red}$ *is invariant under the left multiplication*

$$L_{(z_0,\theta_0)} : (z, \theta) \mapsto (z_0, \theta_0) \circ (z, \theta) = \left(z + z_0, \theta + \theta_0 + \frac{z_0\bar{z} - \bar{z}_0 z}{2i}\right).$$

Proof

$$(L^*_{(z_0,\theta_0)}\alpha)|_{(z,\theta)} = d\left(\theta + \theta_0 + \frac{\bar{z}z_0 - \bar{z}_0 z}{2i}\right) + \frac{i}{2}\sum_j ((z_j + z_{0j})d\bar{z}_j - (\bar{z}_j + \bar{z}_{0j})dz_j) = \alpha|_{(z,\theta)}.$$

□

In particular, $\mathbb{H}^m_{red}$ preserves the volume form $\alpha \wedge (d\alpha)^m/m!$ on X, hence defines a unitary operator acting on the degree k CR functions on X.

The infinitesimal Heisenberg group action on X can be identified with contact vector field generated by a linear Hamiltonian function $H : \mathbb{C}^m \to \mathbb{R}$.

Lemma 4.2 ([19, Section 3.2]) *For any* $\beta \in \mathbb{C}^m$, *we define a linear Hamiltonian function on* $\mathbb{C}^m$ *by*

$$H(z) = z\bar{\beta} + \beta\bar{z}.$$

The Hamiltonian vector field on $\mathbb{C}^m$ *is*

$$\xi_H = -i\beta\partial_z + i\bar{\beta}\partial_{\bar{z}},$$

and its contact lift is

$$\hat{\xi}_H = -i\beta\partial_z + i\bar{\beta}\partial_{\bar{z}} - \frac{1}{2}(z\bar{\beta} + \beta\bar{z})\partial_\theta.$$

The time t flow $\hat{g}^t$ *on X is given by left multiplication*

$$\hat{g}^t(z, \theta) = (-i\beta t, 0) \circ (z, \theta) = (z - i\beta t, \theta - t\mathrm{Re}(\beta\bar{z})).$$

4.4 Metaplectic Representation

Let $\mathbb{R}^{2m}, \omega = 2\sum_{j=1}^m dx_j \wedge dy_j$ be a symplectic vector space. The space $Sp(m, \mathbb{R})$ consists of linear transformation $S : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$, such that $S^*\omega = \omega$. In coordinates, we write

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = S \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

In complex coordinates $z_i = x_i + iy_i$, we have then

$$\begin{pmatrix} z' \\ \bar{z}' \end{pmatrix} = \begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix} \begin{pmatrix} z \\ \bar{z} \end{pmatrix} =: \mathcal{A} \begin{pmatrix} z \\ \bar{z} \end{pmatrix},$$

where

$$\begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix} = \mathcal{W}^{-1} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mathcal{W}, \quad \mathcal{W} = \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ -iI & iI \end{pmatrix}. \tag{27}$$

The choice of normalization of $\mathcal{W}$ is such that $W^{-1} = W^*$. Thus,

$$P = \frac{1}{2}(A + D + i(C - B)).$$

We say such $\mathcal{A} \in Sp_c(m, \mathbb{R}) \subset M(2n, \mathbb{C})$. The following identities are often useful.

Proposition 4.3 ([7] Prop 4.17) *Let* $\mathcal{A} = \begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix} \in Sp_c$*, then*

(1) $\begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix}^{-1} = \begin{pmatrix} P^* & -Q^t \\ -Q^* & P^t \end{pmatrix} = K\mathcal{A}^*K$*, where* $K = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$.
(2) $PP^* - QQ^* = I$ *and* $PQ^t = QP^t$.
(3) $P^*P - Q^t\bar{Q} = I$ *and* $P^t\bar{Q} = Q^*P$.

The (double cover) of $Sp(m, \mathbb{R})$ acts on the (downstairs) BF space $\mathcal{H}_k$ via kernel: given $M = \begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix} \in Sp_c$, we have

$$\mathcal{K}_{k,M}(z, w) = \left(\frac{k}{2\pi}\right)^m (\det P)^{-1/2} \exp\left\{k\frac{1}{2}\left(z\bar{Q}P^{-1}z + 2\bar{w}P^{-1}z - \bar{w}P^{-1}Q\bar{w}\right)\right\}$$

where the ambiguity of the sign the square root $(\det P)^{-1/2}$ is determined by the lift to the double cover. When $\mathcal{A} = Id$, then $\mathcal{K}_{k,\mathcal{A}}(z, \bar{w}) = \Pi_k(z, \bar{w})$. Similarly, we have the kernel upstairs on X as

$$\hat{\mathcal{K}}_{k,\mathcal{A}}(\hat{z}, \hat{w}) = \mathcal{K}_{k,M}(z, \bar{w})e^{k(i\theta_z - |z|^2/2) + k(-i\theta_w - |w|^2/2)}. \tag{28}$$

A quadratic Hamiltonian function $H : \mathbb{C}^m \to \mathbb{R}$ will generates a one-parameter family of symplectic linear transformations $\mathcal{A}_t = g^t : \mathbb{C}^m \to \mathbb{C}^m$. However, $\mathcal{A}_t$ is only $\mathbb{R}$-linear but not $\mathbb{C}$-linear, i.e. M_t does not preserve the complex structure of $\mathbb{C}^m$. Hence, one need to orthogonal project back to holomorphic sections. To compensate for the loss of norm due to the projection, one need to multiply a factor $\eta_{\mathcal{A}_t}$. This is in the spirit of Proposition 3.9.

Proposition 4.4 *Let $\mathcal{A} : \mathbb{C}^m \to \mathbb{C}^m$ be a linear symplectic map, $\mathcal{A} = \begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix}$, and let $\hat{\mathcal{A}} : X \to X$ be the contact lift that fixes the fiber over* 0*, then*

$$\hat{\mathcal{K}}_{k,\mathcal{A}}(\hat{z}, \hat{w}) = (\det P^*)^{1/2} \int_X \hat{\Pi}_k(\hat{z}, \hat{\mathcal{A}}\hat{u})\hat{\Pi}_k(\hat{u}, \hat{w}) d\,\mathrm{Vol}_X(\hat{u})$$

Proof The contact lift $\hat{\mathcal{A}} : \mathbb{C}^m \times S^1 \to \mathbb{C}^m \times S^1$ is given by $\mathcal{A}$ acting on the first factor:

$$\hat{\mathcal{A}} : (z, \theta) \mapsto (Pz + Q\bar{z}, \theta),$$

one can check that $\hat{\mathcal{A}}^*\alpha = \alpha$. The integral over X is a standard complex Gaussian integral, analogous to [7, Prop 4.31], and with determinant Hessian $1/|\det P|$, hence we have $(\det P^*)^{1/2}/|\det P| = (\det P)^{-1/2}$. □

4.5 Toeplitz Construction of the Metaplectic Representation

As in [5], the metaplectic representation $W_J(S)$ of $S \in Mp(n, \mathbb{R})$ on $\mathcal{H}_J$ can also be constructed by the Toeplitz approach. First, let U_S be the unitary translation operator on $L^2(\mathbb{R}^{2n}, dL)$ defined by $U_S F(x, \xi) := F(S^{-1}(x, \xi))$. The metaplectic representation of S on $\mathcal{H}_J$ is given by ([5], (5.5) and (6.3 b))

$$W_J(S) = \eta_{J,S} \Pi_J U_S \Pi_J, \tag{29}$$

where we define (see [5] (6.1) and (6.3a)),

$$\eta_{J,S} = 2^{-n} \det(I - iJ) + S(I + iJ)^{\frac{1}{2}} \tag{30}$$

and Π_J is the Bargmann–Fock Szegö projector (20).

Also define $\beta_{J,SJS^{-1}} = 2^{-n/2}[\det(SJ + JS)]^{1/4}$. Then, $|\eta_{J,S}| = \beta_{J,SJS^{-1}}$. In fact (see [5], above (6.3a), and (B6))

$$|2^{-n} \det(I - iJ) + S(I + iJ)^{\frac{1}{2}}| = [\det(SJ + JS)]^{1/2} = 2^n \beta^2_{J,SJS^{-1}}.$$

We further record the identities,

$$\det(SJ + JS) = \det(I + J^{-1}S^{-1}JS) = \det(I + S^*S).$$

The following identity gives another explanation of the presence of $(\det P_n)^{-\frac{1}{2}}$ in (9).

Lemma 4.5 (see [5], p. 1388)

$$\eta_{JS}\beta^{-2}_{J,SJS^{-1}} = (\eta_{JS})^{*-1} = \eta_{JS}\ 2^n(\det(I + S^*S))^{-\frac{1}{2}}$$

and (cf. [5], p. 1388),

$$(\eta^*_{J,S})^{-1} = \det((I + iJ) + S(I - iJ)) = 2^n \det(A + D + i(B - C)) = \det P^*.$$

Proof The first equality is proved on p. 1388 of [5]. The second asserts that

$$\beta_{J,SJS^{-1}} = 2^{-n/2}(\det(I + S^*S))^{\frac{1}{4}},$$

which follows from (30) and identity (ii) above. □

Corollary 4.6 $\eta_{J,USU^{-1}} = \eta_{J,S}$ *where* $U \in U(m)$.

Proof This follows from replacing S by USU^{-1} and using that $UJ = JU$. □

4.6 Osculating Bargmann–Fock Space

In this subsection, we first define the osculation Bargmann–Fock space for any fixed point $z \in M$, using the triple (T_zM, ω_z, J_z). Then, we define the preferred local coordinates in a neighborhood U of z and a preferred frame section e_L of L over U, which together determines a coordinate system of the circle bundle $X|_U$ over U. In these special coordinate, the Boutet–Sjöstrand phase can be approximated by the Bargmann–Fock–Heisenberg phase function modulo cubic order terms.

Definition 4.7 Given a point $x \in X_h$ (resp. $z \in M$), we define the *osculating Bargmann–Fock space* at x (resp. z) to be the Bargmann–Fock space of (H_xX, J_x, ω_x) resp. (T_zM, J_z, ω_z). We denote it by $\mathcal{H}_{J_x,\omega_x}$ (resp. $\mathcal{H}_{J_z,\omega_z}$).

If z is a periodic point for g^t, let $\gamma = \bigcup_{0\le s\le t} g^s z$ be the corresponding closed geodesic, and we may apply the metaplectic representation to define $W_{J_z}(Dg^t|_z)$ as a unitary operator on $\mathcal{H}_{J_z,\omega_z}$. There is a square root ambiguity which can be resolved as in [5] but for our purposes it is not very important and for brevity we omit it from the discussion.

Definition 4.8 Let $p \in M$. A coordinate system $(z_1, \dots, z_m)$ on a neighborhood U of p is called *K-coordinates* at p if

$$i \sum_{j=1}^{m} dz_j \wedge d\bar{z}_j = \omega|_p .$$

Let e_L be a local frame and let $\phi(z) = -\log ||e_L(z)||_h^2$, if in a K-coordinates

$$\phi(z) = |z|^2 + \sum_{JK} a_{JK} z^J \bar{z}^K, \quad \text{with } |J| \geq 2, |K| \geq 2. \tag{31}$$

then e_L is called a K-frame.

K-coordinates are defined by Lu–Shiffman in Definition 2.6 of [9]. Existence of K-coordinates and K-frames are proved in [9] (Lemma 2.7). Further, in K-coordinates,

$$\omega = \omega_0 + \sum_{ijk\ell} R_{ijk\ell} z_i \bar{z}_j dz_k \wedge d\bar{z}_\ell + \cdots , \quad \omega_0 = \sum_j dz_j \wedge d\overline{z_j}.$$

The K-frame and K-coordinates together give us 'Heisenberg coordinates':

Definition 4.9 A *Heisenberg coordinate chart* at a point x_0 in the principal bundle X is a coordinate chart $\rho : U \to V$ with $0 \in U \subset \mathbb{C}^m \times S^1$ and $\rho(0) = x_0 \in V \subset X$ of the form

$$\rho(z_1, \dots, z_m, \theta) = e^{i\theta} \frac{e_L^*(z)}{||e_L^*(z)||_{h^k}} ,$$

where e_L is a preferred local frame for $L \to M$ at $P_0 = \pi(x_0)$, and $(z_1, \dots, z_m)$ are K-coordinates centered at P_0. (Note that P_0 has coordinates $(0, \dots, 0)$ and $e_L^*(P_0) = x_0$.)

In these coordinates, the Boutet–Sjöstrand phase $\psi(x, y)$ may be approximated modulo cubic remainder terms by the Bargmann–Fock–Heisenberg phase (26).

The lifted Szegö kernel is shown in [16] and in Theorem 2.3 of [9] to have the scaling asymptotics,

Theorem 4.10 *Let $P_0 \in M$ and choose a Heisenberg coordinate chart about P_0.*

$$k^{-m} \hat{\Pi}_{h^k} \left(\frac{u}{\sqrt{k}}, \frac{\theta_1}{k}, \frac{v}{\sqrt{k}}, \frac{\theta_2}{k} \right) = \hat{\Pi}_{h_z, J_z}^{T_z M}(u, \theta_1, v, \theta_2) \left(1 + k^{-1} A_1(u, v, \theta_1, \theta_2) + \cdots \right),$$

where $\Pi_{h_z, J_z}^{T_z M}$ is the osculating Bargmann–Fock Szegö kernel for $k = 1$ and for the tangent space $T_z M \simeq \mathbb{C}^m$ equipped with the complex structure J_z and Hermitian metric h_z.

Here we identify the coordinates $(u, \theta_1, v, \theta_2)$ with linear coordinates on $T_z M \times S^1 \times T_z M \times S^1$.

5 Proof of Theorem 2.2

In this section we study the rescaled Weyl sum

$$\Pi_{k,f}^{E}(z,z) := \sum_{j} f(k(\mu_{k,j}-E))\Pi_{k,j}(z,z).$$

Our purpose is to prove Theorem 2.2. By comparison with interface asymptotics [19], we now need to consider the Hamiltonian flow for long times.

The main idea of the proof is that aside from the holonomy factor (the value of the phase at the critical point), the data of the principal term in Theorem 2.2 localizes at the periodic point. That is, the data come from the derivative of the first return map and do not involve data along the orbit. Too see this, we use the quadratic Taylor approximation of the phase $\psi(x,\hat{g}^t y)+\psi(y,x)$ in (t,y) around a periodic point (T,x). First, we approximate the phase ψ by its osculating Bargmann–Fock approximation ψ_0 at x. Further we approximate $\hat{g}^t$ by its linear approximation $D\hat{g}^t$. We also need to determine the quadratic approximation to the holonomy term of the phase coming from the θ variable. This part of the calculation is apriori non-local. But we show in Proposition 5.6 that the Hessian of the holonomy term $\hat{\theta}_w(T)$ vanishes at the periodic point. After these Taylor approximations, the calculation is essentially reduced to the linear Bargmann–Fock case of Sect. 4.

5.1 *Stationary Phase Integral Expression*

Let $z\in M$ and $x\in X$ such that $\pi(x)=z$. Let $f\in\mathcal{S}(\mathbb{R})$ with Fourier transform $\hat{f}(t)=\int f(x)e^{itx}\frac{dx}{2\pi}$ compactly supported. We combine the definition (15) with two compositions of the Boutet de Monvel–Sjoestrand parametrix (24) to get

$$\begin{aligned}\Pi_{k,f}^{E}(z) &= \int_{\mathbb{R}} \hat{f}(t)e^{-itkE}\hat{U}_k(t,x,x)dt\\ &= \int_{\mathbb{R}}\int_X\int_{S^1}\int_{S^1}\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \hat{f}(t)e^{k\Psi(t,x,y,\sigma_1,\sigma_2,\theta_1,\theta_2)}A_k d\sigma_1 d\sigma_2 d\theta_1 d\theta_2 dy dt + O(k^{-\infty}).\end{aligned}$$

where the phase function is given by,

$$\Psi(t,x,y,\sigma_1,\sigma_2,\theta_1,\theta_2) = -itE + \sigma_1\hat{\psi}(r_{\theta_1}x,\hat{g}^t y) + \sigma_2\hat{\psi}(r_{\theta_2}y,x) - i\theta_1 - i\theta_2 \tag{32}$$

and A_k is a semi-classical symbol. We consider the critical points and the determinant of the Hessian matrix of the phase.

We will work with a K-coordinate and K-frame in a neighborhood U of z. In this coordinate, $z=(0,\dots,0)\in\mathbb{C}^m$, $x=(0,\dots,0;0)\in\mathbb{C}^m\times S^1$, and $y=(w;\theta_w)\in\mathbb{C}^m\times S^1$. We denote $\hat{g}^t y=(w(t);\theta_w(t))$. Since $\theta_w(t)-\theta_w$ only depends on w,t but independent of θ_w, then we define the holonomy phase for flow $\hat{g}^t$:

$$\hat{\theta}_w(t) := \theta_w(t) - \theta_w.$$

Similarly, the holonomy phase $\theta^h_w(t)$ for the horizontal flow $\exp(t\xi^h_H)$ is denoted by

$$\exp(t\xi^h_H)(w;\theta_w) = (g^t w; \theta_w + \theta^h_w(t)). \tag{33}$$

Note that $\hat{\theta}_w(t)$ depends on H, where as $\theta_w(t)$ only depend on H modulo constant,or dH.

Proposition 5.1 *Fix a K-coordinate and K-frame in a neighborhood U at z. Let $\chi : M \to \mathbb{R}$ be a smooth cut-off function supported in U and constant equals to one near z. Then we have*

$$\Pi^E_{k,f}(z) = \int_{\mathbb{R}}\int_M\int_{S^1}\int_{S^1}\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \hat{f}(t)e^{k\Psi'(t,w,\sigma_1,\sigma_2,\theta_1,\theta_2)}\chi(g^t w)\chi(w)S_k d\sigma_1 d\sigma_2 d\theta_1 d\theta_2 dw dt + O(k^{-\infty}).$$

where

$$\Psi'(t,w,\sigma_1,\sigma_2,\theta_1,\theta_2) = -itE + \sigma_1(i\theta_1 - i\hat{\theta}_w(t) - \varphi(w(t)) + \sigma_2(i\theta_2 - \varphi(w)) - i\theta_1 - i\theta_2. \tag{34}$$

Proof Introducing the cut-off function χ in the integral (32) only changes the integral by $O(k^{-\infty})$. Within the support of the cut-off function, we may use the K-coordinates.

