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A. Mani
Gianpiero Cattaneo
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Editors

Algebraic Methods in General Rough Sets

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Algebraic Methods in General Rough Sets

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Preface

Algebraic methods have been extensively used since the very inception of the study of rough sets and related formal approaches to vagueness. Most methods concern logical reasoning, algebraic semantics, granularity, duality, knowledge representation, and computation.

Results and methodologies are scattered across multiple papers in journals, and many connections remain hidden. To remedy this situation, most of the major approaches including the latest trends have been covered in some depth in this research volume. It is intended to serve as a comprehensive up-to-date guide to algebraic approaches to rough sets and reasoning with vagueness, provide strong research directions, and connect algebraic approaches to rough sets with those for other forms of approximate reasoning. Needless to say, it is not intended as an introduction to the subject.

A substantial part of the content of this volume may be found distributed over numerous recent research papers but without all the insightful perspectives, enhancements, and restructuring that has happened due to the efforts of the contributors. We would first of all like to thank all authors: A. Mani, Gianpiero Cattaneo, Davide Ciucci, Bijan Davvaz, Patrik Eklund, María-Ángeles Galán-García, Jouni Järvinen, Piero Pagliani, Sándor Radeleczki, and Ali Shakiba for their effort.

All major semantic approaches are covered in the first seven chapters, while related topics are considered in the remaining three. In the first chapter, commonly used notation and terminology are fixed. In the next chapter, the connection between closure and interior operators in lattices and specific abstract approaches to rough sets is explained in depth by Gianpiero Cattaneo.

In the third chapter, a comprehensive account of algebraic approaches to rough sets from the perspective of an axiomatic approach to granularity is presented by A. Mani. She emphasizes the importance of the connections with granular operator spaces and variants. A novel classification of granular approaches is also part of the chapter.

Piero Pagliani has a refreshing new approach to connections among relation algebras, algebraic modal logic, rough approximations, algebraic geometry, and topology. Collectivization of different kinds of properties has been described in relation to semantic structures like Heyting, Nelson algebras, and variants in fine detail.

In the fifth chapter, Jouni Järvinen and Sándor Radeleczki deal with semantics of irredundant coverings and a subclass of rough approximations defined by tolerance relations on information tables. The chapter also includes algorithms for handling irredundant coverings in computational contexts.

In the sixth chapter, A. Mani considers most duality and representation results that were developed for or have been used in algebraic approaches to rough sets in much detail. She has introduced new adaptations and proven new results on the connections between L-fuzzy sets and general rough sets in the chapter.

If something is less than a complement of something else, then the former is orthogonal to the latter. Pairs of such elements (called orthopairs) have been used to construct algebras such as BZ-Posets, HW-algebras, and variants that serve as semantics for some abstract and concrete approaches to rough and fuzzy sets. These and related representation results are explained in detail in the chapter on orthopairs by Gianpiero Cattaneo and Davide Ciucci.

The second part of the book consists of three short chapters. In the first of these, Patrik Eklund and María-Ángeles Galán-García discuss how algebraic approaches to rough sets can be generalized to monoidal closed categories. Connections with other information models are indicated in brief.

In the ninth chapter, Bijan Davvaz considers the generalization of rough set theoretic concepts to the domain of rings. The connection of set-valued maps and morphisms between rings and general rough sets is also demonstrated.

S-approximation spaces can be seen as a generalization of the neighborhood-based approach in rough sets and Dempster–Shafer theory. Ali Shakibah develops these for decision-making applications in multi-agent scenarios in the last chapter.

Several chapters in this volume were evaluated through the *Second Reader Method*. In this method, authors suggest an expert designated as *second reader* and they work together toward improving the quality of the submission.

We would like to thank all of our second readers and reviewers for their efforts.

We would also like to thank the staff of Springer for their support.

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Gianpiero Cattaneo: Algebraic Methods for Rough Approximation Spaces by Lattice Interior-Closure Operations

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Piero Pagliani: Three Lessons on the Topological and Algebraic Hidden Core of Rough Set Theory

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Jouni Järvinen and Sándor Radeleczki: Irredundant Coverings, Tolerances, and Related Algebras

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Contents

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About the Editors

A. Mani is a researcher in algebra, logic, rough sets, vagueness, mereology, and foundations of mathematics; is a senior member of the International Rough Set Society; and is currently also affiliated with the University of Calcutta. She is a leading researcher on the foundations of rough sets and has published extensively in peer-reviewed international journals on the subject. She has also been part of the scientific and organizing committee of international conferences on rough sets and logic. She has also been involved in teaching and development of courses in machine learning, mathematics, and statistics. She also has many years of experience in providing part-time services in soft computing, GNU/R, and free software. Her homepage is at http://www.logicamani.in.

Gianpiero Cattaneo is an active scientific collaborator of the Department of Computer Science at the University of Milano-Bicocca in Italy. Previously, he retired as a professor of mathematics at the same university, after a career spanning over 45 years. His research areas are discrete time dynamical systems: topological chaos, cellular automata and related languages, and algebraic approach to fuzzy logic and rough sets; axiomatic foundations of quantum mechanics: logico-algebraic approach and realization of reversible gates by quantum computing techniques. He has published enormously in these fields and has been part of the scientific and organizing committee of various conferences such as IQSA, ACRI, RSKT, and FUZIEEE. He has been a visiting professor at various universities such as the mathematics department of Prague Polytechnic, logic and systems science department of London School of Economics, and la Ecole Normale Superieure (ENS) at Lyon. He has also been academically responsible for a number of national and international research projects such as “Information Systems and Parallel Computations, Sub-project: Qualitative Models of Physical Systems” (CNR: 1988–1994), “Mathematical problems on quantum mechanics” (Italian PRIN: 1983–1989), and “Modelling in Economics and Physics” (CSRC, LSE: 1993–1995).

Ivo Düntsch’s main research areas include modal logic and its application to data analysis and machine learning, statistical foundations of rough set theory, lattice theory and universal algebra, qualitative spatial reasoning, and region-based topology. He obtained his PhD under the guidance of S. Koppelberg and his Habilitation at the Freie Universität Berlin in 1989. He has published extensively and has contributed enormously to the research on qualitative spatial reasoning in multiple perspectives and is on the editorial board of journals like the *Transactions on Rough Sets*. He has always been an active collaborator; is an emeritus professor at Brock University, St Catharines, Canada; is a guest professor at Fujian Normal University, Fuzhou, China; and a senior member of the International Rough Set Society.

Introduction

A. Mani, Gianpiero Cattaneo, and Ivo Düntsch

Abstract In this chapter, the content of the book is described in brief and some of the notation and terminology is fixed.

Rough Set Theory (RST) is essentially a mathematical approach for handling vagueness and imprecision in a wide variety of contexts. Originally introduced in the mid eighties of the previous century by Zdislaw Pawlak, the subject has grown tremendously in pure and applied directions from mathematical, algebraic logical, logical, philosophical and computational perspectives. For basic references the reader may refer to the second and third chapter of this volume. In rough set theory,vague and imprecise information is dealt with through binary relations on a set (for some form of indiscernibility) or covers of a set or through more abstract operators in a set-theoretical or more general setting. In this chapter, the content of the book is described in brief and some of the notation and terminology is fixed.

In the next chapter, Gianpiero Cattaneo focuses on a lattice theoretical interpretation of general rough sets and a specific abstract approximation operator approach. Some emphasis is put on the connection with algebraic approaches to modal logic—the elements of the algebra are interpreted as propositions of a modal logic that seek to interpret fragments of reasoning with vague objects. Generalized negation operators such as the intuitionistic (or Brouwer), Pre-Brouwer, and Kleene play a crucial role in the algebraic models considered in the chapter. Overall it has been mostly assumed that the lower approximation operators are definable in terms of the upper approximation or preclusion operators. Algebraic semantics of fuzzy sets including

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the BZ-De Morgan algebras also developed by the author are presented. This leads to a rough set representation of fuzzy sets. Furthermore it is shown that Halmos closure lattices (that model a S5-like modal algebra over a De Morgan algebra instead of a Boolean algebra) are essentially equivalent to Pre-BZ lattices. Connections among Łukasiewicz algebras and BZ algebras are also considered in detail.

In the third chapter, A. Mani presents a comprehensive approach to an axiomatic approach to granularity in general rough sets that can be accommodated within granular operator spaces and generalizations thereof. The approximations considered in the chapter are required to satisfy a minimum of granular properties, and a wide array of algebraic methods that fit into such scheme of things are explained by her. The section on approximations derived from tolerance relations is refreshingly new, and esoteric rough sets that concern partially reflexive relations are approached from an improved perspective. Many gaps in knowledge between cover based rough sets and granular approaches to rough sets have been filled in the section on cover based rough sets. Recent work on quasi-ordered approximation spaces has also been critically presented in the section. A number on innovative algebraic approaches were invented by A. Mani for handling generalized transitive relations. These are presented in the section on granular approach to prototransitive rough sets. She has also included a section on her recent research on antichain based semantics. In this semantics, negation-like operations do not have a central role, and this approach leads to a new class of logics of vagueness. This chapter constitutes more than one-fourth of the present volume.

Piero Pagliani's chapter primarily concerns the topological algebraic logic approach to rough objects formed from point-wise approximations. This is developed from a relation algebraic, topological and algebraic logic perspective. This approach permits the reader to understand fundamental Galois connections and dualities in a clear way. The chapter includes a number of results published earlier by the author, but the presentation is new. Connections among algebraic models constructive, modal and intuitionistic logic are pointed out, and it is argued that in at least one perspective, *the logic of rough sets should be the one modeled by Post algebras of order three, not three-valued Łukasiewicz logic or connected mathematical objects (regular Stone algebras and the like).*

Consider a collection of subsets of a set S each of which satisfy a property τ and whose union is S. Such a collection is said to be *irredundant* with respect to τ, if the removal of any one of the elements of the cover leads to a subcollection whose union is a proper subset of S. Tolerance spaces are generalizations of approximation spaces in which equivalences have been replaced by tolerance relations. In the fifth chapter, Jouni Järvinen and Sándor Radeleczki present their recent research on algebraic semantics of rough objects generated by irredundant normal covers corresponding to a tolerance space. The collection of all rough objects formed by point-wise approximations on a set are shown to form a Kleene algebra. Algorithms for deciding whether a given normal cover is irredundant are also proposed in the chapter. On the whole, this chapter complements the approach to tolerance spaces in the previous three chapters.

Duality and representation results are crucial for progress at many stages in any algebraic approach to general rough sets. In the sixth chapter (*Duality, Representation and Beyond*), A. Mani reviews algebraic approaches to general rough and fuzzy sets from this perspective. Results relating to representation of algebraic models of fragments of rough set models, classical and general rough sets are explained, critically reviewed, new proofs proposed, open problems specified and new directions are suggested. The focus is on discrete and topological dualities as opposed to canonical dualities. Recent duality results in the literature such as those relating to Tarski spaces and preference spaces are adapted for use in general rough contexts by the author. She also proves new results on granular connections between generalized rough and L-fuzzy sets. Examples for related contexts are also included. As the results concern reasoning with vagueness, philosophical aspects of duality and representation have been reflected on in considerable depth in the initial sections.

In the chapter *Algebraic Methods for Orthopairs and induced Rough Approximation Spaces*, Gianpiero Cattaneo and Davide Ciucci study algebraic systems arising from pairs of elements from a partially ordered set (poset) that share some orthogonality between them. The nature of different types of unary negation on a poset have been explained in the initial sections of the chapter. In posets endowed with a De Morgan complementation, an orthopair is simply a pair of elements in which the first component is less than the complement of the second component. The set of all orthopairs can be endowed with minimal BZ poset, Pre-BZ lattice and related structure under constraints on the original poset. Minimal BZ-posets in turn induce rough approximation spaces. It is also shown that the set of orthopairs derived from fuzzy sets cannot be endowed with a Kleene negation, while the set of fuzzy sets with Gödel and Łukasiewicz implications form Heyting Wajsberg algebras. Related concrete representation theorems are proved, and open problems are posed.

The last three chapters concern recent topics that are not directly concerned with algebraic models of facets of reasoning about vagueness.

In the chapter *Rough Objects in Monoidal Closed Categories*, Patrik Eklund and María-Ángeles Galán-García build upon earlier work on monadic rough objects over the category Set. It is shown, in brief, that algebraic semantics can be enriched to work similarly over monoidal closed categories in general and toposes in particular. The connection of the rough information model in such categories with the original relational model are also explained. The proposal is significant for both logical and practical applications. Examples for the latter are provided.

In the chapter *Rough Algebraic Structures Corresponding to Ring Theory*, Bijan Davvaz essentially studies the utility of rough approximation operators over rings. Such questions are motivated by the regular double Stone algebra model of rough sets. Lower and upper approximations of a subset of a ring can be defined through ideals and these have many nice properties. The approximation of an ideal (respectively subring) by another ideal is an ideal (respectively subring). In recent work, the author has generalized classical rough approximations of sets through concept of lower and upper inverse of set-valued maps. These concepts are extended to rings and are shown to have good structure preserving properties. The chapter is of interest in the study of the impact of rough concepts on ring theory.

An S-approximation space is essentially a generalization of the neighborhood approach in rough sets. In the chapter *S-Approximation Spaces*, Ali Shakibah introduces the basics of such spaces and properties of approximations. These are related to three-way decision-making. From a semantic perspective, these structures are of some interest in modal logic approaches to rough sets. Methods of combining S-approximation spaces are also studied. These are of interest in multi agent systems.

1 Notation and Terminology

The notation and terminology used in this book is partly fixed in this note.

- Upper case Roman alphabets will typically denote sets. Thus A, B, C may be sets. Collections of sets or subsets are denoted by script letters like $\mathcal{C}$. $\mathcal{G}$ will be used to denote the granulation in a context.
- Universal sets in a context shall typically be denoted by U, V, S.
- Operators used in superscript form will be denoted by lower case Roman or Greek letters. Thus l, u shall denote lower and upper approximation operators (operations acting on a subset of a power-set to itself).
- Modal operators and other unary/binary operators not written in superscript position shall be denoted by math-blackboard bold \mathbb or \mathsf alphabets. For example, $\mathbb{L}, \mathsf{L}$ can denote unary operators. If Greek letters are used for denoting operators, then Upright Greek letters versions should be preferred (as in \uepsilon).
- Because most of the content relates to order structures, logical or meta-logical conjunction will be denoted by &. In a model S, $(\Phi \& \Psi \longrightarrow \Pi)$ holds will be preferred over $(\Phi, \Psi \longrightarrow \Pi)$.
- Variables (including those that are quantified over sub-collections of sets) are typically denoted by lower case Roman alphabets.

Information tables (also referred to as information systems) are basically representations of structured data tables. When columns for decisions are also included, then they are referred to as *decision tables*. Often rough sets arise from information systems and decision tables.

An *Information Table* $\mathfrak{I}$, is a system of the form

$$\mathfrak{I} := \langle \mathfrak{O}, \mathbb{A}, \{V_a : a \in \mathbb{A}\}, \{f_a : a \in \mathbb{A}\}\rangle$$

with $\mathfrak{O}$, $\mathbb{A}$ and V_a being respectively sets of *Objects*, *Attributes* and *Values* respectively. $f_a : \mathfrak{O} \longmapsto \wp(V_a)$ being the valuation map associated with attribute $a \in \mathbb{A}$. Values may also be denoted by the binary function $\nu : \mathbb{A} \times \mathfrak{O} \longmapsto \wp(V)$

defined by for any $a \in \mathbb{A}$ and $x \in \mathfrak{O}$, $\nu(a, x) = f_a(x)$. Information tables can also be represented as relational systems using higher order constructions.

An information table is *deterministic* (or complete) if

$$(\forall a \in \mathbb{A})(\forall x \in \mathfrak{O})\, f_a(x) \text{ is a singleton.}$$

It is said to be *indeterministic* (or incomplete) if it is not deterministic that is

$$(\exists a \in \mathbb{A})(\exists x \in \mathfrak{O})\, f_a(x) \text{ is not a singleton.}$$

Relations may be derived from information systems by way of conditions of the following form: For $x, w \in \mathfrak{O}$ and $B \subseteq \mathbb{A}$, $(x, w) \in \sigma$ if and only if $(\mathbf{Q}a, b \in B)\, \Phi(\nu(a, x), \nu(b, w),)$ for some quantifier $\mathbf{Q}$ and formula Φ. The relational system $S = \langle \underline{S}, \sigma \rangle$ (with $\underline{S} = \mathbb{A}$) is said to be a *general approximation space*.

In particular if σ is defined by the condition Eq. (1), then σ is an equivalence relation and S is referred to as an *approximation space*.

$$(x, w) \in \sigma \text{ if and only if } (\forall a \in B)\, \nu(a, x) = \nu(a, w) \tag{1}$$

Relative to an information system, many different concepts of *rough objects* have been used in the literature. These are represented in different ways. In classical rough sets, starting from an approximation space consisting of a pair of a set and an equivalence relation over it, approximations of subsets of the set are constructed out of equivalence partitions of the space (these are crisp or definite) that are also regarded as granules in many senses.

A *Partial Algebra* S is a tuple of the form

$$\langle \underline{S}, f_1, f_2, \ldots, f_n, (r_1, \ldots, r_n) \rangle$$

with $\underline{S}$ being a set, f_i's being partial function symbols of arity r_i. The interpretation of f_i on the set $\underline{S}$ should be denoted by $f_i^{\underline{S}}$, but the superscript will be dropped unless that must be emphasized. If predicate symbols enter into the signature, then P is termed a *Partial Algebraic System*. It is a *Total Algebraic System* (or simply an algebraic system), when all the operations are total. The symbol S may also be used for the set $\underline{S}$, whenever the usage is unambiguous in the context.

Definition 1 The following properties of a binary relation R on a set S are well known:

$$(\forall a)(\exists b)\, Rab \tag{Serial}$$

$$(\forall a)(\exists b)\, Rba \tag{Inverse-Serial}$$

$$(\forall a)\, Raa \tag{Reflexive}$$

$$(\forall a, b)\, (Rab \longrightarrow Rba) \tag{Symmetric}$$

$$(\forall a, b, c)\,(Rab\,\&\,Rac \longrightarrow Rbc) \qquad \text{(Euclidean)}$$

$$(\forall a, b, c)\,(Rab\,\&\,Rac \longrightarrow (\exists f)Rbf\&Rcf) \qquad \text{(Strict Up-Confluent)}$$

$$(\forall a, b, c)\,(Rab\,\&\,Rac\,\&\,b \neq c \longrightarrow (\exists f)Rbf\&Rcf) \qquad \text{(Up-Confluent)}$$

Definition 2 A binary relation R on a set S is *weakly-transitive, transitive or proto-transitive* respectively on S if and only if S satisfies

- If Rab, Rbc and $a \neq b \neq c$ holds, then Rac. (i.e. $(R \circ R) \setminus \Delta_S \subseteq R$ (where $\circ$ is relation composition) , or
- Whenever $Rab\,\&\,Rbc$ holds then Rac (i.e. $(R \circ R) \subseteq R$), or
- Whenever Rab, Rbc, Rba, Rcb and $a \neq b \neq c$ holds, then Rac follows, respectively. Proto-transitivity of R is equivalent to $\tau(R) := R \cap R^{-1}$ being weakly transitive.

Definition 3 A binary relation R on a set S is *semi-transitive* on S if and only if S satisfies

- Whenever $\tau(R)ab\&Rbc$ holds, then Rac follows and
- Whenever $\tau(R)ab\&Rca$ holds, then Rcb follows.

$Ref(S)$, $Sym(S)$, $Tol(S)$, $r\tau(S)$, $w\tau(S)$, $p\tau(S)$, $s\tau(S)$, $EQ(S)$ will respectively denote the set of reflexive, symmetric, tolerance, transitive, weakly transitive, pseudo transitive, semi-transitive and equivalence relations on the set S respectively.

By a *pseudo order* is meant an anti-symmetric, reflexive relation. A *quasi-order* is a reflexive, transitive relation, while a partial order is a reflexive, anti-symmetric and transitive relation. The set of pseudo, quasi and partial orders on a set S will be denoted by $\pi O(S)$, $Q(S)$ and $PO(S)$, respectively.

The *successor neighborhoods*, *inverted successor or predecessor neighborhoods*, *multiplicative* and *additive neighborhoods* generated by an element $x \in S$ will, respectively, be denoted as follows:

$$[x] := \{a;\ Rax\} \qquad \text{(Successor)}$$

$$[x]_i := \{a;\ Rxa\} \qquad \text{(Predecessor)}$$

$$[x]_o := \{a;\ Rax\,\&\,Rxa\} \qquad \text{(Multiplicative)}$$

$$[x]_\vee := \{a;\ Rax\ \vee\ Rxa\} \qquad \text{(Additive)}$$

It may be noted that successor neighborhoods have also been denoted by $R^{-1}(x)$ and predecessor neighborhoods by $R(x)$.

A *cover* $\mathcal{C}$ on a set $\underline{S}$ is any sub-collection of $\wp(\underline{S})$. They are sometimes confusingly referred to as *partial covers*. It is said to be *proper* just in case $\bigcup \mathcal{C} = \underline{S}$.

A *neighborhood operator* n on a set $\underline{S}$ is any map of the form $n : \underline{S} \longmapsto \wp\underline{S}$. It is said to be *reflexive* if

$$(\forall x \in \underline{S})\, x \in n(x) \qquad \text{(Nbd:Refl)}$$

The collection of all neighborhoods $\mathcal{N} = \{n(x) : x \in \underline{S}\}$ of $\underline{S}$ will form a proper cover if and only if $(\forall x)(\exists y) x \in n(y)$ (anti-seriality or inverse-seriality). In particular a reflexive relation on $\underline{S}$ is sufficient to generate a proper cover on it. Of course, the converse association does not necessarily happen in a unique way. More details about the connection between neighborhoods and covers can be found in the chapter on duality in this volume.

If $\mathcal{S}$ is a cover of the set $\underline{S}$, then the *neighborhood* of $x \in \underline{S}$ is defined via,

$$nbd(x) := \bigcap \{K : x \in K \in \mathcal{S}\} \qquad \text{(Cover:Nbd)}$$

The *minimal description* of an element $x \in \underline{S}$ is defined to be the collection

$$\mathrm{Md}(x) := \{A : x \in A \in \mathcal{S}, \forall B(x \in B \rightarrow \sim (A \subset B))\} \qquad \text{(Cover:MD)}$$

The *indiscernibility* (or friends) of an element $x \in \underline{S}$ is defined to be

$$Fr(x) := \bigcup \{K : x \in K \in \mathcal{S}\} \qquad \text{(Cover:FR)}$$

The word *indiscernibility* is also used in other senses.

An element $K \in \mathcal{S}$ is said to be *reducible* if and only if

$$(\forall x \in K) K \notin \mathrm{Md}(x) \qquad \text{(Cover:Red)}$$

The collection $\{nbd(x) : x \in S\}$ will be denoted by $\mathcal{N}$. The cover obtained by the removal of all reducible elements is called a *covering reduct*.

2 Ordered Algebras

In a lattice L, an element a^* is a *pseudo-complement* if and only if

$$(\forall x) a \wedge a = 0 \leftrightarrow x \leq a^*$$

Pseudo-complements are unique when they exist and a lattice in which every element has a pseudo-complement is said to be a *pseudo-complemented lattice*. An element a of a pseudo-complemented lattice is *dense* if $a^* = 0$.

In any pseudo-complemented lattice L, it is possible to define the *Glivenko congruence* σ as follows:

$$\sigma ab \text{ if and only if } a^* = b^*$$

A *De Morgan Lattice* or a *Quasi-Boolean Algebra* (ΔML) is an algebra of the form $L = \langle \underline{L}, \vee, \wedge, c, 0, 1 \rangle$ with $\vee, \wedge$ being distributive lattice operations and c

satisfying

$$x^{cc} = x \quad \text{(Complement-1)}$$

$$(x \vee a)^c = x^c \wedge a^c \quad \text{(DeMorgan)}$$

$$(x \leq a \leftrightarrow a^c \leq x^c) \quad \text{(Complement-2)}$$

0 and 1 are the least and greatest elements of the lattice. They can also be read as nullary operations.

It is possible to define a partial unary operation *, via $x^\star := \bigwedge\{x : x \leq x^c\}$ on any ΔML. If it is total, then the ΔML is said to be *complete*. In a complete ΔML L,

$$x^\star \nleq x^c \quad \text{(CQBA1)}$$

$$x^{\star\star} = x \quad \text{(CQBA2)}$$

$$(x \leq a \longrightarrow a^\star \leq x^\star) \quad \text{(CQBA3)}$$

$$x^c = \bigvee\{a : x^\star \nleq a\} \quad \text{(CQBA4)}$$

A ΔML is said to be a *Kleene algebra* if it satisfies

$$x \wedge x^c \leq a \vee a^c \quad \text{(KA)}$$

If $L^+ = \{x \vee x^c : x \in L\}$ and $L^- = \{x \wedge x^c : x \in L\}$, then in a Kleene algebra all of the following hold:

$$(L^-)^c = L^+ \text{ is a filter,}$$

$$(L^+)^c = L^- \text{ is an ideal,}$$

$$(\forall a, b \in L^-)\, a \leq b^c,$$

$$(\forall a, b \in L^+)\, a^c \leq b, \text{ and}$$

$$x \in L^- \leftrightarrow x \leq x^c.$$

A *Heyting algebra* K, is a relatively pseudo-complemented lattice, that is it is a bounded lattice in which the relative pseudo-complementation operation $\Rightarrow$ is well defined:

$$(\forall a, b)\, a \Rightarrow b := \bigvee\{x\,;\, a \wedge x \leq b\} \in K. \quad \text{(RPC)}$$

The following properties of a Heyting algebra are well known:

- Heyting algebras are distributive lattices.
- A complete lattice is a Heyting algebra if and only if finite meets distribute over arbitrary joins.

- The pseudocomplement of an element a in a Heyting algebra is $a \Rightarrow 0$.
- The lattice of congruences $Con(L)$ of a Heyting algebra L corresponds to the lattice of filters $\mathcal{F}(L)$ of L that in turn determines the former because for each $\sigma \in Con(L)$ there exists a $F \in \mathcal{F}(L)$ such that σab if and only $(\exists x \in F)\, a \wedge x = b \wedge x$.
- The quotient $L|\sigma$ of a Heyting algebra by a congruence is a Boolean algebra if and only if the filter corresponding to σ includes all dense elements of L. Such congruences are called *Boolean congruences*.
- The Glivenko congruence is the least Boolean congruence on a Heyting algebra.

A *Quasi-Nelson* algebra Q is a Kleene algebra that satisfies $(\forall a, b)\, a \Rightarrow (a^c \vee b) \in Q$. $a \Rightarrow (a^c \vee b)$ is abbreviated by $a \rightarrow b$ below. Such an algebra satisfies all of the sentences N1–N4:

$$x \rightarrow x = 1 \tag{N1}$$

$$(x^c \vee y) \wedge (x \rightarrow y) = x^c \vee y \tag{N2}$$

$$x \wedge (x \rightarrow y) = x \wedge (x^c \vee y) \tag{N3}$$

$$x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z) \tag{N4}$$

$$(x \wedge y) \rightarrow z = x \rightarrow (y \rightarrow z). \tag{N5}$$

A Nelson algebra is a quasi-Nelson algebra satisfying N5. A Nelson algebra can also be defined directly as an algebra of the form $\langle A, \vee, \wedge, \rightarrow, c, 0, 1\rangle$ with $\langle A, \vee, \wedge, c, 0, 1\rangle$ being a Kleene algebra with the binary operation $\rightarrow$ satisfying N1–N5. Further a *semi-simple Nelson algebra* is a Nelson algebra that satisfies the condition:

$$a \vee a^c = 1 \tag{Nelson-SS}$$

If L is a Heyting algebra, then the lattice

$$N(L) = \{(a, b) : a \wedge b = 0 \,\&\, (a \vee b)^* = 0\}$$

is an *effective lattice* that is also a Nelson algebra.

A *double Heyting algebra* $L = \{L, \wedge, \vee, \rightarrow, \ominus, 0, 1\}$ is an algebra such that $\{L = \{L, \wedge, \vee, \rightarrow, 0, 1\}$ is a complete atomic Heyting algebra and

$$x \ominus x = 0 \tag{2HA2}$$

$$x \vee (x \ominus y) = x \tag{2HA3}$$

$$(x \ominus y) \vee, y = x \vee y \tag{2HA4}$$

$$(x \vee y) \ominus z = (x \ominus z) \vee (y \ominus z) \tag{2HA5}$$

$$z \ominus (x \wedge y) = (z \ominus x) \vee (z \ominus y) \tag{2HA6}$$

It can be shown that completely distributive lattices are double Heyting algebras.

A *pre-rough algebra* is an algebra of the form $S = \langle \underline{S}, \sqcap, \sqcup, \Rightarrow, L, \neg, 0, 1\rangle$ of type (2, 2, 2, 1, 1, 0, 0), which satisfies:

$$\langle \underline{S}, \sqcap, \sqcup, \neg, 0, 1\rangle \text{ is a QBA}$$

$$LL(a) = L(a)\,;\; L(1) = 1\,;\; L(a) \sqcap a = L(a) \tag{PRA0}$$

$$L(a \sqcap b) = L(a) \sqcap L(b)\,;\; L(a \sqcup b) = L(a) \sqcup L(b) \tag{PRA1}$$

$$\neg L \neg L(a) = L(a), ;\; \neg L(a) \sqcup L(a) = 1 \tag{PRA2}$$

$$(L(a) \sqcap L(b) = L(a), \neg L(\neg(a \sqcap b)) = \neg L(\neg a) \longrightarrow a \sqcap b = a) \tag{PRA3}$$

It is possible to define an operation $\Rightarrow$ as follows:

$$a \Rightarrow b := (\neg L(a) \sqcup L(b)) \sqcap (L(\neg a) \sqcup \neg L(\neg b)) \tag{RI0}$$

It is known that the above definition of pre-rough algebra has superfluous conditions. A completely distributive pre-rough algebra is called a *rough algebra*. In all these algebras it is possible to define an operation $\Box$ by setting $\Box(x) = \neg L \neg(x)$ for each element x.

An *MV-algebra* is an algebra of the form $S = \langle \underline{S}, \oplus, ', 0\rangle$ that satisfies the following axioms:

$$(a \oplus b) \oplus c = b \oplus (c \oplus a) \tag{MV1}$$

$$a \oplus 0 = a \tag{MV2}$$

$$a \oplus 0' = 0' \tag{MV3}$$

$$(0')' = 0 \tag{MV4}$$

$$(a' \oplus b)' \oplus b = (a \oplus b')' \oplus a \tag{MV5}$$

The derived term operations $\odot$ and 1 defined as per

$$a \odot b := (a' \oplus b')' \text{ and } 1 = 0',$$

lead to the original definition (equivalent) of MV-algebras as algebras of the form $S = \langle \underline{S}, \oplus, \odot, ', 0, 1\rangle$.

In any MV algebra it is possible to define the lattice operations $\wedge$ and $\vee$ as follows:

$$(\forall a, b)\; a \vee b := (a \odot b') \oplus b$$

$$(\forall a, b)\; a \wedge b := (a \oplus b') \odot b$$

The resulting derived algebra $K = \langle \underline{S}, \wedge, \vee, ', 0, 1\rangle$ is a Kleene algebra.

A *three valued Łukasiewicz algebra* is an algebra of the form $L = \langle \underline{L}, \wedge, \vee, ', \mu, 0, 1 \rangle$ in which the forgetful algebra $\langle \underline{L}, \wedge, \vee, ', 0, 1 \rangle$ is a Kleene algebra and the unary operation μ satisfies the following:

$$\neg a \vee \mu(a) = 1, \tag{L3a}$$

$$\neg a \wedge \mu(a) = \neg a \wedge a, \text{ and} \tag{L3b}$$

$$\mu(a \wedge b) = \mu(a) \wedge \mu(b). \tag{L3c}$$

It can be shown that these are equivalent to semi-simple Nelson algebras and regular double Stone algebras.

Algebraic Methods for Rough Approximation Spaces by Lattice Interior–Closure Operations

Gianpiero Cattaneo

Abstract This chapter deals with the abstract approach to rough sets theory through the equational notion of closure operator in the context of lattice theory, with the associated notion of internal operator as not-closure-not. The involved lattice structures are not necessarily distributive to allow the development of rough theories in the so-called logical-algebraic context of Quantum Mechanics, based on non-distributive lattices of orthomodular type. The chapter is organized into four parts.

In Part I the more general lattice notion of closure operator is introduced, and the induced notion of interior operator is discussed. It is shown that this lattice approach of the inner-closure pairs is categorically equivalent to the non-equational abstract notion of approximation space based on the lower-upper approximation of each lattice element.

Part II deals with three variations of closure operators called respectively, from the more general to the stronger one, as Tarski, Kuratowski and Halmos closures. A characterization of this last is given in terms of Brouwer Zadeh lattice structure.

Part III provides an interpretation of the pairs of internal-closure operators in terms of pairs of necessity-possibility operators in the context of suitable modal logics. A Kripke-like semantic of such kinds of logics is also provided based on a set of possible worlds. The usual approach to the concrete theory of rough sets through Pawlak information systems is investigated in this context.

Part IV treats these internal-closure operators in context of Łukasiewicz algebraic structures, stronger than the Halmos ones. It is shown that the usual fuzzy sets theory is a model of such structures. Finally, the correlation between Łukasiewicz algebraic structures and Nelson algebras is discussed.

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1 Introduction

In this introduction I will make some considerations in order to clarify some preliminary points which are at the base of the present chapter. To this purpose, let us consider a structure $\langle \Sigma, \wedge, \vee, ^{\heartsuit}, ^{\spadesuit}, 0, 1 \rangle$, named for simplicity DD algebra, which is a lattice with respect to the binary operations of meet $\wedge$ and join $\vee$, bounded by the least element 0 and the greatest element 1. This lattice is equipped with two further unary operations, say $^{\heartsuit} : \Sigma \to \Sigma$ and $^{\spadesuit} : \Sigma \to \Sigma$, satisfying some axioms which at the moment it is not important to specify here, since they will be the subject of the following Sect. 2. Our interest is about a so to speak "neutral" formal structure which allows one to make the following meta-theoretical considerations.

Sometimes it happens that different scientific communities work about the same formal structure assigning different names with different interpretations to the same formal terms. Relatively to the just outlined DD structure, in this chapter we encounter three of these different positions about it, which we list and briefly discuss now.

(TL) The first one is the *topological lattice* interpretation in which the elements from Σ are interpreted as objects of a pointless approach to topology (abstract counterpart of subsets of a concrete universe of points). The binary operations $\wedge$ and $\vee$ are interpreted as abstract lattice counterparts of the intersection and union operations on subsets of the concrete universe. The unary operation $^{\heartsuit}$ is interpreted as a *complementation* mapping, in this topological lattice context denoted as $' =^{\heartsuit}$, while the unary operation $^{\spadesuit}$ is interpreted as a *closure* mapping, denoted as $^* =^{\spadesuit}$.

In this context it is possible to introduce another mapping, the *interior* $^o = {}^{\heartsuit\spadesuit\heartsuit} = {}^{\prime * \prime}$.

In the particular case in which $\Sigma = \mathscr{P}(X)$ is the power set of a universe X, we will use the notations $C(A)$ for the closure of a subset A and $I(A)$ for its interior.

(RAS) Another position is the abstract approach to *roughness* in which now the elements from Σ are interpreted as *approximable* elements (i.e., elements which can be approximated) and the unary operation $^{\heartsuit}$ continues to be interpreted as the *complementation* mapping, now denoted as $- = {}^{\heartsuit}$. But it is the unary operation $^{\spadesuit}$, denoted as u and defined by the law $\forall a \in \Sigma, u(a) := a^{\spadesuit}$, which is interpreted as the *upper approximation* map assigning to any abstract approximable element $a \in \Sigma$ its *upper approximation* $u(a) = a^{\spadesuit}$.

In this context the unary operation l defined by the law $l(a) := a^{\heartsuit\spadesuit\heartsuit} = -(u(-(a)))$ is interpreted as the *lower approximation* map.

In the particular case of $\Sigma = \mathscr{P}(X)$ we denote by $U(A)$ the upper approximation of the approximable subset A and by $L(A)$ its lower approximation.

(ML) Finally, the third position is the one in which the elements from Σ are interpreted as realizations of *propositions* of an algebraic approach to modal logic in which the binary operation $\wedge$ is interpreted as algebraic realization of the logical connective AND and the binary connective $\vee$ as algebraic realization of the logical connective OR. The unary connective $^{\heartsuit}$ is interpreted as algebraic realization of the logical connective NOT, denoted for every proposition $a \in \Sigma$ as $\neg(a) = a^{\heartsuit}$, while the unary connective $^{\spadesuit}$ is interpreted as algebraic realization of the modal logical connective of *possibility*, denoted for every proposition $a \in \Sigma$ as $\mu(a) = a^{\spadesuit}$.
In this context the unary operation ν defined by the law $\nu(a) := a^{\heartsuit\spadesuit\heartsuit} = \neg(\mu(\neg(a)))$ is interpreted as the modal logical connective of *necessity*.
In the particular case of $\Sigma = \mathscr{P}(X)$ we use $\mathbf{I}(A)$ for the necessity and $\mathbf{C}(A)$ for the possibility of A.

As to the relationship between the positions (TL) and (ML) we want to be clear in the sense that in the current chapter we are only interested in investigating their relationship as a kind of dictionary correlating the terms of the dichotomy "topological lattice and modal logic" (for instance $a' \leftrightarrow \neg(a)$ or $a^* \leftrightarrow \mu(A)$), and nothing more.

Therefore we will not deal with the subtle but no less important theoretical results of modal logic, such as theorems of completeness or soundness, or other notable issues involving the community of logicians. These topics could be the subject of other chapters of this book and not what will be discussed in this chapter where, as previously emphasized, we will deal only with the terminological vocabulary that correlates these two interpretations of the same algebraic structure, developing some of its possible interesting theoretical results.

Part I: Standard Closures, Induced Interiors and Rough Approximation Spaces (RAS)

After these general considerations about the "philosophy" underlying the present chapter, we anticipate that in the Part I we deal with the notion of *closure operation*, and the induced notion of *interior operation*, in the abstract De Morgan lattice context described in point (TL). Then we investigate the categorical equivalence between *interior–closure operations* of point (TL) and *lower–upper rough approximations* of point (RAS). This De Morgan choice for the purpose of capturing in a very general structure almost all the interesting results that the standard approach to roughness based on a Boolean algebra (or its version on the power set of a concrete universe) usually does (let us recall that a Boolean algebra is an *orthocomplemented* and *distributive* lattice).

This approach is not only interesting in itself since it is formulated in the weakest structure of De Morgan lattice, allowing to obtain theoretical results in a more general context, but also to have as concrete models both the usual fuzzy set theory and that of quantum mechanics in the sharp and unsharp versions. In fact, it must be stressed that these models cannot be led to the particular Boolean case since

– fuzzy set theory is based on a distributive complete lattice structure which **is not** orthocomplemented, but satisfies the weaker notion of *Kleene orthocomplementation* [121] (and see also [17]);
– on the other hand, sharp quantum mechanics is based on a structure of orthocomplemented complete lattice which **is not** distributive, but satisfies the weaker *orthomodularity* condition [9] (see also [29]). Even more general is the situation of unsharp quantum mechanics based on a Kleene orthocomplemented **poset** structure which **is not** a lattice [13, 22, 26, 28], in relation to which, of course, it makes no sense to speak of lattice operations and so, a fortiori, of the corresponding distributive property.

2 Closure Operations in Lattice Context and Upper Approximation Spaces

Let us recall the fundamental notion of *closure operation*, formalized in a lattice context for instance in [8, p. 112], which can also be found in Ward [114] (always in the abstract lattice context) and in Ore [85] (in the concrete context of the complete lattice from the power set of a universe).

Definition 1 A *De Morgan lattice with closure operation* is a structure $\mathfrak{DMC} = \langle \Sigma, \wedge, \vee, ', ^*, 0, 1 \rangle$ where:

(Cl-dM1) The sub-structure $\langle \Sigma, \wedge, \vee, 0, 1 \rangle$ of $\mathfrak{DMC}$ is a (not necessarily distributive) lattice bounded by the least element 0 and the greatest element 1, $\forall a \in \Sigma,\ 0 \leq a \leq 1$. The partial order relation induced by the lattice operations is $a \leq b$ iff $a = a \wedge b$, or equivalently iff $b = a \vee b$.

(Cl-dM2) The sub-structure $\langle \Sigma, \wedge, \vee, ', 0, 1 \rangle$ of $\mathfrak{DMC}$ is a *De Morgan* lattice, i.e., a bounded lattice equipped with a De Morgan unary mapping $' : \Sigma \mapsto \Sigma$ that satisfies for arbitrary $a, b \in \Sigma$ the conditions:

(dM1)	$a = a''$	(involution condition),
(dM2)	$(a \vee b)' = a' \wedge b'$	(De Morgan law).

This mapping $'$ is generically called *complementation mapping*.

(Cl-dM3) The mapping $^* : \Sigma \to \Sigma$, which associates with any element a from Σ the element $a^* \in \Sigma$, is a *closure operation*, that is for any arbitrary pair of elements $a, b \in \Sigma$ it satisfies the properties:

(C1)	$a \leq a^*$	(increasing or extensive)
(C2)	$a^* \vee b^* \leq (a \vee b)^*$	(sub-additive)
(C3)	$a^* = a^{**}$	(idempotent)

For the sake of simplicity we also say that Σ is a *closure lattice* instead of a De Morgan lattice with a closure operation.

Any De Morgan lattice can be equipped with two standard closure operations.

Example 2 Let Σ be a De Morgan lattice Then, the unary operation $^{*_t} : \Sigma \to \Sigma$ associating with any element $a \in \Sigma$ the element $a^{*_t} := 1 \in \Sigma$ is a closure operation called the *trivial closure*.

In particular it is $0^{*_t} = 1$.

Example 3 Let Σ be a De Morgan lattice. Then, the unary operation $^{*_d} : \Sigma \to \Sigma$ associating with any element $a \in \Sigma$ the element $a^{*_d} := a \in \Sigma$ is a closure operation called the *discrete closure*.

In particular it is $0^{*_d} = 0$.

Remark 4 We make now some formal conventions in the case of a complete lattice Σ. First of all, for any family of its elements $\{a_j\}$, with j index running in the index set J, the corresponding meet and join are denoted by $\wedge \{a_j\}$ and $\vee \{a_j\}$, respectively. In the particular case of a family of two elements $\{a, b\}$ this would correspond to the notations $\wedge \{a, b\}$ and $\vee \{a, b\}$, but as done in Definition 1 and as we will do in the sequel, we prefer to keep the standard notations $a \wedge b$ and $a \vee b$. Sometimes, for reasons of simplicity and if this does not generate any confusion, we will also use to write $\wedge a_j$ and $\vee a_j$ instead of $\wedge \{a_j\}$ and $\vee \{a_j\}$.

2.1 About the Complementation Operation in De Morgan Lattices

In this subsection we investigate some important properties of the complementation operation $' : \Sigma \to \Sigma$ in De Morgan lattices. A first interesting result is the following.

Lemma 5 *In any De Morgan lattice one has that* $0' = 1$ *and* $1' = 0$.

We have defined the De Morgan complementation $'$ by the equational condition (dM2). This is not the unique way to make this definition according to the following result whose proof is straightforward.

Proposition 6 *Let Σ be a bounded lattice according to the point (Cl-dM1) of the above Definition 1. Then, under condition (dM1) the following are equivalent among them.*

$$
\begin{array}{lll}
\text{(dM2)} & (a \vee b)' = a' \wedge b' & \text{(De Morgan law)}, \\
\text{(dM2a)} & (a \wedge b)' = a' \vee b' & \text{(dual De Morgan law)}, \\
\text{(dM2b)} & a \leq b \quad \textit{implies} \quad b' \leq a' & \text{(contraposition law)}, \\
\text{(dM2c)} & a' \leq b' \quad \textit{implies} \quad b \leq a & \text{(dual contraposition law)}.
\end{array}
$$

Remark 7 The formulation (dM2) adopted in Definition 1 to introduce the notion of De Morgan complementation has the advantage of being equational, using the symbols $\wedge$ and $\vee$ of the underlying lattice structure. In the case of a distributive lattice without the least element 0 (and so, also without the greatest element 1) the corresponding structure has been called *distributive i-lattice* by J.A. Kalman [61]. The case of a bounded distributive lattice equipped with a De Morgan complementation has been studied by A. Bialynicki–Birula and H. Rasiowa [6] and H. Rasiowa [99] with the name of *De Morgan algebra*.

The formulation (dM2b), on the contrary, is not equational and this could be a theoretical disadvantage, but has the advantage that it allows to introduce a notion of De Morgan complementation $'$ also in the more general context of bounded poset structures $\langle \Sigma, \leq, ', 0, 1 \rangle$. This happens for example in the following articles: Monteiro–Ribeiro [82] (in the abstract poset context) and, in connection with Galois theory, Ore [86] and Everett [42] (both with a poset version of the closure).

Lemma 8 *A De Morgan complementation $' : \Sigma \to \Sigma$ on a lattice Σ is a* bijection*, which is* antitone *[8, p. 3], i.e., it satisfies the two mutually equivalent conditions:*

$$
\begin{array}{ll}
(dM2b) \quad a \leq b \quad \textit{implies} \quad b' \leq a' & (\textit{contraposition law}), \\
(dM2c) \quad a' \leq b' \quad \textit{implies} \quad b \leq a & (\textit{dual contraposition law}).
\end{array}
$$

Antitone bijections are called dual isomorphisms *or also* involutive anti-morphisms.

Proof The De Morgan operation is surjective since $\forall a \in \Sigma, \exists a' \in \Sigma$ s.t. $(a')' = a$ for the (dM1). On the other hand that the (dM2b) is true has been stated in the formulation of Definition 1, point (Cl-dM2). Now, let $\forall a, b \in \Sigma, a \leq b \Rightarrow b' \leq a'$ then putting $a = \alpha'$ and $b = \beta'$ we get by (dM1) $\forall \alpha, \beta \in \Sigma, \alpha' \leq \beta' \Rightarrow \beta \leq \alpha$. The converse is similar. □

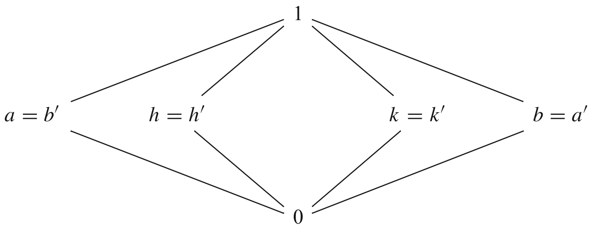


Fig. 1 The six element genuine De Morgan lattice dM6 with two half elements $h \neq k$

An element $h \in \Sigma$ of a De Morgan lattice Σ is said to be a *half element* iff $h = h'$ (note that since $0' = 1$ it is necessary $h \neq 0$). If $h \in \Sigma$ is a half element of Σ then $h \wedge h' = h \neq 0$ and $h \vee h' \neq 1$. The collection of all half elements from Σ, called the *central kernel* of Σ, will be denoted by $N_c{}'(\Sigma) := \{h \in \Sigma : h = h'\}$.

A De Morgan lattice is called *genuine* iff its central kernel contains at least two different half elements $h \neq k$.

Example 9 In Fig. 1 it is shown a genuine De Morgan lattice containing two half elements.

2.2 A Hierarchy of Complementations in De Morgan Lattices

The notion of De Morgan lattice is based on the two conditions (dM1) and (dM2) and in some sense it is the most general definition that can be found in literature. But now we will investigate several cases of structures stronger than this.

2.2.1 Kleene Lattices

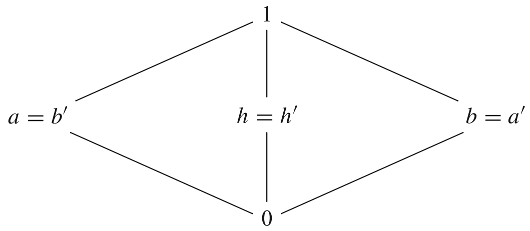
A first strengthening of this minimal notion of De Morgan complementation corresponds to the so-called *Kleene lattice* which is a De Morgan lattice whose complementation mapping satisfies the further condition:

$$\text{(K)} \qquad \forall a, b \in \Sigma,\; a \wedge a' \leq b \vee b'$$

This equational form is equivalent to the non-equational (and so useful in the weaker poset context) condition:

$$\text{(Ka)} \quad \forall a, b \in \Sigma, \;\; a \leq a' \;\text{ and }\; b' \leq b \;\text{ imply }\; a \leq b$$

Fig. 2 The five element genuine modular Kleene lattice K5 with a single half element h



In a Kleene lattice Σ if there exists a half element h, this half element is unique. A Kleene lattice is said to be *genuine* if the central kernel is not empty, that is it admits a single (and so unique) half element: $N'_c(\Sigma) = \{h\}$.

Example 10 Figure 2 shows the Hasse diagram of a five element genuine not distributive (modular) Kleene lattice.

2.2.2 Orthocomplemented Lattices

A further stronger version is the *orthocomplemented lattice*, also *ortholattice* (see [8, p. 52]), in which the De Morgan complementation satisfies the further conditions:

$$\text{(oc-a)} \qquad \forall a \in \Sigma,\ a \wedge a' = 0 \qquad \text{(noncontradiction)}$$

$$\text{(oc-b)} \qquad \forall a \in \Sigma,\ a \vee a' = 1 \qquad \text{(excluded middle)}$$

Trivially, these two conditions (oc-a) and (oc-b) are mutually equivalent between them.

We are particularly interested to three stronger versions of ortholattice according to the following.

(BL) A *Boolean lattice* is an ortholattice Σ which is *distributive*, i.e., it satisfies one of the two mutually equivalent conditions for arbitrary three elements $a, b, c \in \Sigma$:

$$\text{(Bl-1)} \qquad a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c),$$

$$\text{(Bl-2)} \qquad a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

Example 11 (The Boolean (Distributive) Lattice of All Subsets of a Universe) The paradigmatic example of Boolean lattice is the structure $\mathfrak{DL}(X) := \langle \mathscr{P}(X), \cap, \cup, {}^c, \emptyset, X \rangle$ based on the power set $\mathscr{P}(X)$ of a nonempty universe of points X with respect to the set theoretical operations of intersection $\cap$, union $\cup$, and complementation $\forall A \in \mathscr{P}(X)$, $A^c := X \setminus A$.

(fdQL) A *fd-Quantum lattice* is an ortholattice Σ which is *modular*, i.e., it satisfies the following condition for arbitrary three elements $a, b, c \in \Sigma$ (see [8, p. 13], [60, p. 83]):

$$\text{(fd-Ql)} \qquad a \leq c \quad \text{implies} \quad a \vee (b \wedge c) = (a \vee b) \wedge c. \tag{2}$$

(QL) A *Quantum lattice* is an ortholattice Σ which is *orthomodular*, i.e., it satisfies one of the two mutually equivalent conditions for arbitrary two elements $a, b \in \Sigma$ (see [8, p. 53], or [43, 124]):

$$\text{(Ql-1)} \qquad a \leq b \quad \text{implies} \quad a \vee (a' \wedge b) = b,$$

$$\text{(Ql-2)} \qquad a \leq b' \quad \text{implies} \quad a' \wedge (a \vee b) = b.$$

The Quantum (Orthomodular) Lattice of Subspaces of a Hilbert Space

Let us recall that "a (complex) Hilbert space is a vector space over the complex numbers in which there is given a complex valued function of two variables $\langle\phi|\psi\rangle$ such that: (1) For fixed ϕ, $\langle\phi|\psi\rangle$ is a linear function on ψ, (2) $\langle\phi|\psi\rangle = \overline{\langle\psi|\phi\rangle}$, (3) $\langle\phi|\phi\rangle > 0$ unless $\phi = 0$."

Furthermore, setting $\|\phi\| = \sqrt{\langle\phi|\phi\rangle}$, under the distance $d(\phi, \psi) = \|\phi - \psi\|$ the following must be satisfied: (4) $\mathscr{H}$ is a complete metric space (from [71, section 6]; a more complete treatment can be found in [56]).

A *subspace* M of a Hilbert space is any linear manifold (nonempty set closed with respect to the algebraic operations of sum $\psi + \varphi$ and product $\lambda \cdot \psi$ of complex numbers $\lambda \in \mathbb{C}$ times vectors $\psi \in \mathscr{H}$) which is also topologically closed with respect to the metric $d(\psi, \varphi) = \|\psi - \varphi\|$ (for any sequence of vectors $\{\psi_n \in M : n \in \mathbb{N}\}$ from M convergent to a vector $\varphi \in \mathscr{H}$ of the Hilbert space $\mathscr{H}$, i.e., $\lim_{n\to\infty} \|\psi_n - \varphi\| = 0$, it must be $\varphi \in M$). The nonempty condition on M is equivalently formalized by the requirement that the zero vector $\mathbf{0} \in \mathscr{H}$ of the linear part of the Hilbert space must be an element of M, i.e., whatever be the subspace M it is $\mathbf{0} \in M$. There always exist two trivial subspaces of $\mathscr{H}$: the zero dimensional subspace $\{\mathbf{0}\}$ consisting of the unique zero vector and the whole space $\mathscr{H}$.

The collection of all subspaces from $\mathscr{H}$ will be denoted $\mathscr{M}(\mathscr{H})$, which is a partial ordered set $\langle \mathscr{M}(\mathscr{H}), \subseteq, \{\mathbf{0}\}, \mathscr{H} \rangle$ with respect to the set theoretical inclusion $M \subseteq N$ for any pair $M, N \in \mathscr{M}(\mathscr{H})$. This poset is bounded by the least subspace $\{\mathbf{0}\}$ and the greatest subspace $\mathscr{H}$.

It is easy to prove that if $\{M_j \in \mathscr{M}(\mathscr{H}) : j \in J\}$ is any arbitrary collection of subspaces, then their set theoretical intersection is a subspace too which is the g.l.b. with respect to the introduced partial ordering: i.e., $\wedge\{M_j \in \mathscr{M}(\mathscr{H}) : j \in J\} = \cap\{M_j \in \mathscr{M}(\mathscr{H}) : j \in J\}$. Of course, the set theoretical union $\cup\{M_j \in \mathscr{M}(\mathscr{H}) : j \in J\}$ is not a subspace but the l.u.b. exists and it is

$$\vee\{M_j \in \mathscr{M}(\mathscr{H}) : j \in J\} = \cap\{N \in \mathscr{M}(\mathscr{H}) : \cup\{M_j : j \in J\} \subseteq N\}.$$

In conclusion, the structure $\langle \mathcal{M}(\mathcal{H}), \wedge, \vee, \{\mathbf{0}\}, \mathcal{H}\rangle$ is a *complete* (bounded) lattice such that

$$\cup \left\{M_j \in \mathcal{M}(\mathcal{H}) : j \in J\right\} \subseteq \vee \left\{M_j \in \mathcal{M}(\mathcal{H}) : j \in J\right\}. \tag{4}$$

This lattice is **not** distributive. To convince ourself of this, let us consider the following simple example.

Example 12 Consider the three dimensional Hilbert space $\mathbb{C}^3$ of all triples (x_1, x_2, x_3) of complex numbers with the usual operations of vector sum, external product with complex numbers, and inner product

$$\langle (x_1, x_2, x_3)|(y_1, y_2, y_3)\rangle = \sum_{n=1}^{3} \overline{x_n} \cdot y_n.$$

The two one-dimensional subspaces $M_x = \{(x_1, 0, 0) : x_1 \in \mathbb{C}\}$ and $M_y = \{(0, x_2, 0) : x_2 \in \mathbb{C}\}$ are such that $M_x \wedge M_y = M_x \cap M_y = \{(0, 0, 0)\}$, whereas $M_x \vee M_y = \{(x_1, x_2, 0) : x_1, x_2 \in \mathbb{C}\} \supset M_x \cup M_y$, i.e., it is the two-dimensional x, y-plane containing the original two one-dimensional straight lines M_x and M_y.

Furthermore, in the one-dimensional subspace $M_d := \{(x, x, 0) : x \in \mathbb{C}\}$ we have that $M_d \wedge (M_x \vee M_y) = M_d \neq \{\mathbf{0}\} = (M_d \wedge M_x) \vee (M_d \wedge M_y)$, i.e., this lattice is not distributive.

Finally, coming back to $\mathcal{M}(\mathcal{H})$, given a subspace $M \in \mathcal{M}(\mathcal{H})$ its set theoretical complement M^c is not a subspace. This drawback can be overcome introducing for any subspace $M \in \mathcal{M}(\mathcal{H})$ the subset

$$M^{\perp} := \{\varphi \in \mathcal{H} : \forall \psi \in \mathcal{H},\ \langle \varphi|\psi\rangle = 0\},$$

which is a subspace ([56, Theorem 1, p. 40], [51, Theorem 1, p. 24]). Then, the following are standard results of Hilbert space theory:

(OcH-1) $\forall M \in \mathcal{M}(\mathcal{H})\quad M = M^{\perp\perp}$ ([56, Theorem 9, p. 47], [51, Theorem 5, p. 24]),
(OcH-2) $\forall M, N \in \mathcal{M}(\mathcal{H})\quad M \subseteq N$ implies $N^{\perp} \subseteq M^{\perp}$ [51, Theorem 3, p. 24],
(OcH-3) $\forall M \in \mathcal{M}(\mathcal{H})\quad M \wedge M^{\perp} = \{\mathbf{0}\}$ ([56, Corollay 5.1, p. 44], [51, Theorem 7, p. 25]).

From these results it follows that

$$\forall M \in \mathcal{M}(\mathcal{H}),\quad M \vee M^{\perp} = \mathcal{H},\ \text{ with }\ M \cup M^{\perp} \subseteq M \vee M^{\perp}. \tag{5}$$

Hence, the (complete) lattice $\mathfrak{OL}(\mathcal{H}) = \left\langle \mathcal{M}(\mathcal{H}), \wedge = \cap, \vee, {}^{\perp}, \{\mathbf{0}\}, \mathcal{H}\right\rangle$ is orthocomplemented, whose proof of orthomodularity can be found in [36, Theorem 4.18, p. 46], i.e., it is a Quantum lattice.

As to Quantum lattices we must justify the prefix "fd" relative to the modular case. This is due to the following result (see [60, p. 85]):

- A Hilbert space $\mathscr{H}$ is *finite dimensional* (fd) iff the lattice of its subspaces is modular.

Similarly, in [63] one can find the statement:

- One of the simplest models of the modular logic is the lattice of all linear subspaces of an n-dimensional projective space.

The term "quantum" assigned to orthomodular (or modular) lattices is due to the fact that the mathematical foundation of Quantum Mechanics has Hilbert spaces as the basic structure, according to the first and seminal treatment of this argument made by von Neumann in his book [113]. Here *quantum states* are represented by nonzero vectors of a Hilbert space associated to a quantum system. An important role is played by the *yes-no (dichotomic) observables* testing elementary sentences of the form "the value of a measure of the observable lies in the subset Δ of the real line $\mathbb{R}$", for instance "the spin of the particle is up (=1) or is down (=0)" or "the particle is localized in this region of the space $\mathbb{R}^3$". To each of this observable is assigned a subspace of the Hilbert space consisting of the collection of all states with respect to which the elementary sentence is true (probability one of occurrence).

Now we give some examples of *finite* Boolean and Quantum lattices.

Example 13 Figure 3 is the Hasse diagram of a four element Boolean lattice.

Example 14 In Fig. 4 it is shown a fd-Quantum lattice of six elements.

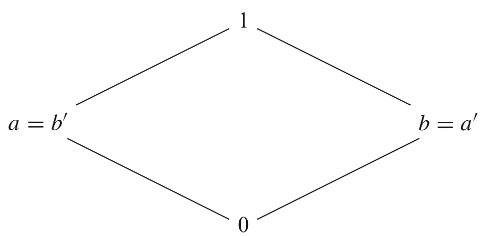


Fig. 3 The Boolean lattice B4, i.e., distributive lattice with standard orthocomplementation

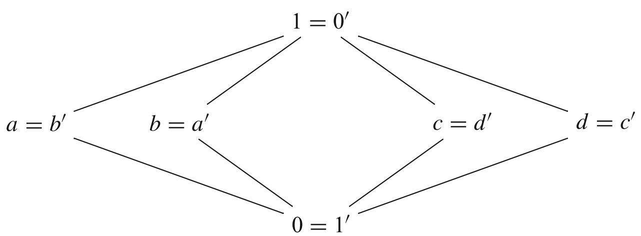


Fig. 4 The six element fd-Quantum lattice QL6

This lattice is not distributive, since $a \wedge (b \vee c) = a \neq 0 = (a \wedge b) \vee (a \wedge c)$. But QL6 is modular since, for instance, the triple $0, b, c$, with $0 \leq c$, is such that $0 \vee (b \wedge c) = b \wedge c = (0 \vee b) \wedge c$; similarly, the triple $a, c, 1$, with $a \leq 1$, is such that $a \vee (c \wedge 1) = a \vee c = (a \vee c) \wedge 1$.

Note that the fd-modular, not distributive, lattice QL6 is the union of the two B4 four element Boolean lattices $\mathfrak{B}_1 = \{0, a, a', 1\}$ and $\mathfrak{B}_2 = \{0, c, c', 1\}$ which in Quantum Mechanics describe the spin along the z-direction and the spin along the x-direction of a $1/2$ spin particle, respectively. Their Boolean behaviour means that singularly they describe a "classical (Boolean)" observable, whereas the fact that their union cannot be covered by a single Boolean lattice leads to the Heisenberg *uncertainty principle* since it means the theoretical impossibility of being *simultaneously* measured by a unique (classical) observable. The two observables are *incompatible*.

We can summarize the behavior of the above introduced ortholattices (BL), (fdQL) and (QL) with the following chain of implications:

$$\boxed{\text{Boolean Lat.}} \Longrightarrow \boxed{\text{fd-Quantum Lat.}} \Longrightarrow \boxed{\text{Quantum Lat.}} \Longrightarrow \boxed{\text{Ortho-Lat.}}$$

This chain of implications must be completed with the following further chain:

$$\boxed{\text{Ortho-Lat.}} \Longrightarrow \boxed{\text{Kleene Lat.}} \Longrightarrow \boxed{\text{De Morgan Lat.}}$$

In the present chapter we adopt the minimal notion of De Morgan complementation since in general all the theoretical results which can be proved do not require (if not explicitly stated) the addition of further conditions, such as (K) or (oc-a,b).

2.3 About the Closure Operation in De Morgan Lattices

A first result on the closure operations is that condition (C3) is in some sense redundant according to the following result.

Lemma 15 *Under condition (C1) the following are equivalent:*

$$(C3) \quad a^* = a^{**} \qquad \text{(idempotent)}$$

$$(C3w) \quad a^{**} \leq a^* \qquad \text{(weak idempotent)}$$

Proof (C3) trivially implies (C3a) as particular case.

Let (C3a) be true, i.e., $a^{**} \leq a^*$. But from (C1) applied to a^* we get $a^* \leq a^{**}$. □

Note that the definition of closure operation given by Definition 1 is equational. Another interesting result is that the sub-additive property (C2) alone is equivalent to a (non-equational) condition according to the following result.

Lemma 16 *In a closure lattice Σ the following two statements are equivalent:*

$$\text{(C2)} \qquad a^* \vee b^* \leq (a \vee b)^* \qquad \text{(sub-additive)}$$

$$\text{(C2a)} \qquad a \leq b \quad \textit{implies} \quad a^* \leq b^* \qquad \text{(isotone)}$$

Proof Let (C2) be true. From $a \leq b$ we get $b = a \vee b$ and so $b^* = (a \vee b)^* \geq a^* \vee b^* \geq a^*$.

Let (C2a) be true. From $a, b \leq a \vee b$ it follows that $a^*, b^* \leq (a \vee b)^*$, i.e., $(a \vee b)^*$ is an upper bound of the pair a, b; but $a^* \vee b^*$ is the least upper bound of the same pair, and so $a^* \vee b^* \leq (a \vee b)^*$. □

Condition (C2a) added to conditions (C1) and (C3) allows one to introduce a notion of closure in the more general context of a poset, which is not necessarily a lattice [42, 82, 86].

Lemma 17 *If Σ is a closure* complete *lattice we have the following extensions of conditions (dM2), (dM2a), and (C2), true for any family $\{a_j\} \subseteq \Sigma$:*

$$\text{(dM2-L)} \quad (\vee \{a_j\})' = \wedge \left\{a'_j\right\};$$

$$\text{(dM2a-L)} \quad (\wedge \{a_j\})' = \vee \left\{a'_j\right\};$$

$$\text{(C2-L)} \quad \vee \left\{a^*_j\right\} \leq (\vee \{a_j\})^*.$$

Proof

(dM2-L) Let us set $k = \wedge \left\{a'_j\right\}$ then $\forall j$, $k \leq a'_j$, from which it follows that $\forall j$, $a_j \leq k'$, that is k' is an upper bound of the family $\{a_j\}$.
Let now c be a generic upper bound of the family $\{a_j\}$, i.e., $\forall j$, $a_j \leq c$, then $\forall j$, $c' \leq a'_j$, i.e., c' is a lower bound of the family $\left\{a'_j\right\}$ but since k is the greatest lower bound of the same family we get $c' \leq k$, that is $k' \leq c$ for any upper bound c of $\{a_j\}$, and this means that k' is the least upper bound of $\{a_j\}$, i.e., $k' = \vee \{a_j\}$, from which $k = (\vee \{a_j\})'$. The proof of the (dM2a-L) is similar to the just proved (dM2-L).

(C2-L) Let us set $h = \vee \{a_j\}$ then $\forall j$, $a_j \leq h$ from which, by (C2a), it follows that $\forall j$, $a^*_j \leq h^*$, i.e., h^* is an upper bound of the family $\left\{a^*_j\right\}$ and so $\vee \left\{a^*_j\right\} \leq h^*$. □

Let us investigate some properties of the now introduced closure operation in the general context of not necessarily complete lattices. First of all, since in general an element of a closure lattice Σ satisfies the increasing condition (C1), it is interesting to single out the subset of *closed elements*, defined as the collection of elements which are equal to their closure.

Formally,

$$\mathscr{C}(\Sigma) := \{c \in \Sigma : c = c^*\}\,. \tag{6}$$

This set is not empty since trivially the greatest element 1 is closed (as consequence of $1 \leq (C1) \leq 1^* \leq 1$).

Condition (C3) says that for any element $a \in \Sigma$ the corresponding a^* is closed; this element is called the *closure* of a.

Remark 18 Let us recall that in [82, Fundamental Theorem 5.1] it is remarked that, given an operation $\varphi : \mathscr{P} \to \mathscr{P}$ on a poset $\mathscr{P}$ which satisfies the only condition (C1), in this context formalized as $\forall a \in \mathscr{P}$, $a \leq \varphi(a)$, if one defines the corresponding family of closed elements formally according to (6), $\mathscr{C}_\varphi(\mathscr{P}) := \{c \in \mathscr{P} : c = \varphi(c)\}$, then the operation φ uniquely determines the family of its closed elements $\mathscr{C}_\varphi(\mathscr{P})$. Moreover it is shown that different mappings from $\mathscr{P}$ to $\mathscr{P}$ satisfying this only condition (C1) can generate the same family of closed elements. But, mappings satisfying condition (C1) are uniquely determined by the family of their closed elements if and only if they satisfy both conditions (C2a) and (C3).

In the context of the power set $\mathscr{P}(X)$ of some universe X the corresponding notion of closure, here formalized in the abstract lattice context by conditions (C1)–(C3) of Definition 1, has been widely treated by E.H. Moore in [83] (see [8, p. 111]). Quoting from [41]: "But probably the explicit and precise concept of closure was introduced into analysis not before the twentieth century, when Friedrich Riesz wrote his pioneering articles [...] (1906) and [...] (1909), E. H. Moore his [...] [83] (1910), and Felix Hausdorff his monograph [...] (1914). To Hausdorff we owe a systematical treatment of topological closure, its relationship to boundary, neighborhoods etc.".

Remark 19 It was A. Tarski who influenced modern mathematical logic by his closure-oriented work about deductive system. Quoting [41]: "The main papers by Tarski from 1923 to 1938 dealing with closure systems in logics are collected in the volume *Logics, Semantics, Metamathematics*, translated by J.H. Woodger [108]. One of the major articles in that collection is entitled *Fundamental concepts of the methodology of the deductive sciences* [107]. Here Tarski focusses on the notion of *deductive(ly closed) system*, by which he means the system of all sentences derivable by certain prescribed logical rules from given axioms [...]. What Tarski establishes first is essentially the definition of *finitary closure operation* (on a countable set of sentences): he shares explicitly the laws of extensivity (C1) and idempotency (C3) and postulates that the closure (the set of consequences via inference rules) of a set A is the union of the closures of all finite subsets of A - a property that, he remarks, implies monotonicity (C2a)." About this argument see also [100, p. 181].

Coming back to the general theory, the following is an interesting result about the structure of the family of closed elements.

Proposition 20 *Let* $\langle \Sigma, \wedge, \vee, {}^*, 0, 1\rangle$ *be a bounded lattice* $\langle \Sigma, \wedge, \vee, 0, 1\rangle$ *with a closure operation* $^* : \Sigma \to \Sigma$. *Then,*

(i) the family $\mathscr{C}(\Sigma) := \{c \in \Sigma : c = c^*\}$ *of its closed elements satisfies the following conditions:*

(PC1) the greatest element 1 *is closed:* $1 \in \mathscr{C}(\Sigma)$*;*
(PC2) $\mathscr{C}(\Sigma)$ *is closed with respect to the meet operation: let* $c, d \in \mathscr{C}(\Sigma)$*, then* $c \wedge d \in \mathscr{C}(\Sigma)$*, i.e.,* $\mathscr{C}(\Sigma)$ *is a meet semi-lattice.*

(ii) Moreover, if Σ *is a* complete *lattice equipped with a closure operation, condition (PC2) must be substituted by the following:*

(PCC2) $\mathscr{C}(\Sigma)$ *is closed with respect to the meet of any arbitrary family of closed elements: let* $\{c_j\} \subseteq \mathscr{C}(\Sigma)$ *then* $\wedge\{c_j\} \in \mathscr{C}(\Sigma)$.

In particular, the structure $\langle \mathscr{C}(\Sigma), \wedge, 1\rangle$ *is a complete meet semi-lattice upper bounded by the greatest element 1 called, according to [37],* closure system.

Proof

(PC1) We have already seen that the greatest element 1 is closed as consequence of (C1) applied to it, $1 \leq 1^* \leq 1$.

(PC2) Let a, b be two closed elements: $a = a^*$ and $b = b^*$. Then, from $a \wedge b \leq \{a, b\}$ and the isotonicity condition (C2a), equivalent to (C2), we obtain that $(a \wedge b)^* \leq \{a^*, b^*\} = \{a, b\}$. So the element $(a \wedge b)^*$ is a lower bound of the pair a, b and from this we get that $(a \wedge b)^* \leq a \wedge b$. But from (C1) it is $(a \wedge b) \leq (a \wedge b)^*$.

(PCC2) can be demonstrated with a slight modification of the proof of the now proved point (PC2). □

The least element 0 in general is not an element of $\mathscr{C}(\Sigma)$ since it may happen that $0^* \neq 0$. But in the case of a complete lattice the element $\hat{0} := \wedge\{c \in \mathscr{C}(\Sigma)\}$ exists in Σ, moreover condition (PCC2) assures that it is an element of $\mathscr{C}(\Sigma)$ which is the least element of this latter.

Example 21 In the universe $X = \{0, 1, 2\}$, let us consider the closure operation on the Boolean lattice of all its subsets $\langle \mathscr{P}(X), \cap, \cup, {}^c, \emptyset, X\rangle$ given by the following table.

$$
\begin{array}{rlrl}
\emptyset^* &= \{0\} & \{0,1,2\}^* &= \{0,1,2\} \\
\{0\}^* &= \{0\} & \{0,1\}^* &= \{0,1,2\} \\
\{1\}^* &= \{0,1\} & \{0,2\}^* &= \{0,1,2\} \\
\{2\}^* &= \{0,2\} & \{1,2\}^* &= \{0,1,2\}
\end{array}
$$

This is a closure according to Definition 1 in which $\emptyset^* = \{0\} \neq \emptyset$, but all the other axioms (C1)–(C3) are easily verified.

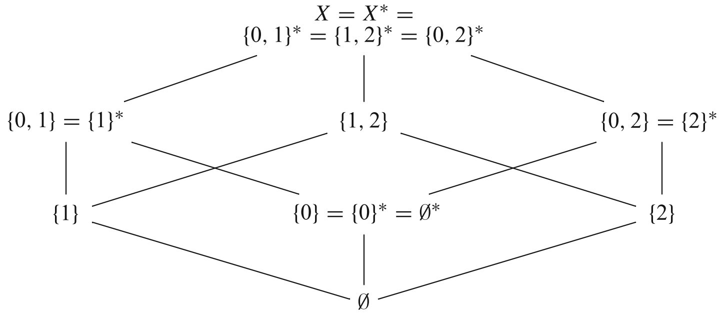


Fig. 5

The Hasse diagram describing this closure, omitting the Boolean complementation $A^c = X \setminus A$, is drawn in Fig. 5:

The corresponding set of closed elements is the pair of subsets $\mathscr{C}(X) = \{\{0\}, X\}$ whose least element is the singleton $\{0\} \neq \emptyset$.

The following result gives an interesting characterization of the closure a^* of every element $a \in \Sigma$ under the complete lattice hypothesis.

Proposition 22 *Let $\langle \Sigma, \wedge, \vee \rangle$ be a nonempty* complete *lattice.*

(TC1) Then, if Σ is equipped with a closure operation * with induced family of closed sets $\mathscr{C}(\Sigma)$ satisfying according, to Proposition 20, conditions (PC1) and (PCC2), then the closure a^* of any element $a \in \Sigma$ can be expressed as

$$a^* = \wedge \{c \in \mathscr{C}(\Sigma) : a \leq c\} \tag{7}$$

(TC2) Conversely, let Σ be equipped with a family of its elements $\mathscr{C}(\Sigma)$ satisfying conditions (PC1) and (PCC2) of Proposition 20, then the mapping $^* : \Sigma \to \Sigma$ associating with any element $a \in \Sigma$ the element $a^* := \wedge\{c \in \mathscr{C}(\Sigma) : a \leq c\}$ is a closure operation in the sense that it satisfies the closure conditions (C1), (C2), and (C3).

Proof Let Σ be a closure complete lattice and for any element $a \in \Sigma$ let us construct the collection $\mathscr{C}(a) := \{c \in \mathscr{C}(\Sigma) : a \leq c\}$. This subset of Σ is not empty since owing to condition (C1) it contains the closed element $1 \in \mathscr{C}(\Sigma)$, with $a \leq 1$.

Let us now suppose that * is a closure operation. Since by (C3) $a^* \in \mathscr{C}(\Sigma)$ and by (C1) $a \leq a^*$, then by definition a^* is an element of $\mathscr{C}(a)$: $a^* \in \mathscr{C}(a)$. Let now $c \in \mathscr{C}(\Sigma)$ be a closed element such that $a \leq c$, then by (C2a) $a^* \leq c^*$, and by the fact that c is a closed element it follows that $a^* \leq c^* = c$ and so $a^* = \wedge \mathscr{C}(a)$.

Conversely, let us suppose that Σ is equipped with a family $\mathscr{C}(\Sigma)$ satisfying conditions (PC1) and (PCC2). By the complete lattice condition the element $a^* := \wedge \{c \in \mathscr{C}(\Sigma) : a \leq c\}$ is well defined as element of Σ and from the condition (PCC2) it is $a^* \in \mathscr{C}(\Sigma)$.

Let us now prove that the correspondence $a \to a^*$ defines a closure operation. First of all also in this situation for any element $a \in \Sigma$ let us define $\mathscr{C}(a) := \{c \in \mathscr{C}(\Sigma) : a \leq c\}$.

(C1) Since a^* is the meet of the family $\mathscr{C}(a)$, and in a complete lattice a is the meet of the family $\Sigma(a) := \{x \in \Sigma : a \leq x\}$, from $\mathscr{C}(a) \subseteq \Sigma(a)$, it follows that $\wedge\Sigma(a) \leq \wedge\mathscr{C}(a)$, i.e., $a \leq a^*$.

(C2a) If $a \leq b$ then $\{c \in \mathscr{C}(\Sigma) : b \leq c\} \subseteq \{d \in \mathscr{C}(\Sigma) : a \leq d\}$, from which one obtains that

$$\wedge \{d \in \mathscr{C}(\Sigma) : a \leq d\} \leq \wedge \{c \in \mathscr{C}(\Sigma) : b \leq c\}\,,\ \text{i.e., } a^* \leq b^*.$$

Let us recall that, according to Lemma 16, (C2a) is equivalent to (C2).

(C3) Applying the just proved (C1) to the element a^* we get $a^* \leq (a^*)^*$. On the other hand, from $a^* \leq a^*$ and $a^* \in \mathscr{C}(\Sigma)$, it follows that $(a^*)^* = \wedge \{d \in \mathscr{C}(\Sigma) : a^* \leq d\} \leq a^*$. So we can conclude that $a^* = (a^*)^*$. □

Point (ii) of Proposition 20 can be completed in the following way.

