On the Symmetries of a Liénard Type Nonlinear Oscillator Equation

R. Mohanasubha, V. K. Chandrasekar, M. Senthilvelan and M. Lakshmanan

Abstract In the contemporary nonlinear dynamics literature, the nonlinear oscillator equation $\ddot{x} + kx\dot{x} + \frac{k^2}{9}x^3 + \tilde{\lambda}x = 0$ is being analyzed in various contexts both classically and quantum mechanically. Classically this nonlinear oscillator equation has been shown to admit three different types of dynamics depending upon the sign and magnitude of the parameter $\tilde{\lambda}$, namely (i) $\tilde{\lambda} = 0$, (ii) $\tilde{\lambda} > 0$ and (iii) $\tilde{\lambda} < 0$. By considering its importance, in this paper, we present the symmetries of its Lagrangian and underlying equation of motion for all the three cases. In particular, we present Lie point symmetries, λ-symmetries, Noether symmetries and telescopic symmetries of this equation. The utility of the symmetries for all the three cases is demonstrated explicitly.

Keywords Nonlinear oscillators · Lie point symmetries · λ-symmetries · Noether symmetries · Telescopic vector fields

1 Introduction

During the past ten years or so considerable interest has been shown on investigating various properties associated with the Liénard type nonlinear oscillator equation,

R. Mohanasubha
Department of Physics, Anna University, Chennai 600025, India
e-mail: subhajeevi@gmail.com

V. K. Chandrasekar
School of Electrical and Electronics Engineering, Centre for Nonlinear Science and Engineering, SASTRA University, Thanjavur 613401, India
e-mail: chandru25nld@gmail.com

M. Senthilvelan · M. Lakshmanan (✉)
Centre for Nonlinear Dynamics, Bharathidasan University, Tiruchirappalli 620024, India
e-mail: lakshman.cnld@gmail.com

M. Senthilvelan
e-mail: senv0000@gmail.com

V. G. Kac et al. (eds.), *Symmetries, Differential Equations and Applications*, Springer Proceedings in Mathematics & Statistics 266,
https://doi.org/10.1007/978-3-030-01376-9_5

$$\Delta(t, x, \dot{x}, \ddot{x}) = \ddot{x} + kx\dot{x} + \frac{k^2}{9}x^3 + \tilde{\lambda}x = 0, \quad (1)$$

where overdot denotes differentiation with respect to t, k and $\tilde{\lambda}$ are arbitrary parameters [4, 8, 12, 15, 22, 24, 27]. Equation (1) arises in the study of equilibrium configurations of a spherical gas cloud acting under the mutual attraction of its molecules and subject to the laws of thermodynamics [3]. Even though a more general equation of this form with time dependent coefficients has been studied long ago considerable interest has been shown on this particular equation when two of the present authors along with Bindu and Pandey have identified it as one of the linearizable equations when a Lie symmetry analysis was carried out on the Liénard type equation $\ddot{x} + f(x)\dot{x} + g(x) = 0$, where $f(x)$ and $g(x)$ are arbitrary functions of x [24]. Originally three of the present authors have proved the integrability of system (1) and demonstrated that this equation admits a conservative non-standard Lagrangian and Hamiltonian description [4]. They have also shown that the frequency of oscillations of this system for $\tilde{\lambda} > 0$ does not depend on the amplitude of oscillations thereby showing that the amplitude dependence of frequency is not necessarily a fundamental property of nonlinear dynamical phenomena [4].

The system (1) admits three different dynamics depending upon the sign of the linear term in it. For example, for the choice $\tilde{\lambda} \leq 0$, the system (1) admits front like solution and $\tilde{\lambda} > 0$ displays explicit sinusoidal periodic solution [4].

This model has further been investigated by several authors under different perspectives [1, 5–7, 9, 15, 16, 22]. For example, it has been demonstrated that the model (1) admits (i) integrating factors [1, 5, 7, 16], (ii) Lagrangian multipliers [22], (iii) $\lambda-$ symmetries [16], (iv) Darboux polynomials [15], and (v) alternate Lagrangian [9]. Equation (1) can be transformed (i) to a free particle equation through invertible point transformation, (ii) to a harmonic oscillator equation through Sundman transformation and (iii) to a linear third order ODE, $w''' + \tilde{\lambda}w = 0$, through a generalized transformation [5–7, 15, 16].

In one of our earlier works, we have constructed a nonstandard Lagrangian, [4]

$$L_1 = \frac{27\tilde{\lambda}^3}{2k^2}\left(\frac{1}{k\dot{x} + \frac{k^2}{3}x^2 + 3\tilde{\lambda}}\right) + \frac{3\tilde{\lambda}}{2k}\dot{x} - \frac{9\tilde{\lambda}^2}{2k^2}, \quad \tilde{\lambda} \neq 0 \quad (2)$$

for this equation. For many of our investigations we stick to the Lagrangian (2) and its associated Hamiltonian (see Eq. (3) below) since when $k \to 0$ both the Lagrangian and Hamiltonian reduce to the linear harmonic oscillator Lagrangian and Hamiltonian, respectively, as the equation of motion does. In a recent work, two of the present authors with Chithiika Ruby have also demonstrated the quantum solvability of its Hamiltonian [8]

$$H = \frac{9\tilde{\lambda}^2}{2k^2}\left(2 - 2\left(1 - \frac{2kp}{3\tilde{\lambda}}\right)^{\frac{1}{2}} + \frac{k^2x^2}{9\tilde{\lambda}} - \frac{2kp}{3\tilde{\lambda}} - \frac{2k^3x^2p}{27\tilde{\lambda}^2}\right), \quad (3)$$

where

$$p = \frac{\partial L}{\partial \dot{x}} = -\frac{27\tilde{\lambda}^3}{2k}\left(\frac{1}{(k\dot{x} + \frac{k^2}{3}x^2 + 3\tilde{\lambda})^2}\right) + \frac{3\tilde{\lambda}}{2k}. \tag{4}$$

By observing that the Hamiltonian (3) can also be equivalently considered in the form

$$H(x, p) = \frac{x^2}{2m(p)} + U(p), \quad -\infty < p \leq \frac{3\tilde{\lambda}}{2k}, \tag{5}$$

where

$$m(p) = \frac{1}{\tilde{\lambda}\left(1 - \frac{2k}{3\tilde{\lambda}}p\right)} \quad \text{and} \quad U(p) = \frac{9\tilde{\lambda}^2}{2k^2}\left(\sqrt{1 - \frac{2k}{3\tilde{\lambda}}p} - 1\right) \tag{6}$$

and recognizing that this form coincides with the position dependent mass Hamiltonian with the difference that the variables x and p are interchanged, the authors went on to quantize the position dependent mass Schroedinger equation in momentum space by augmenting with van Roos ordering. The explicit eigenvalues and eigenvectors have been brought out in an elegant manner.

In this paper we present symmetries of various kinds for Eq. (1) and the nonstandard Lagrangian (2). The reason for consolidating this result is that as far as symmetries are concerned some of the earlier studies are incomplete. For example, eventhough Lie point symmetries are known for this equation for all the three parametric regimes the order reduction procedure has not been done so far for this system. In this paper, we intend to complete it. As far as λ-symmetries are concerned eventhough a detailed investigation has been made on the $\tilde{\lambda} = 0$ case, the analysis has not been carried out for the $\tilde{\lambda} \neq 0$ cases. In this paper we carry out the λ-symmetry analysis for the $\tilde{\lambda} \neq 0$ cases and present two independent λ-symmetries and their associated independent integrals. Similarly eventhough Noether symmetries for the nonstandard Lagrangian (2) with $\tilde{\lambda} = 0$ has been reported it has not been analysed for the $\tilde{\lambda} \neq 0$ cases. We present the Noether symmetries for the remaining two important cases, namely (i) $\tilde{\lambda} > 0$ and (ii) $\tilde{\lambda} < 0$ as well. The telescopic vector fields, which are more generalized vector fields that play important role when the Lie point symmetries and λ-symmetries are absent for a given second order nonlinear ordinary differential equation, are also unknown for this equation. We construct the telescopic vector fields also for Eq. (1).

The plan of the paper is as follows. In Sect. 2, we recall Lie point symmetries of the nonlinear oscillator equation (1) and carry out order reduction procedure for this equation. In Sect. 3, we carry-out the λ-symmetry analysis for the nonlinear oscillator Eq. (1). To begin with, we recall the results that are reported for the case $\tilde{\lambda} = 0$. We then extend the analysis for the cases $\tilde{\lambda} \neq 0$ and give a complete picture. In Sect. 4, we recall Noether's theorem and apply this theorem to the model (1) and derive Noether's symmetries for all the three cases, namely (i) $\tilde{\lambda} > 0$, (ii) $\tilde{\lambda} < 0$ and

(iii) $\tilde{\lambda} = 0$. In Sect. 5, we present the telescopic vector fields for all the three cases. We present our conclusions in Sect. 6.

2 Lie Point Symmetries of Eq. (1)

Let the evolution equation (1) be invariant under the one parameter Lie group of infinitesimal transformations [2, 13, 23]

$$\tilde{t} = t + \varepsilon\xi(t, x) + O(\varepsilon^2), \quad \tilde{x} = x + \varepsilon\eta(t, x) + O(\varepsilon^2), \quad \varepsilon \ll 1, \tag{7}$$

where ξ and η represent the symmetries of Eq. (1) and they are functions of the variables t and x. The associated infinitesimal generator can be written as

$$X = \xi(t, x)\frac{\partial}{\partial t} + \eta(t, x)\frac{\partial}{\partial x}. \tag{8}$$

Equation (1) is invariant under the action of (8) iff

$$X^{(2)}(\Delta)|_{\Delta=0} = 0, \tag{9}$$

where

$$X^{(2)} = \xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial x} + \eta^{(1)}\frac{\partial}{\partial \dot{x}} + \eta^{(2)}\frac{\partial}{\partial \ddot{x}} \tag{10}$$

and $\eta^{(1)}$ and $\eta^{(2)}$ are first and second prolongations respectively, whose explicit expressions can be found in Refs. [2, 13]. For the sake of completeness, we present symmetries and order reduction procedure for each one of the cases separately.

2.1 *Case 1:* $\tilde{\lambda} = 0$

First let us consider the choice $\tilde{\lambda} = 0$. The invariance condition (9) reads ($\ddot{x} = \phi(t, x, \dot{x})$)

$$\xi\frac{\partial\phi}{\partial t} + \eta\frac{\partial\phi}{\partial x} + \eta^{(1)}\frac{\partial\phi}{\partial \dot{x}} - \eta^{(2)} = 0. \tag{11}$$

Solving the invariance condition (11), one obtains the following symmetry generators, namely [24]

$$X_1 = \frac{\partial}{\partial t},\ X_2 = t\left(1-\frac{k}{6}xt\right)\frac{\partial}{\partial t} - \frac{k}{3}x^2t\left(1-\frac{k}{6}xt\right)\frac{\partial}{\partial x},\ X_3 = x\frac{\partial}{\partial t} - \frac{k}{3}x^3\frac{\partial}{\partial x},$$
$$X_4 = xt\frac{\partial}{\partial t} + x^2\left(1-\frac{k}{3}xt\right)\frac{\partial}{\partial x},\ X_5 = -\frac{k}{6}xt^2\frac{\partial}{\partial t} + x\left(1-\frac{k}{3}xt+\frac{k^2}{18}x^2t^2\right)\frac{\partial}{\partial x},$$
$$X_6 = t^2\left(1-\frac{k}{6}xt\right)\frac{\partial}{\partial t} + xt\left(1-\frac{k}{2}xt+\frac{k^2}{18}x^2t^2\right)\frac{\partial}{\partial x},$$
$$X_7 = \frac{k}{2}t^2\left(1-\frac{k}{9}xt\right)\frac{\partial}{\partial t} + \left(1-\frac{k^2}{6}t^2x^2+\frac{k^3}{54}t^3x^3\right)\frac{\partial}{\partial x},$$
$$X_8 = -\frac{k}{6}t^3\left(1-\frac{k}{6}xt\right)\frac{\partial}{\partial t} + t\left(1-\frac{k}{2}xt+\frac{k^2}{9}x^2t^2-\frac{k^3}{108}x^3t^3\right)\frac{\partial}{\partial x}. \tag{12}$$

The nonlinear ODE (1) admits maximal symmetry generators and hence it is linearizable [24]. The symmetry generators constitute $sl(3, R)$ symmetry algebra. Besides several applications, the symmetry generators can also be used to reduce the order of the nonlinear ODE (1). In the following, we demonstrate this procedure by considering the vector field X_3 as an example.