Then phase function Ψ can be written as (within the coordinate patch):

$$\begin{aligned}\Psi &= -itE + \sigma_1(i\theta_1 - i\hat{\theta}_w(t) - i\theta_w + \psi(0, w(t)) - \varphi(w(t)) \\ &\quad + \sigma_2(i\theta_2 + i\theta_w + \psi(w, 0) - \varphi(w)) - i\theta_1 - i\theta_2 \\ &= -itE + \sigma_1(i\widetilde{\theta}_1 - i\hat{\theta}_w(t) - \varphi(w(t)) + \sigma_2(i\widetilde{\theta}_2 - \varphi(w)) - i\widetilde{\theta}_1 - i\widetilde{\theta}_2\end{aligned}$$

where $\widetilde{\theta}_1 = \theta_1 - \theta_w$ and $\widetilde{\theta}_2 = \theta_2 + \theta_w$. We note $\psi(0, w) = 0$ due to the choice of K-frame (31). After the change of variables, we see the phase Ψ does not depend on θ_w. Hence we may perform the θ_w integral, and rewrite $\widetilde{\theta}_i$ as θ_i, to get the reduced phase function Ψ'. □

Proposition 5.2 *The critical points for Ψ' (34) are as following:*
(1) If $z \notin H^{-1}(E)$, there is no critical points.
(2) If $z \in H^{-1}(E)$ but $z \notin \mathcal{P}_E$, then the only critical point corresponds to $t = 0$.
(3) If $z \in H^{-1}(E)$ and $z \in \mathcal{P}_E$, then for each $n \in \mathbb{Z}$, there is a critical point with $t = nT_z$, where T_z is the primitive period of g^t at z.

Proof We will prove that the critical points for Ψ' (32) are given by

$$w = 0,\ w(t) = 0,\ \sigma_1 = \sigma_2 = 1,\ \theta_1 = \hat{\theta}_0(t),\ \theta_2 = 0.$$

Taking derivatives of σ_1 and σ_2, we need to have

$$i\theta_1 - i\hat{\theta}_w(t) - \varphi(w(t) = 0, \quad i\theta_2 - \varphi(w) = 0.$$

Hence

$$\theta_1 = \hat{\theta}_0(t),\ \theta_2 = 0,$$

Thus, we may work in a neighborhood of x from now on.

Taking derivatives in θ_1 and θ_2 and setting them to zero, we get

$$\sigma_1 = 1, \quad \sigma_2 = 1.$$

Taking derivative in t and setting it to zero, we have

$$\frac{\partial \Psi'}{\partial t} = -iE + i\sigma_1 \frac{d\hat{\theta}_w(t)}{dt} = -i(E - \sigma_1 H(0)).$$

Thus, using $\sigma_1 = 1$, we have $E = H(0)$.

Finally, taking derivatives in w, we have

$$\frac{\partial \Psi'}{\partial w} = -i\sigma_1 \partial_w \hat{\theta}_w(t) = -i\partial_w \theta_w(T)$$

where T is a period. Since $\hat{g}^T$ preserves horizontal space, and ∂_w is in the horizontal space at $x = (0; 0)$, hence

$$\partial_w \theta_w(T) = \langle \alpha|_x, (\hat{g}^T|_x)_* \partial_w \rangle = \langle \alpha|_x, \partial_w \rangle = \langle d\theta, \partial_w \rangle = 0.$$

□

5.2 *Determinant of Hessian of* Ψ'

Let T be a period of g^t at z (possibly zero). To compute the contribution at $t = T$, we will do a slight change of variables.

Lemma 5.3 *Define new integration variables*

$$t = T + t',\ w = g^{-t'} w',\ \theta_1 = \theta_1' - \hat{\theta}_{w'}(-t'),\ \theta_2 = \theta_2' + \hat{\theta}_{w'}(-t').$$

Then the Jacobian factor is 1*, and the phase function* Ψ_T *in the new variables is*

$$\Psi_T(t', w', \sigma_i, \theta_i') = -i(T + t')E + \sigma_1(i\theta_1' - i\hat{\theta}_{w'}(T) - \varphi(w'(T)) + \sigma_2(i\theta_2' + \hat{\theta}_{w'}(-t') - \varphi(w'(-t'))) - i\theta_1' - i\theta_2'.$$

(We will drop the prime from now on.)

Proof The Jacobian matrix is block-upper-triangular, with the $w - w'$ block having determinant 1, since g^t preserves the volume form.

The holonomy for flow $\hat{g}^t$ can be written as

$$\hat{\theta}_w(t) = \theta_w(t) - \theta_w(0) = \theta_{w'}(T) - \theta_{w'}(-t') = \hat{\theta}_{w'}(T) - \hat{\theta}_{w'}(-t').$$

□

Lemma 5.4 *The Hessian matrix for* $\Psi_T(t, w, \sigma_i, \theta_i)$ *at* $t = 0, w = 0, \sigma_i = 1, \theta_1 = \hat{\theta}_0(T), \theta_2 = 0$ *is as*

$$Hess\Psi_T = \begin{array}{c} \\ \sigma_1 \\ \theta_1 \\ \sigma_2 \\ \theta_2 \\ t \\ w \end{array} \begin{array}{c} \begin{array}{cccccc} \sigma_1 & \theta_1 & \sigma_2 & \theta_2 & t & w \end{array} \\ \begin{bmatrix} 0 & i & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \partial_{tt}\Psi_T & \partial_{tw}\Psi_T \\ 0 & 0 & 0 & 0 & \partial_{wt}\Psi_T & \partial_{ww}\Psi_T \end{bmatrix} \end{array}.$$

In particular, at this critical point, we have

$$\det Hess\Psi_T = \det \begin{pmatrix} \partial_{tt}\Psi_T & \partial_{tw}\Psi_T \\ \partial_{wt}\Psi_T & \partial_{ww}\Psi_T \end{pmatrix}.$$

Proof The calculation is very similar to that in the proof of Proposition 5.2, and is therefore omitted. □

5.3 Quadratic Approximation to the Phase

To compute the Hessian of the phase function Ψ_T in t and w, suffice to set σ_i, θ_i to their critical value, and compute the Taylor expansion of Ψ_T to second order. Thus, we get

$$\Psi'_T(t, w) := -i(T + t)E - i\hat{\theta}_w(T) - \varphi(w(T)) + i\hat{\theta}_w(-t) - \varphi(w(-t)).$$

We will consider second order Taylor expansion in each term. We write $\simeq$ for equal modulo cubic order term.

Suppose H has Taylor expansion

$$H(w) = E + (\alpha\bar{w} + w\bar{\alpha}) + O(|w|^2).$$

We define the corresponding H_{BF} for the osculating BF space $\mathbb{C}^m \cong T_z M$, as the linear term of H:

$$H_{BF}(w) = \alpha\bar{w} + w\bar{\alpha}.$$

We denote the BF model potential as $\varphi_{BF}(z) = |z|^2$. Let $\hat{g}^t_{BF}$ be the flow generated by H_{BF} on $X_{BF} = \mathbb{C}^m \times S^1$, such that

$$\hat{g}^t_{BF}(w; \theta_w) = (w(t)_{BF}; \theta_w + \hat{\theta}_w(t)_{BF}).$$

Then, we have the following comparison result

Proposition 5.5 *(1)* $\hat{\theta}_w(-t) - tE = \hat{\theta}_w(-t)_{BF} + O_3 = \frac{1}{2}(\alpha\bar{z} + z\bar{\alpha})t.$
(2) $\varphi(w(T)) = |Dg^T w|^2 + O_3.$
(3) $\varphi(w(-t)) = |w(-t)_{BF}|^2 + O_3 = |w + i\alpha t|^2 + O_3.$

Proof (1) $\hat{\theta}_w(-t) = \int_0^t \frac{1}{2} d^c\varphi(\xi_H)|_{w(s)} ds + tH(w)$. Since $d^c\varphi|_w = O(|w|)$ and the integral interval is first order in t, hence

$$\int_0^t \frac{1}{2} d^c\varphi(\xi_H)|_{w(s)} ds = t\frac{1}{2} d^c\varphi(\xi_H)|_w + O_3$$
$$= t\langle \frac{1}{2} d^c\varphi|_w, \xi_H|_0\rangle + O_3 = \int_0^t \frac{1}{2} d^c\varphi_{BF}(\xi_{H_{BF}})|_{w(s)} ds + O_3.$$

And $tH(w) = t(E + H_{BF}(w)) + O_3$. Hence

$$\hat{\theta}_w(-t) - tE = \int_0^t \frac{1}{2} d^c\varphi_{BF}(\xi_{H_{BF}})|_{w(s)} ds + t(E + H_{BF}(w)) - tE + O_3 = \hat{\theta}_w(-t)_{BF} + O_3.$$

Finally, we may use Lemma 4.2 to compute the increment in θ.

(2) Since $\varphi(w) = |w|^2 + O(|w|^3)$ and $w(T) = g^T(w) = g^T(0) + Dg^T w + O(|w|^2) = Dg^T w + O(|w|^2)$, hence

$$\varphi(w(T)) = |Dg^T w|^2 + O_3$$

(3) Since $\xi_H = -i\alpha\partial_z + i\bar{\alpha}\partial_{\bar{z}} + O(|z|)$, we have $w(-t) = w + i\alpha t + O_2$, hence

$$\varphi(w(-t)) = |w + i\alpha t|^2 + O_3 = |w(-t)_{BF}|^2 + O_3.$$

□

Proposition 5.6

$$\hat{\theta}_w(T) = \hat{\theta}_0(T) + O(|w|^3).$$

Proof The proof is rather long, so we break it up into the following two Lemmas.

Lemma 5.7 *There exists a neighborhood $V \subset U$ of z, such that for any $w \in V$, and any path $\gamma : [0, 1] \to V$ from z to w, we have*

$$\hat{\theta}_w(T) = \hat{\theta}_0(T) - \int_\gamma \frac{1}{2} d^c\varphi + \int_{g^T(\gamma)} \frac{1}{2} d^c\varphi.$$

Proof We only give proof for $T = nT_z$, $n > 0$, the $n \le 0$ case is analogous. Let $\{(U_i, e_i, \varphi_i)\}_{i=1}^n$ be a sequence of coordinate patch U_i, such that there exists a partition of $[0, T]$: $0 = t_0 < t_1 < \cdots < t_n = T$, such that U_i covers the ith segment of the orbit $O_i = \{g^s z \mid t_{i-1} \le s \le t_i\}$, and $e_i \in \Gamma(U_i, L)$ are non-vanishing holomorphic sections, and $e^{-\varphi_i} = \|e_i\|^2$. Without loss of generality, we may take $U_1 = U$. We identify index $n + i$ with i.

Since $g^{t_i} z \in U_i \cap U_{i+1}$ for $0 \le i \le n$, hence

$$z \in V := \bigcap_{i=0}^{n} g^{-t_i}(U_i \cap U_{i+1}).$$

For any $w \in V$, let $\gamma : [0, 1] \to V$ be a path from z to w. Let

$$\gamma_0 = \gamma, \quad \gamma_i = g^{t_i}\gamma.$$

Then

$$Im(\gamma_i) \subset U_i \cap U_{i+1}, \forall 0 \le i \le n.$$

Over $U_i \cap U_{i+1}$, define transition function $g_i = \log(e_{i+1}/e_i)$, such that $g_i = a_i + \sqrt{-1}b_i$, with $b_i(g^{t_i}z) \in [0, 2\pi)$. Then we have

$$\|e_{i+1}\| = |g_i|\|e_i\| \Rightarrow e^{-\frac{1}{2}\varphi_{i+1}} = e^{a_i} e^{-\frac{1}{2}\varphi_i} \Rightarrow \varphi_{i+1} - \varphi_i = -2a_i.$$

Over U_i, let $\theta_i = e_i^*/\|e_i^*\|$ be the section in the co-circle bundle X. Then over $U_i \cap U_{i+1}$, we have

$$\log(e_{i+1}^*/e_i^*) = 1/g_i = e^{-a_i - \sqrt{-1}b_i} \Rightarrow \theta_{i+1} - \theta_i \equiv -b_i \mod 2\pi.$$

where we used additive notation for section valued in S^1.

Then, the holonomy can be expressed using Lemma 3.5 in each coordinate patch U_i

$$\hat{\theta}_w(T) = \theta_w(T) - \theta_w = \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \frac{1}{2} \langle d^c\varphi_i, \xi_H\rangle|_{g^s w} ds - (t_{i+1} - t_i)H(w) + b_i(g^{t_i}w).$$

Thus, we may take the difference

$$\hat{\theta}_w(T) - \hat{\theta}_0(T) = \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \frac{1}{2} \langle d^c\varphi_i, \xi_H\rangle|_{g^s w} ds - \int_{t_{i-1}}^{t_i} \frac{1}{2} \langle d^c\varphi_i, \xi_H\rangle|_{g^s z} ds - (t_{i+1} - t_i)(H(w) - H(z)) + \sum_{i=1}^{n} b_i(g^{t_i}w) - b_i(g^{t_i}z)$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \int_0^1 \int_{t_{i-1}}^{t_i} \omega(\partial_t, \partial_s) dt ds - (t_{i+1} - t_i)(H(w) - H(z)) \\
&\quad + \sum_{i=1}^{n} - \int_{\gamma_{i-1}} \frac{1}{2} d^c \varphi_i + \int_{\gamma_i} \frac{1}{2} d^c \varphi_i + \sum_{i=1}^{n} \int_{\gamma_i} db_i \\
&= \sum_{i=1}^{n} \int_0^1 \int_{t_{i-1}}^{t_i} dH(\partial_s) dt ds - (t_{i+1} - t_i)(H(w) - H(z)) \\
&\quad - \int_{\gamma_0} \frac{1}{2} d^c \varphi_1 + \sum_{i=1}^{n} \int_{\gamma_i} \frac{1}{2} d^c (\varphi_i - \varphi_{i+1}) + \int_{\gamma_n} \frac{1}{2} d^c \varphi_{n+1} + \sum_{i=1}^{n} \int_{\gamma_i} db_i \\
&= - \int_{\gamma_0} \frac{1}{2} d^c \varphi_1 + \int_{\gamma_n} \frac{1}{2} d^c \varphi_{n+1} + \sum_{i=1}^{n} \int_{\gamma_i} (d^c a_i + db_i) \\
&= - \int_{\gamma_0} \frac{1}{2} d^c \varphi_1 + \int_{\gamma_n} \frac{1}{2} d^c \varphi_1
\end{aligned}
$$

where in the last step, we used

$$
d^c(a_i + \sqrt{-1} b_i) = d(\sqrt{-1} a_i - b_i) \Rightarrow d^c a_i = -db_i.
$$

□

Lemma 5.8 *For any fixed path* $\gamma : [0, 1] \to U$ *starting from* 0*, and for any* $1 \gg \epsilon > 0$*, we have*

$$
\int_{\gamma([0,\epsilon])} d^c \varphi = \int_0^\epsilon \langle d^c \varphi, \dot{\gamma}(s) \rangle ds = O(\epsilon^3)
$$

Proof If a path $\gamma : [0, 1] \to U$ with $\gamma(0) = 0$ and $\gamma(1) = w$ is a straight-line, then

$$
\int_\gamma d^c \varphi = O(|w|^3).
$$

Indeed, consider the Taylor expansion of $\varphi(z)$ at $z = 0$,

$$
\varphi(z) = |z|^2 + O(|z|^3)
$$

then

$$
d^c \varphi = -2 \sum_i |z_i|^2 d\theta_i + \sum_i (O(|z|^2) dz_i + O(|z|^2) d\bar{z}_i).
$$

However, along a straight line path from 0 to w, θ_i is constant, hence the leading term of $d^c \varphi$ vanishes in the integral. For the remainder term, we have $| \int_\gamma dz_i | = O(|w|)$, hence proving the claim.

Next, we consider a general path as in the statement of the lemma. For each ϵ, we may consider the straight-line path $\beta : [0, \epsilon] \to U$ from 0 to $\gamma(\epsilon)$. From the previous claim, we know $\int_{\beta(\epsilon)} d^c \varphi = O(\epsilon^3)$. Let

$$
\Sigma_\epsilon : [0, \epsilon] \times [0, 1] \to U, (t, u) \mapsto u\gamma(t) + (1 - u)\beta(t).
$$

Then, we may verify that

$$\int_{\gamma([0,\epsilon])} d^c\varphi < C\left|\int_{\Sigma_\epsilon}\omega\right| + O(\epsilon^3) < O(\epsilon^3).$$

where the estimate of $\int_\Sigma \omega = 2\sum_i \int_\Sigma dx_i \wedge dy_i$ can be done by noting for any smooth function f,

$$\int_0^\epsilon f(x)dx - \epsilon\frac{1}{2}(f(0)+f(\epsilon)) = O(\epsilon^3).$$

□

Using above two lemma, we have

$$\hat{\theta}_w(T) = \hat{\theta}_0(T) = -\int_\gamma \frac{1}{2}d^c\varphi + \int_{g^T(\gamma)} \frac{1}{2}d^c\varphi = O(|w|^3) + O(|g^T w|^3) = O(|w|^3).$$

This finishes the proof for Proposition 5.6. □

5.4 Reduction to Osculating BF Model

We continue the calculation of the contribution to the stationary phase integral for period T orbit. The reduced phase function $\Psi'_T(t,w)$ has the following expansion:

$$\begin{aligned}\Psi'_T(t,w) &= -iTE - i\hat{\theta}_0(T) + it\mathrm{Re}(\alpha\bar{w}) - |w+i\alpha t|^2/2 - |Dg^T w|^2/2 + O_3.\\ &= -iTE - i\hat{\theta}_0(T) + iw\bar{\alpha}t - |w|^2/2 - |\alpha t|^2/2 - |Dg^T w|^2/2 + O_3.\end{aligned}$$

We may write the critical value as

$$\Psi'_T(0,0) = \Psi_T|_{crit} = -iTE - i\hat{\theta}_0(T) = -i\theta_0^h(T)$$

using holonomy phase of the horizontal flow (33).