Proposition 23 *Let $\langle \Sigma, \wedge, \vee, {}^* \rangle$ be a* complete *lattice with a closure operator * and let $\leq$ be the partial order relation induced from the lattice operations: $a \leq b$ iff $a = a \wedge b$ (or equivalently iff $b = a \vee b$).*

Let $\leq_{\mathscr{C}}$ be the restriction to $\mathscr{C}(\Sigma)$ of the partial order $\leq$ on Σ (i.e., $\forall c, d \in \mathscr{C}(\Sigma)$, $c \leq_{\mathscr{C}} d$ iff $c \leq d$), then $\mathscr{C}(\Sigma)$ is a complete lattice satisfying the following properties:

(1) If we denote by $\wedge_{\mathscr{C}}$ the meet on $\mathscr{C}(\Sigma)$ with respect to the partial order $\leq_{\mathscr{C}}$ we have that

$$\forall\{c_j\} \subseteq \mathscr{C}(\Sigma), \quad \wedge_{\mathscr{C}}\{c_j\} = \wedge\{c_j\}. \tag{8}$$

(2) If we denote by $\vee_{\mathscr{C}}$ the join on $\mathscr{C}(\Sigma)$ with respect to the partial order $\leq_{\mathscr{C}}$ we have that

(2a) $\forall\{c_j\} \subseteq \mathscr{C}(\Sigma), \quad \vee_{\mathscr{C}}\{c_j\} = \wedge_{\mathscr{C}}\left\{d \in \mathscr{C}(\Sigma) : \forall c_j \leq_{\mathscr{C}} d\right\},$

(2b) $\forall\{c_j\} \subseteq \mathscr{C}(\Sigma), \quad \vee\{c_j\} \leq \vee_{\mathscr{C}}\{c_j\}.$

Proof

(1) The g.l.b. $\wedge c_j$ in Σ means that: (a) $\forall j$, $(\wedge c_j) \leq c_j$ and (b) $\forall a \in \Sigma$ condition $\forall j, a \leq c_j$ implies $a \leq (\wedge c_j)$.

On the other hand, the g.l.b. $\wedge_{\mathscr{C}} c_j$ in $\mathscr{C}(\Sigma)$ means that: (a$_{\mathscr{C}}$) $\forall j$, $(\wedge_{\mathscr{C}} c_j) \leq_{\mathscr{C}} c_j$ and (b$_{\mathscr{C}}$) $\forall d \in \mathscr{C}(\Sigma)$ condition "$\forall j,\ d \leq_{\mathscr{C}} c_j$" implies "$d \leq_{\mathscr{C}} (\wedge_{\mathscr{C}} c_j)$."

Applying (b) to the element $a = (\wedge_{\mathscr{C}} c_j) \in \mathscr{C}(\Sigma) \subseteq \Sigma$, owing to (a$_{\mathscr{C}}$), one gets $(\wedge_{\mathscr{C}} c_j) \leq (\wedge c_j)$. On the other hand, condition (PCC2) of Proposition 45 assures that $\delta := (\wedge c_j) \in \mathscr{C}(\Sigma)$ and so applying to this closed element condition ($b_{\mathscr{C}}$), owing to ($a_{\mathscr{C}}$), we get $(\wedge c_j) \leq (\wedge_{\mathscr{C}} c_j)$. Let us note that in this part of the proof we have freely applied the fact that "$\leq_{\mathscr{C}}$ iff $\leq$."

(2a) Let us set $\alpha = \wedge_{\mathscr{C}} \left\{ d \in \mathscr{C}(\Sigma) : \forall c_j \leq_{\mathscr{C}} d \right\}$ then $\forall c_j \leq_{\mathscr{C}} \alpha$, that is α is an u.b. in $\mathscr{C}(\Sigma)$ of the family $\left\{ c_j \right\}$. Let now h be an u.b. of $\left\{ c_j \right\}$, that is $\forall c_j \leq h$, then $h \in \left\{ d \in \mathscr{C}(\Sigma) : \forall c_j \leq d \right\}$ and so $\alpha \leq h$, i.e., α is the l.u.b. of $\left\{ c_j \right\}$ in $\mathscr{C}(\Sigma)$. Hence we have $\alpha = \vee_{\mathscr{C}} \left\{ c_j \right\}$. The proof of (2b) is straightforward. □

Example 24 Let us consider the Hilbert space $\mathscr{H}$ generating the Quantum Lattice $\mathfrak{QL}(\mathscr{H})$ based on the collection $\mathscr{M}(\mathscr{H})$ of all its subspaces. On the Boolean algebra of its power set $\langle \mathscr{P}(\mathscr{H}), \cap, \cup, {}^c, \emptyset, \mathscr{H} \rangle$ the mapping $A \in \mathscr{P}(\mathscr{H}) \to A^* \in \mathscr{P}(\mathscr{H})$ defined by the rule $A^* = \cap \{ M \in \mathscr{M}(\mathscr{H}) : A \subseteq M \}$ is a closure operation satisfying conditions (C1)–(C3) for which $\emptyset^* = \{\mathbf{0}\} \neq \emptyset$. Trivially, the collection $\mathscr{C}(\mathscr{H})$ of all closed subsets with respect to this closure operation is just $\mathscr{M}(\mathscr{H})$, the collection of all subspaces: $\mathscr{C}(\mathscr{H}) = \mathscr{M}(\mathscr{H})$.

In this case, and according to the just proved Proposition 23, we have that the meet in $\mathscr{C}(\mathscr{H})(= \mathscr{M}(\mathscr{H}))$ of any family of subspaces $\wedge_{\mathscr{C}} \left\{ M_j : j \in J \right\}$ is their set theoretical intersection $\cap \left\{ M_j : j \in J \right\}$. But the join $\vee_{\mathscr{C}} \left\{ M_j : j \in J \right\}$ in $\mathscr{C}(\mathscr{H})(= \mathscr{M}(\mathscr{H}))$ of the same family is not the corresponding set theoretical union since it is the subspace generated by this set theoretical union

$$\cap \left\{ M \in \mathscr{M}(\mathscr{H}) : \cup_{j \in J} M_j \subseteq M \right\}.$$

2.4 Closure Lattices and Upper Rough Approximation Spaces (URAS): Their Categorical Isomorphism

We are now able to give an equivalent formulation of the notion of lattice with closure operation by the notion of *upper rough approximation space*, structure defined in a non-equational way by the following.

Definition 25 An *upper rough approximation space* is a triple $\langle \Sigma, \mathscr{U}(\Sigma), u \rangle$ where

(U-RAS-1) Σ stays for a structure $\left\langle \Sigma, \wedge, \vee, {}', 0, 1 \right\rangle$ of De Morgan lattice;

(U-RAS-2) $\mathscr{U}(\Sigma)$ stays for a structure $\langle \mathscr{U}(\Sigma), \wedge, 1 \rangle$ of upper bounded by 1 meet semi-lattice, under the condition that $\mathscr{U}(\Sigma) \subseteq \Sigma$;

(U-RAS-3) $u : \Sigma \to \Sigma$ is a unary operation on Σ satisfying the following conditions for any $a \in \Sigma$:

(Up1) $a \leq u(a)$;
(Up2) $u(a) \in \mathscr{U}(\Sigma)$;
(Up3) $\forall c \in \mathscr{U}(\Sigma)$, if $a \leq c$ then $u(a) \leq c$.

Note that these three conditions can be compacted in the unique:

$$\forall a \in \Sigma,\ u(a) = \min \{ c \in \mathscr{U}(\Sigma) : a \leq c \}.$$

Elements from Σ are called *approximable* and elements of the subset $\mathscr{U}(\Sigma) \subseteq \Sigma$ are said to be *upper crisp* or also *upper definable*. Furthermore, the mapping $u : \Sigma \to \Sigma$ is called the *upper rough approximation map*.

The elements of the lattice Σ abstractly represent the family of elements which can be approximated by upper crisp elements from $\mathscr{U}(\Sigma)$ (i.e., they are *approximable elements*). The upper crisp approximation is formalized by the upper rough approximation map u associating with any approximable element $a \in \Sigma$ its upper approximation $u(a)$ according to the condition (Up1). Condition (Up2) assures that this upper approximation is upper crisp and the last condition (Up3) says that this upper approximation is the best from the top by upper crisp elements.

Theorem 26 *Let $\left\langle \Sigma, \wedge, \vee, {}', 0, 1 \right\rangle$ be a De Morgan lattice, simply written as Σ. Suppose a closure lattice $\mathscr{A} = \langle \Sigma, {}^* \rangle$ based on Σ.*

Then, the induced structure $\mathscr{A}^{\blacktriangledown} := \langle \Sigma, \mathscr{U}(\Sigma), u \rangle$ where $\mathscr{U}(\Sigma) := \mathscr{C}(\Sigma)$ is the collection of all closed elements of Σ, and $u : \Sigma \to \Sigma$ is the unary operation on Σ defined for any arbitrary element $a \in \Sigma$ as $u(a) := a^*$ (the closure *of a) is an upper rough approximation space.*

Proof Let us set $u(a) = a^*$ and $\mathscr{U}(\Sigma) = \mathscr{C}(\Sigma) = \{c \in \Sigma : c = u(c)\}$. Then, (Up1) is nothing else than the (C1) written as $a \leq u(a)$. (Up2) From (C3), written as $u(a) = u(u(a))$, it follows that $u(a) \in \mathscr{U}(\Sigma)$. (Up3) Let $c \in \mathscr{U}(\Sigma)$, i.e., $c = u(c)$, if $a \in \Sigma$ is such that $a \leq c$ then by (C2a) we get $u(a) \leq u(c) = c$. □

Let us denote by $\mathfrak{C}$(Cl-L) the *category* of closure lattices and by $\mathfrak{C}$(U-RAS) the *category* of upper rough approximation spaces, then the results of Theorem 26 can be summarized by the following correspondence:

$$\mathscr{A} \in \mathfrak{C}(\text{Cl-L}) \xrightarrow{const_1} \mathscr{A}^{\blacktriangledown} \in \mathfrak{C}(\text{U-RAS}) \tag{9}$$

The converse of Theorem 26 also holds.

Theorem 27 *Let $\langle \Sigma, \wedge, \vee, 0, 1 \rangle$ be a De Morgan lattice, simply written as Σ. Suppose an upper rough approximation space $\mathscr{B} = \langle \Sigma, \mathscr{U}(\Sigma), u \rangle$ based on Σ.*

Then, the induced structure $\mathscr{B}^{\blacktriangle} := \langle \Sigma, {}^* \rangle$ based on Σ equipped with the mapping ${}^* : \Sigma \to \Sigma$, associating with any element $a \in \Sigma$ the element $a^* := u(a) \in \Sigma$, is a closure operation, i.e., it satisfies the closure conditions (C1), (C2), and (C3).

Proof Let us set $a^* = u(a) \in \mathscr{U}(\Sigma)$. Then,

(C1) is nothing else than (Up1) written as $a \leq a^*$. (C2a) Let $a \leq b$. Then from (Up2) the element $b^* = u(b) \in \mathscr{U}(\Sigma)$ is such that, by (Up1), $b \leq b^*$ and so $a \leq b \leq b^*$. In this way we have proved that the element $b^* \in \mathscr{U}(\Sigma)$ is such that $a \leq b^*$ and so from (U3) we get $a^* = u(a) \leq b^*$. (C3) From (C1) it is $a^* \leq a^{**}$. Now we apply the (Up3) to the element a^*: if the element $c \in \mathscr{U}(\Sigma)$ is such that $a^* \leq c$, then $a^{**} \leq c$. Putting $c = a^*$ we obtain that $a^{**} \leq a^*$. □

The results of Theorem 27 can be summarized by the correspondence:

$$\mathscr{B} \in \mathfrak{C}(\text{U-RAS}) \xrightarrow{const_2} \mathscr{B}^{\blacktriangle} \in \mathfrak{C}(\text{Cl-L}) \tag{10}$$

The categorical isomorphism between closure lattices and upper rough approximation spaces is given by the following result, whose proof is straightforward.

Theorem 28 *Starting with a closure lattice, forming its upper rough approximation space version and then coming back to the corresponding closure structure, one recovers the original structure we started with. Formally,*

$$\mathscr{A} \in \mathfrak{C}(\text{Cl-L}) \xrightarrow{const_1} \mathscr{A}^{\blacktriangledown} \in \mathfrak{C}(\text{U-RAS}) \xrightarrow{const_2} (\mathscr{A}^{\blacktriangledown})^{\blacktriangle} = \mathscr{A}$$

Conversely, starting from an upper approximation space, forming its closure lattice version and then coming back to the corresponding upper approximation space structure, one recovers the original structure we started with. Formally,

$$\mathscr{B} \in \mathfrak{C}(\text{U-RAS}) \xrightarrow{const_2} \mathscr{B}^{\blacktriangle} \in \mathfrak{C}(\text{Cl-L}) \xrightarrow{const_1} (\mathscr{B}^{\blacktriangle})^{\blacktriangledown} = \mathscr{B}$$

Theorem 28 asserts that the category of De Morgan lattices with closure operation $\mathfrak{C}$(Cl-L) (whose morphisms are structure preserving functions) and the one of De Morgan lattices with upper approximation map $\mathfrak{C}$(U-RAS) (whose morphisms are structure preserving functions) are categorical equivalent (isomorphic) between them, according to the general category theory [68].

Note that this result is not so trivial. Indeed, if we have two categories of algebraic structures, say $\mathfrak{A}$ and $\mathfrak{B}$, with two constructive methods for passing from one to the other, say $\mathfrak{A} \xrightarrow{\text{const}_1} \mathfrak{B}$ and $\mathfrak{B} \xrightarrow{\text{const}_2} \mathfrak{A}$, then in general

$$\mathscr{A} \in \mathfrak{A} \xrightarrow{const_1} \mathscr{A}^{\blacktriangledown} \in \mathfrak{B} \xrightarrow{const_2} (\mathscr{A}^{\blacktriangledown})^{\blacktriangle} \neq \mathscr{A}$$

and

$$\mathscr{B} \in \mathfrak{B} \xrightarrow{\text{const}_2} \mathscr{B}^{\blacktriangle} \in \mathfrak{A} \xrightarrow{\text{const}_1} (\mathscr{B}^{\blacktriangle})^{\blacktriangledown} \neq \mathscr{B}$$

3 Interior Operations Induced from Closures and Lower Approximation Spaces

The notion of interior operation dual to the closure operation with respect to the De Morgan mapping $' : \Sigma \to \Sigma$, which as stressed in Lemma 8 is a bijection on the lattice Σ, is given by the following result.

Theorem 29 *Suppose a closure lattice* $\mathscr{A} = \langle \Sigma, \wedge, \vee, ', ^*, 0, 1 \rangle$. *The mapping* $^o : \Sigma \to \Sigma$ *defined by the law*

$$\forall a \in \Sigma, \ a^o := ((a')^*)' \tag{11}$$

is an interior operation*, i.e., it satisfies the followings:*

(I1)	$a^o \leq a$	*(decreasing)*
(I2)	$(a \wedge b)^o \leq a^o \wedge b^o$	*(sub-multiplicative)*
(I3)	$a^o = a^{oo}$	*(idempotent)*

Analogously to the closure case, the induced structure $\mathscr{I} = \langle \Sigma, \wedge, \vee, ', ^o, 0, 1 \rangle$ *is said to be an* interior lattice *instead of a De Morgan lattice with interior operation.*

Remark 30 Condition (I3) can be more economically expressed in the weak form

(I3w)	$a^o \leq a^{oo}$	(weak idempotent)

Indeed, applying the decreasing condition (I1) to the element a^o we get the inverse of the (I3w): $(a^o)^o \leq a^o$, leading to the condition (I3).

Also in this interior case the equational sub-multiplicative property (I2) alone is equivalent to a non-equational condition according to the following result.

Lemma 31 *In an interior lattice* Σ *the following two statements are equivalent:*

(I2)	$(a \wedge b)^o \leq a^o \wedge b^o$	*(sub-multiplicative)*
(I2a)	$a \leq b$ *implies* $a^o \leq b^o$	*(isotone)*

Note that from Eq. (11) and the De Morgan condition (dM1) it follows that dually the closure can be expressed as

$$\forall a \in \Sigma, \ a^* = ((a')^o)' \tag{12}$$

Suppose an interior lattice $\langle \Sigma, \wedge, \vee, ', ^o, 0, 1 \rangle$. Since in general $a^o \leq a$, the subset of *open elements* is defined as the collection of elements which are equal to their interior.

Formally,

$$\mathscr{O}(\Sigma) = \{h \in \Sigma : h = h^o\}.$$

This set is not empty since the least element 0 is open (from (I1) it is $0 \leq 0^o \leq 0$).

Condition (I3) says that for every $a \in \Sigma$ the element a^o, called the *interior* of a, is open.

Taking into account the relationship between interior and closure expressed by Eq. (11) we have that

$$a \text{ is closed } (a = a^*) \quad \text{iff} \quad a' \text{ is open } (a' = a'^o), \tag{13a}$$

or dually that

$$a \text{ is open } (a = a^o) \quad \text{iff} \quad a' \text{ is closed } (a' = a'^*). \tag{13b}$$

Thus, taking as primitive the notion of closed set $\mathscr{C}(\Sigma)$, we have that $\mathscr{O}(\Sigma) = \{h \in \Sigma : \exists c \in \mathscr{C}(\Sigma) \ s.t \ h = c'\}$.

Of course, the dual of Proposition 20 holds.

Proposition 32 *Let* $\langle \Sigma, \wedge, \vee, {}^o, 0, 1\rangle$ *be a bounded lattice* $\langle \Sigma, \wedge, \vee, 0, 1\rangle$ *with an interior operation* ${}^o : \Sigma \to \Sigma$. *Then,*

(D-i) *the family* $\mathscr{O}(\Sigma) := \{h \in \Sigma : h = h^o\}$ *of all its open elements satisfies the following conditions:*

- *(PO1)* *the least element* 0 *is open:* $0 \in \mathscr{O}(\Sigma)$.
- *(PO2)* $\mathscr{O}(\Sigma)$ *is closed with respect to the join of any pair of open elements, i.e., if* $o, h \in \mathscr{O}(\Sigma)$ *then* $o \vee h \in \mathscr{O}(\Sigma)$.

(D-ii) *Moreover, in the case in which* Σ *is a* complete *lattice, then also the dual of point (ii) of Proposition 20 holds:*

- *(POO2)* $\mathscr{O}(\Sigma)$ *is closed with respect to the join of any arbitrary family of open elements: if* $\{o_j\} \subseteq \mathscr{O}(\Sigma)$ *then* $\vee\{o_j\} \in \mathscr{O}(\Sigma)$.

Hence, the structure $\langle \mathscr{O}(\Sigma), \vee, 0\rangle$ *is a complete join semi-lattice lower bounded by the least element 0.*

This result leads to the dual of Proposition 22.

Proposition 33 *Let* $\langle \Sigma, \wedge, \vee\rangle$ *be a nonempty* complete *lattice.*

(TO1) *Then, if* Σ *is equipped with an interior operation* ^{o} *with induced family of open sets* $\mathscr{O}(\Sigma)$ *satisfying conditions (PO1) and (POO2), then the interior* a^o *of any element* $a \in \Sigma$ *can be expressed as*

$$a^o = \vee \{o \in \mathscr{O}(\Sigma) : o \leq a\} \tag{14}$$

(TO12) *Conversely, let* Σ *be equipped with a family of its elements* $\mathscr{O}(\Sigma)$ *satisfying conditions (PO1) and (POO2), then the mapping* ${}^o : \Sigma \to \Sigma$ *associating with any element* $a \in \Sigma$ *the element* $a^o := \vee\{o \in \mathscr{O}(\Sigma) : o \leq a\}$ *is an interior operation in the sense that it satisfies the interior conditions (I1), (I2), and (I3).*

3.1 Interior Lattices and Lower Rough Approximation Spaces (LRAS): Their Categorical Isomorphism

In the case of an interior lattice we can introduce the dual of the definition 25 according to the following.

Definition 34 An *lower rough approximation space* is a triple $\langle \Sigma, \mathscr{L}(\Sigma), l \rangle$ where

(L-RAS-1) Σ stays for a structure $\langle \Sigma, \wedge, \vee, {}', 0, 1 \rangle$ of De Morgan lattice;
(L-RAS-2) $\mathscr{L}(\Sigma)$ stays for a structure $\langle \mathscr{L}(\Sigma), \vee, 0 \rangle$ of lower bounded by 0 join semi-lattice, under the condition that $\mathscr{L}(\Sigma) \subseteq \Sigma$;
(L-RAS-3) $l : \Sigma \to \Sigma$ is a unary operation on Σ satisfying the following conditions for every $a \in \Sigma$:

(Lo1) $l(a) \leq a$;
(Lo2) $l(a) \in \mathscr{L}(\Sigma)$;
(Lo3) $\forall o \in \mathscr{L}(\Sigma),$ if $o \leq a$ then $o \leq l(a)$.

Note that these three conditions can be compacted in the unique:

$$\forall a \in \Sigma,\ l(a) = \max \{ o \in \mathscr{L}(\Sigma) : o \leq a \} .$$

Elements from Σ are called *approximable* and elements of the subset $\mathscr{L}(\Sigma) \subseteq \Sigma$ are said to be *lower crisp* or also *lower definable*. Furthermore, the mapping $l : \Sigma \to \Sigma$ is called the *lower rough approximation map*.

The dual versions of Theorems 26–28 in this interior context can be summarized in the following unique result.

Theorem 35 *In the context of a De Morgan lattice* $\langle \Sigma, \wedge, \vee, {}', 0, 1 \rangle$*, simply denoted as* Σ*, we have the following.*

(1) *Suppose a De Morgan lattice equipped with an interior operation* $\mathscr{I} = \langle \Sigma, {}^o \rangle$*. Then the induced structure* $\mathscr{I}^{\nabla} := \langle \Sigma, \mathscr{L}(\Sigma), l \rangle$*, where* $\mathscr{L}(\Sigma) = \mathscr{O}(\Sigma)$ *coincides with the set of all open elements of* Σ *and* $l : \Sigma \to \Sigma$ *is the mapping defined for any element* $a \in \Sigma$ *as* $l(a) := a^o$ *(the* interior *of* a*), is a lower rough approximation space.*
(2) *Let* $\mathscr{F} = \langle \Sigma, \mathscr{L}(\Sigma), l \rangle$ *be a lower rough approximation space based on the De Morgan lattice* Σ*, with* $\mathscr{L}(\Sigma) = \{ h \in \Sigma : h = l(h) \}$ *and the mapping* $l : \Sigma \to \Sigma$ *satisfying conditions (Lo1)–(Lo3). Then the induced structure* $\mathscr{F}^{\Delta} := \langle \Sigma, {}^o \rangle$ *based on* Σ *and equipped with the map* ${}^o : \Sigma \to \Sigma$ *associating with any element* $a \in \Sigma$ *the element* $a^o := l(a) \in \Sigma$ *is an interior operation, i.e., it satisfies the interior conditions (I1), (I2), and (I3).*
(3) *The two structures of* interior lattice *and* lower rough approximation space *are categorically equivalent.*
Indeed, if $\mathscr{I}$ *is an interior lattice with induced lower rough approximation space* $\mathscr{I}^{\nabla}$*, then* $(\mathscr{I}^{\nabla})^{\Delta} = \mathscr{I}$*.*

Conversely, if $\mathscr{F}$ is a lower rough approximation space with induced interior lattice $\mathscr{F}^{\triangle}$, then $(\mathscr{F}^{\triangle})^{\nabla} = \mathscr{F}$.

4 Abstract Rough Approximation Spaces (RAS) as Merge of Upper and Lower Approximations

As noticed, the notions of closure and interior are equational, but in order to grasp the intuitive aspects of rough approximations of something we have introduced the equivalent non-equational structures of lower and upper rough approximation spaces.

The abstract notion of rough approximation space as introduced in [14], and further on developed in [16, 20, 21] (with the categorical isomorphism with interior–closure spaces treated in [18, 19]), is the merge of these two structures of lower and upper rough approximation spaces according to the following definition.

Definition 36 Let $\langle \Sigma, \mathscr{U}(\Sigma), u\rangle$ be an upper rough approximation space based on a De Morgan lattice Σ, with $\mathscr{U}(\Sigma) = \mathscr{C}(\Sigma)$ the collection of all closed elements of the closure isomorphic structure, and let $\langle \Sigma, \mathscr{L}(\Sigma), l\rangle$ be the induced dual, by De Morgan complementation, lower rough approximation space, with $\mathscr{L}(\Sigma) = \mathscr{O}(\Sigma)$ the collection of all open elements of the interior isomorphic structure.

Then, the merge of these two structures $\langle \Sigma, \mathscr{L}(\Sigma), \mathscr{U}(\Sigma), r\rangle$, where $r : \Sigma \mapsto \mathscr{L}(\Sigma) \times \mathscr{U}(\Sigma) = \mathscr{O}(\Sigma) \times \mathscr{C}(\Sigma)$ is the mapping which associates with any element $a \in \Sigma$ its *lower–upper (open–closed) rough approximation* $r(a) = (l(a), u(a)) = (a^o, a^*)$, with $l(a) = a^o \in \mathscr{O}(\Sigma)$, $u(a) = a^* \in \mathscr{C}(\Sigma)$, and $l(a) = a^0 \leq a \leq u(a) = a^*$ (see conditions (I1) and (C1)), is the induced *abstract rough approximation space*.
(From now on we use only the formulae $\mathscr{O}(\Sigma)$ and $\mathscr{C}(\Sigma)$ instead of the equivalent $\mathscr{L}(\Sigma)$ and $\mathscr{U}(\Sigma)$.)

In this rough approximation space context, an element e is said to be *crisp* (or *exact*, also *sharp*) iff $r(e) = (e, e)$, i.e., its rough approximation is the trivial one, and this is equivalent to ask that $e \in \mathscr{E}(\Sigma) := \mathscr{C}(\Sigma) \cap \mathscr{O}(\Sigma)$, i.e., it is a *clopen* element. Thus, the set of all clopen elements is the collection of crisp elements.

As previously pointed out, the non-equational conditions defining the lower and the upper approximation maps, (Lo1)–(Lo3) and (Up1)–(Up3), capture the intuitive aspects of an expected rough approximation as the best approximation of the element a from the bottom (resp., top) by lower (resp., upper) crisp elements.

The diagram of Fig. 6 summarizes the results about *abstract rough approximation spaces* by the behavior of the *rough approximation map*.

Let us quote the following statement from [15], concerning the meta-theoretical discussion about roughness principles, which in the original version involves the rough approximation of *subsets* A from the power set $\mathscr{P}(X)$ of some universe X,

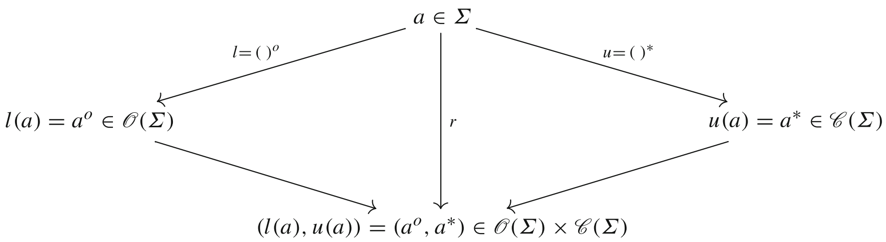


Fig. 6 The rough approximation map induced from interior-closure operations

here translated into the general abstract case of rough approximation of *elements a* from a De Morgan lattice Σ.

An important remark relative to these considerations has to do with some aspects, in general hidden in rough set literature, which can be listed in the following *Conditions for Rough Set Theory* [. . .].

(RC1) The *Condition of Roughness Coherence*: the lower approximation $l(a)$ of the element a must be less or equal than a itself and the upper approximation $u(a)$ must be greater or equal than a: $l(a) \leq a \leq u(a)$.
The comparison is made by the partial order relation $\leq$ [induced by] the De Morgan lattice Σ [. . .].

(RC2) The *Condition of Crispness*: it is possible to single out [two] classes of [lower crisp and upper crisp elements $\mathscr{O}(\Sigma)$ and $\mathscr{C}(\Sigma)$, respectively,] such that either the lower and the upper approximations are crisp, i.e., they describe precise concepts (or properties).

(RC3) The *Condition of Best Approximation*: the lower and upper approximations are not only crisp, but they also give the *best approximation* of any [element a] by crisp elements [from the bottom and the top, respectively].

Conditions (RC1), (RC2), and (RC3) are, of course, formal definitions of *coherence*, *crispness*, and *best approximation*, respectively.

A different discussion is centered on the fact that a formal theory in order to describe roughness *must* satisfy these three conditions, that now assume the role of the following *meta-theoretical* requirement:

(RMTP) A formal theory about roughness based on a De Morgan lattice Σ must satisfy the three conditions (RC1), (RC2) and (RC3).

This should be an interesting element of debate inside the rough set community since, at least at the level of the covering case [see Sect. 15.1] or of incomplete information systems [see Sect. 16.1], one can found some approaches to rough set theory in which only the coherence principle is satisfied, but neither the crispness and so nor the best approximation conditions are considered. This corresponds to the acceptance of [the following weaker] meta-theoretical principle of roughness:

(w-RMTP) In order to have a mathematical theory describing roughness [. . .] the only condition (RC1) is required to be satisfied.

This means that in some context rough set theory is characterized by the weak requirement that an approximation satisfies the unique condition of coherence that the lower approximation must be less or equal to the element which is approximated, and in its turn this element must be less or equal to its upper approximation. No requirement about

> crispness and best approximation is taken into account in the development of the theory. And this is a meta-theoretical decision also if hidden in many cases.
>
> On the other hand the more complete approach, from my point of view, is the one that also in some more generic cases furnishes the right approach to rough approximation [...] according to the (RMTP) principle. This means that it must satisfy not only the coherence principle, but also it must discriminate what is crisp (precise) [...] as an a priori knowledge about the system, and then define a rough approximation not only by the minimal coherence condition, but also by a pair of crisp best approximations from the bottom and the top, respectively.

A similar discussion has been proposed in [39] (and see also its sequel [40]).

4.1 Rough Approximation Spaces by Orthopairs

Now we consider another way to represent rough approximations by open–open pairs, different from the one schematized by the diagram of Fig. 6 consisting in open–closed pairs. To this aim let us introduced a binary relation of *orthogonality* on the De Morgan lattice structure $\langle \Sigma, \wedge, \vee, {}', 0, 1 \rangle$ according to the following definition:

$$\forall a, b \in \Sigma \quad a \perp b \quad \text{iff} \quad a \leq b' \tag{15}$$

Let us list the formal properties satisfied by this orthogonality relation $\perp$ on Σ, comparing them with the corresponding properties characterizing the partial order relation $\leq$ on Σ which as well known are of being: reflexive, anti-symmetric, and transitive.

The binary relation of orthogonality satisfies the following:

(og-1a)	$\forall a \in \Sigma,\ a \perp 0$	(0-irreflexive)
(og-1b)	$\forall a \in \Sigma,\ a \perp a \text{ iff } a \leq a'$	(1-kernel irreflexive)
(og-2)	$\forall a, b \in \Sigma,\ a \perp b \text{ implies } b \perp a$	(symmetric)
(og-3)	$\forall a, b, c \in \Sigma,\ a \leq b \text{ and } b \perp c \text{ imply } a \perp c$	($\perp$ -absorbing)

Note that in the case of an *ortholattice* for which conditions (oc-a,b) hold, the (og-1b) must be substituted by the condition

(ol-1b) $\quad \forall a \in \Sigma,\ a \perp a$ implies $a = 0 \quad$ (irreflexive).

If given an ortholattice Σ we define as $\Sigma_0 := \Sigma \setminus \{0\}$, then the restriction $\perp_0$ on this latter of the orthogonality relation defined on Σ by Eq. (15) satisfies the following:

(ol_0-1)	$\forall a, b \in \Sigma_0,\ a \perp_0 b \text{ implies } a \neq b$	(irreflexive)
(ol_0-2)	$\forall a, b \in \Sigma_0,\ a \perp_0 b \text{ implies } b \perp_0 a$	(symmetric)
(ol_0-3)	$\forall a, b, c \in \Sigma_0,\ a \leq b \text{ and } b \perp_0 c \text{ imply } a \perp_0 c$	($\perp$ -absorbing)

Note that if one introduces on Σ_0 the binary relation $\mathscr{S} = \text{Not} \perp$, then the above two conditions (ol$_0$-1) and (ol$_0$-2) are translated in the following involving $\mathscr{S}$, respectively:

$$\text{(sim-1)} \quad \forall a \in \Sigma_0, \quad a\mathscr{S}a \qquad \text{(reflexive)}$$

$$\text{(sim-2)} \quad \forall a, b \in \Sigma_0, \quad a\mathscr{S}b \quad \text{implies} \quad b\mathscr{S}a \qquad \text{(symmetric)}$$

Reflexive and symmetric binary relations are called *similarity relations* (after Poicaré [96]), also used in the context of modal logic semantic (see [32]), or *tolerance relations* (after Zeeman [123]) in the context of incomplete information systems (see [116]). A *similarity space* or *tolerance space* is then a pair $(X, \mathscr{S})$ based on a universe of points X equipped with a similarity (tolerance) relation $\mathscr{S}$.

In a De Morgan lattice with closure operation the open element $a^{\sim} = e(a) := a^{*\prime} \in \mathscr{O}(\Sigma)$ is defined as the *exterior* of $a \in \Sigma$.

Given the definition of orthogonality (15), we have that the usual relationship between interior and closure of an element $\forall a \in \Sigma$, $a^o \leq a^* = (a^{\sim})'$ can be equivalently formulated in the following way:

$$\forall a \in \Sigma, \quad a^o \perp a^{\sim} \qquad \text{(i.e., } [l(a) = a^o] \perp [a^{\sim} = e(a)])$$

In this way, we can consider the open–open pair

$$\forall a \in \Sigma, \quad (l(a), e(a)) \in \mathscr{O}(\Sigma) \times \mathscr{O}(\Sigma) \;\text{ with }\; l(a) \perp e(a), \tag{16a}$$

equivalently

$$\forall a \in \Sigma, \quad (a^o, a^{\sim}) \in \mathscr{O}(\Sigma) \times \mathscr{O}(\Sigma) \;\text{ with }\; a^o \perp a^{\sim}. \tag{16b}$$

Hence, we can introduce the *rough approximation map by orthopairs* (also $\perp$-*rough approximation map*) as the mapping $r_\perp : \Sigma \to \mathscr{O}(\Sigma) \times \mathscr{O}(\Sigma)$, associating with any element $a \in \Sigma$ the orthopair $r_\perp(a) := (a^o, a^{\sim}) = (l(a), e(a)) \in \mathscr{O}(\Sigma) \times \mathscr{O}(\Sigma)$, under the orthogonality condition $a^o \perp a^{\sim}$ (i.e., $l(a) \perp e(a)$) [14, 24, 25].

The $\perp$-*rough approximation space* based on the De Morgan closure lattice Σ is then the structure $\langle \Sigma, \mathscr{O}(\Sigma), r_\perp \rangle$.

Let us stress that any De Morgan complementation $a \in \Sigma \to a' \in \Sigma$, as involutive (i.e., (dM1)) antimorphism (i.e., (dM2b)), is a bijection on Σ which allows the identification $a \leftrightarrow a'$. In particular we can identify $a^* \leftrightarrow (a^*)' = a^{\sim}$, from which the further identification $(a^o, a^*) \leftrightarrow (a^o, a^{\sim})$ can be considered, with $(a^o, a^*) \in \mathscr{O}(\Sigma) \times \mathscr{C}(\Sigma)$ whereas $(a^o, a^{\sim}) \in \mathscr{O}(\Sigma) \times \mathscr{O}(\Sigma)$. Hence, the rough approximations $r(a) = (a^o, a^*)$ based on ordered pairs $a^o \leq a^*$ depicted in Fig. 6 can be bijectively identified with the rough approximations $r_\perp(a) = (a^o, a^{\sim})$ based on orthopairs $a^o \perp a^{\sim}$ depicted in Fig. 7.

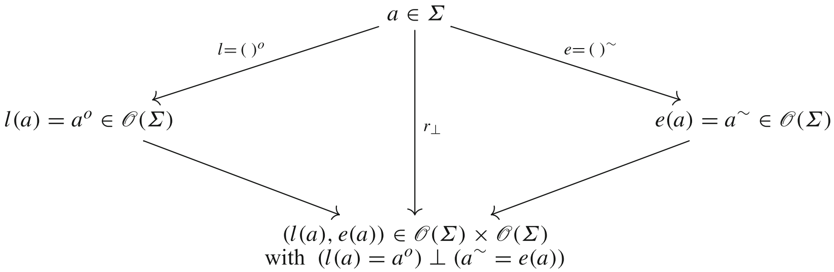


Fig. 7 The rough approximation map by orthopairs

5 Galois Connections, Similarity Relations and Induced Closure Operations: An Elementary Introduction

In this section I treat, without going into excessive details, the theory of Galois connection [42, 86], whose fundamental points are now outlined making reference to the Birkhoff book [8]. The main interest is to show that they generate closure operation as particular structures.

6 Formal Concepts Induced by Galois Connections

Let us consider a binary relation $\mathscr{G}$ on the cartesian product $X \times V$ of two nonempty sets, i.e., $\mathscr{G} \subseteq X \times V$, with $x\mathscr{G}v$ denoting the fact that $(x, v) \in \mathscr{G}$.

In this section we adopt the terminology of Formal Concept Analysis (FCA). Let us recall that FCA is formulated on the notion of a formal context, which is a binary relation $\mathscr{G}$ between a set of objects X and a set V of properties or attributes. The set of objects of a formal concept is referred to as the extension, and the set of properties as the intension. They uniquely determine each other. When $x\mathscr{G}v$ we say that the object x possesses the property v, or that the property v is possessed by the object x [45, 115].

The binary relation $\mathscr{G}$ induces set-theoretical operators from sets of objects to sets of properties, and from sets of properties to sets of objects, respectively. A *formal concept* is defined as a pair (A, Z) of a set of objects $A \subseteq X$ and a set of properties $Z \subseteq V$ connected by the two set-theoretical operators.

Precisely, for any pair (A, Z) of subsets with $A \subseteq X$ and $Z \subseteq V$, according to [8, p. 122] let us define the *polar* $A^{\#}$ of A and the *polar* $Z^{\dagger}$ of Z as follows:

$$A^{\#} := \{v \in V : \forall a \in A,\ a\mathscr{G}v\} \quad \text{and} \quad Z^{\dagger} := \{x \in X : \forall z \in Z,\ x\mathscr{G}z\} \tag{17}$$

So, $A^{\#} \subseteq V$ is a subset of V whose elements $v \in V$ can be characterized by the property written in the compact form $A\mathscr{G}v$, whereas $Z^{\dagger} \subseteq X$ is a subset of X whose elements $x \in X$ are characterized by the property written in the compact form $x\mathscr{G}Z$. Formally,

$$A^{\#} := \{v \in V : A\mathscr{G}v\} \quad \text{and} \quad Z^{\dagger} := \{x \in X : x\mathscr{G}Z\} \tag{18}$$

In the case of a single object $x \in X$ the corresponding collection of all properties possessed by it, i.e., $\{x\}^{\#}$, in literature is also denoted by $x\mathscr{G}$. Similarly in the case of a single property $v \in V$ the collection of all objects which possesses this property, i.e., $\{v\}^{\dagger}$, is usually denoted by $\mathscr{G}v$. Under these assumptions, for any subset $A \in \mathscr{P}(X)$ and any subset $Y \in \mathscr{P}(V)$ we have the following.

$$A^{\#} = \{v \in V : A \subseteq \mathscr{G}v\} = \bigcap_{a \in A} a\mathscr{G}$$

$$Y^{\dagger} = \{x \in X : Y \subseteq x\mathscr{G}\} = \bigcup_{y \in Y} \mathscr{G}y$$

The so-introduced two mappings $^{\#} : \mathscr{P}(X) \mapsto \mathscr{P}(V)$ and $^{\dagger} : \mathscr{P}(V) \mapsto \mathscr{P}(X)$ define a *Galois connection* between the Boolean algebras of subsets $\mathscr{P}(X)$ and $\mathscr{P}(V)$ since they satisfies the characteristic conditions:

$$\text{(GC1)} \quad A \subseteq B \quad \text{implies} \quad B^{\#} \subseteq A^{\#}$$

$$\text{(GC2)} \quad Z \subseteq Y \quad \text{implies} \quad Y^{\dagger} \subseteq Z^{\dagger}$$

$$\text{(GC3)} \quad A \subseteq (A^{\#})^{\dagger} \quad \text{and} \quad Z \subseteq (Z^{\dagger})^{\#}$$

Moreover, setting $P_v(a) := a\mathscr{G}v$ from "$\forall a \in A_1 \cup A_2,\ P_v(a)$" iff "$\forall a \in A_1,\ P_v(a)$ and $\forall a \in A_2,\ P_v(a)$" iff "$A_1^{\#} \cap A_2^{\#}$" (and respectively for $P_x(z) := x\mathscr{G}z$) it follows

$$\text{(GC4)} \quad (A_1 \cup A_2)^{\#} = A_1^{\#} \cap A_2^{\#} \quad \text{and} \quad (Y_1 \cup Y_2)^{\dagger} = Y_1^{\dagger} \cap Y_2^{\dagger}$$

This condition (GC4) can be extended to arbitrary families of subsets of X, and arbitrary families of subsets of V.

We now show as the notion of formal contexts provides a common framework for the study of rough set theory and formal concept analysis, if rough set theory is formulated on the basis of two universes X and V (for a discussion about this argument see [117], with associated references). Precisely, the polar $A^{\#}$, as subset of V, gives rise to the new subset, $(A^{\#})^{\dagger} \subseteq X$, and the polar $Z^{\dagger}$, as subset of X, gives rise to the new subset, $(Z^{\dagger})^{\#} \subseteq V$. In this way we can introduce two mappings:

- the first is a transformation of the power set $\mathscr{P}(X)$ on itself, $^{*} : \mathscr{P}(X) \mapsto \mathscr{P}(X)$, defined by the correspondence $A \to A^{*} := (A^{\#})^{\dagger}$, and
- the second is a transformation of the power set $\mathscr{P}(V)$ on itself, $^{\circledast} : \mathscr{P}(V) \mapsto \mathscr{P}(V)$, defined by the correspondence $Z \to Z^{\circledast} := (Z^{\dagger})^{\#}$.

The main result is summarized by the following.

Theorem 37 *The transformations*

$$A \to A^* := (A^{\#})^{\dagger} \quad on\ \mathscr{P}(X)$$

$$Z \to Z^{\circledast} := (Z^{\dagger})^{\#} \quad on\ \mathscr{P}(V)$$

are both closure operators, the first on the Boolean complete lattice $\mathscr{P}(X)$ and the second on the Boolean complete lattice $\mathscr{P}(V)$, both as particular concrete cases of de Morgan complete lattices with closure operation. Hence, following the discussion of Sect. 2.3 we have the following:

(i) The collections of closed sets relatively to these closures are respectively

$$\mathscr{C}(X) := \{C \in \mathscr{P}(X) : C = C^*\} = \{C \in \mathscr{P}(X) : C = C^{\#\dagger}\}$$

$$\mathscr{C}(V) := \{K \in \mathscr{P}(V) : K = K^{\circledast}\} = \{K \in \mathscr{P}(V) : K = K^{\dagger\#}\}$$

Let us recall that these two families are nonempty ($X \in \mathscr{C}(X)$ and $V \in \mathscr{C}(V)$) and closed with respect to arbitrary intersection, i.e., closure systems, according to Proposition 20.

(ii) Moreover, according to the general theory developed in Sect. 3, the transformation

$$A \to A^o := A^{c*c} = A^{c\#\dagger c}$$

is an interior operation on $\mathscr{P}(X)$ and the transformation

$$Z \to Z^{\circledcirc} := Z^{c\circledast c} = Z^{c\dagger\# c}$$

is an interior operation on $\mathscr{P}(V)$.

(iii) Finally, and according to Sect. 4, the pair $r_X(A) = (A^o, A^*)$ is the rough approximation of A in $\mathscr{P}(X)$ and $r_V(Z) = (Z^{\circledcirc}, Z^{\circledast})$ is the rough approximation of Z in $\mathscr{P}(V)$.

Let us denote by $\emptyset_X$ (resp., $\emptyset_V$) the empty set of X (resp., V). Then, in general $\emptyset_X$ (resp., $\emptyset_V$) is not a closed element of $\mathscr{C}(X)$ (resp., $\mathscr{C}(V)$).

Example 38 Let $X = \{a, b\}$, $V = \{0, 1\}$, and $\mathscr{G}_1 = \{(a, 0), (a, 1)\}$. Then, $(\emptyset_X)^{\#} = V$ and so $(\emptyset_X)^* = V^{\dagger} = \{a\} \neq \emptyset_X$. Dually, for $\mathscr{G}_2 = \{(a, 0), (b, 0)\}$ one has that $(\emptyset_V)^{\circledast} = \{0\} \neq \emptyset_V$.

Following [45, 115], from a formal context one can introduce the notion of formal concept.

Definition 39 A *formal concept* in the *context* $(X, V, \mathscr{G})$ is a pair $(A, W) \in \mathscr{P}(X) \times \mathscr{P}(V)$ such that $A = W^{\dagger}$ and $W = A^{\#}$. Hence,

(i) (A, W) is a formal concept iff $(A, W) = (W^{\dagger}, A^{\#})$.

A first result is the following one.

Proposition 40 *If the pair* (A, W) *is a formal concept then* $(A, W) = (A^*, W^{\circledast})$. *In other words, if* (A, W) *is a formal concept then* A *is* X*-closed* ($A \in \mathscr{C}(X)$) *and* W *is* V*-closed* ($W \in \mathscr{C}(V)$).

Proof Since $A = W^{\dagger}$ and $W = A^{\#}$ imply $A = W^{\dagger} = (A^{\#})^{\dagger} = A^*$ and $W = A^{\#} = (W^{\dagger})^{\#} = W^{\circledast}$, the statement is proved. □

The converse of this proposition does not work, as the following example shows.

Example 41 Let $X = \{a, b, c\}$, $V = \{0, 1\}$, and $\mathscr{G} = \{(b, 0)\}$. Then the subset $\{b\}$ is X-closed since $\{b\}^{\#} = \{0\}$ from which it follows $\{b\}^{\#\dagger} = \{0\}^{\dagger} = \{b\}$ and trivially V is V-closed. Hence, $(\{b\}, V) \in \mathscr{C}(X) \times \mathscr{C}(V)$, but $\{b\}^{\#} \neq V$.

Proposition 42 *Let* $\mathscr{L}$ *be the collection of all formal concepts from the context* $(X, V, \mathscr{G})$. *Then, the binary relation* $\sqsubseteq$ *on* $\mathscr{L} \times \mathscr{L}$ *defined by*

$$(A_1, W_1) \sqsubseteq (A_2, W_2) \quad \textit{iff} \quad A_1 \subseteq A_2, \quad (\textit{or equivalently iff} \;\; W_2 \subseteq W_1) \tag{19}$$

is a partial order relation with respect to which $\mathscr{L}$ *is a lattice whose meet and join operations of any pair* (A_1, W_1) *and* (A_2, W_2) *of formal concepts are the following:*

$$(A_1, W_1) \sqcap (A_2, W_2) = (A_1 \cap A_2, (W_1 \cup W_2)^{\circledast}) \tag{20a}$$

$$(A_1, W_1) \sqcup (A_2, W_2) = ((A_1 \cup A_2)^*, W_1 \cap W_2) \tag{20b}$$

This lattice is complete since the above meet and join operations can be extended to arbitrary families of formal concepts.

Proof Let $A_1 \subseteq A_2$, from $W_1 = A_1^{\#}$ and $W_2 = A_2^{\#}$ it follows, by (GC1), that $W_2 = A_2^{\#} \subseteq\subseteq A_1^{\#} = W_1$. Now it is easy to prove that Eq. (19) defines a partial order.

Moreover, from the definition of formal concepts $A_1 = W_1^{\dagger}$ and $A_2 = W_2^{\dagger}$ from which it follows $(A_1 \cap A_2)^{\#} = (W_1^{\dagger} \cap W_2^{\dagger})^{\#} = (GC4) = (W_1 \cup W_2)^{\dagger\#}$.
On the other hand, by Proposition 40 A_1, A_2 are both X-closed and so, by the property (i) of Theorem 37 of $\mathscr{C}(X)$, also $A_1 \cap A_2$ is closed: $(A_1 \cap A_2)^* = A_1 \cap A_2$. Hence, $((W_1 \cup W_2)^{\circledast})^{\#} = (W_1 \cup W_2)^{\dagger\#\dagger} = (GC4) = (W_1^{\dagger} \cap W_2^{\dagger})^{\#\dagger} = (A_1 \cap A_2)^* = A_1 \cap A_2$. In conclusion the pair $(A_1 \cap A_2, (W_1 \cup W_2)^{\circledast})$ is a formal concept. It is straightforward to prove that Eq. (20) define (complete) lattice meet and join operations. □

As stated at the beginning of this section, I would have only made an introduction to the fundamental properties of formal concept analysis as a particular application of Galois connection theory. People who want to approach this topic can access the work of Yao first mentioned (see also [98, 118–120]). Furthermore, Düntsch and Gediga pointed out that the set-theoretical operators used in formal concept analysis and rough set theories have been considered in modal logics, and therefore referred to them as modal-style operators [38, 47, 48]. They have also demonstrated that modal-style operators are useful in data analysis.

7 Tarski Closure and Interior Operations

In the notion of closure introduced by Definition 1 there is no requirement about the least element 0. But, for instance in the Ore paper [86] in which a closure operation is defined on a poset according to conditions (C1), (C2a), and (C3), one can found the statement "when a zero element exists in the partially ordered set it is customary the make the fourth axiomatic assumption:

$$\text{(C0)} \quad \text{the zero element is closed :} \quad 0^* = 0.\text{"}$$

Similarly, in the context of the power set of a universe, Ore says "by an additional definition one prescribes that the void set is closed" [85].

Let us recall that in the context of the power set $\mathscr{P}(X)$, Boolean lattice of subsets of a universe X, an operation $^* : \mathscr{P}(X) \to \mathscr{P}(X)$ which satisfies all conditions (C1), (C2), (C3), and (C0), determines a structure $\langle \mathscr{P}(X), ^* \rangle$ which in [77] has been called *(F)-space*. As to the axiom (C0) let us quote the following Monteiro statement from [77]: "It must be remarked that the axioms (C1), (C2) and (C3) are the characteristic axioms of the notion of *closure property*; see for example [8]. Axiom (C0) enters in the proof of the following property, which is an essential property [. . .]: a point a of the universe X cannot have the point a itself as accumulation point. Into the systems which verify the axioms (C1), (C2) and (C3), it is generally possible to suppose that the axiom (C0) is also verified, we can consequently identify the notion of (F)-space with the notion of *closure property*."

In the paper [82] Monteiro and Ribeiro stressed that "(C3) is the condition α) that Appert [2] has studied in his Thesis. An interesting and unexpected example of this (F)-space is the one which can be recognized taking the ensemble of meaningful propositions of any deductive discipline and for closure operation the one which assigns to any set of such propositions the set of its consequences (by the rules of inference of the involved discipline). See for this subject Tarski [107]." To tell the truth in the quoted Tarski papers Axiom (C0) is not explicitly formulated also if it seems to be reasonable to add, in the interpretation discussed in Remark 65 of Sect. 2 about closure operation as "the set of consequences via interference rules," the additional condition that the set of consequences obtained by an empty set of axioms is empty, i.e., condition (C0).

These considerations lead to investigate as interesting argument closure operations which satisfy besides the usual conditions (C1), (C2), and (C3), of closure introduced in Definition 1 the further condition (C0) requiring that also the zero element must be closed. This will be the argument of the present section in which closures of this kind, for the reasons now discussed, are called *Tarski closure* *operations*. As *Tarski closure lattice* we mean a De Morgan lattice equipped with a Tarski closure.

Coherently with the considerations now made, we note that if Σ is a De Morgan lattice equipped with a closure operation $^* : \Sigma \to \Sigma$ satisfying the only conditions (C1)–(C3), but not the (C0), then it is always possible to modify the closure definition of * in a new definition ^{*_T} in order to have a Tarski closure according to the following definition:

$$\forall a \in \Sigma, \quad a^{*_T} := \begin{cases} a^* & \text{if } a \neq 0 \\ 0 & \text{if } a = 0 \end{cases}$$

Of course, conditions (C1)–(C3) continue to be satisfied by the new closure for every pair of elements a, b both different from 0, while conditions (C1) and (C3) are valid when $a = 0$. Condition (C2) is also satisfied when one or both elements a, b is 0.

Example 43 Let us consider the trivial closure $^{*_t} : \Sigma \to \Sigma$ defined by the law $\forall a \in \Sigma, a^{*_t} = 1$, introduced in Example 2, with respect to which $0^{*_t} = 1$. Then the modified closure, denoted as $a^{*_{tT}}$ and given by

$$\forall a \in \Sigma, \quad a^{*_{tT}} := \begin{cases} 1 & \text{if } a \neq 0 \\ 0 & \text{if } a = 0 \end{cases}$$

is a Tarski closure.

Example 44 In Fig. 8 it is applied the procedure now discussed for transform a not Tarski closure on a Kleene lattice with unique half element b (at the left side where $0^* = b \neq 0$) into a Tarski closure (at the right side where $0^* = 0$).

In the Tarski complete lattice case the statement of Proposition 20 must be modified in the following manner.

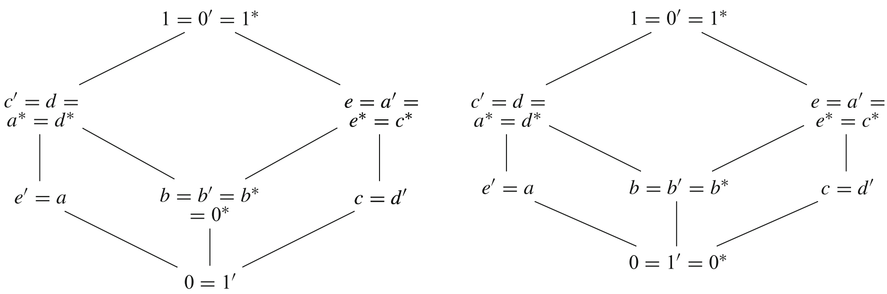


Fig. 8 Tarski closure realization at the right side of a not Tarski closure at the left side

Proposition 45 *Let Σ be a De Morgan* complete *lattice equipped with a Tarski closure operation. Then, the family $\mathscr{C}(\Sigma) := \{c \in \Sigma : c = c^*\}$ of its closed elements satisfies the following conditions.*

(PC1a) *The least element* 0 *and the greatest element* 1 *are both closed:* $0, 1 \in \mathscr{C}(\Sigma)$*;*

(PCC2) *$\mathscr{C}(\Sigma)$ is closed with respect to the meet of any arbitrary family of closed elements: let $\{c_j\} \subseteq \mathscr{C}(\Sigma)$ then $\wedge\{c_j\} \in \mathscr{C}(\Sigma)$.*

So, the structure $\langle \mathscr{C}(\Sigma), \wedge, \vee, ', ^*, 0, 1 \rangle$ is a pre topological lattice of closed elements *in the sense that it satisfies the two abstract lattice conditions (PC1a) and (PCC2) of a topology of closed elements, without the further important topological condition of being closed with respect to finite join (it is not a join semi-lattice).*

With respect to Tarski closure, Proposition 22 must be completed in the point (TC2) asserting that besides condition (C1)–(C3) condition (C0) is also satisfied; indeed, this follows from the definition of $0^* = \wedge \{c \in \mathscr{C}(\Sigma) : 0 \leq c\}$, the fact that by (PC1a) it is $0 \in \mathscr{C}(\Sigma)$, and from the trivial condition $0 \leq 0$. The results of Proposition 23 continue to apply in their original statements.

As in the case of a closure operation, on the basis of a Tarski closure, one can apply Eq. (11) in order to obtain the dual (with respect to the De Morgan complementation) *Tarski interior operation* which is characterized by the conditions (I1) $a^o \leq a$ (decreasing), and (I2) sub-multiplicative with the equivalent version of isotonicity "$a \leq b$ implies $a^o \leq b^o$", and (I3) $a^{oo} = a^o$ (idempotency), plus the following dual of the closure condition (C0):

$$\text{(I0)} \quad \text{the greatest element is open}: \quad 1^o = 1 \qquad \text{(normalization)}.$$

The decreasing condition (I1) leads one to introduce the collection of Tarski open elements $\mathscr{O}(\Sigma) := \{h \in \Sigma : h = h^o\}$ and in the Tarski complete lattice case the statement dual of Proposition 45 holds.

Proposition 46 *Let Σ be a De Morgan* complete *lattice equipped with a Tarski interior operation. Then, the family $\mathscr{O}(\Sigma) := \{h \in \Sigma : h = h^o\}$ of its open elements satisfies the following conditions.*

(PO1a) *The least element* 0 *and the greatest element* 1 *are both open:* $0, 1 \in \mathscr{O}(\Sigma)$*;*

(POO2) *$\mathscr{O}(\Sigma)$ is closed with respect to the join of any arbitrary family of open elements: let $\{h_j\} \subseteq \mathscr{O}(\Sigma)$ then $\wedge\{h_j\} \in \mathscr{O}(\Sigma)$.*

So, the structure $\langle \mathscr{O}(\Sigma), \wedge, \vee, ', ^o, 0, 1 \rangle$ is a pre topological lattice of open elements *in the sense that it satisfies the two abstract lattice conditions (PO1a) and (POO2) of a topology of open elements, without the further important topological condition of being closed with respect to finite meet (it is not a meet semi-lattice).*

The Tarski versions of closed and open elements $\mathscr{C}(\Sigma)$ and $\mathscr{O}(\Sigma)$ in general do not coincide, neither one is a subset of the other. Thus, it is worthwhile to consider also the set of all *clopen* elements, $\mathscr{E}(\Sigma) := \mathscr{C}(\Sigma) \cap \mathscr{O}(\Sigma)$, which in this Tarski context contains both the least element 0 and the greatest element 1 of the lattice Σ.

The notion of rough approximation space (RAS) introduced in Definition 36 of Sect. 4 can be specified in the case of Tarski interior-closure operations from a complete lattice case according to the following.

(RAS-T) Let Σ be a De Morgan *complete* lattice equipped with a Tarski closure $u(a) = a^*$ and induced Tarski interior $l(a) = a^o = a'^{*'}$. Then the *Tarski rough approximation space* is the structure $\langle \Sigma, \mathscr{O}(\Sigma), \mathscr{C}(\Sigma), r \rangle$ where:

(1) Σ is the collection of all *approximable elements*;
(2) $\mathscr{O}(\Sigma) = \{d \in \Sigma : d = d^o = l(d)\}$ is the join *complete* semi-lattice of all *lower crisp elements*, containing both the elements $0, 1 \in \mathscr{O}(\Sigma)$ (pre topological lattice of open elements);
(3) $\mathscr{C}(\Sigma) = \{c \in \Sigma : c = c^* = u(c)\}$ is the meet *complete* semi-lattice of all *upper crisp elements*, containing both the elements $0, 1 \in \mathscr{C}(\Sigma)$ (pre-topological lattice of closed elements);
(4) $r : \Sigma \to \mathscr{O}(\Sigma) \times \mathscr{C}(\Sigma)$ is the *rough approximation map* associating with any approximable element $a \in \Sigma$ the open–closed crisp pair $r(a) = (a^o, a^*) = (l(a), u(a))$, consisting of the lower crisp approximation $l(a) = a^o \in \mathscr{O}(\Sigma)$ expressed by Eq. (14) and the upper crisp approximation $u(a) = a^* \in \mathscr{C}(\Sigma)$ of a expressed by Eq. (7), satisfying the meta-theoretical principle (RMTP) of roughness coherence (RC1), crispness (RC2), and best approximation (RC3).

The following result will be useful in the sequel.

Lemma 47 *Let $l : \Sigma \to \Sigma$ be a Tarski interior operation with associated collection of open elements $\mathscr{O}(\Sigma)$. Then,*

$$\forall a \in \Sigma,\ \forall o \in \mathscr{O}(\Sigma), \quad o \leq a \quad \textit{iff} \quad o \leq l(a) \tag{21}$$

Proof Let $o \leq a$, then by isotonicity $l(o) \leq l(a)$, but from the hypothesis that o is open $o = l(o) \leq l(a)$. Conversely, let $o \leq l(a)$, then for decreasing property (I1) it follows that $o \leq l(a) \leq a$. □

Part II: Closure-Interior Operations as Variations of the Standard Version

Once stated the categorical isomorphism between closure lattices and upper approximation spaces, it is possible to introduce a hierarchy of closure operators starting from the standard closure version given in Definition 1, or better from the Tarski closure admitting the (C0) condition regarding the closure of the zero element.

This is just the case we are investigating with: the formalization of the closure (and induced interior) operator gives the hint to explore other axiomatic lattice versions of the condition (C1)–(C3) (or (I1)–(13)) in order to fit the richness of situations which can be described by them.

All the closure versions which we shall treat in the present part are characterized, in particular, by the assumption (C0) of closure of the zero element.

8 Kuratowski Closure and Induced Interior Operations

A *Kuratowski (topological) closure operation* on a De Morgan lattice Σ is a unary operation * on Σ which satisfies the conditions (C0), (C1), and (C3) of Tarski closure definition, but in which the sub-additive condition (C2) is substituted by the stronger property:

(C2K) $\quad a^* \vee b^* = (a \vee b)^* \quad$ (additive)

About this argument we can cite from Rasiowa–Sikorski [100]: "The closure operation * satisfies the conditions (C0), (C1), (C2K) and (C3). These Axioms are due to Kuratowski [65]. For a detailed exposition of the theory of topological spaces see e.g. Kelley [62], Kuratowski [66, 67]." In the introduction of his paper, Nakamura [84] says: "The concept of closure was axiomatized by F. Riesz and Kuratowski [66] on the field of sets, and Terasaka [110] generalized it onto abstract Boolean algebras. The object of this note is to extend it onto general lattices. Incidentally, 'combinations of topologies' of G. Birkhoff [7] are treated from more general point of view." Quoting M. Ward [114]: "These axioms are satisfied by Kuratowski's closure operator over a Boolean algebra with points [66]." For a more recent treatment see also the Birkhoff book [8, p. 116]. Finally, according to Halmos "a *closure operator* is a normalized [(C0)], increasing [(C1)], idempotent [(C3)], and additive [(C2K)] mapping [. . .]. The first systematic investigation of the algebraic properties of closure operators was carried out by McKinsey and Tarski [73]."
In agreement with all these quotations, in the sequel we shall call *Kuratowski closure operation* any topological closure on a De Morgan lattice.

A De Morgan lattice equipped with a Kuratowski closure operation is simply called a *Kuratowski closure lattice*.

Note the following result.

Proposition 48 *Under conditions* (C1) *and* (C3) *the additive property* (C2K) *is equivalent to the simultaneous verification of the two conditions:*

(C2Ka) $a \le b$ *implies* $a^* \le b^*$ *(isotonicity).*
(C2Kb) $a = a^*$ *and* $b = b^*$ *imply* $(a \vee b)^* = a \vee b$ *(topological closure).*

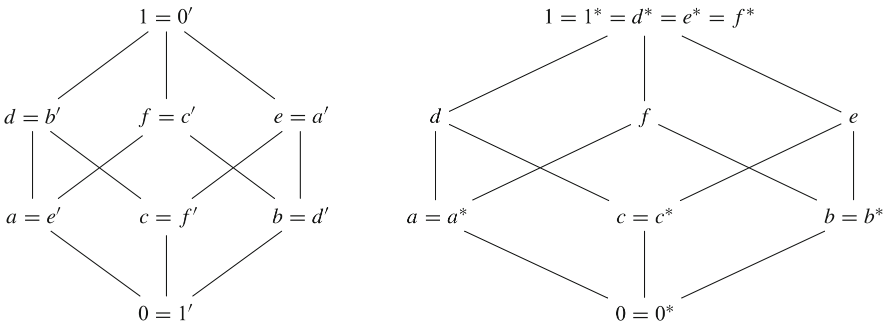


Fig. 9 At the left side the Boolean lattice and at the right side the Tarski no Kuratowski closure

Comparing the (C2Ka,b) with the (C2a), according to Lemma 16 equivalent to the (C2), it follows that any Kuratowski closure is a Tarski closure, but the converse is not true.

Example 49 In the double Hasse diagrams of Fig. 9 we show a Tarski closure lattice which is not Kuratowski since, for instance, $a^* \vee b^* = f \neq (a \vee b)^* = f^* = 1$. Generally, $x^* \vee y^* < (x \vee y)^*$ for $x, y = a, b, c$ with $x \neq y$, whereas $x^* \vee y^* = (x \vee y)^*$ for $x \leq y$.

Given a Kuratowski (i.e., topological) closure operator, the *Kuratowski* (i.e., topological) *interior operator* dually defined according to Eq. (11) satisfies conditions (I0), (I1), (I3), plus the following property which substitutes the condition (I2):

(I2K) $\quad a^o \wedge b^o = (a \wedge b)^o \quad$ (multiplicative).

Moreover, we have the dual of Proposition 48.

Proposition 50 *Under conditions* (I1) *and* (I3) *the multiplicative property* (I2K) *is equivalent to the simultaneous satisfaction of the two conditions:*

(I2Ka) $a \leq b$ *implies* $a^o \leq b^o$ *(isotonicity).*
(I2Kb) $a = a^o$ *and* $b = b^o$ *imply* $(a \wedge b)^o = a \wedge b$ *(topological interior).*

Also in this case, in the sequel we shall call *Kuratowski interior operation* any topological interior defined on a De Morgan lattice. A De Morgan lattice equipped with a Kuratowski interior operation is simply called a *Kuratowski interior lattice*.

From the point of view of pointless topology of a Kuratowski lattice Σ, we have the following "completion" of the results of Propositions 32 and 20 about interior and closure operations, respectively.

Proposition 51 *Let Σ be a Kuratowski lattice. Then,*

(1) The family $\mathscr{O}(\Sigma)$ of all open elements, besides conditions (PO1a) *and* (PO2) *(or* (POO2) *in the case of a complete lattice), satisfies the further condition:*

(PO3) *$\mathscr{O}(\Sigma)$ is also closed with respect to the lattice meet operation: let $o, h \in \mathscr{O}(\Sigma)$, then also $o \wedge h \in \mathscr{O}(\Sigma)$.*

Therefore, in the case of a Kuratowski complete *lattice we have that the collection $\mathscr{O}(\Sigma)$ is just a true pointless lattice topology of open elements, in the sense that it contains the two elements* $0, 1$ *but besides the closure with respect to arbitrary join of open elements, it is also closed with respect to finite meet of open elements.*

(2) The family $\mathscr{C}(\Sigma)$ of all closed elements, besides conditions (PC1a) *and* (PC2) *(or* (PCC2) *in the case of a complete lattice), satisfies the further condition:*

(PC3) *$\mathscr{C}(\Sigma)$ is also closed with respect to the lattice join operation: let $c, d \in \mathscr{C}(\Sigma)$, then also $c \vee d \in \mathscr{C}(\Sigma)$.*

Therefore, in the case of a Kuratowski complete *lattice we have that the collection $\mathscr{C}(\Sigma)$ is just a true pointless lattice topology of closed elements, in the sense that it contains the two elements* $0, 1$ *but besides the closure with respect to arbitrary meet of closed elements, it is also closed with respect to finite join of closed elements.*

This leads to the following characterization of rough approximation spaces isomorphic to Kuratowski complete lattices.

(RAS-K) Let Σ be a De Morgan *complete* lattice equipped with a Kuratowski closure $u(a) = a^*$ and induced Kuratowski interior $l(a) = a^o = a'^{*\prime}$. Then the *Kuratowski rough approximation space* is the structure $\langle \Sigma, \mathscr{O}(\Sigma), \mathscr{C}(\Sigma), r \rangle$ where

(1) Σ is the collection of all *approximable elements*;
(2) $\mathscr{O}(\Sigma) = \{d \in \Sigma : d = d^o = l(d)\}$ is the pointless topological lattice of all *lower crisp elements* (i.e., open elements);
(3) $\mathscr{C}(\Sigma) = \{c \in \Sigma : c = c^* = u(c)\}$ is the pointless topological lattice of all *upper crisp elements* (i.e., closed elements);
(4) $r : \Sigma \to \mathscr{O}(\Sigma) \times \mathscr{C}(\Sigma)$ is the *rough approximation map* associating with any approximable element $a \in \Sigma$ the open–closed crisp pair $r(a) = (a^o, a^*) = (l(a), u(a))$, consisting of the lower crisp approximation $l(a) = a^o \in \mathscr{O}(\Sigma)$ defined by Eq. (14) and the upper crisp approximation $u(a) = a^* \in \mathscr{C}(\Sigma)$ of a defined by (7), satisfying the meta-theoretical principle (RMTP) of roughness coherence (RC1), crispness (RC2), and best approximation (RC3).

Remark 52 In literature one can find the definition of *Čech closure* as a weaker notion with respect to the Kuratowski one, in the sense that conditions (C0), (C1), and (C2K) are satisfied but not the idempotency condition (C3) (see [31], and also [105]). Concrete examples of Čech closures will be encountered in the cases of global coverings of a concrete universe discussed in Sect. 15.2 and of incomplete information systems discussed in Sect. 16.1.

Now we are interested to the converse of Proposition 50, but we formalize it in the most interesting case for future developments of a complete lattice.

Proposition 53 *Let* $\langle \Sigma, \wedge, \vee, ', 0, 1 \rangle$ *be a De Morgan* complete *lattice, simply denoted by* Σ. *Then,*

(1) if Σ *is equipped with an algebraic topology of open elements* $\mathscr{O}(\Sigma)$*, i.e., a subset of* Σ *satisfying conditions (PO1a), (POO2), and (PO3), then introduced for any element* $a \in \Sigma$ *the corresponding (open) element*

$$a^o := \vee \{o \in \mathscr{O}(\Sigma) : o \leq a\}$$

one has that the mapping $^o : \Sigma \to \Sigma$, $a \to a^o$ *is a Kuratowski interior operation, i.e., all conditions (I0), (I1), (I2K), and (I3), are satisfied.*
In other words, any abstract topological space *of open elements* $\langle \Sigma, \mathscr{O}(\Sigma) \rangle$ *induces in a canonical way a* Kuratowski interior complete lattice $\langle \Sigma, {}^o \rangle$;

(2) if Σ *is equipped with an algebraic topology of closed elements* $\mathscr{C}(\Sigma)$*, i.e., a subset of* Σ *satisfying conditions (PC1a), (PCC2), and (PC3), then introduced for any element* $a \in \Sigma$ *the corresponding (closed) element*

$$a^* := \wedge \{c \in \mathscr{C}(\Sigma) : a \leq c\}$$

one has that the mapping $^* : \Sigma \to \Sigma$, $a \to a^*$ *is a Kuratowski closure operation, i.e., all conditions (C0), (C1), (C2K), and (C3), are satisfied.*
In other words, any abstract topological space *of closed elements* $\langle \Sigma, \mathscr{C}(\Sigma) \rangle$ *induces in a canonical way a* Kuratowski closure complete lattice $\langle \Sigma, {}^* \rangle$.

8.1 Kuratowski Interior and Closure Operations on the Power Set of a Concrete Universe X

The Kuratowski approach to topology based on a concrete universe of points X has been developed by him in [65] and then further on completed in [66] (and see also [100, p. 13]). Adopting the formal denotation from H. Rasiowa and R. Sikorski, a *topological space by an internal operation* is formalized by a mapping $I : \mathscr{P}(X) \to \mathscr{P}(X)$ associating with any subset $A \in \mathscr{P}(X)$ the subset $I(A) \in \mathscr{P}(X)$, called the *interior* of A, in such a way that the following properties, called in [62, p. 43] *Kuratowski axioms*, are satisfied (corresponding to the abstract lattice case of Theorem 29, with condition (I2) substituted by (I2K) and the addition of condition (I0)):

(I0-X) $I(X) = X$,
(I1-X) $I(A) \subseteq A$,
(I2-X) $I(A \cap B) = I(A) \cap I(B)$,
(I3-X) $I(I(A)) = I(A)$.

Given the interior operator $I : \mathscr{P}(X) \to \mathscr{P}(X)$ satisfying the above conditions (I0-X)–(I3-X), if the orthocomplement of A in $\mathscr{P}(X)$ is denoted by $-A := X \setminus A$ then the corresponding closure operation $C : \mathscr{P}(X) \to \mathscr{P}(X)$ is defined by the law $\forall A \in \mathscr{P}(X)$, $C(A) := -(I(-(I(A))))$, this last simply written as $-I-(A)$, which trivially satisfies the following:

(C0-X) $C(\emptyset) = \emptyset$,
(C1-X) $A \subseteq C(A)$,
(C2-X) $C(A \cup B) = C(A) \cup C(B)$,
(C3-X) $C(C(A)) = C(A)$.

In this way we can consider the concrete structure $\mathfrak{K}(X) := \langle \mathscr{P}(X), \cap, \cup, -, I, C, \emptyset, X \rangle$ consisting of the Boolean algebra of subsets $\mathfrak{B}(X) := \langle \mathscr{P}(X), \cap, \cup, -, \emptyset, X \rangle$ based on the power set $\mathscr{P}(X)$ of the universe X, equipped with a Kuratowski interior operation I and induced Kuratowski closure $C = -I-$.

As usual the inclusions (I1-X) and (C1-X) allow one to introduce the collections

$$\mathscr{O}(X) := \{O \in \mathscr{P}(X) : O = I(O)\} \text{ and } \mathscr{C}(X) := \{K \in \mathscr{P}(X) : K = C(K)\}$$

of *open* and *closed sets*, respectively.

According to the general results of Proposition 51, the family $\mathscr{O}(X)$ (resp., $\mathscr{C}(X)$) is a real topology of open (resp., closed) sets for X since

(Top1) the empty set and the whole space are both open (resp., closed) sets,
(Top2) it is closed with respect to the union (resp., intersection) of any arbitrary family of open (resp., closed) sets, and
(Top3) it is closed with respect to the intersection (resp., union) of any *finite* family of open (resp., closed) sets.

Let us recall the relationships between the two families $\mathscr{O}(X)$ and $\mathscr{C}(X)$: K is closed iff K^c is open, and O is open iff O^c is closed.

Relatively to the above notions of concrete Kuratowski interior and closure operations and the induced definitions of open and closed sets, we have that:

(RAS-KX) The *Kuratowski rough approximation space* is the structure $\mathfrak{RAS}(X) := \langle \mathscr{P}(X), \mathscr{O}(X), \mathscr{C}(X), r \rangle$ where $r : \mathscr{P}(X) \to \mathscr{O}(X) \times \mathscr{C}(X)$ is the mapping associating with any subset A of X the open–closed pair of subsets of X given by $r(A) := (I(A), C(A))$ which satisfies the canonical rough meta-theoretical principle **(RMTP)** of coherence, crispness, and best approximation.

As to the coherence, this is the consequence of (I1-X) and (C1-X), $\forall \in \mathscr{P}(X)$, $I(A) \subseteq A \subseteq C(A)$; moreover, conditions (I3-X) and (C3-X) assure the crispness conditions $I(A) \in \mathscr{O}(X)$ and $C(A) \in \mathscr{C}(X)$. Formally, we have the diagram of Fig. 10.

Let us take into account that we are also interested to the *exterior* of the subset A of X defined as $E(A) := X \setminus C(A) \in \mathscr{O}(X)$, which is open as set theoretical complement of a closed set.

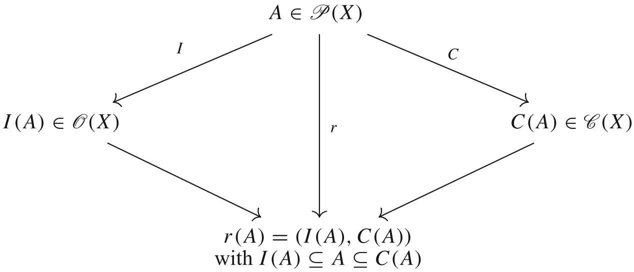


Fig. 10 Rough approximation on $\mathscr{P}(X)$ of the subset A

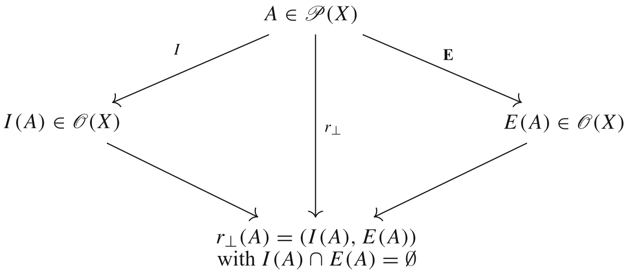


Fig. 11 Ortho-rough approximation on $\mathscr{P}(X)$ of the subset A

If the standard way to give a rough representation of an approximable subset A of X consists of a topological open-closed pair $r(A) = (I(A), C(A)) \in \mathscr{O}(X) \times \mathscr{C}(X)$, under the *inclusion* relation between an open-closed pair $I(A) \subseteq C(A)$ depicted in Fig. 10, as discussed in Sect. 4.1 there is an equivalent way of giving this representation by means of the so-called *ortho-pairs*. This corresponds to consider a rough approximation space $\mathfrak{RAS}(X)_\perp := \langle \mathscr{P}(X), \mathscr{O}(X), r_\perp \rangle$ where the *ortho-rough approximation map* is the application $r_\perp : \mathscr{P}(X) \to \mathscr{O}(X) \times \mathscr{O}(X)$ associating with any subset A of X its ortho-rough approximation $r_\perp(A) := (I(A), E(A))$, under the *orthogonality* relation $I(A) \cap E(A) = \emptyset$. Formally, we have the diagram of Fig. 11.