Substituting the expression $\xi = x$ and $\eta = \frac{-k}{3}x^3$ in the characteristic equation $\frac{dt}{\xi} = \frac{dx}{\eta} = \frac{d\dot{x}}{\eta^{(1)}}$ and integrating the resultant equation one finds the invariants u and v as $u = t - \frac{3}{kx}$, and $v = \frac{3x}{k} + \frac{x^3}{\dot{x}}$. The second-order invariant can be derived from the relation $w = \frac{dv}{du}$. Evaluating and simplifying the resultant equation, we arrive at $\frac{dv}{du} = \frac{(kx^3+3x\dot{x})^2}{9\dot{x}^2} = \frac{k^2}{9}v^2$. Integrating this first order differential equation we find $v = -\frac{9}{9I_1+k^2u}$, where I_1 is an integration constant. Substituting the expressions u and v in this solution and rewriting the resultant equation for $\dot{x}$, we end up with

$$\dot{x} - \frac{kx(9I_1x + k^2tx - 3k)}{3(9I_1 + k^2t)} = 0. \tag{13}$$

Integrating Eq. (13) we obtain the general solution of the MEE equation in the following form,

$$x(t) = \frac{6(9I_1 + k^2t)}{kt(18I_1 + k^2t) + 6I_2}, \tag{14}$$

where I_2 is the second integration constant. In a similar manner, one can carry out the order reduction procedure for the rest of the vector fields. Since the procedure is repetitive, one can move on to investigate the other two cases.

2.2 Case 2: $\tilde{\lambda} > 0$

In this case the corresponding symmetry generators are [24]

$$
\begin{aligned}
X_1 &= \frac{\partial}{\partial t},\;\; X_2 = -\frac{k}{3\tilde{\lambda}}\cos\sqrt{\tilde{\lambda}}t\frac{\partial}{\partial t} + \left(\cos\sqrt{\tilde{\lambda}}t - \frac{k}{3\sqrt{\tilde{\lambda}}}x\sin\sqrt{\tilde{\lambda}}t\right)\frac{\partial}{\partial x},\\
X_3 &= \left(\sin 2\sqrt{\tilde{\lambda}}t + \frac{k}{3\sqrt{\tilde{\lambda}}}x\cos 2\sqrt{\tilde{\lambda}}t\right)\frac{\partial}{\partial t}\\
&\quad + x\left(\sqrt{\tilde{\lambda}}\cos 2\sqrt{\tilde{\lambda}}t - \frac{2k}{3}x\sin 2\sqrt{\tilde{\lambda}}t - \frac{k^2}{9\sqrt{\tilde{\lambda}}}x^2\cos 2\sqrt{\tilde{\lambda}}t\right)\frac{\partial}{\partial x},\\
X_4 &= \left(\cos 2\sqrt{\tilde{\lambda}}t - \frac{k}{3\sqrt{\tilde{\lambda}}}x\sin 2\sqrt{\tilde{\lambda}}t\right)\frac{\partial}{\partial t}\\
&\quad - x\left(\sqrt{\tilde{\lambda}}\sin 2\sqrt{\tilde{\lambda}}t + \frac{2k}{3}x\cos 2\sqrt{\tilde{\lambda}}t - \frac{k^2}{9\sqrt{\tilde{\lambda}}}x^2\sin 2\sqrt{\tilde{\lambda}}t\right)\frac{\partial}{\partial x},\\
X_5 &= x\sin\sqrt{\tilde{\lambda}}t\frac{\partial}{\partial t} + x^2\left(\sqrt{\tilde{\lambda}}\cos\sqrt{\tilde{\lambda}}t - \frac{k}{3}x\sin\sqrt{\tilde{\lambda}}t\right)\frac{\partial}{\partial x},\\
X_6 &= x\cos\sqrt{\tilde{\lambda}}t\frac{\partial}{\partial t} - x^2\left(\sqrt{\tilde{\lambda}}\sin\sqrt{\tilde{\lambda}}t + \frac{k}{3}x\cos\sqrt{\tilde{\lambda}}t\right)\frac{\partial}{\partial x},\\
X_7 &= x\frac{\partial}{\partial t} - \left(\frac{3\tilde{\lambda}x}{k} + \frac{k}{3}x^3\right)\frac{\partial}{\partial x},\\
X_8 &= -\frac{k}{3\tilde{\lambda}}\sin\sqrt{\tilde{\lambda}}t\frac{\partial}{\partial t} + \left(\sin\sqrt{\tilde{\lambda}}t + \frac{k}{3\sqrt{\tilde{\lambda}}}x\cos\sqrt{\tilde{\lambda}}t\right)\frac{\partial}{\partial x}.
\end{aligned}
\tag{15}
$$

The vector fields (15) can again be shown to form an $sl(3, R)$ algebra.

In the following, we demonstrate the usefulness of the symmetry vector fields by considering the vector field X_7. The associated characteristic equation reads

$$
\frac{dt}{x} = \frac{dx}{-(\frac{kx^3}{3} + \frac{3\tilde{\lambda}x}{k})} = \frac{d\dot{x}}{-\frac{\dot{x}(k^2x^2 + k\dot{x} + 3\tilde{\lambda})}{k}}. \tag{16}
$$

Now integrating the above Eq. (16) we find the invariants u and v to be of the following forms:

$$
u = \frac{\tan^{-1}\left(\frac{kx}{3\sqrt{\tilde{\lambda}}}\right) + \sqrt{\tilde{\lambda}}t}{\sqrt{\tilde{\lambda}}},\;\; v = \frac{x(k^2x^2 + 3k\dot{x} + 9\tilde{\lambda})}{\dot{x}}. \tag{17}
$$

The second-order invariant reads $\frac{dv}{du} = 9\tilde{\lambda} + \frac{v^2}{9}$. Integrating the later we find $v = 9\sqrt{\tilde{\lambda}}\tan(9I_1\sqrt{\tilde{\lambda}} + \sqrt{\tilde{\lambda}}u)$, where I_1 is an integration constant. Substituting the expressions u and v in this solution and rewriting the resultant equation for $\dot{x}$, we end up with

$$\dot{x} + \frac{k^2x^3 + 9\tilde{\lambda}x}{3\left(kx - 3\sqrt{\tilde{\lambda}}\tan\left(\sqrt{\tilde{\lambda}}(9I_1 + t) + \tan^{-1}\left(\frac{kx}{3\sqrt{\tilde{\lambda}}}\right)\right)\right)} = 0. \tag{18}$$

Integrating Eq. (18) we obtain the general solution of (1) with positive values of $\tilde{\lambda}$ in the following form,

$$x(t) = \frac{3\sqrt{\tilde{\lambda}}\sin(\sqrt{\tilde{\lambda}}(9I_1 + t))}{I_2\sqrt{\tilde{\lambda}} - k\cos(\sqrt{\tilde{\lambda}}(9I_1 + t))}, \tag{19}$$

where I_2 is the second integration constant. One may extend the order reduction procedure for the remaining vector fields too in a similar fashion. Now we move on to the third case.

2.3 *Case 3: $\tilde{\lambda} < 0$*

In this case, we find the equation is invariant under the following forms of symmetry generators: [24]

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial t}, \quad X_2 = e^{2\sqrt{\tilde{\lambda}}t}\left[\left(1 - \frac{k}{3\sqrt{\tilde{\lambda}}}x\right)\frac{\partial}{\partial t} + \left[\left(\frac{k}{9\sqrt{\tilde{\lambda}}}x^3 - \frac{2k}{3}x^2 + \sqrt{\tilde{\lambda}}x\right)\frac{\partial}{\partial x}\right],\\
X_3 &= e^{-2\sqrt{\tilde{\lambda}}t}\left[\left(1 + \frac{k}{3\sqrt{\tilde{\lambda}}}x\right)\frac{\partial}{\partial t} - \left(\frac{k}{9\sqrt{\tilde{\lambda}}}x^3 + \frac{2k}{3}x^2 + \sqrt{\tilde{\lambda}}x\right)\frac{\partial}{\partial x}\right],\\
X_4 &= x\frac{\partial}{\partial t} - \left(\frac{k}{3}x^3 - \frac{3\tilde{\lambda}}{k}x\right)\frac{\partial}{\partial x}, \quad X_5 = e^{\sqrt{\tilde{\lambda}}t}\left[x\frac{\partial}{\partial t} - \left(\frac{k}{3}x^3 - \sqrt{\tilde{\lambda}}x^2\right)\frac{\partial}{\partial x}\right],\\
X_6 &= e^{-\sqrt{\tilde{\lambda}}t}\left[x\frac{\partial}{\partial t} - \left(\frac{k}{3}x^3 + \sqrt{\tilde{\lambda}}x^2\right)\frac{\partial}{\partial x}\right], \quad X_7 = e^{\sqrt{\tilde{\lambda}}t}\left[\frac{\partial}{\partial t} - \left(\sqrt{\tilde{\lambda}}x - \frac{3\tilde{\lambda}}{k}\right)\frac{\partial}{\partial x}\right],\\
X_8 &= e^{-\sqrt{\tilde{\lambda}}t}\left[\frac{\partial}{\partial t} + \left(\sqrt{\tilde{\lambda}}x + \frac{3\tilde{\lambda}}{k}\right)\frac{\partial}{\partial x}\right].
\end{aligned} \tag{20}$$

To obtain the general solution for this case, we consider the vector field X_6. Solving the characteristic equation associated with this vector field

$$\frac{dt}{xe^{-\sqrt{\tilde{\lambda}}t}} = \frac{dx}{-e^{-\sqrt{\tilde{\lambda}}t}\left(\frac{kx^3}{3}+\sqrt{\tilde{\lambda}}x^2\right)}$$
$$= \frac{d\dot{x}}{\frac{1}{3}e^{-\sqrt{\tilde{\lambda}}t}(kx^2(\sqrt{\tilde{\lambda}}x-3\dot{x})-3(-\tilde{\lambda}x^2+\sqrt{\tilde{\lambda}}x\dot{x}+\dot{x}^2))} \quad (21)$$

we obtain the invariants u and v of the form

$$u = \frac{-\log(kx+3\sqrt{\tilde{\lambda}})+\sqrt{\tilde{\lambda}}t+\log(x)}{\sqrt{\tilde{\lambda}}}, \quad (22)$$

$$v = \frac{3(-k\sqrt{\tilde{\lambda}}x^2+2kx\dot{x}-3\tilde{\lambda}x+3\sqrt{\tilde{\lambda}}\dot{x})}{2(kx^4+3\sqrt{\tilde{\lambda}}x^3+3x^2\dot{x})}. \quad (23)$$