The leading term of the stationary integral can be obtained by the following model result on BF space.

Proposition 5.9 *Let $H = \alpha\bar{z} + z\bar{\alpha}$. Let $\mathcal{A}: \mathbb{C}^m \to \mathbb{C}^m$ be a symplectic linear map, $\mathcal{A}w = Pw + Q\bar{w}$. Suppose ξ_H is invariant under $\mathcal{A}$. Then*
(1)

$$\begin{aligned}&(\det P^*)^{1/2}\left(\frac{k}{2\pi}\right)^{2m}\int_{\mathbb{C}^m} e^{k(itw\overline{\alpha}-|w|^2/2-t^2|\alpha|^2/2-|\mathcal{A}w|^2/2)}d\,\mathrm{Vol}_{\mathbb{C}^m}(w)\\ &= \hat{\mathcal{K}}_{k,\mathcal{A}}((0;0),\hat{g}^t(0;0))\end{aligned}$$

$$= (\det P)^{-1/2} \left(\frac{k}{2\pi}\right)^m e^{-kt^2(|\alpha|^2 - \bar{\alpha}P^{-1}Q\bar{\alpha})/2}$$

where the metaplectic representation kernel $\hat{\mathcal{K}}_{k,\mathcal{A}}(\hat{z}, \hat{w})$ is defined in (28).

(2)

$$\int_{\mathbb{R}} \hat{\mathcal{K}}_{k,\mathcal{A}}((0;0), \hat{g}^t(0;0))dt = \left(\frac{k}{2\pi}\right)^{m-1/2} (\det P)^{-1/2}(\bar{\alpha}P^{-1}\alpha).$$

Proof (1) We note that

$$\left(\frac{k}{2\pi}\right)^m e^{k(-|\mathcal{A}w|^2/2)} = \hat{\Pi}_k(0, (\mathcal{A}w; 0)),$$

and

$$\left(\frac{k}{2\pi}\right)^m e^{k(itw\overline{\alpha} - |w|^2/2 - t^2|\alpha|^2/2)} = \hat{\Pi}_k(\hat{g}^{-t}(w;0), 0) = \hat{\Pi}_k((w;0), \hat{g}^t(0;0)).$$

Hence by Proposition 4.4, we have

$$\begin{aligned} &(\det P^*)^{1/2} \left(\frac{k}{2\pi}\right)^{2m} \int_{\mathbb{C}^m} e^{k(itw\overline{\alpha} - |w|^2/2 - t^2|\alpha|^2/2 - |\mathcal{A}w|^2/2)} d\,\mathrm{Vol}_{\mathbb{C}^m}(w) \\ &= (\det P^*)^{1/2} \int_{\mathbb{C}^m} \hat{\Pi}_k(0, (\mathcal{A}w; 0))\hat{\Pi}_k((w;0), \hat{g}^t(0;0))dw \\ &= \hat{\mathcal{K}}_{k,\mathcal{A}}((0;0), \hat{g}^t(0;0)). \end{aligned}$$

And the last line follows by $\hat{g}^t(0;0) = (-i\alpha t; 0)$ and definition for $\hat{\mathcal{K}}_{k,\mathcal{A}}$.

(2) Next, we use the fact that ξ_H is preserved by $\mathcal{A}$, i.e.

$$\begin{pmatrix} -i\alpha \\ i\bar{\alpha} \end{pmatrix} = \begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix} \begin{pmatrix} -i\alpha \\ i\bar{\alpha} \end{pmatrix}$$

Thus

$$\alpha = P\alpha - Q\bar{\alpha} \tag{35}$$

hence

$$|\alpha|^2 - \bar{\alpha}P^{-1}Q\bar{\alpha} = |\alpha|^2 - \bar{\alpha}P^{-1}(P\alpha - \alpha) = \bar{\alpha}P^{-1}\alpha$$

Then, we have

$$\left(\frac{k}{2\pi}\right)^m (\det P)^{-1/2} \int_{\mathbb{R}} e^{-k\frac{1}{2}t^2(\bar{\alpha}P^{-1}\alpha)} dt = \left(\frac{k}{2\pi}\right)^{m-1/2} (\bar{\alpha}P^{-1}\alpha)^{-1/2}(\det P)^{-1/2}$$

□

Combining all the steps before, we have proven the following proposition.

Proposition 5.10 *Let $z \in M$ be a periodic point for the flow ξ_H and $H(z) = E$, then*

$$\Pi^E_{k,f}(z,z) = \sum_{n\in\mathbb{Z}} \hat{f}(nT_z) e^{-ikn\theta^h_z} \mathcal{G}_n \left(\frac{k}{2\pi}\right)^{m-1/2} (1 + O(k^{-1}))$$

where if $Dg^{nT_z}|_z$ in K-coordinate at z can be written as $\begin{pmatrix} P_n & Q_n \\ \bar{Q}_n & \bar{P}_n \end{pmatrix}$, then

$$\mathcal{G}_n = (\det P_n)^{-1/2} (\bar{\alpha} P_n^{-1} \alpha)^{-1/2}.$$

6 Proof of Proposition 1.6

The issue at hand is the regularity of the measures $\mu_k^{z,1,E}$ defined on test functions $f \in \mathcal{S}(\mathbb{R})$ with $\hat{f} \in C_0^\infty(\mathbb{R})$ in Theorem 2.2. It is only an interesting question when $z \in \mathcal{P}_E$. In this case,

$$\int_{\mathbb{R}} f \mu_k^{z,1,E} = \left(\frac{k}{2\pi}\right)^{m-1/2} \sum_{n\in\mathbb{Z}} \hat{f}(nT_z)\, \mathcal{G}_n(z) e^{-ink\theta^h_z} + O(k^{m-3/2}).$$

Unravelling the Fourier transform gives that, in the sense of distributions,

$$d\mu_k^{z,1,E}(x) = \left(\frac{k}{2\pi}\right)^{m-1/2} \sum_{n\in\mathbb{Z}} e^{inT_z x}\, \mathcal{G}_n(z) e^{-ikn\theta^h_z} dx + O(k^{m-3/2}).$$

The proposition asserts first that this series converges absolutely and uniformly when the orbit through z is real hyperbolic. To prove this we need to consider the behavior of the matrix element $\bar{\alpha} P_n^{-1} \alpha$ and the determinant $\det P_n$ as $n \to \infty$, where as in (7)

$$P_n := P_J S^n P_J : T_z^{(1,0)} M \to T_z^{(1,0)} M.$$

We first develop the symplectic linear algebra introduced in Sect. 3.1.

6.1 *Matrix Elements and Determinants of Positive Definite Symplectic Matrices*

We are interested in $P_J S P_J$ with $P_J = \frac{1}{2}(I - iJ)$. We also use the notation $\langle \alpha, \beta \rangle = \bar{\beta}^t \cdot \alpha$ for the sesquilinear inner product.

First we prove

Proposition 6.1 *If S is positive definite symmetric symplectic, with invariant vector ξ and $\alpha = P_J\xi$, and if the spectrum of S is $\{e^{\lambda_j}, e^{-\lambda_j}\}_{j=1}^n$ with $\lambda_j \geq 0$ then*

$$\begin{cases} (i) \ \ [P_J S P_J]^{-1}\alpha = \alpha, \\ (ii) \ \det P_J S P_J|_{T_0^{1,0}\mathbb{R}^{2n}} = \prod_{j=1}^n [\cosh \lambda_j]. \end{cases}$$

Proof The proof is through a series of lemmas:

Lemma 6.2 *If S is positive definite symplectic, then*

$$P_J S P_J = \frac{1}{2} P_J (S + S^{-1}) = \frac{1}{2}(S + S^{-1}) P_J$$

Proof

$$\begin{aligned} P_J S P_J &= \tfrac{1}{4}(I - iJ)S(I - iJ) = \tfrac{1}{4}[S - iJS - iSJ - JSJ] \\ &= \tfrac{1}{4}[S + S^{-1}] - \tfrac{i}{4}[J[S + S^{-1}] = \tfrac{1}{4}\left((S + S^{-1}) - iJ(S + S^{-1})\right) = \tfrac{1}{2} P_J (S + S^{-1}). \end{aligned}$$

since $JSJ = -S^{-1}$ if S is symmetric. Also,

$$J(S + S^{-1}) = JS + SJ = (S^{-1} + S)J$$

so that $P_J(S + S^{-1}) = (S + S^{-1})P_J$. □

Lemma 6.3 *Let S be positive definite symmetric symplectic and e_j be eigenvectors of S for eigenvalues $\lambda_1, \ldots, \lambda_n$. Consider the basis $P_J e_k$ of $H_J^{1,0}$. Then*

$$[P_J S P_J] P_J e_k = \cosh(\lambda_j) P_J e_k,$$

and $[P_J S P_J]^{-1} = P_J [S + S^{-1}]^{-1} P_J$.

Proof Follows from the previous lemma and the fact that $(S + S^{-1})$ commutes with P_J:

$$[P_J S P_J] P_J e_k = \frac{1}{2} P_J (S + S^{-1}) e_k = \frac{1}{2}(e^{\lambda_j} + e^{-\lambda_j}) P_J e_k = \cosh(\lambda_j) P_J e_k.$$

□

Statement (i) of the Proposition follows from the fact that

$$[P_J S P_J]\alpha = \frac{1}{2}(1 + 1)\alpha = \alpha.$$

Statement (ii) follows from the fact that the eigenvalues of $P_J S P_J$ are $\cosh \lambda_j$ by Lemma 6.3. □

6.2 Strong Hyperbolicity Hypothesis

Let z be a periodic point of the Hamiltonian flow g^t. Under this hypothesis, we have the following result.

Proposition 6.4 *If* $\dim_{\mathbb{C}} M = m > 1$*, and* z *be a periodic point with primitive period* T*, satisfying the strong hyperbolic hypothesis. Then*

$$\sum_{n \in \mathbb{Z}} |\mathcal{G}_n(z)| < \infty.$$

Proof Let the spectrum of $S := Dg^T$ be $\{e^{\lambda_j}, e^{-\lambda_j}\}_{j=1}^m$, with $\lambda_1 = 0$ and $\lambda_j > 0$ for $j = 2, \ldots, n$. Then, recall that

$$\mathcal{G}_n(z) = [\det(P_J S^n P_J) \langle (P_J S^n P_J)^{-1} \alpha, \alpha \rangle]^{-1/2}.$$

Then, from previous section, we have $\det(P_J S^n P_J) = \prod_{j=1}^n \cosh(n\lambda_j)$, and $\langle (P_J S^n P_J)\alpha, \alpha \rangle = \langle \alpha, \alpha \rangle$ independent of n. Since $\lambda_j > 0$ for $j = 2, \ldots, m$, hence

$$|\mathcal{G}_n| = |\det(P_J S^n P_J) \langle \alpha, \alpha \rangle|^{-1/2} < C e^{-|n| \sum_j \lambda_j}$$

for some positive constant C. Thus the sum $\sum_{n \in \mathbb{Z}} |\mathcal{G}_n(z)|$ converges exponentially fast. □

6.3 Proof of Proposition 1.6

By Proposition 6.4, the family of measures

$$d\nu_T(\lambda) := \sum_{|n| \leq T} \rho_T(nT(z)) e^{-i\lambda n T_z} e^{-ikn\theta_z^h(T_z)} \mathcal{G}_n(z) d\lambda, \quad (T \in \mathbb{R}_+)$$

converges in the weak* sense of distributions on the space $\mathcal{S}(\mathbb{R})$ of Schwartz functions to the limit distribution,

$$d\nu(\lambda) := \sum_{n \in \mathbb{Z}} e^{-i\lambda n T_z} e^{-ikn\theta_z^h(T_z)} \mathcal{G}_n(z) d\lambda,$$

since the coefficients $\mathcal{G}_n(z)$ are bounded in n and by dominated convergence,

$$\int_{\mathbb{R}} f(\lambda) d\nu_T(\lambda) = \sum_{|n| \leq T} \rho_T(nT(z)) \hat{f}(nT(z)) \mathcal{G}_n(z) \to \sum_{n \in \mathbb{Z}} \hat{f}(nT(z)) \mathcal{G}_n(z),$$

where the sum on the right side converges absolutely.

7 Proof of Theorem 1.7

In this section we apply Theorem 2.2 and a Tauberian theorem to prove Theorem 1.7. We are concerned with the Weyl sums,

$$\Pi_{k,[E_1,E_2]}(z) = \int_{E_1}^{E_2} d\mu_k^{z,1,E} = \sum_{j:k(\mu_{kj}-H(z))\in[E_1,E_2]} \Pi_{k,j}(z).$$

The basic idea is to convolve $\mathbf{1}_{[E_1,E_2]}$ with a well-chosen Schwartz test function depending on (h, T), apply Theorem 2.2 and then estimate the remainder.

We consider both families of measures of (3), μ_k^z and $\mu_k^{z,1,E}$. The main difference is the range of eigenvalues involved. The measures μ_k^z have a fixed compact support, the range $H(M) = [H_{\min}, H_{\max}]$ of H, and the mean level spacing between the k^m point masses μ_{kj} is k^{-m}. The measures $\mu_k^{z,1,E}$ are scaled versions,

$$\mu_k^{z,1,E}[-M, M] = \sum_{j:|\mu_{jk}-E|<\frac{M}{k}} \Pi_{kj}(z),$$

and the mean level spacing between the point masses is k^{-m+1}. Of course,

$$\sum_{j:|\mu_{jk}-E|<\frac{M}{k}} \Pi_{kj}(z) = \mu_k^{z,1,E}[-M, M] = \mu_k^z\left[\frac{-M}{k}, \frac{M}{k}\right], \tag{36}$$

As a preliminary, we quote a result from [19, Theorem 3]:

Theorem 7.1 *Let E be a regular value of H and $z \in H^{-1}(E)$. If ϵ is small enough, such that the Hamiltonian flow trajectory starting at z does not return to z for time $|t| < 2\pi\epsilon$, then for any Schwarz function $f \in \mathcal{S}(\mathbb{R})$ with $\hat{f}$ supported in $(-\epsilon, \epsilon)$ and $\hat{f}(0) = \int f(x)dx = 1$, and for any $\alpha \in \mathbb{R}$ we have*

$$\int_{\mathbb{R}} f(x)d\mu_k^{z,1,\alpha}(x) = \left(\frac{k}{2\pi}\right)^{m-1/2} e^{-\frac{\alpha^2}{\|\xi_H(z)\|^2}} \frac{\sqrt{2}}{2\pi\|\xi_H(z)\|}(1 + O(k^{-1/2})).$$

There is a further integrated version of the Weyl law with remainder,

$$\#\left\{j : |\mu_{kj} - E| \le \frac{M}{k}\right\} = \frac{2M}{(2\pi)^n}\mathrm{Vol}(h^{-1}(E))k^{m-1} + o(k^{m-1}). \tag{37}$$

The constraint in the sum (36) is a 'codimension one' condition localizing around $H^{-1}(E)$. The extra integration in (37) gives an extra factor of $k^{-\frac{1}{2}}$ in the stationary phase expansion. Note that $\int_M \Pi_{kj}(z)dV(z) = \mathrm{Mult}(\bar{}_{\mathrm{kj}})$ (the multiplicity of the eigenvalue, generically equal to 1), so the integrated Weyl law does not deal with non-uniform weights $\Pi_{kj}(z)$. The integrated Weyl law (essentially contained in [3]).

The remainder estimate requires the use of a semi-classical Tauberian theorem for a sequence $\mu_k^{z,1,E}$ of measures. Before getting started, let us note some basic facts about this sequence. First, $\mu_k^{z,1,E}$ is not normalized to be a probability measure, but it is finite and could be normalized by dividing by its mass $\Pi_{h^k}(z) \simeq k^m + O(k^{m-1})$). In the following discussion, we divide by the mass. Second, note that $\Pi_{h^k}(z)^{-1} d\mu_k^{z,1,E}$ is a centered re-scaling of $\Pi_{h^k}(z)^{-1} d\mu_k^z$ (3). That is $D_k \tau_E d\mu_k^{z,1E} = d\mu_k^z$ where the dilation operator is defined by $D_k \nu(I) = \nu(kI)$ for any interval I and measure ν. Also $\tau_E f(x) = f(x - E)$. Now, μ_k^z is supported in $H(M)$ (the range of $H : M \to \mathbb{R}$), hence $\mu_k^{z,1,E}$ is supported in $k(H - E)(M)$. In [19] we studied $\Pi_{h^k}(z)^{-1} \mu_k^{z,\frac{1}{2},E} := D_{\sqrt{k}} \Pi_{h^k}(z)^{-1} \mu_k^{z,1,E}$, whose support is $\sqrt{k}(H - E)(M)$ and proved that it tends to a Gaussian. In particular, its Fourier transform is continuous at 0, and by Levy's continuity theorem (or by direct analysis), the sequence $\Pi_{h^k}(z)^{-1} \mu_k^{z,1/2,E}$ is tight. By comparison, $\Pi_{h^k}(z)^{-1} \mu_k^{z,1,E}$ is not tight, and indeed the $\Pi_{h^k}(z)^{-1} \mu_k^{z,1,E}([a, b]) \simeq k^{-\frac{1}{2}}$, so that the mass is spreading out to infinity and it does not weak* converge on $C_b(\mathbb{R})$.