8.1.1 From Topological Spaces to Kuratowski Interior-Closure Operations

In this subsection we anticipate some considerations relatively to the approach to topology in order to coincide with the development which will be done in Sect. 15.3 of Part III. In this regard, let us recall the very important notion of *base* for a topology. In literature one can find different definitions of this concept, obviously equivalent to each other (for instance in [100]), but we adopt one that presents a slight change to that found in [62, theorem 11, p. 47] according to the following.

Definition 54 A family $\beta = \{B_j \in \mathscr{P}(X) : j \in J\}$ (indexed by the index set J) of subsets of X is said to be a *base* for a topology of X iff

(Co1) $\emptyset \in \beta$,
(Co2) $X = \cup\{B_j : j \in J\}$,
(Co3) for any pair B_i and B_j of subsets from β a collection $\{\hat{B}_k : k \in K\}$ (with $K \subseteq J$) of subsets from β exists such that $B_i \cap B_j = \cup\{\hat{B}_k : k \in K\}$.

The two conditions (Co1) and (Co2) define a *covering* of X, where the further condition (Co3) is the one which characterizes *topological covering*. A subset O of X is an *open set* iff it is the union of some sets of the base, i.e., iff $O = \cup\{B_k \in \beta : k \in K \subseteq J\}$. The collection of all open sets induced from the base β will be denoted by $\mathscr{O}_\beta(X)$ and it is easy to prove that it satisfies the required conditions (PO1a), (POO2), and (PO3), for a topology of open sets.

Now we can apply the results of Proposition 53 for defining the *interior* of any subset A of X. As expected the following result holds.

Proposition 55 *Let a concrete universe X be equipped with a topological covering (i.e., a base for a topology) β with induced family of open sets $\mathscr{O}_\beta(X)$. If for any subset A of X one defines the subset*

$$I(A) := \cup\{O \in \mathscr{O}_\beta(X) : O \subseteq A\} = \cup\{B \in \beta : B \subseteq A\} \tag{22}$$

then the mapping $I : \mathscr{P}(X) \to \mathscr{P}(X)$ $A \to I(A)$ is a Kuratowski interior operation.

8.2 *Kuratowski Interior and Closure Operations on the Lattice of Fuzzy Sets on a Concrete Universe*

Following Zadeh [121], let us recall that in a given universe of points X a *fuzzy set* is a mapping $f : X \to [0, 1]$ whose collection will be denoted as $\mathscr{F}(X) := [0, 1]^X$.

A particular subset of $\mathscr{F}(X) := [0, 1]^X$ is the collection of all *crisp sets*, i.e., the two values functions $\chi : X \to \{0, 1\}$ whose collection is $\{0, 1\}^X$. In particular if for any subset $A \in \mathscr{P}(X)$ we denote by χ_A the *characteristic function* of A defined for every $x \in X$ as $\chi_A(x) = 1$ if $x \in A$, and $= 0$ otherwise, we have that $\{0, 1\}^X = \{\chi_A : A \in \mathscr{P}(X)\}$. Two particular crisp sets are $\mathbf{0} := \chi_\emptyset$, defined as $\forall x \in X, \mathbf{0}(x) = 0$, and $\mathbf{1} := \chi_X$, defined as $\forall x \in X$, $\mathbf{1}(x) = 1$. In the sequel we are interested, for every fuzzy set $f \in \mathscr{F}(X)$ to the following subsets of X called the *certainty-yes*, *certainty-no*, and *possibility* domains of f, respectively:

$$A_1(f) := \{x \in X : f(x) = 1\} \qquad A_0(f) := \{x \in X : f(x) = 0\} \tag{23a}$$

$$A_p(f) := \{x \in X : f(x) \neq 0\} \tag{23b}$$

On $\mathscr{F}(X)$ some useful binary operations can be introduced. They are defined for any pair of fuzzy sets $f, g \in \mathscr{F}(X)$ and every point of the universe $x \in X$ by the following laws:

$$(f \wedge g)(x) := \min\{f(x), g(x)\} \tag{24a}$$

$$(f \vee g)(x) := \max\{f(x), g(x)\} \tag{24b}$$

The useful two unary operations are defined for every $f \in \mathscr{F}(X)$ and every $x \in X$ by the laws:

$$f'(x) := (\mathbf{1} - f)(x) \quad \text{and} \quad f^o(x) = \chi_{A_1(f)}(x) \tag{25}$$

Now it is easy to prove that

(1) the structure $\mathfrak{K}(\mathscr{F}(X)) := \langle \mathscr{F}(X), \wedge, \vee, ', \mathbf{0}, \mathbf{1} \rangle$ is the Kleene complete distributive lattice of all fuzzy sets which is not De Morgan since there exists the *genuine* (i.e., not crisp) fuzzy set **1/2** $\in \mathscr{F}(X)$ defined for every $x \in X$ by $(\mathbf{1/2})(x) := \frac{1}{2}$ for which the Kleene condition holds: (K) $\forall f, g \in \mathscr{F}(X)$, $(f \wedge f') \leq \mathbf{1/2} \leq (g \vee g')$. The Kleene complementation is not an orthocomplementation since the half fuzzy set **1/2** is such that $\mathbf{1/2} \wedge \mathbf{1/2}' = \mathbf{1/2} \neq \mathbf{0}$ and $\mathbf{1/2} \vee \mathbf{1/2} = \mathbf{1/2} \neq \mathbf{1}$.
This structure was highlighted for the first time by Zadeh in his paper [121], where however he recognized only the De Morgan properties (dM1) and (dM2) of the operation $'$, but not the validity of (K).

(2) The unary operation $^o : \mathscr{F}(X) \to \mathscr{F}(X),\ f \to f^o := \chi_{A_1(f)}$ is a Kuratowski interior.
We only prove the condition (I2K), all the other conditions are straightforward to prove. First of all, let us note that trivially for any pair of subsets $A, B \in \mathscr{P}(X)$ it is $\chi_A \wedge \chi_B = \chi_{A\cap B}$. Let $f, g \in \mathscr{F}(X)$ be two fuzzy sets, then $f^o \wedge g^o = \chi_{A_1(f)} \wedge \chi_{A_1(g)} = \chi_{A_1(f)\cap A_1(g)}$. Now, we have to consider $(f \wedge g)^o = \chi_{A_1(f\wedge g)}$; but $x \in A_1(f \wedge g)$ iff $\min\{f(x), g(x)\} = 1$ iff $f(x) = 1$ and $g(x) = 1$ iff $x \in A_1(f) \cap A_1(g)$. Therefore, $(f \wedge g)^o = \chi_{A_1(f\wedge g)} = \chi_{A_1(f)\cap A_1(g)} = f^o \wedge g^o$.

In this way we have obtained the Kleene distributive complete lattice with Kuratowski interior operation of all fuzzy sets $\mathfrak{F}(\mathscr{F}(X)) := \langle \mathscr{F}(X), \wedge, \vee, ', {}^o, \mathbf{0}, \mathbf{1} \rangle$.

Note that the orthogonality relation between pairs of fuzzy sets $f, g \in \mathscr{F}(X)$ is the following, where the *sum* $(f + g)$ of two fuzzy sets is defined as usual $\forall x \in X$, $(f + g)(x) := f(x) + g(x)$, which in general is not a fuzzy set:

$$f \perp g \quad \text{iff} \quad f \leq g' \quad \text{iff} \quad (f + g) \in \mathscr{F}(X). \tag{26}$$

From this it follows that the orthogonality between pairs of crisp sets $\chi_a, \chi_B \in \{0,1\}^X$ is the following

$$\chi_A \perp \chi_B \quad \text{iff} \quad A \cap B = \emptyset. \tag{27}$$

The closure of any fuzzy set $f \in \mathfrak{F}(\mathscr{F}(X))$ is the crisp set $f^* := f'^{o\prime} = \chi_{A_p(f)}$ characteristic function of the possibility domain. Of course, the mapping $^* : \mathscr{F}(X) \to \mathscr{F}(X)$, $f \to f^* = \chi_{A_p(f)}$ is a Kuratowski closure operation.

The collection of open fuzzy sets with respect to the Kuratowski interior, $\mathscr{O}(\mathscr{F}(X)) := \{f \in \mathscr{F}(X) : f = f^o\} = \{0,1\}^X$, is just the set of all crisp sets. Similarly, the collection of all closed fuzzy sets with respect to the Kuratowski interior $\mathscr{C}(\mathscr{F}(X)) := \{f \in \mathscr{F}(X) : f = f^*\} = \{0,1\}^X$ also in this case is the set of all crisp sets. Hence, we are in a situation in which the collections of open and closed elements coincide, $\mathscr{O}(\mathscr{F}(X)) = \mathscr{C}(\mathscr{F}(X))$. Topologically, we have to do with a family of *clopen fuzzy sets* simply denoted as $\mathscr{E}(\mathscr{F}(X)) := \mathscr{O}(\mathscr{F}(X)) = \mathscr{C}(\mathscr{F}(X))$. The study of Kuratowski lattices whose collections of open and closed elements coincide will be the argument of the forthcoming section under the name of Halmos interior–closure lattices.

We conclude this subsection about fuzzy sets depicting the diagram of the rough approximation map in the case of fuzzy set theory in Fig. 12.

Recalling the definition of exterior as the complement of the closure, we have that the exterior of a fuzzy f is $f^{\sim} = f^{*\prime} = \chi_{A_0(f)}$, i.e., the characteristic function of the certainty-no (also *impossibility*) domain of f. Therefore, the corresponding ortho-rough approximation of the fuzzy set f is depicted by Fig. 13, once taken into account the orthogonality relationship $A_1(f) \cap A_0(f) = \emptyset$ between the certainty-yes and the impossibility domains of f.

Owing to the one-to-one and onto correspondence between characteristic functions (crisp sets) from $\{0,1\}^X$ and subsets from the power set $\mathscr{P}(X)$, denoted as $\chi_A \longleftrightarrow A$, one can identify the ortho-rough approximation of a fuzzy set $r_\perp(f) = (\chi_{A_1(f)}, \chi_{A_0(f)}) \in \mathscr{E}(\mathscr{F}(X)) \times \mathscr{E}(\mathscr{F}(X))$ and the ortho-pair of subsets of the universe X, $(A_1(f), A_0(f)) \in \mathscr{P}(X) \times \mathscr{P}(X)$ with $A_1(f) \cap A_0(f) = \emptyset$.

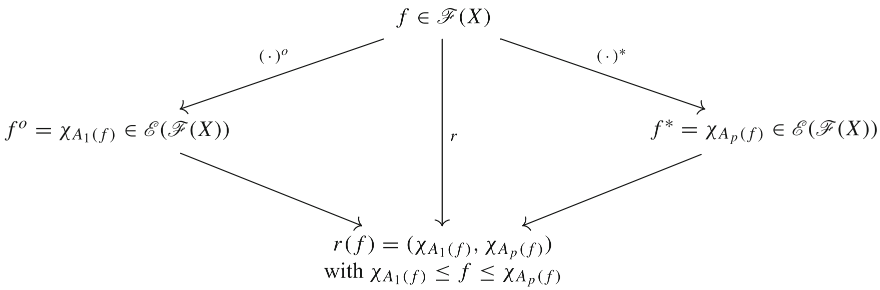


Fig. 12 Rough approximation of the fuzzy set f

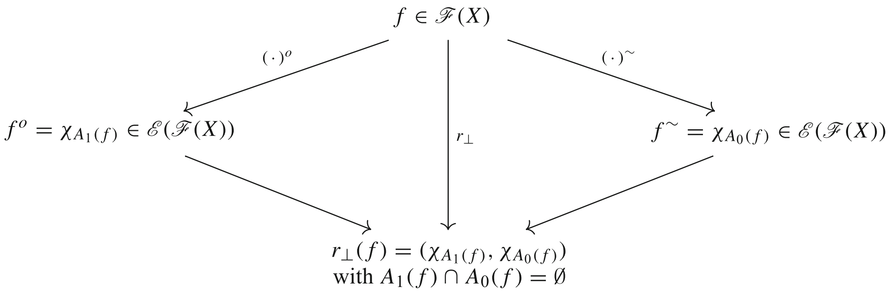


Fig. 13 Ortho-rough approximation of the fuzzy set f

9 The Halmos Closure and Induced Interior Operations

In this section we will investigate a further strengthening of the Kuratowski (and so also of the Tarski) closure operation according to the following definition.

Definition 56 A *Halmos closure* operation on a De Morgan lattice Σ is a mapping $^* : \Sigma \mapsto \Sigma$ which satisfies the following conditions for arbitrary $a, b \in \Sigma$:

$$(C1) \quad a \leq a^* \quad \text{(increasing)}$$

$$(C2K) \quad a^* \vee b^* = (a \vee b)^* \quad \text{(additive)}$$

$$(sC3) \quad a^{*\prime *} = a^{*\prime} \quad \text{(closure interconnection)}$$

A *Halmos closure lattice* is any De Morgan lattice equipped with a Halmos closure operation.

This definition has been introduced by Halmos in [52] (collected in [53]) on the basis of a Boolean algebra structure as the algebraic formalization of the *existential quantifier*, where it is pointed out that "The concept of existential quantifier occurs implicitly in a brief announcement of some related work of Tarski and Thompson [109]". In the context of the abstract approach to roughness theory we prefer to generalize this structure to the more general case of de Morgan lattices, keeping the terminology of closure instead of the one of quantifier.

From this definition it seems that the convention adopted before to always assume the closure of the zero element is not required explicitly. But in the following result this impression fails to comply with.

Lemma 57 *Under conditions* (C1) *and* (sC3) *the closure of the zero elements follows.*

$$[\text{(C1) and (sC3)}] \quad \Longrightarrow \quad \text{(C0)} \quad 0^* = 0 \qquad \text{(normalized)}$$

Proof From (C1) applied to the element 1 and the fact that this is the greatest element of the lattice, we get $1 \leq 1^* \leq 1$, i.e., $1^* = 1$. From (sC3) applied to the element 1 it follows that $1^{*\prime *} = 1^{*\prime} = 1'$. Taking into account Lemma 5, i.e., $1' = 0$, we obtain $1^{*\prime *} = 0$, and from $1^* = 1$ it follows that $1'^* = 0$ and from $1' = 0$ we conclude that $0^* = 0$. □

Let us now prove an interesting result which will be useful for the next proposition.

Lemma 58 *Condition* (C2K) *implies isotonicity of closure operation. Formally:*

$$\text{(C2K)} \quad \Longrightarrow \quad (a \leq b \ \textit{ implies } \ a^* \leq b^*).$$

Proof Let $a \leq b$, i.e., $b = a \vee b$, then $b^* = (a \vee b)^* = \text{(C2K)} = a^* \vee b^* \geq a^*$. □

The following result assures that Halmos closure is a Kuratowski closure.

Proposition 59 *Let Σ be a De Morgan lattice with an operation $^* : \Sigma \mapsto \Sigma$ which satisfies conditions* (C0), (C1)*, and* (C2K)*. Then the following implication holds:*

$$\text{(sC3)} \quad \forall a \in \Sigma,\ a^{*\prime *} = a^{*\prime} \quad \textit{implies} \quad \text{(C3)} \quad \forall a \in \Sigma,\ a^* = a^{**}$$

In other words, any Halmos closure operation is a Kuratowski closure operation, and so a fortiori a Tarski closure operation. That is,

$$\boxed{\textit{Halmos closure}} \Longrightarrow \boxed{\textit{Kuratowski closure}} \Longrightarrow \boxed{\textit{Tarski closure}} \qquad (28)$$

Proof Applying condition (C1) to the element a^* we get $a^* \leq a^{**}$. From (C1) $a \leq a^*$ and (sC3) we obtain $a \leq a^{*\prime *\prime}$, which applied to $a^{*\prime}$ leads to $a^{*\prime} \leq a^{*\prime *\prime *\prime}$. On the other hand, from $a \leq a^{*\prime *\prime}$, by Lemma 58 the isotonicity of closure holds, and so $a^* \leq a^{*\prime *\prime *}$ follows. Using the contraposition law of De Morgan complementation we get $a^{*\prime *\prime *\prime} \leq a^{*\prime}$. Therefore, $a^{*\prime} = a^{*\prime *\prime *\prime}$, i.e., $a^* = a^{*\prime *\prime *} = (a^{*\prime *\prime})^* = \text{(sC3)} = (a^{*\prime\prime})^* = a^{**}$. □

Example 60 There are Kuratowski closure which are not Halmos, as shown by the double Hasse diagrams of Fig. 14. The Hasse diagram at the left side corresponds to an orthocomplemented lattice which is not distributive ($(a \wedge b) \vee c = c \neq f = (a \vee c) \wedge (b \vee c)$). At the right side we have a Kuratowski closure which is not Halmos ($b^{*\prime} = a \neq d = b^{*\prime *}$).

Proposition 61 *In any De Morgan lattice, under conditions* (C1) *and* (sC3)*, the following two are equivalent:*

$$\begin{array}{lll} \text{(C2a)} & a \leq b \ \textit{ implies } \ a^* \leq b^* & \text{(isotone)} \\ \text{(C2K)} & (a \vee b)^* = a^* \vee b^* & \text{(additive)} \end{array}$$

Proof That (C2K) implies isotonicity as been proved in Lemma 58.

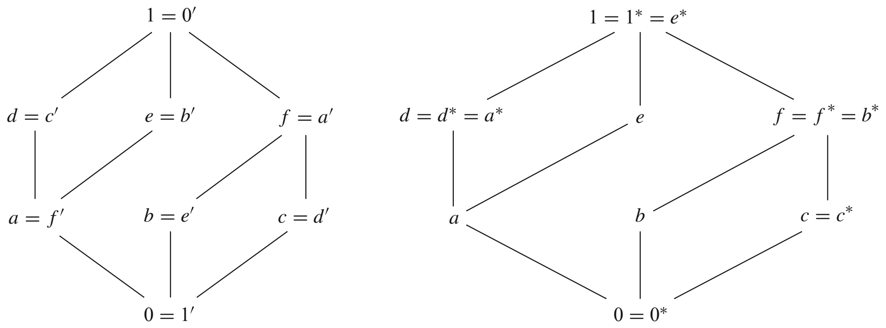


Fig. 14 At the left side the orthocomplemented not distributive lattice and at the right side the Kuratowski not Halmos closure

Conversely, let (C1) and (sC3) be true. Then if (C2a) holds from $a^*, b^* \leq (a \vee b)^*$, by contraposition, $(a \vee b)^{*\prime} \leq a^{*\prime}, b^{*\prime}$ follows, i.e., $(a \vee b)^{*\prime}$ is a lower bound of the pair $a^{*\prime}, b^{*\prime}$. Let c be any other lower bound of the same pair: $c \leq a^{*\prime}, b^{*\prime}$; then from (C2a) it follows $c^* \leq a^{*\prime *}, b^{*\prime *}$ and applying the contraposition law $c^{*\prime} \geq a^{*\prime *\prime}, b^{*\prime *\prime} = \text{(sC3)} = a^*, b^* \geq a, b$, i.e., $c^{*\prime}$ is an upper bound of the pair a, b from which it follows that $a \vee b \leq c^{*\prime}$, which (using isotonicity (C2a)) implies $(a \vee b)^* \leq c^{*\prime *}$ and by contraposition $c^{*\prime *\prime} \leq (a \vee b)^{*\prime}$. From (C1) we get $c \leq c^* = \text{(sC3)} = c^{*\prime *\prime} \leq (a \vee b)^{*\prime}$.

In conclusion, we have obtained that $(a \vee b)^{*\prime}$ is a lower bound of the pair $a^{*\prime}, b^{*\prime}$; moreover if c is a generic lower bound of the same pair $a^{*\prime}, b^{*\prime}$ the necessarily $c \leq (a \vee b)^{*\prime}$. That is $(a \vee b)^{*\prime} = a^{*\prime} \wedge b^{*\prime}$, from which it follows that $(a \vee b)^* = (a^* \vee b^*)$, which is the condition (C2K). □

Note that if one introduces the dual notion of interior operation $a^o := a'^{*\prime}$ then the condition (sC3) characterizing Halmos closure as different from Kuratowski (or Tarski) closure can be equivalently expressed according to the following:

Lemma 62 *In a De Morgan lattice equipped with a mapping* $a \to a^*$ *and the dual* $a \to a^o := a'^{*\prime}$*, without any required condition on these mappings, the following statements are mutually equivalent among them:*

$$\text{(sC3)} \quad \forall a \in \Sigma,\ a^{*\prime} = a^{*\prime *}$$

$$\text{(sC3a)} \quad \forall a \in \Sigma,\ a^o = a^{o*}$$

$$\text{(sC3b)} \quad \forall a \in \Sigma,\ a^* = a^{*o}$$

Proof Indeed, applying (sC3), $\forall a \in \Sigma$, $a^{*\prime} = a^{*\prime *}$, to the element a' we get $\forall a \in \Sigma$, $a'^{*\prime} = a'^{*\prime *}$, i.e., $a^o = a^{o*}$, which is (sC3a). Conversely, applying (sC3a), $\forall a \in \Sigma$, $a^o = a^{o*}$, to the element a' we obtain $a'^o = a'^{o*}$, from which it follows $a''^{*\prime} = a''^{*\prime *}$, i.e., $a^{*\prime} = a^{*\prime *}$, which is (sC3). On the other hand, $a^{*\prime} = a^{*\prime *}$ iff $a^* = a^{*\prime *\prime}$ iff $a^* = a^{*o}$. □

Note that (sC3a) means that in Halmos structures the interior a^o of any element a is closed and (sC3b) that the closure a^* of any element a is open.

Lemma 63 *Under the conditions of Lemma 62 the further equivalence holds:*

$$\text{(sC3b)} \quad \forall a \in \Sigma,\ a^* = a^{*o} \qquad \text{(mixed interconnection)}$$

$$\text{(sI3)} \quad \forall a \in \Sigma,\ a^{o\prime} = a^{o\prime o} \qquad \text{(interior interconnection)}$$

Proof Indeed, making use of the definition of interior, condition (sC3b) assumes the form $a^{\prime o\prime} = a^{\prime o\prime o}$, true for any a; hence applying this result to the element a' we get $a^{o\prime} = a^{o\prime o}$. Conversely, from (sI3) $a^{o\prime} = a^{o\prime o}$, using the definition $a^o = a^{\prime *\prime}$ we get $a^{\prime *\prime\prime} = a^{\prime *\prime\prime o}$, i.e., $a^{\prime *} = a^{\prime * o}$. Applying this last result to the element a' we have that $a^* = a^{*o}$. □

Proposition 64 *Let* $\mathscr{H} = \langle \Sigma, \wedge, \vee, {}', {}^*, 0, 1\rangle$ *be a Halmos closure lattice. The interior operator* ${}^o : \Sigma \to \Sigma$ *defined by the law* $\forall a \in \Sigma,\ a^o := a^{\prime *\prime}$ *can be characterized by the following properties*

$$\text{(I1)} \quad a^o \leq a \qquad \text{(decreasing)}$$

$$\text{(I2K)} \quad a^o \wedge b^o = (a \wedge b)^o \qquad \text{(multiplicative)}$$

$$\text{(sI3)} \quad a^{o\prime o} = a^{o\prime} \qquad \text{(interior interconnection)}$$

where the dual of condition (sC3)*, interconnecting closure with De Morgan negation, has been substituted by the equivalent formulation* (sI3)*, interconnecting interior with De Morgan negation. The induced structure* $\mathscr{H}^{\Delta} = \langle \Sigma, \wedge, \vee, {}', {}^o, 0, 1\rangle$ *is a* Halmos interior lattice.

Moreover, using the equivalence between (sC3b) and (sC3) stated in Lemma 62 and the implication of Proposition 59, the further implication holds:

$$\text{(sC3b)} \quad \forall a \in \Sigma,\ a^* = a^{*o} \quad \textit{implies} \quad \textit{(I3)} \quad \forall a \in \Sigma,\ a^o = a^{oo}$$

That is, any Halmos interior operator is a Kuratowski interior operator, and so also a Tarski interior operator.

Analogously to the closure case expressed by Lemma 57, from (I1) and (sI3) it is possible to prove that the condition (I0) of normalization, $1 = 1^o$, holds.

If as usual we denote by $\mathscr{C}(\Sigma)$ (resp., $\mathscr{O}(\Sigma)$) the collection of all closed (resp., open) elements, condition (sC3) can also be expressed saying that for any $a \in \Sigma$ the element $a^{*\prime}$ is closed and condition (sI3) that for any $a \in \Sigma$ the element $a^{o\prime}$ is open.

Remark 65 Condition (sC3b) can be more economically expressed as

$$\text{(sC3w)} \quad \forall a \in \Sigma,\ a^* \leq a^{*o}$$

Indeed, condition (I1) applied to the element a^* leads to $\forall a \in \Sigma, a^{*o} \leq a^*$.

The following property characterizes De Morgan lattices with Halmos closure relatively to De Morgan lattices with Kuratowski closure.

Proposition 66 *Let Σ be a De Morgan lattice with a topological (i.e., Kuratowski) closure operation. Then the following are equivalent:*

1. *the topological closure operation satisfies condition (sC3), i.e., it is Halmos;*
2. *the collection of closed elements and the collection of open elements coincide: $\mathscr{C}(\Sigma) = \mathscr{O}(\Sigma)$.*

Proof Let (sC3) be true. Then if $a \in \mathscr{C}(\Sigma)$ we have that $a = a^* =$ (sC3b) $= a^{*o}$, i.e., $a \in \mathscr{O}(\Sigma)$. On the other hand, if $a \in \mathscr{O}(\Sigma)$ then $a = a^o =$ (sC3a) $= a^{o*}$, i.e., $a \in \mathscr{C}(\Sigma)$.

Let * be a Kuratowski closure and let $\mathscr{C}(\Sigma) = \mathscr{O}(\Sigma)$. From condition (C3) of idempotency of Kuratowski closure it follows that a^* is closed and so, by hypothesis, it is also open, i.e., $a^* = a^{*o}$, which is condition (sC3b) equivalent to (sC3). □

Lemma 67 *Let Σ be a Halmos closure lattice. Then, under condition $\mathscr{C}(\Sigma) = \mathscr{O}(\Sigma)$ we have that the collection of all clopen elements is defined as $\mathscr{E}(\Sigma) := \mathscr{C}(\Sigma) \cap \mathscr{O}(\Sigma) = \mathscr{C}(\Sigma) = \mathscr{O}(\Sigma)$.*

Proof Let $\mathscr{C}(\Sigma) = \mathscr{O}(\Sigma)$, then $\mathscr{E}(\Sigma) = \mathscr{C}(\Sigma) \cap \mathscr{O}(\Sigma) = \mathscr{C}(\Sigma) \cap \mathscr{C}(\Sigma) = \mathscr{C}(\Sigma) = \mathscr{O}(\Sigma)$. □

In the present Halmos case the results about the categorical isomorphism between closure lattices and upper approximation spaces given by Theorems 26–28 and the corresponding categorical isomorphism between interior lattices and lower approximation spaces expressed by Theorem 35 can be formulated in the following way.

Theorem 68 *Any Halmos closure–interior lattice $\mathscr{H}^{\blacktriangle} = \langle \Sigma, ^*, ^o \rangle$ consisting of a De Morgan lattice Σ equipped with a Halmos closure * and induced Halmos interior $^o = {}'^{*}{}'$ operations, with set of closed elements $\mathscr{C}(\Sigma)$ and open elements $\mathscr{O}(\Sigma)$ coinciding between them and forming the set of crisp (clopen) elements $\mathscr{E}(\Sigma) = \mathscr{C}(\Sigma) = \mathscr{O}(\Sigma)$,*

is categorically equivalent

to the abstract approximation space for rough theories $\mathscr{H}^{\blacktriangledown} = \langle \Sigma, \mathscr{E}(\Sigma), l, u \rangle$ equipped with an upper approximation map $u(a) = a^*$ and a lower approximation map $l(a) = a^o$ satisfying the further condition that $\mathscr{E}(\Sigma) = \mathscr{C}(\Sigma) = \mathscr{O}(\Sigma)$.

Proof If Σ is a Halmos closure-interior lattice, then by Proposition 59 it is also a Kuratowski closure-interior lattice, and so a fortiori also a Tarski closure-interior lattice. Thus from Theorems 26 and 27, the point (1) of the dual Theorem 35, and the just proved Proposition 66, we obtain an abstract rough approximation space for which the sets of upper and lower crisp elements coincide.

Conversely, let us consider a rough approximation space whose set of upper and lower crisp elements coincide. By Theorem 27 the upper part generates a closure operator satisfying conditions (C1), (C2), and (C3). In Sect. 3.1 we have seen that the zero element 0 is open, i.e., $0 \in \mathscr{O}(\Sigma)$, and owing to the condition $\mathscr{C}(\Sigma) = \mathscr{O}(\Sigma)$ we obtain that $0 \in \mathscr{C}(\Sigma)$, i.e., the condition (C0) of Tarski closure. From the condition (C3) it follows that for any element a the corresponding closure is closed, i.e., $a^* \in \mathscr{C}(\Sigma)$, but from the hypothesis $\mathscr{C}(\Sigma) = \mathscr{O}(\Sigma)$ it follows that $a^* \in \mathscr{O}(\Sigma)$, i.e., $a^{*o} = a^o$, which is the condition (sC3b) equivalent to (sC3).
Now, condition (C2) is equivalent to the isotonicity (C2a) of Sect. 2 and so, by Proposition 61 we obtain that also condition (C2K) is verified. □

Let us recall that in the interpretation of open and closed elements from Σ as lower and upper crisp approximations, respectively, the elements from $\mathscr{E}(\Sigma)$ are simultaneously upper and lower crisp, and so they have been defined as *exact*, or also tout court *crisp*.

The rough approximation space equivalent to Halmos closure–interior lattice according to the just proved Theorem 68 is a particular case of rough approximation space characterized by the condition that the two families of upper $\mathscr{C}(\Sigma)$ and lower $\mathscr{O}(\Sigma)$ crisp elements collapse in a unique class $\mathscr{E}(\Sigma)$ of crisp elements.

This is an abstraction, as we will see later in Sects. 16 and 16.1, of the Pawlak approach to rough set theory based on (complete) information systems generating partitions of a concrete universe X, introduced in the paper [89]. For this reason, in order to distinguish this class of rough approximation spaces from the more general ones characterized by $\mathscr{C}(\Sigma) \neq \mathscr{O}(\Sigma)$ we adopt the terminology to call them as *rough approximation space of type P*, where "P" stays for "Pawlak type".

The rough approximation spaces in which the condition $\mathscr{C}(\Sigma) \neq \mathscr{O}(\Sigma)$ holds, discussed in Sect. 4, are the abstraction of the situation consisting of the covering of a concrete universe X generated by an incomplete information system, as introduced in the papers [122] and [116], and which will be discussed later in Sect. 16.1.

The diagram of Fig. 15 summarizes the rough approximation map in the case of a Halmos closure space.

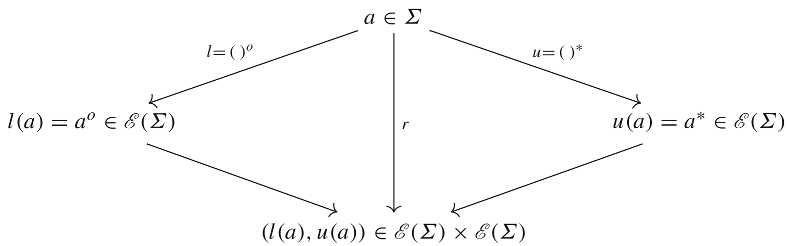


Fig. 15 The rough approximation map induced from a Halmos closure characterized by the condition $\mathscr{E}(\Sigma) = \mathscr{C}(\Sigma) = \mathscr{O}(\Sigma)$

This leads to the following characterization of rough approximation spaces isomorphic to Halmos complete lattices.

(RAS-H) Let Σ be a De Morgan *complete* lattice equipped with a Halmos closure $u(a) = a^*$ and induced Halmos interior $l(a) = a^o = a'^{*'}$.
Then the *Halmos rough approximation space* is the structure $\langle \Sigma, \mathscr{E}(\Sigma), r \rangle$ where

(1) Σ is the collection of all *approximable elements*;
(2–3) $\mathscr{E}(\Sigma) = \mathscr{O}(\Sigma) = \mathscr{C}(\Sigma)$ is the pointless topological complete lattice of all *crisp elements* (i.e., clopen elements);
(4) $r : \Sigma \to \mathscr{E}(\Sigma) \times \mathscr{E}(\Sigma)$ is the *rough approximation map* associating with any approximable element $a \in \Sigma$ the clopen–clopen crisp pair $r(a) = (a^o, a^*) = (l(a), u(a))$, consisting of the lower crisp approximation $l(a) = a^o \in \mathscr{E}(\Sigma)$ defined by Eq. (14) and the upper crisp approximation $u(a) = a^* \in \mathscr{E}(\Sigma)$ of a defined by (7), satisfying the meta-theoretical principle (RMTP) of roughness coherence (RC1), crispness (RC2), and best approximation (RC3).

9.1 Pre BZ Structures from Halmos Closure Operators

In this subsection we explore another unary operation which can be induced from any Halmos closure operation whose algebraic behavior shares some aspects of an algebraic version of intuitionistic (Brouwer) negation. First of all we prove some preliminary results.

Lemma 69 *In a De Morgan lattice structure equipped with a unary operation $a \to a^*$ on which no property is required to hold, once defined the unary operation $a \to a^\sim$ by the condition $a^\sim := a^{*\prime}$ (equivalently $a^* = a^{\sim\prime}$) the following are equivalent:*

$$\text{(sC3)} \quad \forall a \in \Sigma,\ a^{*\prime *} = a^{*\prime} \qquad \text{(closure interconnection)}$$

$$\text{(IR)} \quad \forall a \in \Sigma,\ a^{\sim\prime} = a^{\sim\sim} \qquad \text{(Brouwer interconnection)}$$

Proof Let (sC3b), equivalent by Lemma 62 to (sC3), be true. Then $a^{\sim\prime} = a^{*\prime\prime} = a^* = (sC3b) = a^{*\prime *\prime} = a^{\sim\sim}$, i.e., the interconnection rule (IR) is true.

Let the interconnected rule (IR) be true. If we apply the definition $a^* = a^{\sim\prime}$ to the element a' we have that $a'^* = a'^{\sim\prime}$, from which $a'^{*\prime} = a'^{\sim}$ follows. Using this equality and (IR) we obtain $a^* = a^{\sim\prime} = \text{(IR)} = a^{\sim\prime\sim} = a^{*\prime\sim} = a^{*\prime *\prime}$, from which (sC3) follows, $a^{*\prime} = a^{*\prime *}$. □

Proposition 70 *The following categorical equivalence holds.*

(CB) *Let* $\mathscr{A} = \langle \Sigma, \wedge, \vee, ', {}^*, 0, 1 \rangle$ *be a Halmos closure lattice, then introduced the corresponding* exterior *operation* $a^\sim = a^{*\prime}$ *the following are satisfied for every pair* $a, b \in \Sigma$*:*

(B1) $a \leq a^{\sim\sim}$ (weak double negation law)
(B2) $a \leq b$ *implies* $b^\sim \leq a^\sim$ (B-contraposition law)

Moreover, the further property holds:

$$\textit{(IR)} \quad \forall a \in \Sigma,\ a^{\sim\prime} = a^{\sim\sim} \quad \text{(interconnection rule)}$$

That is, the structure $\mathscr{A}^\blacklozenge = \langle \Sigma, \wedge, \vee, ', {}^\sim, 0, 1 \rangle$ *is a pre Brouwer Zadeh (BZ) lattice.*

(BC) *Conversely, if* $\mathscr{B} = \langle \Sigma, \wedge, \vee, ', {}^\sim, 0, 1 \rangle$ *is a De Morgan lattice equipped with an exterior operation* $^\sim$ *satisfying conditions (B1), (B2), and the interconnection rule (IR), i.e., it is a pre Brouwer Zadeh (BZ) lattice, then* $a^* = a^{\sim\prime}$ *defines a Halmos closure operation and so the structure* $\mathscr{B}^\bigstar = \langle \Sigma, \wedge, \vee, ', {}^*, 0, 1 \rangle$ *is a Halmos closure lattice.*

(CE) *The two structures of* Halmos closure lattices *and* pre BZ lattices *are categorically equivalent. Indeed,*

(CE1) *if* $\mathscr{A}$ *is a Halmos closure lattice with induced pre BZ lattice* $\mathscr{A}^\blacklozenge$*, then* $(\mathscr{A}^\blacklozenge)^\bigstar = \mathscr{A}$*.*
(CE2) *Conversely, if* $\mathscr{B}$ *is a pre BZ lattice with induced Halmos closure lattice* $\mathscr{B}^\bigstar$*, then* $(\mathscr{B}^\bigstar)^\blacklozenge = \mathscr{B}$*.*

Proof First of all, by Lemma 69 conditions (sC3) and (IR) are equivalent without any requirement on the mapping $a \to a^\sim$.

Let us now consider a Halmos closure operation $a \to a^*$. Then, defined $a^\sim = a^{*\prime}$, one proves the following.

(B1) From (C2) we have $a \leq a^* = (sC3) = a^{*\prime*\prime} = a^{\sim\sim}$.

(B2) According to the closure isotonicity condition (C2a) (which according to Proposition 61 is equivalent to (C2K)) we have that $a \leq b$ implies $a^* \leq b^*$, applying to this latter inequality the contraposition law for De Morgan negation $b^{*\prime} \leq a^{*\prime}$, i.e., $b^\sim \leq a^\sim$.
Conversely, let us consider a De Morgan lattice Σ equipped with a unary operation $^\sim : \Sigma \to \Sigma$ satisfying conditions (B1), (B2), and (IR). Then, defined $a^* = a^{\sim\prime}$, one obtains

(C1) $a \leq (B1) \leq a^{\sim\sim} = (IR) = a^{\sim\prime} = a^*$, which is condition (C1).

(C2K) Owing to Proposition 61, in order to prove this condition it is sufficient to prove that condition (B2) implies the isotonicity of the mapping $a \to a^* = a^{\sim\prime}$. Indeed, $a \leq b$ by (B2) implies $b^\sim \leq a^\sim$ which by contraposition implies $a^{\sim\prime} \leq b^{\sim\prime}$, i.e., $a^* \leq b^*$. □

Proposition 71 *Let Σ be a lattice equipped with a unary mapping $^\sim : \Sigma \to \Sigma$. Then the following two conditions are equivalent:*

$$\text{(B2)} \qquad a \le b \ \text{ implies } \ b^\sim \le a^\sim \qquad \text{(B-contraposition law)}$$

$$\text{(sB-dM1)} \qquad (a \vee b)^\sim \le a^\sim \wedge b^\sim \qquad \text{(sub B-De Morgan law)}$$

Proof Let (B2) be true. From $a, b \le a \vee b$ it follows that $(a \vee b)^\sim \le a^\sim, b^\sim$, i.e., $(a \vee b)^\sim$ is a lower bound of the pair $a^\sim, b^\sim$; but $a^\sim \wedge b^\sim$ is the greatest lower bound of the same pair and so $(a \vee b)^\sim \le a^\sim \wedge b^\sim$.

Conversely, let (sB-dM1) be true. If $a \le b$, i.e., $b = a \vee b$, then $b^\sim = (a \vee b)^\sim \le a^\sim \wedge b^\sim \le a^\sim$. □

Proposition 72 *In a lattice equipped with a unary operation $^\sim$ satisfying the only condition (B1) the following are equivalent:*

$$\text{(B2)} \qquad a \le b \quad \text{implies} \quad b^\sim \le a^\sim \qquad \text{(B-contraposition law)}$$

$$\text{(B-dM1)} \qquad (a \vee b)^\sim = a^\sim \wedge b^\sim \qquad \text{(first B-De Morgan law)}$$

Proof Let (B-dM1) be true. Then (B-dM1) implies $(a \vee b)^\sim \le a^\sim \wedge b^\sim$. But in Proposition 71 we have already shown that $(a \vee b)^\sim \le a^\sim \wedge b^\sim$ is equivalent to the B-contraposition condition (B2), without any requirement on $^\sim$.

Let now (B2) be true. From $a, b \le a \vee b$ and (B2) we get $(a \vee b)^\sim \le a^\sim, b^\sim$, that is $(a \vee b)^\sim$ is a lower bound of the pair $a^\sim, b^\sim$. Let us suppose that c is a generic lower bound of the same pair $c \le a^\sim, b^\sim$, then from (B1) and (B2) we get $a, b \le a^{\sim\sim}, b^{\sim\sim} \le c^\sim$, i.e., $c^\sim$ is an upper bound of the pair a, b and so, since $a \vee b$ is the least upper bound of the same pair, we get $a \vee b \le c^\sim$, from which always by (B1) and (B2) $c \le c^{\sim\sim} \le (a \vee b)^\sim$ follows. Therefore, we have sown that $(a \vee b)^\sim$ is the greatest lower bound of the pair $a^\sim, b^\sim$, i.e., $(a \vee b)^\sim = a^\sim \wedge b^\sim$. □

Lemma 73 *Under conditions (B1) and (B2) the following holds:*

$$\forall a, \quad a^\sim = a^{\sim\sim\sim} \tag{29}$$

corresponding to the fact that "in Heyting's logic we have a law of triple negation" [44]. From this result it follows the identity

$$\text{(B4)} \qquad 1 = 0^\sim \qquad \text{(coherence condition)}$$

Moreover, the following implication holds for any element a:

$$\text{(IR)} \quad a^{\sim\prime} = a^{\sim\sim} \ \text{ implies } \ \text{(wIR)} \quad a^\sim \le a'$$

Lastly, from the only condition (B2) we have that

$$\text{(wB-dM2)} \qquad a^\sim \vee b^\sim \le (a \wedge b)^\sim$$

which can be considered a weak form of the second B-De Morgan law, dual of (B-dM1).

Proof Condition (B1) applied to the element $a^{\sim}$ leads to the inequality $a^{\sim} \leq a^{\sim\sim\sim}$. On the other hand, applying to (B1), $a \leq a^{\sim\sim}$, condition (B2) we get $a^{\sim\sim\sim} \leq a^{\sim}$.

From the condition that 0 is the least element we get $0 \leq 1^{\sim}$, from which by (B2) it follows $1^{\sim\sim} \leq 0^{\sim}$ and using (B2) we obtain $0^{\sim\sim} \leq 1^{\sim\sim\sim} = (29) = 1^{\sim}$, and so by (B1) applied to the least element 0 we obtain $0 \leq 0^{\sim\sim} \leq 1^{\sim}$; if to this result we apply condition (B2) it follows $1^{\sim\sim} \leq 0^{\sim}$, but (B1) applied to the greatest element 1 leads to $1 \leq 1^{\sim\sim} \leq 0^{\sim} \leq 1$, i.e., $1 = 0^{\sim}$.

From (B1), $a \leq a^{\sim\sim}$, by contraposition we get $a^{\sim\sim\prime} \leq a'$, but owing to (IR) we obtain $a^{\sim\sim\sim} \leq a'$ and using (29) this leads to $a^{\sim} \leq a'$.

From $a \wedge b \leq a, b$, by (B2), we get $a^{\sim}, b^{\sim} \leq (a \wedge b)^{\sim}$, i.e., $(a \wedge b)^{\sim}$ is an upper bound of the pair $a^{\sim}, b^{\sim}$. But $a^{\sim} \vee b^{\sim}$ is the least upper bond of the same pair and so $a^{\sim} \vee b^{\sim} \leq (a \wedge b)^{\sim}$. □

Remark 74 The *exterior* unary operation $a^{\sim} = a^{*\prime}$, according to Proposition 70, satisfies the properties (B1) and (B2), algebraic version of a *pre Brouwer negation*. This is an algebraic realization of a pre Brouwer negation, which is not Brouwer, since it lacks the important property of noncontradiction, accepted as true by intuitionistic logic [57] (see also [17]), formalized as

$$\text{(B3)} \qquad \forall a \in \Sigma, \quad a \wedge a^{\sim} = 0.$$

Hence, the structure $\langle \Sigma, \wedge, \vee, ', ^{\sim}, 0, 1 \rangle$ induced from a Halmos closure consists of a De Morgan lattice equipped with the further pre Brouwer complementation satisfying the interconnection rule (IR). According to the definition introduced in [25] this structure has been called *pre Brouwer Zadeh (BZ)* lattice (also called *minimal Brouwer Zadeh (BZ)* lattice).

Using this terminology the results of Proposition 70 can be compacted in the following categorical isomorphism:

$$\boxed{\text{Halmos closure lattice}} \quad \Longleftrightarrow \quad \boxed{\text{pre BZ lattice}} \tag{30}$$

An interesting behavior regards the set $\mathscr{E}(\Sigma)$ of all exact (or crisp) elements of the pre BZ lattice induced from a Halmos closure lattice characterized by the property that $\mathscr{C}(\Sigma) = \mathscr{O}(\Sigma)$. First of all we prove the following.

Lemma 75 *Let* $\langle \Sigma, \wedge, \vee, ', ^{*}, 0, 1 \rangle$ *and* $\langle \Sigma, \wedge, \vee, ', ^{\sim}, 0, 1 \rangle$ *be the two isomorphic structures of Halmos closure lattice and pre BZ lattice linked by the relationships* $a^{\sim} = a^{*\prime}$ *and* $a^{*} = a^{\sim\prime}$. *Then the following are equivalent for a single element* $\alpha \in \Sigma$*:*

$$\text{(Ex1)} \quad \alpha = \alpha^{*} \iff \text{(Ex2)} \quad \alpha = \alpha^{o} \iff \text{(Ex3)} \quad \alpha = \alpha^{\sim\sim}.$$

Proof (Ex1) $\Longrightarrow$ (Ex2). Let α be an element of Σ such that $\alpha = \alpha^*$. Since a Halmos closure lattice is a De Morgan lattice, by Lemma 62 it is true the (sC3b), $\forall a \in \Sigma$, $a^* = a^{*o}$; then for the particular α it is $\alpha = \alpha^* = \alpha^{*o}$. But from $\alpha = \alpha^*$ it follows that $\alpha^o = \alpha^{*o}$. Therefore, we get $\alpha = \alpha^o$.

(Ex2) $\Longrightarrow$ (Ex1). Let α be an element of Σ such that $\alpha = \alpha^o$. Always by Lemma 62 it is true the (sC3a), $\forall a \in \Sigma$, $a^o = a^{o*}$; then for the particular α it is $\alpha = \alpha^o = \alpha^{o*}$. But from $\alpha = \alpha^o$ it follows that $\alpha^* = \alpha^{o*}$. Therefore, we get $\alpha = \alpha^*$. So we have proved the equivalence "(Ex1) $\Longleftrightarrow$ (Ex2)".

(Ex3) $\Longrightarrow$ (Ex1). Let $\alpha = \alpha^{\sim\sim}$ be true for a fixed $\alpha \in \Sigma$. Then, from (IR) $\forall a \in \Sigma\ a^{\sim\sim} = a^{\sim\prime}$, we get $\alpha = \alpha^{\sim\prime} = \alpha^*$.

(Ex2) $\Longrightarrow$ (Ex3). Let $\alpha = \alpha^o$ be true, then by the just proved equivalence we have that $\alpha = \alpha^* = \alpha^{\sim\prime} = \text{(IR)} = \alpha^{\sim\sim}$. ◻

From this lemma and the fact that Halmos closure lattices are characterized by the property that, according to Lemma 67, the set of crisp elements satisfies the equalities $\mathscr{E}(\Sigma) = \mathscr{C}(\Sigma) \cap \mathscr{O}(\Sigma) = \mathscr{C}(\Sigma) = \mathscr{O}(\Sigma)$, it follows the chain of identities:

$$\mathscr{E}(\Sigma) = \{e \in \Sigma : e = e^*\} = \{e \in \Sigma : e = e^o\} = \{e \in \Sigma : e = e^{\sim\sim}\} \tag{31}$$

Theorem 76 *Let $\mathfrak{BZ} = \langle \Sigma, \wedge, \vee, {}', {}^\sim, 0, 1\rangle$ be the pre BZ lattice isomorphically induced from a Halmos closure lattice $\mathfrak{H} = \langle \Sigma, \wedge, \vee, {}', {}^*, 0, 1\rangle$, and let us consider its subset $\mathscr{E}(\Sigma)$. Then,*

(1) For every $e \in \mathscr{E}(\Sigma)$ it is $e' = e^\sim$, from which it follows that $e' \in \mathscr{E}(\Sigma)$. From this result we have that the mapping ${}'^e : \mathscr{E}(\Sigma) \to \mathscr{E}(\Sigma)$, $e \to e'^e := e'$, restriction to $\mathscr{E}(\Sigma)$ of the mapping ${}' : \Sigma \to \Sigma$, is well defined and satisfies the two conditions (dM1) and (dM2) of a De Morgan complementation.

(2) $\mathscr{E}(\Sigma)$ is closed with respect to the restriction to it of the binary lattice operations $\wedge$ and $\vee$ of Σ, denoted by $\wedge_e$ and $\vee_e$ respectively. Moreover for any pair of elements $e, h \in \mathscr{E}(\Sigma)$ it is $e \wedge_e h = e \wedge h$ and $e \vee_e h = e \vee h$, leading to the result that $\langle \mathscr{E}(\Sigma), \wedge, \vee, 0, 1\rangle$ is a bounded lattice.

The subset $\mathscr{E}(\Sigma)$ of Σ is the support of the De Morgan lattice system $\langle \mathscr{E}(\Sigma), \wedge, \vee, {}', 0, 1\rangle$, where owing to the points (1) and (2) we have used the simplified notations $\wedge$, $\vee$, and ${}'$.

Furthermore, the restriction to $\mathscr{E}(\Sigma)$ of both the interior and closure operations from the Halmos closure lattice $\mathfrak{H}$ coincide with the identity: $\forall e \in \mathscr{E}(\Sigma)$, $e^* = e^o = e$, that is ${}^* \restriction \mathscr{E}(\Sigma) = {}^o \restriction \mathscr{E}(\Sigma) = id$. In other words, they are meaningless.

Proof

(1) Let $e \in \mathscr{E}(\Sigma)$, that is, by (31), $e = e^{\sim\sim}$. Then, from this we get $e' = (e^\sim)^{\sim\prime} = \text{(IR)} = e^{\sim\sim\sim} = (29) = e^\sim$, that is $e' = e^\sim$. But, always from (29) $e^\sim = (e^\sim)^{\sim\sim}$ and from (31) it follows that $e^\sim \in \mathscr{E}(\Sigma)$, that is $e' = e^\sim \in \mathscr{E}(\Sigma)$. From the fact that ${}'^e = {}' \restriction \mathscr{E}(\Sigma)$ i.e., it is the restriction of ${}'$ to $\mathscr{E}(\Sigma)$, and the fact

that $'$ satisfies the De Morgan conditions (dM1) and (dM2), it follows that also $'^{e}$ satisfies these two conditions.

(2) Let $e, h \in \mathscr{E}(\Sigma)$, i.e., by (31) $e^{\sim\sim} = e$ and $h^{\sim\sim} = h$. Then, $(e \vee h)^{\sim\sim} =$ (IR) $= (e \vee h)^{\sim\prime} =$ (B-dM1) $= (e^{\sim} \wedge h^{\sim})' =$ (dM2) $= e^{\sim\prime} \vee h^{\sim\prime} =$ (IR) $= e^{\sim\sim} \vee h^{\sim\sim} = e \vee h$. Therefore, we have proved the identity $(e \vee h)^{\sim\sim} = e \vee h$, and so $(e \vee h) \in \mathscr{E}(\Sigma)$ by (31). This means that a fortiori $e \vee_e h = e \vee h$.

Let us now consider two elements $\alpha, \beta \in \mathscr{E}(\Sigma)$. Then we have just proved that $\alpha \vee_e \beta = \alpha \vee \beta$, and so $(\alpha \vee_e \beta)' = (\alpha \vee \beta)'$, from which, by (dM2), we get $\forall \alpha, \beta \in \mathscr{E}(\Sigma)$, $(\alpha \vee_e \beta)' = \alpha' \wedge \beta'$, where $\alpha' \wedge \beta'$ is the g.l.b. of the pair α', β' as elements of Σ; in other words, $\forall c \in \Sigma$ s.t. $c \leq \alpha', \beta'$ it is $c \leq \alpha' \wedge \beta'$. This implies that $\forall \gamma \in \mathscr{E}(\Sigma) \subseteq \Sigma$, condition $\gamma \leq \alpha', \beta'$ implies a fortiori that $\gamma \leq \alpha' \wedge \beta'$, i.e., $\alpha' \wedge \beta'$ is the g.l.b of the pair α', β' in $\mathscr{E}(\Sigma)$, written as $\forall \alpha, \beta \in \mathscr{E}(\Sigma)$, $\alpha' \wedge_e \beta' = \alpha' \wedge \beta'$. Applying this identity to the two elements $e = \alpha'$ and $h = \beta'$ we have $e \wedge_e h = e \wedge h$.

Trivially, from (31) $\forall e \in \mathscr{E}(\Sigma)$ $e^* = e^o = e$. □

Some comments on the condition (B3) "$\forall a \in \Sigma$, $a \wedge a^{\sim} = 0$", introduced in Remark 74, which is not satisfied in a pre BZ lattice. First of all, let us note that condition (B3), as shown from the following example, is independent from the other two Brouwer conditions (B1) and (B2). On the other hand both conditions (wIR) and (IR) are independent from the whole structure of De Morgan lattice equipped with a Brouwer complementation.

Example 77 The Hasse diagrams of Fig. 16 depict a three elements linear lattice consisting of the distinct elements $\{0, d, 1\}$ equipped with a Kleene negation with half element d (at the left side) and a pre Brouwer complementation (at the right side). This mapping $^{\sim}$ satisfies the conditions (B1) and (B2) of a pre Brouwer complementation, but not (B3) since $h \wedge h^{\sim} = 1 \wedge 1^{\sim} = h \neq 0$. Also condition (wIR) is not satisfied since $h' < h^{\sim}$, and so owing to Lemma 73 also condition (IR) does not hold; but this can be directly proved since $h^{\sim\prime} < h^{\sim\sim}$.

Note that if one try to define as usual $a^* := a^{\sim\prime}$, then one obtains the following Hasse diagram which has nothing to do with any kind of closure operation:

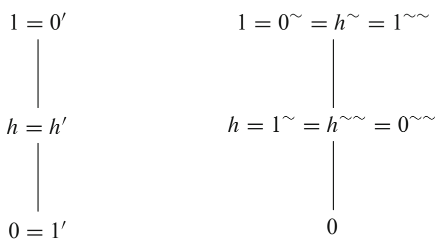
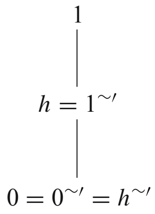


Fig. 16 A Kleene (and so a fortiori De Morgan) Brouwer lattice in which (wIR) condition is not satisfied



In particular $h^* = h^{\sim\prime} < h$ and $1^* = 1^{\sim\prime} < 1$.

The condition (B3) can be formulated in some equivalent statements according to the following results.

Lemma 78 *In any De Morgan lattice equipped with a generic unary operation $a \to a^*$, defined the unary operations $a \to a^{\sim} := a^{*\prime}$ and $a \to a^o := a^{\prime *\prime}$, the following are equivalent for arbitrary element a:*

$$(C4) \qquad a^* \vee a' = 1 \qquad \text{(weak De Morgan excluded middle by closure)}$$

$$(B3) \qquad a \wedge a^{\sim} = 0 \qquad \text{(Brouwer noncontradiction)}$$

$$(I4) \qquad a^o \wedge a' = 0 \qquad \text{(weak De Morgan noncontradiction by interior)}$$

Proof Indeed, $a' \vee a^* = 1$ iff $a \wedge a^{*\prime} = 0$. Applying this latter to a' we get $a' \wedge a^{\prime *\prime} = 0$ from which $a' \wedge a^o = 0$ follows. Conversely, $a' \wedge a^o = 0$ implies $a' \wedge a^{\prime *\prime} = 0$, from which by (dM1) and (dM2a) $a \vee a^{\prime *} = 1$, which applied to a' leads to $a' \vee a^* = 1$. □

Let us stress that the notion of Brouwer Zadeh (BZ) lattice introduced in the papers [22, 25] (with the application to axiomatic unsharp quantum mechanics in [23]) is based on a Kleene lattice (Kleene complementation satisfying conditions (dM1), (dM2), and (K)) equipped with a Brouwer complementation (satisfying conditions (B1), (B2), and (B3)), with these two complementations related by the interconnection rule (IR). Inside a BZ lattice the interconnection rule (IR) implies the weak interconnection rule (wIR).

In this chapter we adopt the weaker convention of defining as BZ lattice any structure $\langle \Sigma, \wedge, \vee, ', {}^{\sim}, 0, 1 \rangle$ of bounded lattice equipped with a De Morgan negation $'$ (without requiring the Kleene condition (K)) and a full Brouwer complementation $^{\sim}$, interconnected by the (IR) rule.

Condition (IR) is independent from all the other axioms defining a BZ lattice structure as the following example shows.

Example 79 Figure 17 shows the Hasse diagram of a Kleene lattice equipped with a Brouwer complementation. All the axioms (dM1), (dM2), (K), (B1), (B2), and (B3) are satisfied but $c^{\prime\sim} = 0 \neq 1 = c^{\sim\sim}$. On the other hand the (wIR) is satisfied since for all elements $x \neq 0$ it is $x^{\sim} = 0 \leq x'$, whereas $0^{\sim} = 0'$.

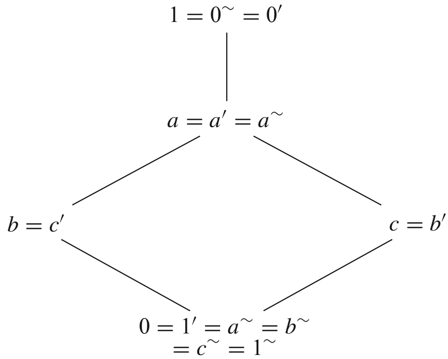


Fig. 17 Hasse diagram of a quasi BZ lattice

Part III: Semantics of Modal Logics

In the introduction to this chapter I discussed the possible "dictionary" relationship between the language of closure lattices of point (TL) and the language of modal logic of point (ML), with the analysis of this relationship as the unique interest of this Chapter. In this Part III we enter in some details about this topic.

Let us start from the De Morgan lattice structure $\langle \Sigma, \wedge, \vee, ', 0, 1 \rangle$ where elements of Σ may be thought of as realizing *propositions* in the *algebraic semantic model* for a set of formulas of some logic [100]—including 'truth' and 'falseness' (1 and 0)—where the algebraic operations of meet ($\wedge$) and join ($\vee$) describe the logical connectives of AND and OR. The conditions (dM1) and (dM2b) characterizing De Morgan lattices are the *almost* minimal in order to furnish the role of negation and so the De Morgan operation ($'$) describes the logical connective of a *generalized* NOT denoted as $\neg(a) = a'$.

Moreover, the unary operation of interior $^o : \Sigma \to \Sigma$ will be interpreted as the algebraic realization of a modal *necessity* connective, translated by the dictionary in the formal notation of $\nu(a) = a^o$. So Eq. (12) defining closure in terms of interior and complementation corresponds to the modal connective of *possibility* denoted as $\mu(a) = \neg\nu\neg(a) = a'^{o}{}'$; this according to the interpretation of "possible=NOT-necessarily-NOT" (condition Df$\Diamond$ in [32, p. 7], that is possibility is definable in terms of necessity and negation).

Ultimately, the structure $\langle \Sigma, \wedge, \vee, ', ^o, ^*, 0, 1 \rangle$, in the interpretation (TL) of lattice with respect to the meet $\wedge$ and join $\vee$ binary operations, unary operations of complementation $'$, interior ^{o} and closure *, is "translated" into the structure $\langle \Sigma, \wedge, \vee, \neg, \nu, \mu, 0, 1 \rangle$ of the interpretation (ML) of algebraic semantic of some modal logic system with respect to the binary connectives AND and OR, the unary connectives of negation $\neg$, necessity ν and possibility μ. The lattice bounds 0 and 1 corresponds to the "truth" and "falsehood" of the logical environment. More precisely, an *interior lattice* $\mathfrak{IL}(\Sigma) = \langle \Sigma, \wedge, \vee, ', ^o, 0, 1 \rangle$ can be considered as an *algebraic model* of a modal logic system $\mathfrak{AM}(\Sigma) = \langle \Sigma, \wedge, \vee, \neg, \nu, 0, 1 \rangle$ based on a De Morgan lattice (instead of a Boolean lattice [32, p. 212]), with the closure-possibility induced according to Eq. (12).

In order to have a better understanding of the algebraic realization of the modal system in terms of interior-necessity operation, let us consider a structure $\mathfrak{IM}(\Sigma) = \langle \Sigma, \wedge, \vee, \to, \neg, \nu, 0, 1\rangle$, which is an interior-necessity lattice $\mathfrak{AM}(\Sigma) = \langle \Sigma, \wedge, \vee, \neg, \nu, 0, 1\rangle$ equipped with a further binary connective of *implication* $\to$ assigning to any pair of lattice elements $a, b \in \Sigma$ another lattice element $a \to b \in \Sigma$ satisfying, according to [54, 55], one of the *minimal implicative conditions*, called *law of entailment*, relating this implication connective to the partial order relation of the lattice according to the assumption:

$$(E^*) \qquad a \to b = 1 \quad \text{iff} \quad a \le b \tag{32}$$

The partial order relation $\le$ algebraically describes a binary *implication relation* and the now introduced condition expresses the fact that "if a proposition a implies a proposition b, then the conditional proposition $a \to b$ is universally true, and conversely. [. . .] Here 1 is the lattice unit element, which corresponds to the *universally true proposition*" [55].

The validity of (E^*), denoting by $\nu(a) = a^o$ the modal necessity (interior) of a, leads to the following property whose proof strongly depends from the assumptions (I0) and (I1):

$$\nu(a \to b) = 1 \quad \text{iff} \quad a \le b \tag{33}$$

Indeed let $a \le b$. Then according to (32) $a \to b = 1$, from which we have $\nu(a \to b) = \nu(1) =$ (I0) $= 1$. Conversely, let $\nu(a \to b) = 1$. Then, $1 = \nu(a \to b) \le$ (I1) $\le a \to b \le 1$, i.e., $a \to b = 1$ which, according to (32), assumes the form $a \le b$.

10 A Brief Discussion About the Syntactical–Semantical Dichotomy of Formal Languages

Sometimes it happens that the algebraic realization of a logic by a suitable lattice structure is identified with the logic itself, without any distinction between the semantical aspect and the syntactical one.

For instance, in the Birkhoff–von Neumann seminal paper "The Logic of Quantum Mechanics" of 1936 [9] one can find the statement: "Our main conclusion, based on admittedly heuristic arguments, is that one can reasonably expect to find a calculus of propositions [of quantum mechanics] which is formally indistinguishable from the calculus of linear subspaces with respect to *set products*, *linear sum*, and *orthocomplement*–and resembles the usual calculus of propositions with respect to *and*, *or*, and *not*."

After a long silence about this argument, Mackey in a 1957 article [71] says: "All quantum mechanical systems have [...] the same logic. This logic **is** the partially ordered set of all closed subspaces of a separable infinite dimensional Hilbert space" [my bold]. The first passage from the concrete Hilbert space case to a lattice abstraction can be found in a 1962 Varadarajan paper: "A partially ordered orthocomplemented set is called a logic" [111], or in his successive book: "This leads one to introduce axiomatically a class of partially ordered sets and to study their properties. We call these systems *logics*. The basic assumption [is that this logic] is a lattice under a partial ordering" [112, Chapter VI, p. 105]. We also quote from a Bugajska and Bugajski paper: "we assume the [quantum] logic to be an orthomodular complete ortholattice [...]. Let $\mathscr{L}$ denote the logic of a given physical system. The fundamental properties of $\mathscr{L}$ are [given by the] Postulate 1.– $\mathscr{L}$ is an orthomodular σ-orthoposet [...] whose extended logic $\tilde{\mathscr{L}}$ is a complete, atomic ortholattice" [12].

We conclude this list of citations with a formally correct sentence from Goldblatt: "Some physicist maintain that from a quantum-theoretical standpoint, the propositions pertaining to a physical system exhibit a non-standard logical-structure, and indeed that their *associated algebra* is an orthomodular lattice, rather than a Boolean algebra, as in the case of classical systems. Consequently a new area of logical investigation has grown up under the name of *quantum logic*, of which one aspect is the study of propositional logic characterised by the class of orthomodular lattice" [49].

This quote implies the correct way of understanding a logic as subdivided into a *syntactic* and a semantic part.

Let us take a look at these two aspects briefly. First of all we have a *formal language* consisting in a set $\mathscr{F}$ of well formed formulae (wffs) constructed in the usual way on the basis of (1) a set of propositional variables, (2) a set of connectives, (3) parentheses (and). Quoting always from [49]:

(Sin) The *syntactical* approach examines formal relationships between wffs, and focuses on the notion of *consequence* or *derivability* of formulae.

(Sem) The *semantical* approach [...] has as its goal the assignment of meanings or interpretations to wffs, and the setting out of conditions under which a wff is to be true or false.

10.1 The Syntactical Approach to S4 Modal Logic

Let us start this subsection with a statement of Goldblatt from [49]: "Given a formal language, an axiom system S can be defined as an ordered pair $\langle \mathfrak{A}, \mathfrak{R} \rangle$ where $\mathfrak{A}$ is a set of wffs of the language, called *axioms*, and $\mathfrak{R}$ is a set of *rules of inference* that govern operations allowing certain formulae to be derived from others. A wff α is said to be a *theorem* of S [...] if there exists in S a proof of α i.e. a finite sequence of wffs whose last member is α, and such that each member of the sequence is either an axiom, or derivable from earlier members by one of the rules in $\mathfrak{R}$.

A *logic* [...] can be tough of as a set $\mathscr{L}$ of formulae closed under the application of certain inferential rules to its members. The members of $\mathscr{L}$ are called $\mathscr{L}$-theorems. [...]. For example, if $S = \langle \mathfrak{A}, \mathfrak{R} \rangle$ is an axiom system, then an S-logic can be defined as any set of wffs that includes the axiom set $\mathfrak{A}$ and is closed under the rules of $\mathfrak{R}$. In general the intersection $\mathscr{L}$s of all S-logics will be an S-logic, whose members are precisely those wffs for which there are proofs in S. This is often described by saying that S is an *axiomatization* of $\mathscr{L}$s, or that $\mathscr{L}$s is *generated* by S."

A complete discussion about this argument can be found in [100]. In this book the *formalized language* $\mathscr{L}$ of *modal logic* is defined as follows:

> the set of binary connectives contains the disjunction sign $\cup$, the conjunction sign $\cap$, and the implication sign $\Rightarrow$; the set of unary connectives contains the negation sign $-$ and another sign denoted by $\mathbf{I}$ and called the necessity sign.
>
> [...] if α is a formula in $\mathscr{L}$, then $\mathbf{I}\alpha$ is also a formula in $\mathscr{L}$. The formula $\mathbf{I}\alpha$ should be read *it is necessary that* α. Formalized languages $\mathscr{L}$ of the type just mentioned will be called *modal languages* of the zero order. [...] modal languages $\mathscr{L}$ constitute the domain of definition of a non-classical logic called the *modal logic* [100, p. 469].

Now we schematize the procedure completely treated in [100, section 7, chapter XI] to describe modal logic as a *formalized theory*. First of all,
(S) We fix a set $\mathscr{A}$ of formulas [i.e., $\mathscr{A} \subseteq \mathscr{L}$] (called the *logical axioms* of a modal logic) which can be divided in two parts:

(ML1) the set of axioms whose induced Lindenbaum–Tarski algebra will be the one of complemented lattice (whose list is of no interest in this general exposition); and

(ML2) the following set of axioms characterizing modal logic, where α, β are formulas in $\mathscr{L}$:

(M1) $((\mathbf{I}\alpha \cap \mathbf{I}\beta) \Rightarrow \mathbf{I}(\alpha \cap \beta)$,

(M2) $(\mathbf{I}\alpha \Rightarrow \alpha)$,

(M3) $(\mathbf{I}\alpha \Rightarrow \mathbf{II}\alpha)$,

(M4) $\mathbf{I}(\alpha \cup -\alpha)$.

The admitted *rules of inference* are the following two [100, p. 470]:

$$(R_{mp}) \qquad \frac{\alpha,\ \alpha \Rightarrow \beta}{\beta} \qquad \text{(modus ponens)}$$

$$(R_{\mu}) \qquad \frac{(\alpha \Rightarrow \beta)}{(\mathbf{I}\alpha \Rightarrow \mathbf{I}\beta)}$$

Then, putting together two sentences from [100, p. 183] and [100, p. 470]: "For every set S of formulas in $\mathscr{L}$ we define $C(S)$ as the set of all formulas (in $\mathscr{L}$) which are derivable from S by the rules of inference R_{mp}, R_{μ} and [axioms] in $\mathscr{A}$. The consequence operation C defined by (S) in $\mathscr{L}$ is said to be *determined by the rules of inference* R_{mp}, R_{μ} *and the set* $\mathscr{A}$ *of logical axioms*. [...] By definition, the modal logic [...] assigns, to every modal language $\mathscr{L}$ of zero order [...] a consequence operation C in $\mathscr{L}$ [i.e., $C : \mathscr{L} \to \mathscr{L}$] defined as follows: for any set S of formulas in that language [$\mathscr{L}$], the set $C(S)$ [of all consequences] is the least set containing all formulas in S and all the logical axioms in $\mathscr{A}$ just mentioned, and closed under the rules of inference (R_{mp}) and (R_μ)."

Summarizing,

> The deductive system $\mathfrak{S} = \langle \mathscr{L}, C \rangle$ will be called [...] the *modal propositional calculus* based on $\mathscr{L}$. Besides the modal propositional [... calculus] we shall also examine *modal theories of the zero order* $\mathfrak{T} = \langle \mathscr{L}, C, \mathscr{A} \rangle$. According to the general definition [...] formulas in the set $C(\mathscr{A})$ are called *theorems* of the theory [100, p. 471].

In this way, "the set of all formulas $\mathscr{F}$ in $\mathscr{L}$ [...] should now be understood as an algebra

$$\langle \mathscr{F}, \cap, \cup, \Rightarrow, -, \mathbf{I} \rangle \tag{34}$$

with three binary operations $\cup$, $\cap$, $\Rightarrow$ and two unary operations $-$, $\mathbf{I}$ [100, p. 471]."

> Despite the fact that some of the operations on $\mathscr{F}$ are denoted by the same symbols $\cap$, $\cup$ used, in general, for lattice operations, the algebra $\mathscr{F}$ is not a lattice with respect to $\cup$ and $\cap$. For instant, if α and β are two distinct formulas, then $(\alpha \cup \beta)$ and $(\beta \cup \alpha)$ are two distinct formulas in $\mathscr{F}$. This implies that the operation $\cup$ does not satisfy the commutative law. Similarly, the operation $\cap$ does not satisfy the commutative law in $\mathscr{F}$. Also the associative laws [...] do not hold in $\mathscr{F}$ [100, p. 209].

10.2 From Syntactic to Semantic: The Lindenbaum–Tarski Algebra Induced by S4 Modal Logic

> In particular, if $\mathfrak{T}$ is a modal theory of the zero [...] order, then the relation $\approx$, defined in $\mathscr{F}$ as follows:
>
> $$\alpha \approx \beta \quad \text{iff} \quad (\alpha \Rightarrow \beta) \text{ and } (\beta \Rightarrow \alpha) \text{ are theorems in } \mathfrak{T}, \tag{35}$$
>
> is a congruence with respect to the operations $\cup, \cap, \Rightarrow, -$. If $\alpha \approx \beta$, i.e., $(\alpha \Rightarrow \beta)$ and $(\beta \Rightarrow \alpha)$ are theorems in $\mathfrak{T}$, then also formulas $(\mathbf{I}\alpha \Rightarrow \mathbf{I}\beta)$ and $(\mathbf{I}\beta \Rightarrow \mathbf{I}\alpha)$ are theorems in $\mathfrak{T}$ by the rule of inference (R_μ), i.e., $\mathbf{I}\alpha \approx \mathbf{I}\beta$. This proves that the equivalence relation $\approx$ is also a congruence with respect to the operation $\mathbf{I}$, i.e., it is a congruence in the algebra (34). [...] the set $\mathfrak{U}(\mathfrak{T}) = \mathscr{F}/\approx$ can be conceived as an algebra
>
> $$\langle \mathfrak{U}(\mathfrak{T}), \wedge, \vee, \to, \neg, I \rangle \tag{36}$$
>
> called the *algebra of the theory* $\mathfrak{T}$ [...], which is a *topological Boolean algebra* [100, p. 471–72].

Without going into excessive technical details, for the knowledge of which we refer to the Rasiowa Sikorski book, we summarize what is of most interest to us. Mainly that, once denoted by $\|\alpha\|$ the equivalence class containing α, the algebra $\mathfrak{U}(\mathfrak{T})$ is a Boolean algebra (*distributive* ortholattice) with respect to the relationships defined for any formulas α, β

$$\|\alpha\| \wedge \|\beta\| = \|\alpha \cap \beta\|\,, \quad \|\alpha\| \vee \|\beta\| = \|\alpha \cup \beta\| \tag{37a}$$

$$\|\alpha\| \to \|\beta\| = \|\alpha \Rightarrow \beta\|\,, \quad \neg\,\|\alpha\| = \|-\alpha\| \tag{37b}$$

$$1 = \|\alpha \cup -\alpha\|\,. \tag{37c}$$

This Boolean algebra is further equipped with a Kuratowski interior operation I defined by the equality $I\,\|\alpha\| := \|\mathbf{I}\alpha\|$, which satisfies all the properties (I0), (I1), (I2K), and (I3) of a Kuratowski interior operation on a lattice. This algebraic structure of Kuratowski interior distributive lattice is called the *Lindenbaum-Tarski algebra* generated by the S4 modal theory.

The mapping from the generic algebra $\langle \mathscr{F}, \cap, \cup, \Rightarrow, -, \mathbf{I}\rangle$ to the topological Boolean algebra

$$\langle \mathfrak{U}(\mathfrak{T}), \wedge, \vee, \to, \neg, I\rangle$$

$$v_c : \mathscr{F} \to \mathfrak{U}(\mathfrak{T}),\ a \to v_c(a) := \|a\|$$

taking into account Eq. (37), is an algebraic homomorphism since

$$v_c(\alpha \cap \beta) = \|\alpha \cap \beta\| = \|\alpha\| \wedge \|\beta\| = v_c(\alpha) \wedge v_c(\beta)$$
$$v_c(\alpha \cup \beta) = \|\alpha \cup \beta\| = \|\alpha\| \vee \|\beta\| = v_c(\alpha) \vee v_c(\beta)$$
$$v_c(\alpha \Rightarrow \beta) = \|\alpha \Rightarrow \beta\| = \|\alpha\| \to \|\beta\| = v_c(\alpha) \to v_c(\beta)$$
$$v_c(-\alpha) = \|-\alpha\| = \neg\,\|\alpha\| = \neg v_c(\alpha)$$
$$v_c(\mathbf{I}\alpha) = \|\mathbf{I}\alpha\| = I\,\|\alpha\| = I v_c(\alpha).$$

For these algebraic properties of homomorphism the map v_c is called the *canonical valuation* [100, p. 474].

Let $\langle \mathscr{A}, \wedge, \vee, \to, \neg, I\rangle$ be an abstract topological Boolean algebra, then a *valuation* is any homomorphism $v : \mathscr{F} \to \mathscr{A}$ from the algebra of formulas $\mathscr{F}$ in $\mathscr{A}$. "A valuation is said to be a *model* for a theory $\mathfrak{T} = \langle \mathscr{L}, C, \mathscr{A}\rangle$ provided v is a model for the set $\mathscr{A}$ of axioms of $\mathfrak{T}$" [100, p. 474]. Note that at the same page of [100] one can find the following results:

- If $v(\alpha \Rightarrow \beta) = 1$ (i.e., $v(\alpha) \to v(\beta)$=1), then $v(\alpha) \leq v(\beta)$ (which is the law of entailment (E*)). Hence,
- $I(v(\alpha)) \leq I(v(\beta))$, i.e., $v(\mathbf{I}\alpha \Rightarrow \mathbf{I}\beta) = 1$ (i.e., $Iv(\alpha) \to Iv(\beta) = 1$)

This is what is needed for the continuation of the chapter, and so we stop here. Obviously, since everything has been done in the context of modal logic S4, if we want to take into account other modal logics such as S5 or non-distributive situations, it will be enough to change some of the axioms appropriately.

11 Tarski Interior and Closure Operations as Algebraic Models of Logical Necessity and Possibility Modal Connectives

After this discussion about the S4 modal logic based on a formalized language of the zero order, in this section we treat the pure semantical aspect linked to the Tarski interior-necessity operation, induced from a Tarski closure-possibility operation as discussed in Sect. 7. Using the law of entailment (E^*), the properties defining the Tarski interior-necessity operation $\nu : \Sigma \to \Sigma$ (see Theorem 29 plus condition (I0) of Sect. 7) can be rewritten in the following way:

(I0-L)	$\nu(1) = 1$	(N condition)
(I1-L)	$\nu(a) \to a = 1$	(decreasing)
(I2-L)	$\nu(a \wedge b) \to (\nu(a) \vee \nu(b)) = 1$	(sub-multiplicative)
(I3-L)	$\nu(a) \to \nu(\nu(a)) = 1$	(idempotent)

This reformulation of the properties characterizing Tarski interior operation as modal necessity can be interpreted in the following way (see also [21]).

