Colonel Blotto Game with Coalition Formation for Sharing Resources

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Abstract. In this paper, we consider a 4-player, two stage Colonel Blotto game in which one player, the attacker, simultaneously participates in three disjoint Colonel Blotto games against three defenders. During the first stage of the game, the defenders can choose to form independent coalitions by transferring resources (troops, funds, computing resources, etc.) among each other if the transfer benefits the defenders involved. In the second stage, the attacker observes these transfers among defenders and then allocates a portion of his overall resources to fight against each defender. We find that the formation of coalitions depends on both the ratios of resources between the attacker and the defenders and on each defender's total battlefield value to resource ratio. For one parameter region, we completely characterize the subgame-perfect Nash equilibrium. For another parameter region, we show that there are parameters of the game for which transfers occur and provide a computational method to calculate those transfers.

Keywords: Constant sum game with resource constraints · Colonel Blotto game · Coalition formation in games

1 Introduction

The Colonel Blotto game, first proposed by Borel in 1921 [2,3], is a classic constant-sum model of resource allocation between two budget constrained players. In this game, two players, Colonels A and B, have resource levels X_A and X_B, respectively. Each player allocates his resources across a finite number of battlefields. Whichever player allocates the most resources to a single battlefield wins that battle. The winner of the game is the player that wins the most battlefields.

The Colonel Blotto game has diverse applications within military and security domains, where agencies allocate limited resources across various geographic

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locations to counter adversarial threats. In addition, the model is useful to analyze situations including network resource games [12,20], cyber-security games [5,11,16], economic contests [7,14], and political contests [8,15,19].

While the Colonel Blotto game seems relatively straightforward, it has proven difficult to solve. Borel's original formulation was first solved by Borel and Ville in 1938 for two players, three battlefields, and symmetric resource allocation [4]. In 1950, Gross and Wagner [9] solved the game for symmetric resource allocation and more than three battlefields. They also solved the case of two battlefields and asymmetric resource allocation. However, the Colonel Blotto game remained unsolved for asymmetric player resources and an arbitrary number of battlefields until Roberson's seminal work in 2006 [17]. Roberson advanced the field significantly, with many follow-on works that extended his solution to specific applications.

In this work, we consider a four-player, two-stage Colonel Blotto game in which one player, the attacker, simultaneously participates in three disjoint Colonel Blotto games against three defenders. During the first stage of the game, the defenders can choose to form an alliance, or coalition, by transferring a single-dimensional resource (troops, funds, computing resources, etc.) from one defender to another. This transfer between two defenders only occurs when the transfer will not decrease both defenders' payoffs in the final stage. The attacker observes these transfers between defenders and then allocates a portion of his overall resources to the final stage Colonel Blotto game against each defender.

Similar to [10,11,13], we consider a model of noncooperative alliances in which only individually rational *ex ante* transfers of resources are allowed. As such, the model does not rely on any assumption of commitment to a coalition nor the *ex post* division of payoffs.

We find that the formation of coalitions, based on the transfer of resources, depends on both the ratios of resources between the attacker and a defender and on each defender's total battlefield value to resource ratio. In one case that we study, only resource rich defenders are willing to transfer resources. However, in another case, somewhat counter-intuitively, the most resource rich defender does not necessarily transfer resources to other defenders. Instead, defenders that have a lower total battlefield value to resource ratio are those that tend to be willing to transfer resources.

Other authors have considered coalition formation in Colonel Blotto games. In [14], Kovenock and Roberson consider the same game that we've described above but with only two defenders. In addition, they only consider cases where transfers between defenders strictly improve the payoff of each defender. The authors characterize the attacker's resource division strategy for multiple regions of the resource budget and calculate parameters for when a transfer of resources between defenders occurs. However, they do not calculate the amount of resource transfer. In [11], Gupta et al. also consider a multi-stage, one attacker, two defender complete information Colonel Blotto game. In their formulation, in addition to transferring resources, the two defenders can choose to add additional battlefields, at some cost per battlefield. The authors find the subgame-perfect

Nash equilibrium (SPNE) for this game for certain parameter regions. Finally, in [10], Gupta et al. consider a change to the information structure from [11], where the attacker can not observe the resource transfer between the two defenders. They find that in some parameter regions the SPNE remains unchanged, while in other regions the SPNE is significantly different from [11].

As far as we are aware, this work is the first attempt to extend this multi-stage Colonel Blotto game setting to more than two defenders. In the two defenders case, there is only one possible coalition formation. In the N defender case, the number of possible coalitions are $\frac{N(N-1)}{2} = O(N^2)$. Thus, the seemingly simple extension to the previous case requires us to investigate a large number of possible coalition formations. We view our work as an attempt to identify situations where the analysis can be simplified and understand regimes where coalitions can be formed.

1.1 Outline of the Paper

In Sect. 2, we present a brief overview of the Classical Colonel Blotto game and review the pertinent results from [17]. Following that, we formalize the multi-stage model used throughout this paper in Sect. 3 and present the main results. In Sect. 4, we derive the best response of the attacker. Sections 5 and 6 are devoted to computing the equilibrium transfers among the defenders under two assumptions on the strength of the attacker. Finally, in Sect. 7, we conclude with an analysis of the work and highlight directions for future work.

2 The Classical Colonel Blotto Model

In this section, we introduce the classic asymmetric resource, homogeneous battlefield value Colonel Blotto game (CBG) and appropriate notations. In the classic CBG, two players, call them A and B, simultaneously allocate their forces, X_A and X_B, across a finite number, n, of homogeneous battlefields with value v. Battlefield values are homogeneous; therefore, we have $v_j = v_k \; \forall j, k \in \{1, \ldots, n\}$. If a player sends a higher level of force to battlefield j, then that player wins that battlefield and receives a payoff of v_j. If the player sends a lower level of force to battlefield j, then that player loses and receives a payoff of 0. Each player's total payoff in the game is the sum of the payoffs across the battlefields. Without loss of generality, assume $X_A \leq X_B$, so that player B is the "stronger" player. In the case of a tie, we follow [17] and assume that player B wins the battlefield.

More formally, we can define the classic CBG similarly to the definition in [6]. The classic CBG$\big\{\mathcal{P}, \{\mathcal{X}\}_{i\in\mathcal{P}}, \{X_i\}_{i\in\mathcal{P}}, \mathcal{N}, \{v_j\}_{j=1}^{n}, \{U_i\}_{i\in\mathcal{P}}\big\}$ is defined by six components: *(a)* the players in the set $\mathcal{P} \triangleq \{A, B\}$, *(b)* the strategy spaces $\mathcal{X}_i$ for $i \in \mathcal{P}$, *(c)* the available resource X_i for $i \in \mathcal{P}$, *(d)* the set of n battlefields, $\mathcal{N}$, *(e)* the homogeneous value of each battlefield, $v_j = v_k \; \forall j, k \in \mathcal{N}$, and *(f)* the utility function U_i for each player $i \in \mathcal{P}$.