Theorem 1.7 not only gives the leading order term but also the order of the remainder. As is well-known from work of Duistermaat–Guillemin, Ivrii, Safarov and others, obtaining a sharp remainder term requires the use of something similar to Fourier transform methods and in particular Fourier Tauberian theorems. As mentioned before, Theorem 1.7 is analogous to Safarov's non-classical pointwise Weyl asymptotics for the spectral function of a Laplace operator Δ, or more precisely, asymptotics on intervals $[\lambda, \lambda + 1]$ for $\sqrt{-\Delta}$. The Q-notation is adopted from [14, 15]. Since we are working on phase space, Q involves closed orbits rather than loops in configuration space. However, we need to use a semi-classical Tauberian theorem rather than the homogeneous Tauberian theorem of [15], i.e. we are considering a sequence of measures $\mu_k^{z,1,E}$ on a fixed interval rather than a fixed measure on expanding intervals $[0, \lambda]$.

Semi-classical Tauberian theorems have been known for a long time. It is a classical fact that to obtain sharp remainder estimates, one must make use of the Fourier transform of the measures on long time intervals $[-T, T]$. A Tauberian theorem of the needed type is proved in [12], adapting the statement of Safarov's non-classical Weyl asymptotics to a semi-classical problem. This theorem does not quite apply to our setting for various reasons: (i) It assumes the sequence of measures have fixed compact support; (ii) it assumes the 'weights' or masses of the point masses are uniform. On the contrary, the 'weights' $\Pi_{k,j}(z)$ of $\mu_k^{z,1,E}$ are highly non-uniform in a way that is inconsistent with the hypotheses of the Tauberian Theorem of [12]. Consider the graph of the weights $\Pi_{k,j}(z)$ as a function of μ_{kj}, i.e. the coefficients of the point masses of μ_k^z (3). On average the weights are of order 1 since there are k^m terms and the total sum is $\Pi_k(z) \simeq \mathrm{Vol(M}, \omega)\mathrm{k^m}$. But the weights are highly non-uniform:

(1) they peak when $\mu_{kj} \simeq H(z)$; indeed, it is shown inf [19, Theorem 1] that μ_k^z tends weakly to $\delta_{H(z)}$.
(2) By [19, Theorem 2], $\sum_{j:|\mu_{kj} - H(z)| < Mk^{-\frac{1}{2}}} \Pi_{k,j}(z) \sim Mk^m$ while the number of terms is of order $k^{m-\frac{1}{2}}$. Thus, on average, $\Pi_{k,j}(z)$ is of size $k^{\frac{1}{2}}$ in this eigenvalue range.

(3) Further, $\Pi_{k,j}(z) \lesssim k^{-C}$ when $|H(z) - \mu_{kj}| \geq Ck^{-\frac{1}{2}} \log k$. Hence, the weights decay rapidly when μ_{kj} lies outside of the range $|H(z) - \mu_{kj}| \leq Ck^{-\frac{1}{2}} \log k$. Consequently, the sequence of dilated measures $\mu_k^{z,1,E}$ concentrates in the sets $[-k^{\frac{1}{2}} \log k, k^{\frac{1}{2}} \log k]$.

Since we need to modify the Tauberian Theorem of [12] to accommodate the strong peaking of the weights around $H(z)$, we go through the modified proof in detail.

7.1 Mollifiers and Convolution

We use the following notation: Let $\rho_1 \in C_0^\infty(-1, 1)$ satisfy $\rho_1(t) = 1$ on $[-\frac{1}{2}, \frac{1}{2}]$, $\rho_1(-t) = \rho_1(t)$. We may also assume $\mathcal{F}\rho_1(\tau) \geq 0$ and $\mathcal{F}\rho_1(\tau) \geq \delta_0 > 0$ for $|\tau| \leq \epsilon_0$, where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the standard Fourier transform and its inverse,

$$\hat{f}(x) := (\mathcal{F}f)(x) = (2\pi)^{-1} \int f(t)e^{-itx}dt, \quad \check{f}(x) = (\mathcal{F}^{-1}f)(x) = \int f(t)e^{itx}dx$$

Then set,

$$\rho_T(\tau) = \rho_1\left(\frac{\tau}{T}\right), \quad \theta_T(x) := \hat{\rho}_T(x) = T\hat{\rho}_1(xT). \tag{38}$$

In particular, $\int \theta_T(x)dx = 1$ and $\theta_T(x) > T\delta_0$ for $|x| < \epsilon_0/T$. Let

$$\sigma_k^{z,1,E}(x) = \mu_k^{z,1,E}(-\infty, x].$$

7.2 Tauberian Theorem for $\mu_k^{z,1,E}$

In this section we determine the asymptotics of

$$\sigma_k^{z,1,E}(E_2) - \sigma_k^{z,1,E}(E_1) = \int_{E_1}^{E_2} d\mu_k^{z,1,E}(x) = \sum_{j:\frac{E_1}{k} \leq \mu_{jk} - E \leq \frac{E_2}{k}} \Pi_{k,j}(z).$$

We recall that the mean level spacings of $k(\mu_{k,j} - E)$ is k^{-m+1} so that the number of terms in the sum is of order k^{m-1}. The plan is to mollify the measures by convolution with θ_T (38), so that it suffices to determine the asymptotics of

$$\begin{aligned} &\sigma_k^{z,1,E} * \theta_T(E_2) - \sigma_k^{z,1,E} * \theta_T(E_1) \\ &+ \left(\sigma_k^{z,1,E}(E_2) - \sigma_k^{z,1,E}(E_1)\right) - \left(\sigma_k^{z,1,E} * \theta_T(E_2) - \sigma_k^{z,1,E} * \theta_T(E_1)\right) \end{aligned} \tag{39}$$

Since

$$\sigma_k^{z,1,E} * \theta_T(E_2) - \sigma_k^{z,1,E} * \theta_T(E_1) = \int_{E_1}^{E_2} \theta_{h,T} * d\mu_k^{z,1,E}(\lambda),$$

we have

$$\begin{aligned}&\left(\sigma_k^{z,1,E}(E_2) - \sigma_k^{z,1,E}(E_1)\right) - \left(\sigma_k^{z,1,E} * \theta_T(E_2) - \sigma_k^{z,1,E} * \theta_T(E_1)\right)\\ &= \int_{E_1}^{E_2} (\theta_T * d\mu_k^{z,1,E} - d\mu_k^{z,1,E}).\end{aligned} \tag{40}$$

First we consider the top terms of (39).

Proposition 7.2 *Assume that $H(z) = E$, $z \in \mathcal{P}_E$. Then*

$$\frac{d}{dx}(\sigma_k^{z,1,E} * \theta_T)(x) = \left(\frac{k}{2\pi}\right)^{m-1/2} \sum_{n\in\mathbb{Z}} \rho_T(nT_z) e^{-ixnT_z} e^{-ikn\theta_z^h(T_z)} \mathcal{G}_n(z) + O_T(k^{m-3/2}) \tag{41}$$

and

$$\begin{aligned}&\sigma_k^{z,1,E} * \theta_T(E_2) - \sigma_k^{z,1,E} * \theta_T(E_1)\\ &= k^{m-\frac{1}{2}} \int_{E_1}^{E_2} \sum_{|nT_z|\le T} \rho_T(nT_z) e^{-i\lambda n T_z} e^{-ikn\theta_z^h(T_z)}\, \mathcal{G}_n(z) d\lambda + O(k^{m-1})\,.\end{aligned}$$

Proof

$$\begin{aligned}\frac{d}{dx}(\sigma_k^{z,1,E} * \theta_T)(x) &= \int \theta_T(x-y) d\mu_k^{z,1,E}(y)\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \rho_T(-t) e^{-it(x-y)} \sum_j \delta_{k(\mu_{k,j}-E)}(y) \Pi_{k,j}(z) dy dt\\ &= \int_{\mathbb{R}} \rho_T(t) e^{-itx} \sum_j e^{itk(\mu_{k,j}-E)} \Pi_{k,j}(z) dt\\ &= \int_{\mathbb{R}} \rho_T(t) e^{-itx-itkE} U_k(t,z,z) dt\\ &= \left(\frac{k}{2\pi}\right)^{m-1/2} \sum_{n\in\mathbb{Z}} \rho_T(nT_z) e^{-ixnT_z} e^{-ikn\theta_z^h(T_z)} \mathcal{G}_n(z)(1+O(k^{-1})).\end{aligned}$$

where the last line follows from Theorem 2.2 to $f(y) = \theta_T(x-y)$. □

Corollary 7.3 *Under the strong hyperbolicity hypothesis (Definition 1.5), there exists constants $\gamma_0(z)$, $C_1(T,z)$, such that*

$$\frac{d}{dx}(\sigma_k^{z,1,E} * \theta_T)(x) \le \left(\frac{k}{2\pi}\right)^{m-1/2} \gamma_0(z) + C_1(T,z) k^{m-3/2}.$$

Proof We start from (41), and let $T \to \infty$. By Proposition 6.4, the sum in (41) with ρ_T replaced by 1 converges absolutely. □

We now employ a semi-classical Fourier Tauberian theorem to estimate (40). In fact, since we already semi-classically scaled $d\mu_k^z$ by k, we do not need to scale again. We only refer to the Tauberian as semi-classical because it applies to a sequence $\mu_k^{z,1,E}$ of measures on a fixed interval rather than to a fixed measure on a dilated family of intervals as in the homogeneous Tauberian theorem.

The Tauberian theorem states:

Proposition 7.4 *There exist constant $\gamma(z)$, $C(T, z)$ such that, for any $T > 0$,*

$$\int_{E_1}^{E_2} (\theta_T * d\mu_k^{z,1,E} - d\mu_k^{z,1,E}) \leq \frac{\gamma(z)}{T} k^{m-\frac{1}{2}} + C(T, z) k^{m-3/2}.$$

Together with Proposition 7.2 this gives

Corollary 7.5 *For any $T > 0$, there exist $\gamma_0(z, \tau)$, γ, $C_1(T, z, \tau) > 0$ so that*

$$\sigma_k^{z,1,E}(E_2) - \sigma_k^{z,1,E}(E_1)$$
$$= \left(\frac{k}{2\pi}\right)^{m-\frac{1}{2}} \int_{E_1}^{E_2} \sum_{|nT_z| \leq T} \rho_T(nT_z) e^{-i\lambda n T_z} e^{-ikn\theta_z^h(T_z)} \mathcal{G}_n(z) d\lambda + \frac{1}{T} O(k^{m-\frac{1}{2}}) + O_T(k^{m-3/2}).$$

7.3 *Proof of Proposition 7.4*

As mentioned above, the hypotheses of [12, Theorem 3.1] do not hold in our setting. Hence we must extract from [12, Theorem 3.1] the key elements that pertain to our setting.

We have,

$$\begin{aligned}
\int_{E_1}^{E_2} (\theta_T * d\mu_k^{z,1,E} - d\mu_k^{z,1,E}) &= \int_{\mathbb{R}} (\mu_k([E_1, E_2] - \tau) - \mu_k[E_1, E_2])\, \theta_T(\tau) d\tau \\
&= T \int_{\mathbb{R}} (\mu_k([E_1, E_2] - \tau) - \mu_k[E_1, E_2]))\, \hat{\rho}_1(\tau T) d\tau \\
&= T \int_{|\tau| \leq \frac{1}{T}} (\mu_k([E_1, E_2] - \tau) - \mu_k[E_1, E_2]))\, \hat{\rho}_1(\tau T) d\tau \\
&+ T \int_{|\tau| > \frac{1}{T}} (\mu_k([E_1, E_2] - \tau) - \mu_k[E_1, E_2]))\, \hat{\rho}_1(\tau T) d\tau \\
&=: I_1 + I_2.
\end{aligned}$$

Evidently, the key objects to estimate are the increments

$$\mu_k([E_1, E_2] - \tau) - \mu_k([E_1, E_2])$$

The key point is to prove the analogue of [12, Proposition 3.2]:

Proposition 7.6 *There exist constants $\gamma_1(z)$ and $C_1(T, z)$ such that, for any $T > 0$,*

$$|(\mu_k([E_1, E_2] - \tau) - \mu_k[E_1, E_2]))| \leq \gamma_1(z)\left(\frac{1}{T} + |\tau|\right)k^{m-\frac{1}{2}} + C_1(T, z)O(k^{m-3/2})$$

We now show that Proposition 7.6 implies Proposition 7.4.

Proof First, observe that Proposition 7.6 implies,

$$I_1 \leq \sup_{|\tau| \leq \frac{1}{T}} |\mu_k([E_1, E_2] - \tau) - \mu_k([E_1, E_2])|,$$

and Proposition 7.6 immediately implies the desired bound of Proposition 7.4 for $|\tau| \leq \frac{1}{T}$. For I_2 one uses that $\hat{\rho}_1 \in \mathcal{S}(\mathbb{R})$. Since $T \int_{|\tau| \geq \frac{1}{T}} \hat{\rho}_1(\tau T) d\tau \leq 1$, Proposition 7.6 implies,

$$I_2 \lesssim k^{m-\frac{1}{2}} \gamma_1(z) T \int \left(\frac{1}{T} + |\tau|\right) \hat{\rho}_1(T\tau) d\tau + C_1(T, z) O(k^{m-3/2}) T \int_{|\tau| > \frac{1}{T}} \hat{\rho}_1(\tau T) d\tau$$

If one changes variables to $r = T\tau$ one also gets the estimate of Proposition 7.4. □

We now prove Proposition 7.6.

Proof We need to estimate $(\mu_k[E_1, E_2] - \tau) - \mu_k[E_1, E_2]))$. The estimate depends both on the position of $[E_1, E_2]$ relative to the center of mass at 0 and on the position of τ. We recall the the total mass of $\mu_k = \mu_k^{z,1,E}$ on the complement of $[-\sqrt{k}\log k, \sqrt{k}\log k]$ is rapidly decaying in k. Hence we may assume that at least one of the following occurs:

- $[E_1, E_2] \cap [-\sqrt{k}\log k, \sqrt{k}\log k] \neq \emptyset$, i.e. $E_1 \geq -\sqrt{k}\log k$, $E_2 \leq \sqrt{k}\log k$.
- $[E_1, E_2] - \tau \cap [-\sqrt{k}\log k, \sqrt{k}\log k] \neq \emptyset$, i.e. $E_1 - \tau - \sqrt{k}\log k$, $E_2 - \tau \leq \sqrt{k}\log k$.

The proof is broken up into 3 cases: (1) $|\tau| \leq \frac{\epsilon_0}{T}$, (2) $\tau = \frac{\ell}{T}\epsilon_0$, (3) $\frac{\ell}{T}\epsilon_0 \leq \tau \leq \frac{\ell+1}{T}\epsilon_0$, for some $\ell \in \mathbb{Z}$.

(1) Assume $|\tau| \leq \frac{\epsilon_0}{T}$. Assume $\tau > 0$ since the case $\tau < 0$ is similar. Write

$$\mu_k([E_1, E_2] - \tau) - \mu_k[E_1, E_2]) = \int_{\mathbb{R}} [\mathbf{1}_{[E_1-\tau, E_2-\tau]} - \mathbf{1}_{[E_1, E_2]}](x) d\mu_k(x).$$

For T sufficiently large so that $\tau \ll E_2 - E_1$,

$$[\mathbf{1}_{[E_1-\tau,E_2-\tau]} - \mathbf{1}_{[E_1,E_2]}](x) = \mathbf{1}_{[E_1-\tau,E_1]} - \mathbf{1}_{[E_2-\tau,E_2]}.$$

We do not expect cancellation between the terms for arbitrary E_1, E_2, τ and therefore must show that each term satisfies the desired estimate. Since they are similar we only consider the $[E_1 - \tau, E_1]$ interval. Since for $|\tau| < \epsilon_0/T$, we have $\theta_T(\tau) > T\delta_0$, thus

$$\begin{aligned}\mu_k([E_1 - \tau, E_1]) &\leq \frac{1}{T\delta_0}\int_{\mathbb{R}} \theta_T(E_1 - x)d\mu_k(x)\\ &\sim \frac{1}{T\delta_0}\frac{d}{dx}(\sigma_k^{z,1,E} * \theta_T)(E_1)\\ &< \frac{\gamma_0(z)}{T\delta_0}k^{m-1/2}\end{aligned}$$

It follows that

$$|\mu_k([E_1, E_2] - \tau) - \mu_k[E_1, E_2])| \leq \frac{2\gamma_0(z)}{T\delta_0}k^{m-1/2}.$$

(2) Assume $\tau = \ell\frac{\epsilon_0}{T}$, $\ell \in \mathbb{Z}$. With no loss of generality, we may assume $\ell \geq 1$. Write

$$\begin{aligned}&\mu_k([E_1, E_2]) - \mu_k\left([E_1, E_2] - \frac{\ell}{T}\epsilon_0\right)\\ &= \sum_{j=1}^{\ell} \mu_k\left([E_1, E_2] - \frac{j-1}{T}\epsilon_0\right) - \mu_k\left([E_1, E_2] - \frac{j}{T}\epsilon_0\right)\end{aligned}$$

and apply the estimate of (1) to upper bound the sum by

$$\frac{2\ell\gamma_0(z)}{T\delta_0}k^{m-1/2} = \frac{2\gamma_0}{\epsilon_0\delta_0}\tau k^{m-1/2}$$

(3) Assume $\frac{\ell}{T}\epsilon_0 \leq \tau \leq \frac{\ell+1}{T}\epsilon_0$ and $|\tau h| \leq \epsilon_1$ with $\ell \in \mathbb{Z}$. Write

$$\begin{aligned}\mu_k([E_1, E_2] + \tau) - \mu_k([E_1, E_2]) &= \mu_k([E_1, E_2] + \tau) - \mu_k([E_1, E_2] + \tfrac{\ell}{T}\epsilon_0)\\ &\quad + \mu_k([E_1, E_2] + \tfrac{\ell}{T}\epsilon_0) - \mu_k([E_1, E_2]).\end{aligned}$$

Apply (1) and (2) , it follows that

$$|\mu_k([E_1, E_2] + \tau) - \mu_k([E_1, E_2])| \leq \frac{2\gamma_0(z)}{\delta_0}\left(\frac{\tau}{\epsilon_0} + \frac{1}{T}\right)\gamma_0(\sigma, \lambda)k^{m-\frac{1}{2}}.$$

□

8 Comparison with BPU

In this section we compare our formula for the leading coefficient in Theorem 2.2 with that in [2]. To do so, we need to introduce the notation and terminology of that article.

