Pursuit Game for an Infinite System of First-Order Differential Equations with Negative Coefficients

Ibragimov Gafurjan, Usman Waziri, Idham Arif Alias and Zarina Bibi Ibrahim

Abstract We consider a pursuit differential game described by an infinite system of 1st-order differential equations with negative coefficients in Hilbert space. The control functions of players are subject to integral constraints. The pursuer attempts to bring the system from a given initial state to another state for a finite time and the evader's purpose is opposite. We obtain a condition of completion of pursuit when the control resource of the pursuer is greater than that of the evader. We study a control problem as well.

Keywords Pursuer · Evader · Infinite system of differential equations · Control strategy

1 Introduction

Differential games in finite dimensional Euclidean spaces were studied by many researchers and developed important methods (see, for instance, [10, 25, 28, 30, 36, 37].)

I. Gafurjan (✉) · I. A. Alias
Department of Mathematics and Institute for Mathematical Research,
Universiti Putra Malaysia, Serdang, Malaysia
e-mail: ibragimov@upm.edu.my

I. A. Alias
e-mail: idham_2@upm.edu.my

U. Waziri · Z. B. Ibrahim
Faculty of Science, Department of Mathematics, Universiti Putra Malaysia,
Serdang, Malaysia
e-mail: usmanwazirimth@yahoo.com

Z. B. Ibrahim
e-mail: zarinabb@upm.edu.my

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There are mainly two constraints on control functions of players: geometric and integral constraints. In-views of the amount of works been done in developing the differential games, the integral constraints have been extensively discussed by many researchers with various approaches (see, for example, [4, 5, 8, 11, 12, 18–21, 26, 27, 29, 31, 34, 35, 39, 42–44]).

One of the powerful tools in studying the control and differential game problems in systems with distributed parameters is the decomposition method. Using this method the control or differential game problem is reduced to ones described by infinite systems of differential equations (see, for example, [2, 6, 7, 9, 13, 32, 40, 41, 45, 46]). We demonstrate briefly the method for the following parabolic equation

$$\frac{\partial z(x,t)}{\partial t} + Az(x,t) = w(x,t), \quad z(x,0) = z_0(x), \tag{1}$$

where $0 \le t \le T$, T is a given positive number, $x = (x_1, \dots, x_n) \in \Omega \subset R^n, n \ge 1$, Ω is a bounded set with piecewise smooth boundary,

$$Az = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial z}{\partial x_j}\right).$$

$a_{ij}(x) = a_{ji}(x)$, $x \in \Omega$, and, for some $c > 0$ and for all

$$(\xi_1, \dots, \xi_n) \in R^n, x \in \Omega, \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \ge c\sum_{i=1}^{n} \xi_i^2.$$

The domain of the operator A is the space of twice continuously differentiable functions with compact support in Ω, denoted by $\overset{\circ}{C}{}^2(\Omega)$. Define inner product

$$(z, y)_A = (Az, y), \quad z, y \in \overset{\circ}{C}{}^2(\Omega).$$

Then $\overset{\circ}{C}{}^2(\Omega)$ becomes incomplete Euclidean space. To obtain a complete Hilbert space associated with the operator A, we complete the space $\overset{\circ}{C}{}^2(\Omega)$ with respect to the norm $||z||_A = \sqrt{(Az, z)}$, $z \in \overset{\circ}{C}{}^2(\Omega)$. We use the fact that the operator A has countably many eigenvalues

$$\lambda_1, \lambda_2, \dots, \quad 0 < \lambda_1 \le \lambda_2 \le \dots, \quad \lim_{k\to\infty} \lambda_k = +\infty,$$

and generalized eigenfunctions $\varphi_1, \varphi_2, \dots$, which is a complete orthonormal system in $L_2(\Omega)$ [33].