The force allocated to each battlefield must be non-negative. Therefore, the strategy space of each player corresponds to the set of feasible allocations across the n battlefields and is given by

$$\mathcal{X}_i = \left\{ \mathbf{x}_i \in \mathbb{R}^n_{\geq 0} \mid \sum_{j=1}^{n} x_{i,j} \leq X_i \right\}, \tag{1}$$

where $x_{i,j}$ is the number of allocated resources by player i to battlefield j. The payoff of player i on battlefield j is defined as:

$$u_{i,j}(x_{i,j}, x_{-i,j}) = \begin{cases} v_j & \text{if } x_{i,j} > x_{-i,j}, \\ \text{t.b.r} & \text{if } x_{i,j} = x_{-i,j}, \\ 0 & \text{if } x_{i,j} < x_{-i,j}, \end{cases} \tag{2}$$

where t.b.r indicates the tie breaking rule and we use the common game theoretic notation $-i$ to refer to all players except player i. We follow [17] and the tie breaking rule is to assume that the stronger player (player with greater resources) wins the battlefield.

Finally, the utility function, U_i, for each player is defined as:

$$U_i(\mathbf{x}_i, \mathbf{x}_{-i}) = \sum_{j=1}^{n} u_{i,j}(x_{i,j}, x_{-i,j}). \tag{3}$$

The classic CBG is a complete information game. All parameters of the game,

$$\text{CBG}\big\{\mathcal{P}, \{\mathcal{X}\}_{i\in\mathcal{P}}, \{X_i\}_{i\in\mathcal{P}}, \mathcal{N}, \{v_j\}_{j=1}^{n}, \{U_i\}_{i\in\mathcal{P}}\big\},$$

are assumed to be common knowledge among all players.

2.1 Strategies of the Players

In the trivial case, $\frac{1}{n}X_B \geq X_A$, there exists a pure strategy equilibrium where player B plays such that $x_{B,j} \geq \frac{1}{n}X_B \geq X_A$, $\mathbf{x}_B \in \mathcal{X}_B$ and wins all of the battlefields. For non-trivial cases, $\frac{1}{n}X_B < X_A \leq X_B$, it is well known that there is no pure strategy equilibrium [17]. Following [17], we define a mixed strategy, or distribution of force, for player i as an n-variate distribution function $P_i : \mathbb{R}^n_{\geq 0} \rightarrow [0, 1]$ with support in $\mathcal{X}_i$, and with one-dimensional marginal distribution functions $\big\{F_i^j\big\}_{j\in\{1,\ldots,n\}}$. A single play of the game for player i corresponds to a random n-tuple drawn from P_i with the set of univariate marginal distribution functions $\big\{F_i^j\big\}_{j\in\{1,\ldots,n\}}$.

2.2 Nash Equilibrium of the Classic Colonel Blotto Game

Roberson completely characterized the unique equilibrium payoffs for the asymmetric resource, homogeneous battlefield value CBG in [17]. Below we summarize his results for the cases that we study in this work.

Lemma 1 (Roberson [17,18]). *For the Classic Colonel Blotto Game,*

$$CBG\big\{\{1,2\}, \{\mathcal{B}_1, \mathcal{B}_2\}, \{r_1, r_2\}, \mathcal{N}, \{v\}, \{U_1, U_2\}\big\},$$

with $n \geq 3$, assume that r_1 and r_2 are such that $\frac{1}{n-1} \leq \frac{r_1}{r_2} \leq n-1$. Then the payoff functions under Nash equilibrium are given by:

$$P^1(CBG) = \begin{cases} nv\left(\frac{2}{n} - \frac{2r_2}{n^2 r_1}\right) & \text{if } \frac{1}{n-1} \leq \frac{r_1}{r_2} < \frac{2}{n}, \\ nv\left(\frac{r_1}{2r_2}\right) & \text{if } \frac{2}{n} \leq \frac{r_1}{r_2} \leq 1, \\ nv\left(1 - \frac{r_2}{2r_1}\right) & \text{if } 1 \leq \frac{r_1}{r_2} \leq \frac{n}{2}, \\ nv\left(1 - \frac{2}{n} + \frac{2r_1}{n^2 r_2}\right) & \text{if } \frac{n}{2} < \frac{r_1}{r_2} < n-1, \end{cases}$$

$$P^2(CBG) = nv - P^1(CBG).$$

If $r_1 = 0$, then $P^1(CBG) = 0$.

For a detailed proof of Lemma 1, see [17,18]. Roberson's result in [17] establishes the existence of the n-variate distributions with support in $\mathcal{B}_1, \mathcal{B}_2$ and with the equilibrium payoffs in Lemma 1. These n-variate distributions are not unique. However, since the game is constant sum, ($P^1(CBG) + P^2(CBG) = nv$), the equilibrium payoffs are unique by ordered interchangeability property of multiple saddle-point equilibria in zero-sum games (we note here that constant sum games are strategically equivalent to zero-sum games).

3 Problem Formulation and Main Results

We consider a 3 + 1 players, two-stage Colonel Blotto game. In this formulation, the first three players (defenders) fight against a common attacker. The initial resource allocation of the three defenders is denoted by β_i, $i \in \{1, 2, 3\}$. Similarly, we use α to denote the total resources of the attacker. The battle between the attacker and Player i takes place on $n_i \geq 3$ battlefields, where each battlefield has equal payoff $v_i > 0$. The description of the two stages are given below.

3.1 Stage One

In this stage, each defender decides on an amount of resources to transfer to the other two defenders, based on whether this transfer of resources will not decrease her expected payoff at the final stage. She also decides whether or not to accept resources from other defenders. We define $t_{i,j}$ as the transfer of resources from defender i to defender j and $t_{j,i}$ as transfer in the opposite direction. Since each defender's resource level, r_i, in the final stage game must be greater than or equal to zero, the total transfer out from defender i must be less than or equal to her starting resource level, β_i.

Thus, the resource level of each defender after transfer is complete is:

$$r_i(t_{i,1}, t_{i,2}, t_{i,3}, t_{1,i}, t_{2,i}, t_{3,i}) = \beta_i + \sum_{j=1}^{3}(t_{j,i} - t_{i,j}),$$

$$t_{i,i} = 0, \qquad \sum_{j=1}^{3} t_{i,j} \leq \beta_i.$$

For notational clarity, we define the strategy vector of transfers to/from defender i as $\mathbf{t}_i = (t_{i,1}, t_{i,2}, t_{i,3}, t_{1,i}, t_{2,i}, t_{3,i})$. We also define $\mathbf{t} = (\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3)$ to represent all of the defenders' strategy vectors. In addition, in a slight abuse of notation, we use r_i to represent $r_i(\mathbf{t}_i)$ and α_i to represent $\alpha_i(\mathbf{t})$.