Let ϕ_τ^h be the horizontal lift of the Hamiltonian flow to X_h (denoted P in [2]). At each point $p \in P$, define $T_p^h P$ to be the horizontal subspace and Λ_p to be the positive definite Lagrangian subspace of $T_p^h P \otimes \mathbb{C}$ (i.e. the type $(1, 0)$ subspace). By the analysis of [3, p. 98] there exists a one-dimensional kernel $\mathcal{W}_p$ of this action, the line of ground states $\mathcal{W}_p \subset H_\infty(T_p^h P)$. A normalized section of the bundle $\mathcal{W} \to P$ defined by $\mathcal{W}_p$ is denoted by e_p. Further denote by $M_\tau : H_\infty(T_p^h P) \to H_\infty(T_p^h P)$ the metaplectic representation of the symplectic group of the horizontal space $H(T_p^h P)$.

Let Ξ denote the Hamilton vector field ξ_H. It is written in [2] that "Ξ acts on $H(T_p^h P)$ and hence on $H_\infty(T_p^h P)$ by via the Heisenberg representation. The action is by translations. The projection from $H_\infty(T_p^h(P))$ to generalized invariant vectors under Ξ is defined by

$$P_\Xi v := \int_{-\infty}^{\infty} e^{it\Xi} v dt$$

the projection from $H_\infty(T_p^h P)$ to the invariant vectors for the flow of Ξ p above z.

Further let Q be a first order pseudo-differential operator on $L^2(P)$ so that $\Pi Q \Pi = D\Pi M_H \Pi$ and so that $[Q, \Pi] = 0$. Let q be the symbol of Q, which generates a contact flow ϕ_t on P. Then the flow maps $\Lambda_p \to \Lambda_{\phi_t(p)}$ and M_τ maps e_p to a multiple of $e_{\phi_t(p))}$. Define $c(t)$ by $\Xi_q e_{\phi_t(p)} = ic(t)e_{\phi_t(p)}$.

Then the formula of [2] for the leading coefficient at a periodic orbit of period τ is

$$C_{\tau,0} = \frac{1}{2\pi^{n+1}} \langle M_\tau^{-1} e_{p_1}, P_\Xi(e_{p_1})\rangle e^{-i\int_0^\tau (\sigma_{sub}(Q)+c(t))dt}.$$

The approach of this paper is to replace $H_\infty(T_p^h)$ by the osculating Bargmann–Fock space, i.e. the Bargmann–Fock space on $H_z^{1,0}M$ which carries a complex structure and Hermitian metric and hence a Gaussian inner product. In effect, the quadratic part of the scaled phase of $U_k(t, z, z)$ replaces the symbol calculus. We do not use Q but the related operator in our setting is $\hat{H}_k$. The P_Ξ operator there corresponds to the dt integral near a period in our approach. We now verify that our formula agrees with theirs to the extent possible.

We would like to compare the expression (9) with the one in [2],

$$\langle M_T^{-1} e_0, P_\Xi e_0\rangle = \langle \eta_{J,Dg^T} \Pi_J U_{Dg^T}^{-1} e_0, \int_{\mathbb{R}} g_*^{BF,\tau} e_0 d\tau\rangle = \eta_{J,Dg^T} \int_{\mathbb{R}} \langle U_{Dg^T}^{-1} e_0, g_*^{BF,\tau} e_0\rangle d\tau$$

where g^τ is the BF translation (Heisenberg representation) of the constant vector field $\xi_H(0)$ by time τ. Here, we dropped the projection operator Π_J, since it is acting on $g_*^{BF,\tau} e_0$, which is holomorphic already.

Let

$$v = e^{-k|z|^2/2}$$

be the (unnormalized) coherent state centered at 0. We first review how Heisenberg group and Metaplectic group acts on it.
(i) Let $w \in \mathbb{C}^m$. Let $\beta(w)$ be translation by w. Then

$$[\beta(w)v](z) = e^{k[z\bar{w}-|z|^2/2-|w|^2/2]} = e^{k[i\mathrm{Im}(z\bar{w})-|z-w|^2/2]}$$

Indeed, it is centered at w, with a non-trivial phase factor $i\mathrm{Im}(z\bar{w})$.

(ii) Let $M = \begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix} \in Sp_c$, with $M^{-1} = \begin{pmatrix} P^* & -Q^t \\ -Q^* & P^t \end{pmatrix}$. Then

$$(Mv)(z) = \frac{1}{(\det P)^{1/2}} e^{k[\frac{1}{2}z\bar{Q}P^{-1}z-\frac{1}{2}|z|^2]}.$$

And for our purpose, we also need

$$(M^{-1}v)(z) = \frac{1}{(\det P^*)^{1/2}} e^{k[-\frac{1}{2}zQ^*(P^*)^{-1}z-\frac{1}{2}|z|^2]}$$

Let $\Xi = -i\alpha\partial_z + i\bar{\alpha}\partial_{\bar{z}}$, the Hamiltonian vector field for $H = \alpha\bar{z} + \bar{\alpha}z$. Then, we can write $P_\Xi v$ as

$$(P_\Xi v)(z) = \int_{\mathbb{R}} \beta(-i\alpha t)v dt = \int_{\mathbb{R}} e^{k[itz\bar{\alpha}-|z|^2/2-|\alpha t|^2/2]} dt$$

It is possible to perform the Gaussian integral, then we get

$$(P_\Xi v)(z) = \sqrt{\frac{2\pi}{k|\alpha|^2}} e^{k[-|z|^2/2-(z\bar{\alpha})^2/2|\alpha|^2)]}$$

We will see, it is better not to evaluate the dt integral first.

Proposition 8.1

$$\langle M^{-1}v, P_\Xi v\rangle = \left(\frac{k}{2\pi}\right)^{-m-1/2} (\bar{\alpha}(P^*)^{-1}\alpha)^{-1/2}(\det P^*)^{-1/2}$$

The power of $\left(\frac{k}{2\pi}\right)$ does not matter, since we did not choose a normalized coherent state. The difference between P and P^* with previous result may be due to the difference of time $+T$ or $-T$ trajectories. Since we will sum time $\{nT \mid n \in \mathbb{Z}\}$ trajectories, the difference does not matter in the end.

Proof

$$
\begin{aligned}
\langle M^{-1}v, P_\Xi v\rangle &:= \int_{\mathbb{C}^m}\int_{\mathbb{R}} \frac{1}{(\det P^*)^{1/2}} e^{k[-zQ^*(P^*)^{-1}z-|z|^2/2]}\overline{e^{k[itz\bar{\alpha}-|z|^2/2-|\alpha t|^2/2]}}dtd\,\mathrm{Vol}(z)\\
&= \int_{\mathbb{R}}\int_{\mathbb{C}^m} e^{k[-it\bar{z}\alpha-zQ^*(P^*)^{-1}z/2-|z|^2-|\alpha t|^2/2]}d\,\mathrm{Vol}(z)dt\\
&= \int_{\mathbb{R}}\int_{\mathbb{C}^m} e^{-\frac{1}{2}k\Psi(t,z)}d\,\mathrm{Vol}(z)dt
\end{aligned}
$$

Let us do the complex Gaussian integral. The phase function is quadratic

$$
\Psi = \begin{pmatrix} t & z^t & \bar{z}^t \end{pmatrix}\begin{pmatrix} |\alpha|^2 & 0 & -i\alpha^t\\ 0 & Q^*(P^*)^{-1} & I\\ -i\alpha & I & 0\end{pmatrix}\begin{pmatrix} t\\ z\\ \bar{z}\end{pmatrix}
$$

We have

$$
\det\begin{pmatrix} |\alpha|^2 & 0 & -i\alpha^t\\ 0 & Q^*(P^*)^{-1} & I\\ -i\alpha & I & 0\end{pmatrix} = \det\begin{pmatrix} |\alpha|^2 & i\alpha^t Q^*(P^*)^{-1} & -i\alpha^t\\ 0 & 0 & I\\ -i\alpha & I & 0\end{pmatrix}
$$

$$
= \det\begin{pmatrix} |\alpha|^2-\alpha^t Q^*(P^*)^{-1}\alpha & i\alpha^t Q^*(P^*)^{-1} & -i\alpha^t\\ 0 & 0 & I\\ 0 & I & 0\end{pmatrix} = (-1)^n(|\alpha|^2-\alpha^t Q^*(P^*)^{-1}\alpha)
$$

Again, we use ξ_H is invariant under M, to get (35), taking conjugate we have

$$
\bar{\alpha}^t = \bar{\alpha}^t P^* - \alpha^t Q^*
$$

Hence

$$
|\alpha|^2-\alpha^t Q^*(P^*)^{-1}\alpha = |\alpha|^2-(\bar{\alpha}^t P^*-\bar{\alpha}^t)(P^*)^{-1}\alpha = \bar{\alpha}^t(P^*)^{-1}\alpha
$$

Thus, doing the complex Gaussian integral, and note that $(-1)^{n/2}$ from determinant Hessian, should cancels with i^n coming from the volume form, we get

$$
\langle M^{-1}v, P_\Xi v\rangle = \left(\frac{k}{2\pi}\right)^{-m-1/2}(\bar{\alpha}(P^*)^{-1}\alpha)^{-1/2}(\det P^*)^{-1/2}.
$$

□

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Scattering Resonances as Viscosity Limits

Maciej Zworski

Abstract Using the method of complex scaling we show that scattering resonances of $-\Delta + V$, $V \in L^\infty_{\rm c}(\mathbb{R}^n)$, are limits of eigenvalues of $-\Delta + V - i\varepsilon x^2$ as $\varepsilon \to 0+$. That justifies a method proposed in computational chemistry and reflects a general principle for resonances in other settings.

1 Introduction and Statement of Results

In this note we show that scattering resonances can be defined as viscosity limits, that is limits of eigenvalues of Hamiltonians suitably regularized as infinity. The detailed proofs are presented in the simplest case of the Schrödinger operator with a compactly supported potential and rely only on standard techniques.

We consider

$$P := -\Delta + V, \quad V \in L^\infty_{\rm comp}(\mathbb{R}^n),$$

where $L^\infty_{\rm comp}$ denotes the spaces of bounded functions vanishing outside of some compact set. (Similarly the subscript $L^\bullet_{\rm loc}$ denotes functions in the space $L^\bullet$ on compact sets.) The scattering resonances are defined as the poles of the meromorphic continuation of resolvent:

$$R_V(z) := (-\Delta + V - z)^{-1} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n), \quad z \in \mathbb{C} \setminus [0, \infty),$$

from the upper half-plane, $\operatorname{Im} z > 0$, through the continuous spectrum, $[0, \infty)$. More precisely,

$$R_V(z) : L^2_{\rm comp}(\mathbb{R}^n) \to L^2_{\rm loc}(\mathbb{R}^n), \tag{1.1}$$

continues meromorphically to the double cover of $\mathbb{C}$ when n is odd and to the logarithmic cover of $\mathbb{C}$ when n is even. The poles of this continuation coincide with the

M. Zworski (✉)
Department of Mathematics, University of California, Berkeley, CA 94720, USA
e-mail: zworski@math.berkeley.edu

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poles of the scattering matrix for the potential V. Their multiplicities (except at the threshold $z = 0$) are given by

$$m(z) := \operatorname{rank} \oint_z R_V(\zeta) d\zeta, \tag{1.2}$$

where the integration is over a small circle around z – see [10, Chapter 3].

Equivalently, we can consider Green's function, that is the integral kernel of $R_V(z)$,

$$R_V(z) f(x) = \int_{\mathbb{R}^n} G(z, x, y) f(y) dy, \tag{1.3}$$

and look at the poles of the continuation of $z \mapsto G(z, x, y)$ for x and y fixed. Another way, based on the method of complex scaling, will be reviewed in Sect. 2.

We now consider a *regularized* operator,

$$P_\varepsilon := -\Delta + V - i\varepsilon x^2, \quad \varepsilon > 0. \tag{1.4}$$

(We write $x^2 := x_1^2 + \cdots + x_n^2$.) It is easy to see (with details reviewed in Sect. 4) that P_ε is an unbounded operator on $L^2(\mathbb{R}^n)$ with a discrete spectrum. We have

Theorem 1 *Suppose that* $\{z_j(\varepsilon)\}_{j=1}^\infty$ *are the eigenvalues of* P_ε. *Then, uniformly on compact subsets of* $\{z : -\pi/4 < \arg z < 7\pi/4\}$,

$$z_j(\varepsilon) \to z_j, \quad \varepsilon \to 0+,$$

where z_j *are the resonances of* P.

A simple one dimensional example illustrating the theorem is given in Fig. 1.

Remarks. 1. A more precise statement involving continuity of spectral projections is given in Sect. 5. The term viscosity is motivated by the viscosity definition of Pollicott–Ruelle resonances given in Dyatlov–Zworski [9] – see Example 3 below.
2. When $\varepsilon < 0$ the spectrum of P_ε is given by complex conjugates of the spectrum of $P_{-\varepsilon}$. Hence we have

$$z_j(\varepsilon) \to \bar{z}_j, \quad \varepsilon \to 0-, \tag{1.5}$$

uniformly on compact subsets of $\{z : -7\pi/4 < \arg z < \pi/4\}$.
3. The term $-i\varepsilon x^2$ is an example of a *complex absorbing potential* and other potentials can also be used – see the discussion below. The proof here requires some analyticity properties of the complex absorbing potential.
4. The restriction to $\arg z > -\pi/4$ when using $-i\varepsilon x^2$ is due to the fact that for $V \equiv 0$ the spectrum of $-\Delta - i\varepsilon x^2$ is given by $\varepsilon^{\frac{1}{2}} e^{-\pi i/4}(2|\ell| + n)$, $\ell \in \mathbb{N}^n$ which is a rescaled spectrum of the Davies harmonic oscillator – see Sect. 3. One can expand the range using $\varepsilon e^{-i\alpha} x^2$, $0 < \alpha < \pi$ in which case we recover resonances with $\arg z > -\alpha/2$.
5. The proof applies with simple modifications to compactly supported *black box* perturbations on $\mathbb{R}^n$ introduced in [25] – see [10, Chapter 4] and [24]. In that case

we need to replace $-i\varepsilon x^2$ by $-i\varepsilon(1-\chi(x))x^2$ where $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ is equal to 1 on a sufficiently large set – see Example 2 below.

The computational method based on calculating eigenvalues of P_ε was introduced in physical chemistry – see Riss–Meyer [19] and Seideman–Miller [20] for the original approach and Jagau et al. [13] for some recent developments and references. However no rigorous mathematical treatments seem to be available and some new interesting open questions can be posed – see Example 4 below.

Fixed complex absorbing potentials have already been used in mathematical literature on scattering resonances. Stefanov [26] showed that semiclassical resonances close to the real axis can be well approximated using eigenvalues of the Hamiltonian modified by a complex absorbing potential. Nonnenmacher–Zworski [16, 17] used fixed complex absorbing potentials to study resonance problems employing gluing techniques of Datchev–Vasy [5, 6]. Yet another application was given by Vasy in [27] where microlocal complex absorbing potentials were used to obtain Fredholm properties and meromorphic continuation of the resolvents (see also [10, Chapter 5]).

We conclude this section with some examples to which Theorem 1 does *not* apply directly but which fit in the same framework.

Example 1 As explained in [23, (c.31)–(c.33)] the theory of Helffer–Sjöstrand [11] applies to the case of potentials which are homogeneous of degree m and satisfy the condition $V(x) = 0,\ x \neq 0 \Longrightarrow \nabla V(x) \neq 0$. That means that resonances of $P = -\Delta + V$ can be defined in $\{z \in \mathbb{C}, \arg z > -\theta_0\}$ for some $\theta_0 > 0$. It is interesting to ask if the viscosity limit gives a global definition in that case.

That is easily seen in the case of quadratic potentials. In fact, suppose that

$$V(x) = \lambda_1^2 x_1^2 + \cdots + \lambda_r^2 x_r^2 - \mu_1^2 x_{r+1}^2 - \cdots - \mu_{n-r}^2 x_n^2, \quad \lambda_j, \mu_\ell > 0.$$

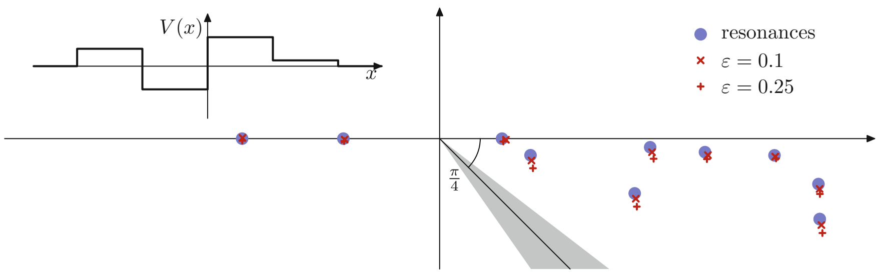

Fig. 1 An illustration of the results of Theorem 1 in the case of a specific potential shown on the left. Resonances are computed using `squarepot.m` [4]. The eigenvalues of P_ε, $\varepsilon = 1/4$ and $\varepsilon = 1/10$ are computed by discretizing the operator using using the first 151 eigenfunctions of the harmonic oscillator $D_x^2 + x^2$. For more numerical illustrations and matlab codes see https://math.berkeley.edu/~zworski/viscap.html

As recalled in Sect. 3 the eigenvalues of P_ε, $\varepsilon > 0$, are given by

$$\sum_{j=1}^{r}(\lambda_j^2 - i\varepsilon)^{\frac{1}{2}}(2k_j + 1) - i\sum_{j=1}^{n-r}(\mu_j^2 - i\varepsilon)^{\frac{1}{2}}(2k_{j+r} + 1), \quad k \in \mathbb{N}_0^n,$$

where the branch of the square root is chosen to be positive on $\mathbb{R}_+$. As $\varepsilon \to 0+$ we obtain the globally defined set of resonances:

$$\sum_{j=1}^{r}\lambda_j(2k_j + 1) - i\sum_{j=1}^{n-r}\mu_j(2k_{j+r} + 1), \quad k \in \mathbb{N}_0^n.$$

Example 2 This example fits in the framework of black box scattering with one dimensional infinity. Consider the modular surface $M = SL_2(\mathbb{Z})\backslash\mathbb{H}^2$ and $\Delta_M \leq 0$ the Laplacian on M. We then put $P = -\Delta_M - \frac{1}{4}$ where $\frac{1}{4}$ guarantees that the the continuous spectrum of P is given by $[0, \infty)$. This is a *black box* Hamiltonian on $\mathcal{H}_0 \oplus L^2([0, \infty))$ in the sense of [25] – see [10, § 4.1]. Traditionally, the resonances of the quotient M are defined as poles of the meromorphic continuation of $(-\Delta_M - s(1-s))^{-1}$ from $\mathrm{Re}\, s > \frac{1}{2}$ to $\mathbb{C}$ and are given by the embedded eigenvalues when $\mathrm{Re}\, s = \frac{1}{2}$ and by the non-trivial zeros of $\zeta(2s)$ where ζ is the Riemann zeta function. The resonances of P are then expressed as $s(1-s)$.