Next, let $C(0, T; H_r(\Omega))$ and $L_2(0, T; H_r(\Omega))$ denote the spaces of continuous and measurable functions defined on $[0, T]$ with the values in

$$H_r(\Omega) = \left\{ f \in L_2(\Omega) \mid f = \sum_{i=1}^{\infty} \alpha_i \varphi_i, \ \sum_{i=1}^{\infty} \lambda_i^r \alpha_i^2 < \infty \right\},$$

respectively, where r is a given number. The space $H_r(\Omega)$ is a Hilbert space with inner product and norm defined as follows: if

$$f = \sum_{i=1}^{\infty} \alpha_i \varphi_i \in H_r(\Omega), \ \ g = \sum_{i=1}^{\infty} \beta_i \varphi_i \in H_r(\Omega),$$

then

$$(f, g) = \sum_{i=1}^{\infty} \lambda_i^r \alpha_i \beta_i, \ \ ||f|| = \left(\sum_{i=1}^{\infty} \lambda_i^r \alpha_i^2 \right)^{1/2}.$$

It was proved [2] that if $w(\cdot) \in L_2(0, T; H_r(\Omega))$, then the initial value problem (1) has a unique solution $z(\cdot) \in C(0, T; H_{r+1}(\Omega))$. Next, represent the functions $z(x, t)$ and $w(x, t)$ as

$$z(x, t) = \sum_{k=1}^{\infty} z_k(t) \varphi_k(x), \ \ w(x, t) = \sum_{k=1}^{\infty} w_k(t) \varphi_k(x), \ \ z_k(\cdot), w_k(\cdot) \in L_2(0, T), \ \ k = 1, 2, \dots,$$

and substitute them into the Eq. (1), and then equate the coefficients at $\varphi_k(x)$ to obtain

$$\dot{z}_k + \lambda_k z_k = w_k, \ \ z_k(0) = z_{k0}, \ k = 1, 2, \dots,$$

where $w_k, z_k, z_{k0} \in R^1$, $k = 1, 2, \dots$, w_k, are control parameters, $z_{k0} = (z_0, \varphi_k)$. Thus, we have obtained an infinite system of differential equations. Usually, the control function is subjected to geometric or integral constraint. The geometric and integral constraints for the control function $w \in H(0, T; H_r(\Omega))$ of the form

$$||w(x, t)|| \le \rho, \ \ \int_0^T ||w(x, t)||^2 dt \le \rho^2,$$

respectively, can be written as follows

$$\left(\sum_{k=1}^{\infty} \lambda_k^r w_k^2(t) \right)^{1/2} \le \rho, \ \ \sum_{k=1}^{\infty} \lambda_k^r \int_0^T w_k^2(t) dt \le \rho^2,$$

respectively.

Hence, there is an important connections between control problems described by PDE and those described by infinite system of differential equations. Control and differential game problems described by infinite system of differential equations are of independent interest and can be investigated within one theoretical framework independently of those described by PDE assuming that the coefficients $\lambda_k, k = 1, 2, \ldots$, are any real numbers. Of course, in the case where λ_k are any real numbers, we must give adequate definitions of state space, solution of infinite system of differential equations. Also, we have to prove the existence-uniqueness of solution in the state space.

There are several works devoted to control or differential game problems described by infinite system of differential equations (see, for example, [1, 3, 14, 16, 17, 22–24, 38]).

In the paper [14] a differential game problem described by the following infinite system of differential equations

$$\dot{z}_k + \lambda_k z_k = -u_k + v_k,\ z_k(0) = z_{k0},\ k = 1, 2, \ldots, \tag{2}$$

where $z_k, u_k, v_k \in \mathbb{R}^1$, and $\lambda_k, k = 1, 2, \ldots$, are positive numbers, was studied when integral constraints are subjected to control functions of the players.

