Chapter 8
Beliefs and Values in Upper Secondary School Students' Mathematical Reasoning

Åke Hestner and Lovisa Sumpter

Abstract This study focuses on upper secondary school students' mathematical reasoning when in pairs solving a task where values are part of the context. In particular, the focus is on arguments for decisions students put forward during their solution attempts and explanations and descriptions in stimulated recall interviews. Three themes of beliefs were identified: expectations, motivation, and emotions. Similar expectations were indicated as in previous studies (e.g. there should be an algorithm to solve the task). The main differences found were about motivation and emotion. Here, the students were more positive compared to previous studies saying such types of mathematical problems including values add a new dimension to problem-solving.

8.1 Introduction

Previous research has stressed the important role of beliefs when students are trying to solve mathematical tasks (e.g. Lester, Garofalo, & Kroll, 1989; Philippou & Christou, 1998; Schoenfeld, 1992). For instance, students can be constrained by their beliefs (Schoenfeld, 1992; Wong, Marton, Wong, & Lam, 2002), such that students would continue with unsuccessful strategies when working with non-routine tasks based on the idea that certain tasks are connected to certain algorithms (Lerch, 2004). But studies also show that beliefs can assist a student to be persistent (Carlson, 1999), or that a student who express confidence and control is more likely to continue and therefore succeed (Hannula, 2006). Previous studies have looked at different types of beliefs that was indicated in student's arguments for the choices made when solving different types of mathematical tasks (Jäder, Sidenvall, & Sumpter, 2017; Sumpter, 2013). Independent of the task, a routine task or a non-routine task, similar themes of beliefs were indicated: expectations, motivations, and one specific emotional belief, security. Also, the indicated

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beliefs seemed to interplay and this in a negative way (e.g. the only way for me to solve this task is to find the algorithm since that is the safest way). In addition, the students tried to solve the tasks with imitative reasoning, this independent if it was a routine task or a non-routine task. Such behaviour seems to reoccur in different countries (Dìaz-Obando, Plasencia-Cruz, & Solano-Alvarado, 2003; Furinghetti & Morselli, 2009), even though one could argue that beliefs are contextually bound (Francisco, 2013): students expect mathematical tasks in school to be solvable by memorised algorithms.

In this present study, we would like to study upper secondary school students when working with a task very different from what normally can be found in textbooks and other materials. We will use values as a mean to see if different themes of beliefs can be generated or different variations of beliefs within these themes. The research questions posed are as follows: (1) What characterises the reasoning students use when solving a problem-solving task addressing societal values? and (2) what are the indicated beliefs in students' mathematical reasoning when solving such a task?

8.2 Background

Beliefs research has provided several definitions of beliefs (Furinghetti & Pehkonen, 2002), which means that there is no consensus. Instead, depending on the research question different theoretical frameworks will provide different sets of analytical tools. This study builds upon previous research and therefore the same definition will be used. Beliefs are here defined as "an individual's understandings that shape the ways that the individual conceptualizes and engages in mathematical behaviour generating and appearing as thoughts in mind." (Sumpter, 2013, p. 1118). In order to study students' beliefs but also acknowledging that beliefs are attributed (Speer, 2005), we use *Beliefs Indications* (BI). BI is "a theoretical concept and part of a model aiming to describe a specific phenomenon, i.e. the type of arguments given by students when solving school tasks in a lab setting." (Sumpter, 2013, p. 1116). We would here like to extend this definition of BI to go beyond "arguments given in task solving situations" and therefore we propose BI to include arguments and explanations given by students in other situations too such as interview sessions. We argue that although beliefs are here seen as something indicated, the results are interesting if they can help predict and explain behaviour.

In the present study, values will be used as a tool to possibly generate different set of indicated beliefs about problem-solving. Values can be seen as mediated from beliefs and attitudes and are expressed when an individual is doing active choices between different alternative with different values attached to them (Clarkson, Bishop, FitzSimons, & Seah, 2000). The choices are related to something being right or wrong (McLeod, 1992), and often closely related to motivation (Hannula, 2012). Although values are part of both the individual actors within the educational system (e.g. a student or a teacher), they can also be manifested in other ways (Bishop, 2012). We will here adopt the values written in the political texts that govern Swedish mathematics education. In the curriculum for upper secondary school, the first chapter is called "Fundamental values and tasks of the school" stipulating that:

> The national school system is based on democratic foundations. The Education Act (2010:800) stipulates that education in the school system aims at students acquiring and developing knowledge and values. It should promote the development and learning of students, and a lifelong desire to learn. Education should impart and establish respect for human rights and the fundamental democratic values on which Swedish society is based. (Skolverket, 2013, p. 4)

These values are listed further down in the texts such as equality and equity. In the syllabus for mathematics, we can read that regarding problem-solving the students should be able to use mathematical models to solve problems concerning both situations connected to future potential profession but also everyday life. This is here linked to the ability to follow and perform mathematical reasoning.