If we choose the fundamental domain of $SL_2(\mathbb{Z})$ to be $\{x + iy : |x| \leq 1, x^2 + y^2 \geq 1\}$ then the Laplacian in the cusp $y > 1$ is $y^{-2}(\partial_x^2 + \partial_y^2)$. The Hamiltonian on $L^2([0, \infty)_r)$ is given by $-\partial_r^2$, $r = \log y$ – see [10, § 4.1, Example 3]. In the language of Theorem 1 (see Remark 5) and in (x, y) coordinates

$$P_\varepsilon = -\Delta_M - \tfrac{1}{4} - i\varepsilon(1 - \chi(y))(\log y)^2, \tag{1.6}$$

where $\chi \in \mathcal{C}_c^\infty([0, \infty))$, $\chi(y) \equiv 1$ for $y < \frac{3}{2}$ and $\chi(y) \equiv 0$ for $y > 2$. The operator P_ε has discrete spectrum for $\varepsilon > 0$ and the eigenvalues converge to the resonances of P uniformly on compact subsets of $\mathrm{Im}\, z > -\pi/4$. Equivalently if we define Σ_ε

$$s(\varepsilon) \in \Sigma_\varepsilon \Longleftrightarrow s(\varepsilon)(1 - s(\varepsilon)) \in \sigma(P_\varepsilon)$$

The limit points of Σ_ε, $\varepsilon \to 0+$, in $\mathrm{Re}\, s < \frac{1}{2}$, $|s| > C$ are given by the nontrivial zeros of $\zeta(2s)$. (Strictly speaking, when using the black box formalism we should consider $P_\varepsilon = -\Delta_M - \frac{1}{4} - i\varepsilon(1 - \chi(y))(\log y)^2\Pi_0$ where Π_0 is the projection onto the zero mode of $-\partial_x^2$, $x \in \mathbb{R}/\mathbb{Z}$. In our case the absorbing potential has a mild effect on higher modes so the projection can be dropped.)

Example 3 Suppose that X is a compact manifold and V is a vector field on X generating an Anosov flow, $\varphi^t = \exp tV$. That means that the tangent space to X has a continuous decomposition $T_xX = E_0(x) \oplus E_s(x) \oplus E_u(x)$ which is invariant, $d\varphi_t(x)E_\bullet(x) = E_\bullet(\varphi_t(x))$, $E_0(x) = \mathbb{R}V(x)$, and for some C and $\theta > 0$ fixed

$$|d\varphi_t(x)v|_{\varphi_t(x)} \leq Ce^{-\theta|t|}|v|_x, \quad v \in E_u(x), \quad t < 0,$$
$$|d\varphi_t(x)v|_{\varphi_t(x)} \leq Ce^{-\theta|t|}|v|_x, \quad v \in E_s(x), \quad t > 0. \tag{1.7}$$

where $|\bullet|_y$ is given by a smooth Riemannian metric on X. A class of examples is given by $X = T^1M$ where M is a negatively curved Riemannian manifold and φ^t is the geodesic flow in its unit tangent bundle X.

If $\Delta_g \leq 0$ is the Laplacian for some metric on X then – see [9] – the limit set of the spectrum of

$$P_\varepsilon = V/i + i\varepsilon\Delta_g, \quad \varepsilon \to 0+$$

is a discrete set given by the *Pollicott–Ruelle* resonances – see [9] for the definition and references. Adding the Laplacian corresponds to taking a *viscosity* regularization and that explains our terminology. Another interpretation is given in terms of Brownian motion: the pullback by the flow flow $x(t) := \varphi_t(x(0))$, is given by $e^{itP_0}f(x) = f(x(t))$, $\dot{x}(t) = -V_{x(t)}$, $x(0) = x$. For $\varepsilon > 0$ the evolution equation is replaced by the Langevin equation:

$$e^{-itP_\varepsilon}f(x) = \mathbb{E}[f(x(t))], \quad \dot{x}(t) = -V_{x(t)} + \sqrt{2\varepsilon}\dot{B}(t), \quad x(0) = x,$$

where $B(t)$ is the Brownian motion corresponding to the metric g on X. Hence considering P_ε corresponds to a stochastic perturbation of the deterministic flow. In the case of scattering resonances the same interpretation can be proposed on the Fourier transform side.

The assumption that the flow satisfies (1.7) is crucial as otherwise the limit set is typically not discrete. The simplest example is given by $X = \mathbb{S}^1 \times \mathbb{S}^1$, $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$, and $V = \partial_{x_1} + \alpha\partial_{x_2}$, $\alpha \notin \mathbb{Q}$, $\Delta_g = \partial_{x_1}^2 + \partial_{x_2}^2$. In that case the limit set of the spectrum of P_ε is the lower half plane. Other limit sets are possible, for instance in the case of the geodesic flow on $\mathbb{S}^2$, $X = T^1\mathbb{S}^2 \simeq SO(3)$. The spectrum of P_0 is given by $\mathbb{Z}$ (with infinite multiplicities) and if we take Δ_g to be the Casimir operator then the limit set of the spectrum of P_ε as $\varepsilon \to 0+$ is $\mathbb{Z} - i[0, \infty)$. For yet another example see [9, § 1].

Example 4 We expect that viscosity definition of resonances remains valid, in a small angle near the real axis, for all *dilation analytic* potentials – see [11] and references given there and Sect. 2 below for a review of complex scaling. It would be interesting to find a Schrödinger operator P for which the limit set of the spectrum of P_ε, $\varepsilon \to 0$ is not discrete. Candidates are given by potentials which are *not* dilation analytic, for instance,

$$-\partial_x^2 + \frac{\sin x}{x}, \quad x \in \mathbb{R}.$$

Notation. We use the following notation: $f = \mathcal{O}_\ell(g)_H$ means that $\|f\|_H \leq C_\ell g$ where the norm (or any seminorm) is in the space H, and the constant C_ℓ depends on ℓ. When either ℓ or H are absent then the constant is universal or the estimate is scalar, respectively. When $G = \mathcal{O}_\ell(g) : H_1 \to H_2$ then the operator $G : H_1 \to H_2$ has its norm bounded by $C_\ell g$. Also when no confusion is likely to result, we denote the operator $f \mapsto gf$ where g is a function by g.

2 Review of Complex Scaling

The complex scaling method changes the original Hamiltonian $P = P_0$ to a non-self-adjoint Hamiltonian $P_{0,\theta}$ such that $P_{0,\theta} - z : H^2 \to L^2$ is a Fredholm operator when $\arg z > -2\theta$. It was introduced by Aguilar–Combes [1], Balslev–Combes [2] and Simon [21]. For a review of practical applications of this method in computational chemistry see Reinhardt [18]. As the method of *perfectly matched layers* (PML) it has reappeared in numerical analysis – see Berenger [3]. The presentation here follows the geometric approach of Sjöstrand–Zworski [25]. Eventually the proof that the viscosity eigenvalues converge to scattering resonances is a straightforward application of the methods of [25] (see also [24, § 7.2] for a more detailed presentation and [10, § 4.5] for an approach to complex scaling based on the continuation of the Green function $G(z, x, y)$ in (1.3) in variables x and y).

Suppose that $\Omega \subset \mathbb{C}^n$ is an open subset and that

$$P(z, D_z) = \sum_{|\alpha| \leq m} a_\alpha(z) D_z^\alpha, \quad D_{z_j} := \tfrac{1}{i}\partial_{z_j}, \quad D_z^\alpha = D_{z_1}^{\alpha_1} \cdots D_{z_n}^{\alpha_n}, \tag{2.1}$$

is a differential operator with holomorphic coefficients. For instance we can have $P(z, D_z) = \sum_{j=1}^n D_{z_j}^2 - i\varepsilon z_j^2$.

Suppose that $\Omega \subset \mathbb{C}^n$ is an open subset and that $\Gamma \subset \Omega$ is a *maximal totally real* submanifold. That means that Γ is a smooth real submanifold of dimension n such that

$$\forall\, x \in \Gamma, \quad T_x\Gamma \cap i T_x\Gamma = \{0\}. \tag{2.2}$$

Here we identify $T_x\Gamma$ with a real subspace of $\mathbb{C}^n$. The condition (2.2) means that there exists a *complex linear* change of variables $A : \mathbb{C}^n \to \mathbb{C}^n$ such that $A(T_x\Gamma) = \mathbb{R}^n \subset \mathbb{C}^n$. Locally, Γ can be represented using real coordinates:

$$\mathbb{R}^n \supset U \ni x \mapsto f(x) = (f_1(x), \cdots, f_n(x)) \in \Gamma \subset \Omega \subset \mathbb{C}^n. \tag{2.3}$$

Composing the matrix $\partial_x f(x) := (\partial_{x_j} f_k(x))_{1\leq k,j\leq n}$ with A we obtain an invertible matrix $\mathbb{R}^n \to \mathbb{R}^n$. That means that

$$\det \left(\frac{\partial f_k(x)}{\partial x_j} \right)_{1\leq k,j\leq n} \neq 0. \tag{2.4}$$

Conversely, if (2.4) holds, then $\partial f(x)$ is an injective complex linear matrix and for any sets $U, V \subset \mathbb{C}^n$, $\partial f(x)(U) \cap \partial f(x)(V) = \partial f(x)(U \cap V)$. Hence,

$$\begin{aligned} T_x\Gamma \cap i T_x\Gamma &= \partial f(x)(\mathbb{R}^n) \cap i\partial f(x)(\mathbb{R}^n) = \partial f(x)(\mathbb{R}^n) \cap \partial f(x)(i\mathbb{R}^n) \\ &= \partial f(x)(\mathbb{R}^n \cap i\mathbb{R}^n) = \{0\}, \end{aligned}$$

and (2.4) implies (2.2). The volume form on Γ is obtained by pushing forward the standard volume form on $\mathbb{R}^n$ by f. That of course depends on the choice of f (in what follows the uniformity will be guaranteed by (2.8) below).

Example. As a simple illustration consider $n = 2$ and $f(x_1, x_2) = (x_1 + ix_2, 0) \in \mathbb{C}^2$. Then

$$\partial_x f(x) = \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}, \quad T_x f(\mathbb{R}^2) = \mathbb{C} \oplus \{0\} \subset \mathbb{C}^2.$$

The tangent space is not totally real and condition (2.4) is violated. To introduce the next topic we also note we cannot restrict operators, P, with holomorphic coefficients to $f(\mathbb{R}^2)$ in a way that for holomorphic functions, u, $(Pu)|_{f(\mathbb{R}^2)} = (P_{f(\mathbb{R}^n)})(u|_{f(\mathbb{R}^n)})$. As an example consider $P = \partial_{z_2}$ and $u = z_2$.

The point of introducing totally real submanifolds Γ is the fact that an operator, P, with holomorphic coefficients can be restricted to an operator with complex smooth coefficients on Γ, P_Γ, in such a way that for u holomorphic near Γ, $Pu|_\Gamma = P_\Gamma(u|_\Gamma)$.

The differential operator $P(z, D_z)$ given in (2.1) defines a unique P_Γ a differential operator on Γ as follows. Using (2.3) we can identify a small neighbourhood of any $z_0 \in \Gamma$ with $U \subset \mathbb{R}^n$. Then $u \in \mathcal{C}^\infty(\Gamma \cap f(U))$ can be identified with $u \circ f \in \mathcal{C}^\infty(U)$. We then have

$$(P_\Gamma u) \circ f(x) = \sum_{|\alpha| \le m} (a_\alpha \circ f)(x)(({}^t\partial_x f(x)^{-1} D_x)^\alpha (u \circ f)(x). \tag{2.5}$$

It is easy to see that this definition is independent of the choice of f and that the condition (2.4) is crucial.

The key fact is the standard result about continuation of solutions to $P_\Gamma u$. The proof based on [14, 15, 22] can be found in [25, Lemma 3.1] and (in more detail) [24, Lemma 7.2]. With the notation above we have the following:

Lemma 1 *Suppose that* $W \subset \mathbb{R}^n$ *is open and that* $F : [0, 1] \times W \ni (s, x) \mapsto F(s, x) \in \mathbb{C}^n$, *is a smooth proper map satisfying for all* $s \in [0, 1]$

$$\det \partial_x F(s, x) \neq 0, \quad \textit{and } x \mapsto F(s, x) \textit{ is injective.}$$

In addition assume that there exists a compact set $K \subset W$ *such that*

$$x \in W \setminus K \implies F(0, x) = F(s, x), \quad 0 \le s \le 1,$$

and that $F([0,1]\times W)\subset\Omega$ *with* $P(z,D_z)$ *a differential operator with holomorphic coefficients in* Ω.

Now assume that for $\Gamma_s := F(\{s\}\times W)$, P_{Γ_s} *is an elliptic differential operator in the sense that*

$$\Big|\sum_{|\alpha|=m} a_\alpha(z)\zeta^\alpha\Big| \geq C|\zeta|^m, \quad (z,\zeta)\in T^*\Gamma_s.$$

If $u_0\in\mathcal{C}^\infty(\Gamma_0)$ *and* $P_{\Gamma_0}u_0$ *extends to a holomorphic function on* Ω, *then for every* $s\in[0,1]$ *there exists a holomorphic function,* U_s *defined near* Γ_s *such that, for some* ε,

$$U_0|_{\Gamma_0}=u_0, \quad |s-s'|<\varepsilon \implies U_s=U_{s'} \text{ on the intersection of their domains.}$$

In other words, the function u_0 defined on Γ_0 extends to a possibly multivalued function U in a neighbourhood of $f([0,1]\times W)$.

The lemma will be applied to a family of deformations of $\mathbb{R}^n$ in $\mathbb{C}^n$. Our goal is to restrict the operator $P_\varepsilon=-\Delta-i\varepsilon x^2+V$, $\varepsilon\geq 0$, to the corresponding totally real submanifolds. For that the deformation has to avoid the support of V and we choose r_0 such that $\operatorname{supp} V\subset B(0,r_0)$. We then construct

$$[0,\pi)\times[0,\infty)\ni(\theta,t)\longmapsto g_\theta(t)\in\mathbb{C} \tag{2.6}$$

which is $\mathcal{C}^\infty$, is injective on $[0,\infty)$ for every fixed θ and satisfies

$$g_\theta(t)=t \text{ for } 0\leq t\leq r_0, \tag{2.7}$$
$$0\leq \arg g_\theta(t)\leq\theta, \quad \partial_t g_\theta\neq 0, \tag{2.8}$$
$$\arg g_\theta(t)\leq \arg\partial_t g_\theta(t)\leq \arg g_\theta(t)+\varepsilon_0, \tag{2.9}$$
$$g_\theta(t)=e^{i\theta}t \text{ for } t\geq T_0 \text{ where } T_0 \text{ depends only on } \varepsilon_0 \text{ and } r_0. \tag{2.10}$$

(To construct such a function choose $\chi\in\mathcal{C}^\infty(\mathbb{R})$ such that $\chi(t)=0$ for $t\leq r_0+1$ and $\chi(t)=1$ for $t\geq T_0-1$, and $0\leq\chi'(t)\leq\varepsilon_0/(t\theta)$, where the last condition can be met once $T_0-r_0\gg e^{\theta/\varepsilon_0}$. We then put $g_\theta(t)=te^{i\theta\chi(t)}$.) We now define the totally real submanifolds, Γ_θ, as images of $\mathbb{R}^n$ under the maps

$$f_\theta:\mathbb{R}^n\to\mathbb{C}^n, \quad f_\theta(x):=g_\theta(|x|)x/|x|, \quad \Gamma_\theta:=f_\theta(\mathbb{R}^n). \tag{2.11}$$

For $\varepsilon\geq 0$ and $0\leq\theta<\pi$ we put

$$\begin{aligned} -\Delta_\theta &:= (-\Delta_z)|_{\Gamma_\theta}, \quad x_\theta := z|_{\Gamma_\theta},\\ Q_{\varepsilon,\theta} &:= -\Delta_\theta - i\varepsilon x_\theta^2, \quad P_{\varepsilon,\theta}:=Q_{\varepsilon,\theta}+V. \end{aligned} \tag{2.12}$$

Parametrizing Γ_θ by $(t,\omega) \in [0,\infty) \times \mathbb{S}^{n-1}$, $(t,\omega) \mapsto g_\theta(t)\omega$, we have

$$-\Delta_\theta = \left(g'_\theta(t)^{-1}D_t\right)^2 - i(n-1)g_\theta(t)^{-1}g'_\theta(t)^{-1}D_t + g_\theta(t)^{-2}D_\omega^2, \tag{2.13}$$

where $D_t = \partial_t/i$ and $D_\omega^2 = -\Delta_{\mathbb{S}^{n-1}}$. The symbol is given by

$$\sigma(-\Delta_\theta) = g'_\theta(t)^{-1}\tau^2 + g_\theta(t)^{-2}w^2, \quad (t,\omega;\tau,w) \in T^*([0,\infty)\times\mathbb{S}^{n-1}).$$

The basic result based on ellipticity at infinity is

$$-2\theta + \delta < \arg z < 2\pi - 2\theta - \delta, \ \ |z| \geq \delta \implies (-\Delta_\theta - z)^{-1} = \mathcal{O}_\varepsilon(1) : L^2(\Gamma_\theta) \to H^2(\Gamma_\theta).$$

This follows from [25, Lemmas 3.2–3.5] applied with $P = -\Delta$. As will be reviewed in Sect. 4 this shows that $P_{0,\theta} - z : H^2 \to L^2$ is a Fredholm operator in this range of values of z and that the eigenvalues are independent of θ.