3.2 Stage Two (Final Stage)

Once the transfers are complete in the first stage, the attacker decides on the amount of resources, α_i, to allocate to each final stage battle such that:

$$\sum_{i=1}^{3} \alpha_i(\mathbf{t}) \leq \alpha, \quad \alpha_i(\mathbf{t}) \geq 0.$$

In the final stage of the game each defender battles with the attacker in an independent classic Colonel Blotto game using the resource allocation determined in stage one. The set of players for each battle is $\mathcal{P} \triangleq \{A, B_i\}$, where A represents the attacker and B_i represents defender i. Each individual battle takes place over a set of battlefields, $\mathcal{N}_i = \{1, \ldots, n_i\}$, belonging to defender i. For defender i, each battlefield has homogeneous value $v_k = v_{k'}\ \forall k, k' \in \mathcal{N}_i$. The strategy space of the attacker and defender in this game is, respectively,

$$\mathcal{A}_i = \left\{ \boldsymbol{\alpha}_i \in \mathbb{R}^{n_i}_{\geq 0} \mid \sum_{j \in \mathcal{N}_i} \alpha_{i,j} \leq \alpha_i \right\}, \quad \mathcal{R}_i = \left\{ \mathbf{r}_i \in \mathbb{R}^{n_i}_{\geq 0} \mid \sum_{j \in \mathcal{N}_i} r_{i,j} \leq r_i \right\}.$$

Using the notation introduced in Sect. 2, each defender battles the attacker in a CBG given by:

$$\text{CBG}\big\{\{A, B_i\}, \{\mathcal{A}_i, \mathcal{R}_i\}, \{\alpha_i, r_i\}, \mathcal{N}_i, v_i, \{U_i\}_{i \in \mathcal{P}}\big\}.$$

In shorthand notation, we refer to this individual final stage game as CBG^{i}.

We annotate the overall two-stage game described in this section as:

$$2\text{CB}\big\{\{A, \{B_i\}\}, \{\alpha, \{\beta_i\}\}, \{\mathcal{N}_i\}, \{v_i\}\big\} \text{ with } i \in \{1, 2, 3\}$$

and refer to this overall two-stage game as 2CB.

In the overall game 2CB, the payoff to defender i is her payoff in the game CBG^{i}. The attacker's overall payoff is the sum of his payoffs in the individual CBG^{i} games.

In this work, we consider a small subset of the possible parameter regions. We focus on games where if all players play according to the SPNE in the first stage, then the resource allocation at the final stage is such that $\frac{2}{n_i} < \frac{\alpha_i}{r_i} < \frac{n_i}{2}$ for all $i \in \{1, 2, 3\}$ or $\alpha_i = 0$ for some $i \in \{1, 2, 3\}$. The two specific cases that we consider are:

1. $\frac{2}{n_i} < \frac{\alpha_i}{r_i} < 1 \quad \forall i \in \{1, 2, 3\}$,
2. $\frac{2}{n_i} < \frac{r_i}{\alpha_i} < 1 \quad \forall i \in \{1, 2, 3\}$.

We also introduce the following notation for clarity in presentation of the results. Let $D = \{1, 2, 3\}$ represent the set of defenders. Also, let $K_i = n_i v_i$, which is the total battlefield value (value of the game) CBG^{i}.

3.3 Main Results

In this subsection, we present the main results of the paper and briefly discuss the results. For clarity, we present the proofs of these results and a more detailed discussion in Sects. 4–6.

The proof outline for all three of the theorems we present here is similar and relies on using backwards induction to find the subgame-perfect Nash equilibrium (SPNE). Starting at the final stage, games, $\mathrm{CBG^i}$, we use the results from [17] to calculate the Nash equilibrium (NE) of the final stage. We then calculate the attacker's optimal resource allocation in response to the stage one resource transfers between the defenders. Finally, we rank order the defenders based on the starting resource levels and calculate the defenders' optimal resource allocation to find the NE of the subgame starting at stage one.

We first consider the case where the attacker is the weakest player in the game. As such, the resource levels of the defenders and attacker are such that $\alpha < \min\{\beta_1, \beta_2, \beta_3\}$. In addition, assume that the vector of resource transfers, $\mathbf{t}$, is such that the game remains in case 1, $\frac{2}{n_i} < \frac{\alpha}{r_i} < 1 \quad \forall i \in \{1, 2, 3\}$. In general, one can think of the ratio $\frac{K_i}{\beta_i}$ as the relative strength of each defender. Without loss of generality, we index defenders by inverse relative strength:

$$\frac{K_1}{\beta_1} \geq \frac{K_2}{\beta_2} \geq \frac{K_3}{\beta_3}. \tag{4}$$

Theorem 1. *Consider a two-stage game, 2CB, where the parameters of the game are such that:*

1. $\alpha < \min\{\beta_1, \beta_2, \beta_3\}$,
2. $\frac{2}{n_i} < \frac{\alpha}{r_i} < 1 \quad \forall i \in \{1, 2, 3\}$,
3. $\beta_2(K_1 + K_3) - K_2(\beta_1 + \beta_3) \geq 0$,
4. $\frac{K_1}{\beta_1} > \frac{K_2}{\beta_2}$.

then there is a family of SPNEs such that:

$$\begin{aligned}
&\alpha_1^* = \alpha, \\
&\alpha_2^* = 0, \quad \alpha_3^* = 0, \\
&t_{1,2}^* = t_{1,3}^* = 0, \\
&t_{2,1}^* + t_{2,3}^* \geq 0, \\
&t_{3,1}^* + t_{3,2}^* \geq 0, \\
&t_{2,1}^* + t_{2,3}^* < t_{3,2}^* + \frac{\beta_2(K_1 + K_3) - K_2(\beta_1 + \beta_3)}{K_1 + K_2 + K_3}, \\
&t_{3,1}^* + t_{3,2}^* < t_{2,3}^* + \frac{\beta_3(K_1 + K_2) - K_3(\beta_1 + \beta_2)}{K_1 + K_2 + K_3}.
\end{aligned}$$

From Theorem 1, we see that when the attacker is the weakest player in the game his optimal strategy is to battle with only one defender. However, the defender that he battles with is not necessarily the weakest defender in terms of overall resources, but the defender that is weakest in terms of resources to total battlefield value. Another observation is that defenders with less resources are willing to transfer resources to a defender with a higher resource level, as long as that transfer doesn't result in the defender making the transfer becoming the relatively weakest player.

We next consider the case where the attacker is much stronger than all of the defenders combined. We assume that $\alpha > \sum_{i=1}^{3} \beta_i$. In addition, assume that the vector of resource transfers, $\mathbf{t}$, is such that the game remains in case 2, $\frac{2}{n_i} < \frac{r_i}{\alpha_i} < 1 \quad \forall i \in \{1, 2, 3\}$.

Theorem 2. *Consider a two-stage game, 2CB, where the parameters of the game are such that:*

1. $\frac{2}{n_i} < \frac{r_i}{\alpha_i} < 1 \quad \forall i \in \{1, 2, 3\}, \forall \mathbf{t}$,
2. $\beta_i - \beta_k > 2\sqrt{\frac{K_i}{K_k}}\sqrt{\beta_i \beta_k} + \sqrt{\frac{\beta_k}{K_k}}\sqrt{K_j \beta_j}$

then there is a positive transfer from defender i to defender k, $t_{i,k} > 0$.