The second-order invariant can be found from the relation $w = \frac{dv}{du}$. In this case, we find

$$\frac{dv}{du} = -\frac{(kx+3\sqrt{\tilde{\lambda}})^2(kx^2-3\sqrt{\tilde{\lambda}}x+3\dot{x})}{3x^2(kx^2+3(\sqrt{\tilde{\lambda}}x+\dot{x}))} = -\left(\frac{k^2}{3}+2\sqrt{\tilde{\lambda}}v\right). \quad (24)$$

Integrating Eq. (24), we find $v = e^{-2\sqrt{\tilde{\lambda}}u}I_1 - \frac{k^2}{6\sqrt{\tilde{\lambda}}}$, where I_1 is an integration constant. Substituting the expressions u and v in this solution and rewriting the resultant equation for $\dot{x}$, we end up with

$$\dot{x} + \frac{x(e^{2\sqrt{\tilde{\lambda}}t}(kx-3\sqrt{\tilde{\lambda}})-6I_1(k\sqrt{\tilde{\lambda}}x+3\tilde{\lambda}))}{3(e^{2\sqrt{\tilde{\lambda}}t}-6I_1\sqrt{\tilde{\lambda}})} = 0. \quad (25)$$

Integrating Eq. (25), we obtain the general solution of Eq. (1) with negative $\tilde{\lambda}$ in the following form

$$x(t) = \frac{3(\sqrt{\tilde{\lambda}}e^{2\sqrt{\tilde{\lambda}}t}-6I_1\tilde{\lambda})}{3I_2\sqrt{\tilde{\lambda}}e^{\sqrt{\tilde{\lambda}}t}+6I_1\sqrt{\tilde{\lambda}}k+ke^{2\sqrt{\tilde{\lambda}}t}}, \quad (26)$$

where I_2 is the second integration constant. One may verify that the remaining vector fields can also be used to derive the above general solution of the given Eq. (1).

3 λ-Symmetries

Recently efforts have been made to generalize the classical Lie algorithm and obtain integrals and general solution of nonlinear ODEs, in particular equations which lack Lie point symmetries. One such generalization is the λ-symmetry approach [17].

The method of finding λ-symmetries for a second-order ODE has been discussed in depth by Muriel and Romero [18] and the advantage of finding such symmetries has also been demonstrated by them. They also have developed an algorithm to determine integrating factors and integrals from λ-symmetries for second-order ODEs [19]. The relation among λ-symmetries, Lie point symmetries and local-nonlocal transformations for Liénard I and II-type equations was studied in Ref. [25]. The vector fields associated with λ-symmetries are being denoted as v instead of X just to differentiate λ-symmetries from Lie point symmetry vector fields.

A vector field v is a λ-symmetry of the second-order equation if there exists a function such that

$$v^{[\lambda,(2)]}(\Delta(t,x,\dot{x},\ddot{x})) = 0 \;\; when \;\; \Delta(t,x,\dot{x},\ddot{x}) = 0, \tag{27}$$

where $v^{[\lambda,(2)]}$ is given by

$$v^{[\lambda,(2)]} = \xi(t,x)\frac{\partial}{\partial t} + \eta^{[\lambda,(0)]}(t,x)\frac{\partial}{\partial x} + \eta^{[\lambda,(1)]}(t,x,\dot{x})\frac{\partial}{\partial \dot{x}} + \eta^{[\lambda,(2)]}(t,x,\dot{x},\ddot{x})\frac{\partial}{\partial \ddot{x}}, \tag{28}$$

with

$$\eta^{[\lambda,(0)]} = \eta(t,x), \tag{29}$$

$$\eta^{[\lambda,(1)]} = (D_t+\lambda)\eta^{[\lambda,(0)]}(t,x) - (D_t+\lambda)(\xi)\dot{x}, \tag{30}$$

$$\eta^{[\lambda,(2)]} = (D_t+\lambda)\eta^{[\lambda,(1)]}(t,x,\dot{x}) - (D_t+\lambda)(\xi)\ddot{x}. \tag{31}$$

In the above prolongation formula if we put $\lambda = 0$, we end up with standard Lie prolongation expressions. Solving the invariance condition (27) we can determine the functions ξ, η and λ for the given equation. We note here that three unknowns ξ, η and λ have to be determined from the invariance condition (27). The procedure is as follows.

Let us suppose that the second-order Eq. (1) has Lie point symmetries. In this case, the λ-function can be determined in a more simple way without solving the invariance condition (27) as follows. If X is a Lie point symmetry of (1) and $Q = \eta - \dot{x}\xi$ is its characteristics, then $v = \frac{\partial}{\partial x}$ is a λ-symmetry of (1) for $\lambda = \frac{D[Q]}{Q}$ [25]. The λ-symmetry satisfies the invariance condition [19]

$$\phi_x + \lambda\phi_{\dot{x}} = D[\lambda] + \lambda^2. \tag{32}$$

Once the λ-symmetry is determined, we can obtain the first integrals in two different ways. In the first way, we can calculate the integral directly from the λ-symmetry using the four step algorithm given below. In the second way, we can find the integrating factor μ from λ-symmetry directly. With the help of integrating factors and λ-symmetries we can obtain the first integrals by integrating the system of Eq. (34) given below. In the following, we enumerate both the procedures.

(A) Method of finding the first integral directly from λ-symmetry [19]

The method of finding the integral directly from λ-symmetry is as follows:

1. Find a first integral $w(t, x, \dot{x})$ of $v^{[\lambda,(1)]}$, that is a particular solution of the equation $w_x + \lambda w_{\dot{x}} = 0$, where the subscript denotes partial derivative with respect to that variable and $v^{[\lambda,(1)]}$ is the first-order λ-prolongation of the vector field v.
2. Evaluate $D[w]$ and express it in terms of (t, w) as $D[w] = F(t, w)$.
3. Find a first integral G of $\partial_t + F(t, w)\partial_w$.
4. Evaluate $I(t, x, \dot{x}) = G(t, w(t, x, \dot{x}))$.

(B) Method of finding integrating factors from λ [19]

If X is a Lie point symmetry of (1) and $Q = \eta - \dot{x}\xi$ is its characteristics, then $v = \partial_x$ is a λ-symmetry of (1) for $\lambda = D[Q]/Q$ and any solution of the first-order linear system

$$D[\mu] + \left(\phi_{\dot{x}} - \frac{D[Q]}{Q}\right)\mu = 0, \quad \mu_x + \left(\frac{D[Q]}{Q}\mu\right)_{\dot{x}} = 0, \tag{33}$$

is an integrating factor of (1). Here D represents the total derivative operator and it is given by $\frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \phi\frac{\partial}{\partial \dot{x}}$.

Solving the system of equations (33) one can get μ. Once the integrating factor μ is known then a first integral I such that $I_{\dot{x}} = \mu$ can be found by solving the system of equations

$$I_t = \mu(\lambda\dot{x} - \phi), \quad I_x = -\lambda\mu, \quad I_{\dot{x}} = \mu. \tag{34}$$

From the first integrals, we can write the general solution of the given equation.

In the following we apply the above method to Eq. (1)

3.1 Case 1: $\tilde{\lambda} = 0$

Bhuvaneshwari et al. had studied the λ-symmetries for Eq. (1) with $\tilde{\lambda} = 0$ [1]. They have found the λ-symmetries from the Lie point symmetries by using the relation $\lambda = \frac{D[Q]}{Q}$, where $Q = \eta - \dot{x}\xi$. For this purpose they considered the Lie point symmetries X_2 and X_4 from Eq. (12). The expressions for Q turns out to be

$$Q_1 = \frac{1}{18}(k^2t^2x^3 - 6ktx^2 + 3kt^2x\dot{x} - 18t\dot{x}), \;\; Q_2 = x^2\left(1 - \frac{k}{3}xt\right) - tx\dot{x}. \tag{35}$$

The two λ-functions are of the form

$$\lambda_1 = \frac{\left(1 - \frac{2}{3}ktx - \frac{k}{6}t^2\dot{x} + \frac{k^2}{18}t^2x^2\right)}{t\left(1 - \frac{k}{6}tx\right)}, \ \lambda_2 = \frac{\dot{x}}{x} - \frac{kx}{3}. \tag{36}$$

The associated λ-symmetry is $v = \frac{\partial}{\partial x}$.

3.1.1 First Integrals from λ_1 and λ_2

By following the above discussed procedure, we have found

$$w(t, x, \dot{x}) = \frac{\left(1 - \frac{1}{3}ktx + \frac{k}{6}t^2\dot{x} + \frac{k^2}{18}t^2x^2\right)}{kt^2\left(1 - \frac{k}{6}tx\right)}. \tag{37}$$

In the second step, we obtain determining equation for w as

$$D[w] = ktw^2 - \frac{3w}{t} \tag{38}$$

using λ_1. In the third step, we obtain the function $G(t, w)$ as

$$G(t, w) = \frac{1}{t^3 w} - \frac{k}{t}. \tag{39}$$

In the final step, we found the integral I_1 as

$$I_1 = \frac{k}{6}t + \frac{\left(1 - \frac{k}{6}tx\right)}{\left(t\dot{x} - x + \frac{k}{3}tx^2\right)}. \tag{40}$$

In the same way, we have found the function w for λ_2 as

$$w(x, \dot{x}) = \frac{\dot{x}}{x} + \frac{k}{3}x. \tag{41}$$

In the second step, we get the determining equation as

$$D[w] + w^2 = 0. \tag{42}$$

We get the function G in the third step as

$$G = t - \frac{1}{w}. \tag{43}$$

As the final step we get the integral as

$$I_2 = t - \frac{x}{\dot{x} + \frac{k}{3}x^2}. \tag{44}$$

From the integrals I_1 and I_2, we can write the general solution as

$$x(t) = \frac{t + I_2}{\frac{k}{6}t^2 + \frac{k}{3}tI_2 - I_1 I_2}. \tag{45}$$

3.1.2 Integrating Factors from λ_1 and λ_2

We can also find the integrating factors from λ_1 and λ_2 using the relation (33). Substituting the function λ_1 in Eq. (33) we get

$$\mu_{1x} + \left(\frac{\left(1 - \frac{2}{3}ktx - \frac{k}{6}t^2\dot{x} + \frac{k^2}{18}t^2x^2\right)}{t\left(1 - \frac{k}{6}tx\right)} \mu_1 \right)_{\dot{x}} = 0. \tag{46}$$

The characteristic equation associated with Eq. (46) is given by

$$\frac{dx}{1} = \frac{d\dot{x}}{\frac{\left(1 - \frac{2}{3}ktx - \frac{k}{6}t^2\dot{x} + \frac{k^2}{18}t^2x^2\right)}{t\left(1 - \frac{k}{6}tx\right)}} = \frac{d\mu_1}{-\frac{kt^2}{6\left(t - \frac{1}{6}kt^2x\right)}\mu_1}. \tag{47}$$

Integrating (47) we find the integrals to be of the form

$$C_1 = \frac{\left(1 - \frac{1}{3}ktx + \frac{k}{6}t^2\dot{x} + \frac{k^2}{18}t^2x^2\right)}{kt^2\left(1 - \frac{k}{6}tx\right)}, \quad C_2 = \left(t - \frac{1}{6}kt^2x\right)\mu_1. \tag{48}$$

From the above, we obtain the general solution as

$$\mu_1 = -\frac{C_1\left[t\left(\frac{\left(1 - \frac{1}{3}ktx + \frac{k}{6}t^2\dot{x} + \frac{k^2}{18}t^2x^2\right)}{kt^2\left(1 - \frac{k}{6}tx\right)}\right)\right]}{\left(t - \frac{1}{6}kt^2x\right)}. \tag{49}$$

Choosing the function C_1 appropriately we get

$$\mu_1 = -\frac{k^2t\left(1 - \frac{k}{6}tx\right)}{6\left(1 - \frac{1}{3}ktx + \frac{k}{6}t^2\dot{x} + \frac{k^2}{18}t^2x^2\right)^2}. \tag{50}$$

We find that the expression (50) also satisfies the first equation in (33) as well and thus forms a compatible solution to the system of equation (33).