In most reasoning research, reasoning is thought of as high quality thinking but is seldom defined (Lithner, 2008; Sumpter, 2013). Since we would like to talk about reasoning that includes non-mathematical arguments, we need to use a broad definition that goes beyond logical thinking and therefore reasoning is defined as:

> […] the line of thought adopted to produce assertions and reach conclusions in task-solving. It is not necessarily based on formal logic, thus not restricted to proof, and may even be incorrect as long as there are some kind of sensible (to the reasoner) reasons backing it. (Lithner, 2008, p. 257)

Reasoning is a sequence, correct or incorrect, that starts with a task and results with a conclusion including, potentially, the result "no conclusion". We see this sequence having the following four steps (Lithner, 2008): (1) A (sub-)task is met, which is denoted task situation (TS); (2) A strategy choice (SC) is made where "choice" is seen in a wide sense (choose, recall, construct, discover, guess, etc.); (3) The strategy is implemented (SI); and, (4) A conclusion (C) is obtained. The characterisation of reasoning types is the results of the analysis of explicit or implicit arguments for strategy choice, implementation (Lithner, 2008) and conclusion (Hedefalk & Sumpter, 2017). There are two main categories of reasoning: Imitative Reasoning (IR) which means that the task solver applies a recalled or externally provided solution method, and Creative Mathematical founded Reasoning (CMR) where a solution method is constructed by the solver (Lithner, 2008). Dependent how the solver express different types of arguments for the solution, the CMR can be identified as Local CMR or Global CMR where the latter includes verification and control for the whole task situation. IR is a family of different types of reasoning (see Bergqvist, Lithner, and Sumpter (2008) and Lithner (2008) for a longer description and discussion).

8.3 Methods

The task was designed with three aspects in mind. First, the task aims to stimulate CMR, hence a non-routine task/problem-solving task. Secondly, the mathematical resources needed to be an area that has already been covered in the first course of upper secondary mathematics and most likely also at lower secondary level so that the chosen students should be able to solve the task although not know a certain procedure and/or algorithm. Here, the mathematical properties required to solve the task, besides problem-solving, were proportional reasoning and the understanding of natural numbers and divisors of such. Thirdly, we wanted to tap into values. In this case, we decided upon a realistic problem (c.f. Gutstein, 2006). The students were given a statement from an official source, UN-UNDP, which states how money is distributed between individuals: the richest one fifth of the world population shares 75% of the world's resources and the poorest one-fifth shares 2% of the resources. The students were asked to describe this with numbers where they were able to choose any optional number of people and things. They were also encouraged to draw a picture. As a second step, the students were asked to formulate a statement of a distribution they would prefer to see.

Three pairs of students participated, all from an upper secondary school studying a programme with a medium-intensity mathematical course, the Arts Programme. It is a higher education preparatory programme, although only taking one course in mathematics called Mathematics 1b. In this sense, it is not a mathematically intense programme. The decision to allow students working in pairs was made since it worked well in previous studies in order to stimulate the mathematical talk during the problem-solving session (e.g. Jäder et al., 2017); this is compared to when the students were alone in the lab setting (e.g. Sumpter, 2013). The analysis focus on the arguments for the decisions that were made during the problem-solving sessions and therefore the students were filmed in a lab-setting, but the films were also used for stimulated recalled interviews. Each couple worked for about 50 min. A few days later, an interview was made with each of the students individually (about 20 min). In this study, we follow the ethical guidelines and rules given by CODEX. A fourth couple was first asked to participate, but decided to withdraw before the data collection. A decision was made to not replace this couple since the data from the first three couples were rich.