In the present paper, we study a pursuit differential game problems described by (2) in the case of negative coefficients λ_k, $k = 1, 2, \ldots$. Pursuer tries to bring the state of the system from an initial state z^0 to another given one z^1 for a finite time. Previous studies of differential games described by infinite system of differential equations have only dealt with the case $z^1 = 0$. We obtain sufficient conditions of completion of pursuit.

2 Statement of Problem

Consider the following Hilbert space

$$l_r^2 = \left\{ \alpha = (\alpha_1, \alpha_2, \ldots) |\ \sum_{k=1}^{\infty} |\lambda_k|^r \alpha_k^2 < \infty \right\},$$

where, r is a real number and $\lambda_1, \lambda_2, \ldots$, is a bounded sequence of negative numbers, with inner product and norm defined by

$$\langle \alpha, \beta \rangle_r = \sum_{k=1}^{\infty} |\lambda_k|^r \alpha_k \beta_k,\ \alpha,\ \beta \in l_r^2,\ \ ||\alpha|| = \left(\sum_{k=1}^{\infty} |\lambda_k|^r \alpha_k^2 \right)^{1/2}.$$

Let

$$L_2(0, T, l_r^2) = \left\{ w(\cdot) = (w_1(\cdot), w_2(\cdot), \ldots) |\ \|w(\cdot)\|_{L_2(0,T,l_r^2)} < \infty,\ w_k(\cdot) \in L_2(0, T) \right\},$$

where $T > 0$ is a given sufficiently big number,

$$\|w(\cdot)\|_{L_2(0,T,l_r^2)} = \left(\sum_{k=1}^{\infty} |\lambda_k|^r \int_0^T w_k^2(t)dt \right)^{1/2},$$

We examine control and pursuit differential game problems described by the following infinite system of differential equations

$$\dot{z}_k + \lambda_k z_k = -u_k + v_k, \ z_k(0) = z_k^0, \ k = 1, 2, \ldots, \tag{3}$$

where $z_k, u_k, v_k \in \mathbb{R}^1, \ k = 1, 2, \ldots; u = (u_1, u_2, \ldots)$ is the control parameter of pursuer and $v = (v_1, v_2, \ldots)$ is that of evader, $z^0 = (z_1^0, \ z_2^0, \ldots) \in l_{r+1}^2$.

Let

$$S(\rho_0) = \left\{ w(\cdot) \in L_2(0, T, l_r^2) | \ \|w(\cdot)\|_{L_2(0,T,l_r^2)} \le \rho_0 \right\},$$

where ρ_0 is a given positive number.

Definition 1 Functions $w(\cdot) \in S(\rho_0)$, $u(\cdot) \in S(\rho)$, and $v(\cdot) \in S(\sigma)$ are called admissible control, admissible control of pursuer, and admissible control of evader, respectively, where ρ and σ are given positive numbers.

It's assumed that $\rho > \sigma$.

Definition 2 Let $w(\cdot) \in S(\rho_0)$. A function $z(t) = (z_1(t), z_2(t), \ldots)$, $0 \le t \le T$, with $z_k(0) = z_k^0, \ k = 1, 2, \ldots$, is called solution of the initial value problem

$$\dot{z}_k(t) + \lambda_k z_k(t) = w_k(t), \ z_k(0) = z_k^0, \ k = 1, 2, \ldots, \tag{4}$$

if $z_k(t)$, $k = 1, 2, \ldots$, are absolutely continuous and almost everywhere on $[0, T]$ satisfy the Eq. (4).

Let $C(0, T; l_{r+1}^2)$ be the space of continuous functions $z(t) = (z_1(t), z_2(t), \ldots) \in l_{r+1}^2$ defined on $[0, \ T]$. We need the following proposition [15].

Proposition 1 *If $w(\cdot) \in S(\rho)$, then infinite system of differential equations (4) has the only solution $z(t) = (z_1(t), z_2(t), \ldots)$, $0 \le t \le T$, in the space $C(0, T; l_{r+1}^2)$, where*

$$z_k(t) = e^{\beta_k t} \left(z_k^0 + \int_0^t w_k(s) e^{-\beta_k s} ds \right), \ k = 1, 2, \ldots,$$

with $\beta_k = -\lambda_k > 0$.