(MT0) Condition (I0-L), $\nu(1) = 1$, describes the N modal principle according to [32, p. 20]. But in the form "If $a = 1$, then $\nu(a) = 1$" it is the version of the *modal rule of inference* RN which "means $\vdash \nu(a)$ whenever $\vdash a$" [32, pp. 14, 15], [46, p. 136].

(MT1) $\nu(a) \to a$ is universally true, algebraic version of the T modal principle "*if necessary a, then a*" [32, p. 6], [46, p. 136], and [100, p. 470 as (M_2)].

(MT2) Condition (I2-L) is the algebraic version of the modal M principle [32, p. 20]. Moreover, the necessity isotonicity condition (I2a), equivalent to (I2), assumes the form "$a \to b = 1$ implies $\nu(a) \to \nu(b) = 1$", algebraic version of the *modal rule of inference* RM [32, p. 17], in [100, p. 470] written as $\frac{(a \to b)}{(\nu(a) \to \nu(b))}$.

(MT3) Condition (I3-L) is the algebraic version of modal principle S4 "whatever is necessary is necessary necessary" [32, p. 18], [46, p. 137], and [100, p. 470 as (M_3)].

Moreover, from the interpretation of Tarski closure as modal possibility, $\mu(a) = a^*$, the following can be stated:

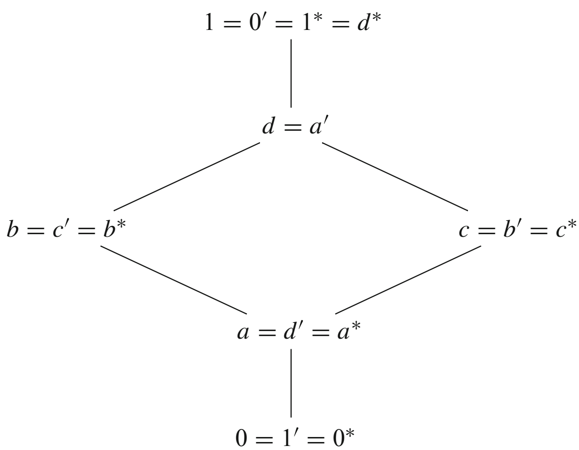
(MT4) Condition (Up1) of Theorem 26 (or equivalently condition (C1) of closure) together with condition (Lo1) of Theorem 35 (or equivalently condition (I1) of interior) lead to the inequality $\nu(a) \leq \mu(a)$ expressed by (E^*) as $\nu(a) \to \mu(a) = 1$, i.e., algebraic version of the modal principle D "whatever is necessary is possible" [32, p. 10].

(MT5) Equation (12), written in modal notation as $\mu(a) = \neg(\nu(\neg(a)))$, corresponds to the modal schema (Df□) which "embodies the idea that what is possible is just what is not–necessary–not" [32, p. 7], [46, p. 134].

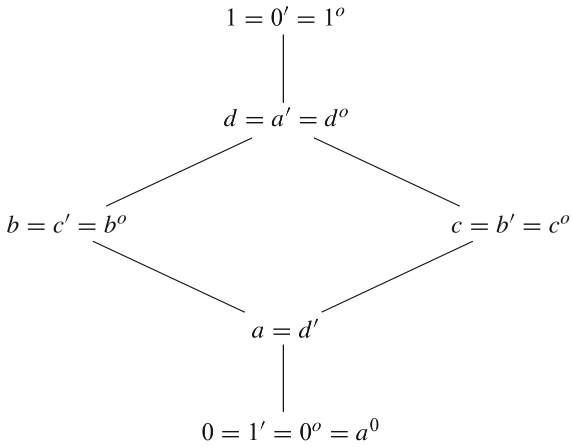
We can collect all these considerations in the following statements, which summarize the main properties of the algebraic model of the modal system induced from the Tarski interior operation.

- We have considered a not necessarily distributive De Morgan lattice as the most general algebraic environment in which to develop the theory. This obviously does not exclude the possibility of considering as starting point of the algebraic semantic of modal logic a Boolean lattice (classical modal logic) [32, p. 212] or an orthomodular lattice (quantum modal logic).
- The Tarski interior operation, interpreted as modal necessity, satisfies the properties of a *S4-like modal necessity* operation, in the sense that the algebraic versions of modal principles N, T, M, D (with the modal schema Df□), and the peculiar modal principle 4, hold.
- Furthermore, particular versions of (I0-L) and (I2-L) realize the modal inference rules RN and RM, respectively.
- The term "S4-like" rises from the fact that the inequality "$\forall a, b \in \Sigma,\ \nu(a) \wedge \nu(b) \leq \nu(a \wedge b)$" (dual of the condition (I2), "$\forall a, b \in \Sigma,\ \nu(a \wedge b) \leq \nu(a) \wedge \nu(b)$"), expressed according to (E^*) as the tautology $(\nu(a) \wedge \nu(b)) \to \nu(a \wedge b) = 1$, algebraic version of the *modal C principle* [32, p. 20], in [100, p. 470 denoted as (M_1)], in general does not hold, as the counter-Example 80 shows.
  This modal C principle characterizes the algebraic approach to modal logic induced from Kuratowski interior operation interpreted as modal necessity, as we will see in Sect. 12.

Example 80 Let us consider the following Hasse diagram depicting a Tarski closure lattice.



The Hasse diagram depicting the corresponding Tarski interior lattice is the following one:



In this last $\exists b, c \in \Sigma$ s.t. $(b \wedge c)^o = a^0 = 0 < a = b^o \wedge c^o$, i.e., $\exists b, c \in \Sigma$ s.t. $\nu(b \wedge c) < \nu(b) \wedge \nu(c)$, which is a negation of modal C principle.

We have the following result which turns out to be very useful in the sequel.

Proposition 81 *The universally true proposition*

$$(K) \qquad \nu(a \to b) \to (\nu(a) \to \nu(b)) = 1, \tag{38}$$

is the algebraic formulation of modal principle K expressing, according to [32, p. 7], [46, p. 136]. The distributivity of necessity with respect to the conditional *implies the isotonicity condition of necessity "$a \leq b$ implies $\nu(a) \leq \nu(b)$", equivalent to the universally true proposition:*

$$(M) \qquad \nu(a \wedge b) \to (\nu(a) \wedge \nu(b)) = 1 \tag{39}$$

that corresponds to the algebraic version of modal principle M (see the above point (MT2)).

Proof Let $a \leq b$, then in particular, according to (33), one has that $\nu(a \to b) = 1$. But according to (32) the hypothesis (K), $\nu(a \to b) \to (\nu(a) \to \nu(b)) = 1$, can be formulated as $\nu(a \to b) \leq (\nu(a) \to \nu(b))$ and so from the previous result it follows that $1 = \nu(a \to b) \leq (\nu(a) \to \nu(b)) \leq 1$, i.e., $\nu(a) \to \nu(b) = 1$ and so, according to (32), $\nu(a) \leq \nu(b)$.
On the other hand, we have seen that isotonicity of necessity is equivalent to the sub-multiplicative (I2), $\nu(a \wedge b) \leq \nu(a) \wedge \nu(b)$, which according to (32), can be expressed as the tautology $\nu(a \wedge b) \to \nu(a) \wedge \nu(b) = 1$. □

As second result we have.

Proposition 82 *The isotonicity condition of necessity "$a \leq b$ implies $\nu(a) \leq \nu(b)$" implies*

$$[\nu(a \to b) = 1 \ \ \textit{and} \ \ \nu(a) = 1] \quad \textit{implies} \quad \nu(b) = 1$$

Proof From (33), the condition $[\nu(a \to b) = 1$ and $\nu(a) = 1]$ can be written as $[a \leq b$ and $\nu(a) = 1]$, which, by the isotonicity of necessity, leads to $1 = \nu(a) \leq \nu(b) \leq 1$, i.e., $\nu(b) = 1$. □

Thus, we can further on add the following comments to the previous summary about the S4-like modal system.

- According to Proposition 81 in the algebraic model based on a Tarski interior operation the modal principle M is a necessary condition for the validity of the K principle, but the unique minimal condition (E*) about the implication connective is not sufficient to assure the validity of (K), differently from the standard S4 Lewis modal system which requires that this principle holds (see [32, p. 131]).
- In order to have a satisfaction of this modal K principle it is necessary to consider the stronger notion of Kuratowski interior with an implication connective which satisfies two further conditions besides (E*), as we discuss in Sect. 12, obtaining in this case a real (also if non Boolean) S4 Lewis modal system.

12 Algebraic Model of Modal Logic Induced by Kuratowski Closure–Interior Operations

Since Kuratowski interior operation is a fortiori Tarski interior operation, the considerations made in Sect. 11 on the Tarski lattice structure as algebraic model of the S4-like modal algebra, characterized by the validity of modal principles N, T, M, and 4, can be applied also to the Kuratowski case.

In this algebraic approach to modal logic, denoting as usual the necessity operator as interior, $\nu(a) = a^o$, the translate conditions (I0-L), (I1-L), and (I3-L) continue to be true whereas the characterizing Kuratowski multiplicative condition (I2K) can be split into the algebraic formulation of two modal principles:

$$\text{(I2M)} \quad \nu(a \wedge b) \leq \nu(a) \wedge \nu(b) \qquad \text{(M modal principle)}$$

$$\text{(I2C)} \quad \nu(a) \wedge \nu(b) \leq \nu(a \wedge b) \qquad \text{(C modal principle)}$$

or, using (E*), to the two modal formulations which can be found in [32, p. 20] (whereas the second one is the axiom (M_1) of the modal logic described in [100, p. 470]):

$$\nu(a \wedge b) \to (\nu(a) \wedge \nu(b)) = 1 \qquad \text{(M modal principle)}$$

$$(\nu(a) \wedge \nu(b)) \to \nu(a \wedge b) = 1 \qquad \text{(C modal principle)}$$

Also in the Kuratowski case, from the general condition $\forall a \in \Sigma$, $\nu(a) \leq a$ it is possible to single out the collection of all *open* elements $o \in \Sigma$ s.t. $\nu(o) = o$, denoted as $\mathscr{O}(\Sigma)$.

Let us recall that Proposition 81 proves that

$$\text{K modal principle} \quad \Rightarrow \quad \text{M modal principle}$$

In order to get some other result of this kind one must consider the Kuratowski context with an implication connective satisfying some further conditions, according to the following result.

Proposition 83 *Let us consider a Kuratowski interior operator on the lattice Σ. If the implication connective $\to$, besides condition (E*) satisfies the two further properties:*

(1) the modus ponens *propositional tautology [100, p. 163] (and see also [54]):*

$$(MP) \quad \forall a, b \in \Sigma,\ (a \wedge (a \to b)) \to b = 1$$

(2) the strict pseudo-complement *condition [54]:*

$$(sPC) \quad \forall a, b \in \Sigma,\ \forall o \in \mathscr{O}(\Sigma),\quad a \wedge o \leq b \quad \textit{implies} \quad o \leq a \to b\,,$$

then the C modal principle implies the K modal principle.

Proof The modus ponens condition (MP), using (E*), can be formulated as $a \wedge (a \to b) \leq b$, from which by the necessity isotonicity (I2a) we get $\nu(a \wedge (a \to b)) \leq \nu(b)$, and from the modal C principle (I2C) it follows that $\nu(a) \wedge \nu(a \to b) \leq \nu(b)$. Since according to the idempotent condition (I3) the element $\nu(a \to b)$ is open, applying to the just proved inequality the strict pseudo-complement condition (sPC) we obtain $\nu(a \to b) \leq \nu(a) \to \nu(b)$, i.e., the K modal principle $\nu(a \to b) \to (\nu(a) \to \nu(b)) = 1$. □

Thus, we have shown that in a structure $\langle \Sigma, \wedge, \vee, \to, \neg(= {}'), \nu(= {}^o), 0, 1 \rangle$ consisting of a Kuratowski necessity-interior lattice equipped with an implication binary connective $\to$ satisfying conditions (E*), (MP), and (sPC), one has that

$$\text{C modal principle} \Rightarrow \text{K modal principle} \Rightarrow \text{M modal principle}$$

Note that from modus ponens condition ($a \wedge (a \to b) \leq b$) and the strict pseudo-complement condition (sPC) in the case of a Kuratowski necessity-interior *complete* lattice the following holds (see (S4) from [54]):

$$a \to b = \sup\{o \in \mathscr{O}(\Sigma) : a \wedge o \leq b\} \tag{40}$$

The following is an interesting result which will be applied in the sequel.

Proposition 84 *Let Σ be a Kuratowski necessity-interior lattice and let $\to_m$ be an implication connective such that $\forall a, b \in \Sigma$, $a \to_m b \in \mathscr{O}(\Sigma)$. Then, the condition*

$$\text{(sPCL)} \quad \forall a, b \in \Sigma, \ \forall o \in \mathscr{O}(\Sigma), \quad a \wedge o \leq b \quad \textit{iff} \quad o \leq a \to_m b$$

is equivalent to the fulfillment of the two conditions

(sPC) $\quad \forall a, b \in \Sigma$, $\forall o \in \mathscr{O}(\Sigma)$, $a \wedge o \leq b$ *implies* $o \leq a \to_m b$,
(MP) $\quad \forall a, b \in \Sigma$, $a \wedge (a \to_m b) \leq b$.

Moreover, condition (sPCL) *implies the law of entailment* (E^*).
In the complete *lattice case Eq.* (40) *assumes the form*

$$a \to_m b = \max\{o \in \mathscr{O}(\Sigma) : a \wedge o \leq b\} \tag{41}$$

Proof Let (sPCL) be true, then condition (sPC) is trivially verified. On the other hand putting $o = a \to_m b \in \mathscr{O}(\Sigma)$, from $o = a \to_m b \leq a \to_m b$ and the implication $\Leftarrow$ of (sPCL) it follows that $a \wedge (a \to_m b) \leq b$, i.e., condition (MP).

Conversely, let (sPC) and (MP) be true. If $o \leq a \to_m b$ is true, then $a \wedge o \leq a \wedge (a \to_m b) \leq \text{(MP)} \leq b$. That is, we have proved that $o \leq a \to_m b$ implies $a \wedge o \leq b$. This result together with (sPC) lead to (sPCL).

The (sPCL) condition under the choice $o = 1 \in \mathscr{O}(\Sigma)$ leads to E^*. □

Corollary 85 *Let $\to_m$ be an implication connective such that $\forall a, b \in \Sigma$, $a \to_m b \in \mathscr{O}(\Sigma)$. Then,* (sPCL) *implies* (sPC), (MP)*, and* E^*.

Furthermore, the only condition (sPCL) *implies the modal principle* C*, which in turn implies the modal principles* K *and* M.

The above notion (sPC) of strict pseudo-complementation takes the term "strict" from the name assigned by Hardegree to the open element of a Kuratowski lattice: "a lattice element o is strict (open) just in case $o = \nu(o)$" [54]. For this reason it is said that condition (sPC) pertains to S4 strict conditional since ν is an S4 necessity operator (interior operator) and this condition holds for all "strict" elements $o = \nu(o)$.

The reason of this weakening of the usual notion of pseudo-complementation (see [100, p. 123]), in the present context denoted by $\to_c$ and formally defined by

$$\text{(PC)} \qquad \forall a, b, x \in \Sigma, \quad a \wedge x \leq b \quad \text{implies} \quad x \leq a \to_c b,$$

rises from the fact that under the *classical implicative conditions* (MP) and (PC) the (not necessarily Kuratowski) lattice Σ is automatically distributive (see [54]), against our point of view of describing also some non-distributive situation.

Let us recall that if Σ is a *complete* lattice, then quoting [54]: "the conditional operation [i.e., $\to_c$] is uniquely specified by conditions (MP) and (PC) so that

$$\text{(IC)} \quad a \to_c b = \sup\{x \in \Sigma : a \wedge x \leq b\}."$$

With some slight changes to the proof of Proposition 84, the following result can be proved.

Proposition 86 *Let Σ be a Kuratowski necessity-interior lattice and let $\to_c$ be a generic implication connective. Then, the condition*

$$\text{(PCL)} \quad \forall a, b, x \in \Sigma, \quad a \wedge x \leq b \quad \textit{iff} \quad x \leq a \to_c b$$

is equivalent to the fulfillment of the two conditions

(PC) $\forall a, b, x \in \Sigma$, $a \wedge c \leq b$ *implies* $c \leq a \to_c b$,
(MP) $\forall a, b \in \Sigma$, $a \wedge (a \to_c b) \leq b$.

Moreover, condition (PCL) *implies the law of entailment* (E*).
In the complete *lattice case the (IC) assumes the form*

$$\text{(ICL)} \qquad a \to_c b = \max\{x \in \Sigma : a \wedge x \leq b\}$$

The following result will be useful in the proof of the next proposition.

Lemma 87 *Let $\nu : \Sigma \to \Sigma$ be a Kuratowski necessity operation with corresponding collection of open elements $\mathcal{O}(\Sigma)$. Then*

$$\forall a \in \Sigma, \forall o \in \mathcal{O}(\Sigma), \quad o \leq a \quad \textit{iff} \quad o \leq \nu(a) \tag{42}$$

Proof Let $o \leq a$, then by isotonicity $\nu(o) \leq \nu(a)$, but from the hypothesis that o is open $o = \nu(o) \leq \nu(a)$.

Conversely, let $o \leq \nu(a)$, then for decreasing it follows that $o \leq \nu(a) \leq a$. □

Proposition 88 *Let us consider a Kuratowski necessity-interior lattice equipped with an implication connective $\to$ satisfying conditions (E*), (MP), and (sPCL). If one introduces the unary operator $\sim : \Sigma \to \Sigma$ associating with any element $a \in \Sigma$ the open element in $\mathcal{O}(\Sigma)$ defined as*

$$\sim a := \nu(a \to 0) \tag{43}$$

then the following hold for arbitrary $a, b \in \Sigma$:

(B2) $a \leq b$ *implies* $\sim b \leq \sim a$ (contraposition law),
(B3) $a \wedge \sim a = 0$ (noncontradiction law).

In the particular case of an implication connective $\to_m$ such that $\forall a, b \in \Sigma$ it is $a \to_m b \in \mathcal{O}(\Sigma)$ the unique condition (sPCL) *implies* (B2) *and* (B3).

Proof (For the proof we adapt to the present situation, with some light modifications, the proof of [100, p. 62].)

(B3) Condition (sPCL) under the particular case of $b = 0$ assumes the form: $\forall a \in \Sigma$, $\forall o \in \mathcal{O}(\Sigma)$, $o \leq a \to 0$ iff $a \wedge o = 0$. From the fact that $\nu(a \to 0)$ is open and that $\sim a = \nu(a \to 0) \leq a \to 0$ it follows that $a \wedge \sim a = 0$.

(B2) Let $a \leq b$; then from the just proved (B1) $b \wedge \sim b = 0$ we get that a fortiori $a \wedge \sim b = 0$, with $\sim b$ open, and so from (sPC), $\sim b \leq a \to 0$, and from Lemma 87 we obtain $\sim b \leq \nu(a \to 0) = \sim a$. □

Proposition 89 *Under the condition of Proposition 88, the restriction of Eq. (43) to the lattice $\mathcal{O}(\Sigma)$ of open elements generates a mapping $\sim: \mathcal{O}(\Sigma) \to \mathcal{O}(\Sigma)$ satisfying for arbitrary $o, h \in \mathcal{O}(\Sigma)$ the following properties:*

(B1) $o \leq \sim\sim o$ (weak double negation law),
(B2) $o \leq h$ *implies* $\sim h \leq \sim o$ (contraposition law),
(B3) $o \wedge \sim o = 0$ (noncontradiction law).

Proof In Proposition 43 we have stressed that $\sim a$, as interior of an element from Σ, is open and so the restriction to $\mathcal{O}(\Sigma)$ of the mapping $\sim$ produces a correspondence from $\mathcal{O}(\Sigma)$ into $\mathcal{O}(\Sigma)$.

Since (B2) and (B3) are true for any pair $a, b \in \Sigma$, then they are also true for arbitrary pairs $o, h \in \mathcal{O}(\Sigma) \subseteq \Sigma$. Let us consider (B1); from (B3) we have that for any $o \in \mathcal{O}(\Sigma)$ it is $\sim o \wedge o = 0$, with o open, and so by (sPC) this implies $o \leq \sim o \to 0$, which by isotonicity implies $o = \nu(o) \leq \nu(\sim o \to 0) = \sim (\sim o)$, i.e., the (BO1). □

That is the unary operation $\sim: \mathcal{O}(\Sigma) \to \mathcal{O}(\Sigma)$ satisfies the properties of an *intuitionistic* (or *Brouwer (B)*) *negation* as discussed by Heyting in [58], where, after claimed that "the main differences between classical and intuitionistic logics are in the properties of the negation", the intuitionistic negation is characterized by the properties whose algebraic versions are just conditions (B1), (B2), and (B3) (see [17], also for a deep discussion about the algebraic model of this kind of negation). Furthermore in [58] it is explicitly asserted that "in the theory of negation the principle of excluded middle $o \vee \sim o = 1$ fails", and that "we have meet many examples for which [$\sim\sim o \leq o$] fails", and that "$\sim (a \wedge b) = \sim a \vee \sim b$ cannot be asserted".

Let us note that under condition (B1), the contraposition law (B2) is equivalent to the so-called *intuitionistic De Morgan law* "$\sim (o \vee h) = \sim o \wedge \sim h$."

12.1 The Classical Case of Necessity–Interior Kuratowski Operation on Boolean Algebras

We discuss now the case in which the lattice Σ is distributive and the De Morgan negation $\neg = {}'$ on Σ satisfies the *complementation conditions* $\forall a \in \Sigma$, $a \wedge \neg a = 0$ (*noncontradiction law*) and $a \vee \neg a = 1$ (*excluded middle law*), i.e., $\neg$ is an *orthocomplementation* and thus Σ is a Boolean algebra.

Proposition 90 *If Σ is a Boolean algebra the classical implication connective*

$$a \to_c b = \neg a \vee b \tag{44}$$

satisfies

(a) the pseudo-complement condition (PCL):
$\forall a, b, x \in \Sigma$, $a \wedge x \leq b$ *iff* $x \leq a \to_c b$ *[100, p. 56],*
(b) the law of entailment (E^*): $a \to_c b = 1$ iff $a \leq b$.
(c) the modus ponens *condition (MP): $\forall a, b \in \Sigma$, $a \vee (a \to_c b) \leq b$.*
According to (E^*) condition (MP) can be written as $(a \vee (a \to_c b)) \to_c b = 1$. This universally valid sentence usually is also written as

$$\frac{a,\ a \to_c b}{b}$$

Proof Let us prove (PCL), then by Proposition 86 both conditions (MP) and (E^*) are true.
Let $a \wedge x \leq b$ for arbitrary $a, b, x \in \Sigma$. Then $x =$ (excluded middle) $= x \wedge (\neg a \vee a) =$ (distributivity) $= (x \wedge \neg a) \vee (x \wedge a) \leq \neg a \vee b$, i.e., $x \leq (a \to_c b)$. Conversely, if $x \leq \neg a \vee b$ then $a \wedge x \leq a \wedge (\neg a \vee b) =$ (distributivity) $= (a \wedge \neg a) \vee (a \wedge b) =$ (noncontradiction) $= a \wedge b \leq b$. □

Proposition 91 *Under the condition that $\langle \Sigma, \wedge, \vee, \neg, \nu, 0, 1 \rangle$ is a Boolean algebra equipped with a Kuratowski necessity-interior operator $\nu = {}^o$ one has that the so-called* necessity implication connective

$$a \to_m b := \nu(\neg a \vee b) = \nu(a \to_c b) \tag{45}$$

satisfies (sPCL), (MP), and (E^*) conditions.

Proof (sPCL) Let $a \wedge o \leq b$ for $a, b \in \Sigma$ and $o \in \mathcal{O}(\Sigma)$. In Proposition 90 we have shown that for any triple $a, b, x \in \Sigma$ condition (PCL) holds: $a \wedge x \leq b$ iff $x \leq (a \to_c b)$, which in the case of $x = o \in \mathcal{O}(\Sigma)$ leads to $a \wedge o \leq b$ iff $o \leq (a \to_c b)$, and applying Lemma 87 to this result we obtain $a \wedge o \leq b$ iff $o \leq \nu(a \to_c b)$.

Conditions (MP) and (E^*) are true owing to Corollary 85. □

Corollary 92 *Let $\left\langle \Sigma, \wedge, \vee, ', \nu, 0, 1 \right\rangle$ be a Boolean algebra with Kuratowski interior-necessity operator ν. According to Proposition 91 if we introduce the implication binary connective (45), $a \to_m b := \nu(\neg a \vee b) = \nu(a \to_c b)$, satisfying conditions (E^*), (MP), and (sPC), then we have a "spurious" quasi Brouwer–Zadeh (BZ) situation in the sense that:*

(BZ1) the mapping $\neg : \mathcal{O}(\Sigma) \to \mathcal{C}(\Sigma)$ is the algebraic version of a De Morgan negation transforming open elements in closed ones,

(BZ2) *the mapping* $\sim: \mathcal{O}(\Sigma) \to \mathcal{O}(\Sigma)$, *defined for any open element* $o \in \mathcal{O}(\Sigma)$ *according to*

$$\sim o = o \to_m 0 = \nu(\neg(o))$$

is the algebraic version of a Brouwer (intuitionistic) negation transforming open elements in open elements,

(BZ3) *the two negations are linked by the further condition:*

$$\textit{(wIC)} \quad \sim o \leq \neg o \quad \textit{(weak interconnection law)}$$

Proof Let us note that $\sim o = \nu(\neg(o)) = o'^{o} = o''^{*\prime} = o^{*\prime} = \neg(\mu(o))$ i.e., this Brouwer (intuitionistic) negation is the *impossibility* = *Not* possibility. On the other hand, $\sim o = \nu(\neg(o)) \leq \text{(I1)} \leq \neg o$. □

Summarizing the results about the abstract Boolean algebra case,

- if the structure $\langle \Sigma, \wedge, \vee, \neg, 0, 1\rangle$ is a *Boolean algebra*, interpreting the Kuratowski interior operator on Σ as algebraic version of the modal necessity operator, $\nu(a) = a^o$, and introducing the necessity implication operator $a \to_m b = \nu(\neg a \vee b)$, one has that according to Proposition 91 the conditions (E*), (MP), and (sPC), are true.
- Then, we have an algebraic model of a modal system which satisfies the principles N, T, C (from which it follows K and M), and 4, i.e., we have the Lewis system S4 which in [32, p. 131] is denoted as KT4.
- In relation to this topic, it is interesting to quote the following Monteiro statement from [78]: "The propositional modal calculus S4 has been considered for the first time by Lewis and Langford [69]. A first geometric interpretation of this calculus was indicated by Tang [106]. McKinsey [72], under the influence of the ideas of Tarski, has highlighted the algebraic characteristics of this calculation [...]. We will call *Lewis algebra* a couple formed by a Boolean algebra and a [closure operation C (in our notation μ)]. This notion has been introduced by Terasaka [110], and successively by McKinsey and Tarski [73], as a generalization of the notion of topological space, which can also be called: topological Boolean algebra or closure algebra.
  We can say that from the algebraic point of view the propositional calculus S4 is a *free Lewis algebra*. If p represents a proposition, then Cp [in our notation $\mu(p)$] represents the proposition *p is possible*.
  Starting from the C [in our notation μ] operator we can define the operator I [in our notation ν] through the formula $Ia = -C - a$ [in our notation $\nu(a) = \neg\mu\neg(a)$], which verifies the conditions [(I0)–(I3)]. If p represents a proposition, then Ip [in our notation $\nu(p)$] represents the proposition *p is necessary*."

12.2 The Classical Case of Necessity–Interior Kuratowski Operation on the Power Set of a Universe

In the concrete Boolean algebra $\langle \mathscr{P}(X), \cap, \cup, -, \emptyset, X\rangle$ based on the power set of the universe X, the algebraic version of the *implication connective* corresponding to Eq. (44) is

$$A \to_c B := -A \cup B.$$

which, according to Proposition 90, satisfies the pseudo-complement condition (PCL), the modus ponens condition (MP), and the law of entailment (E*).

If, as discussed in Sect. 8.1, the Boolean algebra $\mathscr{P}(X)$ is equipped with an interior operation $I : \mathscr{P}(X) \to \mathscr{P}(X)$ with associated collection of open sets $\mathscr{O}(X)$, then according to Eq. (45) we can introduce on $\mathscr{P}(X)$ another binary operation, interpreted as *necessity implication connective*, defined for any pair of subsets $A, B \in \mathscr{P}(X)$ as the open set:

$$A \to_m B = I(-A \cup B) \in \mathscr{O}(X) \tag{46}$$

This implication connective, in agreement with the results just proved for the abstract case and according to the discussion of the example in [100, p. 59]

(IC-1) "For any subsets A, B, C of X, the inclusion $A \cap C \subseteq B$ holds iff $C \subseteq -A \cup B$."

(IC-2) If $O \in \mathscr{O}(X)$ is open, the inclusion $A \cap O \subseteq B$ holds iff $O \subseteq -A \cup B$.

(IC-3) Since the power set $\mathscr{P}(X)$ is a complete lattice, "for any $A, B \in \mathscr{O}(X)$, the set $A \to_m B = I(-A \cup B)$ is the greatest open set O satisfying the condition $A \cap (A \to_m B) \subseteq B$" [100, p. 59, equation (3)].

So according to [100, p. 125] $A \to_m B$ is the *pseudo-complement* of A *relative* to B. The *pseudo-complement* of A, denoted as $\sim A$, is obtained by Eq. (46) setting $B = \emptyset$:

$$\sim A := A \to_m \emptyset = I(-A) \in \mathscr{O}(X) \tag{47}$$

Since the lattice $\mathscr{P}(X)$ contains the empty set as zero element, according to [100, p. 58], one has the *pseudo-Boolean algebra*

$$\langle \mathscr{P}(X), \cap, \cup, \to_m, \sim\rangle .$$

As shown in [100, p. 61–62], and always according to the general theory now developed, the pseudo-complementation mapping $\sim \mathscr{O}(X) \to \mathscr{O}(X)$, $A \to \sim A$, is also a *Brouwer complementation* in the sense that the following properties algebraically characterizing the intuitionistic (Brouwer) negation are satisfied:

(B1-X)	$A \subseteq \sim\sim A$	(weak double negation)
(B2-X)	$A \subseteq B$ implies $\sim B \subseteq \sim A$	(contraposition)
(B3-X)	$A \cap \sim A = \emptyset$	(noncontradiction)

In agreement with [78] we can quote the following statement which in the original version involves the possibility-closure operation but which we "translate" into a necessity-interior statement: "if $(X, \mathscr{O}(X))$ is a topological space and $I : \mathscr{P}(X) \to \mathscr{P}(X)$ the correspondent Kuratowski interior operator; then, the system $(\mathscr{P}(X), I)$ is a (complete) Lewis algebra. All the subalgebras of $(\mathscr{P}(X), I)$ will be called *topological Lewis algebras*. The question then arises of knowing whether all the abstract Lewis algebras are isomorphic to a topological Lewis algebra. The answer to this question is positive [73]."

12.2.1 A Remark About Fuzzy Sets

Let us consider the Kleene distributive (complete) lattice $\langle \mathscr{F}(X), \wedge, \vee, ', 0, 1 \rangle$ of all the fuzzy sets on the universe X, equipped with the Kuratowski interior ^{o} and closure * operations introduced in Sect. 8.2. The corresponding to (46) algebraic version of the necessity implication connective $f \to_m g$ is the crisp set defined for every $x \in X$ by $(f \to_m g)(x) := [(f' \vee g)^o](x) = \chi_{A_1(f' \vee g)}(x)$, where $A_1(f' \vee g) = A_0(f) \cup A_1(g)$. Extensively,

$$\forall x \in X, \quad (f \to_m g)(x) := \begin{cases} 1 & \text{if } f(x) = 0 \text{ or } g(x) = 1 \\ 0 & \text{otherwise} \end{cases} \tag{48}$$

In this fuzzy set case in general the condition $f \to_m g = \mathbf{1}$ is not equivalent to $f \leq g$, that is the *law of entailment* (E*) is not true for $\mathscr{F}(X)$.

Example 93 Let us consider the universe $X = \mathbb{R}$. The two fuzzy sets $f = (1/2)\chi_{[0,1)} + \chi_{[1,2]} + (1/2)\chi_{(2,3]}$ and $g = \chi_{[0,2]} + (1/2)\chi_{(2,3]}$ are such that $f \leq g$, but $f \to_m g = \chi_{(-\infty,2]} + \chi_{[3,\infty)} \neq \mathbf{1}$.

12.3 Algebraic Model of Modal Logic Induced by Halmos Closure-Interior Operations

Let us recall that taking into account the equivalences proved in Lemma 62 between conditions (sC3) and (sC3b), and adopting the (ML) notations $\mu(a) = a^*$, $\nu(a) = a^o$, the statement of Proposition 66 can be restated in the following manner translating it in the ν notation:

Proposition 94 *An operation* $\nu : \Sigma \to \Sigma$ *on a De Morgan lattice* $\langle \Sigma, \wedge, \vee, ', 0, 1 \rangle$ *is a Halmos interior operations iff the following two conditions hold:*

(H1) the operation ν *is a Kuratowski interior operation, i.e., for arbitrary* $a, b \in \Sigma$*,*

(I0)	$\nu(1) = 1$	(normalization)
(I1)	$\nu(a) \le a$	(decreasing)
(I2)	$\nu(a) \wedge \nu(b) = \nu(a \wedge b)$	(multiplicative)
(I3w)	$\nu(a) \le \nu(\nu(a))$	(weak idempotent)

(H2) the further condition making a link between closure and interior holds:

$$\text{(sC3b)} \qquad \forall a \in \Sigma,\ \mu(a) = \nu(\mu(a)).$$

From the algebraic model of modal logic the schema (sC3b) corresponds to the modal 5 principle. Quoting [32, p. 6]: "The import of 5 is that what is possible [$\mu(a)$] is necessarily possible: *if possibly a, then necessarily possibly a*" [$\mu(a) = \nu(\mu(a))$].

From the point of view of algebraic model of modal logic we have the following:

- The Halmos interior lattice $\langle \Sigma, \wedge, \vee, -, \nu = {}^o, 0, 1 \rangle$ is an algebraic model of a S5-like modal system, based on a De Morgan lattice instead of on a Boolean one, in the sense that modal principles N, T, M, C, and 5, are satisfied.
- Indeed, as seen in Sect. 12 relatively to Kuratowski necessity-interior operation, (I0) realizes the modal principle N; (I1) is the algebraic version of modal T principle; (I2) of modal M and C principles; and (sC3b), in the equivalent weak formulation (sC3w) of Sect. 9, i.e., $a^* \le a^{*o}$, is the algebraic realization of modal 5 principle, written in modal notation as $\mu(a) \le \nu(\mu(a))$, i.e., $\mu(a) \to \nu(\mu(a)) = 1$ "what is possible is necessarily possible: *if possibly a, then necessarily possibly a*" [32, p. 6].

Note that modal principle 5 implies the two modal principles B and 4.

- Indeed, from principle 5, $\mu(a) \le \nu(\mu(a))$, making use of closure increasing (C1) $a \le \mu(a)$, it follows the modal B principle: $a \le \nu(\mu(a))$, i.e., $a \to \nu(\mu(a)) = 1$ [32, p. 16].

Furthermore, as shown in Proposition 64, modal principle 5 (formalized as (sC3b)) implies modal principle 4:

- $\nu(a) \le \nu(\nu(a))$, i.e., $\nu(a) \to \nu(\nu(a)) = 1$ [32, p. 18].

13 Kripke–Style Models of Tarski, Kuratowski, and Halmos Closure Distributive Lattices

In the present Part III let us consider a particular structure $\langle \Sigma, \wedge, \vee, -, 0, 1\rangle$ of which I propose two interpretations.

(Syn) The first is a *syntactic* interpretation as described in Sect. 10.1 in which Σ is the collection of all *well formed formulae* (wffs) (also *sentences*) constructed in a usual way: (1) a suitable collection of propositional variables, (2) the connectives $\wedge$, $\vee$, and $-$ of conjunction AND, disjunction OR, and negation NOT, respectively, as signs used to build these sentences. The constant signs 0 and 1 describing truth and falseness. In this sense, the structure under examination, now referred as $\mathfrak{L}(\Sigma) = \langle \Sigma, \wedge, \vee, -, 0, 1\rangle$, is a *formalized language of the zero order* or a *formalized language of a propositional calculus* [100, pp. 166–167].

(Sem) The second is a *semantical interpretation* in which the structure, now denoted as $\mathfrak{B}(\Sigma) = \langle \Sigma, \wedge, \vee, -, 0, 1\rangle$, is a *Boolean algebra* considered as the algebraic realization of propositions of a hidden formalized language, which is neglected during the development of semantics. Of course, in this semantical algebraic context signs $\wedge$, $\vee$, and $-$, correspond to the distributive lattice operations of *meet*, *join*, and *orthocomplementation*, respectively. 0 and 1 being the *least* and the *greatest* elements of the lattice. In other words, as explained in Sect. 11 for a more general lattice structure, the Boolean algebra $\mathfrak{B}(\Sigma)$ may be thought of as algebraic realization of logical sentences, with 0 representing falseness and 1 truth, and the involved operations considered as algebraic realizations of the logical connectives AND ($\wedge$), OR ($\vee$), and NOT (for $-$).

Each of us can select appropriately among these "syntactic-semantic" alternatives as per their cultural interests in relation to these topics. To stay on the most general situation, we will use the symbol $\mathfrak{U}(\Sigma)$ to denote the only two possible alternatives $\mathfrak{U}(\Sigma) = \mathfrak{L}(\Sigma)$ and $\mathfrak{U}(\Sigma) = \mathfrak{B}(\Sigma)$.

Let us now considered a nonempty set X with associated its power set $\mathscr{P}(X)$ formed by all its possible subsets. The structure $\mathfrak{P}(X) = \langle \mathscr{P}(X), \cap, \cup, ^c, \emptyset, X\rangle$, with $\cap$ and $\cup$ the set theoretical union and intersection operations, respectively, and ^{c} the set theoretical complement ($\forall A \in \mathscr{P}(X)$, $A^c = X \setminus A$), is a standard realization of a Boolean algebra, which as lattice is *complete and atomic* whose atoms are the singletons $\{x\}$ of X.

From a pure semantical point of view, and generalizing what seen in Sect. 8.1, points of X are interpreted as *possible worlds* or also *semantical states*, applied to the particular Kuratowski case, argument which will be treated in Sect. 15.3.

Given the abstract Boolean structure $\mathfrak{U}(\Sigma)$, both in syntactic and semantic interpretation, borrowing a usual terminology, we will define as

- *Boolean frame* the pair $\mathfrak{F}(X) = (X, \mathscr{P}(X))$ formed by a set of X points, or worlds or states, and the Boolean algebra over $\mathscr{P}(X)$.
- A *model* based on $\mathfrak{F}(X)$, is an ordered triple $\mathscr{M} = \langle X, \mathscr{P}(X), v\rangle$, where the mapping $v : \Sigma \to \mathscr{P}(X)$ is a *world valuation* assigning to any propositional variable $a \in \Sigma$ the subset of possible worlds $v(a) \in \mathscr{P}(X)$ under the condition of being a homomorphism.
  The subset $v(a) \in \mathscr{P}(X)$ is interpreted as the collection of all possible world (states) in which the proposition a is true.

The homomorphism properties of the valuation v from a given model $\mathscr{M}$ are obviously the following:
Let $v(a) = A$ and $v(b) = B$, then

(Hom-1) $v(a \wedge b) = v(a) \cap v(b) = A \cap B$, the statement a AND b is true in all the semantical states in which both the statements a and b are simultaneously true.
(Hom-2) $v(a \vee b) = v(a) \cup v(b) = A \cup B$, the statement a OR b is true in all the semantical states in which at least one of two the statements either a or b is true.
(Hom-3) $v(\neg a) = v(a)^c = (A)^c$, the statement $\neg a$ is true in all state in which a is not true.

Let us recall that, as seen in Eq. (44) of Proposition 90, in any abstract Boolean algebra it is possible to introduce the *classical implication connective* $a \to_c b := \neg a \vee b$, whose induced valuation v according to the rules just introduced is the following:

$$v(a \to_c b) = v(\neg a \vee b) = v(a)^c \cup v(b) = A^c \cup B \tag{49}$$

Of course, fixed a Boolean frame $\mathfrak{F}(X) = (X, \mathscr{P}(X))$ based on the universe X of possible worlds, there are as many models $\mathscr{M}_j$ as there are possible homomorphisms $v_j : \Sigma \to \mathscr{P}(X)$. Let us denote by $\hom(\Sigma, \mathscr{P}(X))$ the collection of all such homomorphisms, the subscript j is an index belonging to the index set J in such a way that $\hom(\Sigma, \mathscr{P}(X)) := \{v_j : j \in J\}$.

Following Goldblatt [49], once fixed a model $\mathscr{M} = \langle X, \mathscr{P}(X), v\rangle$, we can formalized this interpretation introducing the notation

$$\mathscr{M} \models_x a \quad \text{iff} \quad x \in v(a) \text{ in the model } \mathscr{M}. \tag{50}$$

In this way, "read a is true (holds) at x in $\mathscr{M}$ for $\mathscr{M} \models_x a$" [49]. With respect to this notation the above homomorphism conditions (Hom-1)–(Hom-3) assume the form:

$$\text{(Hom-M1)} \quad \mathscr{M} \models_x (a \wedge b) \text{ iff } \mathscr{M} \models_x a \text{ and } \mathscr{M} \models_x b,$$

$$\text{(Hom-M2)} \quad \mathscr{M} \models_x (a \vee b) \text{ iff } \mathscr{M} \models_x a \text{ or } \mathscr{M} \models_x b,$$

$$\text{(Hom-M3)} \quad \mathscr{M} \models_x \neg a \text{ iff } \mathscr{M} \not\models_x a.$$

At this point the problem is to give an extension of the above valuation mappings when the Boolean algebra Σ is equipped with a further unitary operation of closure-possibility $^* : \Sigma \to \Sigma$ (with $\mu = {}^*$), and related unary operator of interior-necessity $^o : \Sigma \to \Sigma$ (with $\nu = {}^o$), in the Tarski, Kuratowski, and Halmos three cases. This means to assign to any proposition $a \in \Sigma$ the corresponding valuations $v(a^*) \in \mathscr{P}(X)$ and $v(a^o) \in \mathscr{P}(X)$ as two suitable subsets A^* and A^o of the universe X, respectively.

But before entering into this problem let us discuss about three possible approaches.

(KS) The first approach is the orthodox *Kripke semantic* assignment of subsets of possible worlds by suitable binary relations R, usually called of *accessibility*. Without entering into excessive details, which can be found in the textbooks of modal logic (such as [32] or [46] for an introductory approach, and [10]), we only say that the Kripke semantic defines the necessity modal operator ν, "by the condition that $\nu(a)$ is true at x in $\mathscr{M}$ only when a is true in all worlds accessible to x in this model": $\mathscr{M} \models_x \nu(a)$ iff $\mathscr{M} \models_{x'} a$ for all x' s.t. $x' R x$. Analogously, "possibility $\mu(a)$ is true at x in $\mathscr{M}$ when a is true in some world accessible to x in this model": $\mathscr{M} \models_x \nu(a)$ iff $\mathscr{M} \models_{x'} a$ for some x' s.t. $x' R x$.

Clearly these semantical assignments depend on the properties of the relation R, which in the standard model $\mathscr{M} = \langle X, R, v\rangle$ can be serial, reflexive, symmetric, transitive, Euclidean, [32, p. 80] and some appropriate combinations of them for giving for instance similarity, quasi-ordering, equivalence relations [32, p. 83].

At any rate, the valuation in the semantical approach to modal logics by binary relations R is a mapping

$$v_R : \Sigma \to \mathscr{P}(X). \tag{51}$$

(OS) The second approach is the semantical analysis based on *orthoframes* defined as pairs $\mathfrak{F} = \langle X, \perp\rangle$ consisting of a nonempty set X of possible worlds, also called the *carrier* of $\mathfrak{F}$, and an *orthogonality relation* $\perp$ on X, i.e., $\perp \subseteq X \times X$ is irreflexive and symmetric [49]. In this approach to any subset A of X it is associated its orthocomplement $A^\perp := \{x \in X : \forall a \in A,\ x \perp a\}$. Then, it turns out that in general $A \subseteq A^{\perp\perp}$ and so we can single out the $\perp$-*closed* subsets as those subsets C for which $C = C^{\perp\perp}$. Note that in [50] $\perp$-closed subsets are called $\perp$-regular.

Following Goldblatt [49], "$\mathscr{M} = \langle X, \perp, v\rangle$ is an *orthomodel* on the frame $\mathfrak{F} = \langle X, \perp\rangle$ iff v is a function assigning to each proposition a a $\perp$-closed subset $v(a)$ of X." Once introduced the notation of Eq. (50), conditions (Hom-M1)–(Hom-M3) are assumed as characterizing this semantical approach.

A valuation, as a mapping $v : \Sigma \to \mathscr{P}(X)$ assigning to any $a \in \Sigma$ a subset of possible worlds $v(a) \in \mathscr{P}(X)$ in which a is true, seems in some

way to mimic a behavior similar to that of the orthodox Kripke semantics seen in the previous point. Also if the name of Kripke semantic cannot be correctly assign to it, we can at least assume that it gives rise to a *Kripke-style semantic*. This name is what we have adopted in a paper of ours [26]. The $\perp$-semantic is the argument of study of Sect. 14, where a generic irreflexive and symmetric relation is called a *preclusion relation*, denoted by #, in order to distinguish this general case from the real orthogonality relation $\perp$ on vector spaces.
At any rate, the valuation in the semantical approach to modal logics by orthogonality relation $\perp$ is a mapping

$$v_\perp : \Sigma \to \mathscr{P}(X). \tag{52}$$

(CS) The last approach will be developed in Sect. 15 and will concern the generation of interior operations by appropriate coverings of the universe, with consequent induced closure operations. In particular we will consider generic coverings, topological coverings, and partition coverings as generators of Tarski, Kuratowski and Halmos interior operations. As usual, interior operations in the (TL) approach described in the introduction can be also interpreted as modal necessity connectives of the (ML), approach producing S4-like, S4, and S5 algebraic models of modal logics.
The valuations in the semantical approach to modal logics by coverings can be summarized in a mapping, where γ denotes a generic covering of X,

$$v_\gamma : \Sigma \to \mathscr{P}(X). \tag{53}$$

All these three approaches present the common behavior of having an evaluation mapping v that assigns to each proposition a a subset $v(a) \in \mathscr{P}(X)$ of the universe of possible worlds X. This fact could lead to an arbitrary conclusion to say that we are always in the presence of a Kripke semantical approach. A second, more correct, possibility consists in attributing the term of Kripke semantics only to the first (KS) approach and in any case believe that in the other two cases (OS) and (CS) we are dealing with a Kripke-like, Kripke-style, similar Kripke, semantics.

14 Binary Relation of Preclusion on the Universe X and Induced Quantum Logic

In the present section we introduce a procedure in order to generate interior–closure pairs of subsets from the power set $\mathscr{P}(X)$ of the universe X based on a *preclusion*, also *discernibility* (i.e., irreflexive and symmetric) binary relation on X, according to the (OS) approach outlined in Sect. 13.

Let us recall the formalisation of a similarity relation $S \subseteq X \times X$ according to the two following conditions:

$$\text{(Sim1)} \quad \forall x \in X,\ (x,x) \in S \qquad \text{(reflexive)}$$

$$\text{(Sim2)} \quad \forall x, y \in X,\ \ (x,y)S \text{ implies } (y,x)S \qquad \text{(symmetric)}$$

A preclusion relation # can always be obtained as negation of a similarity relation S according to $x\#y$ iff Not$(x, y)S$. Since the reflexivity condition (Sim1) of S can be expressed as the condition "$x = y$ implies $(x, y)S$", the negation of this condition is "$x\#y$, implies $x \neq y$." Hence, the preclusion (discernibility) relation is formalized by the two conditions

$$\text{(Pre1)} \quad \forall x, y \in X,\ x\#y \text{ implies } x \neq y \qquad \text{(irreflexive)}$$

$$\text{(Pre2)} \quad \forall x, y \in X,\ x\#y \text{ implies } y\#x \qquad \text{(symmetric)}$$

A *preclusion space* is any pair $(X, \#)$. Then, for any subset A of X we can define its *preclusion complement* (or #-complement), as the subset:

$$A^{\#} := \{x \in X : \forall a \in A,\ x\#a\} \tag{54}$$

So an element of the universe belongs to $A^{\#}$ iff it is distinguishable from all the elements of A.

Example 95 In any universe X the equality between elements, $a = b$, is trivially an equivalence relation, stronger version of a similarity relation. The induced preclusion relation is the relation of being different, $a \neq b$. For any subset $A \in \mathscr{P}(X)$ the corresponding $\neq$-complement is just the set theoretical complement of A: $A^{\neq} = X \setminus A = A^c$. Hence, trivially the family of $\neq$-closed subset (i.e., those subsets which coincide with their bi $\neq$-complement), $\mathscr{C}(X, \neq) := \{M \in \mathscr{P}(X) : M = (M^{\neq})^{\neq} = (M^c)^c\}$ coincides with the power set $\mathscr{P}(X)$, and so this family turns out to be a Boolean lattice with respect to the usual set theoretical operations of intersection, union, and complementation.

The preclusion relation of Example 95 can be applied to any universe X, in particular to the universe $\mathbb{R}^2$ consisting of all the pairs of real numbers. But in this last case there is another interesting preclusion relation.

Example 96 The set $\mathbb{R}^2$ can be equipped with a structure of real linear space with respect to the real linear combinations of its elements, in this case called *vectors*. Precisely,

(H1) if $\mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2) \in \mathbb{R}^2$ and if $\alpha, \beta \in \mathbb{R}$ then $\alpha\mathbf{x} + \beta\mathbf{y} := (\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2) \in \mathbb{R}^2$. This linear space is two dimensional since the pair of vectors $\mathbf{u}_1 := (1, 0) \in \mathbb{R}^2$ and $\mathbf{u}_2 := (0, 1) \in \mathbb{R}^2$ are linearly independent ($\nexists \alpha \in \mathbb{R}$, s.t. $\mathbf{u}_1 = \alpha\mathbf{u}_2$) and such that any other vector of $(x_1, x_2) \in \mathbb{R}^2$ can be expressed as their linear combination: $(x_1, x_2) = x_1\mathbf{u}_1 + x_2\mathbf{u}_2$.

(H2) But the real linear space $\mathbb{R}^2$ can be equipped also with a scalar product associating with any pair of its vectors $\mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2) \in \mathbb{R}^2$ the real number $\langle \mathbf{x}|\mathbf{y}\rangle := x_1 y_1 + x_2 y_2 \in \mathbb{R}$.

(H3) The *norm* or *modulus* of a vector $\mathbf{x} = (x_1, x_2)$ is then defined as $\|\mathbf{x}\| := \sqrt{\langle \mathbf{x}|\mathbf{x}\rangle} = \sqrt{(x_1)^2 + (x_2)^2}$.

In this inner product space (also *Hilbert space*) one can introduce the *orthogonality* binary relation:

$$\text{Let } \mathbf{x}, \mathbf{y} \in \mathbb{R}^2, \quad \text{then } \mathbf{x} \perp \mathbf{y} \quad \text{iff} \quad \langle \mathbf{x}|\mathbf{y}\rangle = 0. \tag{55}$$

Then it is a standard result of any inner product space, not only of $\mathbb{R}^2$, the so-called Schwarz inequality $|\langle \mathbf{x}|\mathbf{y}\rangle| \leq \|\mathbf{x}\| \cdot \|\mathbf{y}\|$ (see [56, p. 9]). Since the inner product furnishes a real number, once denoted $\mathbb{R}^2_0 := \mathbb{R}^2 \setminus \{(0, 0)\}$ we can derive the inequalities

$$\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^2_0, \quad -1 \leq \frac{\langle \mathbf{x}|\mathbf{y}\rangle}{\|\mathbf{x}\| \cdot \|\mathbf{y}\|} \leq 1 \tag{56}$$

and so there exists an unique *angle* $\vartheta \in [0, \pi)$ between the two vectors s.t. $\langle \mathbf{x}|\mathbf{y}\rangle = \|\mathbf{x}\| \cdot \|\mathbf{y}\| \cdot \cos\vartheta$. Hence, the orthogonality relation expressed by Eq. (55) assumes the following intuitive form:

$$\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^2_0, \quad \mathbf{x} \perp \mathbf{y} \quad \text{iff} \quad \vartheta = \pi \tag{57}$$

For any subset A from $\mathbb{R}^2$ if we denote by $A^\perp := \{\mathbf{x} \in \mathbb{R}^2 : \forall \mathbf{a} \in A,\ \mathbf{x} \perp \mathbf{a}\}$ its $\perp$-*complement*, we can have the two cases:

($\mathbb{R}^2$-a) If $A = \{\mathbf{a}_1, \mathbf{a}_2\}$ consists of two collinear vectors, $\exists\lambda \in \mathbb{R}$ s.t. $\mathbf{a}_2 = \lambda\mathbf{a}_1$, then $A^\perp = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} \perp \mathbf{a}_1\}$, i.e., it is the one dimensional subspace consisting of all the vectors orthogonal to $\mathbf{a}_1$ (or equivalently to $\mathbf{a}_2$). For instance if $A = \{(1, 1),\ (3, 3)\}$ then
$A^\perp = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = -x_1\}$ which is the straight line (one dimensional subspace), forming the diagonal of the II-IV quadrant of $\mathbb{R}^2$. This result can be extended to any family A consisting of vectors that all lie on the same straight line.

($\mathbb{R}^2$-b) If $A = \{\mathbf{a}_1, \mathbf{a}_2\}$ is formed by two vectors which are not collinear, $\nexists\lambda \in \mathbb{R}$ s.t. $\mathbf{a}_2 = \lambda\mathbf{a}_1$, then $A^\perp = \mathbb{R}^2$. This result can be extended to any family of vectors in which at least two of them are not collinear.

Summarizing, for any subset of vectors A from $\mathbb{R}^2$ its ortho-complement $A^\perp$ can only be of two types, either a one-dimensional subspace or the whole $\mathbb{R}^2$ which in any case is also a subspace. However, in any case we have a subspace of $\mathbb{R}^2$ even if A is any generic subset.

The Two Dimensional Complex Case All these results can be extended to the two dimensional complex Hilbert space $\mathbb{C}^2$ of vectors (x_1, x_2) where in this case the two components $x_1, x_2 \in \mathbb{C}$ are complex numbers. Of course the linear combinations of point (H1) involve complex numbers $\alpha, \beta \in \mathbb{C}$. The scalar product of point (H2) is now $\langle \mathbf{x}|\mathbf{y}\rangle = \overline{x_1}\, y_1 + \overline{x_2}\, y_2 \in \mathbb{C}$, where for a generic complex number $z = a + ib$ it is $\overline{z} = a - ib$.

The Schwarz inequality is always true, but we can only say that the quantity $\frac{\langle \mathbf{x}|\mathbf{y}\rangle}{\|\mathbf{x}\|\cdot\|\mathbf{y}\|} \in \mathbb{C}$ is a complex number which belongs to the closed circle of center $\mathbf{0} = (0, 0)$ and radius 1. In any case the relation of orthogonality $\mathbf{x} \perp \mathbf{y}$ iff $\langle \mathbf{x}|\mathbf{y}\rangle = 0$ is always extended to the present complex case.

Proposition 97 *Let $(X, \#)$ be a* preclusion space *and let us consider the structure*

$$\mathfrak{P}(X, \#) = \left\langle \mathscr{P}(X), \cap, \cup, {}^c, {}^\#, \emptyset, X \right\rangle$$

on the power set $\mathscr{P}(X)$ of the universe X. Then, the following hold.

(1) The sub-structure $\mathfrak{B}(X, \#) = \langle \mathscr{P}(X), \cap, \cup, \emptyset, X\rangle$ is a distributive, atomic complete lattice with respect to the set theoretical intersection $\cap$ and union $\cap$, bounded by the least element $\emptyset$ and the greatest element X. The partial order relation induced from this lattice structure is the standard set theoretical inclusion $\subseteq$ and the atoms are the singletons $\{x\}$ for any $x \in X$.

(2) The unary operation ${}^c : \mathscr{P}(X) \to \mathscr{P}(X)$ associating with any subset $A \in \mathscr{P}(X)$ its set theoretical complement *$A^c = X \setminus A \in \mathscr{P}(X)$ is a standard orthocomplementation: (C1) $A = A^{cc}$ (involution); (C2) $A \subseteq B$ implies $B^c \subseteq A^c$ (contraposition); (C3) $A \cap A^C = \emptyset$ (noncontradiction) and $A \cup A^c = X$ (excluded middle).*

(3) The unary operation ${}^\# : \mathscr{P}(X) \to \mathscr{P}(X)$ associating with any subset $A \in \mathscr{P}(X)$ the preclusion complement *$A^\# \in \mathscr{P}(X)$ defined according to Eq. (54) is a Brouwer orthocomplementation:*

(B1) $A \subseteq A^{\#\#}$ (weak involution)*;*
(B2) $A \subseteq B$ implies $B^\# \subseteq A^\#$ (#-contraposition)*;*
(B3) $A \cap A^\# = \emptyset$ (noncontradiction).

In general $A \cup A^\# \neq X$ (the excluded middle is not *verified).*

(4) The two complementations are linked by the weak interconnection rule*:*

$$\text{(wIR)} \qquad \forall A \in \mathscr{P}(X), \quad A^\# \subseteq A^c.$$

The condition (wIR) is equivalent to the following

$$\text{(wIR-a)} \qquad \forall A \in \mathscr{P}(X), \quad A^{\#\#} \subseteq A^{\#c}.$$

Proof The first two points (1) and (2) are trivial consequence of standard set theory. Let us consider the other two points.

(B1) $\forall \alpha \in A^\#$ (by (54)) is such that $\forall a \in A$, $\alpha \# a$, that is $\forall a \in A$ is such that $\forall \alpha \in A^\#$, $\alpha \# a$, which can be written as $\forall a \in A$ is such that

$a \in \{x \in X : \forall \alpha \in A^{\#},\ \alpha \# x\} = (A^{\#})^{\#}$, i.e., $A \subseteq A^{\#\#}$. (B2) Let $A \subseteq B$, then $B^{\#} = \{x \in X : \forall b \in B,\ b\#x\} \subseteq \{x \in X : \forall a \in A,\ a\#x\} = A^{\#}$.
(B3) $\Longleftrightarrow$ (wIR). Let $x \in A^{\#}$ then (by Eq. (54)) $\forall a \in A$, $a\#x$; but from (Pre1) $\forall a \in A$, $a \neq x$, that is $x \in A^c$. So we have proved that $A^{\#} \subseteq A^c$ which is trivially equivalent to $A^{\#} \cap A = \emptyset$.

As to the condition (wIR-a), let (wIR) be true. Then applying the element $A^{\#}$ to it we get $A^{\#\#} \subseteq A^{\#c}$. Conversely, let (wIR-a) be true. Then, $A^{\#} = A^{\#\#\#} \subseteq A^{\#\#c}$. But from condition (B1) $A \subseteq A^{\#\#}$ we get, using the contraposition law, $A^{\#\#c} \subseteq A^c$. □

Remark 98 In the proof of this proposition it is evident that the two conditions (B3) of Brouwer noncontradiction and (wIR) of weak interconnection rule are equivalent, and so the condition (B3) turns out to be redundant, and therefore eliminable from the formulation of point (3) of Proposition 97. However we have kept this condition explicit in order to underline its importance in characterizing the Brouwer negation.

Take into account that the weak interconnection rule (wIR) cannot be substitute by the stronger interconnection rule (IR) $A^{\sim\sim} = A^{\sim\prime}$, which in general does not hold in the preclusion context.

Example 99 For instance in Example 96 of the two dimensional vector space $\mathbb{R}^2$, we have seen that the subset $A = \{(1, 1), (3, 3)\}$ has the $\perp$-complement $A^{\perp} = \{(x_1, x_2) = x_2 = -x_1\}$. From this result it follows that $A^{\perp\perp} = \{(y_1, y_2) : y_1 = y_2\}$, with $A \subseteq A^{\perp\perp}$, but $A^{\perp c} = \mathbb{R}^2 \setminus A^{\perp}$ which is trivially different from $A^{\perp\perp}$, with $A^{\perp\perp} \subseteq A^{\perp c}$.

Let us stress that the above concrete lattice structure $\langle \mathscr{P}(X), \cap, \cup, {}^c, {}^{\#}, \emptyset, X \rangle$ is a model of the abstract lattice structure $\langle \Sigma, \wedge, \vee, {}', {}^{\sim}, 0, 1 \rangle$ according to the realizations depicted by the correspondences of Table 1.

The abstract lattice structure $\langle \Sigma, \wedge, \vee, {}', {}^{\sim}, 0, 1 \rangle$ is called *quasi Brouwer Boolean (BB) lattice* (also *algebra*) iff it is a distributive lattice Σ equipped with an orthocomplementation $'$ (i.e., a Boolean algebra according to Sect. 2.2) and a Brouwer negation $^{\sim}$, with these two linked by the weak interconnection rule (wIR) $\forall a \in \Sigma, a^{\sim} \leq a'$, or the equivalent (wIR-a) $\forall a \in \Sigma, a^{\sim\sim} \leq a^{\sim\prime}$.

Coming back to the concrete quasi BB algebra induced from a preclusion space one can introduce the following.

Table 1 Concrete realization on $\mathscr{P}(X)$ of the abstract quasi BB lattice Σ

Abstract Lattice		Concrete Universe X
$a \in \Sigma$	$\Longrightarrow$	$A \in \mathscr{P}(X)$
$a \leq b$	$\Longrightarrow$	$A \subseteq B$
$a \wedge b$	$\Longrightarrow$	$A \cap B$
$a \vee b$	$\Longrightarrow$	$A \cup B$
a'	$\Longrightarrow$	$A^c = X \setminus A$
$a^{\sim}$	$\Longrightarrow$	$A^{\#}$
0 and 1	$\Longrightarrow$	$\emptyset$ and X

Definition 100 Let $\mathfrak{P}(X, \#) = \langle \mathscr{P}(X), \cap, \cup, {}^c, {}^\#, \emptyset, X \rangle$ be the quasi BB lattice induced from the preclusion space $(X, \#)$. Then making use of the two negations ^{c} and $^\#$ another unusual negation can be introduced $^\flat : \mathscr{P}(X) \to \mathscr{P}(X)$ associating with any subset A the subset $A^\flat := A^{c\#c}$ satisfying the properties:

(AB1) $A^{\flat\flat} \subseteq A$ (weak anti-involution);
(AB2) $A \subseteq B$ implies $B^\flat \subseteq A^\flat$ (AB contraposition);
(AB3) $A \cup A^\flat = X$ (excluded middle).

Since this negation is dual with respect to the Brouwer complementation, it is called *anti-Brouwer complementation*.

Remark 101 Let us consider the abstract context of quasi BB lattice $\langle \Sigma, \wedge, \vee, {}', {}^\sim, 0, 1 \rangle$, where we can also use the notations of the algebraic realization of a logic $\neg a = a'$ and $\sim a = a^\sim$ for denoting the Boolean negation and the Brouwer negation, respectively. Recalling that in the general BZ context (see Sect. 9.1) the Brouwer negation, defined as $a^\sim := a^{*\prime}$, i.e., $\sim a = \neg\mu(A)$, is the *impossibility* connective as negation of the possibility $\mu(a) = a^*$.

In this algebraic context the dual of the Brouwer negation, called in [25] *anti-Brouwer negation*, is denoted by $\flat a = a^\flat$ and formally defined as $\flat a := \neg \sim \neg a$. From the fact that, according to the point (3) of Proposition 97, the Brouwer negation satisfies the abstract version of conditions (B1)–(B3), it follows that the anti-Brouwer negation satisfies the abstract versions of conditions (AB1)–(AB3).

Recall that taking into account the above Table 1 we have the following realizations $\neg a \Rightarrow A^c$ and $\sim a \Rightarrow A^\#$, and so $\flat a = \neg \sim \neg a \Rightarrow A^\flat = A^{c\#c}$, the latter according to Definition 100.

Note that McKinsey and Tarski in [74] called with the name of *Brouwer complementation* a complementation satisfying the abstract versions of conditions (AB1)–(AB3), denoting it with the symbol $\rceil a$, in other words it is $\rceil a = \flat a$. This leads to a terminological confusion of calling in [74] as Brouwer complementation what in [25] has been called anti-Brouwer complementation.

On the other hand, Monteiro in [81] defines $\lceil a = \mu(-a)$ and $\rceil a = -\lceil -a$. So in this Monteiro notation $\lceil a = \mu(-a) = -(-\mu) - a = - \sim -a = \flat a$. In other words, in the Monteiro approach it was denoted by $\lceil$ what in [74] was denoted by $\rceil a$, increasing also in this case the formal confusion.

We can prove the following results.

Proposition 102 *In any concrete quasi BB lattice $\mathfrak{P}(X, \#)$ one has the following relationships relating the three complementations.*

$$\forall A \in \mathscr{P}(X), \quad A^\# \subseteq A^c \subseteq A^\flat \tag{58}$$

Further the following chain of inclusions holds.

$$\forall A \in \mathscr{P}(X), \quad A^{c\#} \subseteq A^{\flat\flat} \subseteq A \subseteq A^{\#\#} \subseteq A^{\#c} \tag{59}$$

where in particular $A^{\flat\flat} = A^{c\#\#c}$.

Proof The inclusion $\forall A \in \mathscr{P}(X)$, $A^{\#} \subseteq A^{c}$ is nothing else than the weak interconnection rule (wIR); applying this inequality to the element A^c we get $A^{c\#} \subseteq A^{cc} = A$ from which, by contraposition, $A^c \subseteq A^{c\#c} = A^{\flat}$.

The inclusion $\forall A \in \mathscr{P}(X)$, $A \subseteq A^{\#\#}$ is condition (B1) of the Brouwer negation. Applying $A^{\#}$ to the (wIR) $\forall A \in \mathscr{P}(X)$, $A^{\#} \subseteq A^c$ we get $A^{\#\#} \subseteq A^{\#c}$.

Applying A^c to (B1) $\forall A \in \mathscr{P}(X)$, $A \subseteq A^{\#\#}$ we obtain $A^c \subseteq A^{c\#\#}$, and by contraposition $A^{c\#\#c} \subseteq A$, but $A^{c\#\#c} = A^{c\#cc\#c} = A^{\flat\flat}$. Lastly, we have proved that $\forall A \in \mathscr{P}(X)$, $A^{\#\#} \subseteq A^{\#c}$; applying A^c to this result we get $A^{c\#\#} \subseteq A^{c\#c}$ and by contraposition we have that $A^{c\#} \subseteq A^{c\#\#c} = A^{c\#cc\#c} = A^{\flat\flat}$. □

Proposition 103 *Let* $\mathfrak{P}(X, \#) = \langle \mathscr{P}(X), \cap, \cup, {}^c, {}^{\#}, \emptyset, X \rangle$ *be the quasi BB lattice structure based on the power set of X and generated by the preclusion space* $(X, \#)$. *Then,*

(1) the mapping

$$I : \mathscr{P}(X) \to \mathscr{P}(X), \quad A \to I(A) := A^{c\#\#c}$$

is a Tarski interior operator, *i.e.,*

$$(I0) \qquad X = I(X) \qquad (normalized = N principle)$$

$$(I1) \qquad I(A) \subseteq A \qquad (decreasing = T principle)$$

$$(I2) \qquad I(A \cap B) \subseteq I(A) \cap I(B) \qquad (submultiplicative = M principle)$$

$$(I3) \qquad I(A) = I(I(A)) \qquad (idempotent = S4 principle)$$

(2) the mapping

$$C : \mathscr{P}(X) \to \mathscr{P}(X), \quad A \to C(A) := A^{\#\#}$$

is a Tarski closure operator, *i.e.,*

$$(C0) \qquad \emptyset = C(\emptyset) \qquad (normalized = P principle)$$

$$(C1) \qquad A \subseteq C(A) \qquad (increasing = T principle)$$

$$(C2) \qquad C(A) \cup C(B) \subseteq C(A \cup B) \qquad (subadditive = M principle)$$

$$(C3) \qquad C(A) = C(C(A)) \qquad (idempotent = S4 principle)$$

According to the general discussion of Sect. 11, this Tarski interior (resp., closure) operation can be interpreted as algebraic realization of an S4-like modal necessity (resp., possibility), in this case defined on a Boolean algebra, satisfying the algebraic versions of the modal principles N (resp., P), T, M, and the peculiar 4, but not the Lewis required modal principle K. Setting $-A = X \setminus A = A^c$ *the two expected relationship hold:*

(Df□) $\quad I(A) = -C-(A)$ *and (Df◇)* $\quad C(A) = -I-(A)$.

Proof We prove the point (2) regarding the closure operation; the proof of the point (1) is obtained by duality.

(C0) For every subset $A \in \mathscr{P}(X)$, $\emptyset \subseteq A^{\#}$ and by weak involution (B1) and #-contraposition (B2) one gets $A \subseteq A^{\#\#} \subseteq \emptyset^{\#}$, which in the particular case of $A = X$ leads to $X \subseteq X^{\#\#} \subseteq \emptyset^{\#} \subseteq X$, i.e., $\emptyset^{\#} = X$. On the other hand, for every subset $A \in \mathscr{P}(X)$, $A^c \subseteq X$ and by the (wIR) $\emptyset \subseteq X^{\#} \subseteq X^c \subseteq A^{cc} = A$, which in the particular case of $A = \emptyset$ leads to $X^{\#} = \emptyset$. Hence, $\emptyset = (X)^{\#} = (\emptyset^{\#})^{\#}$, i.e., condition (C0) of closure.

(C1) Condition (B1) in point (3) of Proposition 97, $A \subseteq A^{\#\#}$, under the definition $C(A) := A^{\#\#}$ assumes the form $A \subseteq C(A)$ of condition (C1) of closure.

(C2) Applying to the inclusion $A \subseteq B$ the condition (B2) of the #-contraposition law we obtain $B^{\#} \subseteq A^{\#}$, and with another application of the #-contraposition we get $A^{\#\#} \subseteq B^{\#\#}$, i.e., the condition (C2) of closure $C(A) \subseteq C(B)$.

(C3) Finally applying Eq. (29) of Lemma 73, $C(C(A)) = (A^{\#\#\#})^{\#} = (A^{\#})^{\#} = C(A)$, which is the idempotence condition (C3) of closure. □

From the fact that according to (I1) one has that $I(A) \subseteq A$, as usual it is possible to single out the collection of all #-*open* sets defined as follows:

$$\mathscr{O}(X, \#) := \{O \subseteq X : O = I(O) = O^{c\#\#c}\}.$$

According to the general results of Proposition 46 the set $\mathscr{O}(X, \#)$ is a *pre topology of open sets*, in the sense that it contains the empty set and the whole universe and is closed with respect to arbitrary set theoretical union (but it is not closed with respect to finite intersection).

Since according to (C1) one has that $A \subseteq C(A)$, it is possible to introduce the collection of all #-*closed* sets defined as follows:

$$\mathscr{C}(X, \#) := \{K \subseteq X : C = C(K) = K^{\#\#}\}.$$

According to the general results of Proposition 45 the set $\mathscr{C}(X, \#)$ is a *pre topology of closed sets*, in the sense that it contains the empty set and the whole universe and is closed with respect to arbitrary set theoretical intersection (but it is not closed with respect to finite union).

The collection of all #-clopen sets is then:

$$\mathscr{C}\mathscr{O}(X, \#) = \mathscr{C}(X, \#) \cap \mathscr{O}(X, \#).$$

Both the empty set $\emptyset$ and the whole universe X are #-clopen. In the sequel, if there is no confusion, we simply speak of open, closed, and clopen sets instead of #-open, #-closed, and #-clopen sets.

In Proposition 97 we have seen that the weak interconnection rule (wIR-a), $\forall A \in \mathscr{P}(X)$, $A^{\#\#} \subseteq A^{\#c}$, characterizes the quasi BB lattice structure $\mathfrak{P}(X, \#)$ based on the power set $\mathscr{P}(X)$. Moreover Example 99 shows that the (strong) interconnection rule (IR), $\forall A \in \mathscr{P}(X)$, $A^{\#\#} = A^{\#c}$, in general does not hold in this kind of structure.

This makes it possible to introduce another form of interior-necessity (resp., closure-possibility) operation different from the one dealt with in Proposition 103 according to the following results.

Proposition 104 *Let* $\mathfrak{B}(X,\#) = \langle \mathscr{P}(X),\ \cap,\ \cup,\ ^c,\ ^\#,\ \emptyset,\ X \rangle$ *be the quasi BB lattice structure based on the power set of X and generated by the preclusion space* $(X,\#)$*. Then, introduced the mappings*

$$\nu : \mathscr{P}(X) \to \mathscr{O}(X), \quad \nu(A) := A^{c\#} \qquad \text{(B-necessity)} \tag{60a}$$

$$\mu : \mathscr{P}(X) \to \mathscr{C}(X), \quad \mu(A) := A^{\#c} \qquad \text{(B-possibility)} \tag{60b}$$

with $\nu(A) = -\mu - (A)$ *and* $\mu(A) = -\nu - (A)$*, the following hold:*

(BIC0)	$\nu(X) = X, \quad \mu(\emptyset) = \emptyset$	(N – P principles)
(BIC1)	$\nu(A) \subseteq A \subseteq \mu(A)$	(T and D principles)
(BI2)	$\nu(A \cap B) = \nu(A) \cap \nu(B)$	(M and C principles for necessity)
(BC2)	$\mu(A \cup B) = \mu(A) \cup \mu(B)$	(M and C principles for possibility)
(BCI3)	$A \subseteq \nu(\mu(A)), \quad \mu(\nu(A)) \subseteq A$	(B principles)

The B-operation ν *(resp., B-operation* μ*) can be interpreted as algebraic realization of a B-like modal necessity (resp., possibility) defined on a Boolean algebra, satisfying the algebraic versions of the modal principles N (resp., P), T, M and C, and the peculiar B, but not the principle K.*

In general, ν and μ do not satisfy the following principles:

$$\nu(\nu(A)) = \nu(A), \quad \mu(\mu(A)) = \mu A \qquad \text{(4 principles)}$$

$$\nu(A) = \nu(\mu(A)), \quad \mu(A) = \mu(\nu(a)) \qquad \text{(5 principles)}$$

Example 105 If $\# = \{(a,b),(b,a)\}$ is a preclusion relation on the universe $X = \{a,b,c,h,k\}$, then

$$\{a\}^\# = \{b\}, \quad \{b\}^\# = \{a\}, \quad \emptyset^\# = X, \quad \text{and} \quad A = \emptyset \ \text{ in all other cases.}$$

From $\mu(\{a\}) = \{a\}^{\#c} = \{b\}^c = \{a,c,h,k\}$ we get $\mu(\mu(\{a\})) = \mu(\{a,c,h,k\}) = \{a,c,h,k\}^{\#c} = \emptyset^c = X$. Therefore, $\mu(\mu(\{a\})) = X \neq \{a,c,h,k\} = \mu(\{a\})$.

On the other hand, $\nu(\mu(\{a\})) = \nu(\{a,c,h,k\}) = \{a,c,h,k\}^{c\#} = \{b\}^\# = \{a\}$. But $\nu(\{a\}) = \{a\}^{c\#} = \{b,c,h,k\}^\# = \emptyset$. Therefore, $\nu(\mu(\{a\})) = \{a\} \neq \emptyset = \nu(\{a\})$.

Made these considerations on the algebraic realizations of some modal logics (S4 and B), looking at the right side of Eq. (59) we have two possible candidates for defining a closure (upper rough approximation) of the set A, denoting them as $A^{\circledast} := A^{\#\#}$ and $A^* := A^{\#c}$, with induced interiors (lower rough approximations) of

A, denoted as $A^{\circledcirc} := A^{c\circledast c} = A^{\flat\flat}$ and $A^o := A^{c*c} = A^{c\#}$. With this notations the above chain of inclusions of (59) assumes now the form:

$$A^o \subseteq A^{\circledcirc} \subseteq A \subseteq A^{\circledast} \subseteq A^* \tag{61}$$

This allows one to consider as possible candidates for a lower–upper rough approximation of the subset A the two pairs

$$r_\#(A) := (A^{\circledcirc}, A^{\circledast}) \quad \text{and} \quad R_\#(A) := (A^o, A^*).$$

These rough approximations are constrained to the inclusions expressed by Eq. (61) in such a way that the approximation $r_\#(A)$ of A turns out to be better than the approximation $R_\#(A)$ of the same set. But there are some other interesting properties which differentiate these two rough approximations.

In general the rough approximation map $R_\#$ is not idempotent and so it does not satisfy both the conditions (RC2) of crispness and (RC3) of best approximation; in other words only the *weak* meta-theoretical principle (w-RMTP) discussed in Sect. 4 is satisfied in the case of this approximation. On the contrary, as consequence of the Tarski interior (lower)–closure (upper) approximation proved in Proposition 103, the rough approximation map $r_\#$ satisfies all the principles of coherence, crispness, and best approximation required by the meta-theoretical principle (RMTP) of Sect. 4.