The data both from the problem-solving sessions and the interviews were transcribed, and a description and an interpretation of the problem-solving sessions were made. The task situations (TSs) were identified using an appropriate grain size (c.f. Bergqvist et al., 2008). For each sequence starting with a TS, the central decisions were identified together with the argumentation for these decisions. To be able to identify, analyse and report patterns within the data, we used thematic analysis (c.f. Braun & Clarke, 2006), where the focus was on Beliefs Indication (BI). BI:s could be explicit statements in the transcripts but also hidden in students' behaviour (for a longer discussion about BI, see Jäder et al. (2017) or Sumpter (2013)). The analysis was made by the first author using the second author as a validator of the analysis. A third person functions as an additional validator when needed. Passages when the BI was not clear were left out. The three themes of BI:s from Sumpter (2013), security, motivation and expectation, were used as a basis for deductive analysis, but the analysis was also inductive in order to explore new themes. The themes were checked against each other and back to the original data, this since themes had to "cohere together meaningfully, while there should be clear and identifiable distinctions between themes" (Braun & Clarke, 2006, p. 91). As a last step of the analysis, the reasoning for each TS was analysed using Lithner's (2008) framework. Here, just as

Jäder et al. (2017) the analysis only aimed to characterise the TS using the main type of reasoning, that is, CMR and IR. But compared to Jäder et al. (2017), we also study whether the IR and/or CMR were local (i.e. a separate sequence or global).

All together, we employed the same analysis structure as Sumpter (2013): (1) BI is used as a general initial coding scheme; (2) the four steps of reasoning function as a representational scheme; and, (3) two tools for conducting two different types of analyses (i.e. thematic analysis of BI:s and Lithner's (2008) framework about reasoning).

8.4 Results

Here, we will present some of the results of the analysis. First, we have the results of the analysis of the reasoning, see Table 8.1:

As we can see in Table 8.1, all three couples attempted to use CMR when solving the task: they did not only try IR. However, the three couples differ in the global strategy: A and C looked for algorithms, although producing local CMR, whereas B used CMR as a global strategy as well after starting with IR.

All three pairs started with IR and all three pairs also expressed arguments which could indicate a belief about expectation: there should be an algorithm. This is here exemplified with couple B:

Student 1:	23 times… there is probably a really super clever way of calculating this
Student 2:	Yes, but if we…
Student 1:	Or we can just take 23 times 6
Student 2:	But that can be…?
Student 1:	But does everyone has to have the same? Why should it be equal, [I] think that is stupid. You know…
Student 2:	But if we write a… you know, an equation
Student 1:	Yes
Student 2:	That must be the easiest way to solve it

Although this pair expressed expectations in line with "there should be an algorithm", here a "super clever way", the pair did not stay in the search. Instead, the mathematical talk moved to the context and about values. This could be contrasted with student 1 from couple C who explained the strategy choice in the interview:

Table 8.1 Different types of reasoning used by students when solving the given task

Student pair	IR	CMR	Global CMR	Arrive with correct conclusion
A	Yes	Yes	No	No
B	Initially	Yes	Yes	Yes
C	Yes	Yes	No	No

Table 8.2 Sub-themes of BI connected to emotions regarding mathematical task; *n*

Theme (*n*)	Example
The problem was fun because it was open (3)	I thought it was fun… It was like, you know, as long as you were in the boundaries. A bit like that, that's how it felt. But it was almost more fun 'cause… uhm…. [You] Could do a bit what you wanted [Interview Student 2; couple B]
Creative problems are fun and inspiring (1)	You have to be creative too… like if you… [laughter] you could something fun of it, like… we, I could sit [working] forever [Interview Student 2; couple B]
Fun task within range (1)	It was mainly a fun task, I reckon. Not too easy and not too difficult [Interview Student 1; couple C]
It is important it is fun (1)	It was fun… 'cause that is important [Interview Student 2; couple B]
To work with problem-solving is fun (1)	Like, I think it is fun with problem-solving in general, I think it is fun [Interview Student 1; couple B]