Unlike the weakest attacker case, we observe from Theorem 2 that resource transfers only occur from a defender with a higher resource level to a defender with a lower resource level. In addition, if the difference in the resource levels between two defenders is higher than a certain threshold, then we observe that a transfer of some resources is a dominant strategy for those defenders. Finally, recall from the problem formulation in Sect. 3.1 that our model allows defender k to choose whether or not to accept a transfer from defender i. In the proof of Lemma 5 in Sect. 6, we find that defender k is always willing to accept resources from defender i whenever defender i is willing to transfer those resources. We also show that there are parameter regions where one defender is willing to accept resources, but other defenders are not willing to transfer. As a result, there is no coalition formation in this situation.

Finally, we consider a specific parameter configuration of the game and show that, in equilibrium, the strongest defender in terms of initial resource allocation, defender 1, transfers resources to at least one other defender and that there is no transfer between defenders 2 and 3.

Theorem 3. *Consider a two-stage game, 2CB, where the parameters of the game are such that:*

1. $\frac{2}{n_i} < \frac{r_i}{\alpha_i} < 1 \quad \forall i \in \{1, 2, 3\}, \forall \mathbf{t}$,
2. $\beta_1 - \beta_2 > 2\sqrt{\frac{K_1}{K_2}}\sqrt{\beta_1 \beta_2} + \sqrt{\frac{\beta_2}{K_2}}\sqrt{K_3 \beta_3}$,
3. $\beta_1 - \beta_3 > 2\sqrt{\frac{K_1}{K_3}}\sqrt{\beta_1 \beta_3} + \sqrt{\frac{\beta_3}{K_3}}\sqrt{K_2 \beta_2}$,
4. $\frac{\beta_1 + \beta_2}{2} - \beta_3 \leq 2\sqrt{\frac{K_2}{K_3}}\sqrt{\beta_2 \beta_3} + \sqrt{\frac{\beta_3}{K_3}}\sqrt{K_1 \frac{\beta_1 + \beta_2}{2}}$,

5. $\frac{\beta_1+\beta_3}{2} - \beta_2 \leq 2\sqrt{\frac{K_3}{K_2}}\sqrt{\beta_2\beta_3} + \sqrt{\frac{K_1}{K_2}}\sqrt{\frac{\beta_1+\beta_2}{2}\frac{\beta_2+\beta_3}{2}}$,
6. $\beta_1 > \beta_2 \geq \beta_3$.

Then $t^*_{2,3} = t^*_{3,2} = 0$, $t^*_{2,1} = t^*_{3,1} = 0$, $t^*_{1,2}, t^*_{1,3} \geq 0$, *where* $t_{1,2}$ *and* $t_{1,3}$ *are solutions to defender 1's optimization problem:*

$$\begin{aligned} \max_{t_{1,2},t_{1,3}} \quad & \phi_1(\mathbf{t_1}) = \frac{\sqrt{K_1 r_1}}{2\alpha}\left(\sum_{j=1}^{3}\sqrt{K_j r_j}\right) \\ \text{subject to} \quad & 0 \leq t_{1,2} < \frac{\beta_1-\beta_2}{2},\ 0 \leq t_{1,3} < \frac{\beta_1-\beta_3}{2}. \end{aligned} \tag{5}$$

From Theorem 3, one can immediately notice that if defender 1 has significantly more resources than defenders 2 and 3, and if defenders 2 and 3 have a relatively similar level of resources then it is in the strongest defender's best interest to form a coalition and transfer resources to the other two defenders. At the same time, the weaker defenders have no incentive to transfer resources. Combining observations from Theorems 2 and 3, we note that the expected payoffs of all three players increases.

In the subsequent sections, we prove the results stated above. We first compute the attacker's resource allocation and Nash equilibrium payoffs of the players in the final stage game assuming the knowledge of the transfer. Thereafter, in Sects. 5 and 6, we proceed to solve the stage 1 game for the cases stated in the theorems above.

4 Best Response of the Attacker

In this section we calculate the attacker's optimal resource allocation in response to the resource transfers between the defenders. As the final stage payoffs are given by Lemma 1 in Sect. 2.2, we are left to solve for the stage one optimal resource allocation of the attacker and the optimal transfers by the defenders. We first solve the attacker's problem by using the best response strategies of the attacker to the observed post-transfer resource allocations $r_i = \beta_i + \sum_{j=1}^{3}(t_{j,i} - t_{i,j})$. This will provide the attacker's optimal resource allocation, $\alpha^*_i(\mathbf{t})$, to each separate final stage game, CBG^{i}.

Proposition 1. *Consider a two-stage game, 2CB. For an admissible resource transfer strategy,* $\mathbf{t}$*, the attacker's optimal payoff maximizing strategy is:*

1. *The case* $\frac{2}{n_i} < \frac{\alpha}{r_i} < 1 \quad \forall i \in \{1,2,3\}$*: Let* $I = \left\{ i \;\middle|\; i \in \max_{i=1,2,3} \frac{K_i}{r_i} \right\}$, $\Delta_{|I|} = \left\{ \mathbf{p} \mid p_i \geq 0, \sum_{i\in I} p_i = 1 \right\}$, $\mathbf{p} \in \Delta_{|I|}$.

$$\alpha^*_i(\mathbf{t}) = \begin{cases} \alpha & \text{if } i \in I, |I| = 1, \\ 0 & \text{if } i \notin I, \\ \alpha p_i & \text{if } |I| > 1,\ i \in I. \end{cases}$$

2. The case $\frac{2}{n_i} < \frac{r_i}{\alpha}\left(\sum_{i=1}^{3}\sqrt{\frac{K_j r_j}{K_i r_i}}\right) < 1 \quad \forall i \in \{1,2,3\}$*:*

$$\alpha_i^*(\mathbf{t}) = \frac{\alpha}{\sum_{j=1}^{3}\sqrt{\frac{K_j r_j}{K_i r_i}}}.$$

Before proving Proposition 1, we first state a well known result from optimization theory and prove an auxiliary lemma.

Lemma 2 (Optimization Over a Simplex [1]). *In a constrained optimization problem with the objective of maximizing* $f(\mathbf{x})$*, consider the case where the constraint set is a simplex*

$$\mathcal{X} = \left\{\mathbf{x} \mid x_i \geq 0, \sum_{i=1}^{n} x_i = r\right\}$$

where $r > 0$ *is a given scalar. Then the necessary condition for* $\mathbf{x}^*$ *to be a local maximum is*

$$x_i^* > 0 \implies \frac{\partial f(\mathbf{x}^*)}{\partial x_i} \geq \frac{\partial f(\mathbf{x}^*)}{\partial x_j}, \quad \forall j. \tag{6}$$

If $f(\mathbf{x})$ *is concave, then* (6) *is also sufficient for the global optimality of* $\mathbf{x}^*$*.*

Lemma 3. *Let* $N = 3$ *be the number of defenders in the game. For an attacker with a payoff function that is the summation of strictly-increasing single-variable functions,*

$$\pi(\boldsymbol{\alpha}) = \sum_{i=1}^{N} \pi_i(\alpha_i),$$

the attacker exhausts his entire budget, $\sum_{i=1}^{N} \alpha_i = \alpha$*, at the optimum.*