Note that this existence-uniqueness theorem for the system (4) was proved for any finite interval $[0, \ T]$. Therefore, we investigate the system (3) and (4) on $[0, \ T]$.

Definition 3 A function

$$U(t, v) = (U_1(t, v), U_2(t, v), \ldots), \ \ U : [0, \ T] \times l_r^2 \to l_r^2,$$

with the components of the form

$$U_k(t, v) = w_k(t) + v_k(t), \ k = 1, 2, \ldots,$$

is referred to as the strategy of pursuer, if, for any admissible control of evader $v(\cdot) = (v_1(\cdot), v_2(\cdot), \ldots)$, the system (3) has the only solution at $u(t) = U(t, v)$, where $w(\cdot) = (w_1(\cdot), w_2(\cdot), \ldots) \in S(\rho - \sigma)$.

We are given another state $z^1 = (z_1^1, \ z_2^1, \ldots) \in l^2_{r+1}$.

Definition 4 We say that the game (3) can be completed for the time θ ($\theta \leq T$), if there exists a strategy U of pursuer such that, for any admissible control of evader, $z(\tau) = z^1$ at some time τ, $0 \leq \tau \leq \theta$.

Pursuer tries to bring the state of the system (3) from z^0 to z^1, and the purpose of evader is opposite. Formulate the problems.

Problem 1 Find a condition on the states $z^0, z^1 \in l^2_{r+1}$ such that the state $z(t)$ of the system (4) can be transferred from the initial position z^0 to the final position z^1 for a finite time.

Problem 2 Find a condition on the states $z^0, z^1 \in l^2_{r+1}$, for which pursuit can be completed in the game (3) for a finite time.

3 Control Problem

In this section, we study a control problem for transferring the system $z(t)$ from the initial position z^0 to the final position z^1.

For the system (4), we study the control problem: find a time θ such that

$$z(0) = z^0, \ z(\theta) = z^1. \tag{5}$$

First, we analysis the following series

$$E(t) = E_1(t) + E_2(t), \ t > 0, \tag{6}$$

where

$$E_1(t) = 2\sum_{k=1}^{\infty} \beta_k^r |z_k^0|^2 \phi_k(t), \quad E_2(t) = 2\sum_{k=1}^{\infty} \beta_k^r |z_k^1|^2 \psi_k(t), \tag{7}$$

$$\phi_k(t) = \frac{2\beta_k}{1 - e^{-2\beta_k t}}, \ \psi_k(t) = \frac{2\beta_k}{e^{2\beta_k t} - 1}, \ k = 1, 2, \ldots.$$

Lemma 1 *Let $z^0, z^1 \in l^2_{r+1}$. If, in addition, $z^0, z^1 \in l^2_r$, then the series $E(t)$ converges at any $t > 0$.*

Proof Let $z^0, z^1 \in l_r^2$. To show that the series (6) converges, we show that the series $E_1(t)$ and $E_2(t)$ converge. Since β_k is a bounded sequence of positive numbers, therefore $\beta = \sup_k \beta_k < \infty$. Since $\beta_k \le \beta$, then it is not difficult to show that

$$\phi_k(t) = \frac{2\beta_k}{1 - e^{-2\beta_k t}} \le \frac{2\beta}{1 - e^{-2\beta t}},$$

which implies that

$$E_1(t) \le \frac{4\beta}{1 - e^{-2\beta t}} \sum_{k=1}^{\infty} \beta_k^r |z_k^0|^2.$$

The series on the right hand side of this inequality is convergent since $z^0 \in l_r^2$. Thus, the series $E_1(t)$ is convergent.