It is worth noting that the *Axiomatic Foundations of Quantum Mechanics* in its so-called *sharp* (also *crisp*) version disregards the rough approximation approach and is rather interested in the structure of the *pre topological* family of #-closed subsets $\mathscr{C}(X, \#)$.

To this purpose let us state the following results.

Proposition 106 *Let* $(X, \#)$ *be a preclusion space. With respect to the ordering of set theoretical inclusion the structure based on the collection of all #-closed subsets:*

$$\mathfrak{C}(X, \#) := \langle \mathbb{C}(X, \#), \subseteq, {}^\#, \emptyset, X \rangle$$

is a complete lattice in which for any family $\{M_i\}$ *of #-closed sets from* $\mathbb{C}(X, \#)$

i) the greatest lower bound (g.l.b.), written $\wedge M_j$*, exists and turns out to be the set theoretical intersection*

$$\wedge M_j = \cap M_j$$

ii) the least upper bound (l.u.b.), written $\vee M_j$*, exists and it is*

$$\vee M_j = \cap \left\{ K \in \mathscr{C}(X, \#) : \cup M_j \subseteq K \right\}$$

This lattice is equipped with the mapping

$$^{\#} : \mathscr{C}(X,\#) \mapsto \mathscr{C}(X,\#),\ M \to M^{\#}$$

which is well posed since the application to $\mathscr{C}(X,\#)$ *of Eq.* (29) *in Lemma 73 can be written as* $M^{\#} = (M^{\#})^{\#\#}$*, i.e.,* $M^{\#} \in \mathscr{C}(X,\#)$*. This mapping is a standard orthocomplementation, i.e., we have that for arbitrary* $M, N \in \mathscr{C}(X,\#)$

(oc-1)	$M = M^{\#\#}$	*(double negation law)*
(oc-2)	$(M \vee N)^{\#} = M^{\#} \cap N^{\#}$	*(de Morgan law)*
(oc-3)	$M \cap M^{\#} = \emptyset$ *and* $M \vee M^{\#} = X$	*(noncontradiction and excluded middle laws)*

Summarizing, $\mathscr{C}(X,\#)$ *is a* pre topology of closed subsets *for the universe* X *and the pair* $(X, \mathscr{C}(X,\#))$ *is a* pre topological space *(as usual in order to have a real topological space* $\mathscr{C}(X,\#)$ *should also be closed with respect to the* finite *set theoretical union, but this does not generally happen).*

In general $\vee_j M_j$ *contains and does not coincide with the set theoretical union:*

$$\forall \left\{M_j \in \mathscr{C}(X,\#) : j \in J\right\},\quad \cup M_j \subseteq \vee M_j \quad \textit{with} \quad \begin{cases} \cup M_j \notin \mathscr{C}(X,\#) \\ \vee M_j \in \mathscr{C}(X,\#) \end{cases}$$

Similarly,

$$\forall M \in \mathscr{C}(X,\#), \quad M \cup M^{\perp} \subseteq M \vee M^{\#} = X.$$

Let us note that $\vee M_j = (\cup M_j)^{\#\#}$; moreover, as usual, the De Morgan law (oc-2) is equivalent to its dual De Morgan law (oc-2a) $(M \cap N)^{\#} = M^{\#} \vee N^{\#}$ which in its turn is equivalent to the contraposition law (oc-2b) $M \subseteq N$ implies $N^{\#} \subseteq M^{\#}$.

The family of all structures $\mathscr{C}(X,\#)$ can be divided into three subclasses:

(CM) $\mathscr{C}(X,\#)$ has a structure of Boolean algebra. That is one of the following equivalent identities is satisfied (see the hierarchy discussed in Sect. 2.1):

(Bl-1) $\forall M, N, Q \in \mathscr{C}(X,\#), \quad M \cap (N \vee Q) = (M \cap N) \vee (M \cap Q)$

(Bl-2) $\forall M, N, Q \in \mathscr{C}(X,\#), \quad M \vee (N \cap Q) = (M \vee N) \cap (M \vee Q).$

This corresponds to the case of *classical mechanics* since as stated from Mackey in [71] "the *logic of classical mechanics* is a Boolean algebra - the Boolean algebra of all [...] subsets of phase space" (i.e., a *classical logic*). Similarly, quoting [43]: "By a Boolean logic we mean a Boolean algebra of propositions in which the Boolean lattice operations of join, meet and orthocomplementation correspond to the logical operations of disjunction, conjunction, and negation respectively. As is well known the ordering may be interpreted as a logical *relation of implication* between propositions."

(QM) $\mathscr{C}(X, \#)$ has a structure of orthomodular lattice, i.e., one of the following equivalent identities is satisfied (see the hierarchy discussed in Sect. 2.1):

$$\text{(Ql-1)} \quad \forall M, N \in \mathscr{C}(X, \#), \ \ M \subseteq N \ \text{ implies } \ M \vee (M^{\perp} \cap N) = N$$

$$\text{(Ql-2)} \quad \forall M, N \in \mathscr{C}(X, \#), \ \ M \perp N \ \text{ implies } \ M^{\perp} \cap (M \vee N) = N$$

This corresponds to the case of *quantum mechanics* and $\mathscr{C}(X, \#)$ is the so-called *logic of quantum mechanics* (*quantum logic*).

(GQ) $\mathscr{C}(X, \#)$ is not Boolean. The quantum mechanics case is a subcase of this situation which corresponds to some *generalized quantum case*.

Example 107 Consider the preclusion relation $\# = \{(a, b), (b, a)\}$ on the universe $X = \{a, b, c, h, k\}$ that was discussed in Example 105. From

$$\{a\}^{\#} = \{b\}, \quad \{b\}^{\#} = \{a\}, \quad \emptyset^{\#} = X, \quad \text{and} \quad A = \emptyset \text{ in all other cases.}$$

we get that the corresponding collection of #-closed elements is the Boolean algebra $\mathscr{C}_B(X, \#) = \{\emptyset, \{a\}, \{b\}, X\}$, with the orthocomplementation expressed by the identities $\emptyset^{\#} = X$, $\{a\}^{\#} = \{b\}$, $\{b\}^{\#} = \{a\}$, and $X^{\#} = \emptyset$. The corresponding Hasse diagram is given by Fig. 18.

Note that the quasi Brouwer Boolean (BB) algebra $\langle \mathscr{P}(X, \#), \cap, \cup, {}^{c}, {}^{\#}, \emptyset, X \rangle$, with for instance $\{a\}^{c} = \{b, c, h, k\}$, induces the Boolean algebra $\langle \mathscr{C}_B(X, \#), \wedge = \cap, \vee, {}^{\#}, \emptyset, X \rangle$, with $\{a\}^{\#} = \{b\}$. Moreover, $\{a\} \wedge \{b\} = \{a\} \cap \{b\} = \emptyset$, whereas $\{a\} \vee \{b\} = X \supset \{a, b\} = \{a\} \cup \{b\}$. Furthermore, for instance, $\{a\}^{\#} = \{b\} \subset \{b, c, h, k\} = \{a\}^{c}$.

This is an example in which the Boolean behaviour of $\mathscr{P}(X, \#)$ it is in some manner inherited from $\mathscr{C}(X, \#)$. But this is not always the case, as the example shows.

Example 108 Let us consider the finite universe $X = \{a, b, c, d, h, k\}$ with the preclusion relation $\# = \{(a, b), (b, a), (c, d), (d, c)\}$. The corresponding collection

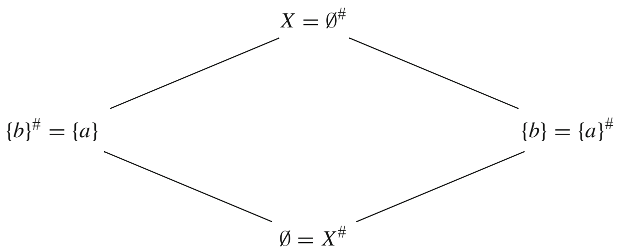


Fig. 18 Hasse diagram of the Boolean algebra $\mathscr{C}_B(X, \#)$ of #-closed sets

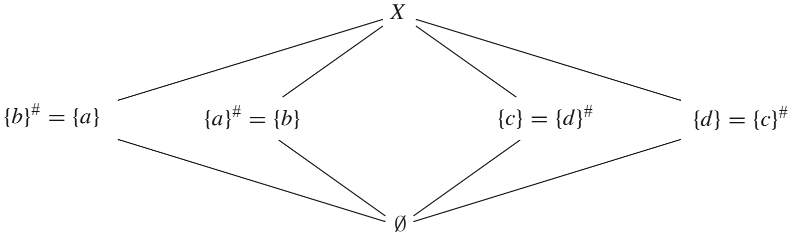


Fig. 19 Hasse diagram of the Quantum lattice $\mathscr{C}_Q(X, \#)$ of #-closed sets

of #-closed elements is the *Quantum lattice* $\mathscr{C}_Q(X, \#) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, X\}$, with the orthocomplementation expressed by the identities $\emptyset^\# = X$, $\{a\}^\# = \{b\}$, $\{b\}^\# = \{a\}$, $\{c\}^\# = \{d\}$, $\{d\}^\# = \{c\}$, and $X^\# = \emptyset$. The corresponding Hasse diagram is given by Fig. 19.

We have seen that the (GQ) case for definition is not a Boolean structure, but in any case it turns out to be the set theoretical union of all its Boolean subalgebras containing the two elements $\emptyset$ and X. Indeed, for any $M \in \mathscr{C}(X, \#)$ the subalgebra $\mathscr{B}(M) := \{\emptyset, M, M^\perp, X\}$ is trivially Boolean and their collection completely covers X. Obviously this is only part of the Boolean subalgebras of $\mathscr{C}(X, \#)$ because some more complicated structures of this kind may exist. Anyway, borrowing some terminology from differential topology we say that $\mathscr{C}(X, \#)$ is a *Boolean manifold* whose *local charts* are all the possible Boolean subalgebras $\mathscr{B}(X, \#)$ of $\mathscr{C}(X, \#)$.

As to the point QM relating to the quantum mechanical situation, let us report the following rather long statement of Hardgree from [54]:

> A physical theory may be characterized as consisting of a *semi-interpreted language* [...]. Every such a language is a triple $\langle \mathbb{E}, \mathscr{H}, v\rangle$, with the following identifications: $\mathbb{E}$ is a set of *elementary sentences* all having the form 'magnitude A has a value in Δ', or 'A lies in Δ' for short; synonymous with 'magnitude' are 'variable', 'observable'. $\mathscr{H}$ is a set of states or a *states spaces* which is the analogous of the set of 'possible worlds' in modal logic. Finally, v is a *satisfaction function* which assigns to each elementary sentence p in $\mathbb{E}$ the set $v(p)$ of states which satisfy (verify) p. I will generally write $\|p\|$ in place of $v(p)$, and I refer to semi-interpreted language simply as language.
>
> In addition to the elementary sentences of a language, which are purely syntactic in character, the set of *elementary propositions* of a language $\langle \mathbb{E}, \mathscr{H}, v\rangle$ is the image $v[\mathbb{E}]$ of $\mathbb{E}$ under v. In other words, if p is an elementary sentence in $\mathbb{E}$, then the corresponding elementary proposition is the set $v(p)(= |p|$ of states which satisfy p.
>
> The *propositional algebra* of a language is then the set $v(\mathbb{E})$ of elementary propositions together with the logical operations and relations peculiar to that language.
>
> It must be emphasized, however, that the logical operations of language $\mathscr{L}$ are not necessarily displayed in the syntax of $\mathscr{L}$. Indeed, the languages which concern us all have a uniformly flat and uninteresting syntactic structure: the sentences are all atomic, having the form 'A lies in Δ'. [...] In particular, the logical connectives are not syntactical defined as is customary in logical calculi, but are instead defined semantically.

Then Hardegree enters in the specific Hilbert space description.

In the case of the language $\mathscr{L}_{qm}$ of QM [...] the set of pure quantum states form a complex separable Hilbert space $\mathscr{H}$, and the elementary propositions are the subspaces (closed linear manifolds) of this space. With respect to the relation of set inclusion, the subspaces of $\mathscr{H}$ form a complete atomic orthomodular, but not distributive lattice. Inasmuch as the lattice of elementary quantum propositions is not distributive, the logical structure of QM is not Boolean. [...] First of all, since the meet (infimum) operation on the lattice of subspaces of $\mathscr{H}$ is set intersection, the quantum language is closed under conjunction. [...]

On the other hand, the quantum language is closed under neither exclusion negation nor exclusion disjunction. In place of these classical logical operations, $\mathscr{L}_{qm}$ has analogs we may call quantum negation and quantum disjunction, which are represented respectively by the *orthocomplement operation* and the *join (supremum) operation* on the lattice of subspaces of $\mathscr{H}$. The orthocomplement of a subspace M is the orthogonal complement $M^\perp$ [...]. Since a given vector x may fail to be an element of either M or $M^\perp$, the quantum negation differs from the classical exclusion negation, being instead a species of *choice negation*. A choice negation is characterized by the fact that a sentence p and its choice negation $p^\perp$ may *both* fail to be true at the same time.

However, the semantic unusualness of the quantum language is not so much a function of the quantum negation as it is a function of the quantum disjunction. Being definable, via De Morgan law, in term of conjunction and quantum (choice) negation, the quantum disjunction represents what may be called a *choice disjunction* and has certain remarkable semantic characteristics. [...] Semantically, this means that the quantum disjunction of p and q can be true at a state x while at the same time neither p nor q is true in x.

In particular the *formal analog* of the law of excluded middle is valid in concrete QL: for any subspace M, $M \vee M^\perp = \mathscr{H}$. In other words, every state satisfies the quantum disjunction $p \vee p^\perp$, but has we have already observed there are states which satisfies neither p nor its negation q [i.e., $\exists x$ s.t. $x \in M \vee M^\perp \setminus (M \cup M^\perp)$].
Thus, the quantum logical version of the 'law of excluded middle' has nothing like the traditional *semantic* significance of this law: it does not exclude a 'middle' possibility.

[Compare this statement with Eqs. (4) and (5).]

15 Tarski, Kuratowski, and Halmos Interior–Closures Operations Induced from Coverings

15.1 Covering of the Universe as Model of Tarski Interior with Induced Closure Operations

Let us consider a *covering* of the universe X, i.e., a family $\gamma = \{K_i \in \mathscr{P}(X) : i \in I\}$ of subsets of X (indexed by the index set I) which satisfies the two conditions:

(Co1) $\emptyset \in \gamma$,
(Co2) $X = \cup\{K_i : i \in I\}$.

To any subset A of the universe X it is possible to assign the subset (where we denote for simplicity by K the generic subset of the covering γ):

$$A^o := \cup\{K \in \gamma : K \subseteq A\} \tag{62a}$$

Then, it is easy to prove that the mapping $^o : \mathscr{P}(X) \to \mathscr{P}(X),\ A \to A^o$ is a *Tarski interior operator* on the power set $\mathscr{P}(X)$ in such a way that A^o is the *interior* of A. The corresponding family $\mathscr{O}_\gamma(X) := \{O \in \mathscr{P}(X) : O = O^o\}$, according to Proposition 32, satisfies conditions (PO1) and (POO2) of being a *pre-topological space* of *open subsets* of the power set $\mathscr{P}(X)$. Trivially, $\gamma \subseteq \mathscr{O}_\gamma(X)$, i.e., any element of the covering is open.

Hence, by Eq. (12) of Sect. 3, one immediately have that for every $A \in \mathscr{P}(X)$ the subset $A^* = ((A^c)^o)^c$ is the *closure* of A with the mapping $^* : \mathscr{P}(X) \to \mathscr{P}(X),\ A \to A^*$ which satisfies the condition of being a *Tarski closure operator* on $\mathscr{P}(X)$. Let us note that $A^* = ((A^c)^o)^c = [\cup\{K \in \gamma : K \subseteq A^c\}]^c = \cap\{K^c : K \in \gamma \text{ and } A \subseteq K^c\}$ and so, introducing the *anti-covering* of X as the collection $\gamma' := \{H \in \mathscr{P}(X) : \exists K \in \gamma \text{ s.t. } H = K^c\}$, we have that

$$A^* = \cap\left\{H \in \gamma' : A \subseteq H\right\} \tag{62b}$$

Thus, the family $\mathscr{C}_\gamma(X) = \{C \in \mathscr{P}(X) : C = C^*\}$, according to Proposition 22, turns out to be a *pre topological space* of *closed subsets* of the power set $\mathscr{P}(X)$. In this case we have the following formulation corresponding to (RAS-T) of Sect. 7.

(RAS-T)$_\gamma$ The *Tarski rough approximation space* induced by the covering γ is given by the structure $\mathfrak{R}_\gamma = \left\langle \mathscr{P}(X), \mathscr{O}_\gamma(X), \mathscr{C}_\gamma(X), r_\gamma \right\rangle$ with

(1) the power set $\mathscr{P}(X)$ as the collection of all approximable subsets of X, which is an atomic complete lattice whose atoms are the singletons $\{x\}$;
(2) $\mathscr{O}_\gamma(\Sigma)$ as the collection of all lower crisp subsets of X (pre topological space of open subsets);
(3) $\mathscr{C}_\gamma(X)$ as the collection of all upper crisp subsets of X (pre topological space of closed subsets);
(4) $r_\gamma : \mathscr{P}(X) \to \mathscr{O}_\gamma(X) \times \mathscr{C}_\gamma(X)$ as the rough approximation map assigning to any subset A of X the open-closed pair of subsets $r_\gamma(A) = (A^o, A^*)$, with $A^0 \subseteq A \subseteq A^*$, i.e., A is approximated from the bottom by the open set A^o and from the top by the closed set A^*, according to the meta-theoretical principle (RMTP) discussed at Sect. 4 consisting in the satisfaction of the conditions of roughness coherence (RC1), of crispness (RC2), and of best approximation by crisp sets (RC3).

If one want to stress that the interior A^o and the closure A^* can be considered as the lower rough approximation and the upper rough approximation of the subset A of the universe X, then it is possible to set them as $l_\gamma(A) = A^o$ for denoting the *lower* and $u_\gamma(A) = A^*$ for denoting the *upper approximation*. Adopting this notation and making use of the two families of pre-topological open $\mathscr{O}_\gamma(X)$ and closed subset $\mathscr{C}_\gamma(X)$ the above definitions of interior (lower approximation) and closure (upper approximation) of A can be equivalently formulated in the following way:

$$l_\gamma(A) = \cup\left\{O \in \mathscr{O}_\gamma(X) : O \subseteq A\right\} = A^o \tag{63a}$$

$$u_\gamma(A) = \cap\left\{C \in \mathscr{C}_\gamma(X) : A \subseteq C\right\} = A^* \tag{63b}$$

Example 109 Let $X = \{x_1, x_2, x_3\}$ be a (finite) universe of three points. Let us denote the non trivial subsets of X in the following way: $A = \{x_1\}$, $C = \{x_2\}$, $B = \{x_3\}$, $D = \{x_1, x_2\}$, $F = \{x_1, x_3\}$, and $E = \{x_2, x_3\}$. The Hasse diagram of the Boolean lattice $\mathscr{P}(\{x_1, x_2, x_3\})$ is depicted in Fig. 20.

Let us consider the covering $\gamma = \{\emptyset, D = \{x_1, x_2\}, F = \{x_2, x_3\}\}$, then the interior operator can be depicted by the Hasse diagram of Fig. 21.

The lattices of open and closed sets are depicted in the Hasse diagrams of Fig. 22.

The family of open sets $\mathscr{O}_\gamma(X) = \{\emptyset, D = \{x_1, x_2\}, E = \{x_2, x_3\}, X\}$ satisfies conditions (PO1) and (POO2), this last condition concerns the closure with respect to the arbitrary union operation $\cup$, but does not satisfies condition (PO3), i.e., the closure with respect to finite intersection operation $\cap$: indeed, $D \cap E = \{x_2\} \notin \mathscr{O}_\gamma(X)$. Thus,

- the family $\mathscr{O}_\gamma(X)$ is a *pre-topology* for the universe X, which is not a topology.

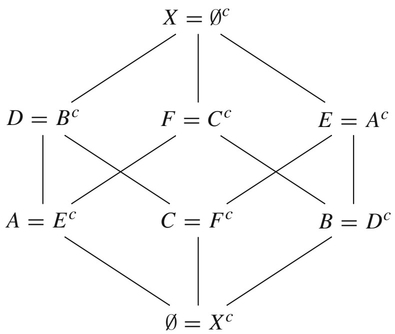


Fig. 20 The Boolean lattice $\mathscr{P}(\{x_1, x_2, x_3\})$

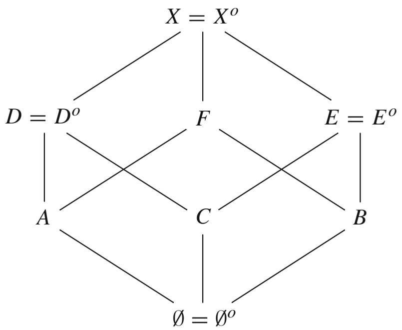


Fig. 21 Interior Boolean lattice induced from the covering γ of the Boolean lattice of Fig. 20

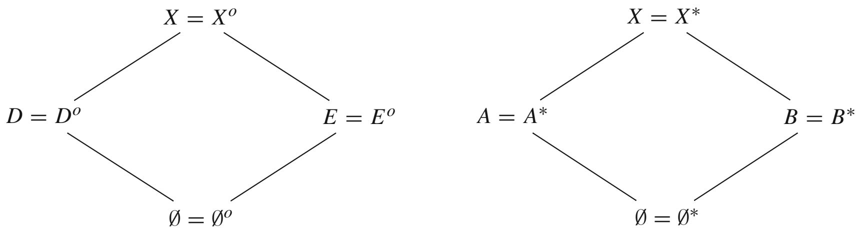


Fig. 22 Open (at the left side) and Closed (at the right side) subsets induced from the covering γ of the Boolean lattice of Fig. 20

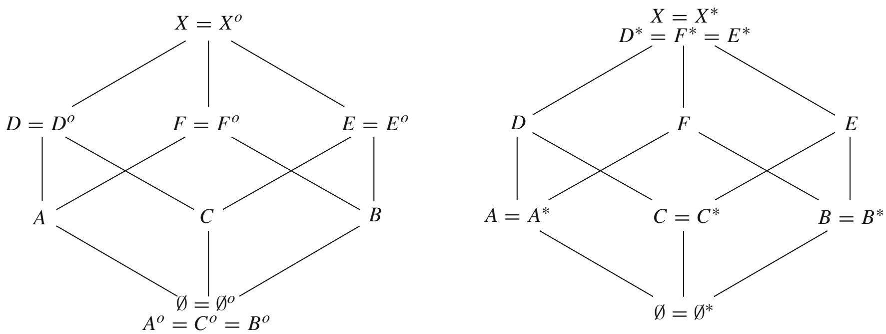


Fig. 23 Interior and closure Boolean lattices induced from the covering γ_1

Note that with respect to the lattice of open subsets at the left side of Fig. 22 the join of any two elements, in this case denoted by $\vee_o$, coincides with the original join of the corresponding elements of the lattice. This does not happen in the case of the meet: for instance $D \wedge_o E = \emptyset \subseteq C = D \cap E$.

In this example we have the case of a starting Boolean lattice with a covering which induces the two lattices of open and closed subsets which are both Boolean. This is not the general case, as the following example based on the same universe, but equipped with a different covering, shows.

Example 110 Let us consider the same universe $X = \{x_1, x_2, x_3\}$ of the previous Example 109, but with associated the different covering $\gamma_1 = \{\emptyset, D = \{x_1, x_2\}, F = \{x_2, x_3\}, E = \{x_3, x_1\}\}$. The interior and closure Boolean complete lattices induced by this covering are drawn in Fig. 23.

Note that the Hasse diagram at the right side is a concrete case of the abstract Tarski closure lattice which is not Kuratowski presented in Fig. 9 of Example 49. Instead of the points $a, b, c, \ldots$ of the abstract lattice Σ here one has the concrete subsets $A, B, C, \ldots$ of the power set $\mathscr{P}(\{x_1, x_2, x_3\})$.

The open and closed subsets induced from the covering γ_1 are very different and depicted in the Hasse diagrams of Fig. 24.

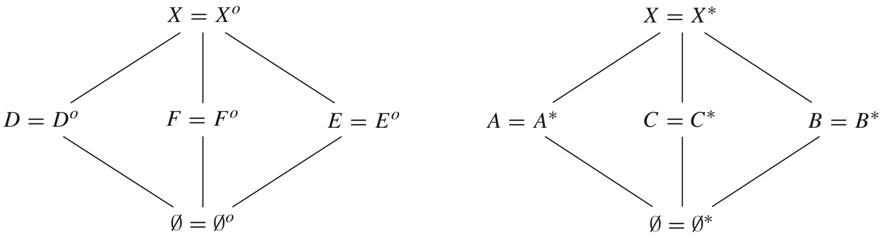


Fig. 24 Open (at the left side) and Closed (at the right side) subsets induced from the covering γ_1

Both the lattices of Fig. 24 are examples of the five-elements modular *non distributive* lattice M_5 [8, p. 13]. Also in this example,

- the family $\mathscr{O}_{\gamma_1}(X) = \{\emptyset, \{x_1, x_2\}, \{x_2, x_3\}, \{x_1, x_3\}, X\}$ is a pre topology of open sets which is *not* a topology, since for instance $\{x_1, x_2\} \cap \{x_2, x_3\} = \{x_2\} \notin \mathscr{O}_{\gamma_1}(X)$.
- Similarly, the family $\mathscr{C}_{\gamma_1}(X) = \{\emptyset, \{x_1\}, \{x_2\}, \{x_3\}, X\}$ is a pre topology of closed sets which is *not* a topology, since for instance $\{x_1\} \cup \{x_2\} = \{x_1, x_2\} \notin \mathscr{C}_{\gamma_1}(X)$.

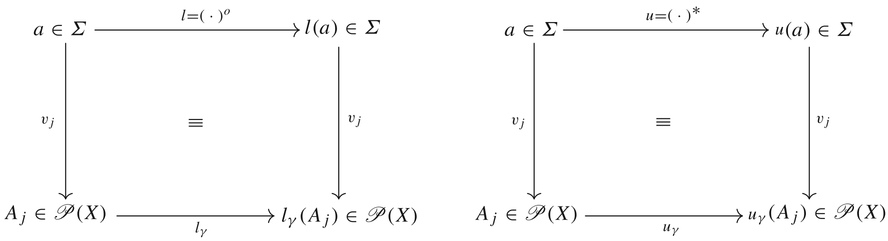
In the present covering case of the universe X, the pre-topological valuation of the necessity and possibility connectives from the Boolean lattice Σ are the following:

$$v_j(a^o) = v_j(a)^o = {A_j}^o = l_\gamma(A_j) \quad \text{and} \quad v_j(a^*) = v_j(a)^* = {A_j}^* = u_\gamma(A_j) \tag{64}$$

that is

$$v_j(l(a)) = l_\gamma(v_j(a)) \text{ for } \begin{cases} l(a) = a^o \\ v_j(A) = A_j \end{cases} \qquad v_j(u(a)) = u_\gamma(v_j(a)) \text{ for } \begin{cases} u(a) = a^* \\ v_j(A) = A_j \end{cases}$$

These relations can be summarized by the commutative diagrams:



15.2 Open (Global) Rough Approximation from Coverings

Borrowing some definitions from the partition case (discussed in Sect. 16), in the present covering context it is also possible to introduce the two *open approximations* of any subset A (in [15] called *global approximations*) as

$$l_\gamma^{(g)}(A) := \cup\{K \in \gamma : K \subseteq A\} = A^o \tag{65a}$$

$$u_\gamma^{(g)}(A) := \cup\{H \in \gamma : H \cap A \neq \emptyset\} \neq A^* \tag{65b}$$

In general $u_\gamma^{(g)}(A) \neq A^*$ owing to the particular fact that $u_\gamma^{(g)}(A)$ is an open subset, whereas A^* is a closed subset (in Example 110 one has that $u_\gamma^{(g)}(C) = X \neq C = C^*$). The global lower approximation map $l_\gamma^{(g)} : \mathscr{P}(X) \to \mathscr{O}_\gamma(X)$ is a Tarski interior operator since $\forall A \in \mathscr{P}(X)$, $l_\gamma^{(g)}(A) = A^o$, and so it satisfies the interior axioms (I0), (I1), (I2),and (I3) of Sect. 7. Differently, the global upper approximation map $u_\gamma^{(g)}(A) : \mathscr{P}(X) \to \mathscr{O}_\gamma(X)$, as discussed in [15], satisfies the conditions of Čech closure (C0), (C1), and (C2K), but not the idempotency condition (C3). Let us recall that the lack of idempotency corresponds to the satisfaction of the only weak meta-theoretical situation (w-RMTP) discussed in Sect. 4 according to which the only condition of roughness coherence (RC1) is satisfied, but neither the crispness condition (RC2) nor the best approximation condition (RC3) can be verified.

Example 111 Let us consider the four points universe $X = \{x_1, x_2, x_3, x_4\}$ equipped with the covering $\gamma = \{\emptyset, \{x_1, x_2\}, \{x_2, x_3\}, \{x_4\}\}$. Then, if we consider the subset $A = \{x_1\}$ of X we have that its global upper approximation is $u_\gamma^{(g)}(A) = \{x_1, x_2\}$, from which it follows $u_\gamma^{(g)}(u_\gamma^{(g)}(A)) = u_\gamma^{(g)}(\{x_1, x_2\}) = \{x_1, x_2, x_3\} \neq u_\gamma^{(g)}(A)$. Furthermore, $(u_\gamma^{(g)}(A))^* = \{x_1, x_2\}^{coc} = \{x_3, x_4\}^{oc} = \{x_1, x_2, x_3\} \neq u_\gamma^{(g)}(A)$, i.e., $u_\gamma^{(g)}(A) \notin \mathscr{C}(X)$ it is not closed, in other words it is not upper crisp (condition (Up2) of upper approximation is not satisfied).

Moreover, the following inclusions are always true:

$$\forall A \in \mathscr{P}(X), \quad l_\gamma^{(g)}(A) = A^o \subseteq A \subseteq A^* \subseteq u_\gamma^{(g)}(A) \tag{66}$$

If one defines the *global rough approximation* of A as the pair depending from the covering $r_\gamma^{(g)}(A) := (l_\gamma^{(g)}(A), u_\gamma^{(g)}(A)) \in \mathscr{O}_\gamma(X) \times \mathscr{O}_\gamma(X)$, then the open–closed standard rough approximation $r_\gamma(A) = (A^o, A^*) \in \mathscr{O}_\gamma(X) \times \mathscr{C}_\gamma(X)$, always depending from the covering, furnishes an approximation of A which is better with respect to the open–open global rough approximation $r_\gamma^{(g)}(A)$.

15.3 Topological Covering of the Universe as Model of Kuratowski Interior with Induced Closure Operations

In this subsection we consider a *topological covering* of the universe X, denoted by β, which is a covering satisfying besides the conditions (Co1) and (Co2) of Sect. 15.1 the further condition:

(Co3) for any pair B_i and B_j of subsets of the family β a collection $\{\hat{B}_k : k \in K\} \subseteq \beta$ of elements from the family β exists such that $B_i \cap B_j = \cup\{\hat{B}_k : k \in K\}$.

In topology a family β satisfying these three axioms (Co1), (Co2), and (Co3), is defined as an *open base* [62, pp. 46, 47], [102, p. 99], and any subset $B_i \in \beta$ is called *basic open set*, or from another point of view, *open granule*.

The *interior* of any subset $A \in \mathscr{P}(X)$ is defined as usual by the subset of the universe

$$A^o := \cup\{B_k \in \beta : B_k \subseteq A\}$$

obtaining in this case that the mapping $^o : \mathscr{P}(X) \to \mathscr{P}(X), \quad A \to A^o$ is a *Kuratowski interior operator* since axioms (C0), (C1), (C2K), and (C3), are satisfied. The open sets are then the particular subsets O of the universe X such that $O = O^o$. According to point (1) of Proposition 51, their collection $\mathscr{O}_\beta(X)$ satisfies the conditions (PO1a), (PO2), and the characteristic topological condition (PO3); in other words, it is a real *topology of open sets* for X.

Dually, for any subset $A \in \mathscr{P}(X)$ its closure is the subset $A^* = A^{coc}$ and in this case the mapping $\mathscr{P}(X) \to \mathscr{P}(X)$, $A \to A^*$ is a *Kuratowski closure operator*, and the collection $\mathscr{C}_\beta(X) := \{C \in \mathscr{P}(X) : C = C^*\}$ is a real *topology of closed sets*.

Without entering in deep details, let us note that in the case of a topological covering β it is possible to consider the induced *Kuratowski rough approximation space*, (RAS-K)$_\beta$, as the structure $\mathfrak{R}_\beta = \langle \mathscr{P}(X), \mathscr{O}_\beta(X), \mathscr{C}_\beta(X), r_\beta \rangle$ with $\mathscr{O}_\beta(X)$ (resp., $\mathscr{C}_\beta(X)$) the real topological space of open (resp., closed) subsets of X obtained by β, and the rough approximation of the approximable subset A given by the open–closed pair of subsets $r_\beta(A) = (l_\beta(A), u_\beta(A))$ where $l_\beta(A) = A^o$ (resp., $u_\beta(A)$) is the lower (resp., upper) topological approximation of A.

Example 112 Let $X = \{x_1, x_2, x_3\}$ be a finite three points universe. Let us consider the topological closure (open base) of this universe $\beta = \{\emptyset, \{x_1, x_2\}, \{x_2\}, \{x_2, x_3\}\}$. The interior and closure Boolean complete lattices induced by this topological covering (open base) are depicted by the following Fig. 25.
The open and closed subsets induced from the covering β are given by the Hasse diagrams of Fig. 26.

Different from the case presented by Example 109, whose open and closed lattices depicted by the Hasse diagrams of Fig. 22 represent the pre-topological spaces of open and closed subsets, in the present example

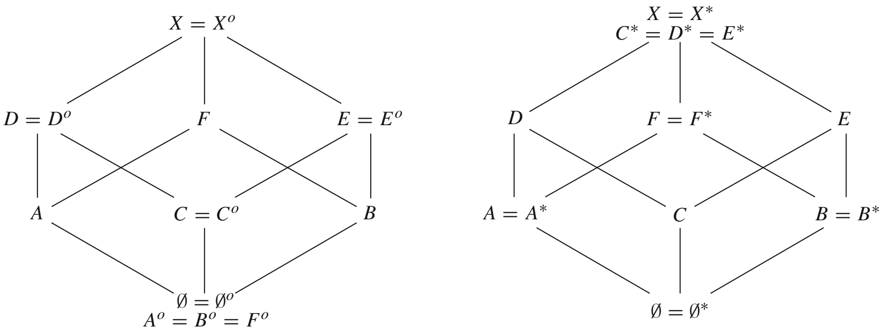


Fig. 25 Interior and closure Boolean lattices induced from the topological covering (open base) β

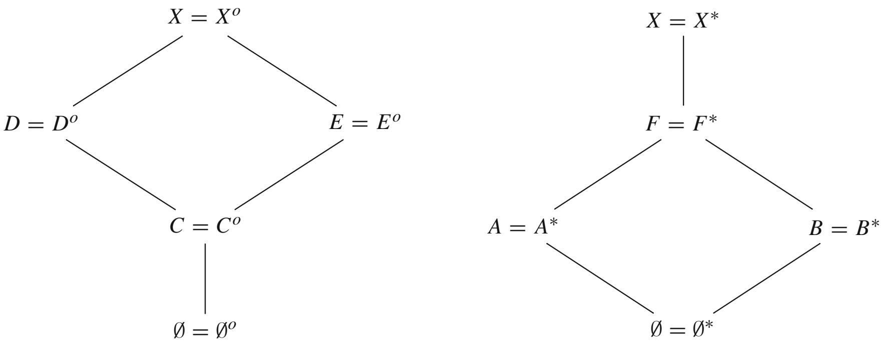


Fig. 26 Open (at the left side) and Closed (at the right side) subsets induced from the topological covering (open base) β

- we have real topological spaces of open and closed subsets. Indeed, $D \wedge_o E = C = D \cap E$ and $A \vee_c B = F = A \cup B$, with $C \in \mathscr{O}_\beta(X)$ open and $F \in \mathscr{C}_\beta(X)$ closed.

16 Partition Covering of the Universe as Model of Halmos Interior with Induced Closure Operations

A *partition covering*, simply *partition*, of the universe X is a covering π of X satisfying the usual conditions (Co1) and (Co2) plus the *disjointness condition*:

(D) for any pair $G_i \in \pi$ and $G_j \in \pi$ of different elements of the partition, $G_i \neq G_j$, it is $G_i \cap G_j = \emptyset$.

Trivially, condition (D) implies condition (Co3), i.e., a partition covering is a particular case of topological covering.

The usual approach to rough set theory as introduced by Pawlak [89, 90, 94] is formally (and essentially) based on a concrete *partition space*, that is a pair (X, π) consisting of a nonempty set X, the *universe* of discourse (with corresponding power set $\mathscr{P}(X)$ whose elements are the *approximable* sets), and a partition $\pi := \{G_i \in \mathscr{P}(X) : i \in I\}$ of X whose elements are the *elementary sets*.

The partition π can be characterized by the induced equivalence relation $\mathscr{R} \subseteq X \times X$, defined as

$$(x, y) \in \mathscr{R} \quad \text{iff} \quad \exists G \in \pi : x, y \in G. \tag{67}$$

In this case x, y are said to be *indistinguishable* with respect to $\mathscr{R}$ and the equivalence relation $\mathscr{R}$ is called an *indistinguishability* relation.

There is a close relationship between partitions and equivalence relations in the sense that any partition generates an equivalence relation and any equivalence relation generates a partition. In particular,

- if $G(x)$ is the elementary set (granule) from the partition π containing the point x, and if $[x] = \{y \in X : (x, y) \in \mathscr{R}\}$ is the equivalence class generated from x by the equivalence relation $\mathscr{R}$,
- then $G(x) = [x]$.

In this indistinguishability context the partition π is considered as the support of some knowledge available on the objects of the universe and so any equivalence class (i.e., elementary set) G is interpreted as a *granule* (or *atom*) of knowledge contained in (or supported by) π.

Of course, a partition space (X, π) generates a topological space whose open base is just the family π, i.e., $\beta = \pi$. Thus proceeding as in Sect. 15.3, for any subset A of X, it is possible to construct the Kuratowski interior and closure operators A^o and A^*, where A^o is the union of elementary sets from π. Hence, also open sets, defined according to the condition $E = E^o$, are the union of elementary sets from the partition π. But the set complement E^c of any open set is also a set theoretical union of elementary sets from π, i.e., an open set. This means that the open set $E = (E^c)^c$ as set theoretical complement of an open set is, from the topological point of view, also closed $E = E^*$, i.e., a *clopen* (simultaneously closed and open) set $E = E^o = E^*$. From this it follows that a partition gives rise to a topology of clopen, formally $\mathscr{O}_\pi(X) = \mathscr{C}_\pi(X)$, and so according to Proposition 66 the closure (resp., interior) operation $A \to A^*$ (resp., $A \to A^o$) is Halmos.

As discussed in Sect. 9, this unique family of clopen, also *crisp*, subsets will be denoted by $\mathscr{E}_\pi(X)(= \mathscr{O}_\pi(X) = \mathscr{C}_\pi(X))$. A *crisp* set from $\mathscr{E}_\pi(X)$ is then a clopen subset obtained as a set theoretical union of elementary subsets from the partition π. Indeed, according to Eqs. (62a) and (62b) applied to the present partition case, we have that

$$E \in \mathscr{E}_\pi(X) \quad \text{iff} \quad E = \cup\{G \in \pi : G \subseteq E\} = \cap\left\{G^c : G \in \pi \text{ and } E \subseteq G^c\right\} \tag{68}$$

From the topological point of view $\mathscr{E}_\pi(X)$ contains both the empty set and the whole space, moreover it is closed with respect to any arbitrary set theoretical union and intersection, i.e., it is an *Alexandroff topology* [1, 3]. In the rough set context, this "clopen" behavior of crisp sets, but not the Alexandroff topological relationship, as realized by Pawlak in his first contributions [90, 91, 93], and successively stressed for instance by Iwinski [59] and Skowron [103].

From another point of view, the collection of all these clopen sets forms a Boolean algebra $\langle \mathscr{E}_\pi(X), \cap, \cup, {}^c, \emptyset, X\rangle$ with respect to set theoretical intersection, union, and complementation. In particular, this Boolean algebra is atomic whose atoms are just the elementary sets G from the partition π.

The *lower approximation map* is defined as the correspondence $l_\pi : \mathscr{P}(X) \mapsto \mathscr{E}_\pi(X)$ associating with any subset A of X its lower approximation defined by the (clopen) crisp set

$$l_\pi(A) := \cup\{E \in \mathscr{E}_\pi(X) : E \subseteq A\} = \cup\{G \in \pi : G \subseteq A\}. \tag{69a}$$

Analogously, the *upper approximation map* is defined by the correspondence $u_\pi : \mathscr{P}(X) \mapsto \mathscr{E}_\pi(X)$ associating with any subset A of X its upper approximation defined by the (clopen) crisp set

$$u_\pi(A) := \cap\{E \in \mathscr{E}_\pi(X) : A \subseteq E\} = \cup\{G \in \pi : G \cap A \neq \emptyset\}. \tag{69b}$$

From these results it follows that

$(\text{RAS})_\pi$ The concrete *rough approximation space* generated by the partition π of the universe X consists of the structure $\mathfrak{R}_\pi := \langle \mathscr{P}(X), \mathscr{E}_\pi(X), r_\pi\rangle$ where:

(1) the set of *approximable* elements, is the Boolean atomic (complete) lattice $\mathscr{P}(X)$ of all subsets A of the universe X, whose atoms are the singletons $\{x\}$ for $x \in X$;
(2) the two sets of lower crisp elements and of upper crisp elements coincide with the Boolean atomic (complete) lattice $\mathscr{E}_\pi(X)$ of all *clopen* subsets of X, whose atoms are the equivalence classes (granules) G of the partition π;
(3) the *rough approximation map* is given by the correspondence $r_\pi : \mathscr{P}(X) \to \mathscr{E}_\pi(X) \times \mathscr{E}_\pi(X)$ associating with any subset A of the universe X the pair of clopen-clopen subsets $r_\pi(A) = (l_\pi(A), u_\pi(A))$, with $l_\pi(A)$ (resp., $u_\pi(A)$) given by the above Eq. (69a) (resp., Eq. (69b)).

The two relationships (69) related to the partition case, in which crisp (clopen) sets are involved, can be compared with the analogous two relationships (65) related to the covering case, in which lower crisp (open) and upper crisp (closed) sets are respectively involved. The main difference is that in the covering case, as generalization of the partition case, in the second equation (65b) there is a difference ($\neq$) whereas in (69b) the two approximations are equal.

Note that for any subset A, the universe turns out to be the set theoretical union of the mutually disjoint sets $X = l_\pi(A) \cup B_\pi(A) \cup E_\pi(A)$, where $B_\pi(A) := u_\pi(A) \setminus l_\pi(A)$ is the *boundary* and $E_\pi(A) := X \setminus u_\pi(A)$ is the *exterior* of A. The triplet of subsets $\pi(A) := \{l_\pi(A), B_\pi(A), E_\pi(A)\}$ is a new (*local*, i.e., depending from A) partition of X induced by A from the original partition π.

16.1 Covering from Incomplete Information System and Partition from Complete Information Systems

Let us give the notion of *Information System* (IS) formalized as a structure $\mathscr{IS} := \langle X, Att, Val, F\rangle$ where: (1) X is a nonempty (in general finite) set of *objects* (*situations*, *entities*, *states*), called the *universe* of the discourse; (2) Att is a nonempty (also in this case in general finite) set of *attributes* which can be evaluated on the objects from X; (3) Val is the set of all *possible values* that can be observed for an attribute a from Att in the case of an object x from X, and (4) F is a mapping, called the *information mapping*, associating with suitable pairs (x, a), consisting of an object $x \in X$ and an attribute $a \in Att$, the value $F(x, a)$ assumed by the object $x \in X$ relatively to the attribute $a \in Att$.

We have to distinguish two cases.

(1) The first case regards the *complete* IS, in the context of the Pawlak approach to rough sets introduced in the seminal papers [89, 90, 92], characterized by the condition that the mapping F is defined on the global collection $X \times Att$ of all object-attribute pairs (x, a). With $Val := \{F(x, a) : x \in X,\ a \in Att\}$ we mean the set of all possible values assumed by the mapping F (in other words, F is surjective). Note that an IS can be equivalently defined as a pair consisting of the universe X and a family $\{f_a : X \mapsto Val(a) \mid a \in Att\}$ of surjective *attribute mappings*, in which the set Att plays the role of set of indices; each f_a (for a fixed "index" $a \in Att$) is a function from the universe X onto its set of values $Val(a) := \{F(x, a) : x \in X\}$ defined by the law: $\forall x \in X,\ f_a(x) := F(x, a)$.

(2) The second case refers to the so-called *incomplete* IS, according to the seminal papers [104, 116, 122], in which the mapping F is partially defined on a subset $(X \times Att)_p$ of the set $X \times Att$ under the *consistency condition*:

(con) For any object x there must exist at least one attribute a_x such that $(x, a_x) \in (X \times Att)_p$, i.e., no object is redundant with respect to the IS in the sense that there must exist at least one information inside the IS which can be obtained about it (otherwise this object can be suppressed by the IS without any loss of information).

Similarly to the complete case, an incomplete information systems can also be formalized by a family of mappings indexed by the index set Att, where for every "index" a the (surjective) mapping $f_a : X(a) \mapsto Val(a)$ is defined on $X(a) := \{x \in X : (x, a) \in (X \times Att)_p\}$ by the law that associates with any object $x \in X(a)$ the value $f_a(x) := F(x, a)$ (where in this incomplete case $Val(a) := \{F(x, a) : x \in X(a)\}$).

Sometimes we use the *null* symbol $*$ to denote the fact that the value possessed by an object x with respect to the attribute a is unknown: formally we set $F(x, a) = *$ when $(x, a) \notin (X \times Att)_p$, or equivalently $f_a(x) = *$ when $x \notin X(a)$. For a given attribute $a \in Att$ let us denote $val(a) := Val(a) \cup \{*\}$ and so f_a can now be considered as the surjective application $f_a : X \to val(a)$, defined on the whole space of objects X.

Furthermore, for any fixed object $x \in X$ we can define its *attribute domain of definition* $Att(x) := \left\{a \in Att : (x, a) \in (X \times Att)_p\right\}$ as the collection of all attributes with respect to which the information function can be applied (furnishes information) relatively to the object x.

16.2 The Case of Incomplete IS

Let us consider an incomplete IS and a family $\mathscr{A}$ of attributes from Att ($\mathscr{A} \subseteq Att$), then a binary relation of *$\mathscr{A}$-indiscernibility* $I_{\mathscr{A}} \subseteq X \times X$ about objects from X can be introduced according to:

(ind) let $x, y \in X$, then

$$(x, y) \in I_{\mathscr{A}} \quad \text{iff} \quad \forall a \in \mathscr{A}, (f_a(x) = f_a(y) \text{ or } f_a(x) = * \text{ or } f_a(y) = *).$$

This binary relation $I_{\mathscr{A}}$ on the set of objects X is *reflexive* and *symmetric*, but in general non transitive, called *similarity* relation (after Poincaré [95]) in the context of Kripke semantics of modal logic (see [32, p. 83]) or *tolerance* relation (after Zeeman [123]) in the context of incomplete IS [97]. From the modal logic viewpoint, making reference to the author of the original semantical approach, i.e., S. Kripke [64], "a *normal model structure* (nms) is an ordered [pair] $(X, \mathscr{R})$, where X is a non empty set, and $\mathscr{R}$ a reflexive relation defined on X. If $\mathscr{R}$ is transitive, we call the nms a S4 *model structure*; if $\mathscr{R}$ is symmetric, we call it a BROUWERsche *model structure*; if $\mathscr{R}$ is an equivalence relation, we call it a S5 *model structure*."

For any fixed object x of the universe, let us construct the (elementary) *open* (or *indiscernibility*) *granule* generated by this object $K_{\mathscr{A}}(x) := \{y \in X : (x, y) \in I_{\mathscr{A}}\}$, whose collection $\gamma_{\mathscr{A}} = \{K_{\mathscr{A}}(x) : x \in X\} \cup \{\emptyset\}$ is trivially a covering of X (as a consequence of the reflexivity condition it is $x \in K_{\mathscr{A}}(x)$).

The following result expresses the *$\mathscr{A}$-discernibility* binary relation $D_{\mathscr{A}} \subseteq X \times X$ as negation of indiscernibility: $D_{\mathscr{A}} = \neg I_{\mathscr{A}}$. Formally, $(x, y) \in D_{\mathscr{A}}$ iff $(x, y) \notin I_{\mathscr{A}}$, which is *irreflexive* and *symmetric*.

Proposition 113 *Let $x, y \in X$, then*

$$(x, y) \in D_{\mathscr{A}} \quad \textit{iff} \quad \exists a_0 \in \mathscr{A} : f_{a_0}(x) \neq f_{a_0}(y) \textit{ with } f_{a_0}(x) \neq * \textit{ and } f_{a_0}(y) \neq *.$$

Proof $(x, y) \notin I_{\mathscr{A}}$ iff $\neg[\forall a \in \mathscr{A}, (f_a(x) = f_a(y)$ or $f_a(x) = *$ or $f_a(y) = *)]$ iff $\exists a_0 \in \mathscr{A}$: $(f_{a_0}(x) \neq f_{a_0}(y)$ and $f_{a_0}(x) \neq *$ and $f_{a_0}(y) \neq *)$. □

Example 114 Let the universe be $X = \{x, y, z\}$ and let us consider the incomplete information table based on it and the set of attributes $Att = \{a, b, c\}$:

	a	b	c
x	yes	0	$*$
y	yes	$*$	h
z	$*$	1	$*$

Let us now construct the similarity granules with respect to the whole set of attributes $\mathscr{A} = Att$, omitting for simplicity the subscript Att to each of them.

Case x: Besides the trivial case $(x, x) \in I$, i.e., $x \in K(x)$, we have the two cases: (1) For the object $y \in X$, $f_a(x) = f_a(y)$, $f_b(y) = *$, $f_c(x) = *$ and so any attribute satisfies the above condition (ind) concluding that $(x, y) \in I$, i.e., $y \in K(x)$. (2) For the object $z \in X$, $f_b(x) = 0 \neq 1 = f_b(z)$, and so $(x, z) \notin I$, i.e., $z \notin K(x)$.

Case y: Besides the trivial case $(y, y) \in I$, i.e., $y \in K(y)$, we have the two cases: (1) For the object $x \in X$, $f_a(y) = f_a(x)$, $f_b(y) = *$, $f_c(x) = *$ concluding that $(y, x) \in I$, i.e., $x \in K(y)$. (2) For the object $z \in X$, $f_a(z) = *$, $f_b(y) = *$, $f_c(z) = *$ concluding that $(y, z) \in I$, i.e., $z \in K(y)$.

Case z: Also in this final case, besides the trivial $(z, z) \in I$, i.e., $z \in K(z)$, we have the two cases: (1) $f_b(z) = 1 \neq 0 = f_b(x)$ and so $x \notin K(z)$. (2) For the object $y \in X$, $f_a(z) = *$, $f_b(y) = *$, $f_c(z) = *$ concluding that $(z, y) \in I$, i.e., $y \in K(z)$.

Summarizing, the similarity granules with respect to the whole set of attributes are $K(x) = \{x, y\}$, $K(y) = \{x, y, z\} = X$, $K(z) = \{z, y\}$, generating the covering $\gamma = \{\emptyset, \{x, y\}, \{y, z\}, X\}$ of X, which is the pre topology of open sets discussed in Example 109 (see also the left side of Fig. 22).

If for any subset $\mathscr{A} \subseteq Att$ of attributes, for any fixed object $x \in X$ one defines

$$\mathscr{A}(x) := \left\{a \in \mathscr{A} \subseteq Att : (x, a) \in (X \times Att)_p\right\},$$

Table 2 Flats incomplete information system

Flat	Price	Rooms	Down-Town	Furniture
x_1	High	2	Yes	*
x_2	High	*	Yes	No
x_3	*	2	Yes	No
x_4	Low	*	No	No
x_5	Low	1	*	No
x_6	*	1	Yes	*

then the following is an equivalent way to define the $\mathscr{A}$-indiscernibility relation in incomplete IS:

(ind)

$$\text{let } x, y \in X, \text{ then } (x, y) \in I_{\mathscr{A}} \text{ iff } [\forall a \in \mathscr{A}(x) \cap \mathscr{A}(y),\ f_a(x) = f_a(y)]$$

$$\text{or } [\forall a \in \mathscr{A} \setminus (\mathscr{A}(x) \cap \mathscr{A}(y)), \text{ either } f_a(x) = * \text{ or } f_a(y) = *].$$

Example 115 In Example 114 it is interesting the case of the pair (y, z), where we have that $\mathscr{A}(y) = \{a, c\}$ and $\mathscr{A}(x) = \{b\}$. Thus, $\mathscr{A}(y) \cap \mathscr{A}(z) = \emptyset$ and so there is no attribute to test the first part of (ind) whereas for all the attributes of $Att \setminus (\mathscr{A}(y) \cap \mathscr{A}(z)) = Att$ it is $f_a(z) = *$, $f_b(y) = *$, and $f_c(z) = *$.

Example 116 Table 2 gives the representation of an incomplete IS based on a set X of 6 objects describing flats and involving 4 possible attributes about flats.

Let us set P=Price, R=Rooms, DT=Down-Town, and F=Furniture, we have the following attribute domains of definition associated with any flat:

$$Att(x_1) = \{P, R, DT\} \qquad Att(x_4) = \{P, DT, F\}$$
$$Att(x_2) = \{P, DT, F\} \qquad Att(x_5) = \{P, R, F\}$$
$$Att(x_3) = \{R, DT, F\} \qquad Att(x_6) = \{R, DT\}$$

If one considers the set of all attributes (i.e., $\mathscr{A} = Att$) and the similarity relation of indiscernibility (ind) just introduced, fixing for instance the flat x_1, in order to have the granule generated by it we must consider the following cases relative to the other flats

Case x_2. $Att(x_1) \cap Att(x_2) = \{P, DT\}$, with respect to which $f_P(x_1) = f_P(x_2)$ and $f_{DT}(x_1) = f_{DT}(x_2)$; moreover, $f_F(x_1) = *$ and $f_R(x_2) = *$. Concluding that condition (ind) is satisfied, i.e., $(x_1, x_2) \in I_{\mathscr{A}}$.

Case x_3. $Att(x_1) \cap Att(x_3) = \{R, DT\}$, with respect to which $f_R(x_1) = f_R(x_3)$ and $f_{DT}(x_1) = f_{DT}(x_3)$; moreover, $f_P(x_3) = *$ and $f_F(x_1) = *$. Concluding that also in this case $(x_1, x_3) \in I_{\mathscr{A}}$.

Case x_4. $Att(x_1) \cap Att(x_4) = \{P, DT\}$, with respect to which $\exists P$ such that $f_P(x_1) \neq f_P(x_4)$ and so $(x_1, x_4) \notin I_{\mathscr{A}}$.

Concluding this analysis for all the remaining cases x_5 and x_6 and considering all the other flats as fixed, we obtain the following open granules (where we omit for simplicity the subscript $\mathscr{A}$):

$$K(x_1) = \{x_1, x_2, x_3\} \quad K(x_2) = \{x_1, x_2, x_3, x_6\} \quad K(x_3) = \{x_1, x_2, x_3\}$$
$$K(x_4) = \{x_4, x_5\} \quad K(x_5) = \{x_4, x_5, x_6\} \quad K(x_6) = \{x_2, x_5, x_6\}$$

whose collection (plus the empty set) γ constitutes a covering of X.

16.2.1 The Incomplete IS Similarity Approach to Covering RAS

Now, we can apply to the covering $\gamma_{\mathscr{A}}$ the results of Sect. 15.1 corresponding to the construction of a rough approximation space (RAS) based on the power set $\mathscr{P}(X)$ of the universe X equipped with the Tarski interior–closure pair of operators, and consequent rough approximation map. That is, to any approximable subset A of the universe X we can assign its interior A^o and closure $A^* = ((A^c)^o)^c$ expressed as

$$A^o := \cup\{K_{\mathscr{A}}(x) \in \gamma_{\mathscr{A}} : K_{\mathscr{A}}(x) \subseteq A\} \tag{70a}$$

$$A^* := \cap\left\{(K_{\mathscr{A}}(x))^c : K_{\mathscr{A}}(x) \in \gamma_{\mathscr{A}} \text{ and } A \subseteq (K_{\mathscr{A}}(x))^c\right\} \tag{70b}$$

In this way one obtains a Tarski interior operator $l_{\mathscr{A}} : \mathscr{P}(X) \to \mathscr{P}(X),\ A \to l_{\mathscr{A}}(A) := A^o$ and a Tarski closure operator $u_{\mathscr{A}} : \mathscr{P}(X) \to \mathscr{P}(X),\ A \to u_{\mathscr{A}}(A) := A^*$.

The collection $\mathscr{O}_{\mathscr{A}}(X)$ of all *open* sets, i.e., subsets $O \in \mathscr{P}(X)$ such that $O = O^o$, and the collection $\mathscr{C}_{\mathscr{A}}(X)$ of all *closed* sets, i.e., subsets $C \in \mathscr{P}(X)$ such that $C = C^*$, constitute the *pre-topologies* of open and closed sets for the universe X, respectively, as consequence of the facts that $\mathscr{O}_{\mathscr{A}}(X)$ is closed with respect to arbitrary union and $\mathscr{C}_{\mathscr{A}}(X)$ with respect to arbitrary intersection.

Example 117 In the incomplete IS discussed in Example 116 given by Table 2, whose indiscernibility relation $I_{\mathscr{A}}$ is obtained by the class $\mathscr{A} = Att$ of all attributes, the corresponding family of open sets is

$$\mathscr{O}_{Att}(X) = \{\ \emptyset,\ \{x_4, x_5\},\ \{x_1, x_2, x_3\},\ \{x_2, x_5, x_6\},\ \{x_4, x_5, x_6\},\ \{x_1, x_2, x_3, x_6\},$$
$$\{x_2, x_4, x_5, x_6\},\ \{x_1, x_2, x_3, x_4, x_5\},\ \{x_1, x_2, x_3, x_5, x_6\},\ X\}$$

The corresponding family of closed sets is

$$\mathscr{C}_{Att}(X) \equiv \{\ \emptyset,\ \{x_4\},\ \{x_6\},\ \{x_4, x_5\},\ \{x_1, x_3\},\ \{x_1, x_2, x_3\},\ \{x_4, x_5, x_6\}$$
$$\{x_1, x_3, x_4\},\ \{x_1, x_2, x_3, x_6\},\ X\}$$

Both these two families are pre topologies, of open sets the first ($\{x_4, x_5\} \cap \{x_2, x_5, x_6\} = \{x_5\}$ which is not open) and of closed sets ($\{x_4\} \cup \{x_6\} = \{x_4, x_6\}$ which is not closed) the second.

Therefore, the concrete (in the sense that it is based on the "concrete" universe X) rough approximation space (RAS) is the structure $\langle \mathscr{P}(X), \mathscr{O}(X), \mathscr{C}_\gamma(X), r \rangle$ where $\mathscr{P}(X)$ is the collection of all *approximable* sets, the family $\mathscr{O}(X)$ of open sets is the collection of all *lower crisp* subsets, the family $\mathscr{C}_\gamma(X)$ of closed sets is the collection of all *upper crisp* subsets, and finally for any approximable subset A of X the open–closed pair $r_{\mathscr{A}}(A) = (l_{\mathscr{A}}(A), u_{\mathscr{A}}(A)) = (A^o, A^*) \in \mathscr{O}(X) \times \mathscr{C}_\gamma(X)$ is the rough approximation of A relatively to the family $\mathscr{A}$ of attributes satisfying the roughness meta-theoretical principle (RMTP) of roughness coherence, crispness, and best approximation by crisp elements.

16.2.2 The Incomplete IS Similarity Approach to Global RAS

In Sect. 15.2 we have seen as another rough approximation space (RAS), called global, can be obtained by a closure of the universe. Precisely, in this global case the open lower and upper approximations of A induced by the covering $\gamma_{\mathscr{A}}$ according to Eq. (65) are now given by

$$l_{\mathscr{A}}^{(g)}(A) := \cup\{K_{\mathscr{A}}(x) \in \gamma_{\mathscr{A}} : K_{\mathscr{A}}(x) \subseteq A\} \tag{71a}$$

$$u_{\mathscr{A}}^{(g)}(A) := \cup\{K_{\mathscr{A}}(x) \in \gamma_{\mathscr{A}} : K_{\mathscr{A}}(x) \cap A \neq \emptyset\} \tag{71b}$$

where $l_{\mathscr{A}}^{(g)}(A) = A^o$, i.e., it is just the Tarski interior of Eq. (70a) whereas $A^* \subseteq u_{\mathscr{A}}^{(g)}(A)$. Therefore $l_{\mathscr{A}}^{(g)} : \mathscr{P}(X) \to \mathscr{O}(X)$ is a Tarski lower approximation map and $u_{\mathscr{A}}^{(g)} : \mathscr{P}(X) \to \mathscr{O}(X)$ is a Čech upper approximation map, producing the global rough approximation map $r_\gamma^{(g)} : \mathscr{P}(X) \to \mathscr{O}(X) \times \mathscr{O}(X)$ associating with any set A of the universe its open-open global rough approximation $r_\gamma^{(g)}(A) = (A^o, u^{(g)}(A))$ which according to (66) furnishes a worst approximation of A relatively to the Tarski open-closed one $r(A) = (A^o, A^*)$. Furthermore, the global rough approximation space (RAS) satisfies the weak roughness meta-theoretical principle (w-RMTP) consisting in the unique condition of roughness coherence (RC1).

16.2.3 The Incomplete IS Similarity Approach to Local RAS

Let us stress that there is another possibility, called "*local*" in [15], to define a pair of lower and upper approximations of the subset A of the universe X from an incomplete information system (IS), which is typically linked to the structure of incomplete IS, and not to the more general structure of covering. formally defined as the following:

$$l^{(l)}_{\mathscr{A}}(A) := \{x \in X : K_{\mathscr{A}}(x) \subseteq A\} \tag{72a}$$

$$u^{(l)}_{\mathscr{A}}(A) := \{x \in X : K_{\mathscr{A}}(x) \cap A \neq \emptyset\} \tag{72b}$$

The local upper (resp. lower) approximation map $u^{(l)}_{\mathscr{A}}$ (resp., $l^{(l)}_{\mathscr{A}}$) is a Čech closure (resp., interior) operator in the sense that it satisfies conditions (C0) the element 0 is closed, (C1) increasing, and (C2K) additive condition of closure (resp., (I0) the element 1 is open, (I1) decreasing, and (I2K) multiplicative condition of interior). In general, idempotency does not hold and so both the lower and upper approximation maps $l^{(l)}_{\mathscr{A}}$ and $u^{(l)}_{\mathscr{A}}$, respectively, satisfy the weak rough meta-theoretical principle (w-RMTP).

Trivially, these give rise to a different approximation of the generic approximable subset A of the universe X according to the inequalities:

$$l^{(l)}_{\mathscr{A}}(A) \subseteq l_{\mathscr{A}}(A) \subseteq A \subseteq u_{\mathscr{A}}(A) \subseteq u^{(l)}_{\mathscr{A}}(A) \tag{73}$$

From this it follows that the "global" rough approximation $r_{\mathscr{A}}(A) := (l_{\mathscr{A}}(A),\ u_{\mathscr{A}}(A))$ is better than the "local" one $r^{(l)}_{\mathscr{A}}(A) := (l^{(l)}_{\mathscr{A}}(A),\ u^{(l)}_{\mathscr{A}}(A))$.

Example 118 In the IS described by Table 2 of Example 116 let us consider the subset of flats $A = \{x_1, x_2, x_3, x_4\}$. The two corresponding Tarski lower (interior) and upper (closure) approximations are

$$l(\{x_1, x_2, x_3, x_4\}) = \{x_1, x_2, x_3\} \quad \text{and} \quad u(\{x_1, x_2, x_3, x_4\}) = X$$

The local lower and upper approximations of the same set A are $l^{(l)}(A) = \{x_1, x_3\}$, with the strict inclusion $l^{(l)}(A) \subset l(A)$, and $u^{(l)}(A) = X = u(A)$. In this way, the open–closed rough approximation $r(A) = (\{x_1, x_2, x_3\},\ X)$ is evidently better than the local rough approximation $r^{(l)}(A) = (\{x_1, x_3\},\ X)$.

On the other hand, with respect to the set $B = \{x_1, x_4\}$, the lower approximations are $l(B) = l^{(l)}(B) = \emptyset$, whereas the upper approximations are the closed one $u(B) = \{x_1, x_3, x_4\}$, the local one $u^{(l)}(B) = \{x_1, x_2, x_3, x_4, x_5\}$, and the third global one $u^{(g)}(B) = X$, with the strict inclusions $u(B) \subset u^{(l)}(B) \subset u^{(g)}(B)$. In this way, also in this case the open–closed rough approximation $r(B) = (\emptyset,\ \{x_1, x_3, x_4\})$ is better than the local one $r^{(l)}(B) = (\emptyset,\ \{x_1, x_2, x_3, x_4, x_5\})$, which in its turn is better than the global one $r^{(g)}_{\gamma}(B) = (\emptyset,\ X)$.

The following two subsets of the universe show an intriguing relationship between local and global approximations. Indeed, let us consider the set $C = \{x_2, x_4, x_5, x_6\}$ then $l^{(l)}(C) = \{x_4, x_5, x_6\}$, $l^{(g)}(C) = \{x_2, x_4, x_5, x_6\}$ and $u^{(l)}(C) = u^{(g)}(C) = X$. So we have the chain of inclusions

$$l^{(l)}(C) \subset l^{(g)}(C) = C \subset u^{(g)}(C) = u^{(l)}(C)$$

On the other hand, in the case of the set $D = \{x_1, x_2, x_3\}$ we have that $l^{(l)}(D) = l^{(g)}(D) = D$, whereas $u^{(l)}(D) = \{x_1, x_2, x_3, x_6\}$, $u^{(g)}(D) = \{x_1, x_2, x_3, x_5, x_6\}$ and so in this case the chain of inclusions is

$$l^{(l)}(D) = l^{(g)}(D) = D \subset u^{(l)}(D) \subset u^{(u)}(D)$$

The last two chains of inclusions verified in the previous example are particular cases of the following general result.

Proposition 119 *The following chain of inclusions is true for any subset H of the universe X in the case of an incomplete IS for any subfamily $\mathscr{A}$ of attributes:*

$$l_{\mathscr{A}}^{(l)}(H) \subseteq l_{\mathscr{A}}^{(g)}(H) \subseteq H \subseteq u_{\mathscr{A}}^{(l)}(H) \subset u_{\mathscr{A}}^{(u)}(H) \tag{74}$$

Proof As usual, for the sake of simplicity, let us omit the subscript $\mathscr{A}$ in the present proof. Let $x \in l^{(l)}(H)$, then from definition (72) it must be $K(x) \subseteq H$. This means, by definition (71), that a fortiori $K(x) \subseteq l^{(g)}(H)$, and since $x \in K(x)$, we have that $x \in l^{(g)}(H)$.

Conversely, let $x \in u^{(l)}(H)$, then the similarity granule $K(x)$, by (72), is such that $K(x) \cap H \neq \emptyset$. But from (71) it follows that $K(x) \subseteq u^{(g)}(H)$, and since $x \in K(x)$, we have that $x \in u^{(g)}(H)$. □

16.3 The Case of Complete IS

In the case of complete IS, as particular cases of incomplete IS, the above indiscernibility relation on objects induced by a family $\mathscr{A}$ of attributes reduces to the following binary relation which we denote with the symbol $R_{\mathscr{A}}$ in order to distinguish this complete IS case from the incomplete one:

(c-ind) let $x, y \in X$, then $(x, y) \in R_{\mathscr{A}}$ iff for every attribute $a \in \mathscr{A}$ one has $f_a(x) = f_a(y)$.

This is an equivalence (reflexive, symmetric, and transitive) relation on X, in this case called *indistinguishability* relation in order to distinguish the complete IS case from the incomplete one, such that the family $\mathscr{O}_{\mathscr{A}}(X)$ of open sets and the family $\mathscr{C}_{\mathscr{A}}(X)$ of closed sets coincide. Once denoted by $\mathscr{E}_{\mathscr{A}}(X) := \mathscr{O}_{\mathscr{A}}(X) = \mathscr{C}_{\mathscr{A}}(X)$ the common collection of *clopen* sets, the following holds:

- In the context of a complete IS, for any family of attributes $\mathscr{A} \subseteq Att$, it is $l_{\mathscr{A}}(A) = l_{\mathscr{A}}^{(l)}(A)$ and $u_{\mathscr{A}}(A) = u_{\mathscr{A}}^{(l)}(A)$. Moreover, $l_{\mathscr{A}}(A)$ defines a Halmos interior operator and $u_{\mathscr{A}}(A)$ defines a Halmos closure operator according to Sect. 16.

Summarizing,

- incomplete IS are models of the Tarski interior–closure operators by coverings, whereas complete IS are models of the Halmos interior–closure operators by partitions in the context of concrete rough approximation spaces.

Part IV: Algebraic Methods for Many Valued Logics. Variations of Halmos Closure: Stone and Łukasiewicz Closures

In the previous Parts I and II we have investigated three standard closure operators, Tarski, Kuratowski, and Halmos closures, with associated algebraic modal logic interpretations of the involved axioms as S4-like, S4, and S5 systems, respectively. Furthermore, the equivalent ways of defining the Halmos lattice structure as pre Brouwer Zadeh (BZ) lattice has been investigated.

In this final Part IV we introduce two closure operators based on a Halmos closure lattice, called Stone and Łukasiewicz, respectively, showing their interpretation as algebraic realizations of many valued logics.

17 The Stone Closure Lattices with Corresponding Full Brouwer Zadeh (BZ) Lattices

The introduction of the notion of Stone closure operator, argument of the present section, is consequence of the following result.

Proposition 120 *In any De Morgan lattice equipped with a unary operator $a \to a^*$, if the derived unary operator $a \to a^{\sim} := a^{*\prime}$ satisfies the only rule of interconnection*
(IR) $\forall a,\, a^{\sim\prime} = a^{\sim\sim}$
then, the following statements are mutually equivalent among them for any element a:

(S1) $a^* \vee a^{*\prime} = 1$ (modal excluded middle of [25])
(S2) $a^* \wedge a^{*\prime} = 0$ (modal noncontradiction of [25])
(S) $a^{\sim} \vee a^{\sim\sim} = 1$ (Stone condition)

Moreover, under the further condition (C1) of the operator * or equivalently the condition (wIR) of the operator $^{\sim}$, they imply the Brouwer noncontradiction law (B3), or one of the equivalent its formulations (C4) or (I4) stated in Proposition 78. Formally, for any element a:

$$\textit{under } [(wIR)\Leftrightarrow(C1)],\ \textit{condition } (S) \quad \textit{implies} \quad (B3)\quad \forall a,\ a \wedge a^{\sim} = 0.$$

Proof (S1) ⇔ (S2). Indeed, $a^* \wedge a^{*\prime} = 0$ iff $(a^* \wedge a^{*\prime})' = 1$ iff $a^{*\prime} \vee a^* = 1$.

(S1) ⇔ (S). From (S1), $a^* \vee a^{*\prime} = 1$, using the definition $a^* = a^{\sim\prime}$, we get $a^{\sim\prime} \vee a^{\sim\prime\prime} = 1$, from which, by the interconnection rule (IR), it follows that $a^{\sim\sim} \vee a^{\sim} = 1$, i.e., (S). Conversely, from (S), $a^{\sim} \vee a^{\sim\sim} = 1$, by the (IR) we get $a^{\sim} \vee a^{\sim\prime} = 1$ which, using the definition $a^{\sim} = a^{*\prime}$, leads to $a^{*\prime} \vee a^{*\prime\prime} = a^{*\prime} \vee a^* = 1$, i.e., (S1). Finally, let us consider the condition (S2), equivalent to condition (S). From (C1), $a \leq a^*$, and (S1), $a^* \wedge a^{*\prime} = 0$, it follows that $0 \leq a \wedge a^{*\prime} \leq a^* \wedge a^{*\prime} = 0$, i.e., $a \wedge a^{*\prime} = 0$, and from $a^{\sim} = a^{*\prime}$ we get $a \wedge a^{\sim} = 0$. □

Definition 121 A *Stone closure lattice* (resp., *algebra*) is a structure $\mathfrak{SCL} = \langle \Sigma, \wedge, \vee, {}', {}^*, 0, 1 \rangle$, where

(SC-1) the sub-structure $\langle \Sigma, \wedge, \vee, {}', 0, 1 \rangle$ is a De Morgan lattice (resp., distributive lattice);

(SC-2) the mapping $^* : \Sigma \to \Sigma$ is a unary operator satisfying the conditions:

(C1)	$a \leq a^*$	(increasing = modal T)
(C2K)	$a^* \vee b^* = (a \vee b)^*$	(additive = modal M and C)
(sC3)	$a^* = a^{*\prime*\prime}$	(closure interconnection = modal 5)
(S1)	$a^* \vee a^{*\prime} = 1$	(Stone = modal excluded middle)

Therefore, a *Stone closure lattice* is a Halmos closure lattice which satisfies besides conditions (C1), (C2K), and (sC3), the condition (S1) equivalent to the Stone condition (S) $a^{\sim} \vee a^{\sim\sim} = 1$. In this case the closure operation is called *Stone closure*.

Let us recall that, according to Lemma 57, condition (C0) is derived from the axioms (C1) and (sC3) of a Halmos closure, and so it is true also in the present case of a Stone closure.

In [27] one can find the following.

Definition 122 A *generalized Łukasiewicz (gŁ) algebra* is a distributive De Morgan lattice equipped with a closure operation satisfying the conditions (C0), (C1), (C2K), and (S1).

Hence, we have that any Stone closure is a gŁ lattice satisfying the further condition (sC3) of modal 5 principle.

$$\text{Stone} = (\text{C0}), \underbrace{(\text{C1}), (\text{C2K}), (sC3)}_{\text{Halmos}} + (\text{S1}) = \underbrace{(\text{C0}), (\text{C1}), (\text{C2K}), (\text{S1})}_{\text{g-Łukasiewicz}} + (\text{sC3}) \tag{75}$$

Therefore,

$$\boxed{\text{generalized Łukasiewicz algebras}} \Longrightarrow \boxed{\text{Stone closure algebras}} \tag{76}$$

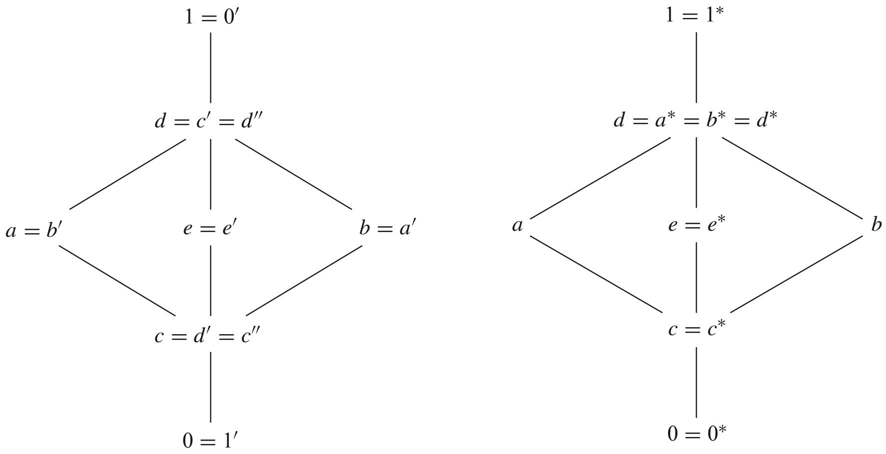


Fig. 27 Non distributive lattice equipped with a Kleene (non standard) negation at the left side and a Halmos no Stone closure at the right

In the part involving the Halmos closure we have inserted also the condition (C0), which we have seen is a consequence of the axioms (C1) and (sC3). This in order to compare the corresponding behavior of Stone closure with respect to generalized Łukasiewicz closure in which this condition is independent from the other three axioms.

In general Halmos closures are not Stone, as Example 123 shows.

Example 123 Let us consider the Hasse diagrams of two non-distributive lattices described in Fig. 27. On the left side of the figure we have a Kleene lattice and at the right side a Halmos closure lattice.

The lattice drawn in the left side is not distributive $(a \vee e) \wedge b = b \neq c = (a \wedge b) \vee (e \wedge b)$, the negation $'$ is Kleene but it is not standard ($a \wedge a' \neq 0$ and $a \vee a' \neq 1$). The closure * at the right side is Halmos but it is not Stone ($a^* \wedge a^{*\prime} = c \neq 0$ and $b^* \vee b^{*\prime} = d \neq 1$). This Halmos is such that condition (C4), equivalent to (B3), of Proposition 78 is not satisfied ($a' \vee a^* = d \neq 1$).