The crucial property is

Lemma 2 *Let* $R_0(z) = (-\Delta - z)^{-1} : L^2 \to H^2$, *Im* $z > 0$, *be the free resolvent and let* $R_0(z)$ *also denote its analytic continuation across* $[0,\infty)$ *as an operator* $L^2_{\mathrm{comp}} \to H^2_{\mathrm{loc}}$.

Suppose that $\chi \in \mathcal{C}^\infty_{\mathrm{c}}(B(0,r_0))$ *so that* χ *is defined on* Γ_θ. *Then for* $-2\theta < \arg z < 2\pi - 2\theta$, $\theta < \pi$,

$$\chi R_0(z)\chi = \chi(-\Delta_\theta - z)^{-1}\chi. \tag{2.14}$$

Proof We recall the main features of the proof which is implicit in [25, § 3]. It is sufficient to establish the identity (2.14) for $0 < \arg z < 2\pi - 2\theta$ as it then follows by analytic continuation. It is also enough to show that in this range of z and $0 \leq \theta_1 < \theta_2 \leq \theta$, $|\theta_1 - \theta_2| \ll 1$,

$$\chi(-\Delta_{\theta_1} - z)^{-1}\chi = \chi(-\Delta_{\theta_2} - z)^{-1}\chi. \tag{2.15}$$

For that we show that for $f \in L^2(B(0,r_0)) \subset L^2(\Gamma_{\theta_j})$ there exists U holomorphic in a neighbourhood $\Omega_{\theta_1,\theta_2}$ of

$$\bigcup_{\theta_1\leq\theta\leq\theta_2} (\Gamma_\theta \setminus B(0,r_0)) \subset \mathbb{C}^n$$

such that

$$U|_{\Gamma_{\theta_j}}(x) = [(-\Delta_{\theta_j} - z)^{-1}\chi f](x) \ \text{ for } x \in \Gamma_{\theta_j} \setminus B(0,r_0). \tag{2.16}$$

The unique continuation property for second order elliptic operators then shows that

$$\chi(\Delta_{\theta_1} - z)^{-1}\chi f = \chi(\Delta_{\theta_2} - z)^{-1}\chi f,$$

proving (2.14).

To show the existence of U such that (2.16) holds we use Lemma 1 applied to a modified family of deformations. The key is to show that a holomorphic extension, U, of the solution to

$$(-\Delta_{\theta_1} - z)u_1 = \chi f, \quad u_1 \in L^2(\Gamma_{\theta_1}), \quad u_1 = U|_{\Gamma_1}$$

satisfies $u_2 := U|_{\Gamma_2} \in L^2(\Gamma_{\theta_2})$ (the equation $(-\Delta_{\theta_2} - z)u_2 = \chi f$ is automatically satisfied). That means that $u_2 = (-\Delta_{\theta_2} - z)^{-1}(\chi f)$ proving (2.16).

The modified family of contours is obtained as follows. Fix $T \gg 1$ and choose $\chi \in \mathcal{C}_c^\infty((2,5);[0,1])$ equal to 1 near $[3,4]$. Then define

$$\begin{aligned} g_{\theta_1,\theta_2,T}(t) &:= g_{\theta_1}(t) + \chi(t/T)(g_{\theta_2}(t) - g_{\theta_1}(t)), \\ \Gamma_{\theta_1,\theta_2,T} &:= \{g_{\theta_1,\theta_2,T}(t)\omega \,:\, t \in [0,\infty),\ \omega \in \mathbb{S}^{n-1}\} \subset \mathbb{C}^n. \end{aligned} \tag{2.17}$$

We can apply Lemma 1 to the family of totally real submanifolds interpolating between Γ_{θ_1} and $\Gamma_{\theta_1,\theta_2,T}$: $[0,1] \ni s \longmapsto \Gamma_{\theta_1,\theta_1+s(\theta_1,\theta_2),T}$. That implies that there exists a holomorphic function U^T defined in a neighbourhood of the union of these submanifolds and such that $u_1 = U^T|_{\Gamma_{\theta_1}}$. Changing T we obtain a family of functions agreeing on the intersections of their domains and that gives U defined in the neighbourhood $\Omega_{\theta_1,\theta_2}$. To see that $U|_{\Gamma_{\theta_2}} \in L^2(\Gamma_{\theta_2})$ it suffices to show that (Fig. 2)

$$\|U^T|_{\Gamma_{\theta_1,\theta_2,T}}\|_{L^2(\Gamma_{\theta_1,\theta_2,T})} \leq C_0 \|u_1\|_{L^2(\Gamma_{\theta_1} \cap \{T \leq |z| \leq 6T\},)}, \tag{2.18}$$

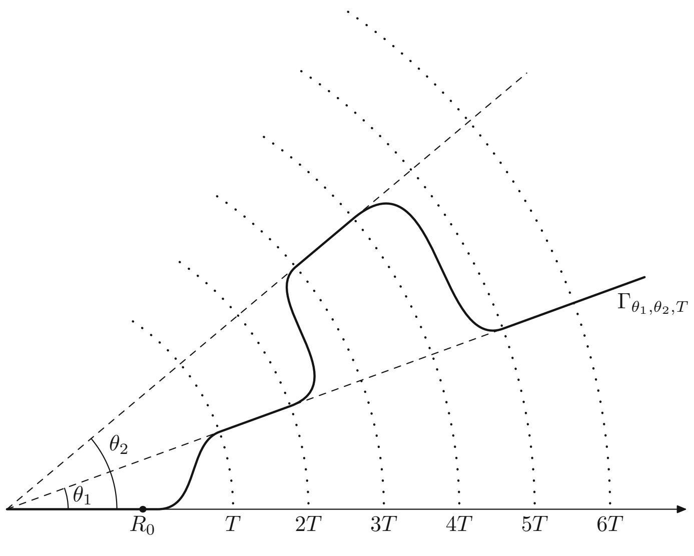

Fig. 2 The deformed totally real submanifold $\Gamma_{\theta_1,\theta_2,T}$ interpolating between Γ_{θ_1} and Γ_{θ_2}

where C_0 is independent of T. (We apply (2.18) with $T = 2^j$ and sum over j.)
To see (2.18)

$$\Omega_1(T) = \{z \in \mathbb{C}^n : 2T \le |z| \le 5T\} \cap \Gamma_{\theta_1,\theta_2,T} \supset \Gamma_{\theta_1,\theta_2,T} \setminus \Gamma_{\theta_1},$$
$$\Omega_2(T) = \{z \in \mathbb{C}^n : T \le |z| \le 6T\} \cap \Gamma_{\theta_1,\theta_2,T}, \quad \Omega_2(T) \setminus \Omega_1(T) \subset e^{i\theta_1}\mathbb{R}^n.$$

We claim that for T large and $u \in \mathcal{C}^\infty(\Gamma_{\theta_1,\theta_2,T})$,

$$\|u\|_{L^2(\Omega_1(T))} \le C\|(-\Delta_{\Gamma_{\theta_1,\theta_2,T}} - z)u\|_{L^2(\Omega_2(T))} + C\|u\|_{L^2(\Omega_2(T)\setminus\Omega_1(T))}. \tag{2.19}$$

For $|\theta_2 - \theta_1| \ll 1$, this estimate is a perturbation of a standard semiclassical elliptic estimate: treating $h := 1/T$ as a semiclassical parameter, uniform ellipticity of $-e^{-2i\theta}h^2\Delta - z$ shows that for $v \in \mathcal{C}^\infty(\mathbb{R}^n)$,

$$\|v\|_{L^2(\{2\le|x|\le5\})} \le C\|(-e^{-2i\theta}h^2\Delta - z)v\|_{L^2(\{1\le|x|\le6\})} + C\|v\|_{L^2(\{1\le|x|\le2\}\cup\{5\le|x|\le6\})}.$$

(This can be seen applying the inverse from [28, Theorem 4.29] to χv where $\chi \in \mathcal{C}^\infty_c((1,6))$ is equal to 1 on $[2,5]$.) The properties of $\Omega_j(T)$ then imply (2.18) completing the argument. □

3 The Davies Harmonic Oscillator

The operator $H_{\varepsilon,\gamma} := -\Delta + e^{-i\gamma}\varepsilon x^2$, $\varepsilon > 0$, $0 \le \gamma < \pi$, was used by Davies [7] to illustrate properties of non-normal differential operators. We recall the following basic result:

Lemma 3 *The operator $H_{\varepsilon,\gamma}$ is an unbounded operator on L^2 with the discrete spectrum given by*

$$\sigma(H_{\varepsilon,\gamma}) = e^{-i\gamma/2}\sqrt{\varepsilon}(n + 2|\mathbb{N}_0^n|), \quad |\alpha| = \alpha_1 + \cdots + \alpha_n. \tag{3.1}$$

If $\Omega \Subset \{z : -\gamma < \arg z < 0\} \setminus e^{-i\gamma/2}[0,\infty)$*, then for some constant* $C_1 = C_1(\Omega)$*,*

$$\frac{1}{C_1}e^{\varepsilon^{-\frac12}/C_1} \le \|(H_{\varepsilon,\gamma} - z)^{-1}\|_{L^2\to L^2} \le C_1 e^{C_1\varepsilon^{-\frac12}}, \ z \in \Omega. \tag{3.2}$$

In addition for any $\delta > 0$ *there exists a constant* C_2 *such that, uniformly in* $\varepsilon > 0$*,*

$$\|(H_{\varepsilon,\gamma} - z)^{-1}\|_{L^2\to L^2} \le C_2/|z|, \quad \delta < \arg z < 2\pi - \gamma - \delta, \quad |z| > \delta. \tag{3.3}$$

Proof By rescaling $y = \sqrt{\varepsilon}x$ this operator in unitarily equivalent to $-\varepsilon\Delta_y + e^{-i\gamma}y^2$, that is a semiclassical, $h = \sqrt{\varepsilon}$, quadratic operator. For the analysis of the spectrum and upper bounds on the resolvent for general quadratic operators see Hitrik–

Sjöstrand–Viola [12] and references given there – in particular we obtain (3.1) and the upper bound in (3.2). The lower bound in (3.2) follows from general arguments for operators with analytic coefficients – see [8, § 3] and the bound (3.3) from (semiclassical) ellipticity of $-h^2\Delta_y + e^{-i\gamma}y^2 - z$ for $\delta < \arg z < 2\pi - \gamma - \delta$, $|z| > \delta$. □

We now consider the special case of $H_{\varepsilon,\pi/2} = Q_{\varepsilon,0}$ and of its deformation $Q_{\varepsilon,\theta}$ – see (2.12). The facts we need are given in the next two lemmas. The first is the analogue of Lemma 2:

Lemma 4 *In the notation of Lemma 2,* $0 \le \theta \le \pi/8$, $\varepsilon > 0$, *and* $-2\theta < \arg z < 3\pi/2 + 2\theta$ *we have*

$$\chi(Q_{\varepsilon,0} - z)^{-1}\chi = \chi(Q_{\varepsilon,\theta} - z)^{-1}\chi. \tag{3.4}$$

In particular, for $0 \le \theta \le \pi/8$, *the spectrum is independent of* θ *and given by* $\sqrt{\varepsilon}e^{-i\pi/4}(n + 2|\mathbb{N}_0^n|)$.

Proof We follow the argument in the proof of Lemma 2 and use the notation introduced there. Hence it is enough to prove that $0 \le \theta_1 < \theta_2 \le \pi/8$ and $|\theta_1 - \theta_2|$ small it is enough to show that (Fig. 3)

$$\chi(Q_{\varepsilon,\theta_1} - z)^{-1}\chi = \chi(Q_{\varepsilon,\theta_2} - z)^{-1}\chi.$$

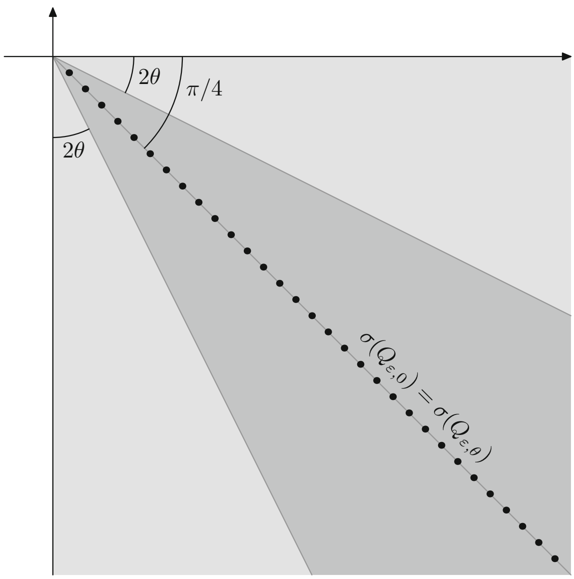

Fig. 3 A visualization of the spectrum of $Q_{\varepsilon,0} = -\Delta - i\varepsilon x^2$ which is equal to the spectrum of the deformed operator $Q_{\varepsilon,\theta}$. The lightly shaded region is the numerical range of $Q_{\varepsilon,0}$ and the darker shaded region, the numerical range of $-e^{-2i\theta}\Delta - ie^{2i\theta}\varepsilon x^2$. The estimates for the resolvents of $Q_{\varepsilon,\theta}$ improve outside of that region

We only need to establish this for $z \in e^{i(-2\theta_1+\pi/2)}(1,\infty)$ as then the result follows by analytic continuation. The only difference is an estimate which replaces (2.19): for $\tau > 1$,

$$\begin{gathered} \|u\|_{L^2(\Omega_1(T))} \leq C\|(Q_{\Gamma_{\theta_1,\theta_2,T}} - ie^{-2\theta_1}\tau)u\|_{L^2(\Omega_2(T))} + C\|u\|_{L^2(\Omega_2(T)\setminus\Omega_1(T))}, \\ Q_{\theta_1,\theta_2,T} := -\Delta_{\Gamma_{\theta_1,\theta_2,T}} - i\varepsilon(x|_{\Gamma_{\theta_1,\theta_2,T}})^2 \end{gathered} \tag{3.5}$$

uniformly for $T \gg 1$. To see this we first note that for $\varepsilon > 0$, $Q_{\theta_1,\theta_2,T} - z$, $z \in \mathbb{C}$, is a Fredholm operator (since it is elliptic and near infinity it is equal to $e^{-2i\theta}H_{\varepsilon,\pi/2-4\theta}$).

To obtain an estimate we notice that for $t > T$ and $g_{\theta_1,\theta_2,T}$ defined in (2.17),

$$g'_{\theta_1,\theta_2,T}(t) = \chi(t/T)e^{i\theta_2} + (1-\chi(t/T))e^{i\theta_1} + (t/T)\chi'(t/T)(e^{i\theta_2} - e^{i\theta_1}),$$

so that from (2.8) and (2.10),

$$\theta_1 - C|\theta_2 - \theta_1| \leq \arg g'_{\theta_1,\theta_2,T}(t) \leq \theta_2.$$

Also, $\theta_1 \leq \arg g_{\theta_1,\theta_2,T}(t) \leq \theta_2$. Hence,

$$\operatorname{Re}\langle (e^{2i\theta_1}Q_{\theta_1,\theta_2,T} - i\tau)u, u\rangle \geq \|Du\|^2/C$$

where we used the fact that for for $0 \leq \theta \leq \pi/8$, $\operatorname{Re}(-ie^{4\theta}) \geq 0$. The imaginary part is then estimated as follows,

$$-\operatorname{Im}\langle (e^{2i\theta_1}Q_{\theta_1,\theta_2,T} - i\tau)u, u\rangle \geq \tau\|u\|_{L^2(\Gamma_{\theta_1,\theta_2,T})} - \mathcal{O}(|\theta_2 - \theta_1|)\|Du\|^2.$$

We conclude that when $|\theta_2 - \theta_1|$ is small enough

$$\|(Q_{\Gamma_{\theta_1,\theta_2,T}} - ie^{-2i\theta_1}\tau)u\| \geq (\|u\| + \|Du\|)/C,$$

This and the Fredholm property imply that

$$(Q_{\theta_1,\theta_2,T} - ie^{-2i\theta_1}\tau)^{-1} = \mathcal{O}(1) : L^2(\Gamma_{\theta_1,\theta_2,T}) \to H^1(\Gamma_{\theta_1,\theta_2,T}).$$

that is the operator is invertible with bounds independent of T. From this (3.5) follows by a standard localization argument: we choose $\chi_T \in \mathcal{C}^\infty(\Omega_2(T), [0,1])$, such that $\chi_T = 1$ on $\Omega_1(T)$ with derivative bounds independent of T. We then apply the inverse above to $(Q_{\theta_1,\theta_2,T} - ie^{-2i\theta_1}\tau)\chi_T u$ with the commutator terms estimated by $\|u\|_{L^2(\Omega_2(T)\setminus\Omega_1(T))}$. □

The next lemma shows how complex scaling dramatically improves the exponential bound (3.2):

Lemma 5 *Suppose that* $0 \leq \theta \leq \pi/8$ *and that* $\Omega \Subset \{z : -2\theta < \arg z < 3\pi/2 + 2\theta\}$. *Then there exists* $C = C(\Omega)$ *(in particular independent of* $\varepsilon > 0$*) such that*

$$\|(Q_{\varepsilon,\theta}-z)^{-1}\|_{L^2\to L^2}\leq C,\quad z\in\Omega.$$

Proof Let $\chi_j\in\mathcal{C}_c^\infty([0,\infty))$ be equal to 1 on $[0,r_0]$ and satisfy $\chi_j=1$ on $\operatorname{supp}\chi_{j+1}$, $j=0,1$. Parametrizing Γ_θ by $F_\theta:[0,\infty)_t\times\mathbb{S}^{n-1}\to\Gamma_\theta$, $F_\theta(t,\omega)=g_\theta(t)\omega$ (with g_θ given in (2.6)) we define functions $\chi_j^h\in\mathcal{C}_c^\infty(\Gamma_\theta)$ as

$$\chi_j^h\circ F_\theta(t,\omega):=\chi_j(th),\quad 0<h\leq 1.$$

In view of (2.10) and (2.13) we see that for h small enough

$$\begin{aligned}Q_{\varepsilon,\theta}(1-\chi_1^h)&=(-e^{-2i\theta}\Delta_x-i\varepsilon e^{2i\theta}x^2)(1-\chi_1^h)\\&=e^{-2i\theta}H_{\varepsilon,\gamma}(1-\chi_1^h),\quad\gamma:=\pi/2-4\theta,\quad x=t\omega.\end{aligned}$$

In view of (3.3) we have

$$(1-\chi_2^h)e^{2i\theta}(H_{\varepsilon,\gamma}-e^{2i\theta}z)^{-1}(1-\chi_2^h)=\mathcal{O}_\delta(1):L^2(\Gamma_\theta)\to H^2(\Gamma_\theta),\tag{3.6}$$

for

$$-\delta<2\theta+\arg z<2\pi-\gamma-\delta=3\pi/2+4\theta-\delta,\quad|z|>\delta,$$

and in particular for $z\in\Omega$. We stress that the bounds are independent of ε.