To determine the integrating factor associated with λ_2 directly we first solve the second equation in (33), that is

$$\mu_{2x} + \left(\frac{\dot{x}}{x} - \frac{k}{3}x\right)\mu_{2\dot{x}} + \frac{1}{x}\mu_2 = 0. \tag{51}$$

The characteristic equation associated with the above equation can be written as

$$\frac{dx}{1} = \frac{d\dot{x}}{\frac{\dot{x}}{x} - \frac{k}{3}x} = \frac{d\mu_2}{\frac{\mu_2}{x}}. \tag{52}$$

Integrating (52) we find the integral as

$$\mu_2 = -\frac{x}{\left(\dot{x} + \frac{k}{3}x^2\right)^2}. \tag{53}$$

We find that the above expression also satisfies the first equation in (33).

3.2 Case 2: $\tilde{\lambda} \neq 0$

In the earlier case where $\tilde{\lambda} = 0$, we fixed the λ-symmetries from the set of Lie point symmetries itself. For the two cases $\tilde{\lambda} > 0$ and $\tilde{\lambda} < 0$ we derive the λ-symmetries by solving the associated invariance condition which has not been considered so far for this equation. To determine the λ-symmetry for Eq. (1), we solve the following determining equation

$$D[\lambda] + \lambda^2 + \lambda kx + k\dot{x} + \frac{k^2}{3}x^2 + \tilde{\lambda} = 0. \tag{54}$$

To obtain a particular solution of Eq. (54), we assume an ansatz

$$\lambda = a_1\dot{x} + a_2, \tag{55}$$

where a_1 and a_2 are functions of x.

Substituting (55) in (54) and solving the resultant equation, we find

$$\lambda_1 = \frac{\dot{x}}{x} - \frac{kx}{3}. \tag{56}$$

Now we use the above said procedure and obtain the first integral. The calculations are given below.

In the first step, we setup the determining equation for $w(t, x, \dot{x})$, that is

$$w_x + \left(\frac{\dot{x}}{x} - \frac{kx}{3}\right) w_{\dot{x}} = 0. \tag{57}$$

A particular solution of (57) is

$$w(t, x, \dot{x}) = \frac{kx}{3} + \frac{\dot{x}}{x}. \tag{58}$$

In the second step, we express $D[w]$ in terms of (t, w) as $D[w] = F(t, w)$. In this case, we find

$$D[w] = -(w^2 + \tilde{\lambda}). \tag{59}$$

In the third step, we fix the function $G(t, w)$ as

$$G(t, w) = \frac{\sqrt{\tilde{\lambda}}t + \tan^{-1}\left(\frac{w}{\sqrt{\tilde{\lambda}}}\right)}{\sqrt{\tilde{\lambda}}}. \tag{60}$$

Now replacing w with the expression (58) we obtain the first integral in the form

$$I(t, x, \dot{x}) = \frac{\tan^{-1}\left(\frac{kx^2+3\dot{x}}{3\sqrt{\tilde{\lambda}}x}\right) + \sqrt{\tilde{\lambda}}t}{\sqrt{\tilde{\lambda}}}. \tag{61}$$

By recalling the formula $\arctan(x) = \frac{1}{2}i[\ln(1 - ix) - ln(1 + ix)]$ and simplifying the resultant equation we obtain the first integral as

$$I_1 = e^{-2\sqrt{-\tilde{\lambda}}t} \frac{\left(\dot{x} + \frac{k}{3}x^2 + x\sqrt{-\tilde{\lambda}}\right)}{\left(\dot{x} + \frac{k}{3}x^2 - x\sqrt{-\tilde{\lambda}}\right)}. \tag{62}$$

To prove the integrability of Eq. (1), we are in need of one more λ-symmetry. To obtain it, we assume a more general ansatz for λ which is of the form

$$\lambda_2 = \frac{a_1(t, x)\dot{x} + a_2(t, x)}{a_3(t, x)\dot{x} + a_4(t, x)}. \tag{63}$$

where a_1, a_2, a_3 and a_4 are arbitrary functions of t and x and to be determined. Substituting the above ansatz in the λ-determining Eq. (54) and solving the resultant equation, we obtain

$$\lambda_2 = \frac{\frac{k\dot{x}}{3} - \left(\frac{kx}{3} + \sqrt{-\tilde{\lambda}}\right)^2}{\frac{kx}{3} + \sqrt{-\tilde{\lambda}}}. \tag{64}$$

We note here that while solving the Eq. (54) with the ansatz (63) we also obtain (56) as another particular solution. We do not mention it here as we have already dealt with it. Following the above said procedure now we find the integral associated with $\tilde{\lambda_2}$. To begin it, we set up the determining equation for $w(t, x, \dot{x})$ as

$$w_x + \frac{\frac{k\dot{x}}{3} - \left(\frac{kx}{3} + \sqrt{-\tilde{\lambda}}\right)^2}{\frac{kx}{3} + \sqrt{-\tilde{\lambda}}} w_{\dot{x}} = 0. \tag{65}$$

A particular solution of the above equation is

$$w(t, x, \dot{x}) = \frac{kx^2 + 3\sqrt{-\tilde{\lambda}}x + 3\dot{x}}{3(kx + 3\sqrt{-\tilde{\lambda}})}. \tag{66}$$

The total derivative of $w(t, x, \dot{x})$ reads

$$D[w] = \sqrt{-\tilde{\lambda}}w - kw^2. \tag{67}$$

In the third step, we determine the function $G(t, w)$ as

$$G(t, w) = -\frac{i \log\left(\frac{e^{i\sqrt{\tilde{\lambda}}t}\left(kw - i\sqrt{\tilde{\lambda}}\right)}{w}\right)}{\sqrt{\tilde{\lambda}}}. \tag{68}$$

Now replacing the variable w by (66) we obtain the integral associated with $\tilde{\lambda}_2$ in the form

$$I(t, x, \dot{x}) = -\frac{i \log\left(\frac{e^{i\sqrt{\tilde{\lambda}}t}(k(3\dot{x} + kx^2) + 9\tilde{\lambda})}{kx^2 + 3\left(\sqrt{-\tilde{\lambda}}x + \dot{x}\right)}\right)}{\sqrt{\tilde{\lambda}}}. \tag{69}$$

After rearranging the integral in more elegant form, we obtain

$$I_2 = -\frac{6}{k}e^{\sqrt{-\tilde{\lambda}}t}\left(\frac{\tilde{\lambda} + \frac{k}{3}\dot{x} + \frac{k^2}{9}x^2}{\dot{x} + \frac{k}{3}x^2 + x\sqrt{-\tilde{\lambda}}}\right). \tag{70}$$

From the integrals I_1 and I_2, we can write the solution of Eq. (1) for $\tilde{\lambda} > 0$ and $\tilde{\lambda} < 0$. First let us consider the case $\tilde{\lambda} > 0$.

3.2.1 Case 2: $\tilde{\lambda} > 0$

For $\tilde{\lambda} > 0$, integrals (62) and (70) are complex. To get the real integrals, we consider the following combinations of the integrals

$$\tilde{I}_1 = \frac{4}{kI_1^2I_2^2} = \frac{(3\dot{x} + kx^2)^2 + 9\tilde{\lambda}x^2}{(3k\dot{x} + k^2x^2 + 9\tilde{\lambda})^2}, \tag{71}$$

$$\tilde{I}_2 = -\frac{2e^{i\delta}}{k|I_1I_2|} = e^{i(\sqrt{\tilde{\lambda}}t+\delta)}\left(\frac{3\dot{x} + kx^2 - 3i\sqrt{\tilde{\lambda}}x}{3k\dot{x} + k^2x^2 + 9\tilde{\lambda}}\right), \tag{72}$$

where δ is phase constant. Now the integrals $\tilde{I}_1$ and $|\tilde{I}_2|$ can be considered as two real integrals of Eq. (1) for $\tilde{\lambda} > 0$. The solution for Eq. (1) from the two integrals (62) and (70) can be written as

$$x(t) = \frac{A\sin(\sqrt{\tilde{\lambda}}t + \delta)}{1 - \frac{k}{3\sqrt{\tilde{\lambda}}}A\cos(\sqrt{\tilde{\lambda}}t + \delta)}, \qquad 0 \le A < \frac{3\sqrt{\tilde{\lambda}}}{k}, \tag{73}$$

where $A = 3\sqrt{\tilde{\lambda}\tilde{I}_1}$ and δ is an arbitrary constant.

3.2.2 Case 3: $\tilde{\lambda} < 0$

For $\tilde{\lambda} < 0$, integrals (62) and (70) are real from which we can straightforwardly write the general solution as

$$x(t) = \frac{3\sqrt{|\tilde{\lambda}|(\tilde{I}_1e^{2\sqrt{|\tilde{\lambda}|t}} - 1)}}{k\tilde{I}_1\tilde{I}_2e^{\sqrt{|\tilde{\lambda}|t}} + k(1 + \tilde{I}_1e^{2\sqrt{|\tilde{\lambda}|t}})}, \tag{74}$$

where I_1 and I_2 are constants.

3.2.3 Integrating Factors from λ_1 and λ_2

To find the integrating factors from λ_1 and λ_2, we consider the second equation in (33) and obtain

$$\mu_{1x} + \left(\frac{\dot{x}}{x} - \frac{k}{3}x\right)\mu_{1\dot{x}} + \frac{1}{x}\mu_1 = 0, \tag{75}$$

$$\mu_{2x} + \left(\frac{\frac{k\dot{x}}{3} - \left(\frac{kx}{3} + \sqrt{-\tilde{\lambda}}\right)^2}{\frac{kx}{3} + \sqrt{-\tilde{\lambda}}}\right)\mu_{1\dot{x}} + \frac{k}{3\left(\frac{kx}{3} + \sqrt{-\tilde{\lambda}}\right)}\mu_2 = 0. \tag{76}$$

The characteristic equations associated with the above equations can be written as

$$\frac{dx}{1} = \frac{d\dot{x}}{\frac{\dot{x}}{x} - \frac{k}{3}x} = \frac{d\mu_1}{\frac{\mu_1}{x}}, \tag{77}$$

$$\frac{dx}{1} = \frac{d\dot{x}}{\frac{\frac{k\dot{x}}{3} - \left(\frac{kx}{3} + \sqrt{-\tilde{\lambda}}\right)^2}{\frac{kx}{3} + \sqrt{-\tilde{\lambda}}}} = \frac{d\mu_2}{\frac{3\mu_2\left(\frac{kx}{3} + \sqrt{-\tilde{\lambda}}\right)}{k}}. \tag{78}$$

Solving the above characteristic equations and choosing the constants appropriately, we obtain the solutions of the above equations as

$$\mu_1 = -\frac{18\sqrt{-\tilde{\lambda}}xe^{-2\sqrt{-\tilde{\lambda}}t}}{(kx^2 - 3\sqrt{-\tilde{\lambda}}x + 3\dot{x})^2}, \tag{79}$$

$$\mu_2 = -\frac{18e^{\sqrt{-\tilde{\lambda}}t}(k\sqrt{-\tilde{\lambda}}x - 3\tilde{\lambda})}{k(kx^2 + 3(\sqrt{-\tilde{\lambda}}x + \dot{x}))^2}. \tag{80}$$

The above integrating factors also satisfy the first equation of Eq. (33).