We can see that $\psi_k(t) \le \frac{1}{t}$, $t > 0$, $k = 1, 2, \ldots$. Then

$$E_2(t) \le \frac{2}{t} \sum_{k=1}^{\infty} \beta_k^r |z_k^1|^2.$$

The series on the right hand side of this inequality is convergent since $z^1 \in l_r^2$. Thus, the series $E_2(t)$ is convergent. This completes the proof of Lemma 1.

We'll need some properties of $E(t)$.

Property 1 *$E(t)$ has the following properties:*

(i) $E(t)$ *is decreasing on* $(0, +\infty)$*;*
(ii) $E(t) \to +\infty$ *as* $t \to 0^+$*;*
(iii) $E(t) \to 4 \sum_{k=1}^{\infty} \beta_k^{r+1} |z_k^0|^2$ *as* $t \to +\infty$.

Proof The first property follows from the fact that $\psi_k(t)$ and $\phi_k(t)$, $k = 1, 2, \ldots$, are decreasing on $(0, +\infty)$.

The proof of the property (ii) follows from the observations that $\psi_k(t) \to +\infty$ and $\phi_k(t) \to +\infty$, as $t \to 0^+$ for each k.

Finally, we prove the property (iii). According to Lemma 1, $E(t)$ is convergent for any $t > 0$. We fix $t_0 > 0$. Since $E(t_0)$ is convergent, then for any $\varepsilon > 0$, there exists a positive integer N such that

$$F(t_0) = \sum_{k=N+1}^{\infty} \beta_k^r \left(2|z_k^0|^2 \phi_k(t_0) + 2|z_k^1|^2 \psi_k(t_0)\right) < \frac{\varepsilon}{3}, \tag{8}$$

and also

$$\sum_{k=N+1}^{\infty} 4\beta_k^{r+1} |z_k^0|^2 < \frac{\varepsilon}{3} \tag{9}$$

since $z^0 \in l^2_{r+1}$. Then, $F(t) < \frac{\varepsilon}{3}$ for all $t \geq t_0$ since the functions $\psi_k(t)$ and $\phi_k(t)$ are decreasing on $(0, +\infty)$ for each k.

On the other hand, there exists number $T_1 > 0$ such that, for all $t > T_1$,

$$\left| 2\sum_{k=1}^{N} \beta_k^r \left(|z_k^0|^2 \phi_k(t) + |z_k^1|^2 \psi_k(t) \right) - 4\sum_{k=1}^{N} \beta_k^{r+1} |z_k^0|^2 \right| < \frac{\varepsilon}{3}, \tag{10}$$

since the sum consists of a finite number of summands and

$$\lim_{t\to+\infty} \phi_k(t) = 2\beta_k, \quad \lim_{t\to+\infty} \psi_k(t) = 0, \quad k = 1, 2, \ldots$$

Thus, by (8)–(10)

$$\begin{aligned} \left| E(t) - 4\sum_{k=1}^{\infty} \beta_k^{r+1} |z_k^0|^2 \right| &\leq \left| 2\sum_{k=1}^{N} \beta_k^r \left(|z_k^0|^2 \phi_k(t) + |z_k^1|^2 \psi_k(t) \right) - 4\sum_{k=1}^{N} \beta_k^{r+1} |z_k^0|^2 \right| \\ &\quad + 2\sum_{k=N+1}^{\infty} \beta_k^r \left(|z_k^0|^2 \phi_k(t) + |z_k^1|^2 \psi_k(t) \right) + 4\sum_{k=N+1}^{\infty} \beta_k^{r+1} |z_k^0|^2 \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

This proves property (iii).