Noting that

$$(-\Delta_\theta-z)^{-1}=\mathcal{O}(1):L^2(\Gamma_\theta)\to H^2(\Gamma_\theta),\quad z\in\Omega,\tag{3.7}$$

(for $0\leq\theta\leq\pi/8$, $-2\theta<\arg z<2\pi-2\theta$) we now put

$$T_{\varepsilon,\theta}^h(z):=\chi_0^h(-\Delta_\theta-z)^{-1}\chi_1^h+(1-\chi_1^h)e^{2i\theta}(H_{\varepsilon,\gamma}-e^{2i\theta}z)^{-1}(1-\chi_2^h),$$

so that $(Q_{\varepsilon,\theta}-z)T_{\varepsilon,\theta}^h(z)=I+K_{\varepsilon,\theta}^h(z)$, where

$$\begin{aligned}K_{\varepsilon,\theta}^h(z):=&-i\varepsilon x_\theta^2\chi_0^h(-\Delta_\theta-z)^{-1}\chi_1^h-[\Delta_\theta,\chi_0^h](-\Delta_\theta-z)^{-1}\chi_1\\&+[\Delta_\theta,\chi_1^h]e^{2i\theta}(1-\chi_2^h)(H_{\varepsilon,\gamma}-e^{2i\theta}z)^{-1}(1-\chi_2^h).\end{aligned}$$

Since $[\Delta_\theta,\chi_j^h]=\mathcal{O}(h):H^1(\Gamma_\theta)\to L^2(\Gamma_\theta)$ and $x_\theta^2\chi_1^h=\mathcal{O}(h^{-2}):L^2(\Gamma_\theta)\to L^2(\Gamma_\theta)$, we conclude from (3.6) and (3.7) that for $z\in\Omega$,

$$K_{\varepsilon,\theta}^h(z)=\mathcal{O}(h^{-2}\varepsilon)+\mathcal{O}(h):L^2(\Gamma_\theta)\to L^2(\Gamma_\theta).$$

Hence by choosing h first, we see that for $\varepsilon<\varepsilon_0(h)$, $I+K_{\varepsilon,\theta}^h(z)$ has a uniformly bounded inverse and $0\leq\varepsilon<\varepsilon_0$

$$(Q_{\varepsilon,h}-z)^{-1}=T_{\varepsilon,\theta}^h(z)(I+K_{\varepsilon,\theta}^h(z))^{-1}=\mathcal{O}(1):L^2(\Gamma_\theta)\to L^2(\Gamma_\theta),\quad z\in\Omega.$$

In view of Lemma 4 we know that for $z \in \Omega$, $(Q_{\varepsilon,h} - z)^{-1}$ exists for $\varepsilon > \varepsilon_0$ and that gives the bound for all values ε. □

4 Meromorphic Continuation

In this section we will review the meromorphy of the resolvent $R_V(z)$, see (1.1), in a way connecting it to the resolvent of P_ε given in (1.4), $\varepsilon \geq 0$. For that we define

$$R_\varepsilon(z) = (-\Delta - i\varepsilon x^2 - z)^{-1}, \quad R_{V,\varepsilon}(z) = (-\Delta - i\varepsilon x^2 + V - z)^{-1}, \quad \varepsilon \geq 0. \tag{4.1}$$

For $\varepsilon > 0$, these operators are meromorphic for $z \in \mathbb{C}$ as operators on L^2. For $\varepsilon = 0$, $R_0(z)$ is holomorphic in the sense of (1.1) on the double cover of $\mathbb{C} \setminus \{0\}$ when n is odd and on the logarithmic cover when n is even – see for instance [10, § 3.1]. We are only concerned with continuation to $\arg z \geq -\pi/4$.

In what follows we apply the usual arguments for meromorphic continuation – see [[10], §§ 2.2, 3.2] – but with $R_0(z)$ replaced by $R_\varepsilon(z)$. The complex scaling is needed to establish the crucial Lemma 7 which involves only the unscaled resolvent, $R_\varepsilon(z)$.

Let $\rho \in \mathcal{C}_c^\infty(\mathbb{R}^n; [0, 1])$ be equal to 1 on a neighbourhood of supp V. We have

Lemma 6 *For* $\varepsilon \geq 0$

$$z \mapsto (I + V R_\varepsilon(z)\rho)^{-1}, \quad -\pi/4 < \arg z < 7\pi/4,$$

is a meromorphic family of operators on $L^2(\mathbb{R}^n)$ *for with poles of finite rank. Then*

$$m_\varepsilon(z) := \frac{1}{2\pi i} \operatorname{tr} \oint_z (I + V R_\varepsilon(w)\rho)^{-1} \partial_w (V R_\varepsilon(w)\rho) dw, \tag{4.2}$$

where the integral is over a positively oriented circle enclosing z and containing no poles other than possibly z, satisfies

$$m_\varepsilon(z) = \begin{cases} \frac{1}{2\pi i} \oint_z (w - P_\varepsilon)^{-1} dw, & \varepsilon > 0 \\ m(z), & \varepsilon = 0, \end{cases} \tag{4.3}$$

where $m(z)$ is the multiplicity of the resonance z given by (1.2).

Proof We recall the standard argument (see [10, § 2.2, 3.2] and references given there). For any $\delta > 0$ and uniformly in $\varepsilon \geq 0$,

$$R_\varepsilon(z) = \mathcal{O}_\delta(1/|z|) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n), \quad \delta < \arg z < 3\pi/2 - \delta, \quad |z| > \delta. \tag{4.4}$$

This follows from self-adjointness for $\varepsilon = 0$ and from (3.3) for $\varepsilon > 0$.

For z in (4.4) and $Q_\varepsilon := -\Delta - i\varepsilon x^2$,

$$\begin{aligned}(P_\varepsilon - z) &= (Q_\varepsilon - z)(I + R_\varepsilon(z)V) \\ &= (I + VR_\varepsilon(z)\rho)(I + VR_\varepsilon(z)(1-\rho))(Q_\varepsilon - z).\end{aligned} \tag{4.5}$$

Noting that

$$(I + VR_\varepsilon(z)(1-\rho))^{-1} = I - VR_\varepsilon(z)(1-\rho)$$

we obtain from (4.4) and (4.5) that

$$\begin{gathered}R_{V,\varepsilon}(z) = R_\varepsilon(z)(I + VR_\varepsilon(z)\rho)^{-1}(I - VR_\varepsilon(z)(1-\rho)), \\ \delta < \arg z < 3\pi/2 - \delta, \quad |z| \gg 1,\end{gathered} \tag{4.6}$$

where for large $|z|$, $I + VR_\varepsilon(z)\rho$ is invertible by a Neumann series argument. Since $z \mapsto VR_\varepsilon(z)\rho$ is a holomorphic family of compact operators for $-\pi/4 < \arg z < 3\pi/4$ (see Lemma 3 for the case $\varepsilon > 0$), $z \mapsto (I + VR_\varepsilon(z)\rho)^{-1}$ is a meromorphic family operators in the same range of z. (For $\varepsilon > 0$ the meromorphy is in fact valid for $z \in \mathbb{C}$ – see [10, § C.4].) The formula (4.6) remains valid for that range of z with boundedness on L^2 for $\varepsilon > 0$. For $\varepsilon = 0$ we note that

$$(I - VR_0(z)(1-\rho)), (I + VR_0(z))^{-1} : L^2_{\rm comp} \to L^2_{\rm comp}, \quad R_0(z) : L^2_{\rm comp} \to L^2_{loc},$$

and we obtain the meromorphic continuation of $R_{V,0}(z) : L^2_{\rm comp} \to L^2_{loc}$. Arguing as in the proof of [10, Theorem 3.23] we obtain the multiplicity formula (4.3). (This can also seen using complex scaling as reviewed in the proof of Theorem 2 below.) □

5 Proof of Convergence

The proof of convergence is based on Lemma 6 and on the following lemma in which we use the complex variable techniques of Sects. 2, 3.

Lemma 7 *For $\chi \in \mathcal{C}^\infty_{\rm c}(\mathbb{R}^n)$ consider*

$$T^\chi_\varepsilon(z) := \chi(-\Delta - i\varepsilon x^2 - z)^{-1} x^2 (-\Delta - z)^{-1}\chi, \quad 0 < \arg z < 3\pi/2. \tag{5.1}$$

Then T^χ_ε continues to a holomorphic family of operators

$$T^\chi_\varepsilon(z) : L^2 \to L^2, \quad -\pi/4 < \arg z < 7\pi/4.$$

If $\Omega \Subset \{z : -\pi/4 < \arg z < 3\pi/2\}$ then there exists $C = C_{\Omega,\chi}$ (independent of ε) such that

$$\|T^\chi_\varepsilon(z)\|_{L^2 \to L^2} \le C, \quad z \in \Omega, \quad \varepsilon > 0. \tag{5.2}$$

Proof In the notation of (4.1) we see that for $\delta < \arg z < 3\pi/2 - \delta$, $|z| > \delta$,

$$\chi(R_\varepsilon(z) - R_0(z))\chi = i\varepsilon\chi R_\varepsilon(z)x^2 R_0(z)\chi,$$

where we note that, for in our range of z, $R_0(z)\chi : L^2 \to e^{-c_\delta|x|}L^2$ (by looking, for instance at the explicit formulas for the resolvent, see [10, § 3.1], or by conjugation with exponential weights) and consequently $x^2 R_0(z)\chi : L^2 \to L^2$. Hence

$$T_\varepsilon^\chi(z) = -\frac{i}{\varepsilon}(\chi R_\varepsilon(z)\chi - \chi R_0(z)\chi). \tag{5.3}$$

The right hand side is holomorphic for $-\pi/4 < \arg z < 5\pi/4$ which provides holomorphic continuation of $T_\varepsilon^\chi(z)$, $\varepsilon > 0$.

We now use Lemmas 2 and 4. For that we choose r_0 in the definition of Γ_θ large enough so that $\operatorname{supp}\chi \subset B(0, r_0)$ and take $\theta = \pi/8$. Then we have

$$\begin{aligned} T_\varepsilon^\chi(z) &= -\frac{i}{\varepsilon}\left(\chi(Q_{\varepsilon,\theta} - z)^{-1}\chi - \chi(Q_{0,\theta} - z)^{-1}\chi\right) \\ &= \chi(Q_{\varepsilon,\theta} - z)^{-1}x_\theta^2(Q_{0,\theta} - z)^{-1}\chi, \end{aligned} \tag{5.4}$$

where, in the notation of (2.12), $x_\theta := x|_{\Gamma_\theta}$. We now note that for $z \in \Omega$,

$$(Q_{0,\theta} - z)^{-1}\chi : L^2(\Gamma_\theta) \to e^{-c_\Omega|x|}L^2(\Gamma_\theta). \tag{5.5}$$

This can be seen by conjugation by exponential weights or by constructing a parametrix for $Q_{0,\theta}$ as in the proof of Lemma 4 and using the explicit properties of $(-e^{-2i\theta}\Delta - z)^{-1} = e^{2i\theta}R_0(e^{2i\theta}z)$. From this and Lemma 4 we obtain

$$\|(Q_{\varepsilon,\theta} - z)^{-1}x_\theta^2(Q_{0,\theta} - z)^{-1}\chi\|_{L^2\to L^2} \le C_\Omega, \quad z \in \Omega.$$

Inserting this into (5.4) concludes the proof. □

We can now state a stronger version of Theorem 1 formulated using the projections appearing in (4.2):

Theorem 2 *Suppose that* $-\pi/4 < \arg z < 5\pi/4$ *and that* $m(z) = m \ge 0$, *where* $m(z)$ *is the multiplicity of the resonance of* $P := -\Delta + V$ *at* z *– see* (1.2).

Then there exists ε_0 *and* δ *such that for* $0 < \varepsilon \le \varepsilon_0$, $P_\varepsilon = -\Delta + V - i\varepsilon x^2$ *has* m *eigenvalues in* $D(z, \delta)$*:*

$$\operatorname{tr}\Pi_\varepsilon = m, \quad \Pi_\varepsilon := \frac{1}{2\pi i}\int_{\partial D(z,\delta)}(\zeta - P_\varepsilon)^{-1}d\zeta, \quad \Pi_\varepsilon^2 = \Pi_\varepsilon, \tag{5.6}$$

and for any $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$,

$$\chi\Pi_\varepsilon\chi \in C^\infty([0, \varepsilon_0], \mathcal{L}(L^2(\mathbb{R}^n), L^2(\mathbb{R}^n))). \tag{5.7}$$

Remarks. 1. Notation $f \in C^1([a,b])$ means that f, and f' continuous in $[a,b]$; here $f'(a)$, $f'(b)$ are the left and right derivatives of f at those points. By induction we then define $C^k([a,b])$ and $C^\infty([a,b])$. In view of (1.5) we cannot expect analytic dependence on ε.

2. For $\chi \equiv 1$ on $\operatorname{supp} V$, $m(z) = \operatorname{rank} \chi \Pi_0 \chi$ and (5.7) shows the convergence of resonant states in the case of simple resonances. As the proof shows a stronger statement is obtained by using the complex scaled operators: for $\theta = \pi/8$,

$$\begin{gathered}\Pi_{\varepsilon,\theta} := \tfrac{1}{2\pi i} \textstyle\int_{\partial D(z,\delta)} (\zeta - P_{\varepsilon,\theta})^{-1} d\zeta, \quad P_{\varepsilon,\theta} = -\Delta|_{\Gamma_\theta} - i\varepsilon(x|_{\Gamma_\theta})^2 + V, \\ \Pi_{\varepsilon,\theta} \in C([0,\varepsilon_0); \mathcal{L}^1(L^2(\Gamma_\theta), L^2(\Gamma_\theta))), \quad \Pi_{\varepsilon,\theta}\chi \in \mathcal{C}^\infty([0,\varepsilon_0); \mathcal{L}^1(L^2(\Gamma_\theta), L^2(\Gamma_\theta))),\end{gathered} \tag{5.8}$$

where Γ_θ is the deformation defined in (2.11).

Proof We first note that (4.6) and Lemma 4 imply that for $-\pi/4 \le -2\theta < \arg z < 2\pi - 2\theta$, $\varepsilon \ge 0$,

$$(P_{\varepsilon,\theta} - z)^{-1} = (Q_{\varepsilon,\theta} - z)^{-1}(I + VR_\varepsilon(z)\rho)^{-1}(I - V(Q_{\varepsilon,\theta} - z)^{-1}(1-\rho)). \tag{5.9}$$

Since $z \mapsto (Q_{\varepsilon,\theta} - z)^{-1}$ is a holomorphic family in our range of z's, the Gohberg–Sigal theory – see [10, § C.4] – shows that the poles of $(P_{\varepsilon,\theta} - z)^{-1}$ with $\arg z > -2\theta$ are independent of $0 \le \theta \le \pi/8$ and

$$\operatorname{tr} \frac{1}{2\pi i} \oint (P_{\varepsilon,\theta} - \zeta)^{-1} d\zeta = \operatorname{tr} \frac{1}{2\pi i} \oint (P_\varepsilon - \zeta)^{-1} d\zeta, \quad \varepsilon > 0.$$

If in the definition of Γ_θ we take r_0 large enough so that $\operatorname{supp} \chi \subset B(0, r_0)$ then Lemmas 2 and 4 show that $\chi \Pi_{\varepsilon,\theta} \chi = \chi \Pi_\varepsilon \chi$.

Hence it is enough to prove (5.8). If we assume that z is not a resonance then, in the notation of Lemma 7,

$$\begin{aligned}(I + VR_\varepsilon(z)\rho)^{-1} - (I + VR_0(z)\rho)^{-1} &= i\varepsilon(I + VR_\varepsilon(z)\rho)^{-1} T_\varepsilon^\rho(z)(I + VR_0(z)\rho)^{-1} \\ &= \mathcal{O}_z(\varepsilon \|(I + VR_\varepsilon(z)\rho)^{-1}\|_{L^2 \to L^2}) : L^2 \to L^2.\end{aligned}$$

Hence, for ε small enough z is not an eigenvalue of P_ε. We can now apply the Gohberg–Sigal–Rouché theorem [10, Theorem C.9] to see that the poles of $(I + VR_0(z)\rho)^{-1}$ and $(I + VR_\varepsilon(z)\rho)^{-1}$ coincide with multiplicities. This and (5.9) prove the first statement in (5.8). The second statement follows from differentiation and estimates similar to (5.5). □

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