4 Noether's Theorem and Variational Symmetries

If the given second-order equation has a variational structure then one can also determine the symmetries which leave the action integral invariant. Such symmetries are called variational symmetries. Variational symmetries are important since they provide conservation laws via Noether's theorem [21]. In the following, we recall the method of finding variational symmetries [2, 23].

Noether's theorem states that whenever the action integral $S = \int L(t, x, \dot{x})dt$, where L is the Lagrangian, is invariant under the one parameter continuous group of transformations (7) then the solution of Euler's equation admit the conserved quantity [11, 14],

$$I = (\xi\dot{x} - \eta)\frac{\partial L}{\partial \dot{x}} - \xi L + f, \tag{81}$$

where f is an arbitrary function of t and x. The functions ξ, η and f can be determined from the equation

$$G\{L\} = -\dot{\xi} L + f, \tag{82}$$

where overdot denotes differentiation with respect to time and

$$G\{L\} = \xi \frac{\partial L}{\partial t} + \eta \frac{\partial L}{\partial x} + (\dot{\eta} - \dot{x}\dot{\xi}) \frac{\partial L}{\partial \dot{x}}. \tag{83}$$

Equation (83) can be derived by differentiating the Eq. (81) and simplifying the expressions in the resultant equation. Solving Eq. (83) one can obtain explicit expressions for Noether's symmetries ξ, η and the arbitrary function f. Now substituting these expressions into (81) one can get explicitly the associated integrals of motion.

To derive Noether's symmetries associated with the Lagrangian (2) let us substitute the expression (2) into (81). Doing so we get

$$\begin{aligned} &\eta \left(\frac{-9\tilde{\lambda}^3 x}{(k\dot{x} + \frac{k^2}{3}x^2 + 3\tilde{\lambda})^2} \right) + (\eta_t + \dot{x}\eta_x - \dot{x}(\xi_t + \dot{x}\xi_x)) \left(\frac{-27\tilde{\lambda}^3}{2k(k\dot{x} + \frac{k^2}{3}x^2 + 3\tilde{\lambda})^2} + \frac{3\tilde{\lambda}}{2k} \right) \\ &\quad = -(\xi_t + \dot{x}\xi_x) \left(\frac{27\tilde{\lambda}^3}{2k^2} \left(\frac{1}{k\dot{x} + \frac{k^2}{3}x^2 + 3\tilde{\lambda}} \right) + \frac{3\tilde{\lambda}}{2k}\dot{x} - \frac{9\tilde{\lambda}^2}{2k^2} \right) + f_t + \dot{x} f_x. \end{aligned} \tag{84}$$

Now equating the coefficient of various powers of $\dot{x}$ to zero and solving the resultant equations we obtain three different forms of infinitesimal symmetries for ξ and η depending upon the sign and magnitude of $\tilde{\lambda}$. In the following, we discuss each one of the cases separately.

4.1 Case 1: $\tilde{\lambda} > 0$

Solving the determining equations with $\tilde{\lambda} > 0$, the associated vector fields turn out to be

$$\begin{aligned} X_1 &= \frac{1}{\tilde{\lambda}^{\frac{5}{2}}} \left[\left(\sqrt{\tilde{\lambda}} \sin \sqrt{\tilde{\lambda}} t + \frac{k}{3} \cos \sqrt{\tilde{\lambda}} t x \right) \frac{\partial}{\partial t} \right. \\ &\quad \left. - \left(\frac{3\tilde{\lambda}}{k} + \frac{kx^2}{3} \right) \left(\sqrt{\tilde{\lambda}} \sin \sqrt{\tilde{\lambda}} t + \frac{k}{3} \cos \sqrt{\tilde{\lambda}} t x \right) \frac{\partial}{\partial x} \right], \\ X_2 &= \frac{1}{\tilde{\lambda}^{\frac{5}{2}}} \left[\left(\frac{k}{3} \sin \sqrt{\tilde{\lambda}} t x - \sqrt{\tilde{\lambda}} \cos \sqrt{\tilde{\lambda}} t \right) \frac{\partial}{\partial t} \right. \\ &\quad \left. - \left(\frac{3\tilde{\lambda}}{k} + \frac{kx^2}{3} \right) \left(\frac{k}{3} \sin \sqrt{\tilde{\lambda}} t x - \sqrt{\tilde{\lambda}} \cos \sqrt{\tilde{\lambda}} t \right) \frac{\partial}{\partial x} \right], \\ X_3 &= \frac{1}{\tilde{\lambda}^{\frac{5}{2}}} \left[\left(3\sqrt{\tilde{\lambda}} \sin 2\sqrt{\tilde{\lambda}} t + k \cos 2\sqrt{\tilde{\lambda}} t x \right) \frac{\partial}{\partial t} \right. \end{aligned}$$

$$- \left(\left(\frac{k^2x^2}{3} - 3\tilde{\lambda} \right) \cos 2\sqrt{\tilde{\lambda}}tx + 2k\sqrt{\tilde{\lambda}} \sin 2\sqrt{\tilde{\lambda}}tx^2 \right) \frac{\partial}{\partial x} \Bigg],$$

$$X_4 = \frac{1}{\tilde{\lambda}^{\frac{5}{2}}} \Bigg[\cos\sqrt{\tilde{\lambda}}t \left(k \sin\sqrt{\tilde{\lambda}}tx - 3\sqrt{\tilde{\lambda}} \cos\sqrt{\tilde{\lambda}}t \right) \frac{\partial}{\partial t}$$
$$- \left(\frac{k}{3} \sin\sqrt{\tilde{\lambda}}tx - \sqrt{\tilde{\lambda}} \cos\sqrt{\tilde{\lambda}}t \right) \left(3\sqrt{\tilde{\lambda}} \sin\sqrt{\tilde{\lambda}}tx + k \cos\sqrt{\tilde{\lambda}}tx^2 \right) \frac{\partial}{\partial x} \Bigg],$$

$$X_5 = \frac{\partial}{\partial t}. \tag{85}$$

Substituting each vector field into (81) we obtain the following integrals of motion

$$I_1 = \frac{(3\dot{x} + kx^2)\cos\sqrt{\tilde{\lambda}}t + 3\sqrt{\tilde{\lambda}}x \sin\sqrt{\tilde{\lambda}}t}{\alpha}, \quad I_2 = \frac{(3\dot{x} + kx^2)\sin\sqrt{\tilde{\lambda}}t - 3\sqrt{\tilde{\lambda}}x \cos\sqrt{\tilde{\lambda}}t}{\alpha},$$

$$I_3 = \frac{\left((3\dot{x} + kx^2)^2 - 9\tilde{\lambda}x^2 \right) \sin 2\sqrt{\tilde{\lambda}}t - 6x(3\dot{x} + kx^2)\sqrt{\tilde{\lambda}} \cos 2\sqrt{\tilde{\lambda}}t}{(\alpha)^2},$$

$$I_4 = \frac{k^2((3\dot{x} + kx^2)^2 - 9k^2\tilde{\lambda}x^2)\cos 2\sqrt{\tilde{\lambda}}t + 6k^2x(3\dot{x} + kx^2)\sqrt{\tilde{\lambda}} \sin 2\sqrt{\tilde{\lambda}}t - 9\tilde{\lambda}(k^2x^2 + 6k\dot{x} + 9\tilde{\lambda})}{(\alpha)^2},$$

$$I_5 = \left(\frac{(3\dot{x} + kx^2)^2 + 9\tilde{\lambda}x^2}{(\alpha)^2} \right), \tag{86}$$

where $\alpha = 3k\dot{x} + k^2x^2 + 9\tilde{\lambda}$.

One can select two independent integrals of motions, I_1 and I_2 from the above. The remaining integrals of motions can be written in terms of the integrals of motion I_1 and I_2. For example, in the present case we get

$$I_3 = 2I_1I_2, \quad I_4 = -1 + 2k^2I_1^2, \quad I_5 = (I_1^2 + I_2^2). \tag{87}$$

Using I_1 and I_2 we can construct the general solution in the form

$$x(t) = 3\sqrt{\tilde{\lambda}} \left(\frac{I_1 \sin\sqrt{\tilde{\lambda}}t - I_2 \cos\sqrt{\tilde{\lambda}}t}{1 - k(I_1 \cos\sqrt{\tilde{\lambda}}t + I_2 \sin\sqrt{\tilde{\lambda}}t)} \right). \tag{88}$$

The above solution is obviously equivalent to (73). Since Eq. (1) admits five Noether's symmetries for the case $\tilde{\lambda} > 0$ and so the Lagrangian (2) can be considered as a physically important Lagrangian from Quantum Mechanics point of view.

4.2 Case 2: $\tilde{\lambda} < 0$

Solving the determining equations with $\tilde{\lambda} < 0$, its associated symmetry vector fields turn out to be

$$X_1 = \frac{1}{\tilde{\lambda}^{\frac{5}{2}}}\left[e^{\sqrt{\tilde{\lambda}}t}\left(\left(\sqrt{\tilde{\lambda}} - \frac{kx}{3}\right)\frac{\partial}{\partial t} + \left(\frac{\sqrt{\tilde{\lambda}}}{k} - \frac{x}{3}\right)\left(3\tilde{\lambda} - \frac{k^2x^2}{3}\right)\frac{\partial}{\partial x}\right)\right],\ X_2 = \frac{\partial}{\partial t}$$
$$X_3 = \frac{1}{\tilde{\lambda}^{\frac{5}{2}}}\left[e^{-\sqrt{\tilde{\lambda}}t}\left(\left(\sqrt{\tilde{\lambda}} + \frac{kx}{3}\right)\frac{\partial}{\partial t} + \left(\frac{\sqrt{\tilde{\lambda}}}{k} + \frac{x}{3}\right)\left(3\tilde{\lambda} - \frac{k^2x^2}{3}\right)\frac{\partial}{\partial x}\right)\right],$$
$$X_4 = \frac{1}{\tilde{\lambda}^{\frac{5}{2}}}\left[e^{2\sqrt{\tilde{\lambda}}t}\left(\left(\sqrt{\tilde{\lambda}} - \frac{kx}{3}\right)\frac{\partial}{\partial t} + x\left(\tilde{\lambda} + \frac{k^2x^2}{9} - \frac{2}{3}kx\sqrt{\tilde{\lambda}}\right)\frac{\partial}{\partial x}\right)\right],$$
$$X_5 = \frac{1}{\tilde{\lambda}^{\frac{5}{2}}}\left[e^{-2\sqrt{\tilde{\lambda}}t}\left(\left(\sqrt{\tilde{\lambda}} + \frac{kx}{3}\right)\frac{\partial}{\partial t} - x\left(\tilde{\lambda} + \frac{k^2x^2}{9} + \frac{2}{3}kx\sqrt{\tilde{\lambda}}\right)\frac{\partial}{\partial x}\right)\right]. \quad (89)$$

Substituting each vector field into (81) we obtain the following integrals of motion,

$$I_1 = ke^{a_2t}\left(\frac{3\dot{x} - 3a_2x + kx^2}{\alpha}\right),\ I_2 = ke^{-a_2t}\left(\frac{3\dot{x} + 3a_2x + kx^2}{\alpha}\right),$$
$$I_3 = k^2e^{2a_2t}\left[\frac{(-6ka_2x^3 + k^2x^4 - 18a_2x\dot{x} + 9\dot{x}^2 + x^2(-9\tilde{\lambda} + 6k\dot{x}))}{(\alpha)^2}\right],$$
$$I_4 = k^2e^{-2a_2t}\left[\frac{(6ka_2x^3 + k^2x^4 + 18a_2x\dot{x} + 9\dot{x}^2 + x^2(-9\tilde{\lambda} + 6k\dot{x}))}{(\alpha)^2}\right],$$
$$I_5 = \frac{9\tilde{\lambda}^2}{2}\left(\frac{(3\dot{x} + kx^2)^2 + 9\tilde{\lambda}x^2}{(\alpha)^2}\right). \quad (90)$$

As in the previous case the integrals of motions, I_1 and I_2 are functionally independent from the rest. In other words the remaining integrals of motions can be written in terms of the integrals of motion I_1 and I_2:

$$I_3 = I_1^2, \quad I_4 = I_2^2, \quad I_5 = \frac{9\tilde{\lambda}^2}{2k^2}I_1I_2. \quad (91)$$

Using I_1 and I_2 we can construct the general solution in the form

$$x(t) = \frac{3a_2(I_1e^{2a_2t} - I_2)}{k(I_2 + I_1e^{2a_2t} - 2e^{a_2t})}. \quad (92)$$

The above solution is of front like nature and in this case also we have five Noether's symmetries. The underlying Lagrangian (2) is again a physically important one.