Next since $\dfrac{4}{1 - e^{-2\beta_k t}} > 4, t > 0$, therefore we obtain from (i) and (iii) that

$$E(t) > 4\sum_{k=1}^{\infty} \beta_k^{r+1} |z_k^0|^2, \ t > 0. \tag{11}$$

Property 1 and (11) imply that the equation

$$E(t) = \rho_0^2 \tag{12}$$

has a root $t = \theta$ if and only if

$$\rho_0^2 > 4\sum_{k=1}^{\infty} \beta_k^{r+1} |z_k^0|^2, \tag{13}$$

and this root is unique. Without loss of generality, we can assume that $\theta < T$ since T is sufficiently big number.

The following statement is a solution for the control problem (5).

Theorem 1 *Let inequality (13) be satisfied and $z^0, z^1 \in l_r^2$. Then the system (4) can be transferred from the initial position z^0 to the position z^1 for the time θ.*

Proof Define a control

$$w_k(t) = \begin{cases} -\left[z_k^0 - z_k^1 e^{-\beta_k \theta}\right] \phi_k(\theta) e^{-\beta_k t}, & 0 \le t \le \theta \\ 0, & t > \theta \end{cases}, \quad k = 1, 2, \ldots. \tag{14}$$

Show that this control is admissible. Using Eq. (12), control (14), and the obvious inequality $|x - y|^2 \le 2|x|^2 + 2|y|^2$, we proceed as follows:

$$\begin{aligned}
\sum_{k=1}^{\infty} \beta_k^r \int_0^{\theta} |w_k(s)|^2 ds &= \sum_{k=1}^{\infty} \beta_k^r \int_0^{\theta} \left| -\left[z_k^0 - z_k^1 e^{-\beta_k \theta}\right] \phi_k(\theta) e^{-\beta_k s} \right|^2 ds \\
&\le \sum_{k=1}^{\infty} \beta_k^r \left(2|z_k^0|^2 + 2|z_k^1|^2 e^{-2\beta_k \theta}\right) \phi_k^2(\theta) \int_0^{\theta} e^{-2\beta_k s} ds \\
&= 2 \sum_{k=1}^{\infty} \beta_k^r \left(|z_k^0|^2 \phi_k(\theta) + |z_k^1|^2 \psi_k(\theta)\right) \\
&= E(\theta) = \rho_0^2.
\end{aligned}$$

Show that the system can be transferred from z^0 to z^1 for the time θ. Indeed,

$$\begin{aligned}
z_k(\theta) &= e^{\beta_k \theta} \left(z_k^0 - \left[z_k^0 - z_k^1 e^{-\beta_k \theta}\right] \phi_k(\theta) \int_0^{\theta} e^{-2\beta_k s} ds \right) \\
&= e^{\beta_k \theta} (z_k^1 e^{-\beta_k \theta}) = z_k^1.
\end{aligned}$$

This completes the proof of Theorem 1.

4 Pursuit Differential Game Problem

In this section, we study pursuit differential game described by the Eq. (3). It is assumed that control resources of pursuer is greater than that of evader, that is $\rho > \sigma$.

We obtain from (3) that

$$z_k(t) = e^{\beta_k t} \left(z_k^0 - \int_0^t u_k(s) e^{-\beta_k s} ds + \int_0^t v_k(s) e^{-\beta_k s} ds \right). \tag{15}$$

In view of the previous section we can state that the equation

$$E(t) = 2\sum_{k=1}^{\infty} \beta_k^r \left(|z_k^0|^2\phi_k(t) + |z_k^1|^2\psi_k(t)\right) = (\rho - \sigma)^2 \tag{16}$$

has a root $t = \theta_1$ if and only if

$$(\rho - \sigma)^2 > 4\sum_{k=1}^{\infty} \beta_k^{r+1}|z_k^0|^2, \tag{17}$$

and this root is unique. We can assume, by selecting T if needed that $\theta_1 < T$.