Proof. Fix $\alpha_j = \alpha_j^* \ \forall j$ such that $\sum_{j=1, j\neq i}^{N} \alpha_j^* < \alpha$. Let $\varepsilon > 0$, and take

$$\alpha_i = \alpha - \sum_{\substack{j=1\\ j\neq i}}^{N} \alpha_j^* - \varepsilon < \alpha_i^* = \alpha - \sum_{\substack{j=1\\ j\neq i}}^{N} \alpha_j^*.$$

Then we have

$$\pi(\boldsymbol{\alpha}) = \pi_i(\alpha - \sum_{\substack{j=1\\ j\neq i}}^{N} \alpha_j^* - \varepsilon) + \sum_{\substack{j=1\\ j\neq i}}^{N} \pi_j(\alpha_j^*)$$

$$\pi(\boldsymbol{\alpha}^*) = \pi_i(\alpha - \sum_{\substack{j=1\\ j\neq i}}^{N} \alpha_j^*) + \sum_{\substack{j=1\\ j\neq i}}^{N} \pi_j(\alpha_j^*).$$

By the definition of a strictly increasing function, $\alpha - \sum_{\substack{j=1\\ j\neq i}}^{N} \alpha_j^* > \alpha - \sum_{\substack{j=1\\ j\neq i}}^{N} \alpha_j^* - \varepsilon \implies \pi_i(\alpha - \sum_{\substack{j=1\\ j\neq i}}^{N} \alpha_j^*) > \pi_i(\alpha - \sum_{\substack{j=1\\ j\neq i}}^{N} \alpha_j^* - \varepsilon) \implies \pi(\boldsymbol{\alpha}^*) > \pi(\boldsymbol{\alpha})$. □

We now proceed to prove Proposition 1. At the final stage of the game, the attacker plays a CBG against each individual defender. As such, the expected payoff of each individual CBG^i is given by Lemma 1. The total expected payoff for the attacker is the sum of the individual expected payoffs.

The attacker's reaction curve is the strategy that maximizes his expected payoff against the strategies of the 3 defenders. Therefore, fix the defenders' resource allocation strategy, $\mathbf{t}$. The attacker's expected payoffs as a function of his resource allocation strategy, $\boldsymbol{\alpha}$, for each of the two regions considered are:

$$\text{Case 1: } \pi_1(\boldsymbol{\alpha}) = \sum_{i=1}^{3} K_i \frac{\alpha_i}{2r_i},$$
$$\text{Case 2: } \pi_2(\boldsymbol{\alpha}) = \sum_{i=1}^{3} K_i \left(1 - \frac{r_i}{2\alpha_i}\right).$$

In both cases, one can easily verify that the payoff functions are summations of strictly increasing functions in the individual battle allocations, α_i. Therefore, by Lemma 3, the attacker completely exhausts his resource budget. His budget constraint is then the simplex $\sum_{i=1}^{3} \alpha_i = \alpha, \alpha_i \geq 0$.

For case 1, $\pi_1(\boldsymbol{\alpha})$ is a summation of linear functions of α_i and therefore linear. Since linear functions are also concave, $\pi_1(\boldsymbol{\alpha})$ is a concave function. Lemma 2 provides both the necessary and sufficient conditions for optimality. We then arrive at the desired result for case 1 through a direct application of Lemma 2.

In case 2, $\pi_2(\boldsymbol{\alpha})$ is a summation of concave functions in α_i. Since positive weighted sums of concave functions are concave, $\pi_2(\boldsymbol{\alpha})$ is concave. Similar to case 1, Lemma 2 provides the necessary and sufficient conditions for optimality. Therefore, by Lemma 2:

$$\begin{aligned} \alpha_i > 0 \implies \frac{\partial \pi_2(\boldsymbol{\alpha})}{\partial \alpha_1} &= \frac{\partial \pi_2(\boldsymbol{\alpha})}{\partial \alpha_2} = \frac{\partial \pi_2(\boldsymbol{\alpha})}{\partial \alpha_3} \\ \frac{n_1 v_1 r_1}{2\alpha_1^2} &= \frac{n_2 v_2 r_2}{2\alpha_2^2} = \frac{n_3 v_3 r_3}{2\alpha_3^2} \\ \frac{\alpha_1^2}{n_1 v_1 r_1} &= \frac{\alpha_2^2}{n_2 v_2 r_2} = \frac{\alpha_3^2}{n_3 v_3 r_3}. \end{aligned}$$

So, the attacker's optimal strategy in case 2 is to allocate his resources such that each partial derivative is a constant and equal. By setting this constant to k and using the attacker's budget constraint, we can solve for his optimal resource allocation strategy by algebraic manipulation.

$$k = \frac{\alpha_i^2}{K_i r_i} \implies \alpha_i = \sqrt{K_i r_i}\sqrt{k}. \tag{7}$$

Substituting (7) into the attacker's budget constraint, we have:

$$\sum_{j=1}^{3} \alpha_j = \alpha \implies \sum_{j=1}^{3} \sqrt{K_j r_j}\sqrt{k} = \alpha \implies \sqrt{k} = \frac{\alpha}{\sum_{j=1}^{3} \sqrt{K_j r_j}}. \tag{8}$$

Finally, substituting (8) into (7), we obtain the attacker's optimal resource allocation for case 2. □

In the next two sections, we find an optimal resource transfer strategy for the defenders for each of the two cases that we study. The first case corresponds to an attacker that has a significant disadvantage in resources compared to the defenders. The second case is an attacker that is much stronger than the combined strength of all of the defenders.

5 Weakest Attacker Leads to Proxy Wars

Here we present the proof of Theorem 1 for the case when the attacker has less resources than each of the defenders. We show that in certain situations, the attacker allocates all its resource to fight against one defender, while other defenders carefully choose the amount of resource to transfer to the defender fighting the attacker. This leads to a proxy war situation where some defenders may choose to transfer resources in order to benefit another defender while they themselves avoid fighting.