4.3 *Case 3:* $\tilde{\lambda} = 0$

One may note that in the limit $\tilde{\lambda} = 0$, Eq. (1) becomes the modified Emden equation/second order Ricatti equation which is another Liénard type system which possesses several interesting properties. Interestingly, this system also admits a time independent Lagrangian and Hamiltonian. In the following, we present the Noether's symmetries and their associated constants of motions.

The Lagrangian associated with the MEE equation is,

$$L = \frac{1}{k\dot{x} + \frac{k^2}{3}x^2}. \tag{93}$$

Solving the determining Eq. (83) with $\tilde{\lambda} = 0$, we get

$$\begin{aligned}
X_1 &= x\frac{\partial}{\partial t} - \frac{kx^2}{3}\frac{\partial}{\partial x},\ X_2 = xt\frac{\partial}{\partial t} + \left(x^2 - \frac{ktx^3}{3}\right)\frac{\partial}{\partial x},\ X_3 = \frac{\partial}{\partial t},\\
X_4 &= \left(t - \frac{kt^2x}{2}\right)\frac{\partial}{\partial t} + \left(2x - ktx^2 + \frac{k^2t^2x^3}{6}\right)\frac{\partial}{\partial x},\\
X_5 &= \left(\frac{k^2t^3x}{18} - \frac{kt^2}{6}\right)\frac{\partial}{\partial t} + \left(1 - \frac{2ktx}{3} + \frac{k^2t^2x^2}{6} - \frac{k^3t^3x^3}{54}\right)\frac{\partial}{\partial x}.
\end{aligned} \tag{94}$$

Substituting each vector field into (81) we obtain the following integrals of motion,

$$\begin{aligned}
I_1 &= t - \frac{3x}{kx^2 + 3\dot{x}},\ I_2 = \frac{(-3x + ktx^2 + 3t\dot{x})^2}{(kx^2 + 3\dot{x})^2},\\
I_3 &= \frac{-9k^2t^2x^3 + k^3t^3x^4 - 27x(2 + kt^2\dot{x}) + 6ktx^2(6 + kt^2\dot{x}) + 9t\dot{x}(6 + kt^2\dot{x})}{(kx^2 + 3\dot{x})^2},\\
I_4 &= \frac{(18 - 6ktx + k^2t^2x^2 + 3kt^2\dot{x})}{(kx^2 + 3\dot{x})},\ I_5 = \frac{6\dot{x} + kx^2}{(kx^2 + 3\dot{x})^2}.
\end{aligned} \tag{95}$$

One can easily check that out of the five integrals of motions two are independent and the remaining three can be expressed in terms of the first two, namely

$$I_2 = I_1^2, \quad I_3 = I_1 I_4, \quad I_5 = \frac{1}{9}(I_4 - kI_1^2). \tag{96}$$

We can construct a general solution of the form

$$x(t) = \frac{6(t - I_1)}{kt^2 - 2I_1kt + I_4}, \tag{97}$$

using I_1 and I_4. This case also admits five Noether's symmetries and so the Lagrangian (93) is physically important for $\tilde{\lambda} = 0$ in Eq. (1).

5 Telescopic Vector Fields

Telescopic vector fields are more general vector fields than the ones discussed so far. The Lie point symmetries, contact symmetries and λ-symmetries are all subcases of telescopic vector fields. A telescopic vector field can be considered as a λ-prolongation where the two first infinitesimals can depend on the first derivative of the dependent variable [10, 20, 26]. In the following, we briefly discuss the method of finding telescopic vector fields for a second-order ODE. We then present the telescopic vector fields for Eq. (1).

Let us consider the second-order Eq. (1). The vector field

$$\Omega^{(2)} = \xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial x} + \zeta^{(1)}\frac{\partial}{\partial \dot{x}} + \zeta^{(2)}\frac{\partial}{\partial \ddot{x}} \tag{98}$$

is telescopic if and only if [26]

$$\xi = \xi(t, x, \dot{x}),\ \ \eta = \eta(t, x, \dot{x}),\ \ \zeta^{(1)} = \zeta^{(1)}(t, x, \dot{x}) \tag{99}$$

with $\zeta^{(2)}$ given by

$$\zeta^{(2)} = D[\zeta^{(1)}] - \phi D[\xi] + \frac{\zeta^{(1)} + \dot{x}D[\xi] - D[\eta]}{\eta - \dot{x}\xi}(\zeta^{(1)} - \phi\xi), \tag{100}$$

where ϕ is the given equation ($\ddot{x} = \phi(t, x, x\dot{x})$).

To prove that the telescopic vector fields are more general vector fields, let us introduce two functions g_1 and g_2 in the following forms, namely

$$g_1(t, x, \dot{x}) = \frac{\zeta^{(1)} + \dot{x}\xi_t - \eta_t + \dot{x}(\dot{x}\xi_x - \eta_x)}{\eta - \dot{x}\xi},\ \ g_2(t, x, \dot{x}) = \frac{\dot{x}\xi_{\dot{x}} - \eta_{\dot{x}}}{\eta - \dot{x}\xi}. \tag{101}$$

We can rewrite the prolongations $\zeta^{(1)}$ and $\zeta^{(2)}$ using the above functions g_1 and g_2 as follows:

$$\zeta^{(1)} = D[\eta] - \dot{x}D[\xi] + (g_1 + g_2\phi)(\eta - \dot{x}\xi), \tag{102}$$

$$\zeta^{(2)} = D[\zeta^{(1)}] - \phi x D[\xi] + (g_1 + g_2\phi)(\zeta^{(1)} - \phi\xi). \tag{103}$$

The relationship between telescopic vector fields and previously considered vector fields can be given by the following expressions [10, 26],

$$\zeta^{(1)} = \eta^{(1)} + (g_1 + g_2\phi)(\eta - \dot{x}\xi), \tag{104}$$

$$\zeta^{(2)} = \eta^{(2)} + (g_1 + g_2\phi)(\zeta^{(1)} - \phi\xi). \tag{105}$$

In the above vector fields if we choose $g_1 = g_2 = 0$ and $\xi_{\dot{x}}^2 + \eta_{\dot{x}}^2 = 0$ we get the Lie point symmetries. The choice $g_1 = g_2 = 0$ and $\xi_{\dot{x}}^2 + \eta_{\dot{x}}^2 \neq 0$ gives the contact symmetries. To get λ-symmetries, we should choose $g_1 \neq 0$ and $\xi_{\dot{x}}^2 + \eta_{\dot{x}}^2 = 0$. As a consequence it can be considered as the more general vector field.

Hence the unknowns to be solved in Eq. (32) can also be $(\xi, \eta, \eta^{[\lambda,1]})$ by expressing λ in terms of $(\xi, \eta, \eta^{[\lambda,1]})$. In other words, if the given equation admits the telescopic vector field, then it satisfies the following invariance condition

$$\xi \frac{\partial \phi}{\partial t} + \eta \frac{\partial \phi}{\partial x} + \zeta^{(1)} \frac{\partial \phi}{\partial \dot{x}} - \zeta^{(2)} = 0. \tag{106}$$

In the above expression, ξ, η and $\zeta^{(1)}$ are three unknown functions which we need in order to write the telescopic vector fields of Eq. (1). Since the above expression has three unknowns, it is very difficult to find them systematically. For this purpose, we assume $\xi = 0$ and the remaining two unknown functions can be obtained in the following way. In this case, Eq. (106) turns out to be

$$\eta \frac{\partial \phi}{\partial x} + \zeta^{(1)} \frac{\partial \phi}{\partial \dot{x}} - \zeta^{(2)} = 0. \tag{107}$$

By assuming suitable ansatz for η and $\zeta^{(1)}$ we can find the telescopic vector fields associated with Eq. (1).

5.1 Case 1: $\tilde{\lambda} = 0$

For simplicity, first let us consider the case $\tilde{\lambda} = 0$. Assuming the ansatz

$$\eta = \frac{a_{01} + b_{01}\dot{x} + c_{01}\dot{x}^2}{(d_{01} + e_{01}\dot{x})^m}, \quad \zeta^{(1)} = \frac{a_{11} + b_{11}\dot{x} + c_{11}\dot{x}^2}{(d_{11} + e_{11}\dot{x})^n}, \tag{108}$$

for η and $\zeta^{(1)}$ and substituting them into Eq. (107) and solving the resultant expression we find the following telescopic vector fields for the case $\tilde{\lambda} = 0$:

$$\begin{aligned}
\Omega_1 &= \frac{9x}{(kx^2+3\dot{x})^2}\frac{\partial}{\partial x} + \frac{9\dot{x}-3kx^2}{(kx^2+3\dot{x})^2}\frac{\partial}{\partial \dot{x}} - \frac{18kx\dot{x}}{(kx^2+3\dot{x})^2}\frac{\partial}{\partial \ddot{x}},\\
\Omega_2 &= -\frac{18x(ktx^2+3t\dot{x}-3x)}{(kx^2+3\dot{x})^3}\frac{\partial}{\partial x} + \frac{6(kx^2-3\dot{x})(ktx^2+3t\dot{x}-3x)}{(kx^2+3\dot{x})^3}\frac{\partial}{\partial \dot{x}}\\
&\quad + \frac{36kx\dot{x}(ktx^2+3t\dot{x}-3x)}{(kx^2+3\dot{x})^3}\frac{\partial}{\partial \ddot{x}},\\
\Omega_3 &= \frac{-81t\dot{x}(ktx-2)-27x(ktx(ktx-6)+12)}{(kx^2+3\dot{x})^3}\frac{\partial}{\partial x}\\
&\quad + \frac{9(-9kt^2\dot{x}^2+kx^2(ktx(ktx-8)+18)-18\dot{x})}{(kx^2+3\dot{x})^3}\frac{\partial}{\partial \dot{x}}
\end{aligned}$$

$$+\frac{18k(k^2tx^3(3t\dot{x}+x)-3kx(-3t^2\dot{x}^2+4tx\dot{x}+x^2)-9\dot{x}(t\dot{x}-3x))}{(kx^2+3\dot{x})^3}\frac{\partial}{\partial\ddot{x}},$$

$$\Omega_4=-\frac{18(ktx-3)}{(kx^2+3\dot{x})^2}\frac{\partial}{\partial x}+\frac{6k(x(ktx-6)-3t\dot{x})}{(kx^2+3\dot{x})^2}\frac{\partial}{\partial\dot{x}}+\frac{18k(kx(2t\dot{x}+x)-3\dot{x})}{(kx^2+3\dot{x})^2}\frac{\partial}{\partial\ddot{x}},$$

$$\Omega_5=\frac{18\dot{x}}{(kx^2+3\dot{x})^3}\frac{\partial}{\partial x}-\frac{2kx(kx^2+9\dot{x})}{(kx^2+3\dot{x})^3}\frac{\partial}{\partial\dot{x}}+\frac{2k(k^2x^4+6kx^2\dot{x}-9\dot{x}^2)}{(kx^2+3\dot{x})^3}\frac{\partial}{\partial\ddot{x}}. \quad (109)$$