Theorem 2 *Let (17) be satisfied and $z^0, z^1 \in l_r^2$. Then pursuit can be completed in the game (3) for the time θ_1.*

Proof Construct a strategy for the pursuer. Set

$$u_k(t, v) = \begin{cases} \left[z_k^0 - z_k^1 e^{-\beta_k\theta_1}\right]\phi_k(\theta_1)e^{-\beta_k s} + v_k(t), & 0 \le t \le \theta_1 \\ 0, & t > \theta_1 \end{cases}, \quad k = 1, 2, \ldots \tag{18}$$

Show that strategy (18) is admissible. Applying the Minkowskii inequality, we have

$$\begin{aligned}
\left(\sum_{k=1}^{\infty}\beta_k^r\int_0^{\theta_1}|u_k(s)|^2\,ds\right)^{1/2} &= \left(\sum_{k=1}^{\infty}\beta_k^r\int_0^{\theta_1}\left|\left(z_k^0 - z_k^1e^{-\beta_k\theta_1}\right)\phi_k(\theta_1)e^{-\beta_k s} + v_k(s)\right|^2 ds\right)^{1/2} \\
&\le \left(\sum_{k=1}^{\infty}\beta_k^r\int_0^{\theta_1}\left|\left(z_k^0 - z_k^1e^{-\beta_k\theta_1}\right)\phi_k(\theta_1)e^{-\beta_k s}\right|^2 ds\right)^{1/2} \\
&+ \left(\sum_{k=1}^{\infty}\beta_k^r\int_0^{\theta_1}|v_k(s)|^2\,ds\right)^{1/2} \\
&\le \left(\sum_{k=1}^{\infty}\beta_k^r|z_k^0 - z_k^1e^{-\beta_k\theta_1}|^2\phi_k^2(\theta_1)\int_0^{\theta_1}e^{-2\beta_k s}ds\right)^{1/2} + \sigma. \qquad (19)
\end{aligned}$$

Using the obvious inequality $|x - y|^2 \le 2|x|^2 + 2|y|^2$ and Eq. (16), we obtain form (19) that

$$\begin{aligned}
\left(\sum_{k=1}^{\infty}\beta_k^r\int_0^{\theta_1}|u_k(s)|^2\,ds\right)^{1/2} &\le \left(2\sum_{k=1}^{\infty}\beta_k^r\left(|z_k^0|^2\phi_k(\theta_1) + |z_k^1|^2\psi_k(\theta_1)\right)\right)^{1/2} + \sigma \\
&= E^{1/2}(\theta_1) + \sigma \\
&= \rho - \sigma + \sigma = \rho.
\end{aligned}$$

Thus the strategy (18) is admissible.

Next, we show that pursuit is completed at the time θ_1. Indeed, using (15) and strategy (18), we have

$$
\begin{aligned}
z_k(\theta_1) &= e^{\beta_k\theta_1}\left(z_k^0 - \int_0^{\theta_1}\left(\left(z_k^0 - z_k^1 e^{-\beta_k\theta_1}\right)\phi_k(\theta_1)e^{-\beta_k s} + v_k(s)\right)e^{-\beta_k s}ds + \int_0^{\theta_1} v_k(s)e^{-\beta_k s}ds\right)\\
&= e^{\beta_k\theta_1}\left(z_k^0 - \int_0^{\theta_1}\left(z_k^0 - z_k^1 e^{-\beta_k\theta_1}\right)\phi_k(\theta_1)e^{-2\beta_k s}ds\right)\\
&= e^{\beta_k\theta_1}\left(z_k^0 - z_k^0 + z_k^1 e^{-\beta_k\theta_1}\right) = z_k^1.
\end{aligned}
$$

The proof of the theorem is completed.

5 Conclusion

We have studied a pursuit differential game problem described by infinite system of 1st-order differential equations with negative coefficients in the space l^2_{r+1}. The control functions of players are subjected to integral constraints.

We have obtained a condition for which a control problem is solvable, also we have constructed a control that transfers the system from an initial state z^0 to the final state z^1 for a finite time.

We have obtained a condition of completion of pursuit in the differential game. Moreover, a pursuit strategy has been constructed.

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