The attacker's optimal strategy remains the same as in Proposition 1. For the case when defender 1 is the relatively weakest player, $\frac{K_1}{\beta_1} > \frac{K_2}{\beta_2}$, we know from Proposition 1 that the attacker allocates all of his resources to the battle with defender 1. So, $\alpha_1 = \alpha, \alpha_i = 0 \ \forall i \in \{2, 3\}$. The payoff to the attacker and each defender is a result of Lemma 1 and Proposition 1 and is:

$$\pi(\boldsymbol{\alpha}) = \pi(\alpha_i) = K_i \frac{\alpha}{2r_1},$$

$$\phi_1(\mathbf{t_1}) = K_1 - \pi(\alpha_i) = K_1(1 - \frac{\alpha}{2r_1}),$$

$$\phi_i(\mathbf{t_i}) = K_i \quad \forall i \in \{2, 3\}.$$

Since defender 1's payoff decreases as r_1 decreases, she will never transfer any resources out to other defenders as long as she is the relatively weakest player, $\frac{K_1}{r_1} > \frac{K_2}{r_2} \geq \frac{K_3}{r_3}$. In addition, defender 1 will always accept resources since her payoff increases as r_1 increases. Since in this game resources have no external value, any defender i who is not the relatively weakest player is indifferent to transferring resources since she avoids battle and her payoff does not change. However, defenders 2 and 3 will never transfer out enough resources such that they become the relatively weakest player. Defenders 2 and 3 will also always accept resources since this helps them become relatively stronger and avoid battle. To summarize the above discussion, we have:

1. Defender 1 never transfers resources to other defenders: $t_{1,j} = 0 \quad \forall j$.
2. Defender $i \in \{2, 3\}$ is indifferent to transferring resources out as long as:

$$\frac{K_1}{r_1} = \frac{K_1}{\beta_1 + \sum_{j \in D \setminus \{1\}} t_{j,1}} > \frac{K_i}{r_i} = \frac{K_i}{\beta_i + \sum_{j \in D \setminus \{1,i\}} t_{j,i} - \sum_{j \in D \setminus \{i\}} t_{i,j}}. \quad (9)$$

3. All defenders will always accept resources.

By rearranging (9), we can write the supremum of defenders 2 and 3's maximum transfer amounts to defender 1 as:

$$\begin{aligned} t_{2,1} &= \frac{K_1(\beta_2 + t_{3,2} - t_{2,3})}{K_1 + K_2} - \frac{K_2(\beta_1 + t_{3,1})}{K_1 + K_2}, \\ t_{3,1} &= \frac{K_1(\beta_3 + t_{2,3} - t_{3,2})}{K_1 + K_2} - \frac{K_2(\beta_1 + t_{2,1})}{K_1 + K_2}. \end{aligned} \tag{10}$$

Examining (10), it is apparent that the maximum amount that defender 2 is willing to transfer to defender 1 increases as defender 3 transfers resources to defender 2, decreases in terms of the amount that defender 2 transfers to defender 3, and, most critically, decreases as defender 3 transfers resources to defender 1. Directly solving the system of equations above obtains the desired solution.

$$\begin{aligned} t^*_{2,1} + t^*_{2,3} &< t^*_{3,2} + \frac{\beta_2(K_1 + K_3) - K_2(\beta_1 + \beta_3)}{K_1 + K_2 + K_3}. \\ t^*_{3,1} + t^*_{3,2} &< t^*_{2,3} + \frac{\beta_3(K_1 + K_2) - K_3(\beta_1 + \beta_2)}{K_1 + K_2 + K_3}. \end{aligned}$$

Note, that there is a possibility that defenders 2 and 3 do not transfer resources between each other. Then $t_{2,3} = t_{3,2} = 0$. Since $t_{2,1} + t_{2,3} \geq 0, t_{3,1} + t_{3,2} \geq 0$, this imposes the conditions

$$\beta_2(K_1 + K_3) - K_2(\beta_1 + \beta_3) \geq 0, \tag{11}$$

$$\beta_3(K_1 + K_2) - K_3(\beta_1 + \beta_2) \geq 0. \tag{12}$$

The condition imposed by (12) is satisfied by (4). The condition in (11) is a condition in the statement of the theorem. □

5.1 The Case of No Transfer Between Defenders 2 and 3

Note that since defenders 2 and 3 do not fight against the attacker, the transfers between them does not affect their equilibrium payoffs. Thus, a possible refinement of multiple Nash equilibria would be to assume no transfer between defenders who do not engage with the attacker. In this subsection, we make this assumption and prove two corollaries of Theorem 1 under the assumption of $t_{2,3} = t_{3,2} = 0$. We first have the following auxiliary lemma.

Lemma 4. *Let $z, c > 0$. Then*

$$\frac{x + y}{z + c} \geq \frac{x}{z} \iff zy \geq cx.$$

Proof. The proof follows from algebraic manipulation. □

Corollary 1. *Consider the case of $t_{2,3} = t_{3,2} = 0$. In comparison to the 2-defender case, at equilibrium the maximum possible amount transferred to defender 1,*

$$\sum_{j \in D \setminus \{1\}} t_{i,1},$$

is nondecreasing when defender 3 joins the game (assuming defender 3's relative strength is weaker than that of defender 2). In addition, the maximum expected payoff to defender 1 is nondecreasing.

Proof. Let $t_{2,1}^2$ represent the transfer from player 2 to player 1 in the 2-defender case, and $t_{2,1}^3, t_{3,1}^3$ represent the transfers to player 1 in the 3-defender case. Then the maximum total transfer to defender 1 in each case is

$$t_{2,1}^2 = \frac{\beta_2(K_1) - K_2(\beta_1)}{K_1 + K_2}, \tag{13}$$

$$t_{2,1}^3 + t_{3,1}^3 = \frac{\beta_2(K_1 - K_3) - K_2(\beta_1 + \beta_3)}{K_1 + K_2 + K_3} + \frac{\beta_3(K_1 - K_2) - K_3(\beta_1 + \beta_2)}{K_1 + K_2 + K_3}. \tag{14}$$

By expanding and canceling common terms in (14) we have:

$$t_{2,1}^3 + t_{3,1}^3 = \frac{(K_1\beta_2 - \beta_1 K_2) + (K_1\beta_3 - \beta_1 K_3)}{K_1 + K_2 + K_3}.$$

By definition, $K_i > 0 \ \forall i \in D$ which implies that $K_1 + K_2, K_3 > 0$. Thus, we meet the conditions of Lemma 4; therefore, to show $t_{2,1}^3 + t_{3,1}^3 \geq t_{2,1}^2$, it suffices to show that:

$$(K_1 + K_2)(K_1\beta_3 - \beta_1 K_3) \geq K_3(K_1\beta_2 - \beta_1 K_2). \tag{15}$$

By algebraic manipulation, we can show that (15) is equivalent to:

$$K_1\beta_3 + K_2\beta_3 \geq K_3\beta_2 + K_3\beta_1. \tag{16}$$

Equation (16) always holds true due to the assumed relative strength indexing in (4). Defender 1's payoff is a strictly increasing function of her resource level, r_1, which increases as the amount of resources transferred to her increases. Therefore, her payoff is nondecreasing as defender 3 joins the coalition. □

Corollary 2. *Assume that $t_{2,3} = t_{3,2} = 0$. Then, the maximum amount that defender 2 is willing to transfer to defender 1 decreases or remains constant when defender 3 joins the game.*

Proof. Let $t_{2,1}^2$ represent the case without defender 3, and $t_{2,1}^3$ represent the case with defender 3 in the game. Then

$$t_{2,1}^2 = \frac{\beta_2 K_1 - K_2\beta_1}{K_1 + K_2},$$
$$t_{2,1}^3 = \frac{(\beta_2 K_1 - K_2\beta_1) + (\beta_2 K_3 - K_2\beta_3)}{K_1 + K_2 + K_3}.$$

By definition, $K_i > 0 \ \forall i \in D$ which implies that $K_1 + K_2, K_3 > 0$. Then by Lemma 4

$$t_{i,1}^2 \geq t_{i,1}^3 \iff K_3(\beta_2 K_1 - K_2\beta_1) \geq (K_1 + K_2)(\beta_2 K_3 - K_2\beta_3). \tag{17}$$

Through algebraic manipulation, we can show that (17) is equivalent to:

$$K_1\beta_3 + K_2\beta_3 \geq K_3\beta_2 + K_3\beta_1. \tag{18}$$

Similar to Corollary 1, (18) always holds true due to the assumed relative strength indexing in (4). Therefore, the resource transfer from defender 2 to defender 1 is non-increasing as defender 3 joins the game. □

6 Strongest Attacker Fights Everyone

In this section we present the proofs for Theorem 2 and then identify the equilibrium transfers for a special case in Theorem 3.