The above telescopic vector fields also satisfy the invariance condition (106) with the choice $\tilde{\lambda}=0$. To find the solution from the above admitted telescopic vector fields, one has to follow the standard order-reduction procedure. Let us consider the telescopic vector field Ω_1. The corresponding Lagrange system can be written as

$$\frac{dt}{0}=\frac{dx}{\frac{9x}{(kx^2+3\dot{x})^2}}=\frac{d\dot{x}}{\frac{9\dot{x}-3kx^2}{(kx^2+3\dot{x})^2}}. \quad (110)$$

Solving the above set of equations, we get the characteristics as

$$u=t \text{ and } v=\frac{kx^2+3\dot{x}}{3x}. \quad (111)$$

From the above expression, we get $\frac{dv}{du}$ as

$$\frac{dv}{du}=-v^2. \quad (112)$$

Solution of the above equation is given by

$$v=\frac{1}{u-I_1}. \quad (113)$$

Substituting (111) into (113) and rewriting it, we get a first-order ODE

$$\dot{x}+\frac{x(I_1kx-ktx+3)}{3(I_1-t)}=0. \quad (114)$$

Integrating the above equation, we get the general solution of (1) for the choice $\tilde{\lambda}=0$ as

$$x(t)=\frac{6(I_1-t)}{-6I_2+2I_1kt-kt^2}, \quad (115)$$

where I_1 and I_2 are the integration constants.

5.2 Case 2: $\tilde{\lambda} > 0$

As we did in the previous case, here we obtain the following vector fields for the case $\tilde{\lambda} > 0$:

$$
\begin{aligned}
\Omega_1 &= \frac{9(\sqrt{\tilde{\lambda}}kx\sin(\sqrt{\tilde{\lambda}}t) - 3\tilde{\lambda}\cos(\sqrt{\tilde{\lambda}}t))}{(\alpha)^2}\frac{\partial}{\partial x} \\
&+ \frac{3\sqrt{\tilde{\lambda}}\sin(\sqrt{\tilde{\lambda}}t)(9\tilde{\lambda} - k^2x^2 + 3k\dot{x}) + 18\tilde{\lambda}kx\cos(\sqrt{\tilde{\lambda}}t)}{(\alpha)^2}\frac{\partial}{\partial \dot{x}} \\
&+ \frac{9(\tilde{\lambda}\cos(\sqrt{\tilde{\lambda}}t)(3\tilde{\lambda} - k^2x^2 + 3k\dot{x}) - 2\sqrt{\tilde{\lambda}}kx(2\tilde{\lambda} + k\dot{x})\sin(\sqrt{\tilde{\lambda}}t))}{(\alpha)^2}\frac{\partial}{\partial \ddot{x}}, \\
\Omega_2 &= -\frac{9(\sqrt{\tilde{\lambda}}kx\cos(\sqrt{\tilde{\lambda}}t) + 3\tilde{\lambda}\sin(\sqrt{\tilde{\lambda}}t))}{(\alpha)^2}\frac{\partial}{\partial x} \\
&- \frac{3\sqrt{\tilde{\lambda}}\cos(\sqrt{\tilde{\lambda}}t)(9\tilde{\lambda} - k^2x^2 + 3k\dot{x}) - 18\tilde{\lambda}kx\sin(\sqrt{\tilde{\lambda}}t)}{(\alpha)^2}\frac{\partial}{\partial \dot{x}} \\
&+ \frac{9\tilde{\lambda}\sin(\sqrt{\tilde{\lambda}}t)(3\tilde{\lambda} - k^2x^2 + 3k\dot{x}) + 18\sqrt{\tilde{\lambda}}kx(2\tilde{\lambda} + k\dot{x})\cos(\sqrt{\tilde{\lambda}}t)}{(\alpha)^2}\frac{\partial}{\partial \ddot{x}}, \\
\Omega_3 &= \frac{18\sqrt{\tilde{\lambda}}(\alpha x\cos(2\sqrt{\tilde{\lambda}}t) - 3\sqrt{\tilde{\lambda}}(2kx^2 + 3\dot{x})\sin(2\sqrt{\tilde{\lambda}}t))}{(\alpha)^3}\frac{\partial}{\partial x} \\
&- \frac{54\tilde{\lambda}x\sin(2\sqrt{\tilde{\lambda}}t)d_1 - 6\sqrt{\tilde{\lambda}}\cos(2\sqrt{\tilde{\lambda}}t)a_1}{(\alpha)^3}\frac{\partial}{\partial \dot{x}} \\
&+ \frac{18(\sqrt{\tilde{\lambda}}x\cos(2\sqrt{\tilde{\lambda}}t)b_1 + \tilde{\lambda}\sin(2\sqrt{\tilde{\lambda}}t)c_1)}{(\alpha)^3}\frac{\partial}{\partial \ddot{x}}, \\
\Omega_4 &= \frac{36\sqrt{\tilde{\lambda}}k^2(kx\sin(\sqrt{\tilde{\lambda}}t) - 3\sqrt{\tilde{\lambda}}\cos(\sqrt{\tilde{\lambda}}t))((kx^2 + 3\dot{x})\cos(\sqrt{\tilde{\lambda}}t) + 3\sqrt{\tilde{\lambda}}x\sin(\sqrt{\tilde{\lambda}}t))}{(\alpha)^3}\frac{\partial}{\partial x} \\
&+ \frac{54\tilde{\lambda}k^2x\cos(2\sqrt{\tilde{\lambda}}t)d_1 + 18\tilde{\lambda}k^2x(9\tilde{\lambda} + k^2x^2 + 9k\dot{x}) + 6\sqrt{\tilde{\lambda}}k^2\sin(2\sqrt{\tilde{\lambda}}t)a_1}{(\alpha)^3}\frac{\partial}{\partial \dot{x}} \\
&- \frac{18\sqrt{\tilde{\lambda}}k^2(x\sin(2\sqrt{\tilde{\lambda}}t)b_1 - \sqrt{\tilde{\lambda}}\cos(2\sqrt{\tilde{\lambda}}t)c_1 + \sqrt{\tilde{\lambda}}e_1)}{(\alpha)^3}\frac{\partial}{\partial \ddot{x}}, \\
\Omega_5 &= -\frac{162\tilde{\lambda}\dot{x}}{(\alpha)^3}\frac{\partial}{\partial x} + \frac{18\tilde{\lambda}x(9\tilde{\lambda} + k^2x^2 + 9k\dot{x})}{(\alpha)^3}\frac{\partial}{\partial \dot{x}} - \frac{18\tilde{\lambda}e_1}{(\alpha)^3}\frac{\partial}{\partial \ddot{x}},
\end{aligned} \tag{116}
$$

where $a_1 = (-k^3x^4 + 9k(3\tilde{\lambda}x^2 + \dot{x}^2) + 27\tilde{\lambda}\dot{x})$, $b_1 = (-9\tilde{\lambda}^2 + 2k^3x^2\dot{x} + k^2(7\tilde{\lambda}x^2 + 6\dot{x}^2) + 3\tilde{\lambda}k\dot{x})$, $c_1 = (-k^3x^4 + 6k^2x^2\dot{x} + 15\tilde{\lambda}kx^2 + 9k\dot{x}^2 + 9\tilde{\lambda}\dot{x})$, $d_1 = k(kx^2 + \dot{x}) - 3\tilde{\lambda}$ and $e_1 = (k^3x^4 + 6k^2x^2\dot{x} + 9\tilde{\lambda}kx^2 - 9k\dot{x}^2 - 9\tilde{\lambda}\dot{x})$. The above telescopic vector fields also satisfy the invariance condition (106). To find the solution from the above admitted telescopic vector fields, one has to follow the standard order-reduction procedure. Let us consider the telescopic vector field Ω_5. The corresponding Lagrange system can be written as

$$
\frac{dt}{0} = \frac{dx}{-\frac{162\lambda\dot{x}}{(k^2x^2+3k\dot{x}+9\lambda)^3}} = \frac{d\dot{x}}{\frac{18\lambda x(k^2x^2+9k\dot{x}+9\lambda)}{(k^2x^2+3k\dot{x}+9\lambda)^3}}. \tag{117}
$$

Solving the above set of equations, we get the characteristics as

$$u = t \text{ and } v = \log\left(\frac{81\sqrt{3}(k^2x^2 + 3k\dot{x} + 9\lambda)^9}{\sqrt{(k^2x^2 + 6k\dot{x} + 9\lambda)^9}}\right). \tag{118}$$

From the above expression, we get $\frac{dv}{dt} = 0$. So the function v itself acts as a first integral. Then the integral I_1 takes the form

$$I_1 = \frac{81\sqrt{3}(k^2x^2 + 3k\dot{x} + 9\lambda)^9}{\sqrt{(k^2x^2 + 6k\dot{x} + 9\lambda)^9}}. \tag{119}$$

Rewriting the above expression for $\dot{x}$ and integrating it we obtain the general solution as in Eq. (88).

5.3 Case 3: $\tilde{\lambda} < 0$

For the case $\tilde{\lambda} < 0$, we get the telescopic vector fields by following the procedure discussed in the case $\tilde{\lambda} = 0$. The telescopic vector fields are given by