Proposition 124 *In a Stone closure lattice,* $\mathfrak{SCL} = \langle \Sigma, \wedge, \vee, {}', {}^*, 0, 1 \rangle$*, equipped with the further impossibility operation* $a \to a^{\sim} = a^{*\prime}$*, all the following properties are mutually equivalent among them:*

$$(C4) \quad a^* \vee a' = 1 \qquad (B3) \quad a \wedge a^{\sim} = 0 \qquad (I4) \quad a^o \wedge a' = 0,$$
$$(S1) \quad a^* \vee a^{*\prime} = 1 \qquad (S2) \quad a^* \wedge a^{*\prime} = 0 \qquad (S) \quad a^{\sim} \vee a^{\sim\sim} = 1.$$

Proof A Stone closure lattice is in particular a De Morgan lattice and so, owing to Lemma 78, conditions (C4), (B3), and (I4), are mutually equivalent and also owing to Proposition 120 conditions (S1), (S2), and (S), are mutually equivalent.

Let now (C4) be true: $a^* \vee a' = 1$. From this $a \wedge a^{*\prime} = 0$ follows; applying this latter to a^* we obtain $a^* \wedge a^{**\prime} = 0$. But we have seen that in a Halmos closure lattice Proposition 59 assures that condition (C2), $a^{**} = a^*$, holds and so we obtain that $a^* \wedge a^{*\prime} = 0$, i.e., (S2). Conversely, let (S2) be true: $a^* \wedge a^{*\prime} = 0$. Since in Halmos closures condition (C1), i.e., $a \leq a^*$, holds we have that $0 \leq a \wedge a^{*\prime} \leq a^* \wedge a^{*\prime} = (\text{po1}) = 0$, i.e., $a \wedge a^{*\prime} = 0$, which implies that $a' \vee a^* = 1$. □

Remark 125 In the notion of generalized Łukasiewicz algebra previously discussed it is involved the condition (S1) and we have asserted that this is the definition one can find in [27]. To tell the truth in the definition of [27] it is involved the condition (C4) instead of the equivalent (S1).

From the logical equivalence between the characterizing Stone lattice axiom (S1) and condition (B3) proved in Proposition 124, the further important result follows.

Theorem 126 *In a Stone closure lattice $\mathfrak{SCL} = \langle \Sigma, \wedge, \vee, ', ^*, 0, 1 \rangle$, the impossibility operation $a \to a^\sim = a^{*\prime}$ satisfies all the properties defining an intuitionistic (i.e., Brouwer) negation*

(B1) $\forall a \in \Sigma,\ a \leq a^{\sim\sim}$ (weak double negation law)
(B2) $\forall a, b \in \Sigma,\ a \leq b$ *implies* $b^\sim \leq a^\sim$ (B-contraposition law)
equivalent to the condition (B-dM1) (see Proposition 72)
$\forall a, b \in \Sigma,\ (a \vee b)^\sim = a^\sim \wedge b^\sim$ (first B-De Morgan law)
(B3) $\forall a \in \Sigma,\ a \wedge a^\sim = 0$ (noncontradiction law)

Hence, the structure $\mathfrak{BZL} = \langle \Sigma, \wedge, \vee, ', ^\sim, 0, 1 \rangle$ is a real Brouwer Zadeh (BZ) lattice, i.e., a bounded lattice $\langle \Sigma, \wedge, \vee, 0, 1 \rangle$ equipped with a De Morgan (or Zadeh) negation $a \to a'$ and a Brouwer (or intuitionistic) negation $a \to a^\sim$, with (according to Lemma 69) these two negations linked by the

(IR) $\forall a \in \Sigma,\ a^{\sim\prime} = a^{\sim\sim}$ (interconnection rule)

In other words we have stated that any Stone closure lattice $\mathfrak{SCL}$ induces a Brouwer Zadeh lattice $\mathfrak{BZL}$. But in [25, Appendix] one can find the proof of the converse of this result: any Brouwer Zadeh lattice $\mathfrak{BZL}$ induces a Stone closure lattice $\mathfrak{SCL}$. These two results can be schematized by the categorical isomorphism:

$$\boxed{\text{Stone closure lattices}} \iff \boxed{\text{Brouwer Zadeh (BZ) lattices}} \tag{77}$$

Since a Stone closure lattice is in particular a Halmos closure lattice, all the results proved in Theorem 76 are valid also in this case, with some further property due to the axiom (S1) and its equivalent formulation (B3).

Theorem 127 *Let $\mathfrak{SCL} = \langle \Sigma, \wedge, \vee, ', ^*, 0, 1 \rangle$ be a Stone closure lattice with the induced equivalent BZ lattice $\mathfrak{BZL} = \langle \Sigma, \wedge, \vee, ', ^\sim, 0, 1 \rangle$ by the translation rules $a^\sim = a^{*\prime}$ and $a^* = a^{\sim\prime}$.*
Then, the sub-structure $\mathfrak{CO} = \langle \mathscr{E}(\Sigma), \wedge, \vee, ', 0, 1 \rangle$ based on the collection of all exact (crisp, clopen) elements $\mathscr{E}(\Sigma) = \mathscr{C}(\Sigma) = \mathscr{O}(\Sigma)$ is an orthocomplementation

lattice, i.e., a lattice equipped with a mapping $' : \mathscr{E}(\Sigma) \to \mathscr{E}(\Sigma)$ *satisfying the three conditions:*

(dM1) $\forall e \in \mathscr{E}(\Sigma)$, $a = a''$ (involution)
(dM2) $\forall e, f \in \mathscr{E}(\Sigma)$, $(e \vee f)' = e' \wedge f'$ (first De Morgan law), *equivalent to the*

(dM2a) $\forall e, f \in \mathscr{E}(\Sigma)$, $(e \wedge f)' = e' \vee f'$ (second *De Morgan law), in its turn equivalent to*
(dM2b) $\forall e, f \in \mathscr{E}(\Sigma)$, $e \leq f$ *implies* $f' \leq e'$ (contraposition law).

(oc-a) $\forall e \in \mathscr{E}(\Sigma)$, $a \wedge a' = 0$ (noncontradiction), *equivalent to the*
(oc-b) $\forall e \in \mathscr{E}(\Sigma)$, $a \vee a' = 1$ (excluded middle).

Proof A Stone closure lattice, as particular Halmos closure lattice, owing to Theorem 76, has the sub-structure $\mathfrak{CO}$ of lattice equipped with a negation mapping $e \in \mathscr{E}(\Sigma) \to e' \in \mathscr{E}(\Sigma)$ which satisfies the De Morgan conditions (dM1) and (dM2). But the validity on $\mathfrak{BZL}$ of the axioms (B3) $\forall a \in \Sigma\ a \wedge a^{\sim} = 0$ and the identity $\forall e \in \mathscr{E}(\Sigma)\ e^{\sim} = e'$ lead to the noncontradiction law $\forall e \in \mathscr{E}(\Sigma)\ e \wedge e' = 0$, from which it follows the excluded middle law $\forall e \in \mathscr{E}(\Sigma)\ 1 = (e \wedge e')' = e' \vee e$. □

Remark 128 In the development of this section we have adopted the notational convention relative to the (TL) language of topological lattices. In particular if we consider as primitive the Stone closure a^* of a, then one obtains $a^o = a'^{*'}$ for the Stone interior. Conversely, if the primitive is the Stone interior a^o of a, then the corresponding closure is $a^* = a'^{o'}$. Formally, we can consider the one-to-one correspondence

$$a^* = a'^{o'} \longleftrightarrow a^o = a'^{*'} \tag{78}$$

If we consider the translation of the topological language (TL) into the modal logic language (ML), we have that $\mu(a) = a^* = \neg\nu\neg(a)$ represents the possibility connective and $\nu(a) = a^o = \neg\mu\neg(a)$ the necessity connective.

Similarly to the definition introduced in Remark 98, but in the context of a quasi BZ lattice structure characterized by the weak interconnection rule (wIR), in the case of a BZ lattice $\mathfrak{BZL} = \langle \Sigma, \wedge, \vee, ', ^{\sim}, 0, 1 \rangle$, characterized by the interconnection rule (IR), we can introduce the *anti-Brouwer complement* defined as

$$\forall a \in \Sigma, \quad a^{\flat} := a'^{\sim'}. \tag{79}$$

It is now easy to show that the anti-Brouwer complementation mapping $^{\flat} : \Sigma \to \Sigma$, $a \to a^{\flat}$ on the BZ lattice Σ satisfies the conditions:

(AB1) $\forall a \in \Sigma, a^{\flat\flat} \leq a$, *(anti-weak double negation law)*
(AB2) $\forall a, b \in \Sigma, a \leq b$ implies $b^{\flat} \leq a^{\flat}$,

equivalent to the condition (B-dM2)
$\forall a, b \in \Sigma, (a \wedge b)^\flat = a^\flat \vee b^\flat$ *(second B-De Morgan law)*
(AB3) $\forall a \in \Sigma, a \vee a^\flat = 1$ *(excluded middle law)*

Remark 129 Before making the considerations of the present remark, let us recall the identification between modal operators:

$$\mu(a) = \neg\nu\neg(a) \longleftrightarrow \nu(a) = \neg\mu\neg(a). \tag{80}$$

Now, applying the definition $\forall a, a^\flat = a'^{\sim\prime}$, to the element $a = \alpha'$ we get that $\forall \alpha, \alpha'^\flat = \alpha^{\sim\prime}$, from which we obtain that $\forall \alpha, \alpha^\sim = \alpha'^{\flat\prime}$. On the other hand, the identities $a^\flat = a'^{\sim\prime}$ and $a^o = a'^\sim$ lead to $a^\flat = a^{o\prime}$. So, in this case we have the identification:

$$a^\sim = a^{*\prime} = a^{\prime\flat\prime} \longleftrightarrow a^\flat = a^{o\prime} = a^{\prime\sim\prime} \tag{81a}$$

In the (MP) language, and corresponding notations, the complement $\sim(a) = \neg\mu(a) = a^{*\prime}$ represents the impossibility as non-possible (or Brouwer negation), and $\lceil(a) = \neg\nu(a) = a^{o\prime}$ represents the contingency as non-necessary (or anti-Brouwer negation). Therefore, the above identification can be rewritten as follows:

$$\sim(a) = \neg\lceil\neg(a) \longleftrightarrow \lceil(a) = \neg\sim\neg(a). \tag{81b}$$

So we can consider the two structures,

$\mathfrak{SCL} = \langle \Sigma, \wedge, \vee, \neg, \mu, \sim, 0, 1 \rangle$ (Stone closure lattice with Brouwer negation)

$a\mathfrak{SIL} = \langle \Sigma, \wedge, \vee, \neg, \nu, \lceil, 0, 1 \rangle$ (Stone interior lattice with anti-Brouwer negation)

which can be identified according to the identifications (80) and (81b),

$$\mathfrak{SCL} \longleftrightarrow a\mathfrak{SIL}.$$

18 Łukasiewicz Closure Lattices and BZ Lattices Satisfying the Brouwer Second De Morgan Law

In this section we consider another version of closure operator whose definition, introduced in [27] under the condition of lattice distributivity, is the following one.

Definition 130 A *Łukasiewicz (Ł) closure lattice* is a structure $\mathfrak{LCL} = \langle \Sigma, \wedge, \vee, ', ^*, 0, 1 \rangle$ such that

(Ł1) the sub-structure $\langle \Sigma, \wedge, \vee, ', 0, 1 \rangle$ is a bounded De Morgan lattice;
(Ł2) the mapping $^* : \Sigma \to \Sigma$ is a unary operation satisfying the conditions:

$(C1)$	$a \leq a^*$	(increasing = modal T)
$(C2M)$	$a^* \wedge b^* = (a \wedge b)^*$	(multiplicative = dual modal M and C)
$(sC3)$	$a^* = a^{*\prime *\prime}$	(closure interconnection = modal 5)
$(C4)$	$a' \vee a^* = 1$	(Stone = modal excluded middle)

This unary operation is called *Łukasiewicz closure operation*. Let us recall that condition (C0), i.e., $0 = 0^*$, is deduced from conditions (C1) and (sC3).

From this definition it is possible to consider a Łukasiewicz closure as different from a Stone closure by the fact that it substitutes the distributivity of the join by possibility, i.e., condition (C2K), with the distributivity of the meet by possibility, i.e., condition (C2M). But the following result states a link between these two closures.

Proposition 131 *Under axioms (C1) and (sC3), we can state that:*

$$(C2M)\ a^* \wedge b^* = (a \wedge b)^* \quad \Longrightarrow \quad (C2K)\ a^* \vee b^* = (a \vee b)^*.$$

Proof Let $a \leq b$, then $a = a \wedge b$ from which $a^* = (a \wedge b)^*$ follows. Applying to this latter the condition (C2M) we obtain that $a^* = a^* \wedge b^*$, i.e., $a^* \leq b^*$. In other words, (C2M) implies (C2a) and so, making use of Proposition 61, we conclude that (C2M) implies (C2K). □

Conditions (C1), (C2K), (sC3), and (C4) are the ones which defines a Stone closure, and so Łukasiewicz closure lattices can be redundantly considered as Stone closure lattices for which the further condition (C2M) holds:

$$\text{Łukasiewicz} = \underbrace{(C1), (C2K), (sC3), (C4) \Leftrightarrow (S1)}_{Stone} + \text{(C2M)} \tag{82}$$

As a consequence of this result we have the following implication:

$$\boxed{\text{Łukasiewicz closure lattices}} \quad \Longrightarrow \quad \boxed{\text{Stone closure lattices}} \tag{83}$$

The Hasse diagram of the below Example 132 shows a not distributive Stone closure lattice which is not Łukasiewicz.

Example 132 Let us consider the not distributive ($(c \wedge d) \vee a = a \neq c = (c \vee a) \wedge (d \vee a)$) orthocomplemented lattice equipped with a Stone closure drawn in Fig. 28. This lattice is Stone since for any element x one has that $x^* \wedge x^{*\prime} = 0$. The condition (C2M) is not satisfied since $a^* \wedge b^* = 1 \neq 0 = (a \wedge b)^*$, i.e., this is a Stone closure which is not Łukasiewicz.

Remark 133 The Stone closure equipped with this further (C2M) condition has been introduced and widely discussed in [30] as algebraic semantic of a stronger

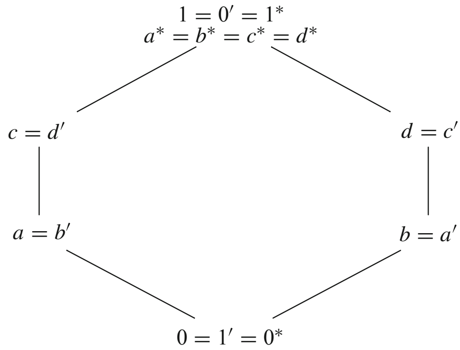


Fig. 28 Six element not distributive Stone closure, without condition (C2M)

version of the S5-like modal logic, where condition (C2M) is denoted as (MD_μ). Term "like" as usual denoting the fact that the involved lattice structure is not necessarily Boolean and "stronger" in the sense that all the algebraic versions of S5 modal logic principles are verified plus the unusual further modal principle (MD_μ)=(CM2). In [30] this structure has been called MDS5 *closure lattice*, but according to the just given discussion MDS5 closure lattices are nothing else than Łukasiewicz closure lattices.

Lemma 134 *In a De Morgan lattice, once introduced two unary operators $a \to a^*$ and $a \to a^\sim$ linked by the relationships $a^* = a^{\sim\prime}$ and $a^\sim = a^{*\prime}$, the following statements are equivalent for any pair of elements a, b:*

$$\text{(B-dM2)} \qquad a^\sim \vee b^\sim = (a \wedge b)^\sim \qquad \text{(second B-De Morgan law)}$$

$$\text{(C2M)} \qquad a^* \wedge b^* = (a \wedge b)^* \qquad \text{(closure multiplicative condition)}$$

Proof $a^\sim \vee b^\sim = (a \wedge b)^\sim \;\Leftrightarrow\; (a^\sim \vee b^\sim)' = (a \wedge b)^{\sim\prime}$, where the implication $\Leftarrow$ is consequence of property (dM1) of the De Morgan negation. Now applying property (dM2) of the De Morgan negation the following identity holds $(a^\sim \vee b^\sim)' = a^{\sim\prime} \wedge b^{\sim\prime}$, and so $a^{\sim\prime} \wedge b^{\sim\prime} = (a \wedge b)^{\sim\prime}$, i.e., $a^* \wedge b^* = (a \wedge b)^*$. □

So, as consequence of Lemma 134 one obtains that the structures of "Łukasiewicz closure lattice" and of "BZ lattices satisfying the (B-dM2) condition (i.e., BZ^{dM} lattices)" are equivalent between them:

$$\boxed{\text{Łukasiewicz closure lattices}} \iff \boxed{\mathrm{BZ}^{dM} \text{ lattices}} \tag{84}$$

18.1 The Distributive Case

In [27, definition 4.2] the notion of *Łukasiewicz algebra* is introduced according to the following definition.

Definition 135 A structure $\langle \Sigma, \wedge, \vee, {}', {}^*, 0, 1 \rangle$ is a *Łukasiewicz algebra* iff $\langle \Sigma, \wedge, \vee, {}', 0, 1 \rangle$ is a distributive De Morgan lattice and the following axioms are satisfied by the closure operation:

$(C0)$	$0 = 0^*$	(0 closure = modal P)
$(C1)$	$a \leq a^*$	(increasing = modal T)
$(C2M)$	$a^* \wedge b^* = (a \wedge b)^*$	(multiplicative = modal M and C for possibility)
$(C4)$	$a' \vee a^* = 1$	(Stone = modal excluded middle)

But from the crucial property of the uniqueness of the Boolean complement in distributive lattices, in the appendix A of [27] the following result has been proved:

- **Lemma A2**. Any Łukasiewicz algebra, making use of all the axioms (C0), (C1), (C2M), and (C4), satisfies the condition (sC3) $\forall a \in \Sigma$, $a^* = a^{*\prime *\prime}$.

This lemma is obtained with a slight modification of an analogous result proved by Monteiro in [80] in a system stronger than the Łukasiewicz one. This lemma allows one to state that any Łukasiewicz algebra is nothing else than a Łukasiewicz closure lattice of Definition 130 satisfying the further condition of *distributivity*.

Proposition 136 *The following categorical isomorphism holds:*

$$\boxed{\textit{Distributive Łukasiewicz closure lattices}} \iff \boxed{\textit{Łukasiewicz algebras}} \tag{85}$$

Proof Let Σ be a distributive Łukasiewicz closure lattice. Then in particular conditions (C1) and (sC3) hold, and so, applying Lemma 57, condition (C0), $0 = 0^*$, follows.

Conversely, let Σ be a Łukasiewicz algebra. Then, Σ is distributive and conditions (C1), (C2M), and (C4) hold. Moreover, lemma A2 implies that condition (sC3) is satisfied. □

The following example shows not only the independence in a distributive Boolean lattice of axiom (C0) from all the other axioms (C1), (C2M), and (S1), but also the importance of axiom (C0) in proving lemma A2.

Example 137 In the Hasse diagram of Fig. 29 we show a distributive Boolean lattice which satisfies conditions (C1), (C2M), and (S1), but not (C0). That is, it is not a Łukasiewicz algebra.

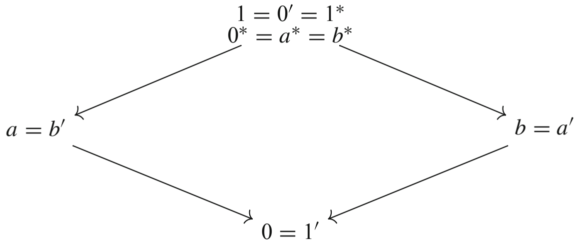


Fig. 29 Classical Boolean lattice without the condition $0 = 0^*$ (indeed, $0 < 0^*$)

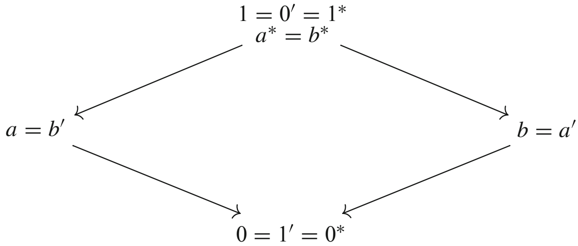


Fig. 30 Classical Stone closure lattice which is not a Łukasiewicz algebra

This lattice is Boolean because it is distributive with the negation $a \to a'$ which satisfies besides the conditions (dM1) and (dM2) also the condition that for every element x it is $x \wedge x' = 0$ and $x \vee x' = 1$. Trivially, $0 < 0^*$, and so condition (C0) is not satisfied. From the fact that (C0) is not true, the above lemma A2 cannot be proved since the condition (C0) is essential in its proof. Indeed, we have $1^{*\prime *} = 1 \neq 0 = 1^{*\prime}$.

The following is an interesting property of Łukasiewicz algebras, whose proof can be found in [80, theorem 4.3].

Theorem 138 *In a Łukasiewicz algebra Σ the De Morgan negation $'$ is indeed a Kleene negation, that is the axiom "(K) $\quad \forall a, b \in \Sigma,\ a \wedge a' \leq b \vee b'$" is satisfied.*

A modification of Example 137 shows a classical Stone closure lattice which is not a Łukasiewicz algebra.

Example 139 Figure 30 shows the Hasse diagram of a classical Stone closure lattice which is not a Łukasiewicz algebra.

This is a classical Stone closure: classical in the sense that the orthocomplemented lattice sub-structure is distributive (Boolean lattice). This classical Stone closure is not Łukasiewicz since condition (C2M), $\forall x, y, x^* \wedge y^* = (x \wedge y)^*$, does not hold: indeed in the particular case of $x = a$ and $y = b$ it is $a^* \wedge b^* = 1 \neq 0 = (a \wedge b)^*$.

Let us note that a fortiori the lack of condition (C2M) does not allow to prove lemma A2.

With a merge of two different classical Stone closure lattices of Example 139 we now give another example of a (non distributive) Stone closure lattice which is not a (non distributive) Łukasiewicz closure lattice.

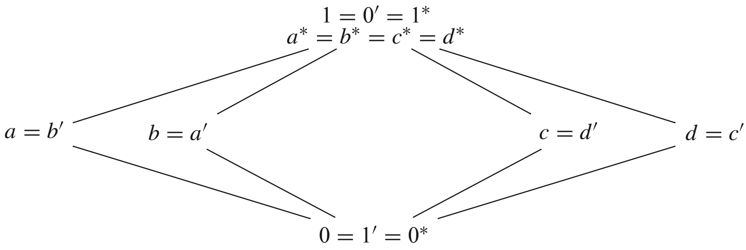


Fig. 31 Stone closure on a modular lattice describing two one-half spins

Example 140 The Hasse diagram drawn in Fig. 31 shows a non distributive, precisely modular, Stone closure lattice which is not a Łukasiewicz closure.

This orthocomplemented lattice is modular (if $x \leq z$, then $x \vee (y \wedge z) = (x \vee y) \wedge z$ holds), condition which implies orthomodularity (if $x \leq z$, then $x \vee (x' \wedge z) = z$ holds), i.e., it is a fd-*quantum lattice*. As in the case of Example 139 the unary operator * is a Stone closure which is not Łukasiewicz.

19 Fuzzy Set Theory as Paradigmatic Concrete Model of Łukasiewicz Algebra

Given a concrete universe of points X, in several sections of the chapter we have investigated some properties of the collection $\mathscr{F}(X)$ of all fuzzy sets based on X. Formally, we recall that

$$f \in \mathscr{F}(X) \quad \text{iff} \quad \forall x \in X,\ 0 \leq f(x) \leq 1. \tag{86}$$

We have also considered the *crisp or exact sets*, whose collection has been denoted by $\mathscr{E}(X)$, defined as the mappings $u : X \to [0, 1]$ (i.e., $u \in \mathscr{F}(X)$) which are idempotent with respect to the product of functions. Formally,

$$u \in \mathscr{E}(X) \quad \text{iff} \quad u \in \mathscr{F}(X) \ \text{ and } \ u^2 = u \tag{87a}$$

$$\text{iff} \quad \forall x \in X,\ 0 \leq u(x) \leq 1 \ \text{ and } \ u^2 = u \tag{87b}$$

From this definition it immediately follows that $\mathscr{E}(X) = \{0, 1\}^X$, i.e., it is the collection of all $\{0, 1\}$ (or two) valued functions defined on X. A little explanation relatively to the adopted formalism. Given two generic fuzzy sets $f, g \in \mathscr{F}(X)$ their product is still a fuzzy set, denoted by $f \cdot g \in \mathscr{F}(X)$ and defined by the law $\forall x \in X$, $(f \cdot g)(x) := f(x) \cdot g(x)$. In the particular case of $f = g$ it is customary to set $f \cdot f = f^2$.

As well known, crisp sets coincide with characteristic functions of some subset $A \in \mathscr{P}(X)$ of X, defined by the rule $\chi_A(x) := 1$ if $x \in A$, and $= 0$ otherwise (which as mappings belonging to $\{0, 1\}^X$ are crisp sets), in the sense that the following holds:

$$u \in \mathscr{E}(X) \quad \text{iff} \quad \exists A \in \mathscr{P}(X) \text{ s.t. } u = \chi_A. \tag{88}$$

Two particular examples of crisp sets are the *identically zero function* $\mathbf{0}$ and the *identically one function* $\mathbf{1}$, defined for every $x \in X$ by the laws $\mathbf{0}(x) := 0$ and $\mathbf{1}(x) := 1$, respectively. In other words $\mathbf{0} = \chi_\emptyset$, the characteristic function of the empty set, and $\mathbf{1} = \chi_X$, the characteristic function of the whole universe.

Trivially, the assignment to any characteristic function $\chi_A \in \mathscr{E}(X)$ of the subset $A \in \mathscr{P}(X)$ is not only a bijection, which we denote by the notation $\chi_A \longleftrightarrow A$, but also an isomorphism between the two Boolean algebras $\left\langle \{0, 1\}^X, \wedge, \vee, {}', \mathbf{0}, \mathbf{1} \right\rangle$ and $\langle \mathscr{P}(X), \cap, \cup, {}^c, \emptyset, X \rangle$. Indeed,

$$\begin{aligned} \chi_A \wedge \chi_B = \chi_{(A \cap B)} &\longleftrightarrow A \cap B \\ \chi_A \vee \chi_B = \chi_{(A \cup B)} &\longleftrightarrow A \cup B \\ (\chi_A)' = \chi_{A^c} &\longleftrightarrow A^c \\ \mathbf{0} = \chi_\emptyset &\longleftrightarrow \emptyset \\ \mathbf{1} = \chi_X &\longleftrightarrow X \end{aligned}$$

A *genuine*, that is not crisp, fuzzy set is the *one-half fuzzy set* $\mathbf{1/2} : X \to [0, 1]$, defined for $\forall x \in X$ by the law $\mathbf{1/2}(x) := \frac{1}{2}$.

Finally, for any given fuzzy set $f \in \mathscr{F}(X)$ in the sequel we are interested to the following subsets of X generated by f:

$$\begin{aligned} A_1(f) &:= \{x \in X : f(x) = 1\} && \text{(the certainty domain)} \\ A_0(f) &:= \{x \in X : f(x) = 0\} && \text{(the impossibility domain)} \\ A_p(f) &:= A_0(f)^c = \{x \in X : f(x) \neq 0\} && \text{(the possibility domain)} \end{aligned}$$

Now we verify that it is possible to give to $\mathscr{F}(X)$ a structure of distributive bounded lattice equipped with a Łukasiewicz closure operator, i.e., $\mathscr{F}(X)$ is a Łukasiewicz algebra.

(Ł-F1) The set $\mathscr{F}(X)$ is a *poset* with to the partial order relation defined as follows:

$$f \leq g \quad \text{iff} \quad \forall x \in X, \; f(x) \leq g(x) \tag{89}$$

Trivially, this poset is bounded by the least fuzzy set $\mathbf{0}$ and the greatest fuzzy set $\mathbf{1}$, since with respect to this partial ordering the above equation (86) becomes:

$$\forall f \in \mathscr{F}(X),\ \mathbf{0} \leq f \leq \mathbf{1}.$$

The poset $\mathscr{F}(X)$ turns out to be a *distributive lattice* with respect to the two lattice operations defined for any pair of fuzzy sets $f, g \in \mathscr{F}(X)$ and any point of the universe $x \in X$ as follows:

$$(f \wedge g)(x) = \min\{f(x), g(x)\} \tag{90a}$$

$$(f \vee g)(x) = \max\{f(x), g(x)\} \tag{90b}$$

This lattice is *complete* since these two lattice operations can be extended to any arbitrary family of fuzzy sets $\{f_j\} \subseteq \mathscr{F}(X)$.

(Ł-F2) The *De Morgan negation* on $\mathscr{F}(X)$ is given by the correspondence $f \in \mathscr{F}(X) \to f' := (\mathbf{1} - f) \in \mathscr{F}(X)$, where

$$\forall x \in X,\ f'(x) := 1 - f(x).$$

This is indeed a Kleene negation since the following conditions are easily verified:

(dM1) $\forall f \in \mathscr{F}(X)$, $f = f''$ (involution or double negation law);
(dM2b) $\forall f, g \in \mathscr{F}(X)$, $f \leq g$ implies $g' \leq f'$ (contraposition law);
(K) $\forall f, g \in \mathscr{F}(X)$, $f \wedge f' \leq \mathbf{1/2} \leq g \vee g'$ (Kleene condition).

In particular we have that $\mathbf{1/2} \wedge (\mathbf{1/2})' = \mathbf{1/2} \neq \mathbf{0}$ (the noncontradiction principle does not hold) and $\mathbf{1/2} \vee (\mathbf{1/2})' = \mathbf{1/2} \neq \mathbf{1}$ (the excluded middle principle does not hold). In this case we say that the Kleene negation is *genuine* since the presence of the genuine fuzzy set **1/2** forbids the possibility that $'$ is an orthocomplementation.

(Ł-F3) The *closure operator* on $\mathscr{F}(X)$ is given by the correspondence $f \in \mathscr{F}(X) \to f^* \in \mathscr{F}(X)$, where

$$\forall x \in X,\ f^*(x) := \chi_{A_p(f)}(x). \tag{91}$$

That is, it is the characteristic function of the *possibility domain* of the fuzzy set f.

It is easy to prove that * satisfies all the conditions (C0), (C1), (C2M), and (C4), of Definition 135 characterizing a Łukasiewicz algebra. Let us recall that f^* is interpreted as *possibility* of the fuzzy set f, denoted by $\mu(f)$, in the algebraic model of modal logic.

As to the condition (S1), equivalent to (C4), first of all let us consider that for a given subset $A \in \mathscr{P}(X)$ of the universe X it is $(\mathbf{1} - \chi_A) = \chi_{A^c}$. From this we get that $f^* \vee f^{*\prime} = f^* \vee (\mathbf{1} - f^*) = \chi_{A_p(f)} \vee \chi_{A_p(f)^c} = \mathbf{1}$.

Therefore,

- *the structure* $\langle \mathscr{F}(X), \wedge, \vee, ', ^*, \mathbf{0}, \mathbf{1}, \mathbf{1/2}\rangle$ *is a Łukasiewicz algebra containing the half element* **1/2**. *Note that* $(\mathbf{1/2})' = \mathbf{1/2}$ *and* $(\mathbf{1/2})^* = \mathbf{1}$.

The *interior operator* on $\mathscr{F}(X)$ is then defined in the usual way for a generic fuzzy set f as "necessity= Not-possibility-Not":

$$f^o = f^{'*'} = \chi_{A_1(f)}. \tag{92}$$

That is, it is the characteristic function of the *certainty domain* of the fuzzy set f. If we adopt the logical notation of interior of a fuzzy set as $\nu(f) = f^o$, then we have that the collection of all open elements with respect to this interior is $\mathscr{O}(X) := \{f \in \mathscr{F}(X) : f = \nu(f)\} = \{0, 1\}^X$. So this family coincides with the above defined family $\mathscr{E}(X)$ of all crisp sets, i.e., $\mathscr{O}(X) = \mathscr{E}(X)$. As expected,

$$\nu(f) = f^o = \chi_{A_1(f)} \le f \le \chi_{A_p(f)} = f^* = \mu(f).$$

(BZ^{dM}-F) The *impossibility operator* on $\mathscr{F}(X)$, given by the correspondence $f \in \mathscr{F}(X) \to f^\sim \in \mathscr{F}(X)$, is defined for any fuzzy set f as "impossibility = Not-possibility":

$$\forall x \in X,\ f^\sim(x) = f^{*'}(x) = \chi_{A_0(f)}(x). \tag{93}$$

That is, it is the characteristic function of the *impossibility domain* of f (and is the fuzzy set explicitly formulated in Sect. 8.2 by Eq. (86)).

Note that for a fixed subset A of X the impossibility of its characteristic function χ_A is $(\chi_A)^\sim = \chi_{A^c}$, and so one has that for any fuzzy set f it is $f^{\sim\sim} = \chi_{A_p(f)}$, the characteristic function of the *possibility domain*, which is just $f^{\sim\sim} = f^*$ (see Eq. (91)).
Therefore, as consequence of Proposition 131 applied to the present case which assures that any Łukasiewicz closure lattice, and so any Łukasiewicz algebra, is a Stone closure lattice, and of Theorem 126 applied to the present case (with the conclusion (77)) we have that

- *the Łukasiewicz algebra of all fuzzy sets on X,*

$$\langle \mathscr{F}(X), \wedge, \vee, ', ^*, \mathbf{0}, \mathbf{1}, \mathbf{1/2}\rangle,$$

induces a BZ De Morgan (BZ^{dM}) distributive lattice structure $\langle \mathscr{F}(X), \wedge, \vee, ', ^\sim, \mathbf{0}, \mathbf{1}, \mathbf{1/2}\rangle$ *(as consequence of Lemma 134).*

This means that all the standard Brouwer negation properties are satisfied:

(B1) $\forall f \in \mathscr{F}(X)$, $f \leq f^{\sim\sim}$ (weak double negation law);

(B2) $\forall f, g \in \mathscr{F}(X)$, $f \leq g$ implies $g^{\sim} \leq f^{\sim}$ (B-contraposition law) [equivalent to the first B-De Morgan law (B-dM1) $\forall f, g \in \mathscr{F}(X)$, $(f \vee g)^{\sim} = f^{\sim} \wedge g^{\sim}$]; furthermore, it holds

(B-dM2) $\forall f, g \in \mathscr{F}(X)$, $(f \wedge g)^{\sim} = f^{\sim} \vee g^{\sim}$ (second B-De Morgan law);

(B3) $\forall f \in \mathscr{F}(X)$, $f \wedge f^{\sim} = \mathbf{0}$.

Finally, the strong interconnection rule trivially holds:

(IR) $\forall f \in \mathscr{F}(X), \quad f^{\sim c} = f^{\sim\sim}$.

Since for any fuzzy set $f \in \mathscr{F}(X)$ all the above unary operations f^* (Eq. (91)), f^o (Eq. (92)), and $f^{\sim}$ (Eq. (93)) define crisp sets, we have the further identifications:

$$f^* = \chi_{A_p(f)} \longleftrightarrow A_p(f) = \{x \in X : f(x) \neq 0\}$$

$$f^o = \chi_{A_1(f)} \longleftrightarrow A_1(f) = \{x \in X : f(x) = 1\}$$

$$f^{\sim} = \chi_{A_0(f)} \longleftrightarrow A_0(f) = \{x \in X : f(x) = 0\}$$

Łukasiewicz algebras, as particular case of (distributive) Stone closure lattices and so also of (distributive) Halmos closure lattices, have the set of all exact elements characterized by the identities expressed by Eq. (31). From this it follows that the fuzzy set f is exact in Stone closure theory ($f = f^{\sim\sim}$) iff f is crisp in fuzzy set theory ($\exists A \in \mathscr{P}(X)$ s.t. $f = \chi_A$, i.e., $f \in \mathscr{E}(X)$).

19.1 Rough Set Representations of Fuzzy Sets

After this discussion and according to the general theory, in the fuzzy sets case the *rough approximation* of f is the interior (necessity)-closure(possibility) pair

$$r(f) = (f^o, f^*) = (\chi_{A_1(f)}, \chi_{A_p(f)}), \quad \text{with} \quad A_1(f) \subseteq A_p(f). \tag{94}$$

The *orthogonality relation* between fuzzy sets, $f \perp g$ iff $f \leq g'$, see Sect. 4.1, has the form $f + g \leq \mathbf{1}$, i.e., the sum $f + g \in \mathscr{F}(X)$ is a fuzzy set. Formally,

$$\forall f, g \in \mathscr{F}(X), \quad f \perp g \ \text{ iff } \ f + g \in \mathscr{F}(X) \tag{95}$$

In the particular case of two crisp sets $\chi_A, \chi_B \in \mathscr{E}(X)$ we have that

$$\chi_A \perp \chi_B \quad \text{iff} \quad A \cap B = \emptyset \tag{96}$$

In the context of fuzzy set theory we prefer to represent the rough approximation of f in an equivalent way as the interior(necessity)-exterior (impossibility) *orthopair* as follows.

$$r_\perp(f) := (f^o, f^\sim) = \left(\chi_{A_1(f)}, \chi_{A_0(f)}\right), \quad \text{with} \;\; A_1(f) \cap A_0(f) = \emptyset. \tag{97}$$

This orthopair representation belongs to the general collection of all orthopairs of crisp sets, denoted as

$$(\mathscr{E}(X) \times \mathscr{E}(X))_\perp := \{(\chi_A, \chi_B) \in \mathscr{E}(X) \times \mathscr{E}(X) : \chi_A \perp \chi_B\}.$$

Then, since there is a bijective identification between crisp sets and subsets, $\chi_A \in \mathscr{E}(X) \longleftrightarrow A \in \mathscr{P}(X)$, we can summarize all this discussion about the orthopair rough approximation of a fuzzy set f by the diagram of Fig. 32, where the orthopair of crisp sets $r_\perp(f) = (\chi_{A_1(f)}, \chi_{A_0(f)})$ is identified with the orthopair of subsets $ext(f) = (A_1(f), A_0(f))$, called the *extension* of f. This orthopair representation belongs to the collection of all orthopairs of subsets of the universe X, denoted as

$$(\mathscr{P}(X) \times \mathscr{P}(X))_\perp := \{(A_1, A_0) \in \mathscr{P}(X) \times \mathscr{P}(X) : A_1 \cap A_0 = \emptyset\}.$$

Formally, the identification is

$$r_\perp(f) = (\chi_{A_1(f)}, \chi_{A_0(f)}) \;\longleftrightarrow\; ext(f) = (A_1(f), A_0(f)) \tag{98}$$

All this can be summarized by the diagram of Fig. 32:

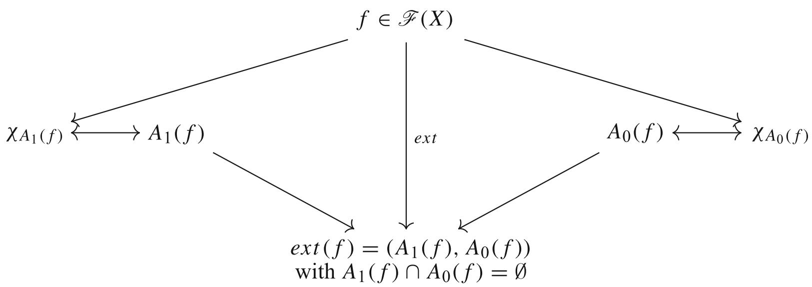


Fig. 32 The extensional representation $ext(f)$ of the fuzzy set f by orthopairs of fuzzy sets, $\chi_{A_1(f)} \perp \chi_{A_0(f)}$, bijectively identified with orthopairs of subsets of the universe

19.2 From the Algebra of Fuzzy Sets to the Algebra of Orthopairs

The result of the representation of a fuzzy set f by its extension as the orthopair $ext(f) = (A_1(f), A_0(f))$ of subsets of the universe X can be described as the mapping

$$\text{ext} : \mathscr{F}(X) \to (\mathscr{P}(X) \times \mathscr{P}(X))_\perp, \; f \to ext(f) = (A_1(f), A_0(f)) \tag{99}$$

with respect to which we have the following result.

Proposition 141 *The extensional mapping ext* : $\mathscr{F}(X) \to (\mathscr{P}(X) \times \mathscr{P}(X))_\perp$ *is surjective.*
The restriction of ext to the family $\mathscr{E}(X)$ *of all crisp sets is a bijection* $\chi_A \twoheadrightarrow (A, A^c)$ *from* $\mathscr{E}(X)$ *onto* $(\mathscr{P}(X), \mathscr{P}(X))_{\perp,c} := \{(A, A^c) : A \in \mathscr{P}(X)\} \subseteq (\mathscr{P}(X) \times \mathscr{P}(X))_\perp$*, subset of* $(\mathscr{P}(X) \times \mathscr{P}(X))_\perp$ *consisting of all* crisp orthopairs.

Proof For every orthopair $(A_1, A_0) \in (\mathscr{P}(X) \times \mathscr{P}(X))_\perp$ there exists the fuzzy set $f_{1,0} = \frac{1}{2}\left(\chi_{A_1} + \chi_{A_0^c}\right) = \frac{1}{2}\chi_{(A_1 \cup A_0)^c} + \chi_{A_1}$ s.t. $A_1(f_{1,0}) = A_1$ and $A_0(f_{1,0}) = A_0$.

Trivially, for any pair (A, A^c) the crisp set χ_A is such that ext$(\chi_A) = (A, A^c)$. Moreover, if $(A, A^c) \neq (B, B^c)$, i.e., if $A \neq B$, then $\chi_A \neq \chi_B$. □

The results of this proposition can be represented by the diagrams of Fig. 33.

We have seen that the family of all fuzzy sets $\mathscr{F}(X)$ has a structure of Łukasiewicz algebra

$$\left\langle \mathscr{F}(X), \wedge, \vee, {}', {}^*, \mathbf{0}, \mathbf{1}, \mathbf{1/2} \right\rangle,$$

with half element **1/2** and induced impossibility operator $f^\sim = f^{*\prime}$ which confers to $\mathscr{F}(X)$ an algebraic structure of Brouwer Kleene (BK) distributive (complete) lattice.

On the contrary, $(\mathscr{P}(X) \times \mathscr{P}(X))_\perp$ lacks an algebraic structure of any kind. In this subsection we will proceed according to a constructive procedure to assign a precise algebraic operation to orthopairs in such a way that it corresponds to the algebraic operation on fuzzy sets.

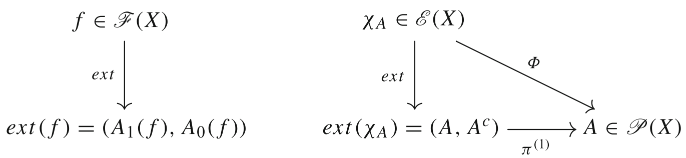


Fig. 33 At the left side it is represented the extension of a fuzzy set by an orthopair of subsets. At the right side we have the corresponding extensional representation of a crisp set, identified with a subset of the universe, where $\pi^{(1)}$ is the canonical projection on the first component

$$\begin{array}{ccc} f \in \mathscr{F}(X) & \xrightarrow{ext} & (A_1(f), A_0(f)) \\ \downarrow{\scriptstyle (\cdot)'} & & \downarrow{\scriptstyle -(\cdot)} \\ f' \in \mathscr{F}(X) & \xrightarrow[ext]{} & -(A_1(f), A_0(f)) = \\ & & = (A_0(f), A_1(f)) \end{array} \qquad \begin{array}{ccc} f \in \mathscr{F}(X) & \xrightarrow{ext} & (A_1(f), A_0(f)) \\ \downarrow{\scriptstyle (\cdot)^*} & & \downarrow{\scriptstyle \Diamond(\cdot)} \\ f^* \in \mathscr{F}(X) & \xrightarrow[ext]{} & \Diamond(A_1(f), A_0(f)) = \\ & & = (A_p(f), A_p(f)^c) \end{array}$$

Fig. 34 Commutative diagrams for the construction of $ext(f')$, at the left side, and of $ext(f^*)$, at the right side

Let us start with the explicit construction in order to obtain in the $(\mathscr{P}(X)^2)_\perp := (\mathscr{P}(X) \times \mathscr{P}(X))_\perp$ context the unary operations corresponding to the operations $'$ and * of the structure $\mathscr{F}(X)$. First of all, let us note that $A_1(f') = \{x \in X : f(x) = 0\} = A_0(f)$ and $A_0(f') = \{x \in X : f(x) = 1\} = A_1(f)$, and so $ext(f') = (A_1(f'), A_0(f')) = (A_0(f), A_1(f))$.

Similarly, from Eq. (91), we have that $A_1(f^*) = \{x \in X : f(x) \neq 0\} = A_p(f)$ and $A_0(f^*) = \{x \in X : f(x) = 0\} = A_p(f)^c$. From these results we are able to construct the commutative diagrams of Fig. 34.

In other words, we have obtained:

$$ext(f') = -(A_1(f), A_0(f)) = (A_0(f), A_1(f)) \tag{100a}$$

$$ext(f^*) = \Diamond(A_1(f), A_0(f)) = (A_p(f), A_p(f)^c) \tag{100b}$$

Now, without any diagram, we list all the connectives in $(\mathscr{P}(X)^2)_\perp$ obtained by the corresponding connectives in $\mathscr{F}(X)$, denoting by $\sqcap$ the one corresponding to $\wedge$ and by $\sqcup$ the one corresponding to $\vee$. We prove the only first case. From $(f \wedge g)(X) = \min\{f(x), g(x)\}$ we have that $A_1(f \wedge g) = \{x \in X : f(x) = 1 \text{ and } g(x) = 1\} = A_1(f) \cap A_1(g)$, but $A_0(f \wedge g) = \{x \in X : f(x) = 0 \text{ or } g(x) = 0\} = A_0(f) \cup A_0(g)$. Hence,

$$ext(f \wedge g) := (A_1(f), A_0(f)) \sqcap (A_1(g), A_0(g)) = \tag{101a}$$

$$= (A_1(f) \cap A_1(g), A_0(f) \cup A_0(g)) \tag{101b}$$

$$ext(f \vee g) := (A_1(f), A_0(f)) \sqcup (A_1(g), A_0(g)) = \tag{101c}$$

$$= (A_1(f) \cup A_1(g), A_0(f) \cap A_0(g)) \tag{101d}$$

Moreover,

$$ext(\mathbf{0}) := (\emptyset, X), \ ext(\mathbf{1}) := (X, \emptyset) \text{ and } ext(\mathbf{1/2}) := (\emptyset, \emptyset) \tag{101e}$$

On the basis of these results related to the connectives of $(\mathscr{P}(X)^2)_\perp$, considered as primitives and expressed by Eqs. (100) and (101), or by a direct computation from Eqs. (92) and (93), we can obtain the further realizations:

$$ext(f^o) := \square(A_1(f), A_0(f)) = (A_1(f), A_1(f)^c) \tag{102a}$$

$$ext(f^\sim) :=\sim (A_1(f), A_0(f)) = (A_0(f), A_0(f)^c) \tag{102b}$$

Note that all the extensions expressed by Eqs. (100b) and (102) are particular cases of the extensions of crisp sets:

$$\forall \chi_A \in \mathscr{E}(X), \quad ext(\chi_A) = (A, A^c).$$

This result allows the identification between the collection of all crisp orthopairs and the power set of X:

$$(A, A^c) \in (\mathscr{P}(X)^2)_{\perp,e} \longleftrightarrow \mathscr{P}(X) \ni A$$

In this way, once denoted by $\alpha_{\mathbf{0}} = ext(\mathbf{0})$, $\alpha_{\mathbf{1}} = ext(\mathbf{1})$, and $\alpha_{\mathbf{1/2}} = ext(\mathbf{1/2})$, we have now the correspondence between the two structures

$$\begin{aligned} \mathfrak{F}(X) = \left\langle \mathscr{F}(X), \wedge, \vee, {}', {}^*, \mathbf{0}, \mathbf{1}, \mathbf{1/2} \right\rangle \implies \\ \mathfrak{P}(X)_\perp = \left\langle (\mathscr{P}(X)^2)_\perp, \sqcap, \sqcup, -, \lozenge, \alpha_{\mathbf{0}}, \alpha_{\mathbf{1}}, \alpha_{\mathbf{1/2}} \right\rangle \end{aligned} \tag{103}$$

where we know that $\mathfrak{F}(X)$ is a Łukasiewicz algebra.

Let us now associate with any fuzzy set $f : X \to [0, 1]$, whose range is the unit compact interval $[0, 1]$ of the real line $\mathbb{R}$, the three valued fuzzy set $f_3 : X \to \{0, 1/2, 1\}$ defined by the rule

$$\forall x \in X, \;\; f_3(x) := \begin{cases} 0 & \text{if } f(x) = 0 \\ 1/2 & \text{if } f(x) \neq 0 \text{ and} f(x) \neq 1 \\ 1 & \text{if } f(x) = 1 \end{cases}$$

Denoting by $\mathscr{F}_3(X) := \{0, 1/2, 1\}^X$ the collection of all three-valued fuzzy sets on the universe X, we have that $f_3 \in \mathscr{F}_3(X)$. If now we introduce the further subset of the universe X determined by the fuzzy set $f \in \mathscr{F}(X)$,

$$A_u(f) := \{x \in X : f(x) \neq 0, 1\} \qquad \text{(uncertainty domain)}$$

then we have that

$$f_3 = \chi_{A_1(f)} + (1/2)\chi_{A_u(f)}.$$

Then, it is easy to verify the following equalities: $A_1(f_3) = A_1(f)$ and $A_0(f_3) = A_0(f)$. So, we can define the *extension* of the three valued fuzzy set f_3 as

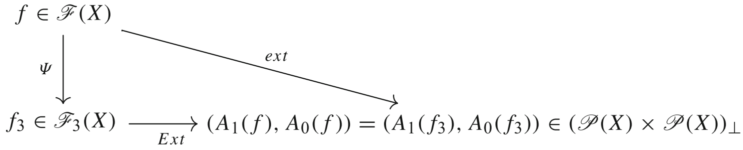


Fig. 35 Graphical relationship between the extension $ext(f)$ of the fuzzy set f and the extension $Ext(f_3)$ of its three valued representation f_3

$Ext(f_3) := (A_1(f_3), A_0(f_3))$. Once introduced the mapping $\Psi : \mathscr{F}(X) \to \mathscr{F}_3(X)$ defined by the correspondence $f \in \mathscr{F}(X) \to \Psi(f) := f_3 \in \mathscr{F}_3(X)$, we have the equality:

$$\forall f \in \mathscr{F}(X), \quad \text{ext}(f) = (A_1(f), A_0(f)) = (A_1(f_3), A_0(f_3)) = \text{Ext}(f_3).$$

In other words,

$$\forall f \in \mathscr{F}(X), \quad ext(f) = Ext(\Psi(f)).$$

These results can be summarized by the diagram of Fig. 35:

The structure $\langle \mathscr{F}_3(X), \wedge, \vee, ', ^*, \mathbf{0}, \mathbf{1}, \mathbf{1/2} \rangle$, where the involved operations are the restriction to $\mathscr{F}_3(X)$ of the just ones defined on $\mathscr{F}(X)$ ($\mathscr{F}_3(X)$ is closed with respect to all these operations), turns out to be a Łukasiewicz algebra satisfying the further condition:

$$\forall f_3 \in \mathscr{F}_3(X), \quad f_3 \wedge f_3' = f_3^* \wedge f_3' \tag{104a}$$

or in the modal logic notation:

$$\forall f_3 \in \mathscr{F}_3(X), \quad f_3 \wedge \neg f_3 = \mu(f_3) \wedge \neg f_3. \tag{104b}$$

19.3 Łukasiewicz Algebra of Orthopairs of Subsets from the Universe X

The extension mapping $ext : \mathscr{F}(X) \to (\mathscr{P}(X), \mathscr{P}(X))_\perp$, which assigns in a surjective way to any fuzzy set $f \in \mathscr{F}(X)$ its *extension* $ext(f) = (A_1(f), A_0(f))$, with $A_1(f) \cap A_0(f)$, is indeed a mapping from the Łukasiewicz algebra $\langle \mathscr{F}(X), \wedge, \vee, ', ^*, \mathbf{0}, \mathbf{1} \rangle$ onto the collection of all orthopairs $(\mathscr{P}(X) \times \mathscr{P}(X))_\perp$ which at the moment has no lattice structure. Let us now equipped this set of orthopairs with the following operations.

$$(A_1, A_0) \sqcap (B_1, B_0) := (A_1 \cap B_1, A_0 \cup B_0) \qquad \text{(lattice meet)}$$

$$(A_1, A_0) \sqcup (B_1, B_0) := (A_1 \cup B_1, A_0 \cap B_0) \qquad \text{(lattice join)}$$

$$-(A_1, A_0) := (A_0, A_1) \qquad \text{(Kleene negation)}$$

$$\Diamond(A_1, A_0) := (A_1, X \setminus A_1) \qquad \text{(modal possibility)}$$

$$\mathbb{O} := (\emptyset, X) \qquad \text{(least element)}$$

$$\mathbb{I} = (X, \emptyset) \qquad \text{(greatest element)}$$

$$\mathbb{H} := (\emptyset, \emptyset) \qquad \text{(half element)}$$

The partial order relation induced by the lattice operations is the following:

$$(A_1, A_0) \sqsubseteq (B_1, B_0) \quad \text{iff} \quad A_1 \subseteq B_1 \text{ and } B_0 \subseteq A_0$$

The obtained structure $\mathbb{A}(X) = \langle (\mathscr{P}(X) \times \mathscr{P}(X))_\perp, \sqcap, \sqcup, -, \Diamond, \mathbb{O}, \mathbb{I}, \mathbb{H} \rangle$ turns out to be a Łukasiewicz algebra with (unique) half element $\mathbb{H}$ such that $\mathbb{H} = -\mathbb{H}$. A complete treatment of this argument is developed in the Chapter "Rough Objects in Monoidal Closed Categories" of this book, where it is shown that $\mathbb{A}(X)$ has a structure of BB^{dM} algebra with respect to the operations $-(A_1, A_0) = (A_0, A_1)$ and

$$\sim (A_1, A_0) = -\Diamond(A_1, A_0) = (X \setminus A_1, A_1) \qquad \text{(impossibility).}$$

Note that this Łukasiewicz algebra is not three valued, i.e., the condition analogous to the one expressed by Eq. (104b) does not hold since in general (and as consequence of the fact that in general $A_1 \cup A_0 \neq X$):

$$(A_1, A_0) \sqcap -(A_1, A_0) = (\emptyset, A_1 \cup A_0) \neq (\emptyset, X) = \Diamond(A_1, A_0) \sqcap -(A_1, A_0)$$

Moreover, the extensional mapping $ext : \mathscr{F}(X) \to (\mathscr{P}(X) \times \mathscr{P}(X))_\perp,\ f \to ext(f) = (A_1(f), A_0(f))$ is a *surjective homomorphism* of Łukasiewicz algebras, whose restriction to the collection $\mathscr{F}_3(X)$ of all three valued fuzzy sets $Ext : \mathscr{F}_3(X) \to (\mathscr{P}(X) \times \mathscr{P}(X))_\perp,\ f_3 \to Ext(f_3) = (A_1(f_3), A_0(f_3)) = (A_1(f), A_0(f))$ is only one-to-one.

Example 142 Let $X = [0, 10]$ be the closed interval of the real line $\mathbb{R}$. Let us consider the three valued fuzzy set $f_3 = (1/2)\chi_{[1,2)} + \chi_{[2,5]} + (1/2)\chi_{(5,9]}$. Then, $f_3' = \chi_{[0,1)} + (1/2)\chi_{[1,2)} + (1/2)\chi_{(5,9]} + \chi_{(9,10]}$ and $f_3^* = \chi_{[1,9]}$. From these we get $f_3 \wedge f_3' = (1/2)\chi_{[1,2)\cup(5,9]}$ and $f_3^* \wedge f_3' = (1/2)\chi_{[1,2)\cup(5,9]}$, and so $f_3 \wedge f_3' = f_3^* \wedge f_3'$.

On the other hand, $Ext(f_3) = ([2, 5], [0, 1) \cup (9, 10])$, $Ext(f_3') = ([0, 1) \cup (9, 10], [2, 5])$, and $Ext(f_3^*) = ([1, 9], [0, 1) \cup (9, 10])$. Hence, $Ext(f_3) \sqcap Ext(f_3') = (\emptyset, [0, 1) \cup [2, 5] \cup (9, 10]$ and $Ext(f_3^*) \sqcap Ext(f_3') = (\emptyset, [0, 1) \cup [2, 5] \cup (9, 10])$, i.e., $Ext(f_3) \sqcap Ext(f_3') = Ext(f_3^*) \sqcap Ext(f_3')$.

Note that $Ext(f_3') = ([0,1) \cup (9,10], [2,5]) = -Ext(f_3)$, but

$$Ext(f_3^*) = ([1,9], [0,1) \cup (9,10]) \neq ([2,5], X) = \Diamond Ext(f_3)$$

so the closure operation is not preserved.

19.4 Łukasiewicz Algebraic Structure in the Pawlak Partition Space Case

Always following the complete treatment performed in Chapter "Rough Objects in Monoidal Closed Categories" of this book, let us consider a partition space (X, π) essential structure of the Pawlak approach to rough set theory, with the cardinality of X *finite*: $|X| < \infty$. Then the power set $\mathscr{P}(X, \pi)$ is a Brouwer Boolean (BB) algebra with respect to the operations

$$\neg A := X \setminus A \qquad \text{and} \qquad \approx A = \cup \left\{ G \in \pi : G \subseteq H^c \right\} \tag{105}$$

We have just seen that the collection of all orthopairs from the universe $\mathbb{A}(X) := \left\{ (A_1, A_0) \in \mathscr{P}(X)^2 : A_1 \cap A_0 \right\}$ has a structure of Ł algebra with respect to the pair of operations $-$, $\Diamond$ or, equivalently, of BB^{dM} algebra with respect to the pair of operations $-$, $\sim$.

In this finite partition case we can consider also the collection of all rough representations of subsets of the universe $\mathbb{R}(X) := \left\{ (\nu(A), \approx (A)) \in (\mathscr{P}(X)^2)_\perp : A \in \mathscr{P}(X) \right\}$, where

$$\nu(A) := \cup \{ G \in \pi : G \subseteq A \}, \tag{106}$$

is the *interior-necessity* of A.

In Chapter "Rough Objects in Monoidal Closed Categories" one can find the direct proof that $\mathbb{R}(X)$ has a structure of BK^{dM} algebra with respect to the pair of operations $-$, $\sim$, or, equivalently, of *Łukasiewicz closure distributive lattice* with respect to the pair of operations $-$, $\Diamond$, based on a Kleene algebra with respect to the single operation $-$; for the sake of simplicity this kind of structure will be denoted by ŁK.

Let us recall the definition of the *closure-possibility* of A

$$\mu(A) := \cup \{ G \in \pi : G \cap A \neq \emptyset \} \tag{107}$$

Finally, the collection of all *exact*, also *crisp*, subsets of X is defined as follows

$$\mathscr{E}(X) := \{ E \in \mathscr{P}(X) : \nu(E) = E \} = \{ F \in \mathscr{P}(X) : \mu(F) = F \}. \tag{108}$$

which has a structure of Alexandroff topology with respect to the usual set theoretical operations.

20 Łukasiewicz Algebras and BZ^{dM} Algebras

We can summarize the results obtained in Sect. 19.3 by the following statements.

- The structure $\mathfrak{IA}(X) := \langle \mathbb{A}(X), \sqcap, \sqcup, -, \Diamond, \mathbb{O}, \mathbb{I}, \mathbb{H}\rangle$ is a Ł algebra (with respect to the operations $-, \Diamond$), or equivalently a BB^{dM} algebra (with respect to the operations $-, \sim = -\Diamond$).
- The structure $\mathfrak{IR}(X) := \langle \mathbb{R}(X), \sqcap, \sqcup, -, \Diamond, \mathbb{O}, \mathbb{I}, \mathbb{H}\rangle$ is a ŁK algebra (with respect to the operations $-, \Diamond$), or equivalently a BK^{dM} algebra (with respect to the operations $-, \sim = -\Diamond$).
- Note that in $\mathfrak{IR}(X)$ the Kleene negation $-(\nu(A), \approx A) = (\approx (A), \nu(A))$ cannot be a Boolean negation. So $\mathfrak{IR}(X)$ is not a BB^{dM} algebra.
- Let us recall that in $\mathfrak{IA}(X)$ in general the three valued identity (whose algebraic formulation will be denoted by (L3-b) in the sequel) does not hold, since the general condition $A_1 \cup A_0 \neq X$ leads to

$$(A_1, A_0) \sqcap -(A_1, A_0) \neq \Diamond(A_1, A_0) \sqcap -(A_1, A_0)$$

  So, $\mathfrak{IA}(X)$ cannot be a three valued Łukasiewicz (Ł^3) algebra.
- Consequently, also the ŁK algebra $\mathfrak{IR}(X)$ does not satisfy the three valued identity (L3-b) (since in general $\nu(A) \cup \approx (A) \neq X$).

20.1 Three Valued Łukasiewicz Algebras, BZ^3 Algebras, and Nelson Algebras

We have seen that the collection of all three valued fuzzy sets has a structure of Łukasiewicz algebra written in modal logical notation as

$$\langle \mathscr{F}_3(X), \wedge, \vee, \neg, \mu, \mathbf{0}, \mathbf{1}, \mathbf{1/2}\rangle\,,$$

satisfying the further condition (104b)

$$\forall f_3 \in \mathscr{F}_3(X), \quad f_3 \wedge \neg f_3 = \mu(f_3) \wedge \neg f_3.$$

This condition characterizes a Łukasiewicz algebra as a *three valued Łukasiewicz algebra*, introduced and developed by Moisil in [75, 76] in order to study the three-valued logic of Łukasiewicz [70] on the concrete three valued set $\Sigma = \{0, 1/2, 1\}$. This algebra has been successively axiomatized in an abstract formulation by A. Monteiro [80], with a further contribution of L. Monteiro in [79] (and see also [33–35]). We present here the definition introduced by D. Becchio in [4, 5].

Definition 143 (Becchio [4]) A *three-valued Łukasiewicz algebra ($Ł^3$ algebra)* is a system

$$\langle \Sigma, \wedge, \vee, \neg, \mu, 0, 1 \rangle\,, \text{ where}$$

(L^3-1) the sub-structure $\langle \Sigma, \wedge, \vee, \neg, 0, 1 \rangle$ is a Kleene algebra (distributive Kleene lattice),
(L^3-2) satisfying the characterizing axioms

$$\text{(L3-a)} \qquad \neg a \vee \mu(a) = 1$$
$$\text{(L3-b)} \qquad \neg a \wedge \mu(a) = \neg a \wedge a.$$

In [4] it is given the proof that from these two axioms it follows the further condition (L3-c) $\mu(a \wedge b) = \mu(a) \wedge \mu(b)$, i.e., the above Definition 143 is equivalent to the Monteiro definition of [80] ([34, 35]).

Definition 144 (Monteiro [79]) A *three-valued Łukasiewicz algebra ($Ł^3$ algebra)* is a system

$$\langle \Sigma, \wedge, \vee, \neg, \mu, 0, 1 \rangle$$

which is a Kleene algebra equipped by a unary operation $\mu : \Sigma \to \Sigma$ satisfying the following conditions:

$$\text{(L3-a)} \qquad \neg a \vee \mu(a) = 1$$
$$\text{(L3-b)} \qquad \neg a \wedge \mu(a) = \neg a \wedge a$$
$$\text{(L3-c)} \qquad \mu(a \wedge b) = \mu(a) \wedge \mu(b).$$

The following is an interesting property of three valued Łukasiewicz algebras, whose proof can be found in [80, Theorem 4.3].

Theorem 145 *In a three valued Łukasiewicz algebra Σ the De Morgan complementation $'$ is indeed a Kleene complementation, that is the axiom "(K) $\forall a, b \in \Sigma,\ a \wedge a' \leq b \vee b'$" is satisfied.*

In [27, Thorem 5.6] it is shown that any $Ł^3$ algebra, according to the just given Definition 143, is a Ł algebra according to Definition 135:

$$\boxed{\text{Three valued Łukasiewicz algebras}} \implies \boxed{\text{Łukasiewicz algebras}} \qquad (110)$$

It is interesting to quote from [81]: "Three valued Łukasiewicz algebras play in the study of the trivalent propositional calculus of Łukasiewicz an important and analogous role to that of Boolean algebras in the study of classical propositional calculus."

We have seen that Stone closure lattices and Łukasiewicz lattices (resp., algebras), are categorically equivalent to BZ and BZ^{dM} lattices (resp., algebras), respectively. Now we investigate the categorical equivalence of three valued Łukasiewicz algebras and some kind of BZ structure. At this purpose, and according to [27, Definition 5.2 and Theorem 5.2], we give the following.

Definition 146 A BZ^3 lattice (resp., algebra) is a BZ^{dM} lattice (resp., algebra) Σ that satisfies the further condition:

$$(\mathrm{BZ}^3) \qquad \forall a \in \Sigma, \quad a \vee \sim a = a \vee \neg a.$$

Example 147 A prototypical BZ^3 algebra is based on the three element set $\mathfrak{BZ}^3 = \langle 0, 1/2, 1 \rangle$ equipped with the operations defined for any pair of elements a, b:

$$a \wedge b := \min\{a, b\}, \tag{111a}$$

$$a \vee b := \max\{a, b\}, \tag{111b}$$

$$\neg a := 1 - a, \tag{111c}$$

$$\sim a := 1 \text{ if } a = 0, \quad \text{and } = 0 \text{ otherwise.} \tag{111d}$$

In [27, Theorem 5.7] it is stated the following categorical isomorphism.

Theorem 148 *Ł*^{3} *algebras* $\langle \Sigma, \wedge, \vee, \neg\mu, 0, 1\rangle$ *and* BZ^3 *algebras* $\langle \Sigma, \wedge, \vee, \neg, \sim, 0, 1\rangle$ *are categorically isomorph between them by the definitions*

$$\sim(a) := \neg\mu(a) \quad \textit{and} \quad \mu(a) := \neg\sim(a).$$

That is

$$\boxed{\text{Three valued Łukasiewicz algebras}} \Longleftrightarrow \boxed{\mathrm{BZ}^3\text{algebras}} \tag{112}$$

Example 149 A prototypical Ł^3 algebra is based on the set of three elements $C_0 = \{0, 1/2, 1\}$ whose (distributive) lattice operations are defined for any pair of elements a, b as follows:

$$a \wedge b := \min\{a, b\} \quad \text{and} \quad a \vee b := \max\{a, b\}.$$

The following table shows the primitive operations of negation $\neg$ and possibility μ in the first two columns, then the remaining columns represent the operations of necessity $\nu = neg\mu\neg$, impossibility (Brouwer negation) $\sim = \neg\mu$, and contingency (anti-Brouwer negation) $\flat = \mu\neg = \neg \sim \neg$, respectively.

$a \in C_0$	$\neg(a)$	$\mu(a)$	$\nu(a)$	$\sim(a)$	$\flat(a)$
0	1	0	0	1	1
h	h	1	0	0	1
1	0	1	1	0	0

Of course, the structure $\langle C_0, \wedge, \vee, \neg, \mu, 0, 1\rangle$ is the three element $Ł^3$ algebra, and the structure $\langle C_0, \wedge, \vee, \neg, \sim, 0, 1\rangle$ is the corresponding three element BZ^3 algebra.

Note that the negations $\neg$ and $\flat$ are those introduced by Rasiowa in [99, p. 82], in the development of the three element Nelson algebra denoted by her with C_0, using the symbols $\sim$ (instead of our $\neg$) and $\neg$ (instead of our $\flat$).

In order to increase the notational confusion A. Monteiro in [81] adopts the symbol ∇ instead of our μ and put $\lceil(a) = \nabla\neg(a)$ which corresponds to our $\flat(a) = \mu\neg(a)$.

20.2 Nelson Algebras and Three Valued Łukasiewicz Algebras

In dealing with algebraic methods to describe roughness one can find another line of thought based on Nelson algebras [87, 88], and that is the topic of another chapter of this book. Let us start the discussion on the algebras of Nelson following an interesting article of A. Monteiro [81] where, after the standard Definition 144 of three valued Łukasiewicz algebra, one can find the following definition of Nelson algebra.

Definition 150 A *Nelson algebra* is a system $\mathfrak{NA} = \langle \Sigma, \wedge, \vee, \rightarrow, \neg, 0, 1\rangle$ where

(Nel-1) the sub-structure $\langle \Sigma, \wedge, \vee, \neg, 0, 1\rangle$ is a Kleene algebra (distributive Kleene lattice),

(Nel-2) satisfying the following conditions

$$\text{(N1)} \quad a \rightarrow a = 1$$

$$\text{(N2)} \quad a \wedge (a \rightarrow b) = a \wedge (\neg a \vee b)$$

$$\text{(N3)} \quad (a \rightarrow b) \wedge (\neg a \vee b) = \neg a \vee b$$

$$\text{(N4)} \quad a \rightarrow (b \wedge c) = (a \rightarrow b) \wedge (a \rightarrow c)$$

$$\text{(N5)} \quad a \rightarrow (b \rightarrow c) = (a \wedge b) \rightarrow c.$$

We expose now the results of Lemmas 3.2, 3.3, and Theorem 3.4 of [81] in the form of a single theorem where we translate the Monteiro notation $\lceil a = \nabla\neg(a)$ in the notation $\lceil(a) = \mu\neg(a) = \flat(a)$ (recall the above Example 149).

Theorem 151

(a) If in a three valued Łukasiewicz algebra $\mathfrak{LA} = \langle \Sigma, \wedge, \vee, \neg, \mu, 1\rangle$ *we set*

$$\lceil a := \mu\neg a \quad \textit{and} \quad a \to b := \lceil a \vee b,$$

then the system $\mathfrak{NA} = \langle \Sigma, \wedge, \vee, \to, \neg, 1\rangle$ *is a Nelson algebra such that*

$$\text{(N0)} \quad a \vee \lceil a = 1,$$

moreover one has that

$$\text{(Pos)} \quad \mu(a) = \neg(a \to 0).$$

(b) Let $\mathfrak{NA} = \langle \Sigma, \wedge, \vee, \to, \neg, 1\rangle$ *be a Nelson algebra such that:* (N0) $a \vee \lceil a = 1$, *and if we set*

$$\mu a := \neg a \to 0 = \lceil\neg a,$$

then the system $\mathfrak{LA} = \langle \Sigma, \wedge, \vee, \neg, \mu, 1\rangle$ *is a three valued Łukasiewicz algebra and*

$$\text{(Imp)} \quad a \to b = \lceil a \vee b.$$

(c) Three valued Łukasiewicz algebras can be identified with the Nelson algebras satisfying the condition (N0) $a \vee \lceil a = 1$. *That is*

$$\boxed{\text{Three valued Łukasiewicz algebras}} \iff \boxed{\text{Nelson algebras + (N0)}} \tag{113}$$

From this it follows that

$$\boxed{\text{Three valued Łukasiewicz algebras}} \implies \boxed{\text{Nelson algebras}} \tag{114}$$

Example 152 In the context of the three element $Ł^3$ algebra $C_0 = \{0, 1/2, 1\}$ treated in Example 149, the implication connective of Theorem 151, $a \to b = \lceil a \vee b$ where $\lceil a = \flat a = \mu\neg a$, has the table representation

$\to$	0	h	1
0	1	1	1
h	1	1	1
1	0	h	1

This table coincides with what one can find in the Rasiowa book [99, p. 82] as the three element example of what is defined with the name of *quasi-pseudo-Boolean algebra* (named as Nelson algebra in the A. Monteiro approach). In section 1. **Definition and elementary properties** (p. 68) of the same book the involved algebraic structure contains another implication connective, denoted by $\Rightarrow$, which seems to be a primitive notion of the structure. This contrasts with the demonstration of the (qpB_4) condition which can be found on the same page and which affirms the following equality $a \Rightarrow b = (a \to b) \wedge (\neg b \to \neg a)$. In this way $\Rightarrow$ is a notion which can be derived from the real primitive notions $\to$ and $\neg$. In any case the tabular representation of $\Rightarrow$ in C_0 according to this derived notion assumes the form

$\Rightarrow$	0	h	1
0	1	1	1
h	h	1	1
1	0	h	1

Note that this is the three element table of the implication connective of the Łukasiewicz approach to many-valued logics (see [101, pp. 23 and 36])

$$a \Rightarrow b := \min\{1, 1 - a + b\} = \begin{cases} 1 & \text{if } a \leq b \\ 1 - a + b & \text{otherwise} \end{cases}$$

21 Conclusions and an Open Problem

Concluding our investigation about algebraic methods for describing rough approximations, and summarizing, we can state that in the case of the power set $\mathscr{P}(X, \pi)$ of the universe X equipped with a partition π,

(AM1) the algebraic system $\mathfrak{IR}(X)$ based on the collection of all crisp orthopairs $(\nu(A), \approx A)$, is

- (IR-1) a Kleene distributive lattice equipped with a Łukasiewicz closure, called a ŁK algebra, or equivalently
- (IR-2) a Kleene distributive lattice equipped with a Brouwer negation satisfying the second B-De Morgan condition (B-dM2), called a BK^{dM} algebra.

(AM2) The rough approximation space by orthopairs is the system $\langle \mathscr{P}(X), \mathscr{E}(X), r_{\perp,c} \rangle$ where

- (RAS-1) $\mathscr{P}(X)$ is the collection of all approximable subsets of the universe, (RAS-2) $\mathscr{E}(X)$ is the collection of all definable, also crisp or exact, subsets obtained by the partition π,

(RAS-3) $r_{\perp,c} : \mathscr{P}(X) \to \mathscr{E}(X) \times \mathscr{E}(X)$ is the ortho-rough approximation mapping associating with any approximable subset $A \in \mathscr{P}(X)$ its ortho-rough approximation by crisp sets $r_{\perp,c}(A) := (\nu(A), \approx A)$, under the orthogonality condition $\nu(A) \cap \approx A = \emptyset$.

I want conclude this chapter with an interesting open problem linked to the implication expressed by Eq. (114) of Theorem 151 which states that every three valued Łukasiewicz algebra is a Nelson algebra. On the other hand, Eq. (113) allows us to identify three valued Łukasiewicz algebras with the Nelson algebras satisfying the (N0) condition.

The open problem consists in determining a categorical isomorphism between Nelson's algebras and some generalized form of Łukasiewicz algebras. In other words, is there any generalized algebra of Łukasiewicz categorically equivalent to the algebra of Nelson?

This can be represented by the following reformulation of Eqs. (113) and (114):

$$\boxed{\text{BZ}^{3}\text{-lattice}} \Longleftrightarrow \boxed{\text{3-Łukasiewicz alg.}} \Longleftrightarrow \boxed{\text{Nelson alg. + (N0)}}$$

with corresponding open problems underlined by interrogation marks

$$\boxed{\text{BZ}^{???}\text{-lattice}} \Longleftrightarrow \boxed{\text{(???)-Łukasiewicz alg.}} \Longleftrightarrow \boxed{\text{Nelson alg.}}$$

We can suggest a *conjecture*, based on some reasonable heuristic argumentations, (and that in any case it should be proved) that the structure characterized by the interrogation marks (???) could be Łukasiewicz algebras of Definition 135 of page 131 and BZ^{dM} algebras.

This would make it possible to complete points (IR-1) and (IR-2) of the above (AM1) with the further result:

(IR-3) the algebraic system $\mathfrak{IR}(X)$ based on the collection of all crisp orthopairs $(\nu(A), \approx A)$, can be equivalently formulated as a Nelson algebra.

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Algebraic Methods for Granular Rough Sets

A. Mani

Abstract Granules are the building blocks of concepts of different types in soft computing and rough sets in particular. These granules can be defined in ways that are related to computing or reasoning strategies associated. In this broad perspective, at least three concepts of granular computing have been studied in the literature. The axiomatic approach in algebraic approaches to general rough sets had been introduced in an explicit formal way by the present author. Most of the results and techniques that are granular in this sense are considered critically in some detail in this research chapter by her. It is hoped that this work will serve as an important resource for all researchers in rough sets and allied fields.

1 Introduction

In any general reasoning or computational context, the concepts or objects of primary interest may be definable or be computable in terms of relatively better defined or optimal concepts or objects called granules (or *information granules*). Such contexts are said to be *granular* and concepts of granular computing and reasoning are spoken of. The terms *granular computing* and *information granules* have been around for at least forty years in computational intelligence. Adaptations of these concepts to rough sets have received more attention during the later part of the nineties of the last century.

Few concepts of granularity are known in the literature on computational intelligence in general and rough sets in particular. In the older precision-based approach granules are defined in loose terms as sets of attributes sharing similar functionality or similarity or indistinguishability. In the axiomatic approach due to present author [91], granules are objects that bear some relationship with approximations and their construction. More specifically, all approximations are required to be representable

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by granules in a weak sense, granules are required to be lower definite in a sense, and all pairs of distinct granules are required to be contained in definite objects. This concept of granule has been justified by the wide spectrum of available use cases.

When appropriate concept of granules cannot be identified in a context then related semantics is said to be non granular (non granular rough sets to be precise). Many of the granular and non-granular rough approaches have nice algebraic semantics associated. The validation of the distinction between *granular* and *non-granular* is well reflected in the differences in semantics. Related material has however been either dense or distributed across the literature. In this research exposition semantics of granular rough sets is presented in a coherent way with substantial enhancements and new results by the present author.