Table 8.3 Sub-themes of motivational BI; *n*

Theme (*n*)	Example
You can learn about social issues at the same time you learn mathematics (3)	Yes, it is like good… you remember it… and you get a picture of [things]… how… you learn something while you learn… calculating in math and then you get it double [knowledge] it feels like. It feels good anyway. You remember it [Int K Par A]
A realistic problem (with source) is motivational (3)	I think it is pretty… fun because you can see it like… a small number… can be about the whole world… that you can calculate and write [about] and so… when you solve the problem [Int A Par A]
The context of the task can encourage learning in other subjects/social science (4)	But then I think it will be easier in life… if you talk about other subjects… that if you have it in other subjects you can get a perception in/of mathematics and then [it is] easier in social science when you talk about the world. So when you have the next subject [in line] you can make connections… how everything is related [Int W Par A]

> Well… there is always a trick, surely, because there always is one, but… the only thing is to remember it. But I don't know... uhm… I got stuck concerning that everyone should have the same amount. Because that is not how it always is. [Interview Student 1; couple C]

However, also mentioning the context of the task, the values, the expectation of "the trick", the expected algorithm, and to try to remember which "trick" it should be was stressed by these students.

If we instead look at the BI:s connected to emotions, it is in the interview responses we find most data. We will here focus on the BI expressed about the mathematical task and problem-solving, see Table 8.2:

In Table 8.2, we see that the most common emotion is fun: no other emotions were mentioned regarding the nature of the task. In this section, we also find some motivational beliefs (e.g. "It is important it is fun" and "To work with problem-solving is fun"). Looking at motivational beliefs, the results indicated three sub-themes, see Table 8.3:

The three sub-themes mainly concern how bringing in social aspects into mathematical tasks and connect it to reality means you learn about other thing while learning mathematics. This is, by the students, considered helpful regarding motivation.

As a summary, our analysis resulted in three themes of beliefs, expectations, motivations and emotions, where the latter two are connected through positive emotions. Expectations were mainly about the strategy choice: "finding the right algorithm".

8.5 Discussion

Looking at reasoning, compared to previous studies (Bergqvist et al., 2008;Jäder et al., 2017; Sumpter, 2013), in the present study there were more instances of CMR. One pair even had a global approach in their solution attempt. But it is in the analysis of strategy choices in relation to BI:s, we see some different patterns which might be due to the nature of the task. Although all three couples expressed an expectation that mathematical tasks in school should be solvable by memorised algorithms (c.f. Dìaz-Obando et al., 2003; Furinghetti & Morselli, 2009; Jäder et al., 2017, Sumpter, 2013), one couple had a different approach: global CMR. They did not get stuck in the search of "the algorithm" (c.f. Lerch, 2004). Whether this is indeed a result of the nature of the task or due to other causes such as previous training in problem-solving, we can only speculate. We suggest that this needs to be explored and confirmed by further research. However, one conclusion is, just as Jäder et al. (2017), that it is not enough just to give students a non-routine task and leave it to them to try to learn mathematical problem-solving and mathematical reasoning skills.

Regarding BI:s about motivation and emotion, we can see a few differences compared to what was observed in Jäder et al. (2017) and Sumpter (2013). If we focus on BI:s with emotional attributions, here they were more positive (e.g. students talking primarily about "fun"). In previous studies (Jäder et al., 2017; Sumpter, 2013), the emotions mentioned were more about what was considered safe, and more specific, to choose algorithms that feel safe. Here, the emotional BI:s to some degree also encompassed motivational beliefs signalling a relationship between these two affective factors (c.f. Hannula, 2012). Compared to Jäder et al. (2017) and Sumpter (2013), the motivational BI:s were also more positive. In this sense, the students participating in this study, although not all three were successful in their solution attempts, appeared as constrained in their task solving at least from emotional and motivational point of view (c.f. Sumpter, 2013; Wong et al., 2002). We did not observe the same negative interplay. One possible conclusion is that the proposed task does generate different types of indicated beliefs, but expectations appear more strongly held compared to the other two themes. This also needs to be further investigated. An implication is that in mathematics teaching focusing on problem-solving is not enough to give non-routine tasks without being aware of, and potentially addressing, affective factors.

References

Bergqvist, T., Lithner, J., & Sumpter, L. (2008). Upper secondary students' task reasoning. *International Journal of Mathematical Education in Science and Technology, 39*(1), 1–12.

Bishop, A. J. (2012). From culture to well-being: a partial story of values in mathematics education. *ZDM, 44*(1), 3–8.

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology, 3*, 77–101.

Carlson, M. (1999). The mathematical behavior of six successful mathematics graduate students: Influences leading to mathematical success. *Educational Studies in Mathematics, 40*(3), 237–258.