$$\begin{aligned}
\Omega_1 &= -\frac{9ke^{a_2t}(3\tilde{\lambda} + a_2kx)}{(\alpha)^2}\frac{\partial}{\partial x} \\
&+ \frac{3ke^{a_2t}(9(-\tilde{\lambda})^{3/2} + a_2k^2x^2 + 6\tilde{\lambda}kx - 3a_2k\dot{x})}{(\alpha)^2}\frac{\partial}{\partial \dot{x}} \\
&+ \frac{9\tilde{\lambda}ke^{a_2t}(9\tilde{\lambda}^2 - k^3x^2(a_2x + 2\dot{x}) + k^2x(9a_2\dot{x} - 7\tilde{\lambda}x) + 3\tilde{\lambda}k(5a_2x + 3\dot{x}))}{(3\tilde{\lambda} + a_2kx)(\alpha)^2}\frac{\partial}{\partial \ddot{x}}, \\
\Omega_2 &= \frac{9ke^{a_2(-t)}(a_2kx - 3\tilde{\lambda})}{(\alpha)^2}\frac{\partial}{\partial x} \\
&+ \frac{3ke^{a_2(-t)}(9a_2\tilde{\lambda} + a_2(-k^2)x^2 + 6\tilde{\lambda}kx + 3a_2k\dot{x})}{(\alpha)^2}\frac{\partial}{\partial \dot{x}} \\
&+ \frac{9\tilde{\lambda}ke^{a_2(-t)}(9\tilde{\lambda}^2 + k^3x^2(a_2x - 2\dot{x}) - k^2x(7\tilde{\lambda}x + 9a_2\dot{x}) + 3\tilde{\lambda}k(3\dot{x} - 5a_2x))}{(3\tilde{\lambda} - a_2kx)(\alpha)^2}\frac{\partial}{\partial \ddot{x}}, \\
\Omega_3 &= -\frac{18k^2e^{2a_2t}(a_2k^2x^3 + 6\tilde{\lambda}kx^2 + 3a_2kx\dot{x} + 9(-\tilde{\lambda})^{3/2}x + 9\tilde{\lambda}\dot{x})}{(\alpha)^3}\frac{\partial}{\partial x} \\
&+ \frac{6k^2e^{2a_2t}(a_2k^3x^4 + 9\tilde{\lambda}k^2x^3 - 9k(3a_2\tilde{\lambda}x^2 - \tilde{\lambda}x\dot{x} + a_2\dot{x}^2) - 27\tilde{\lambda}(\tilde{\lambda}x + a_2\dot{x}))}{(\alpha)^3}\frac{\partial}{\partial \dot{x}} \\
&+ \frac{18k^2e^{2a_2t}(k^3x^3(2a_2\dot{x} - \tilde{\lambda}x) + k^2xb_2 + 3\tilde{\lambda}kc_2 + 9\tilde{\lambda}^2(\dot{x} - a_2x))}{(\alpha)^3}\frac{\partial}{\partial \ddot{x}}, \\
\Omega_4 &= \frac{18k^2e^{-2a_2t}(a_2k^2x^3 + 3kx(a_2\dot{x} - 2\tilde{\lambda}x) - 9\tilde{\lambda}(a_2x + \dot{x}))}{(\alpha)^3}\frac{\partial}{\partial x} \\
&+ \frac{6k^2e^{-2a_2t}(-a_2k^3x^4 + 9\tilde{\lambda}k^2x^3 + 9k(3a_2\tilde{\lambda}x^2 + \tilde{\lambda}x\dot{x} + a_2\dot{x}^2) + 27\tilde{\lambda}(a_2\dot{x} - \tilde{\lambda}x))}{(\alpha)^3}\frac{\partial}{\partial \dot{x}}
\end{aligned}$$

$$+\frac{18k^2e^{-2a_2t}(-k^3x^3(\tilde{\lambda}x+2a_2\dot{x})+k^2xd_2+3\tilde{\lambda}ke_2+9\tilde{\lambda}^2(a_2x+\dot{x}))}{(\alpha)^3}\frac{\partial}{\partial\ddot{x}},$$

$$\Omega_5=-\frac{729\tilde{\lambda}^3\dot{x}}{(\alpha)^3}\frac{\partial}{\partial x}+\frac{81\tilde{\lambda}^3x(9\tilde{\lambda}+k^2x^2+9k\dot{x})}{(\alpha)^3}\frac{\partial}{\partial\dot{x}}-\frac{81\tilde{\lambda}^3e_1}{(\alpha)^3}\frac{\partial}{\partial\ddot{x}}, \quad (120)$$

where $a_2=\sqrt{-\tilde{\lambda}}$, $b_2=(7a_2\tilde{\lambda}x^2+6\tilde{\lambda}x\dot{x}+6a_2\dot{x}^2)$, $c_2=(5\tilde{\lambda}x^2+a_2x\dot{x}+3\dot{x}^2)$, $sd_2=(7(-\tilde{\lambda})^{3/2}x^2+6\tilde{\lambda}x\dot{x}-6a_2\dot{x}^2)$, $e_2=(5\tilde{\lambda}x^2-a_2x\dot{x}+3\dot{x}^2)$. Here also one can check that the above telescopic vector fields satisfy the invariance condition (106). To find the solution from the above admitted telescopic vector fields, one has to follow the standard order-reduction procedure. Let us consider the telescopic vector field Ω_1. The corresponding Lagrange system can be written as

$$\frac{dt}{0}=\frac{dx}{-\frac{9ke^{\sqrt{-\lambda}t}(k\sqrt{-\lambda}x+3\lambda)}{(k^2x^2+3k\dot{x}+9\lambda)^2}}=\frac{d\dot{x}}{\frac{3ke^{\sqrt{-\lambda}t}(k^2\sqrt{-\lambda}x^2+6k\lambda x-3k\sqrt{-\lambda}\dot{x}+9(-\lambda)^{3/2})}{(k^2x^2+3k\dot{x}+9\lambda)^2}}. \quad (121)$$

Solving the above set of equations, we get the characteristics as

$$u=t \text{ and } v=\frac{k\sqrt{-\lambda}x^2+3\lambda x+3\sqrt{-\lambda}\dot{x}}{9\sqrt{-\lambda}\lambda-3k\lambda x}. \quad (122)$$

From the above expression, we get $\frac{dv}{du}$ as

$$\frac{dv}{du}=-3\sqrt{-\lambda}\left(\frac{kv^2}{3}+\frac{v}{3}\right). \quad (123)$$

Solution of the above equation is given by

$$v=\frac{e^{I_1}}{e^{\sqrt{-\lambda}u}-e^{I_1}k}. \quad (124)$$

Substituting (122) into (124) and rewriting it we get a first-order ODE,

$$\dot{x}+\frac{e^{I_1}\sqrt{-\lambda}(k^2x^2+9\lambda)-xe^{\sqrt{-\lambda}t}(k\sqrt{-\lambda}x+3\lambda)}{3\sqrt{-\lambda}(e^{\sqrt{-\lambda}t}-e^{I_1}k)}=0. \quad (125)$$

Integrating the above equation, we get the general solution of (1) for the choice $\tilde{\lambda}<0$ as

$$x(t)=\frac{3\sqrt{-\lambda}(-e^{I_1}k+2c_1e^{2\sqrt{-\lambda}t})}{k(2c_1e^{2\sqrt{-\lambda}t}+e^{I_1}k-2e^{\sqrt{-\lambda}t})}, \quad (126)$$

where I_1 and I_2 are the integration constants. Obviously the solution (126) can be rewritten in the form (92).

6 Conclusion

In this paper, we have reviewed four different kinds of symmetries for the Liénard type nonlinear oscillator Eq. (1). It has already been shown that this equation exhibits three different kinds of dynamics depending upon the sign of the parameter $\tilde{\lambda}$. Based on this earlier result we have divided our analysis into three categories while studying the symmetries of this equation. To begin with, we have considered Lie point symmetries of this equation. We have derived the general solution for all the three regimes by considering a vector field in each one of the cases. We then considered λ-symmetries approach to this equation. As we noted earlier, we carried out this calculations for the $\tilde{\lambda} = 0$ case and demonstrated the applicability of λ-symmetries approach in establishing the integrability of this equation. We have then studied the Noether's symmetries of (1) for the parametric choices $\tilde{\lambda} > 0$, $\tilde{\lambda} < 0$ and $\tilde{\lambda} = 0$. The underlying Lagrangian is of non-standard type. However in all the three cases, we found maximal number (five) of Noether's symmetries for the Lagrangian (2). Recently it has been proposed that the physical Lagrangian for a second order differential equation should be the one which admits highest possible number of Noether's symmetries. Our results indicate that even though the Lagrangian is of nonstandard type it can be considered as a physical Lagrangian since it admits maximal number of symmetries. Finally, we have constructed telescopic vector fields for Eq. (1) in the parametric regimes $\tilde{\lambda} > 0$, $\tilde{\lambda} < 0$ and $\tilde{\lambda} = 0$. The method of finding general solution from telescopic vector fields is also explained. Thus we have shown the utility of symmetry analysis in solving the nonlinear ODEs of Liénard type.

Acknowledgements RMS acknowledges the University Grants Commission (UGC), New Delhi, India, for the award of a Dr. D. S. Kothari Post Doctoral Fellowship [Award No. F.4-2/2006 (BSR)/PH/17-18/0004]. The work of M.S. forms part of CSIR research project under Grant No. 03(1397)/17/EMR-II. The work of V.K.C. was supported by the SERB-DST Fast Track scheme for young scientists under Grant No. YSS/2014/000175. The work of M.L. is supported by a DST-SERB Distinguished Fellowship program.

References

1. Bhuvaneswari, A., Kraenkel, R.A., Senthilvelan, M.: Nonlinear Anal. Real World Appl. **13**, 1102–1114 (2012)
2. Bluman, G.W., Anco, S.C.: Symmetries and Integration Methods for Differential Equations. Springer, New York (2002)
3. Chandrasekhar, S.: An Introduction to the Study of Stellar Structure. Dover, New York (1957)
4. Chandrasekar, V.K., Senthilvelan, M., Lakshmanan, M.: Phys. Rev. E **72**, 066203 (2005)
5. Chandrasekar, V.K., Senthilvelan, M., Lakshmanan, M.: Proc. R. Soc. Lond. A **461**, 2451–2476 (2005)
6. Chandrasekar, V.K., Pandey, S.N., Senthilvelan, M., Lakshmanan, M.: Chaos, Solitons Fractals **26**, 1399–1406 (2005)
7. Chandrasekar, V.K., Pandey, S.N., Senthilvelan, M., Lakshmanan, M.: J. Phys. A Math. Gen. **39**, L69–L76 (2006)

8. Chithiika Ruby, V., Senthilvelan, M., Lakshmanan, M.: J. Phys. A Math. Theor. **45**, 382002 (2012)
9. Choudhuri, A.: On the symmetries of the modified Emden-type equation. arXiv:1304.5826v1
10. Gaeta, G.: J. Nonlinear Math. Phys. **16**, 107 (2009)
11. Gelfand, I.M., Fomin, S.V.: Calculus of Variations Translator. In: Silverman, R.A. (ed.). Prentice Hall, Englewood Cliffs (1963)
12. Gubbiotti, G., Nucci, M.C.: J. Nonlinear Math. Phys. **21**, 248–264 (2014)
13. Hydon, P.E.: Symmetry Methods for Differential Equations: A Beginnner's Guide. Cambridge University Press, Cambridge (2000)
14. Lutzky, M.: J. Phys. A Math. Gen. **11**, 249–258 (1978)
15. Mohanasubha, R., Sabiya Shakila, M.I., Senthilvelan, M.: Commun. Nonlinear. Sci. **19**, 799–806 (2014)
16. Muriel, C., Romero, J.L.: λ-symmetries and linearization of ordinary differential equations through nonlocal transformations. Preprint
17. Muriel, C., Romero, J.L.: IMA J. Appl. Math. **66**, 111 (2001)
18. Muriel, C., Romero, J.L.: Theor. Math. Phys. **133**, 1565 (2002)
19. Muriel, C., Romero, J.L.: J. Phys. A Math. Theor. **42**, 365207 (2009)
20. Muriel, C., Romero, J.L.: SIGMA **8**, 106 (2012)
21. Noether, E.: Gttingen Nachrichten Mathematik-physik Klasse **2**, 235 (1918)
22. Nucci, M.C., Tamizhmani, K.M.: J. Nonlinear Math. Phys. **17**, 167 (2010)
23. Olver, P.J.: Applications of Lie Groups to Differential Equations. Springer, New York (1986)
24. Pandey, S.N., Bindu, P.S., Senthilvelan, M., Lakshmanan, M.: J. Math. Phys. **50**, 102701 (2009)
25. Polat, G.G., Özer, T.: Nonlinear Dyn. **85**, 1571–1595 (2016)
26. Pucci, E., Saccomandi, G.: J. Phys. A Math. Gen. **35**, 6145 (2012)
27. Yasar, E.: Mathematical Problems in Engineering, Article ID 916437, 10 pp. (2011)