The literature on algebraic approaches to rough sets is huge. Some relevant papers that touch or cover major trends include

- [4, 82–86, 90–92, 95, 97, 106],
- [5–7, 52, 66, 88–90, 94, 98–100, 103, 105, 159],
- [11, 18–20, 29, 38, 42, 44, 45, 45, 49, 53, 57, 58, 100, 106, 114, 127, 131, 143, 163, 165, 170, 174, 181],
- [22, 41, 47, 60, 63, 70, 76, 78, 111, 112, 115, 117–119, 121, 134, 136, 161],
- [59, 61, 73, 75, 120, 123, 124, 126, 139, 141, 157, 160, 164, 166, 171, 175, 179],
- [1, 10, 31, 40, 46, 48, 50, 61, 64, 103, 105, 113, 116, 133, 140, 154, 167, 173].

Granular computing may be considered from a precision based or axiomatic point of view. Semantics relating to the two differ substantially and not all rough set approximations are granular in a functional way. Algebraic methods for granular rough sets (see for example [91, 98, 99, 105, 106]) differ from those used for other rough sets. In this detailed research chapter, most of the approaches are reviewed or rewritten from this perspective. Focus has also been placed on comparing multiple semantics in specific cases. Cover based approximations and their semantics are also considered in detail from a axiomatic granular approach. Connections with different perspectives of granularity [174] are also pointed it.

Most of the perspectives relative rough sets can be formulated in the framework of granular operator spaces and variants thereof introduced and investigated by the present author in [98, 102, 104, 106]. The frameworks may be seen as special cases of Rough Y-Systems RYS investigated by her in [90, 91, 97], but are optimal for the present purposes.

In general, explicit definitions of rough approximations of a subset A of a universe S used in the literature have at least one of the following forms (for some function $f : S \mapsto \wp(S)$ and formula Φ_{l*} corresponding to approximation operators $l\alpha$, $l\tau$, $l\mu$ and derived commonality and aggregation operations $\otimes$, $\oplus$ respectively and analogously for u. $\mathcal{J}$ being a collection of ideals or filters of subset of $\wp(S)$ with some order structure):

$$A^{l\alpha} = \{x : x \in S \,\&\, \Phi_{l\alpha}(f(x), A)\} \tag{l-Point-wise}$$

$$A^{l\tau} = \{x : x \in S \,\&\, f(x) \in \mathcal{G} \,\&\Phi_{l\tau}(f(x), A, \mathcal{J})\} \tag{l-Co-Granular}$$

$$A^{l\mu} = \otimes\{H : H \in \mathcal{G} \,\&\, \Phi_{l\mu}(H, A)\} \quad \text{(l-Granular)}$$

$$A^{u\alpha} = \{x : x \in S \,\&\, \Phi_{u\alpha}(f(x), A)\} \quad \text{(u-Point-wise)}$$

$$A^{u\tau} = \{x : x \in S \,\&\, f(x) \in \mathcal{G} \,\&\Phi_{u\tau}(f(x), A, \mathfrak{J})\} \quad \text{(u-Co-Granular)}$$

$$A^{u\mu} = \oplus\{H : H \in \mathcal{G} \,\&\, \Phi_{u\mu}(H, A)\} \quad \text{(u-Granular)}$$

This is because in the point-wise approach, neighborhoods of the form $f(x)$ associated with points $x \in S$ (or subsets thereof) are constrained by set theoretical conditions (usually) involving the approximated set A in question to decide the membership of the point in the approximation of the set A. An example of $\Phi_{l\alpha}(f(x), A)$ is $nbd(x) \subseteq A$, where the neighbourhood $nbd(x)$ is the successor neighborhood in a general approximation space.

The concept of co-granular approximations has been defined by the present author in [104]. It is essentially a higher order approach that and has been explained in more detail in the section on granular operator spaces. In general, if rough approximations of a subset $X \subseteq S$ are defined by expressions of the form

$$X^{\oplus} = \{a : \gamma(a) \odot X^{*} \in \mathfrak{J}\}$$

with $\oplus \in \{l, u\}$, $\mathcal{G} \subset \wp(S)$, $\gamma : S \longmapsto \mathcal{G}$ being a map, $* \in \{c, 1\}$ and $\odot \in \{\cap, \cup\}$, then the approximations are said to be *co-granular*.

The granular approximations mentioned are derived from the collection of granules that satisfy some relationship with the set being approximated.

In the literature, most point-wise definitions of approximations are co-granular in the sense implied by the form. Often references to any specific granulation are omitted in the point-wise approach and correctly should be considered without additional assumptions. From the semantic point of view substantial differences exist between co-granular, granular and point-wise approximation based approaches. Some aspects of these differences have been made explicit in [105] and other papers by the present author.

This long research chapter is organized as follows:

- In the following section, different perspectives of granular computing, granules and granulations are discussed.
- In Sect. 3, granular operator spaces, variants thereof and more details of the axiomatic approach are discussed.
- Algebraic semantics of classical rough sets is considered in some detail and breadth next.
- Partial approximation spaces are studied from an improved semantic perspective in Sect. 5.
- In Sect. 6, tolerance spaces are considered in much depth.
- Algebraic semantics of prototransitive rough sets are presented in the next section.
- One approach to approximate algebraic semantics is presented next.

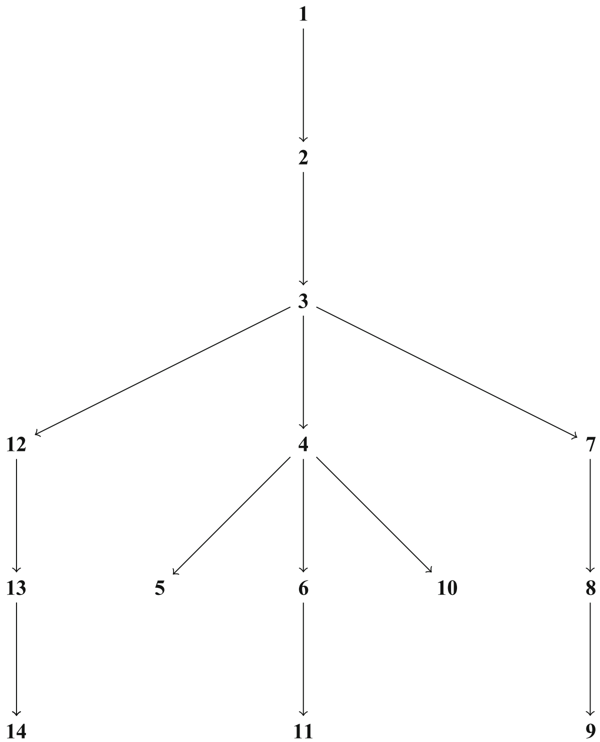

Fig. 1 Dependence between sections

- Algebraic connections with non-monotonic logic and related representation is explained in Sect. 9.
- In the following section, granular connections of cover based rough sets is explored and possible algebraic approaches are reviewed. Quasi order approximation spaces that relate to covers are also studied in the chapter.
- Subsequently choice inclusive algebraic semantics are explained.
- Antichain based semantics is developed in some detail in the following section.
- A specific application of the antichain based approach to quasi-equivalences is explained next
- Open problems and directions are mentioned in Sect. 14.

The dependency between the different sections is in Fig. 1.

From a semantic perspective, all of the following have been considered for granular rough sets:

- TQBA related semantics: This collection of semantics relates to interpretation of rough reasoning from a modal logic perspective and is among the oldest proposals for classical rough sets.
- Super rough order semantics: This is a higher order semantics due to the present author that builds on her results in [83] for classical rough sets. The presented semantics holds for TQBAS as well.

- Pseudo-complemented lattice related semantics: Double Stone and Stone algebras and variants used as a semantics of rough sets fall under this class of semantics.
- Nelson algebra related semantics: Nelson algebras and variants have been used in the semantics of partially ordered sets as approximation spaces and also in proto-transitive rough sets. The latter is presented in some detail. The former semantics is not granular and so has not been included in this chapter.
- Semantics of Esoteric Rough Sets: When partial equivalences (partially reflexive, symmetric and weakly transitive relations) replace the equivalence relation in an approximation space, then under some additional non-triviality conditions these are known as esoteric approximation spaces. Semantics of rough sets over such spaces is also considered in this chapter.
- Higher order semantics for bitten rough sets
- Double Heyting algebras for tolerance spaces
- Heyting algebras for quasi order approximation spaces
- Antichain based semantics
- Choice inclusive semantics
- Approximate Semantics of Proto transitive rough sets and
- PRAX algebras

A number of duality and representation results that relate to this chapter are in the chapter [107] by the present author in this volume. Nelson, Heyting, Lukasiewicz, Post and MV-algebra related semantics of point-wise rough sets, and Grothendieck topologies have been considered in a separate chapter in this volume [122]. In the preclusivity based approach, Pre-BZMV and BZMV-algebras have been used as a semantics of classical rough sets and rough sets over tolerance spaces in particular. The study of ortho pairs in the context of rough sets is closely related. These have also been dealt with in a separate chapter in this volume [17, 21]. Connections with modal logics are stressed in [17].

For some cover based rough sets, associated logics are known but the algebraic semantics are not known in the literature—some directions for filling related gaps are indicated. New results on correspondences between semantics are also proved in this research chapter.

Connections with non-granular rough sets has not been considered and omissions include the modal logic approach [6, 22, 124, 173], point-wise cover based approaches [38, 144], algebras lying between TQBAs and rough algebras [142] and ideal based approaches [105] In most of these cases, the connection with general granular rough sets is not known. Mereological approaches that rely on numeric degrees of membership or inclusion as in [135–137] do not have algebraic models associated. The rough counting based approach of [91] is a promising granular approach that has not been included for reasons of space.

The axiomatic approach [89–91, 97] and simplifications to granular operator spaces[98, 106] invented by the present author are well suited for algebraic semantics of all kinds of granular rough sets and related representation, duality and inverse problems. A major thrust of the chapter will be to reformulate all algebraic semantics in the perspective.

2 Perspectives of Granularity

In the contexts of rough sets, granules (or information granules) are basically collections sharing some properties relating to indiscernibility, similarity or functionality at some levels of discourse.

2.1 Granules and Granulations

A granule may be vaguely defined as some concrete or abstract realization of relatively simpler objects through the use of which more complex problems may be solved. They exist relative to the problem being solved in question. From this broad viewpoint it can be claimed that the basic ideas of granular computing have been in use since the dawn of human evolution. In earlier papers [91, 95], the present author has shown that the methods can be classified into the following three forms:

- Primitive Granular Computing Paradigm: PGCP
- Classical Granular Computing Paradigm: CGCP
- Axiomatic Granular Computing Paradigm: AGCP

These have been referred to as paradigms because of their apparent widespread use and are described in more detail below.

In all theories or theoretical understandings of granularity, the term *granules* refer to parts or building blocks of the computational process and *granulations* to collections of such granules in the context.

Granular computing can also be understood relative to numeric precision or relative to axiomatic frameworks [91] with no reference to numeric precision. The latter problem of defining or rather extracting concepts that qualify as these require much work in the specification of semantic domains and process abstraction. All these are discussed in this section.

2.2 Primitive Granular Computing

Even in the available information on earliest human habitations and dwellings, it is possible to identify a primitive granular computing process at work. This can for example be seen from the stone houses, dating to 3500 BCE, used in what is present-day Scotland (see Fig. 2).

Fig. 2 Dwelling:3500 BCE

If the problem is to construct a certain type of dwelling. Then it can be seen that the original number of requirements may be very large, that these are often modified by available primary building materials like clay, rocks, granite, bricks, unbaked bricks, and compacted clay. Moreover, the tools available to process these can substantially modify the original plan. That is the problem is modified by accessible granules.

This principle is often used for problem solving by people in many contexts. Seasonal variation in production and consumption of vegetables and marine fish in coastal regions of countries like India are well known. The diet of people that depend on these variations may be understood in terms of granularity in the primitive sense. To see this,

- Consider an information table documenting dietary practices with column names for *identity, age, gender, income-group, profession, dietary-item-1, ... , dietary item-n, Favorite foods, Ideal diet,...* or *identity, age, gender, income-group, profession, diet:January, ... , diet:December, Favorite foods, ...* with the valuations in the former for dietary items including information about seasons, expenditure and quantity consumed.
- In both tables, the goal of any subclass of people may be to attain desired ideal diets. But available granules modify approximations of this and the very conception of ideal diet. If the only easily available foods that belong to an ideal diet are too rich in carbohydrates and lacking in protein, then some rethinking of the concept of ideal diet is necessary.
- The main computing process involves optimal choice of available granules. Fish of type-1 may be available in winter, but may be scarce and expensive during other months.

The main features of this and other primitive versions of the paradigm may be seen to be

- Problem requirements are not rigid.
- Concept of granules may be vague.
- Little effort on formalization right up to approximately the middle of the previous century.
- Scope of abstraction is very limited.
- Concept of granules may be concrete or abstract (relative all materialist viewpoints).

2.3 Classical Granular Computing Paradigm

Suppose Alice is painting a toilet sign (of the form indicated in Fig. 3) using brushes of different sizes and a fixed paint palette. Many parts of the sign would be doable with the help of brushes of different sizes. So, in effect, Alice would be able to use many distinct subsets of brushes to paint the sign. Any of these choices is associated with style, the time required to complete the sign and quality. Alice can handle all of these aspects of the entire context in an approximate way through granular strategies without reasoning about every possibility, reasoning about every comparison and reasoning about integrating the result of comparisons.

An example of a granular strategy in the situation can be the following:

- Draw outline of sign using stencils,
- Identify finest and broadest areas,
- Select 1–2 brushes,
- Paint as appropriate,

Fig. 3 WC-Sign

- Check quality of final work,
- Stop or repeat steps using more appropriate brush sizes.

The above strategy is an example of classical granular computing in the context. It is because painting brushes are available in fixed standard sizes. It differs from PGCP in that the form of the sign was preconceived. Brushes are not key determinants of the final concept.

In [91], the precision based granular computing paradigm was traced to Moore and Shannon's paper [109, 152] and named as the *classical granular computing paradigm* CGCP by the present author. This is relative to the information-theoretic perspective. CGCP is often referred to as the granular computing paradigm. CGCP has since been used in soft, fuzzy and rough set theories in different ways. A useful (though dated) overview is in [74].

Granules may be assumed to subsume the concept of information granules—information at some level of precision. In granular approaches to both rough and fuzzy sets, information granules in this sense are more commonly used in practice. Some of the fragments involved in applying CGCP may be:

- Paradigm Fragment-1: Granules can exist at different levels of precision.
- Paradigm Fragment-2: Among the many precision levels, a precision level at which the problem at hand is solvable should be selected.
- Paradigm Fragment-3: Granulations (granules at specific levels or processes) form a hierarchy (later development).
- Paradigm Fragment-4: It is possible to easily switch between precision levels.
- Paradigm Fragment-5: The problem under investigation may be represented by the hierarchy of multiple levels of granulations.

It can be argued that CGCP is of very ancient origin and that it has always been in use after the development of reasonable concepts of precision in human history. The Babylonian method of computing square roots that dates to at least 500 BC is the following:

Babylonian Method

- Problem: To compute $\sqrt{x}$, $x \in R_+$.
- Initialization: Select an arbitrary value x_o close to $\sqrt{x}$.
- Recursion Step: $x_{n+1} = 0.5(x_n + \frac{x}{x_n})$ for $n \in Z_+$
- Repeat previous step till desired accuracy is attained
- This is a quadratically convergent algorithm.
- *Good initialization is necessary for fast convergence*

The Babylonian method is a special case of many other methods including the Newton-Raphson method and the modern Householder's method. The algorithm for the latter is similar

Theorem 1 *If $f : R \longmapsto R$ is a real function that is $r + 1$-times differentiable, then its roots may be approximated by Householder's recursive algorithm. The* *algorithm's performance depends on good initialization and the key recurrence step is*

$$x_{n+1} = x_n + r\frac{(\frac{1}{f})^{(r-1)}(x_n)}{(\frac{1}{f})^{(r)}(x_n)} \quad (1)$$

If a is a zero of f, but not of its derivative, then in a suitable neighborhood of a, the following holds:

$$(\exists \epsilon > 0)\ |x_{n+1} - a| \leq \epsilon |x_n - a|^{r+1} \quad (2)$$

In mathematical contexts, it is possible to indicate concepts of precision in a number of ways (as in the above context of approximation of zeros of a nonlinear real function):

- Fixed values of initialization correspond to bounds on the precision of the solution at different cycles of computation.
- If the precision of the solution desired is alone fixed, then wide variation in initialization would be admissible.
- If the time required for computation is alone fixed or specified by an interval, then again wide variation in precision of initialization would be admissible.

This scenario suggests the following problem: *Can CGCP be classified or graded relative to the ways in which the precision can be categorized?*

2.4 Axiomatic Granular Computing Paradigm

The axiomatic approach to granularity essentially consists in investigations relating to axioms satisfied by granules, the very definitions of granules and associated frameworks. Emphasis on axiomatic properties of granules can be traced to papers in the year 2007. That is, if covers used in constructing approximations are overlooked. Neighborhoods had been investigated by a number of authors (see references in [38, 91, 144, 174]) with emphasis on point-wise approximations. A systematic axiomatic approach to granules and granulations has been due to the present author in [89, 91]. Relatively more specific versions of this approach have rich algebraic semantics associated. Parts of the axiomatic approach developed by the present author for general rough sets have been known in some form in implicit terms. But these were not stressed in a proper way because of the partial dominance of the point-wise approach.

Though Rough Y-Systems are more general, most of the algebraic semantics associated with general rough sets can be formulated in general granular operator spaces. So, in this research chapter, the axiomatic approach will be specialized to the framework.

The stages of development of different granular computing paradigms are as below (this supersedes [91]):

- Classical Primitive Paradigm, CGCP till middle of previous century.
- CGCP in Information Theory: Since Shannon's information theory
- CGCP in fuzzy set theory. It is natural for most real-valued types of fuzzy sets, but even in such domains unsatisfactory results are normal. Type-2 fuzzy sets have an advantage over type-1 fuzzy sets in handling data relating to emotion words, for example, but still far from satisfactory. For one thing linguistic hedges have little to do with numbers. A useful reference would be [176].
- For a long period (up to 2008 or so), the adaptation of CGCP for rough sets has been based solely on precision and related philosophical aspects. The adaptation is described for example in [163, 166, 169]. In [166] the hierarchical structure of granulations is also stressed. This and many later papers on CGCP (like [74, 75, 77]) in rough sets speak of structure of granulations.
- Some Papers with explicit reference to multiple types of granules from a semantic viewpoint include [85, 86, 89, 157, 160].
- The axiomatic approach to granularity initiated in [89] has been developed by the present author in the direction of contamination reduction in [91, 92, 97, 106]. The concept of admissible granules, mentioned earlier, was arrived in the latter paper. From the order-theoretic algebraic point of view, the deviation is in a new direction relative to the precision-based paradigm. The paradigm shift includes a new approach to measures.

2.5 Comparative Actualization in Rough Sets

In the present author's approach, a rough approximation operator can fall under the following categories:

- Granular (in the axiomatic sense)
- Co-Granular (this includes many of the point-wise cover and relation-based approaches) [105]
- Pointwise
- Abstract
- Empirical

This is explained below.

The axiomatic approach has been explained in much depth in this chapter and the papers [89–91, 97] by the present author. In cover based rough sets, three kinds of approximations are mentioned in [174]. Of these the subsystem based approximations would fall under the axiomatic granular approach and are not non granular. This is because in the approach, granulations are necessarily set-theoretically derived from covers (while the approximations remain a simple union of granules).

Co-granularity was defined in [105] as below:

Definition 2 By a *Co-Granular Operator Space By Ideals* GOSI will be meant a structure of the form $S = \langle \underline{S}, \sigma, \mathcal{G}, l_*, u_* \rangle$ with $\underline{S}$ being a set, σ being a binary relation on S, $\mathcal{G}$ a *neighborhood granulation* over S, $\gamma : S \longmapsto \mathcal{G}$ being a map and l_*, u_* being **-lower and *-upper approximation operators* : $\wp(\underline{S}) \longmapsto \wp(\underline{S})$ ($\wp(\underline{S})$ denotes the power set of $\underline{S}$) defined as below for a collection of some-sense ideals $\mathcal{I}, \mathcal{J}$ of $\wp(S)$ ($\underline{S}$ is replaced with S if clear from the context. Lower and upper case alphabets may denote subsets):

$$(\forall X \in \wp(S))\, X^{l_*} = \{a : a \in X \,\&\, \gamma(a) \cap X^c \in \mathcal{I}_\sigma(S)\} \qquad \text{(*-Lower)}$$

$$(\forall X \in \wp(S))\, X^{u_*} = \{a : a \in S \,\&\, \gamma(a) \cap X \notin \mathcal{I}_\sigma(S)\} \cup X \qquad \text{(*-Upper)}$$

In general, if rough approximations of a subset $X \subseteq S$ are defined by expressions of the form

$$X^{\oplus} = \{a : \gamma(a) \odot X^* \in \mathcal{J}\}$$

with $\oplus \in \{l, u\}$, $\mathcal{G} \subset \wp(S)$, $\gamma : S \longmapsto \mathcal{G}$ being a map, $* \in \{c, 1\}$ and $\odot \in \{\cap, \cup\}$, then the approximations will be said to be *co-granular*.

A general definition of point-wise approximations can be proposed in SOPL (or alternatively, in a fixed language) based on the following loose SOPL version : If S is an algebraic system of type τ and $\nu : S \longmapsto \wp(S)$ is a neighborhood map on the universe S, then a *point-wise approximation* $*$ of a subset $X \subseteq S$ is a self-map on $\wp(S)$ that is definable in the form:

$$X^* = \{x : x \in H \subseteq S \,\&\, \Phi(\nu(x), X)\} \qquad (3)$$

for some formula $\Phi(A, B)$ with $A, B \in \wp(S)$.

By *empirical approximations* is meant a set of approximations that have been specified in a concrete empirical context. These may not necessarily be based on known processes or definite attributes. Examples of such approximations have been discussed by the present author in rough contexts in [91, 103]. The first part of the main example from [103] is stated below:

Example 3 This example has the form of a narrative that gets progressively complex.

Suppose Alice wants to purchase a laptop from an on line store for electronics. Then she is likely to be confronted by a large number of models and offers from different manufacturers and sellers. Suppose also that the she is willing to spend less than €x and is pretty open to considering a number of models. This can happen, for example, when she is just looking for a laptop with enough computing power for her programming tasks.

This situation may appear to have originated from information tables with complex rules in columns for decisions and preferences. Such tables are not information systems in the proper sense. Computing power, for one thing is a context dependent function of CPU cache memories, number of cores, CPU frequency, RAM, architecture of chipset, and other factors like type of hard disk storage.

Proposition 4 *The set of laptops $\mathbb{S}$ that are priced less than €x can be totally quasi ordered.*

Proof Suppose $\prec$ is the relation defined according to $a \prec b$ if and only if price of laptop a is less than or equal to that of laptop b. Then it is easy to see that $\prec$ is a reflexive and transitive relation. If two different laptops a and b have the same price, then $a \prec b$ and $b \prec a$ would hold. So $\prec$ may not be antisymmetric. □

Suppose that under an additional constraint like CPU brand preference, the set of laptops becomes totally ordered. That is under a revised definition of $\prec$ of the form: $a \prec b$ if and only if price of laptop a is less than that of laptop b and if the prices are equal then CPU brand of b must be preferred over a's.

Suppose now that Alice has more knowledge about a subset C of models in the set of laptops $\mathbb{S}$. Let these be labeled as *crisp* and let the order on C be $\prec_{|C}$. Using additional criteria, rough objects can be indicated. Though lower and upper approximations can be defined in the scenario, the granulations actually used are harder to arrive at without all the gory details. *Note that idea of a laptop being as good as another is actually about approximations in the scenario*.

This example once again shows that granulation and construction of approximations from granules may not be related to the construction of approximations from properties in a cumulative way.

3 Granular Operator Spaces and Variants

Granular operator spaces and related variants are not necessarily basic systems in the context of application of general rough sets. They are powerful abstractions for handling semantic questions, formulation of semantics and the inverse problem. These may be seen as abstract operator based approach to rough sets enhanced with the axiomatic approach to granularity and without point-wise approximations and negation operations. It is important to stress all these aspects because some connections between the abstract operator approach and general approximation spaces from the perspective of point-wise approximations are well known (see [163]).

Definition 5 A *Granular Operator Space*[98] S is a structure of the form $S = \langle \underline{S}, \mathcal{G}, l, u \rangle$ with $\underline{S}$ being a set, $\mathcal{G}$ an *admissible granulation*(defined below) over S and l, u being operators : $\wp(\underline{S}) \longmapsto \wp(\underline{S})$ ($\wp(\underline{S})$ denotes the power set of $\underline{S}$) satisfying the following ($\underline{S}$ will be replaced with S if clear from the context. Lower and upper case alphabets will both be used for subsets):

$$a^l \subseteq a \,\&\, a^{ll} = a^l \,\&\, a^u \subset a^{uu}$$

$$(a \subseteq b \longrightarrow a^l \subseteq b^l \,\&\, a^u \subseteq b^u)$$

$$\emptyset^l = \emptyset \,\&\, \emptyset^u = \emptyset \,\&\, \underline{S}^l \subseteq S \,\&\, \underline{S}^u \subseteq S.$$

Here, *Admissible granulations* are granulations $\mathcal{G}$ that satisfy the following three conditions (t is a term operation formed from the set operations $\cup, \cap, ^c, 1, \emptyset$):

$$(\forall a \exists b_1, \ldots b_r \in \mathcal{G})\, t(b_1, b_2, \ldots b_r) = a^l$$
$$\text{and } (\forall a)\, (\exists b_1, \ldots b_r \in \mathcal{G})\, t(b_1, b_2, \ldots b_r) = a^u, \qquad \text{(Weak RA, WRA)}$$

$$(\forall b \in \mathcal{G})(\forall a \in \wp(\underline{S}))\, (b \subseteq a \longrightarrow b \subseteq a^l), \qquad \text{(Lower Stability, LS)}$$

$$(\forall a, b \in \mathcal{G})(\exists z \in \wp(\underline{S}))\, a \subset z, b \subset z \,\&\, z^l = z^u = z, \qquad \text{(Full Underlap, FU)}$$

Remarks

- The concept of admissible granulation was defined for RYS in [91] using parthoods instead of set inclusion and relative to RYS, $\mathbf{P} = \subseteq$, $\mathbb{P} = \subset$.
- The conditions defining admissible granulations mean that every approximation is somehow representable by granules in a set theoretic way, that granules are lower definite, and that all pairs of distinct granules are contained in definite objects.

On $\wp(\underline{S})$, the relation $\sqsubset$ is defined by

$$A \sqsubset B \text{ if and only if } A^l \subseteq B^l \,\&\, A^u \subseteq B^u. \tag{4}$$

The rough equality relation on $\wp(\underline{S})$ is defined via $A \approx B$ if and only if $A \sqsubset B \,\&\, B \sqsubset A$.

Regarding the quotient $\wp(\underline{S})| \approx$ as a subset of $\wp(\underline{S})$, the order $\Subset$ will be defined as per

$$\alpha \Subset \beta \text{ if and only if } \alpha^l \subseteq \beta^l \,\&\, \alpha^u \subseteq \beta^u. \tag{5}$$

Here α^l is being interpreted as the lower approximation of α and so on. $\Subset$ will be referred to as the *basic rough order*.

Definition 6 By a *roughly consistent object* will be meant a set of subsets of $\underline{S}$ of the form $H = \{A; (\forall B \in H)\, A^l = B^l, A^u = B^u\}$. The set of all roughly consistent objects is partially ordered by the inclusion relation. Relative this maximal roughly consistent objects will be referred to as *rough objects*. By *definite rough objects*, will be meant rough objects of the form H that satisfy

$$(\forall A \in H)\, A^{ll} = A^{l} \,\&\, A^{uu} = A^{u}. \tag{6}$$

Proposition 7 *$\Subset$ is a bounded partial order on $\underline{S}| \approx$.*

Proof Reflexivity is obvious. If $\alpha \Subset \beta$ and $\beta \Subset \alpha$, then it follows that $\alpha^l = \beta^l$ and $\alpha^u = \beta^u$ and so antisymmetry holds.

If $\alpha \Subset \beta$, $\beta \Subset \gamma$, then the transitivity of set inclusion induces transitivity of $\Subset$. The poset is bounded by $0 = (\emptyset, \emptyset)$ and $1 = (S^l, S^u)$. Note that 1 need not coincide with (S, S). □

The concept of *general granular operator spaces* had been introduced in [102, 106] as a proper generalization of that of granular operator spaces. The main difference is in the replacement of $\subset$ by arbitrary *part of* ($\mathbf{P}$) relations in the axioms of admissible granules and inclusion of $\mathbf{P}$ in the signature of the structure.

Definition 8 A *general granular operator space* (GSP) S shall be a structure of the form $S = \langle \underline{S}, \mathcal{G}, l, u, \mathbf{P} \rangle$ with $\underline{S}$ being a set, $\mathcal{G}$ an *admissible granulation*(defined below) over S, l, u being operators : $\wp(\underline{S}) \longmapsto \wp(\underline{S})$ and $\mathbf{P}$ being a definable binary generalized transitive predicate (for parthood) on $\wp(\underline{S})$ satisfying the following conditions (generalized transitivity can be any proper nontrivial generalization of parthood (see [99]). $\mathbb{P}$ is proper parthood (defined via $\mathbb{P}ab$ if and only if $\mathbf{P}ab \,\&\, \neg \mathbf{P}ba$) and t is a term operation formed from set operations):

$$\mathbf{P}a^l a \,\&\, a^{ll} = a^l \,\&\, \mathbb{P}a^u a^{uu}$$

$$(\mathbf{P}ab \longrightarrow \mathbf{P}a^l b^l \,\&\, \mathbf{P}a^u b^u)$$

$$\emptyset^l = \emptyset \,\&\, \emptyset^u = \emptyset \,\&\, \mathbf{P}\underline{S}^l S \,\&\, \mathbf{P}\underline{S}^u S.$$

$$(\forall x \exists y_1, \ldots y_r \in \mathcal{G})\, t(y_1, y_2, \ldots y_r) = x^l$$

$$\text{and } (\forall x)\, (\exists y_1, \ldots y_r \in \mathcal{G})\, t(y_1, y_2, \ldots y_r) = x^u, \qquad \text{(Weak RA, WRA)}$$

$$(\forall y \in \mathcal{G})(\forall x \in \wp(\underline{S}))\, (\mathbf{P}yx \longrightarrow \mathbf{P}yx^l), \qquad \text{(Lower Stability, LS)}$$

$$(\forall x, y \in \mathcal{G})(\exists z \in \wp(\underline{S}))\, \mathbb{P}xz, \,\&\, \mathbb{P}yz \,\&\, z^l = z^u = z, \qquad \text{(Full Underlap, FU)}$$

It is sometimes more convenient to use only sets and subsets in the formalism as these are the kinds of objects that may be observed by agents and such a formalism would be more suited for reformulation in formal languages. This justifies the severe variation defined in [103]:

Definition 9 A *Higher Rough Operator Space* $\mathbb{S}$ shall be a structure of the form $\mathbb{S} = \langle \underline{\mathbb{S}}, l, u, \leq, \bot, \top \rangle$ with $\underline{\mathbb{S}}$ being a set, and l, u being operators : $\underline{\mathbb{S}} \longmapsto \underline{\mathbb{S}}$

satisfying the following ($\underline{\mathbb{S}}$ is replaced with $\mathbb{S}$ if clear from the context.):

$$(\forall a \in \mathbb{S})\, a^l \leq a \,\&\, a^{ll} = a^l \,\&\, a^u \leq a^{uu}$$
$$(\forall a, b \in \mathbb{S})(a \leq b \longrightarrow a^l \leq b^l \,\&\, a^u \leq b^u)$$
$$\bot^l = \bot \,\&\, \bot^u = \bot \,\&\, \top^l \leq \top \,\&\, \top^u \leq \top$$
$$(\forall a \in \mathbb{S})\, \bot \leq a \leq \top$$
$$\mathbb{S} \text{ is a bounded poset.}$$

Definition 10 A *Higher Granular Operator Space* (SHG) $\mathbb{S}$ shall be a structure of the form $\mathbb{S} = \langle \underline{\mathbb{S}}, \mathcal{G}, l, u, \leq, \vee, \wedge, \bot, \top \rangle$ with $\underline{\mathbb{S}}$ being a set, $\mathcal{G}$ an *admissible granulation*(defined below) for $\mathbb{S}$ and l, u being operators : $\underline{\mathbb{S}} \longmapsto \underline{\mathbb{S}}$ satisfying the following ($\underline{\mathbb{S}}$ is replaced with $\mathbb{S}$ if clear from the context.):

$$(\mathbb{S}, \vee, \wedge, \bot, \top) \text{ is a bounded lattice}$$
$$\leq \text{ is the lattice order}$$
$$(\forall a \in \mathbb{S})\, a^l \leq a \,\&\, a^{ll} = a^l \,\&\, a^u \leq a^{uu}$$
$$(\forall a, b \in \mathbb{S})(a \leq b \longrightarrow a^l \leq b^l \,\&\, a^u \leq b^u)$$
$$\bot^l = \bot \,\&\, \bot^u = \bot \,\&\, \top^l \leq \top \,\&\, \top^u \leq \top$$
$$(\forall a \in \mathbb{S})\, \bot \leq a \leq \top$$

$\mathbf{P}ab$ if and only if $a \leq b$ in the following three conditions. Further $\mathbb{P}$ is proper parthood (defined via $\mathbb{P}ab$ if and only if $\mathbf{P}ab \,\&\, \neg \mathbf{P}ba$) and t is a term operation formed from the lattice operations):

$$(\forall x \exists y_1, \ldots y_r \in \mathcal{G})\, t(y_1, y_2, \ldots y_r) = x^l$$
$$\text{and } (\forall x)\, (\exists y_1, \ldots y_r \in \mathcal{G})\, t(y_1, y_2, \ldots y_r) = x^u, \qquad \text{(Weak RA, WRA)}$$
$$(\forall y \in \mathcal{G})(\forall x \in \underline{\mathbb{S}})\, (\mathbf{P}yx \longrightarrow \mathbf{P}yx^l), \qquad \text{(Lower Stability, LS)}$$
$$(\forall x, y \in \mathcal{G})(\exists z \in \underline{\mathbb{S}})\, \mathbb{P}xz, \,\&\, \mathbb{P}yz \,\&\, z^l = z^u = z \qquad \text{(Full Underlap, FU)}$$

Definition 11 An element $x \in \mathbb{S}$ will be said to be *lower definite* (resp. *upper definite*) if and only if $x^l = x$ (resp. $x^u = x$) and *definite*, when it is both lower and upper definite. $x \in \mathbb{S}$ will also be said to be *weakly upper definite* (resp *weakly definite*) if and only if $x^u = x^{uu}$ (resp $x^u = x^{uu} \,\&\, x^l = x$). Any one of these five concepts may be chosen as a concept of *crispness*.

3.1 Rough Objects

The concept of rough objects must necessarily relate to some of the following:

- object level properties of approximations in a suitable semantic domain,
- object level properties of discernibility in a suitable semantic domain,
- object level properties of indiscernibility in a suitable semantic domain,
- properties of abstractions from approximations,
- properties of abstractions of indiscernibility, or
- some higher level semantic features (possibly constructed on the basis of some assumptions about approximations).

A rough object cannot be known exactly in the rough semantic domain, but can be represented through various means. This single statement hides deep philosophical aspects that are very relevant in practice if realizable in concrete terms. The following concepts of *rough objects* have been either considered in the literature (see [91, 103]) or are reasonable concepts:

RL $x \in \mathbb{S}$ is a lower rough object if and only if $\neg(x^l = x)$.
RU $x \in \mathbb{S}$ is a upper rough object if and only if $\neg(x = x^u)$.
RW $x \in \mathbb{S}$ is a weakly upper rough object if and only if $\neg(x^u = x^{uu})$.
RB $x \in \mathbb{S}$ is a rough object if and only if $\neg(x^l = x^u)$. The condition is equivalent to the boundary being nonempty.
RD *Any pair of definite elements* of the form (a, b) satisfying $a < b$
RP *Any distinct pair of elements* of the form (x^l, x^u).
RIA Elements in an *interval of the form* (x^l, x^u).
RI Elements in an *interval of the form* (a, b) satisfying $a \leq b$ with a, b being definite elements.
ET In esoteric rough sets [85], triples of the form (x^l, x^{lu}, x^u) can be taken as rough objects.
RND A *non-definite element in a RYS*(see [91]), that is an x satisfying $\neg \mathbf{P} x^u x^l$. This can have a far more complex structure when multiple approximations are available.
ROP If a weak negation or complementation ^{c} is available, then orthopairs of the form $(x^l, x^u c)$ can also be taken as representations of *rough objects*.

All of the above concepts of a rough object except for the last two are directly usable in a higher granular operator space.

The positive region of a $x \in \mathbb{S}$ is x^l, while its negative region is x^{uc} – this region is independent from x in the sense of attributes being distinct, but not in the sense of derivability or inference by way of rules. These derived concepts provide additional approaches to specifying subtypes of rough objects and related decision making strategies.

- $POS(x) = x^l$ and $NEG(x) = x^{uc}$ by definition.
- x is *roughly definable* if $POS(x) \neq \emptyset$ and $NEG(x) \neq \emptyset$
- x is *externally undefinable* if $POS(x) \neq \emptyset$ and $NEG(x) = \emptyset$

- x is *internally undefinable* if $POS(x) = \emptyset$ and $NEG(x) \neq \emptyset$
- x is *totally undefinable* if $POS(x) = \emptyset$ and $NEG(x) = \emptyset$

It makes sense to extend this conception in general rough contexts where there exist subsets x for which $x^u \subset x^{uu}$ is possible. In the context of cover based rough sets, the cover is often adjoined to NEG, POS as a subscript as in $NEG_{\mathcal{S}}$ and $POS_{\mathcal{S}}$ respectively.

Definition 12 By the strong negative region associated with a subset $x \in \mathbb{S}$ will be meant the element

$$SNEG(x) = x^{uuc}$$

The reader is invited to construct diagrammatic examples illustrating the above.

3.2 Granularity Axioms

Even when additional lower and upper approximation operators are added to a general granular operator space, the resulting framework will still be referred to as a general granular operator space. In such a framework, granules definitely satisfy some and may satisfy others in the following list of axioms. It is assumed that a finite number of lower ($\{l_i\}_{i=1}^n$) and upper ($\{u_i\}_{i=1}^n$) approximations are used. These have been grouped as per common functionality. These conditions play a central role in defining possible concepts of granules. So they are referred to as axioms.

Representation Related Axioms

The central idea expressed by these axioms is that approximations are formed from granules through set theoretic or more general operations on granules. In classical rough sets, every approximation is a union of equivalence classes (the granules).

$$\forall i,\ (\forall x)(\exists a_1, \ldots a_r \in \mathcal{G})\, a_1 + a_2 + \ldots + a_r = x^{l_i} \text{ and}$$
$$(\forall x)(\exists a_1,\ \ldots\ a_p \in \mathcal{G})\, a_1 + a_2 + \ldots + a_p = x^{u_i} \qquad \text{(Representability, RA)}$$

In the weaker versions below, approximations are assumed to be representable by derived terms instead of through aggregation of granules.

$$\forall i,\ (\forall x \exists a_1, \ldots a_r \in \mathcal{G})\, t_i(a_1,\ a_2, \ldots\ a_r) = x^{l_i} \text{ and}$$
$$(\forall x)(\exists a_1,\ \ldots\ a_r \in \mathcal{G})\, t_i(a_1,\ a_2, \ldots\ a_p) = x^{u_i} \qquad \text{(Weak RA, WRA)}$$

The prefix *sub* can be used to indicate situations, where only a subset of approximations happen to be representable.

$$\exists i,\ (\forall x)(\exists a_1, \ldots a_r \in \mathcal{G})\, a_1 + a_2 + \ldots + a_r = x^{l_i} \text{ and}$$
$$(\forall x)(\exists a_1,\ \ldots\ a_p \in \mathcal{G})\, a_1 + a_2 + \ldots + a_p = x^{u_i} \qquad \text{(Sub RA, SRA)}$$

Further subsetting to lower and upper approximations has been indicated in the following axioms.

$$\forall i,\ (\forall x)(\exists a_1, \ldots a_r \in \mathcal{G}) a_1 + a_2 + \ldots + a_r = x^{l_i} \qquad \text{(Lower RA, LRA)}$$

$$\forall i,\ (\forall x)(\exists a_1,\ \ldots\ a_p \in \mathcal{G})\, a_1 + a_2 + \ldots + a_p = x^{u_i} \qquad \text{(Upper RA, URA)}$$

$$\exists i,\ (\forall x)(\exists a_1, \ldots a_r \in \mathcal{G})\, a_1 + a_2 + \ldots + a_r = x^{l_i} \qquad \text{(Lower SRA, LSRA)}$$

$$\exists i,\ (\forall x)(\exists a_1, \ldots a_p \in \mathcal{G})\, a_1 + a_2 + \ldots + a_p = x^{u_i} \qquad \text{(Upper SRA, USRA)}$$

Crispness Axioms

Something is crisp *in a sense if it is its own approximation in that sense*. This is quite different from claiming that *something is crisp if it cannot be approximated by anything else*. The following crispness axioms of granules concern granules alone.

$$For\ each\ i,\ (\forall a \in \mathcal{G})\, a^{l_i} = a^{u_i} = a \qquad \text{(Absolute Crispness,ACG)}$$

$$\exists i,\ (\forall a \in \mathcal{G}) a^{l_i} = a^{u_i} = a \qquad \text{(Sub Crispness*, SCG)}$$

Crispness Variants: By analogy, the crispness variants LACG, UACG, LSCG, USCG can be defined as for representability.

Mereological Axioms

The axioms for mereological properties of granules is presented next. The axiom of *mereological atomicity* says that no definite elements (relative to any permitted pair of lower and upper approximations) can be proper parts of granules.

$$\forall i,\ (\forall a \in \mathcal{G})(\forall x \in S)(\mathbf{P}xa,\ x^{l_i} = x^{u_i} = x \longrightarrow x = a)$$
$$\text{(Mereological Atomicity, MER)}$$

The axiom of *sub-mereological atomicity* says that no definite elements (relative to at least one specific pair of lower and upper approximations) can be proper parts of granules.

$$\exists i, (\forall a \in \mathcal{G})(\forall x \in S)(\mathbf{P}xa,\ x^{l_i} = x^{u_i} = x \longrightarrow x = a) \qquad \text{(Sub MER,SMER)}$$

The axiom of *inward-mereological atomicity* says that no definite elements (relative to every permitted pair of lower and upper approximations) can be proper parts of granules.

$$(\forall a \in \mathcal{G})(\forall x \in S)(\mathbf{P}xa,\ \bigwedge_i (x^{l_i} = x^{u_i} = x) \longrightarrow x = a) \qquad \text{(Inward MER, IMER)}$$

$$\forall i, (\forall a \in \mathcal{G})(\forall x \in S)(\mathbf{P}xa,\ x^{l_i} = x \longrightarrow x = a) \qquad \text{(Lower MER, LMER)}$$

$$(\forall a \in \mathcal{G})(\forall x \in S)(\mathbf{P}xa,\ \bigwedge_i (x^{l_i} = x) \longrightarrow x = a) \qquad \text{(Inward LMER, ILMER)}$$

MER Variants: The variants UMER, LSMER, USMER, IUMER can be defined by analogy.

Stability Axioms

The basic idea behind stability of granules is that granules should preserve appropriate parthood relations relative to approximations. *Lower stability*, defined below, says that if a granule is part of an object, then the granule should still be part of the lower approximation of the object. In general, the same does not hold for all objects.

$$\forall i,\ (\forall a \in \mathcal{G})(\forall x \in S)(\mathbf{P}ax \longrightarrow \mathbf{P}(a)(x^{l_i})) \qquad \text{(Lower Stability, LS)}$$

$$\forall i, (\forall a \in \mathcal{G})(\forall x \in S)(\mathbf{O}ax \longrightarrow \mathbf{P}ax^{u_i}) \qquad \text{(Upper Stability, US)}$$

$$\text{LS \& US} \qquad \text{(Stability, ST)}$$

$$\exists i,\ (\forall a \in \mathcal{G})(\forall x \in S)(\mathbf{P}ax \longrightarrow \mathbf{P}(a)(x^{l_i})) \qquad \text{(Sub LS, LSS)}$$

$$\exists i,\ (\forall a \in \mathcal{G})(\forall x \in S)(\mathbf{O}ax \longrightarrow \mathbf{P}(a)(x^{u_i})) \qquad \text{(Sub US, USS)}$$

$$\text{LSS \& USS} \qquad \text{(Sub ST, SST)}$$

Overlap Axioms

The possible implications of overlap and underlap relations between granules is captured by these axioms.

$$(\forall x, a \in \mathcal{G})\neg\mathbb{O}xa, \qquad \text{(No Overlap, NO)}$$

$$\forall i, (\forall x, a \in \mathcal{G})(\exists z \in S)\mathbb{P}xz, \mathbb{P}az, z^{l_i} = z^{u_i} = z \qquad \text{(Full Underlap, FU)}$$

$$\forall i, (\forall x, a \in \mathcal{G})(\exists z \in S)\mathbb{P}xz, \mathbb{P}az, z^{l_i} = z \qquad \text{(LFU)}$$

$$\exists i, (\forall x, a \in \mathcal{G})(\exists z \in S)\mathbb{P}xz, \mathbb{P}az, z^{l_i} = z^{u_i} = z \qquad \text{(SFU)}$$

$$\exists i, (\forall x, a \in \mathcal{G})(\exists z \in S)\mathbb{P}xz, \mathbb{P}az, z^{l_i} = z \qquad \text{(Sub LFU)}$$

$$\exists i, (\forall x, a \in \mathcal{G})(\mathbb{P}xz, \mathbb{P}az, z^{l_i} = z^{u_i} = z, \mathbb{P}xb, \mathbb{P}ab$$
$$b^{l_i} = b^{u_i} = b \longrightarrow z = b) \qquad \text{(Unique Underlap, UU)}$$

Idempotence Axioms

Possible idempotence properties of approximation operators relative to granules are captured by these axioms.

$$\forall i, (\forall x \in \mathcal{G})x^{l_i} = x^{l_i l_i} \qquad \text{(Lower Idempotence, LI)}$$

$$\forall i, (\forall x \in \mathcal{G})x^{u_i} = x^{u_i u_i} \qquad \text{(Upper Idempotence, UI)}$$

$$\forall i, (\forall x \in \mathcal{G})x^{u_i} = x^{u_i u_i}, x^{l_i} = x^{l_i l_i}. \qquad \text{(Idempotence, I)}$$

The *pre-similarity* axiom concerns the relation of commonalities between granules and parthood.

$$(\forall x, a \in \mathcal{G})(\exists z \in \mathcal{G})\mathbf{P}(x \cdot a)(z) \qquad \text{(Pre-similarity, PS)}$$

Apparently the three axioms WRA, LS, LU hold in most of the known theories and with most choices of granules. This has been the main motivation for the definition of admissibility of a subset to be regarded as a granule in [91] and in the definition of granular operator spaces.

Granular operator spaces are also related to property systems [121, 124, 145, 146]. The connections are considered in the chapter on representation and duality in the present volume [107] by the present author.

3.3 Specific Cases

A small set of examples, that partially justify the formalism of the axioms are presented in this subsection. More details can be found in [91] and other specific semantics considered in this chapter.

Theorem 13 *In classical rough sets, if $\mathcal{G}$ is the set of partitions, then all of* RA, ACG, MER, AS, FU, NO, PS, I, ST *hold.* UU *does not hold in general.*

In esoteric rough sets [85], partial equivalence relations (symmetric, transitive and partially reflexive relations) are used to generate approximations instead of equivalence relations. In the simplest case, the upper and lower approximations of a subset A of a partial approximation space $\langle S, R\rangle$ are defined via ($[x] = \{a;\ Rxa\}$ being the pseudo-class generated by x)

$$A^l = \bigcup\{[x];\ [x] \subseteq A\};\ \ A^u = \bigcup\{[x];\ [x] \cap A \neq \emptyset\}.$$

For more details, see the section on Esoteric rough sets.

Theorem 14 *In case of esoteric rough sets [85], with the collection of all pseudo-classes being the granules, all of* RA, MER, NO, UU, US *hold, but* ACG *may not.*

Proof RA, NO follow from the definition. It is possible that $[x] \subset [x]^u$, so ACG may not hold. US holds as if a granule overlaps another subset, then the upper approximation of the set would surely include the granule. □

If R is a reflexive relation on a set S and

$$A^l = \cup\{[x] : [x] \subseteq A,\ x \in A\} \text{ and}$$

$$A^u = \cup\{[x] : [x] \cap A \neq \emptyset\, x \in A\},$$

($A^l \subseteq A \subseteq A^u$ for a binary relation R is equivalent to its reflexivity [158, 165]) then $\{[x] : x \in S\}$ is a granulation that satisfies:

Theorem 15 RA, LFU *holds, but none of* MER, ACG, LI, UI, NO, FU *holds in general.*

Proof RA holds by definition, LFU holds as the lower approximation of the union of two granules is the same as the union. It is easy to define an artificial counter example to support the rest of the statement. □

Let $\langle S, (R_i)_i \in K\rangle$ be a multiple approximation space [66], then the strong lower, weak lower, strong upper and weak upper approximations of a set $X \subseteq S$ are defined as follows (modified terminology):

1. $X^{ls} = \bigcap_i X^{li}$,
2. $X^{us} = \bigcup_i X^{ui}$,

3. $X^{lw} = \bigcup_i X^{li}$,
4. $X^{uw} = \bigcap_i X^{ui}$.

Theorem 16 *In a multiple approximation of the above form, taking the set of granules to be the collection of all equivalence classes of the R_is, LSRA, USRA, LSS, USS holds, but all variants of rest do not hold always.*

Proof X^{lw}, X^{us} are obviously unions of equivalence classes. The LSS, USS part for these two approximations respectively can also be checked directly. Counterexamples can be found in [66]. But it is possible to check these by building on actual possibilities. If there are only two distinct equivalences, then at least two classes of the first must differ from two classes of the second. The ls approximation of these classes will strictly be a subset of the corresponding classes, so CG will certainly fail for the (ls, us) pair. Continuing the argument, it will follow that SCG, ACG cannot hold in general. The argument can be extended to other situations. □

3.4 Properties of Point-Wise Approximations

The basic properties of common point-wise rough approximations is summarized in this section for reference.

Let $S = \langle \underline{S}, R \rangle$ be a general approximation space with R being a binary relation. For any subset $A \subseteq \underline{S}$, the lower and upper approximations relative to the predecessor neighborhood are

$$A^{\bar{l}} = \{a : [a]_i \subseteq A \,\&\, a \in S\} \quad \text{(inverse lower)}$$

$$A^{\bar{u}} = \{a : [a]_i \cap A \neq \emptyset \,\&\, a \in S\} \quad \text{(inverse upper)}$$

Note that some authors refer to *predecessor* neighborhoods as *successor* neighborhoods and vice versa [167].

Theorem 17 *The point-wise approximations satisfy all of the following properties ($\top = S$, $\bot = \emptyset$:*

$$(\forall A \in \wp(S))\, A^{\bar{l}} = A^{c\bar{u}c} \quad \text{(L1.S5-Dual)}$$

$$\top^{\bar{l}} = \top \quad \text{(L2. L-Top)}$$

$$(\forall A, B \in \wp(S))\, (A \cap B)^{\bar{l}} = A^{\bar{l}} \cap B^{\bar{l}} \quad \text{(L3)}$$

$$(\forall A, B \in \wp(S))\, A^{\bar{l}} \cup B^{\bar{l}} \subseteq (A \cup B)^{\bar{l}} \quad \text{(L4)}$$

$$(\forall A, B \in \wp(S))\, (A \subseteq B \longrightarrow A^{\bar{l}} \subseteq B^{\bar{l}}) \quad \text{(L5)}$$

$$(\forall A \in \wp(S))\, A^{\bar{u}} = A^{c\bar{l}c} \quad \text{(U1.S5-Dual)}$$

$$\bot^{\bar{u}} = \bot \tag{U2}$$

$$(\forall A, B \in \wp(S))\,(A \cup B)^{\bar{u}} = A^{\bar{u}} \cup B^{\bar{u}} \tag{U3}$$

$$(\forall A, B \in \wp(S))\,(A \cap B)^{\bar{u}} \subseteq A^{\bar{u}} \cap B^{\bar{u}} \tag{U4}$$

$$(\forall A, B \in \wp(S))\, A \subseteq B \longrightarrow A^{\bar{u}} \subseteq B^{\bar{u}} \tag{U5}$$

Proof The proof of L1 and U2 is by direct computation. Rest are proved through set-theoretic arguments of the form: if x is an element of the left hand side, then because of a series of reasons x is an element of the right hand side. □

When additional properties are satisfied by the relation R, then more properties of $\bar{l}$, $\bar{u}$ happen. Converse associations are false in general:

Theorem 18 *The statements below should be read as* if R satisfies the tagged property, then the approximations satisfy the indicated property*:*

$$(\forall X \in \wp(S))\, x^{\bar{l}} \subseteq x^{\bar{u}} \tag{serial, D}$$

$$\emptyset^{\bar{l}} = \emptyset \,\&\, S^{\bar{u}} = S \tag{Serial, L5,U5}$$

$$(\forall X \in \wp(S))\, X^{\bar{l}} \subseteq X \tag{Reflexive, T}$$

$$(\forall X \in \wp(S))\, X \subseteq X^{\bar{u}} \tag{Reflexive, T$'$}$$

$$(\forall X \in \wp(S))\, X \subseteq X^{\bar{u}\bar{l}} \tag{Symmetric, B}$$

$$(\forall X \in \wp(S))\, X^{\bar{l}\bar{u}} \subseteq X \tag{Symmetric, B$'$}$$

$$(\forall X \in \wp(S))\, X^{\bar{l}} \subseteq X^{\bar{l}\bar{l}} \tag{Transitive, 4}$$

$$(\forall X \in \wp(S))\, X^{\bar{u}\bar{u}} \subseteq X^{\bar{u}} \tag{Transitive, 4$'$}$$

$$(\forall X \in \wp(S))\, X^{\bar{l}} \subseteq X^{\bar{l}\bar{u}} \tag{Euclidean, 5}$$

$$(\forall X \in \wp(S))\, X^{\bar{l}\bar{u}} \subseteq X^{\bar{l}} \tag{Euclidean, 5$'$}$$

The properties L5 and U5 are equivalent to D when R is a serial relation.

The next result characterizes classical rough sets and here the point-wise and granular definitions coincide:

Theorem 19 *Any two operators $\bar{l}$, $\bar{u}$ that satisfy all of the properties L1-L5, U1, the six properties of reflexive, transitive and symmetric relations coincide with classical approximation operators.*

An early representation theorem that connects the abstract operator based approach with general approximation spaces was proved in [75]. The result depends crucially on the duality property of the lower and upper approximation property. An improved proof is in [163].

Theorem 20 *If H is a set and l and u are maps* : $\wp(H) \longmapsto \wp(H)$ *that satisfy all of*

$$(\forall A \in \wp(H))\, A^l = A^{cuc} \qquad \text{(l-u duality)}$$

$$\emptyset^u = \emptyset \qquad \text{(U2)}$$

$$(\forall A, B \in \wp(H))\, (A \cup B)^u = A^u \cup B^u \qquad \text{(u-additivity)}$$

then there exists a binary relation R on H, such that l and u coincide with the point-wise approximations generated by predecessor neighborhoods of R. If in addition, the following two conditions of inclusivity and idempotence of u hold, then a reflexive and transitive relation exists that generates the approximation:

$$(\forall A \in \wp(H))\, A \subseteq A^u \qquad \text{(Inclusivity)}$$

$$(\forall A \in \wp(H))\, A^{uu} = A^u \qquad \text{(Idempotence)}$$

$$(\forall A \in \wp(H))\, A \subseteq A^{ucuc} \qquad \text{(Preclusion)}$$

If in addition, the last condition also holds, then a equivalence relation R can be found that generates the approximations.

3.5 Granular Operator Spaces and Property Systems

Data can also be expected to be presented in real life predominantly in terms of approximations and partly in the object-attribute-value way of representing things. In this context it is important to note that the idea of property systems or related basic constructors pursued by different authors [43, 124, 174] was never intended to capture this specific scenario. The examples in [121], in particular, *are abstract ones and the possible problems with basic constructors (when viewed from the perspective of approximation properties satisfied) are issues relating to construction and empirical aspects are missed.*

Definition 21 A *property system*[121, 124, 145, 146] Π is a triple of the form $\langle U, P, R\rangle$ with U being a universe of objects, P a set of properties, and $R \subseteq U \times P$ being a *manifestation relation* subject to the interpretation object a has property b if and only if $(a, b) \in R$. When $P = U$, then Π is said to be a *square relational system* and Π then can be read as a Kripke model for a corresponding modal system.

On property systems, basic constructors that may be defined for $A \subseteq U$ and $B \subseteq P$ are

$$< i >: \wp(U) \longmapsto \wp(P);\ < i > (A) = \{h : (\exists g \in A)\, (g, h) \in R\} \qquad (7)$$

$$< e >: \wp(P) \longmapsto \wp(U); < e > (B) = \{g : (\exists h \in B)\,(g, h) \in R\} \quad (8)$$

$$[i] : \wp(U) \longmapsto \wp(P); [i](A) = \{h : (\forall g \in U)((g, h) \in R \longrightarrow g \in A)\} \quad (9)$$

$$[e] : \wp(P) \longmapsto \wp(U); [e](B) = \{g : (\forall h \in P)((g, h) \in R \longrightarrow h \in B)\} \quad (10)$$

It is known that the basic constructors may correspond to approximations under some conditions and some unclear conditions. Property system are not suitable for handling granularity and many of the inverse problem contexts. The latter part of the statement requires some explanation because suitability depends on the way in which the problem is posed—this has not been looked into comprehensively in the literature.

If all of the data is of the form

Object X is definitely approximated by $\{A_1, \ldots A_n\}$,

with the symbols X, A_i being potentially substitutable by objects, then the data could in principle be written in property system form with the sets of A_is forming the set of properties P—in the situation the relation R attains a different meaning. This is consistent with the structure being not committed to tractability of properties possessed by objects. Granularity would also be obscure in the situation.

If all of the data is of the form

Object X 's approximations are included in $\{A_1, \ldots A_n\}$,

then the property system approach comes under even more difficulties. Granular operator spaces and generalized versions thereof [102] in contrast can handle all this.

3.6 Correspondences Between Granules

Ideas of correspondence between granules are important in a wide variety of rough contexts for the following:

- Semantic correspondences [99]
- Generalized Measures of Rough Sets [92, 95]
- Defining granules, reducts and related structures

In this section, only those aspects relevant for semantics will be considered and most of the content is based on the material of [90]. The results are specialized to higher granular operator spaces (with set-union as aggregation $\oplus$ and set-intersection as commonality operation $\odot$) and proofs have been updated. The concept of a subnatural correspondence SNC is a refined version of the concept in [90] and corresponds to [92]. A map from a higher granular operator space S_1 to another S_2 will be referred to as a *correspondence*. It will be called a *morphism* if and only if it preserves the operations $\oplus$ and $\odot$. $\oplus$-morphisms and $\odot$-morphisms will be used to refer to correspondences that preserve just one of the partial/total operations. Sub-Natural Correspondences (SNC) are intended to capture simpler correspondences that associate granules with elements representable by granules and do not necessarily commit the context to Galois connections. An issue with SNCs is that it fails to adequately capture granule centric correspondences that may violate the injectivity constraint and may not play well with morphisms.

Definition 22 Let If S_1 and S_2 are two higher granular operator space with granulations, $\mathcal{G}_1$ and $\mathcal{G}_2$ respectively, consisting of successor neighborhoods or neighborhoods. A correspondence $\varphi : S_1 \longmapsto S_2$ will be said to be a *Proto Natural Correspondence (PON)* (respectively *Pre-Natural Correspondence (PNC)*) if and only if the second (respectively both) of the following conditions hold:

1. $\varphi_{|\mathcal{G}_1}$ is injective : $\mathcal{G}_1 \longmapsto \mathcal{G}_2$.
2. there is a term function t in the signature of $\wp(S_2)$ such that

$$(\forall [x] \in \mathcal{G}_1)(\exists y_1, \ldots y_n \in \mathcal{G}_2)\, \varphi([x]) = t(y_1, \ldots, y_n).$$

3. the y_is in the second condition are generated by $\varphi(\{x\})$ for each i ($\{x\}$ being a singleton).

An injective correspondence $\varphi : S_1 \longmapsto S_2$ will be said to be a *SNC* if and only if the last two conditions hold.

Note that the base sets of higher granular operator space may be semi-algebras of sets.

Theorem 23 *If φ is a SNC and both $\mathcal{G}_1$ and $\mathcal{G}_2$ are partitions, then the non-trivial cases should be equivalent to one of the following:*

$$(\forall \{x\} \in S_1)\varphi([x]) = [\varphi(\{x\})]. \tag{B1}$$

$$(\forall \{x\} \in S_1)\varphi([x]) = \sim [\varphi(\{x\})]. \tag{B2}$$

$$(\forall \{x\} \in S_1)\varphi([x]) = \bigcup_{y \in [x]} [\varphi(\{y\})]. \tag{B3}$$

$$(\forall \{x\} \in S_1)\varphi([x]) = \sim (\bigcup_{y \in [x]} [\varphi(\{y\})]). \tag{B4}$$

Proof Intersection of two distinct classes is always empty. If $\sim$ is defined, then the second and fourth case will be possible. So these four exhaust all possibilities. □

Below a useful unpublished result is proved:

Theorem 24 *If S_1 is the higher granular operator space corresponding to an approximation space, S_2 is the higher granular operator space corresponding to a tolerance space and $\varphi : S_1 \longmapsto S_2$ is a map such that*

- *$\mathcal{G}_1$ is the partition of S_1 and*
- *$\mathcal{G}_2$ is a system of blocks of S_2,*
- *$\xi(a)$ is the set of blocks including $a \in S_2$*
- *For each x, there is a term function t in the signature of $wp(S_2)$ and for some $y_i \in \xi(\varphi(x))$ such that*

$$(\forall [x] \in \mathcal{G}_1)\, \varphi([x]) = t(y_1, \ldots, y_n).$$

then if φ preserves granules, then $\varphi([x]) \in \xi(\varphi(x))$ or $\mathcal{G}_2$ is a partition.

Proof Since φ preserves granules, it is necessary that

$$(\forall x)\, t(y_1, \ldots, y_n) \in \mathcal{G}_2$$

But as $\mathcal{G}_2$ is a collection of blocks, if $\varphi(x) \in t(y_1, \ldots, y_n)$, then it must be a block containing $\varphi(x)$ or it must be of the form B^c for a block B containing $\varphi(x)$. In the latter case, S_2 would be a union of two disjoint blocks and therefore $\{B, B^c\}$ would be a partition.

To confirm the former case, consider the following terms generated by elements of $\xi(\varphi(x))$:

- $y_1 \cup A$ for any nonempty A is not admissible as then a block would be a subset of another block.
- $y_1 \cap A$ for any nonempty A is not admissible as then a block would be a subset of another block.
- The form $(y_1 \cap y_2^c) \cup (y_3 \cup y_4)^c$ also fails as then a block would be a subset of another block.
- The form y_1^c has already been dealt with.
- So the conclusion follows.

□

Theorem 25 *If S_1 is the higher granular operator space corresponding to an approximation space and S_2 is the higher granular operator space corresponding to a tolerance space with approximations $l\mathcal{B}^*$ and $u\mathcal{B}^*$ and φ is a* SNC *and a $\oplus$-morphism satisfying the first condition above, then all of the following hold:*

1. *$\varphi(x^l) \subseteq (\varphi(x))^{l\mathcal{B}^*}$,*
2. *$\varphi(x^u) \subseteq (\varphi(x))^{u\mathcal{B}^*}$,*
3. *If φ is a morphism, that preserves $\emptyset$ and 1, then equality holds in the above two statements.*

But the converse need not hold in general.

Proof

1. If $A \in S_1$, then

$$\varphi(A^l) = \varphi(\bigcup_{[\{x\}] \subseteq A} [\{x\}]) = \bigcup_{[\{x\}] \subseteq A} \varphi([\{x\}]) = \bigcup_{[\{x\}] \subseteq A} \cap\beta(\{\varphi(\{x\})\}),$$

and that is a subset of $\bigcup_{\varphi([\{x\}])\subseteq\varphi(A)} \cap\beta(\{\varphi(\{x\})\})$. Some of the $\mathcal{B}^*$ elements included in $\varphi(A)$ may be lost if $\varphi(A^l)$ is the starting point.

2. If $A \in S_1$, then

$$\varphi(A^u) = \varphi(\bigcup_{[\{x\}]\cap A\neq\emptyset} [\{x\}]) = \bigcup_{[\{x\}]\cap A\neq\emptyset} \varphi([\{x\}])$$

$$\bigcup_{[\{x\}]\cap A\neq\emptyset} \varphi([\{x\}]) = \bigcup_{[\{x\}]\cap A\neq\emptyset} \cap\beta(\{\varphi(\{x\})\}),$$

and that is a subset of $\bigcup_{y\cap\varphi(A)\neq\emptyset} \cap\beta(y)$. In the last part possible values of y include all of the values in $\varphi(A)$.

3. Because of the conditions on φ, for any $A, B \in S_1$ if $A \cap B = \emptyset$, then $\varphi(A) \cap \varphi(B) = \emptyset$. So a definite element must be mapped into a union of disjoint granules in S_2. Further, for $A \in S_1$ and ξ, η, ζ being abbreviations for $(\cap\beta(\varphi(x)))\cap\varphi(A) \neq \emptyset$, $\varphi([\{x\}]\cap A) \neq \emptyset$ and $\varphi([\{x\}])\cap\varphi(A) \neq \emptyset$ respectively,

$$(\varphi(A))^{u\mathcal{B}^*} = \bigcup_{\xi} \cap\beta(\varphi(x)) = \varphi(\bigcup_{\xi}[\{x\}]) = \varphi(\bigcup_{\zeta}[\{x\}]) = \varphi(\bigcup_{\eta}[\{x\}]),$$

which is $\varphi(A^u))$.

□

Theorem 26 *If S_1 is the higher granular operator space corresponding to an approximation space and S_2 as the higher granular operator space corresponding to a tolerance space with approximations $l\mathfrak{T}$ and $u\mathfrak{T}$ and φ is a SNC and a $\oplus$—morphism satisfying for each singleton $\{x\} \in S_1$, $\varphi([\{x\}]) = [\varphi(x)]$, then all of the following hold:*

1. $\varphi(x^l) \subseteq (\varphi(x))^{l\mathfrak{T}}$,
2. $\varphi(x^u) \subseteq (\varphi(x))^{u\mathfrak{T}}$.

3.7 Relation-Based Rough Sets

Rough objects may be derived from general approximation spaces or covers or they may be derived from abstract ideas of rough approximation. This division is very justified because the approaches have limited connections between them and algebraic methods that can be used also depends on the semantic domains associated with them in distinct ways.

In [30], a critical review of the terminology relating to *information systems* used in rough sets and allied fields has been done. One of the suggestions made has been to avoid the term as it refers to an integrated heterogeneous system that has components for collecting, storing and processing data in closely related fields like artificial intelligence, database theory and machine learning. Information systems or more correctly, information storage and retrieval systems (also referred to as information tables, descriptive systems, knowledge representation system) are basically representations of structured data tables. When columns for decision are also included, then they are referred to as *decision tables*. Often rough sets arise from *information tables* and decision tables. But this need not be the case as has been demonstrated by the present author in [83, 91, 103] and others in abstract approaches [57].

From a mathematical point of view, *information systems and tables* can be described using heterogeneous partial algebraic systems. In rough set contexts, this generality has not been exploited as of this writing. For more on partial algebraic systems the reader may refer to [15, 79].

A *general approximation space* is an algebraic system of the form $S = \langle \underline{S}, R \rangle$ with $\underline{S}$ being a set and R being a binary relation on it. Typically, these are derived from information tables by way of definitions of the form: For $x, w \in \mathfrak{O}$ and $B \subseteq S$, $(x, w) \in R$ if and only if $(\mathbf{Q}a, b \in B)\, \Phi(\nu(a, x), \nu(b, w),)$ for some quantifier $\mathbf{Q}$ and formula Φ. Some common examples of the latter condition are given below:

$(\forall a \in B)\, \nu(a, x) = \nu(a, y)$.
$(\forall a \in B)\, \nu(a, x) \cap \nu(a, y) \neq \emptyset$.
$(\forall a \in B)\, \nu(a, x) \subseteq \nu(a, y)$.
$(\forall a \in B)(\exists z \in B)\, \nu(a, x) \cup \nu(a, y) = \nu(a, z)$.
$(\forall a, b \in B)(\nu(a, x) \cap \nu(b, x) \neq \emptyset \longrightarrow \nu(a, y) \cap \nu(b, y) \neq \emptyset)$.
$(\forall a \in B)(\exists b \in B)\, \nu(a, x) = \nu(b, y)$.
$(\forall a, b \in B)(\nu(a, x) = \nu(b, y) \longrightarrow \nu(a, y) = \nu(b, x))$.
$(\forall a, b \in B)(\nu(a, x) \subset \nu(a, y)\, \&\, \nu(b, x) \subset \nu(b, y) \longrightarrow \nu(a, x) \cap \nu(b, x) = \nu(a, y) \cap \nu(b, y))$.
$(\forall a, b \in B)\, (\nu(a, x) \cap \nu(b, x) \subseteq \nu(a, y))$ or $(\nu(a, x) \cap \nu(b, x) \subseteq \nu(b, y))$.
$(\forall a, b \in B)\, (\nu(a, x) \cap \nu(b, y) \subseteq \nu(a, y))$ or $(\nu(a, y) \cap \nu(b, x) \subseteq \nu(b, y))$.

When the relation R is an equivalence, then instances of the form Rab can be read as *a is indiscernible from b and conversely*. If R is a partial equivalence, then instances of the form Rab can be read as *a is possibly indiscernible from b and conversely*. If R is a tolerance, then instances of the form Rab can be read as *a is similar to b and conversely*. If R is a partial order, then instances of the form Rab can be read as *a's attributes are all present in b*. If R is a quasi order, then instances of the form Rab can be read as *a's attributes are all present in b, but these attributes are possibly insufficient for identifying objects possessing a or b*.

Definition 27 In a general approximation space S, any subset $A \subseteq S$ will be said to be a *R-Block* if and only if it is maximal with respect to the property

$$A^2 \subseteq R \tag{11}$$

The set of all R-blocks of S will be denoted by $\mathcal{B}_R(S)$.

Proposition 28 *If R is reflexive, then* $\mathcal{B}_R(S)$ *is a proper cover of S.*

Proof

- For all $x \in S$, it is necessary that Rxx.
- So $\bigcup \mathcal{B}_R(S)$ must be equal to S

□

Definition 29 A specific mathematical approach to relation-based rough set will be said to be *granular* only if it can be rewritten in the form of a general granular operator space or a higher order granular operator space satisfying additional conditions.

3.8 Properties of Granules

If R is a reflexive relation on a set S and $[x]_i = \{a : Rxa\}$—the predecessor set of points related to x and the lower and upper approximation of a subset $A \subseteq S$ are defined via

$$A^l = \cup\{[x]_i : [x]_i \subseteq A, \ x \in A\} \text{ and}$$

$$A^u = \cup\{[x]_i : [x]_i \cap A \neq \emptyset x \in A\},$$

($A^l \subseteq A \subseteq A^u$ for a binary relation R is equivalent to its reflexivity [158, 165]) then $\{[x]_i : x \in S\}$ is a granulation that satisfies:

Theorem 30 *RA, LFU holds, but none of MER, ACG, LI, UI, NO, FU holds in general.*

Proof RA holds by definition, LFU holds as the lower approximation of the union of two granules is the same as the union. It is easy to define an artificial counter example to support the rest of the statement. □

Let $\langle S, (R_i)_i \in K \rangle$ be a multiple approximation space [66], then the strong lower, weak lower, strong upper and weak upper approximations of a set $X \subseteq S$, when li and ui denote the ith lower and upper approximations respectively, are defined as follows:

1. $X^{ls} = \bigcap_i X^{li}$,
2. $X^{us} = \bigcup_i X^{ui}$,
3. $X^{lw} = \bigcup_i X^{li}$,
4. $X^{uw} = \bigcap_i X^{ui}$.

The terminology used differs substantially from the one used in [66].

Theorem 31 *In a multiple approximation of the above form, taking the set of granules to be the collection of all equivalence classes of the R_is, LSRA, USRA, LSS, USS holds, but all variants of rest do not hold always.*

Proof X^{lw}, X^{us} are obviously unions of equivalence classes. The LSS, USS part for these two approximations respectively can also be checked directly. Counterexamples can be found in [66]. But it is possible to check these by building on actual possibilities. If there are only two distinct equivalences, then at least two classes of the first must differ from two classes of the second. The ls approximation of these classes will strictly be a subset of the corresponding classes, so CG will certainly fail for the (ls, us) pair. Continuing the argument, it will follow that SCG, ACG cannot hold in general. The argument can be extended to other situations. □

Multiple approximations spaces are essentially equivalent to special types of tolerance spaces equipped with the largest equivalence contained in the tolerance, the above could as well have been included with tolerances.

4 Classical Rough Sets

A number of algebraic approaches to classical rough sets may be found in the literature. The algebras seek to capture the semantics at the classical, rough and refined rough semantic domains. Many distinct concepts of rough objects have also been used in the associated semantics. Algebras that seek to model rough sets in the context of the following are known:

RB $x \in \mathbb{S}$ is a rough object if and only if $\neg(x^l = x^u)$. The condition is equivalent to the boundary being nonempty [127]. For classical rough sets, **RL, RU, RW** coincide with **RB**.

RD *Any pair of definite elements* of the form (a, b) satisfying $a < b$. The set of all such elements will be denoted by S_δ.

RP **RP** coincides with **RD** for classical rough sets under specific conditions that have not been articulated in a clear way in the literature. So pairs of the form (x^l, x^u) alone form a separate category of rough objects in general. The set of all such elements will be denoted by $S_{\delta 0}$.

RI Elements in an *interval of the form* (a, b) satisfying $a < b$ with a, b being definite elements. **RIA** coincides with **RI**. In the rough domain these elements get identified into a single object—the roughly equal objects [125]. The set of all such elements will be denoted by S_I.

ROP Because a weak negation or complementation ^{c} is available, orthopairs of the form $(x^l, x^{u}c)$ can also be taken as representations of *rough objects*. The set of all such elements will be denoted by S_O.

RCL Convex Sublattices formed by rough objects in the sense of **RI**—this is due to the present author [83]. The set of all such elements will be denoted by S_C

AC In the antichain based approach, due to the present author [98, 104], the primary objects of interest are antichains of mutually discernible objects. This approach is useful for all general rough set theoretic contexts.

4.1 Boolean Algebra with Operators: As Rough Semantics

A natural semantics in the classical semantic domain in which all objects remain discernible can be constructed from an approximation space $S = \langle \underline{S}, R \rangle$ as follows:

$$\text{Set } \wp(\underline{S}) = \underline{B} \tag{12}$$

$$\text{Form } (\forall x \in \underline{B})\, x^l = \bigcup\{g\,;\; g \subseteq x \,\&\, g \in \mathcal{G}\} \tag{13}$$

$$\text{Form } (\forall x \in \underline{B})\, x^u = \bigcup\{g\,;\; g \cap x \neq \emptyset \,\&\, g \in \mathcal{G}\} \tag{14}$$

$$\text{Define } B = \langle \underline{B}, \cup, \cap, ^c, l, u, 0, 1 \rangle \tag{15}$$

$$0 = \emptyset, \,\&\, 1 = \underline{B} \text{ being 0-ary operations} \tag{16}$$

$$B \text{ is an algebra of type } (2, 2, 1, 1, 1, 0, 0). \tag{17}$$

Theorem 32 *B is an algebra that satisfies the following properties:*

$$\langle \underline{B}, \cup, \cap, ^c, 0, 1 \rangle \textit{ is a Boolean algebra.}$$

$$x^{ll} = x^l;\ \&\, x^{uu} = x^u$$

$$a \subseteq b \longrightarrow a^l \subseteq b^l \,\&\, a^u \subseteq b^u$$

$$0 \subseteq x^l \subseteq x \subseteq x^u \subseteq 1$$

$$(a \cup b)^u = a^u \cup b^u\,;\, \&\, (a \cap b)^u \subseteq a^u \cap b^u$$

$$a^u = a^{clc}\,;\; a^l = a^{cuc}$$

$$(a \cap b)^l = a^l \cap b^l\,;\, \&\, (a \cup b)^l \subseteq a^l \cup b^l$$

$$0^l = 0^u = 0\,;\, \&\, 1^l = 1^u = 1$$

Proof Power sets with the set-operations of union, intersection, complementation and 0-ary operations form a Boolean algebra. The other parts of the theorem can be verified by checking the membership of elements. For example, if $z \in (a \cup b)^u$, then $(\exists h \in \underline{S}) h \in a \cup b$, such that $z \in [h]$ ($[h]$ being the equivalence class generated by h). Because $h \in a \cup b$, $h \in a$ or $h \in b$ must hold. This yields $z \in a^u$ or $z \in b^u$. So $(a \cup b)^u \subseteq a^u \cup b^u$. □

Definition 33 Let $\delta_l = \{x\,;\, x^l = x\}$, $\delta_u = \{x\,;\, x^u = x\}$ and $\delta_{lu} = \{x\,;\, x^l = x = x^u\}$. These are the set of lower-definite, upper definite and definite elements respectively.