Proof of Theorem 2: The payoff to the attacker and each defender in each CBG^{i} is a result of Lemma 1 and is respectively given by:

$$\pi_i(\alpha_i, r_i) = K_i\Big(1 - \frac{r_i}{2\alpha_i}\Big), \qquad \phi_i(\mathbf{t}) = K_i - \pi(\alpha_i) = K_i\Big(\frac{r_i}{2\alpha_i}\Big).$$

The attacker's optimal strategy remains the same as in Proposition 1. Substituting the result of Proposition 1, case 2 into the defender's payoff results in:

$$\phi_i(\mathbf{t}) = \frac{\sqrt{K_i}}{2\alpha}\left(\sqrt{r_i}\sum_{j\in D}\sqrt{K_j r_j}\right). \tag{19}$$

We want to show that there is a parameter range for which it is beneficial for player i to transfer resources to player k and also beneficial for player k to accept those resources. In order to do so, we will show that, for a certain parameter configuration, $\phi_i(\mathbf{t})$ and $\phi_k(\mathbf{t})$ are increasing in $t_{i,k}$.

We first show the following result.

Lemma 5. *If player i is willing to transfer resources to player k, then player k is always willing to accept those resources.*

Proof. One can verify that the partial derivative of the defender i's payoff with respect to resource transfers out, $t_{i,k}$, is

$$\frac{\partial \phi_i(\mathbf{t})}{\partial t_{i,k}} = \frac{K_i}{2\alpha}\left[-1 + \frac{1}{2}\sqrt{\frac{K_k}{K_i}}\left(\frac{r_i - r_k - \sqrt{\frac{r_k}{K_k}}\sqrt{K_j r_j}}{\sqrt{r_i r_k}}\right)\right]. \tag{20}$$

In addition, from defender k's perspective, the partial derivative of her payoff with respect to the transfer in, $t_{i,k}$, is

$$\frac{\partial \phi_k(\mathbf{t})}{\partial t_{i,k}} = \frac{K_k}{2\alpha}\left[1 + \frac{1}{2}\sqrt{\frac{K_i}{K_k}}\left(\frac{r_i - r_k + \sqrt{\frac{r_i}{K_i}}\sqrt{K_j r_j}}{\sqrt{r_i r_k}}\right)\right]. \tag{21}$$

Then, the two defenders will form a coalition if and only if

$$\frac{\partial \phi_i(\mathbf{t})}{\partial t_{i,k}} > 0 \iff r_i - r_k > 2\sqrt{\frac{K_i}{K_k}}\sqrt{r_i r_k} + \sqrt{\frac{r_k}{K_k}}\sqrt{K_j r_j}, \tag{22}$$

$$\frac{\partial \phi_k(\mathbf{t})}{\partial t_{i,k}} > 0 \iff r_i - r_k > -2\sqrt{\frac{K_k}{K_i}}\sqrt{r_i r_k} - \sqrt{\frac{r_i}{K_i}}\sqrt{K_j r_j}, \tag{23}$$

where we used (20) and (21). By definition, $K_i, K_k, K_j > 0 \quad \forall i, j, k \in D$. In addition, by the restrictions imposed in Sect. 3, $r_i, r_k, r_j \geq 0 \quad \forall i, j, k \in D$. Therefore, the right hand side of (22) is always greater than or equal to zero, while the right hand side of (23) is always less than or equal to zero. So, if the condition in (22) holds true, then the condition in (23) must also hold true. The proof of the lemma is thus complete. □

Finally, we now complete the proof of Theorem 2 in terms of the parameters of the game. For fixed $t_{j,l} = 0 \quad \forall (j, l) \neq (i, k)$, if

$$\beta_i - \beta_k > 2\sqrt{\frac{K_i}{K_k}}\sqrt{\beta_i \beta_k} + \sqrt{\frac{\beta_k}{K_k}}\sqrt{K_j \beta_j} \implies \left.\frac{\partial \phi_i(\mathbf{t})}{\partial t_{i,k}}\right|_{t_{i,k}=0} > 0.$$

Thus, there exists small values of $t_{i,k} > 0$ for which the inequalities in (22) and (23) will still hold. This will be true even if defender i and k transfer or receive a small amount of resources from the other defender. This concludes the proof of Theorem 2. □

From Lemma 5, one can immediately notice that the resource rich player can have an incentive to trade resources to a poorer player. However, the required difference in their respective resource levels is not just a function of the two defender's resources and total battlefield values, but also a function of the sum of the other defender's resources and total battlefield values.

By observing (22) and (23) closely, we conclude that there is a region where player k would be willing to accept resources, but player i is not willing to send those resources. This region is defined by:

$$r_i - r_k \in \left(-2\sqrt{\frac{K_k}{K_i}}\sqrt{r_i r_k} - \sqrt{\frac{r_i}{K_i}}\sqrt{K_j r_j}, \sqrt{\frac{K_i}{K_k}}\sqrt{r_i r_k} + \sqrt{\frac{r_k}{K_k}}\sqrt{K_j r_j}\right).$$

In this parameter region, a resource rich player would be willing to accept resources from a poor player, but the poor player would not be willing to transfer those resources.

6.1 The Case of No Transfer Between Defenders 2 and 3

In this subsection, we present the proof of Theorem 3 and identify a parameter region where there is no transfer between defenders 2 and 3 in equilibrium. In what follows, only defender 1 transfers resources to the two defenders.

Proof. We now prove Theorem 3. The proof is divided into three steps:

Step 1: In this step, we observe that the equilibrium transfers (assuming they exist) satisfy $t^*_{1,2} < \frac{\beta_1-\beta_2}{2}$ and $t^*_{1,3} < \frac{\beta_1-\beta_3}{2}$. Indeed, if the transfer $t_{1,i}$ is higher than the upper bound, then $r_1 - r_i < 0$. This implies that the derivative of the expected payoffs with respect to transfer out from defender 1 to defender i is negative. Thus, $t^*_{1,2} < \frac{\beta_1-\beta_2}{2}$ and $t^*_{1,3} < \frac{\beta_1-\beta_3}{2}$ is a dominant strategy for defender 1.