Clarkson, P. C., Bishop, A. J., FitzSimons, G. E.., & Seah, W. T. (2000). Challenges and constraints in researching values. In J. Bana & A. Chapman (Eds.), Mathematics education beyond, *Proceedings of the 23rd annual conference of the mathematics education research group of Australasia* (pp. 188–195), Perth.

Dìaz-Obando, E., Plasencia-Cruz, I., & Solano-Alvarado, A. (2003). The impact of beliefs in student's learning: An investigation with students of two different contexts. *International Journal of Mathematical Education in Science and Technology, 34*(2), 161.

Francisco, J. (2013). The mathematical beliefs and behavior of high school students: Insights from a longitudinal study. *Journal of Mathematical Behavior, 32*(3), 481–493.

Furinghetti, F., & Morselli, F. (2009). Every unsuccessful problem solver is unsuccessful in his or her own way: Affective and cognitive factors in proving. *Educational Studies in Mathematics, 70*, 71–90.

Furinghetti, F., & Pehkonen, E. (2002). Rethinking characterizations of beliefs. In E. Pehkonen, G. C. Leder, & G. Törner (Eds.), *Beliefs: A hidden variable in mathematics education?* (pp. 39–57). Dordrecht: Kluwer.

Gutstein, E. (2006). Reading and writing the world with mathematics: Toward a pedagogy for social justice. New York: Routledge.

Hannula, M. (2006). Affect in mathematical thinking and learning: Towards integration of emotion, motivation and cognition. In J. Maasz & W. Schloeglmann (Eds.), *New mathematics education research and practice* (pp. 209–232). Rotterdam, Netherlands: Sense Publishers.

Hannula, M. S. (2012). Exploring new dimensions of mathematics-related affect: Embodied and social theories. *Research in Mathematics Education, 14*(2), 137–161.

Hedefalk, M., & Sumpter, L. (2017). Studying preschool children's reasoning through epistemological move analysis. In B. Kaur, W. K. Ho, T. L. Toh, & B. H. Choy (Eds.), *Proceedings of the 41st conference of the international group for the psychology of mathematics education*, Singapore, 2017 (Vol. 3, pp. 1–8).

Jäder, J., Sidenvall, J., & Sumpter, L. (2017). Mathematical reasoning and beliefs in non-routine task solving. *International Journal of Science and Mathematics Education, 15*(4), 759–776.

Lerch, C. (2004). Control decisions and personal beliefs: Their effect on solving mathematical problems. *Journal of Mathematical Behavior, 23*, 361–372.

Lester, F. K., Garofalo, J., & Kroll, D. L. (1989). In D. B. McLeod & V. M. Adams (Eds.)., *Affect and mathematical problem solving. A new perspective Self-confidence, interest, beliefs, and metacognition: Key influences on problem-solving behavior* (pp. 75–88). New York, NY: Springer.

Lithner, J. (2008). A research framework for creative and imitative reasoning. *Educational Studies in Mathematics, 67*(3), 255–276.

McLeod, D. B. (1992). Research on affect in mathematics education: A reconceptualization. In D. Grouws (Ed.), *Handbook of research in mathematics teaching and learning* (pp. 575–596). New York: Macmillan Publishing Company.

Philippou, G. N., & Christou, C. (1998). The effects of a preparatory mathematics program in changing prospective teachers' attitudes towards mathematics. *Educational Studies in Mathematics, 35*(2), 189–206.

Schoenfeld, A. (1992). Learning to think mathematically: Problem solving, metacognition and sense-making in mathematics. In D. Grouws (Ed.), *Handbook of research in mathematics teaching and learning* (pp. 334–370). New York, NY: Macmillan Publishing Company.

Skolverket [Swedish National Agency for Education]. (2013). *Curriculum for the upper secondary school*. Stockholm.

Speer, N. (2005). Issues of methods and theory in the study of mathematics teachers' professed and attributed beliefs. *Educational Studies in Mathematics, 58*(3), 361–391.

Sumpter, L. (2013). Themes and interplay of beliefs in mathematical reasoning. *International Journal of Science and Mathematics Education, 11*(5), 1115–1135.

Wong, N.-Y., Marton, F., Wong, K.-M., & Lam, C.-C. (2002). The lived space of mathematics learning. *The Journal of Mathematical Behavior, 21*, 25–47.