Step 2: We next show that if Hypotheses 4, 5, and 6 hold, then no sharing of resources between defender 2 and 3 is a dominant strategy. First, from Lemma 5 there is no transfer of resources from defender 2 to defender 3 if

$$r_2 - r_3 < 2\sqrt{\frac{K_2}{K_3}}\sqrt{r_2 r_3} + \sqrt{\frac{r_3}{K_3}}\sqrt{K_1 r_1}. \tag{24}$$

Using the upper bound on $t^*_{1,2}$ and Hypothesis 4, we upper bound the LHS of the equation above by:

$$r_2 - r_3 < \beta_2 + \frac{\beta_1 - \beta_2}{2} - \beta_3 \leq 2\sqrt{\frac{K_2}{K_3}}\sqrt{\beta_2\beta_3} + \sqrt{\frac{\beta_3}{K_3}}\sqrt{K_1\frac{\beta_1+\beta_2}{2}}.$$

We now proceed to lower bound the RHS of (24) by noting that the first term is at a minimum for $t_{1,2} = 0, t_{1,3} = 0$. Substituting for r_1 and r_3 in the second term in the right hand side and rearranging, we have:

$$\sqrt{\frac{r_3}{K_3}}\sqrt{K_1 r_1} = \sqrt{\frac{K_1}{K_2}}\sqrt{(\beta_3 + t_{1,3})(\beta_1 - t_{1,2} - t_{1,3})}.$$

This term is concave and strictly decreasing in $t_{1,2}$. Therefore, the lower bound occurs when $t_{1,2} = \frac{\beta_1-\beta_2}{2}$. Since the term is concave, it suffices to consider the two bounds on $t_{1,3}$, which are $t_{1,3} = 0$ and $t_{1,3} = \frac{\beta_1-\beta_3}{2}$. Under Hypothesis 6, one can show that the lower bound occurs when $t_{1,3} = 0$. We therefore have:

$$r_2 - r_3 < 2\sqrt{\frac{K_2}{K_3}}\sqrt{\beta_2\beta_3} + \sqrt{\frac{\beta_3}{K_3}}\sqrt{K_1\frac{\beta_1+\beta_2}{2}} \leq 2\sqrt{\frac{K_2}{K_3}}\sqrt{r_2 r_3} + \sqrt{\frac{r_3}{K_3}}\sqrt{K_1 r_1}.$$

Therefore, if Hypotheses 4 and 6 hold, then for any $t_{1,2} < \frac{\beta_1-\beta_2}{2}$ and $t_{1,3} < \frac{\beta_1-\beta_3}{2}$, we have $\partial\phi_2(\mathbf{t})/\partial t_{2,3} < 0$.

A similar result holds if Hypotheses 5 and 6 hold wherein defender 3's expected payoff reduces for positive transfer of resources from defender 3 to defender 2.

Step 3: Fix $t_{1,3} \in (0, \frac{\beta_1-\beta_3}{2})$, define $r_1 = \beta_1 - t_{1,3}$, and assume that $t_{1,3}$ is chosen such that $\partial\phi_1(\mathbf{t})/\partial t_{1,2} > 0$. Now, if defender 1 transfers $t_{1,2}$ amount to defender 2, then $\partial\phi_1(\mathbf{t})/\partial t_{1,2} > 0$ iff

$$r_1 - \beta_2 - 2t_{1,2} > 2\sqrt{\frac{K_1}{K_2}}\sqrt{(r_1 - t_{1,2})(\beta_2 + t_{1,2})} + \sqrt{\frac{(\beta_2 + t_{1,2})}{K_2}}\sqrt{K_3 r_3}.$$

Thus, as $t_{1,2}$ increases from 0, the left side of the equation reduces and the right side of the inequality increases. Thus, at some critical value $\bar{t}_{1,2}(t_{1,3})$ the two sides are equal, and beyond this transfer amount, transferring resource to defender 2 is not beneficial to defender 1. A similar argument holds for the transfer between defender 1 and 3.

Thus, by solving defender 1's optimization problem in (5), we obtain the optimal transfer between the defenders.

□

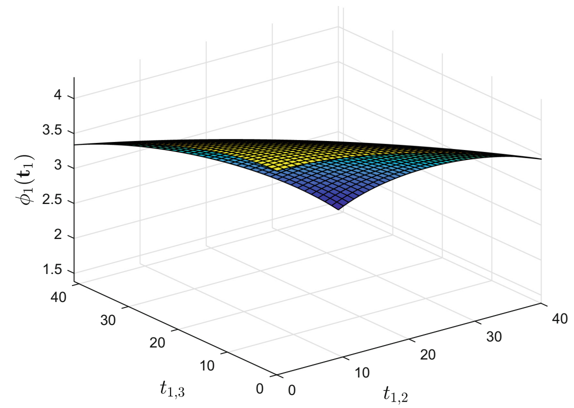

Fig. 1. Payoff of defender 1 for $t_{1,2} \in [0, \frac{\beta_1-\beta_2}{2})$ and $t_{1,2} \in [0, \frac{\beta_1-\beta_3}{2})$ with $\alpha = 160, \beta_1 = 90, \beta_2 = 10, \beta_3 = 8, n_1 = 5, n_2 = 8, n_3 = 6, K_1 = 10, K_2 = 8, K_3 = 6, t_{2,3} = 0, t_{3,2} = 0$.

Figure 1 shows defender 1's payoff versus possible transfers for a configuration of parameters that meet the conditions of Theorem 3. One can observe that defender 1's payoff increases slightly as $t_{1,2}$ and $t_{1,3}$ increase from 0. For the parameter configuration in Fig. 1, we find that $t^*_{1,2} = 0.9390$ and $t^*_{1,3} = 0.2039$.

7 Conclusion

In this paper, we formulated a 4-player, two-stage Colonel Blotto game where defenders can choose to form a coalition by transferring resources in stage one and the attacker observes the transfer among the defenders. This work builds upon the previous work in [10,11], in which the authors considered only two defenders fighting against one attacker. Our goal is to analyze the coalition formation in multi-defender cases, which requires a much more intricate analysis. In the case of two defenders, only one coalition can be formed; on the other hand, if there are N defenders, then there can be $N(N-1)/2$ possible number of coalitions. Our work is an important step in the direction of analyzing the more general case of N defenders.

For certain parameter regions, we've calculated the subgame-perfect Nash equilibrium (SPNE) and identified the parameter regions in which coalitions are formed. Somewhat surprisingly, the resource rich player does not necessarily transfer resources to poorer players when the attacker is the weakest player. In addition, we've shown that in some situations, it is in the best interest of the defenders to add additional weak defenders to the game.

In other parameter regions, we've shown that there are regions where a coalition will form since it is beneficial to transfer resources, but we could not compute the equilibrium transfers due to complex algebraic dependencies. We however note that equilibrium transfers can be computed using computational methods. Unlike the other case, transfer always occurs from the resource rich player to the resource poor player, although there does exist a parameter region where a resource rich player would accept resources from a poorer player.

In the future, we plan to consider the N + 1-player case that considers N defenders and study the equilibrium transfers and payoffs to defenders as more defenders join the coalition. We also plan to consider the case where the attacker has incomplete information and can observe some transfers between defenders but not all transfers.

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