Theorem 34

- $\delta_l(S) = \delta_u(S) = \delta_{lu}(S)$
- *$\delta_{lu}(S)$ with induced set operations forms a Boolean subalgebra of B.*

Proof It is obvious that

$$x^l = x \Leftrightarrow x = \cup\{z : z \in S|R \,\&\, z \subseteq x\}$$
$$\Leftrightarrow x^{lu} = x^l = x = x^u \text{ by definition}$$
$$\text{So } \delta_l(S) = \delta_u(S) = \delta_{lu}(S)$$

- Below, the universal sets are used as superscripts to distinguish between the universes.
- If $a, b \in \delta_l(S)$, then $a \cup^{\wp(S)} b$ is a union of R-classes and so must be a union of R-classes.
- Therefore $a \cup^{\wp(S)} b = a \cup^{\delta_l(S)} b$.
- Two R-classes must be identical if they have non-empty intersection.
- So $a \cap^{\wp(S)} b = a \cap^{\delta_l(S)} b$.
- Complementation, top and bottom are easy to verify.

So $\delta_{lu}(S)$ with induced set operations forms a Boolean subalgebra of B. □

4.2 RD: Semantics of Pairs of Definites

A wide variety of operations have been defined on S_δ in the literature [6, 11, 138]. These relate to aggregation, commonality, complementation, negation and many implications. A key assumption in many papers has been that aggregation and commonality ought to be total operations. In the present author's opinion, this is questionable as these are not always material operations and may also be contaminated from a constructive viewpoint[92]. Natural aggregation and commonality operation on S_δ do exist however:

Definition 35

$$(\forall (a, b) \in S_\delta)(\exists e \in S)(a, b) = (e^l, e^u) \quad \text{(definites)}$$
$$(x^l, x^u) \sqcup (a^l, a^u) = (x^l \cup a^l, x^u \cup a^u) \quad \text{(rough union)}$$
$$(x^l, x^u) \sqcup (a^l, a^u) = (x^l \cup a^l, x^u \cup a^u) \quad \text{(rough union)}$$
$$\neg(x^l, x^u) = (x^{uc}, x^{lc}) \quad \text{(negation)}$$
$$0 = (\emptyset, \emptyset) \quad \text{(bottom)}$$
$$1 = (S, S) \quad \text{(top)}$$

In an approximation space S, an *upper sample* p of x is a subset of x, that satisfies $p^u = x^u$. It is said to be *minimal* if and only if there is no upper sample z of a with $z \subseteq p$.

Theorem 36 *All of the operations in the above definition are well defined.*

Proof

1. To prove that the rough union operation is well defined, it suffices to construct a b satisfying $x^l \cup a^l = b^l \,\&\, x^u \cup a^u = b^u$. This can be done in many ways including the following:
   a. By setting, $b = x \cup a^l \cup ((x \cup a) \cap \partial(x \cup a))$. [5]
   b. By setting, $b = x^l \cup a^l \cup p$: p being a minimal upper sample of $x^l \cup b^l$ [11]
   c. By setting, $h = a^l \cup (a \cap x^{uc}) \cup (a^u \cap x \setminus x^l) \cup (a \cap x^l)$ and taking $b = x \cup h$ [53] Of these, the second method is relatively better structured.
2. To prove that the rough intersection operation is well defined, it suffices to construct a e satisfying $x^l \cap a^l = e^l \,\&\, x^u \cap a^u = e^u$. This can be done in many ways including the following:
   a. By setting, $e = (x \cap a) \cup (x \cap (x^u \cap a^u) \setminus (x \cup a)^u)$. [5]
   b. By setting, $e = (x^l \cap a^l) \cup p$: p being a minimal upper sample of $x^l \cup b^l$ [11]
   c. By setting, $h = a^l \cup (a \cap x^{uc}) \cup (a^u \cap x \setminus x^l) \cup (a \cap x^l)$ and taking $e = x \cap h$ [53]
3. Other parts are easy to prove.

□

Theorem 37

- $\mathbb{S} = \langle S_\delta, \sqcup, \sqcap, \neg, 0, 1\rangle$ *is a complete atomic quasi-Boolean algebra (DeMorgan lattice) and the subcollection of definable elements form a Boolean subalgebra under the induced operations.*
- *The set of atoms of* $\mathbb{S}$ *have the form* $At(\mathbb{S}) = \{(\emptyset, [x]) : x \in S\}$.
- *On* S_δ*, it is possible define a unary L, via* $(\forall (a, b) \in S_\delta) = L(a, b) = (a, a)$.
- *Then* $\mathbb{S} = \langle S_\delta, \sqcup, \sqcap, L, \neg, 0, 1\rangle$ *is a pre-rough algebra and also a topological quasi Boolean algebra.*
- *A derived implication operation:* $a \implies b = (\neg La \sqcup Lb) \sqcap (L\neg a \sqcup \neg L\neg b))$ *is also definable*

If singleton equivalence classes exist, then pairs of definite subsets of the form $(A, A \cup x)$ can be formed. This does not correctly correspond to any rough object as no C satisfying $C^l = A$ and $C^u = A \cup x$ can be found.

The following definition is possible because some of the conditions used to define a pre-rough algebra are superfluous (see [142]).

Definition 38 An *essential pre-rough algebra* will be an algebra of the form

$$E = \left\langle \underline{E}, \sqcap, L, \neg, 0, 1 \right\rangle$$

that satisfies all of the following (with $\sqcup$ being a defined by $(\forall a, b)\; a \sqcup b = \neg(\neg a \sqcap \neg b)$ and $a \leq b$ being an abbreviation for $a \sqcap b = a$.)

$\langle \underline{E}, \sqcap, \sqcup, \neg, 0, 1 \rangle$ is a quasi Boolean algebra.

E1 $L1 = 1$
E2 $(\forall a)\, La \sqcap a = La$
E3 $(\forall a, b)\, L(a \sqcap b) = L(a) \sqcap L(b)$
E4 $(\forall a)\, \neg L \neg La = La$
E5 $(\forall a)\, \neg La \sqcap La = 0$
E6 $(\forall a, b)\, (\neg L \neg a \leq \neg L \neg b \,\&\, La \leq Lb \longrightarrow a \leq b)$

An essential pre-rough algebra is said to be an *essential rough algebra* if $L(E)$ is also complete and completely distributive—that is it satisfies (for any subset X and element a)

$$a \sqcup (\bigsqcap X) = \bigsqcap \{a \sqcup x : x \in X\} \,\&\, a \sqcap (\bigsqcup X) = \bigsqcup \{a \sqcap x : x \in X\}$$

Equivalently, on the set of roughly equal elements (or rough objects) $\wp(S)| \approx$ similar operations can be defined to form an isomorphic semantics. The most minimalist version of this construction is the following ($[A]$ being the set of subsets that are roughly equal to A):

- Let $\underline{E} = \wp(S)| \approx$
- Define $(\forall A, B \in \wp(S))\, [A] \sqcap [B] = [(A \cap B) \cup (A \cap ((A^u \cap B^u) \setminus (A \cap B)^u))]$
- Define $(\forall A \in \wp(S))\, \neg[A] = [A^c] \,\&\, L([A]) = [A^l]$
- Define $0 = [\emptyset]$ and $1 = [S]$
- $E = \langle \underline{E}, \sqcap, L, \neg, 0, 1 \rangle$ is an *essential pre-rough algebra* which is both complete and completely distributive when endowed with the additional definable operation $\sqcup$.

Theorem 39 *Every essential rough algebra is isomorphic to a subalgebra of an essential rough algebra constructed from an approximation space by the above procedure.*

4.2.1 Decontaminating Operations

It can be argued that the operations of aggregation and commonality used in the rough algebra semantics of classical rough sets are contaminated because they use aspects of the classical semantic domain that are not found in the rough semantic domain. This happens because of the mathematical form of the definition. In [92], this problem has been remedied to an extent by the introduction of less contaminated partial algebras by the present author. In the process a new semantic domain that contains the rough semantic domain is created.

The *requirements of a contamination-free semantics* in the rough semantic domain (or Meta-R) for classical rough sets are the following:

- The objects of interest should be roughly equivalent sets (that is rough objects).
- The operations used in the semantics are as contamination-free as is possible.

- The logical constants in the associated logic should be as real (or actualizable) as is possible.

The last two criteria are very closely related. Any one of the two may be expected to determine the other. The first of the three criteria is pretty much compulsory because they determine the domain. The second and third are however relative to the meaning that they may acquire in the intended use of the semantics.

A natural way of realizing the contamination of operations relative basic operations would be through some concept of definability or representability. Taking orders on rough objects as basic predicates, $\sqcup$ can for example be regarded as a non-contaminated operation in pre-rough/rough algebras (as it is definable). From the point of view of representation as a term, $\sqcup$ would be contaminated as higher order constructions would be required. It is also possible to regard the pre-rough/rough algebra or equivalent semantics as being essentially over-determined. Thus it makes sense to weaken the semantics (relative to the properties satisfied) and so the problem would be of suitably weakening the semantics. Key properties that determine the last two requirements relate to level of perception of rough inclusion.

Decontamination can possibly be achieved by using some of the following strategies:

- Replacing $\sqcap$ and $\sqcup$ by other aggregation and commonality operations repectively,
- In practice, agents may not be interested in aggregating all pairs of objects or in determining common parts. So lazy determinations in the form of partial operations are justified. Modeling of human reasoning is one example of such a context.
- In many contexts the bounds may be dependent on the relative bigness or otherwise of the outcome of the specific instance of $\sqcup$ or $\sqcap$. The bigness based cases are not about over-determination of the problem and can be associated with filters, ideals and intervals (or generalizations thereof) of different types in most cases and then would be semantically amenable.

The following concepts of filters and ideals capture the concept of closure under types aggregation and commonality operations and consequence operators.

Definition 40 An arbitrary subset K of $\wp(S)| \approx = Q$ is said to be a *L-Filter* if and only if it satisfies F0 and O1. If in addition it satisfies F1, then it is said to be prime. K is an o-filter if it satisfies F0 alone :

- F0: $(\forall x \in K)(\forall y \in Q)(x \leq y \Rightarrow y \in K)$.
- O1: $(\forall x \in K)Lx \in K$.
- F1: $(\forall a, b \in Q)(1 \neq a \sqcup b \in K \Rightarrow a \in K$ or $b \in K)$.

The dual notions will be that of *U-Ideals, prime U-ideals and o-ideals*. If a L-filter is closed under $\sqcap, \sqcup$, then it will be termed a lattice L-filter. Let $\mathbb{K} = \langle K, \leq, L, U, \neg, 1\rangle$ be the induced partial algebraic system on K.

Proposition 41 *If K is a lattice L-filter, then $\mathbb{K}$ is not a pre-rough algebra, but satisfies:*

1. $\leq$ *is a distributive lattice order.*
2. *Closure under* L, U*, but not under* $\neg$.
3. $Lx \leq x$; $L(a \sqcap b) = La \sqcap Lb$; $LLx = Lx$;
4. $L1 = 1$; $ULx = Lx$; $L(a \sqcup b) = La \sqcup Lb$

Proof If finiteness is assumed, then the lattice is bounded. But there would be no way (in general) of ensuring closure under $\neg, 0$. The three element pre-rough algebra provides the required counterexample. □

Proposition 42 *If* K *is a L-filter, then* $\mathbb{K}$ *satisfies:*

1. $\leq$ *is a join-semilattice lattice order (*$\sqcup$ *is definable).*
2. *Closure under* L, U*, but not under the partial lattice operation* $\sqcap$ *and* $\neg$.
3. $Lx \leq x$; $L(a \sqcap b) \stackrel{w}{=} La \sqcap Lb$; $LLx = Lx$;
4. $L1 = 1$; $ULx = Lx$; $L(a \sqcup b) = La \sqcup Lb$; $x \sqcup (y \sqcap x) \stackrel{w}{=} x$.
5. $x \sqcup (y \sqcap z) \stackrel{w}{=} (x \sqcup y) \sqcap (x \sqcup z)$ *and its dual.*

Proof If $a \sqcap b \in K$, then $L(a \sqcap b) \in K$ by definition and so $La, Lb, a, b \in K$. If $La, Lb \in K$, then it is possible that $La \sqcap Lb \notin K$, which is the reason for the weak equality. □

Theorem 43 *There exists a pre-rough algebra* S *with a nontrivial lattice L-filter* K *satisfying*

$$(\exists a, b \in S \setminus \{1\})(\forall c \in K \setminus \{1\}) a \sqcup b \parallel c.$$

The proof involves a simple construction, but it should be noted that $K \setminus \{1\}$ may or may not be cofinal in $S \setminus \{1\}$. This is important as such a K may be interpreted to consist of *big* elements alone. Such L-filters or lattice L-filters will be said to be *cofine*.

Theorem 44 *Given a pre-rough algebra with no nontrivial lattice L-filters, an infinite number of pre-rough algebras with the same property can be constructed.*

Proof A completely visual proof is possible for proving this. Simply paste a pair of three element pre-rough algebras to the original pre-rough algebra (identifying all the tops and bottoms respectively) and require that the negation of one of the non boundary element is the other. The infinite number of pre-rough algebras follow by recursive application of the process.

A second proof can be through the fact that the product of two pre-rough algebras with the property satisfies the property. □

Definition 45 Given a L-filter K on Q, for any $x, y \in Q$ let

$$x \Cup y = \begin{cases} x \sqcup y & \text{if } x \sqcup y \in K \\ \text{undefined} & \text{otherwise.} \end{cases}$$

$$x \Cap y = \begin{cases} x \sqcap y & \text{if } x \sqcap y \in K \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Further let $x \lhd y$ iff $x = y$ or $x \Cap y = x$ or $x \Cup y = y$.

Proposition 46 *The relation $\lhd$ is a partial order that is not necessarily a lattice order, but is compatible with the operations L, U. Further the restriction of $\lhd$ to K has already been described above.*

Proof Absorption laws can be shown to fail in most pre-rough algebras for an optimal choice of a L-filter. □

Definition 47 By a *operationally contamination-free prerough* algebraic system (or OCPR system) will be meant a partial algebraic system of the form

$$Y = \left\langle \underline{Q}, \lhd, L, U, \Cup, \Cap, 0, 1 \right\rangle,$$

with the operations and relations being as defined above (U is the operation induced by upper approximation operator on Q).

Definition 48 By a *OC-system* (resp *lattice OC-system*) will be meant a pair of the form $\langle Q, K\rangle$ consisting of a pre-rough algebra Q and a L-filter (resp. lattice L-filter) K. If K is cofine, then the system will be said to be *cofine*.

Theorem 49

1. *If K is a lattice L-filter, then $K^+ = \{y :: (\forall x \in K) x \sqcup y = 1\}$ with induced operations from the pre-rough algebra is a lattice L-filter. Such filters will be termed* supremal.
2. *K is a cofine lattice L-filter iff $K^+ = \{1\}$.*
3. *The collection of all supremal lattice L-filters can be Boolean ordered with an order distinct from the order on lattice L-filters.*

Thus starting from a standard rough domain (corresponding to pre-rough algebras), a new rough semantic domain has been arrived at. At least two distinct partial algebras have been defined with one being an extension of a pre-rough algebra, while OC-pre-rough systems constitute a severe generalization. The natural correspondences from a pre-rough algebra to a cofine L-filter (or lattice L-filter) would be forgetful closed morphisms that preserves all operations except for $\neg$.

4.3 Double Stone Algebras

This is a based on the approach of [31, 32, 42]. A *double Stone algebra L* is an algebra of type (2, 2, 1, 1, 0, 0) of the form

$$L = \left\langle \underline{L}, +, \cdot, *, {}^{+}, 0, 1 \right\rangle$$

that satisfies

- $\langle \underline{L}, +, \cdot, 0, 1 \rangle$ is a bounded distributive lattice.
- x^* is the pseudo-complement of x, that is $y \leq x^* \Leftrightarrow y \cdot x = 0$
- x^+ is the dual pseudo-complement of x, i.e. $x^+ \leq y \Leftrightarrow y + x = 1$
- $x^* + x^{**} = 1, x^+ \cdot x^{++} = 0.$

It is possible to replace the second and the third condition by the equations,

- $x \cdot (x \cdot y)^* = x \cdot y^*, \quad x + (x + y)^+ = x + y^+$
- $x \cdot 0^* = x, \quad x + 1^+ = x$
- $0^{**} = 0 \,\&\, 1^{(++)} = 1$

.

A double Stone algebra is *regular* if and only if $x \cdot x^+ \leq y + y^*$ if and only if

$$(x^+ = y^+, x^* = y^* \longrightarrow x = y).$$

Let B be a Boolean algebra and F a filter on it, then let

$$[B, F] = \{(a, b) : a, b \in B, a \leq b, (a \vee b^c) \in F\}.$$

On this,the operations $+, \cdot, ^*, ^+$ are definable via

- $(a, b) + (c, e) = (a \vee c, b \vee e)$
- $(a, b) \cdot (c, e) = (a \wedge c, b \wedge e)$
- $(a, b)^* = (b^c, b^c)$
- $(a, b)^+ = (a^c, a^c)$

An algebra $\mathsf{K} = \langle [B, F], +, \cdot, ^*, ^+ \rangle$ of this form is called a *Katrinak algebra* in [42].In such an algebra B can be identified with $\{(a, a) : a \in B\}$ and F with $\{(a, 1) : (a, 1) \in \mathsf{K}\}$.

Starting from an arbitrary set S it is possible to construct the collection of all fields of subsets $\mathcal{F}(S)$ of it. The Katrinak algebras formed from the elements of $\mathcal{F}(S)$ is also called a *concrete Katrinak algebras*. The following theorem is an adaptation of theorem that had been originally proved in [65].

Theorem 50 *Each concrete Katrinak algebra is a regular double Stone algebra and conversely every regular double stone algebra is isomorphic to a concrete Katrinak algebra.*

Proof

- Let K be a concrete Katrinak algebra constructed over a set S, so that its universal set is $[B, F]$, $B = \langle B, +, \cdot, -, 0, 1 \rangle$ being the Boolean algebra and F being a filter on B such that

$$[B, F] = \{(a, b) \in B \times B : a \leq b \text{and} - b + a \in F\}.$$

- Then, $L = \langle [B, F], +, \cdot, ^*, ^+ \rangle$ is a regular double Stone algebra if it is required that

$$(a, b)^* = (-b, -b), (a, b)^+ = (-a, -a)$$

- Let $\left\langle \underline{L}, +, \cdot, *, ^+, 0, 1 \right\rangle$ be a regular double Stone algebra.
- Define the center $B(L)$ and the dense set of L via, $B(L) = \{x^* : x \in \underline{L}\}$ and $\Delta(L) = \{x : x^* = 0\}$ respectively.
- Then $B(L)$ is a Boolean algebra isomorphic to B in which the operations * and $^+$ coincide with the complementation operation.
- Let $H = (\Delta(L))^{++}$. If $\tau : L \mapsto (B(L), H)$ is a map such that $\tau(x) = (x^{++}, x^{++})$, then it is an isomorphism.

□

In the associated algebraization, a language $\mathfrak{L}$ of rough set logic consisting of a nonempty set of propositional variables P, two binary connectives $\vee$, $\wedge$, two unary connectives $^*, ^+$ (representing negations) and a constant **T** for truth is used. Formulas are constructible in the usual way, so that the set $\mathcal{F}(\mathfrak{L})$ of formulas is a free algebra of type $(2, 2, 1, 1, 0)$ generated over P. A model of $\mathfrak{L}$ then is a pair of the form (W, υ), where W is a set and $\upsilon : P \mapsto \wp(W) \times \wp(W)$ is a valuation, such that if $\upsilon(p) = (A, B)$ then $A \subseteq B$.

Given a model $\mathfrak{M} = (W, \upsilon)$, it's meaning function σ is defined as an extension of the valuation function $\sigma : \mathcal{F}(\mathfrak{L}) \longmapsto \wp W \times \wp W$ such that,

- $\sigma(\mathbf{T}) = (W, W)$
- $\forall p \in P \sigma(p) = \upsilon(p)$
- If $\sigma(\varphi) = (A, B)$ and $\sigma(\psi) = (C, E)$, then
  - $\sigma(\varphi \wedge \psi) = (A \cap C, B \cap E)$
  - $\sigma(\varphi \vee \psi) = (A \cup C, B \cup E)$
  - $\sigma(\varphi^*) = (-B, -B)$
  - $\sigma(\varphi^+) = (-A, -A)$, $-A$ being the complement of A in $\wp(W)$.

Now on $Ran(\sigma) = \{\sigma(\varphi) : \varphi \in \mathcal{F}(\mathfrak{L})\}$, let the operations $+, \cdot, ^*, ^+$ be defined by

- $\sigma(\varphi) \cdot \sigma(\psi) = \sigma(\varphi \wedge \psi)$
- $\sigma(\varphi) + \sigma(\psi) = \sigma(\varphi \vee \psi)$
- $(\sigma(\varphi))^* = \sigma(\varphi^*)$
- $(\sigma(\varphi))^+ = \sigma(\varphi^+)$.

With these operations $Ran(\sigma)$ is a Katrinak algebra and σ is a morphism. The variety generated by it coincides with the variety of regular double stone algebras. Moreover the associated logic has a finitely complete strongly sound inferential base and fails the Beth definability property.

4.4 Super Rough Algebras

This is a higher order semantics of classical rough sets with much potential for generalization to general rough sets. It was initially developed for classical rough sets by the present author in [83]. It is among the difficult approaches primarily because of the algebraic machinery used. The essence of the approach can be generalized to TQBAs satisfying more conditions (this will appear separately). The approach is described after a description of some of the required algebra.

4.4.1 Related Background

Let $H = \langle \underline{H}, \wedge, \vee \rangle$ be a lattice and T a binary reflexive and symmetric relation on it which is 'compatible' in the sense

$$((a, b), (c, e) \in T \longrightarrow (a \wedge c, b \wedge e), (a \vee c, b \vee e) \in T)$$

then T is called a *compatible tolerance* on H. A subset $B \subseteq H$ is called a *block* of T if it is a maximal subset satisfying $B^2 \subseteq T$. Successor neighborhoods of the form $[x]$ will also be referred to as T-associates of x. A sublattice Z of H is called a *convex sublattice* if and only if it satisfies

$$(\forall x, y \in Z)\,(x \leq a \leq y \longrightarrow a \in Z) \tag{18}$$

If C is a subset of H then $\downarrow_l C$, $\uparrow_l C$ will respectively denote the lattice-ideal and filter generated by C. Tolerances can be fully characterized by their associated system of all blocks [23, 34]. For lattices this system can be denoted $H|T$. For finite lattices the result improves to the one presented in [34],

Theorem 51 *If H is a finite lattice, then a collection $\mathcal{S} = \{B_\alpha : \alpha \in I\}$ of subsets of H is such that $\mathcal{S} = H|T$ if and only if*

1. *Every element of $\mathcal{S}$ is a convex sublattice of H.*
2. *$\mathcal{S}$ covers H.*
3. $(\forall C, E \in \mathcal{S})\,(\downarrow_l C = \downarrow_l E \Longleftrightarrow \uparrow_l C = \uparrow_l E)$.
4. *For any two elements $C, A \in \mathcal{S}$ there exist E, F such that* $(\downarrow_l C \vee \downarrow_l A) = \downarrow_l E$, $(\uparrow_l C \vee \uparrow_l A) \leq \uparrow_l E$, $\downarrow_l F \leq (\downarrow_l A \wedge \downarrow_l C)$, *and* $(\uparrow_l C \wedge \uparrow_l A) = \uparrow_l F)$.

A lattice is said to be *semi-join distributive* if it also satisfies $(x \vee y = x \vee z \longrightarrow x \vee y = x \vee (y \wedge z))$. $J(L)$ will denote the set of all join-irreducible elements of a lattice L. A lattice L is said to be *finitely spatial* (resp. *spatial*) if any element of L is a join-irreducible element (resp. complemented join-irreducible element) of L. A lattice L is said to be *lower continuous* if $(a \vee \bigwedge X) = \bigwedge(a \vee x)$ holds for all downward directed subsets for which $\bigwedge X$ exist.

In a poset P a finite sequence of elements which are comparable with their predecessors is said to be a *path*. A path (x_n) is said to be *oriented* if $(\forall i)\, x_i \leq x_{i+1}$

or its converse holds. A poset is said to be *tree-like* if the following conditions hold,

- If $a \leq b$ then there exists an integer $n < \omega$ and $x_0, x_1, \ldots x_n \in P$ such that $a = x_0 \prec x_1 \prec \ldots \prec x_n = b$.
- For any two elements in the poset there exists at most one maximal path from one to the other.

A lattice L is said to be *sectionally complemented* if and only if for any $b < a$ there exists a c such that $b \wedge c = 0$ and $b \vee c = a$.

The set of all convex sublattices of a lattice in particular and a poset in general can be endowed with a lattice structure (w.r.t inclusion) with meet corresponding to set-intersection and join corresponding to

$$A \vee B = A \cup B \cup \{x : (\exists (y, z) \in A \times B \cup B \times A), y \leq x \leq z\}$$

The lattice is algebraic, atomistic, bi-atomic and join-semi distributive. The sublattices of such lattices have been characterized in [9, 149–151]. On a lattice L the following conditions will be abbreviated for convenience,

$$a \wedge (b^* \vee c) = (a \wedge b^*) \vee \bigvee_{i<2} \big(a \wedge (b_i \vee c) \wedge ((b^* \wedge (a \vee b_i) \vee c)\big)$$
$$\text{where } b^* = b \wedge (b_0 \vee b_1) \qquad (\mathsf{S})$$

$$x \wedge (a_0 \vee a_1) \wedge (b_0 \vee b_1) = \bigvee_{i<2} ((x \wedge a_i \wedge (b_0 \vee b_1)) \vee (x \wedge b_i \wedge (a_0 \vee a_1)) \vee$$
$$\bigvee_{i<2} (x \wedge (a_0 \vee a_1) \wedge (b_0 \vee b_1) \wedge (a_0 \vee b_i) \wedge (a_1 \vee b_{i-1})) \qquad (\mathsf{B})$$

$$x \wedge (x_0 \vee x_1) \wedge (x_1 \vee x_2) = (x \wedge x_0 \wedge (x_1 \vee x_2)) \vee (x \wedge x_1 \wedge (x_0 \vee x_2))$$
$$\vee (x \wedge x_2 \wedge (x_0 \vee x_1)) \qquad (\mathsf{U})$$

These conditions necessarily hold in the lattice of convex sublattices of any poset. In [149–151], the following three results are proved.

Theorem 52 *If a lattice L satisfies the condition* S *then it also satisfies the condition,*

$$(\forall a, b, b_0, b_1, c \in J(L)) \text{ if } (a \leq b \vee c \,\&\, b \leq b_0 \vee b_1 \,\&\, a \neq b)$$
$$\text{then } (\exists \overline{b}) (a \leq \overline{b} \vee c \,\&\, \overline{b} < b)$$
$$\text{or } (b \leq (a \vee b_i) \,\&\, a \leq (b_i \vee c) \text{ for some } i < 2) \qquad (S_i)$$

Theorem 53 *If a lattice* L *satisfies the conditions* B *and* U *then it also satisfies the conditions* B_i *and* U_i *defined below.*

$$\mathsf{B}_i \;:\; (\forall x, a_0, a_1, b_0, b_1 \in J(L))\ \textit{if}\ (x \leq a_0 \vee a_1,\ x \leq b_0 \vee b_1)$$
$$\textit{then}\ x \leq a_i\ \textit{or}\ x \leq b_i\ \textit{for some}\ i < 2$$
$$\textit{or}\ x \leq a_0 \vee b_0,\ a_1 \vee b_1\ \textit{or}\ x \leq a_0 \vee b_1,\ a_1 \vee b_0.$$
$$\mathsf{U}_i : (\forall x, x_0, x_1, x_2 \in J(L))\ \textit{if}\ (x \leq x_0 \vee x_1,\ x_0 \vee x_2,\ x_1 \vee x_2)$$
$$\textit{then}\ x \leq x_0\ \textit{or}\ x \leq x_1\ \textit{or}\ x \leq x_2$$

If a lattice satisfies the condition

$$a \wedge (b \vee c \vee e) = (a \wedge (b \vee c)) \vee (a \wedge (b \vee e)) \vee (a \wedge (c \vee e)), \tag{P}$$

then it is said to be *dually 2-distributive*.

Theorem 54 *If* L *is a* complete, lower continuous, finitely spatial and dually 2-distributive *lattice and if it satisfies the conditions* S_i, B_i *and* U_i*, then it satisfies the conditions* S*,* B*, and* U.

For convenience a lattice of the above form which satisfies the three conditions will be called a *long lattice*.

For more on the structure of the lattice of convex sublattices and the lattice of intervals of a lattice the reader is referred to [2, 9, 149–151]. It has been proved in [71, 72] that *two lattices have isomorphic convex sublattices if and only if they have isomorphic interval lattices*. The following result for posets naturally applies to lattices and is relevant for the main duality result.

Let $\mathcal{F}_{cnv}(\Lambda)$ be the set of all convex sublattices and $\mathrm{Int}(\Lambda)$ that of convex intervals of a poset Λ respectively. A poset Φ is said to be *convexly isomorphic* (resp. *interval isomorphic*) to another poset Λ if and only if $\mathcal{F}_{cnv}(\Phi) \simeq \mathcal{F}_{cnv}(\Lambda)$ (resp. $\mathrm{Int}(\Phi) \simeq \mathrm{Int}(\Lambda)$). It has been proved in [71, 72] that every such Λ is constructible from Φ. One of the main results therein is stated below.

Theorem 55 *Let* $A = (\underline{A}, \leq)$ *be any poset. Posets* convexly isomorphic *to* A *are (up to isomorphism) just those which can be obtained by applying the following three constructions successively,*

1. *Construct* $A_1 = (\underline{A}, \leq_1)$*, where* $x \leq_1 y$ *means*

$$x < y\ \textit{and}\ (x, y) \notin P \subset \{(x, y); x, y \in A, x \prec y, x \in \mathsf{Min}(A), y \in \mathsf{Max}(A)\}.$$

2. *Given* A_1*, construct* $A_2 = (\underline{A}, \leq_2)$*; where* $x \leq_2 y$ *holds whenever* $x, y \in C, x \leq_1 y$ *or* $x, y \in D, x \leq_1 y$ *holds for a decomposition* $A = C \cup D$ *of* A *under* $(\forall c \in C, d \in D) c \parallel_1 d$*. Here* $\parallel_1$ *indicates the non comparability of the two elements with respect to the order* $\leq_1$.

3. *Assuming* A_2*, construct* $A_3 = (\underline{A}, \leq_3)$*; where* $x \leq_3 y$ *if and only if* $x \leq 2y$*, or* $(x, y) \in Q$ *for a* $Q \subset \{(x, y); (x, y) \in A^2, x\|_2 y, x \in \mathsf{Min}(A_2), y \in \mathsf{Max}(A_2), \}$*, such that* $(u, v), (v, w) \in Q$ *do not hold simultaneously for any* $u, v, w \in A$.

4.4.2 Main Derivations

Definition 56 In a pre-rough algebra or a topological quasi Boolean algebra of the form $S = \langle \underline{S}, \sqcap, \sqcup, \Rightarrow, L, \neg, 0, 1 \rangle$, let T be a binary relation defined by Tab if and only if $(\exists c \in S)\, L(c) \leq a \leq M(c),\ L(c) \leq b \leq M(c)$. T will be called the *coapproximability relation* on S.

Proposition 57 *The* coapproximability relation *on the pre-rough algebra is a compatible tolerance.*

Proposition 58 *Every block of the tolerance* T *is an interval of the form* $[\bigwedge a_i, \bigvee a_i]$*, whenever the approximation space is finite.*

In the next definition, except for $\sqcap$, $\sqcup$ and L the same operation symbol is used to denote the operation itself in the rough algebra and the super rough set-algebra—the interpretation of the symbol on the set is indicated by superscripts. Thus $\sqcap^{\underline{\mathfrak{R}}}$ means the interpretation of the operation symbol $\sqcap$ over $\underline{\mathfrak{R}}$. Finiteness is not required in the following definition.

Definition 59 A *super rough set-algebra* will be a partial algebra of the form

$$\mathfrak{R} = \left\langle \underline{\mathfrak{R}}, \wedge, \vee, \sqcap^{\underline{\mathfrak{R}}}, \sqcup^{\underline{\mathfrak{R}}}, \neg, L^{\underline{\mathfrak{R}}}, L_T, \downarrow, \uparrow, \underline{S}, \emptyset \right\rangle$$

of type (2, 2, 2, 2, 1, 1, 1, 0, 0) that satisfies:

1. The underlying set of the rough algebra S (with the above indicated operations) is $\underline{S}$.
2. The set of all convex sublattices of S is $\underline{\mathfrak{R}}$.
3. $(\forall A, B \in \mathfrak{R})\, A \sqcap^{\underline{\mathfrak{R}}} B = \{x \sqcap y : x \in A,\ y \in B\}$ if defined in $\mathfrak{R}$.
4. $(\forall A, B \in \mathfrak{R})\, A \sqcup^{\underline{\mathfrak{R}}} B = \{x \sqcup y : x \in A,\ y \in B\}$ if defined in $\mathfrak{R}$.
5. The principal filter and the principal ideal operations with respect to the lattice order on the set of convex sublattices are respectively $\uparrow$ and $\downarrow$.
6. The usual lattice operations on the set of all convex sublattices will correspond to $\wedge$ and $\vee$.
7. $\neg A = \{\neg x : x \in A\}$.
8. $L^{\underline{\mathfrak{R}}}(A) = \{L(x) : x \in A\}$ if defined in $\mathfrak{R}$.
9. $\mathfrak{R} \models \mathsf{S}_i, \mathsf{B}_i, \mathsf{U}_i$
10.

$$L_T(A) = \begin{cases} A & \text{if } A \text{ is a block of } T. \\ \text{undefined else} & \end{cases}$$

A super rough set algebra is correctly a partial algebra with multiple orders and unary operators.

Theorem 60 *In a super rough set-algebra* $\Re$ *as defined above* $\left\langle \underline{\Re}, \sqcup^{\underline{\Re}}, \sqcap^{\underline{\Re}} \right\rangle$ *is a partial distributive lattice which satisfies all weak-equalities in the* ω^* *sense. That is it also satisfies:*

$$(\forall A, B, C)\, A \sqcap (B \sqcap C) \stackrel{\omega^*}{=} (A \sqcap B) \sqcap C$$

$$(\forall A, B, C)\, A \sqcap (B \sqcup C) \stackrel{\omega^*}{=} (A \sqcap B) \sqcup (A \sqcap C)$$

$$(\forall A, B, C)\, A \sqcup (B \sqcap C) \stackrel{\omega^*}{=} (A \sqcup B) \sqcap (A \sqcup C)$$

Recall that $\phi \stackrel{\omega^*}{=} \psi$ *means that if either side is defined, then the other is and the two are equal.*

Theorem 61 *In a super rough set-algebra* $\Re$*, the following properties hold:*

$$\neg(A \sqcap^{\underline{\Re}} B) \stackrel{\omega^*}{=} (\neg A \sqcup^{\underline{\Re}} \neg B) \quad \text{(Weak De Morgan)}$$

$$L^{\underline{\Re}}(A) \stackrel{\omega^*}{=} L^{\underline{\Re}}(L^{\underline{\Re}}(A)) \quad \text{(Weak Idempotence)}$$

$$L^{\underline{\Re}}(A) \stackrel{\omega^*}{=} L^{\underline{\Re}}(A) \sqcap A \quad \text{(Weak Inclusion)}$$

$$L^{\underline{\Re}}(A \sqcap B) \stackrel{\omega^*}{=} L^{\underline{\Re}}(A) \sqcap L^{\underline{\Re}}(B) \quad \text{(Weak-Meet)}$$

$$L^{\underline{\Re}}(A \sqcup B) \stackrel{\omega^*}{=} L^{\underline{\Re}}(A) \sqcup L^{\underline{\Re}}(B) \quad \text{(Weak-Join)}$$

Theorem 62 *Further in a super rough set-algebra* $\Re$*,* $\wedge, \vee$ *are total lattice operations and the following properties hold:*

$$\neg\neg A = A \quad \text{(Double Negation)}$$

$$(L_T(A) = A \longrightarrow (\{x\} \wedge A = \{x\} \longleftrightarrow \{x\} \sqcup \{\bigwedge A\}, = \{x\},\ \{x\} \sqcap \{\bigvee A\} = \{x\})) \quad \text{(Singleton)}$$

$$(\forall A,\ B)(L_T(A) = A \,\&\, L_T(B) = B,\ \&\, (\uparrow A) = (\uparrow B) \longrightarrow (\downarrow A) = (\downarrow B)) \quad \text{(Fixed-1)}$$

$$(\forall A,\ B)(L_T(A) = A, L_T(B) = B,\ (\downarrow A) = (\downarrow B) \longrightarrow (\uparrow A) = (\uparrow B)) \quad \text{(Fixed-2)}$$

$$(L_T(A) = A \longrightarrow L(A) \vee A = A \,\&\, \neg L(\neg A) \vee A = A) \quad \text{(Fixed-3)}$$

$$(L_T(A) = A \longrightarrow L(A) \sqcup A = A \,\&\, \neg L(\neg A) \sqcup A = A) \tag{Fixed-4}$$

$$(L_T(A) = A \longrightarrow (\exists B)\, \neg(A) \wedge B = \neg A \,\&\, L_T(B) = B) \tag{Fixed-5}$$

$$(L_T(A) = A \longrightarrow (\exists B)\, L(A) \wedge B = L(A) \,\&\, L_T(B) = B) \tag{Fixed-6}$$

$$(L_T(A) = A,\ L_T(B) = B,\ A \sqcap B = C \longrightarrow (\exists E)\, E \wedge C = C,\ L_T(E) = E) \tag{Fixed-7}$$

$$(L_T(A) = A,\ L_T(B) = B,\ A \vee B = C \longrightarrow (\exists E)\, E \wedge C = C \,\&\, L_T(E) = E) \tag{Fixed-8}$$

$$(L_T(A) = A,\ L_T(B) = B,\ A \wedge B = C \longrightarrow (\exists E)\, E \wedge C = C \,\&\, L_T(E) = E) \tag{Fixed-9}$$

$$(L_T(A) = A,\ L_T(B) = B,\ A \sqcup B = C \longrightarrow (\exists E)\, E \wedge C = C \,\&\, L_T(E) = E) \tag{Fixed-10}$$

Also for any two fixed points $A,\ B$ of L_T, there exist two other fixed points $E,\ F$ such that

$$((\downarrow A) \vee (\downarrow B)) = (\downarrow E)$$
$$((\uparrow B) \vee (\uparrow A)) \leq (\uparrow E)$$
$$(\downarrow F) \leq ((\downarrow A) \wedge (\downarrow B))$$
$$\textit{and } ((\uparrow B) \wedge (\uparrow A)) = (\uparrow F)$$

The order relation used is the one on the lattice of convex sublattices.

Related abstract representation theorems are also proved in [83] by the present author.

4.5 Nelson Algebras

Nelson, Lukasiewicz, Heyting and Post algebras have been used in the context of rough semantics in [117–119, 124]. These are discussed in a separate chapter in this volume.

The basic construction is

- Let S be an approximation space.
- Form the collection of $\underline{N} = \{(x^l, x^{uc}) : x \in \wp(S)\}$,
- For any $a, b \in \wp(S)$, define $(a^l, a^{uc}) \wedge (b^l, b^{uc}) = (a^l \cap b^l, a^{uc} \cup b^{uc})$
- For any $a, b \in \wp(S)$, define $(a^l, a^{uc}) \vee (b^l, b^{uc}) = (a^l \cup b^l, a^{uc} \cap b^{uc})$

- For any $a, b \in \wp(S)$, define $(a^l, a^{uc}) \Rightarrow (b^l, b^{uc}) = (a^{lc} \cup b^l, a^l \cap b^{uc})$
- For any $a \in \wp(S)$, define $\neg(a^l, a^{uc}) = (a^{uc}, a^l)$ & $\sim (a^l, a^{uc}) = (a^{lc}, a^l)$
- Define $0 = (\emptyset, S)$ and $1 = (S, \emptyset)$.

Theorem 63 *The algebra* $N = \langle \underline{N}, \vee, \wedge, \Rightarrow, \sim, \neg, 0, 1\rangle$ *is a semi-simple Nelson algebra as it also satisfies*

$$(\forall a \in N)\, a \vee \sim a = 1 \qquad \text{(Nelson-SS)}$$

4.6 Properties of Granulations

Theorem 64 *In classical rough sets, if* $\mathcal{G}$ *is the set of partitions, then all of* RA, ACG, MER, AS, FU, NO, PS *hold.* UU *does not hold in general.*

Proof

- RA, ACG, PS, NO follow from the definitions of the approximations and properties of the equivalence partition $\mathcal{G}$.
- Mereological atomicity holds because no crisp element can be properly included in a single class.
- All approximations are unions of disjoint classes and if a class overlaps with another subset of the universe, then the upper approximation of the subset will certainly contain the class by the definition of the latter.

□

Definition 65 By the theory of *Classical RST-RYS*, will be meant a theory $\mathfrak{Th}$ of RYS in which $\oplus, \odot, \ominus$ correspond respectively to set $\cup, \cap, \setminus$ respectively. S is a power-set of some set A, $1 = A$, and the additional granular axioms RA, ACG, MER, FU, NO, PS, ST, I hold.

Theorem 66 *The theory of classical* RST-RYS *is well defined, is not categorical or* κ*-categorical (*κ *being a cardinal) and is consistent.*

Proof For a fixed cardinality of S, multiple non-isomorphic models of classical RST-RYS can be defined. By Thm 2 of [91], it is known that all of RA, ACG, MER, ST, FU, NO, PS hold, but UU does not. So the theory of classical RST-RYS is consistent. □

5 Esoteric Rough Sets

In esoteric rough sets [85], partial equivalence relations (symmetric, transitive and partially reflexive relations) have been used to generate approximations instead of equivalence relations by the present author. A brief description of the main concepts, results and techniques are mentioned in this section. Most of the results proved in the paper concern the so-called *well partial approximation algebras*.

In the simplest case, the upper and lower approximations of a subset A of a partial approximation space $\langle S, R\rangle$ are defined via ($[x] = \{b;\ Rxb\}$ being the pseudo-class generated by x)

$$A^l = \bigcup\{[x];\ [x] \subseteq A\} \tag{lower}$$

$$A^u = \bigcup\{[x];\ [x] \cap A \neq \emptyset\} \tag{upper}$$

Note that predecessor and successor neighborhoods coincide because of symmetry.

The result of relaxing reflexivity is severe from an algebraic point of view. Related semantics do not fall under TQBAs.

Definition 67 A partial approximation $S = \left\langle \underline{S}, R\right\rangle$ is said to be a *well partial approximation space* (ASW) if and only if it satisfies

$$(\forall x)(\exists z)\ Rxz$$

Otherwise it is said to be an *ill-posed partial approximation space*(ASI).

Theorem 68 *The following relation between approximations of subsets of a well partial approximation space hold:*

$$(\forall x)\, x^{ll} \subseteq x^l \subseteq x \tag{ES1}$$

$$(\forall x)\, x^l \subseteq x^{lu} \subseteq x^{ul} \tag{ES2}$$

$$(\forall x)\, x \subseteq x^{ul} = x^{uu} \tag{ES3}$$

$$S^l \subseteq S \tag{ES4}$$

$$\emptyset \subseteq \emptyset^u \tag{ES5}$$

Condition ES3 can fail to hold in a ASI.

Proof The main thing to look out for is neighborhoods that do not include their generator. □

Proposition 69 *A partial approximation space is an ASW if and only if* $S^l = S$

Because of the above properties, the idea of rough object is taken to be an esoteric tuple:

Definition 70 In a partial approximation space S, *esoteric rough tuples* are tuples of the form

$$\left\langle A^l,\ A^{lu}\ A^u\right\rangle$$

for any $A \subseteq S$.

Further,

Definition 71 Two subsets A, B of S, will be said to be *roughly pseudo equal*, $A \doteqdot B$ if and only if

$$A^l = B^l \,\&\, A^u = B^u$$
$$A^{lu} = B^{lu}$$

Proposition 72 *The rough pseudo-equality relation $\doteqdot$ defined is an equivalence relation on $\wp(S)$. The class generated by a subset A is denoted by $[A]_r$.*

Proposition 73 *Let A, B be two subsets of a partial approximation space S, then there exists a subset C such that $A^{lu} \cup B^{lu} = C^{lu}$.*

Definition 74 A subset A of a partial approximation space S is said to be an *almost definite set* if and only if it is the case that

$$A^l = A^{lu} = A^u$$

An almost definite set is a *definite set* if and only if it is the case that

$$A^l = A^{lu} = A^u = A.$$

If $\mathbf{E}(S)$ and $\mathbf{F}(S)$ are the set of all almost definite and definite subsets of S then

$$\mathbf{E}(S) \subseteq \mathbf{F}(S)$$

Example 75 Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ be a set and a partial equivalence R on it be given by

$$R = \{(1,\ 2), (2,\ 1), (3,\ 3), (4,\ 4), (5,\ 5), (4,\ 5), (5,\ 4), (6,\ 9), (9,\ 6), (6,\ 7), (7,\ 6), (7,\ 9), (9,\ 7)\}.$$

Consider the subsets

$$A = \{2,\ 3\},\ B = \{1, 5, 6, 8, 9\} \,\&\, F = \{1,\ 4\}$$

The approximations of the sets are as in Table 1. In the table, sets of the form $\{1, 2\}$ have been abbreviated as 12. The neighborhood granules generated by the elements are as in Table 2.

Table 1 Approximations

Set	l	ll	u	lu	uu
A	23	23	23	23	23
B	1	1	145,679	1	145,679
F	1	1	145	1	145

Table 2 Neighborhoods

S	1	2	3	4	5	6	7	8	9
$[x]$	2	1	3	45	45	79	69		67

A *new rough approach to mark and recapture* methods is proposed by the present author below. A detailed paper will appear separately.

Example 76 A practical example for partial approximation spaces is afforded by the problem of estimating fishes in a pond or a river. Sampling techniques like the *mark and recapture method* [147] and improvements thereof [148] for estimating the number of fishes in a pond are well known. These methods are widely used in epidemiology and many similar situations.

The basic steps are:

- Collect an initial set of fishes.
- Mark and release them into the pond.
- Draw one or more samples from the resulting population.
- Estimate various statistics from the samples.
- Make decisions subject to assumptions about the process.

In all these cases, the issue of discernibility can be seen from the perspective of partial approximation spaces. If modern cameras and identification algorithms are used without any observable physical marking, then it is possible that marked objects in some or many of the second or later stage samples are not correctly identified. This leads to a partial approximation spaces in the context and related theory can be of much relevance in reasoning in the context for automated decision making.

5.1 Exceptional Sets

One of the main strategies used in the study is this assumption: *if approximations of a set (or sets) satisfies a property P, but not Q that is a specialization of P, while Q (and therefore P in a trivial sense) holds for approximations in classical rough sets that is not satisfied in an approximation space, then this scenario must be due to the existence of certain kinds of subsets*. This leads to a classification of such sets and a new approach to the main questions.

Definition 77 A set K of a partial approximation space S will be said to *R-isolated* if and only if

$$(\forall a, b)\,(Rab\,\&\,a \in K \longrightarrow b \in K).$$

Table 3 Isolated sets

Name	$\in A$	$\notin A$	Rxx	$\neg Rxx$	Rxw
3 : 1	$\{a\}$	$\{b, c\}$	$\{b\}$	$\{a, c\}$	$\{ab, ac\}$
3 : 2	$\{a, b\}$	$\{c\}$	$\{\}$	$\{a, b, c\}$	$\{ab, ac\}$
3 : 3	$\{a, b\}$	$\{c\}$	$\{b\}$	$\{a, c\}$	$\{ab, ac\}$
3 : 4	$\{a, b\}$	$\{c\}$	$\{c\}$	$\{a, b\}$	$\{ab, ac\}$
3 : 5	$\{a\}$	$\{b, c\}$	$\{a\}$	$\{b, c\}$	$\{ab, ac\}$
3 : 6	$\{a\}$	$\{b, c\}$	$\{b, c\}$	$\{a\}$	$\{ab, ac\}$
2 : 1	$\{a\}$	$\{b\}$	$\{\}$	$\{a, b\}$	$\{ab\}$
2 : 2	$\{a\}$	$\{b\}$	$\{b\}$	$\{a\}$	$\{ab\}$
2 : 3	$\{a\}$	$\{b\}$	$\{a\}$	$\{b\}$	$\{ab\}$

The relationship of the collection $\mathcal{F}(A)$ of all isolated sets which are not disjoint from a given set A can be expected to actually define the approximations of the set A. In [85], it is shown that it suffices to restrict ourselves to one, two and three element isolated sets alone. These are classified into different types on the basis of the relationship of the elements to A, R and S. This is done in coherence with the different types of possible approximations. Table 3 summarizes the possibilities:

Definition 78 A three element isolated set $\{a, b, c\}$ not contained in a set A will be said to be $\frac{3}{1}$-*exceptional for* A if and only if all of the following are true.

(i) $a \in A \,\&\, \neg b, c \in A$
(ii) $\neg Raa \,\&\, Rbb \,\&\, \neg Rcc$
(iii) $Rab \,\&\, Rac$

Definition 79 A set K will be said to be exceptional for a set A if and only if all of the following hold:

(i) K is isolated.
(ii) $(\exists x \in K)\, \neg Rxx$
(iii) $K \nsubseteq A \,\&\, K \cap A \neq \emptyset$

Using the above concepts of exceptional sets, a fine grained classification of subsets of a partial approximation space is obtained. One of the interesting concepts used in the paper [85] is that of a fully featured subset—that is intended to capture the idea of a subset with a maximal set of nontrivial properties. There is a typo in the minor proposition on bijection between two fully featured subsets in the paper [85]:

Definition 80 A set of the form

$$K = K_0 \cup \{x_1, y_1, y_2, z_1, z_2, a_1, b_1, c_1, f_1, f_2, g_1, h_1\}$$

satisfying all of

1. K_0 is a nonempty subset of S with at least two elements that are not subsets of any other exceptional subsets of K.
2. Elements of K_0 are distinct from the others listed.

In the following $(A : ES(\tau, K))$ should be read as A is a type τ-exceptional set for K.

$$(\forall x \in K_0)\, Rxx \tag{Ref}$$

$$(\forall x \in K_0)\, (\forall z)(Rxz \longrightarrow Rzz) \tag{Ref+}$$

$$(\exists x_2, x_3 \in S \setminus K) \left(\{x_1, x_2, x_3\} : ES(\frac{3}{1}, K)\right) \tag{31}$$

$$(\exists z_3 \in S \setminus K) \left(\{z_1, z_2, z_3\} : ES(\frac{3}{2}, K)\right) \tag{32}$$

$$(\exists z_3 \in S \setminus K) \left(\{z_1, z_2, z_3\} : ES(\frac{3}{3}, K)\right) \tag{33}$$

$$(\exists f_3 \in S \setminus K) \left(\{f_1, f_2, f_3\} : ES(\frac{3}{4}, K)\right) \tag{34}$$

$$(\exists g_2, g_3 \in S \setminus K) \left(\{g_1, g_2, g_3\} : ES(\frac{3}{5}, K)\right) \tag{35}$$

$$(\exists h_2, h_3 \in S \setminus K) \left(\{h_1, h_2, h_3\} : ES(\frac{3}{6}, K)\right) \tag{36}$$

$$(\exists a_2 \in S \setminus K) \left(\{a_1, a_2\} : ES(\frac{2}{1}, K)\right) \tag{21}$$

$$(\exists b_2 \in S \setminus K) \left(\{b_1, b_2\} : ES(\frac{2}{2}, K)\right) \tag{22}$$

$$(\exists c_2 \in S \setminus K) \left(\{c_1, c_2\} : ES(\frac{2}{3}, K)\right) \tag{23}$$

will be said to be *fully featured* in S.

Theorem 81 *In case of esoteric rough sets [85], with the collection of all pseudo-classes being the granules, all of* RA, MER, NO, UU, US *hold, but* ACG *may not.*

Proof RA, NO follow from the definition. It is possible that $[x] \subset [x]^u$, so ACG may not hold. US holds as if a granule overlaps another subset, then the upper approximation of the set would surely include the granule. □

5.2 Algebraic Semantics

Many algebraic approaches can be used for studying partial approximation spaces apart from the three in [85]. In this subsection one of the three is presented below:

Theorem 82 *The set* $\mathbf{F}(S)$ *is endowable with a Boolean algebra structure.*

Proof For any $\alpha, \beta \in \mathbf{F}(S)$ if the operations $\wedge$, $\vee$ and ^{c} are defined as below:

$$\alpha \wedge \beta = \bigcup\{[x] : [x] \subseteq \alpha \,\&\, [x] \subseteq \beta\} \quad \text{(meet)}$$

$$\alpha \vee \beta = \bigcup\{[x] : [x] \subseteq \alpha \cup \beta\} \quad \text{(join)}$$

$$\alpha^c = \bigcup\{[x] : [x] \subseteq S \setminus \alpha\} \quad \text{(negation)}$$

then the structure $\mathcal{F}(S) = \langle \underline{\mathbf{F}(S)}, \vee, \wedge, ^c, 0, 1\rangle$ is a Boolean algebra, where $0 = \emptyset$ and $1 = 0^c$. The last operation is not a partial operation as the elements of $\mathbf{F}(S)$ are definite sets. The distributive lattice structure under the defined operations follows from purely set-theoretic considerations. □

Theorem 83 *The set* $\mathbf{E}(S)$ *is not necessarily a lattice under operations defined in the same way as in the above proof.*

Proof It is possible that elements of $\mathbf{E}(S)$ contain isolated elements. For such subsets $\wedge$ and $\vee$ as defined above will not be lattice operations (idempotency will also fail). Counterexamples are easy to construct. □

Given a partial approximation space $\langle \underline{S}, R\rangle$, form its power set $\wp(S)$.

$$\text{Let } (a, b) \notin R \leftrightarrow (a, b) \in F$$

As R is a partial equivalence, F is a partially reflexive and symmetric relation. For any set $H \subseteq S$, let

- $H^\circ = \{x\,;\,(\forall a \in H)\,(x, a) \in F\}$
- $L(H) = H^{c\circ\circ c}$
- $U(H) = H^{\circ\circ}$

Definition 84 The algebra

$$\mathcal{P} = \left\langle \underline{\wp(S)}, \vee, \wedge, ^\circ, ^c, l, u, L, U, \emptyset, S\right\rangle$$

of type (2, 2, 1, 1, 1, 1, 1, 1, 0, 0) will be called a *Neo BZ-Lattice* with the operations l, u being the lower and upper approximations due to R respectively.

Theorem 85 $\mathcal{P} = \left\langle \underline{\wp(S)}, \vee, \wedge, ^{\circ}, ^{c}, l, u, L, U, \emptyset, S \right\rangle$ *is a Boolean algebra with extra operations that satisfies all of the following:*

$$\left\langle \underline{\wp(S)}, \vee, \wedge, ^{c}, \emptyset, S \right\rangle \text{ is a Boolean algebra.}$$

$$H^{\circ} \subseteq H^{\circ\circ\circ}$$

$$(H \subseteq K \longrightarrow (K^{\circ} \setminus K^{*}) \subseteq H^{\circ}), \text{ where } K^{*} = K \wedge K^{\circ}$$

$$H^{\circ} \wedge K^{\circ} \subseteq (H \vee K)^{\circ}$$

$$(H \wedge K)^{\circ} \subseteq H^{\circ} \vee K^{\circ}$$

$$H^{\circ} \setminus H^{*} \subseteq H^{c}$$

Proof

(i) $\vee$ and $\wedge$ are the same as the usual set-theoretic operations of union and intersection.
(ii) If $(a, a) \in F$, $a \in H$ and $\{a\} \cup K$ is an isolated set for some $K \subseteq H^{c}$, then a is in H° and elements R-related to a cannot be in H°. But a subset of K will in general be included in H°. a will then be in $H^{\circ\circ}$. But if no part of K is included in $H^{\circ\circ}$, then $a \notin H^{\circ\circ}$. It is easy to construct examples. $H^{\circ} \subseteq H^{\circ\circ\circ}$ can be verified by considering fully featured sets.

□

Proposition 86 *In general for a subset H, $H^{\circ\circ}$ is not comparable with H.*

Proof Let $x_0 \in H$ be such that

- $(\forall x \in H) \neg (x_0, x) \in R$
- $(x_0, x_0) \notin R$

Then it is that $x_0 \in H^{\circ}$, but it is possible that the element may or may not be in $H^{\circ\circ}$ (depending on how the elements related to x_0 are). The required counterexample is easy. □

The theory can be extended to a more abstract level by using an essentially BZ-lattice abstraction as in [19].

An Abstract Algebraic Approach

The motivations for this approach are similar to those of the double Stone approach in classical rough sets [31, 42] theory. However, there are many differences in the basic technique used. The valuation is into a refined form of sets of tuples of the form (A^{l}, A^{lu}, A^{u}) so that all considerations can be restricted to unions of pseudo-classes. For this purpose an additional operation τ that removes the pseudo classes that are not contained within the set under consideration is used.

Let the language $\mathfrak{L}$ of esoteric rough set logic consist of a nonempty set of propositional variables P, two binary connectives $\vee$, $\wedge$, three unary connectives $*$, $+$, $\flat$ and three constants $\mathbf{T}$, $\mathbf{T}_0$, $\mathbf{F}$ for truth. Formulas are constructible in the usual way, so that the set $\mathcal{F}(\mathfrak{L})$ of formulas is a free algebra of type $(2, 2, 1, 1, 1, 0, 0)$ generated over P. A model of $\mathfrak{L}$ then is a pair of the form (W, υ), where W is a set and $\upsilon : P \mapsto \wp(W) \times \wp(W) \times \wp(W)$ is a valuation, such that if $\upsilon(p) = (A, B, C)$ then $A \subseteq B \subseteq C$. Further it will be assumed that an operation $\tau : \wp(W) \mapsto \wp(W)$ satisfying all of the following is given:

Inclusion $\tau(A) \subseteq A$
Idempotence $\tau(\tau(A)) = \tau(A)$
Monotonicity $(A \subseteq B \longrightarrow \tau(A) \subseteq \tau(B))$
Empty-set $\tau(\emptyset) = \emptyset$
$\perp$ $A^{\perp} = \tau(A^c)$, c being the complementation operation.

Given a model $\mathfrak{M} = (W, \upsilon)$, its meaning function σ will be an extension of the valuation function $\sigma : \mathcal{F}(\mathfrak{L}) \longmapsto \wp(W) \times \wp(W) \times \wp(W)$ such that,

1. $\sigma(\mathbf{T}) = (\tau(W), \tau(W), \tau(W)) = 1$
2. $\sigma(\mathbf{T}_0) = (W, W, W) = 2$
3. $\sigma(\mathbf{F}) = (\emptyset), \emptyset, \emptyset) = 0$
4. $\forall p \in P\ \sigma(p) = \upsilon(p)$
5. If $\sigma(\varphi) = (A, B, C)$ and $\sigma(\psi) = (E, F, G)$, then
   - $\sigma(\varphi \wedge \psi) = (A \cap E, B \cap F, C \cap G)$
   - $\sigma(\varphi \vee \psi) = (A \cup E, B \cup F, C \cup G)$
   - $\sigma(\varphi^*) = (C^{\perp}, C^{\perp}, C^{\perp})$
   - $\sigma(\varphi^+) = (A^{\perp}, A^{\perp}, A^{\perp})$
   - $\sigma(\varphi^\flat) = (B^{\perp}, B^{\perp}, B^{\perp})$

On $Ran(\sigma) = \{\sigma(\varphi) : \varphi \in \mathcal{F}(\mathfrak{L})\}$, let the operations $\oplus$, $\cdot$, *, $^+$ and $^\flat$ be defined as below.

1. $\sigma(\varphi) \cdot \sigma(\psi) = \sigma(\varphi \wedge \psi)$
2. $\sigma(\varphi) \oplus \sigma(\psi) = \sigma(\varphi \vee \psi)$
3. $(\sigma(\varphi))^* = \sigma(\varphi^*)$
4. $(\sigma(\varphi))^+ = \sigma(\varphi^+)$
5. $(\sigma(\varphi)^\flat = \sigma(\varphi^\flat))$.

Theorem 87 *$Ran(\sigma)$ with the defined operations is an algebra satisfying all of the following :*

(i) $\langle Ran(\sigma), +, \cdot, 0, 1\rangle$ *is a bounded distributive lattice.*
(ii) $(a \leq b^* \longrightarrow (a \cdot b) = 0)$
(iii) $a^{***} = a^*$
(iv) $a^{+++} = a^+$
(v) $a^{\flat\flat} \cdot a = a$
(vi) $x^* \oplus x^{**} = 1$
(vii) $x^+ \cdot x^{++} = 0$

(viii) $(x^+ = y^+, x^* = y^*, x^\flat = y^\flat \longrightarrow x \oplus y^* = 1)$
(ix) $x^+ \cdot x^* = x^*$
(x) $2^* = 0$
(xi) $1^{**} = 1$

Proof The following proof is abstract and is not really set theoretical. Connection with sets have been retained for its visual value.

(i) $\oplus$ and $\cdot$ are clearly distributive lattice operations on $Ran(\sigma)$. It is bounded by 0 and 2 and it is essential that 2 covers 1 (lattice-theoretically).
(ii) If $a \leq b^*$, then a is contained in the image of the complement (by τ) of the upper approximation of b, so that the *meet* $\cdot$ of a and b is the image 0 of the triple of empty sets.
(iii) a^{***} is obtained by three applications of an upper approximation followed by a complementation and then by the τ operation in order on the components. But τ essentially forms the union of the largest collection of pseudoclasses that are contained within the complement (component-wise).
(iv) $a^{+++} = a^+$ can be proved in the same way as the above.
(v) $a^\flat$ will consist of pseudoclasses included in complement of the lu applications on the components of a. An application lu on the resulting components will have no effect. The complements of these will contain a component-wise. An application of the τ operation on the components will still contain a. So $a^{\flat\flat} \cdot a = a$.
(vi) x^* is a subset of the complement of the upper approximation of x. x^{**} is essentially the largest union of pseudoclasses contained in the complement of x^*. Now their disjunction ($\oplus$) will not contain the singleton isolated sets alone. This is precisely 1.
(vii) $x^+ \cdot x^{++} = 0$ follows by an argument similar to the one above.
(viii) If $(x^+ = y^+, x^* = y^*, x^\flat = y^\flat$, then obviously it does not mean that $x = y$. But given an arbitrary nonempty pseudo class, it must be the case that it is either in x or y^*. So $x \oplus y^* = 1$.
(ix) x^* is included in x^+
(x) The complement of 2 is 0 and τ of that is still τ.
(xi) τ of the 2 is 1.

□

Definition 88 $Ran(\sigma)$ along with the defined operations is said to be an *esoteric* 2*SA algebra*.

Definition 89 An *abstract esoteric* 2*SA algebra* will be an algebra of the form

$$A = \langle \underline{A}, \vee, \wedge, ^\flat, ^*, ^+, 0, 1, (2, 2, 1, 1, 1, 0, 0) \rangle$$

that satisfies all of the following:

(i) $\langle \underline{A}, \vee, \wedge, 0, 1 \rangle$ is a bounded distributive lattice.
(ii) $a \leq b^* \longrightarrow (a \cdot b) = 0$

(iii) $a^{***} = a^*$

(iv) $a^{+++} = a^+$

(v) $a^{\flat\flat} \wedge a = a$

(vi) $x^* \vee x^{**} = 1$

(vii) $x^+ \wedge x^{++} = 0$

(viii) $(x^+ = y^+ \,\&\, x^* = y^* \,\&\, x^\flat = y^\flat \longrightarrow x \vee y^* = 1)$

(ix) $x^+ \wedge x^* = x^*$

(x) $1^* = 0$

(xi) $1^{**} = 1$

Representation results relating to this abstraction

6 Tolerance Approximation Spaces

There are a few granular approaches to similarity approximation spaces, that is general approximation spaces of the form $S = \langle \underline{S}, T \rangle$ with T being a tolerance relation on S. Possible semantics depend on choice of granulation. Some choices of granulations in the context are the following:

- The collection $\mathcal{B}$ of blocks (maximal subsets B of S that satisfy $B^2 \subseteq T$),
- The collection of successor $\mathcal{N}$ and predecessor $\mathcal{N}_i$ neighborhoods generated by T and
- The collection $\mathcal{T} = \{\cap(\Gamma) : \Gamma \subseteq \mathcal{B}\}$. These will be called the collection of *squeezed blocks*.

Of these $\mathcal{B}$ is the most natural because it is a direct generalization of the concept of a partition of equivalence classes. Related representation theorems can be found in the chapter on duality in this volume [107]. For this case multiple semantic approaches have been developed by the present author [86, 89, 106]. The approach in [106] can possibly be extended to the point-wise contexts as well. The point-wise nongranular approach has been considered in [16, 19, 62, 124, 139, 155] and other papers. The semantics in [86] applies to granulations satisfy a minimal set of conditions.

6.1 Choice Inclusive Approach

The choice inclusive approach invented by the present author for tolerance approximation spaces [87, 89] can be directly generalized to granular operator spaces, general covers on a set and all general approximation spaces. It can also be generalized to the point-wise approximations constructed by reference to granules. Because of this reason, the approach is presented separately in Sect. 11.

6.2 Squeezed Blocks

In classical rough set theory, the negative region of a set is the lower approximation of the complement of the set. This region is disjoint from the upper approximation of the set in question. An analogous property fails to hold in tolerance spaces (TAS). To deal with this different semantic approaches (to tolerance space) involving modified upper approximations has been considered in [86, 157, 160]. The modified upper approximations are formed from upper approximations by *biting off* a part of it to form *bitten upper approximations*. The new approximations turn out to be disjoint from the negative region of the subset and also possess some nice properties.

The squeezed block approach, as a semantics for a specific tolerance space context, was introduced in [160]. The nomenclature is due to the present author. In this approach, taking $\mathfrak{T}$ as the set of granules, the authors define the lower, upper and bitten upper approximation of a $X \subseteq S$ as follows:

$$X^{l_s} = \bigcup\{A : A \subseteq X \,\&\, A \in \mathfrak{T}\} \tag{sq-lower}$$

$$X^{u_s} = \bigcup\{A : A \cap X \neq \emptyset \,\&\, A \in \mathfrak{T}\} \tag{sq-upper}$$

$$X^{u_{sb}} = \bigcup\{A : A \cap X \neq \emptyset \,\&\, A \in \mathfrak{T}\} \setminus (X^c)^l \tag{sqb-upper}$$

Theorem 90 *On the complete Boolean algebra with operators*

$$\left\langle \underline{\wp(S)}, \cup, \cap, l_s, u_{sb}, ^c, \bot, \top \right\rangle$$

on the powerset $\wp(S)$ *(with* $\bot = \emptyset$ *and* $\top = \S$*), all of the following hold:*

$$a^{u_{sb}} = a^{cl_sc} \tag{S5-Dual}$$

$$a \subseteq b \longrightarrow a^{l_s} \subseteq b^{l_s} \tag{Monotone}$$

$$\bot^{l_s} = \bot = \bot^{u_{sb}} \tag{Bottom}$$

$$\top^{l_s} = \top^{u_{sb}} = \top \tag{Top}$$

$$a^{l_s} \subseteq a \subseteq a^{u_{sb}} \quad \text{(Reflexive)}$$

$$x^{l_s} = x^{l_s l_s} \quad \text{(Idempotence)}$$

$$(a \cap b)^{l_s} \subseteq a^{l_s} \cap b^{l_s} \quad \text{(L3)}$$

$$a^{l_s} \cup b^{l_s} \subseteq (a \cup b)^{l_s} \quad \text{(L4)}$$

Proof The proof of the properties is left to the reader. □

Theorem 91 *If the set of definable objects is defined by $\Delta(S) = \{\cup H : H \subseteq \mathfrak{T}\}$, then all of the following hold:*

$$\emptyset, S \in \Delta(S) \quad \text{(Bounds)}$$

$$(\forall A, B \in Delta(S)) A \cup B,\ A \cap B \in \Delta(S) \quad \text{(Closure)}$$

$$\left\langle \underline{\Delta(S)}, \cup, \cap, \emptyset, S \right\rangle \text{ is a complete ring of subsets} \quad \text{(Ring)}$$

In fact $\Delta(S)$ with the induced operations forms an Alexandrov topology [160].

On the set of definable objects $\Delta(S)$, let

- $X \to Z = \bigcup\{B \in \mathfrak{T} \,\&\, X \cap B \subseteq Z\}$
- $X \ominus Z = \bigcap\{B \in \mathfrak{T} \,\&\, X \subseteq Z \cup B\}$.

Then the following theorem provides a topological algebraic semantics ([160]):

Theorem 92 *$\langle \Delta(S), \cap, \cup, \to, \ominus, \emptyset, S\rangle$ is a complete atomic double Heyting algebra. It is also atomistic.*

Proof

- By the previous theorem, $\Delta(S)$ is also infinitely join and infinitely meet distributive lattice.
- The set $P = \bigcup\{B;\ B \in \mathfrak{T} \& X \cap B \subseteq Z\}$ is also the greatest element in the set that satisfies $X \cap P \subseteq Z$.
- The set $Q = \bigcap\{B;\ B \in \mathfrak{T} \& X \subseteq Z \cup B\}$ is also the least element in the set that satisfies $X \subseteq Q \cup Z$.
- The axioms of a complete double Heyting algebra can be verified from this.
- The least granule and definite set containing an element x is $A_x = \bigcap\{A; x \in A \in \mathfrak{T}\}$. It is also the least neighborhood of x in the Alexandrov topology.
- So $\{A_x : x \in S\}$ is the least base for the topology and the set of atoms of the lattice $\Delta(S)$. Therefore, every $Z \in \Delta(S)$ must contain a set of the form A_x. This proves that the double Heyting algebra is atomic.

□

Proposition 93 *$\langle \Delta(S), \cap, \cup, \to, \ominus, \emptyset, S\rangle$ is not regular and does not satisfy weak law of excluded middle.*

Proof

- Let $S = \{a, b, c, e, f, g\}$ and $\mathcal{B} = \{\{a, b, c\}, \{b, c, e\}, \{f, g\}\}$.
- Then $\mathcal{T} = \{\{a, b, c\}, \{b, c, e\}, \{f, g\}, \{b, c\}\}$.
- $\Delta(S) = \mathcal{T} \cup \{\{a, b, c, e\}, \{a, b, c, f, g\}, \{b, c, e, f, g\}, \{b, c, f, g\}, \emptyset, S\}$
- For any X, define $\neg X = X \to \emptyset$ and $X^o = S \ominus X$
- For $\{b\} \neq \{c\}$, $\neg\{b\} = \{c\} = \{f, g\}$ and $\{b\}^o = \{c\}^o = S$. So the algebra is not regular.
- Also $\neg\{a, g\} \cup \neg\neg\{a, g\} = \{b, c, e, f, g\} \neq S$

□

Definition 94 On the collection of rough objects

$$\mathcal{O} = O_s(S) = \{(A^{l_s}, A^{u_{bs}}) : A \subseteq S\},$$

let (for any $(A^{l_s}, A^{u_{bs}}), (B^{l_s}, B^{u_{bs}}) \in \mathcal{O}$),

$$(A^{l_s}, A^{u_{bs}}) \wedge (B^{l_s}, B^{u_{bs}}) = (A^{l_s} \cap B^{l_s}, A^{u_{bs}} \cap B^{u_{bs}}) \quad \text{(Meet)}$$

$$\bigvee (A_i^{l_s}, A_i^{u_{bs}}) = (\bigcup A_i^{l_s}, \bigcup A_i^{u_{bs}}) \quad \text{(Join)}$$

$$\text{Let } \mu(A, B) = \{Z : Z \subseteq S \,\&\, (A^{l_s}, A^{u_{bs}}) \wedge (Z^{l_s}, Z^{u_{bs}}) \leq (B^{l_s}, B^{u_{bs}})\} \quad \text{(H)}$$

$$\text{Let } \nu(A, B) = \{Z : Z \subseteq S \,\&\, (A^{l_s}, A^{u_{bs}}) \leq (Z^{l_s}, Z^{u_{bs}}) \vee (B^{l_s}, B^{u_{bs}})\} \quad \text{(B)}$$

$$(A^{l_s}, A^{u_{bs}}) \to (B^{l_s}, B^{u_{bs}}) = \left(\bigcup_{Z \in \mu(A,B)} Z^{l_s}, \bigcup_{Z \in \mu(A,B)} Z^{u_{bs}} \right) \quad \text{(H1)}$$

$$(A^{l_s}, A^{u_{bs}}) \ominus (B^{l_s}, B^{u_{bs}}) = \left(\bigcap_{Z \in \nu(A,B)} Z^{l_s}, \bigcap_{Z \in \nu(A,B)} Z^{u_{bs}} \right) \quad \text{(B1)}$$

Theorem 95 *The algebra* $\langle \underline{\mathcal{O}}, \vee, \wedge, \to, \ominus, \emptyset, S \rangle$ *is a complete atomic double Heyting algebra.*

Proof

- Note that, for any $A \subseteq S$, $A^{l_s} \in \Delta(S)$ and $A^{u_{bs}} \in Cl(S)$—the set of closed sets in the Alexandrov topology on $(S, \Delta(S))$,

- $\Delta(S)$ and $Cl(S)$ are complete atomic distributive lattices of sets that are infinite meet and join distributive. So their product

$$DC = \langle \Delta(S) \times Cl(S), \vee, \wedge, (\emptyset, \emptyset), (S, S) \rangle$$

is also so.
- $\langle \mathcal{O}, \vee, \wedge, (\emptyset, \emptyset), (S, S) \rangle$ is a complete sublattice of DC and so is also a infinite meet and join distributive complete atomic distributive sublattice of sets.
- On $\mathcal{O}$, the operations as in Def. 94 are induced.
- The rest of the proof is in direct verification.

□

The following theorem summarizes the granular properties in the context.

Theorem 96 *Taking $\mathcal{G}$ to be $\mathcal{T}$ and restricting to lower and bitten upper approximations alone* RA, ACG, NO *do not hold while* LRA, WRA, MER, LACG, ST *do hold.*

Proof If H is a granule, then it is an intersection of blocks. It can be deduced that the lower approximation of H coincides with itself, while the bitten upper approximation is the union of all blocks including H. LRA is obvious, but URA need not hold due to the bitten operation. If a definite set is included in a granule, then it has to be a block that intersects no other block and so the granule should coincide with it. So MER holds. □

6.3 *Higher Order Bited Approach*

In [86], two different approaches for handling rough sets formed by bited granular approximations generated by granulations including blocks in tolerance spaces had been invented by the present author. The most interesting aspect of her approach is in the nature of higher order rough objects used. These are discussed in the present part of a subsection. The notation is considerably simplified from the one used in the paper.

The theory apparently lays emphasis on desired mereological properties at the cost of representation.

Let $Gr(S) \subseteq \wp(S)$ be the collection of granules for a tolerance space defined by some conditions including $\bigcup Gr(S) = S$. A subset X is $\cup$-*granular definable* if and only if $\exists \mathcal{B} \subseteq Gr(S)\ X = \bigcup \mathcal{B}$. In [86], $\cup$-granular definable was *granularly definable*- the latter terminology is not precise enough in the light of [91]. The collection of $\cup$-granular definable sets shall be denoted by $Def_{Gr}(S)$. The lower, upper and bitten approximations of X are defined as below:

$$Gr_*(X) = \bigcup \{A : A \subseteq X,\ A \in Gr(S)\} \quad \text{(Lower)}$$

$$Gr^*(X) = \bigcup \{A : A \cap X \neq \emptyset,\ A \in Gr(S)\} \quad \text{(Upper)}$$

$$POS_{Gr}(X) = Gr_*(X) \quad \text{(Positive Region)}$$

$$NEG_{Gr}(X) = Gr_*(X^c) \quad \text{(Negative Region)}$$

$$Gr_b^*(X) = Gr^*(X) \setminus NEG_{Gr}(X) \quad \text{(Bitten Upper.)}$$

Clearly, the positive and negative region are ∪-granular definable. But in this scheme of things $Gr^*(X) \cap NEG_{Gr}(X) \neq \emptyset$ is possible. The *bitten upper approximation* has been defined to overcome this problem. Relative this the boundary is given by

$$BN_{Gr}(X) = Gr_b^*(X) \setminus Gr_*(X) = S \setminus (POS_{Gr}(X) \cup NEG_{Gr}(X)).$$

$Gr(S)$ may be taken to be the set of T-relateds or the set of blocks of T or something else. For example, if $Gr(S)$ is the set of all sets of the form T_x with $T_x = \{y;\ (x, y) \in T,\ y \in S\}$ then it is a proper cover of S. The upper and lower approximations of a subset X are then:

$$Gr_*(X) = \bigcup\{Y \in Gr(S);\ Y \subseteq X\} taglower$$

$$Gr^*(S) = \bigcup\{Y \in Gr(S);\ Y \cap, X \neq \emptyset\} \quad \text{(upper)}$$

$$Gr_b^*(X) = Gr^*(X) \setminus Gr_*(X) \quad \text{(bitten upper)}$$

The tuple $\langle S,\ Gr(S),\ T,\ Gr_*,\ Gr_b^*\rangle$ is called a *bitten approximation system* (**BAS**).

The properties of the approximations are as follows (Table 4):

Table 4 Bited approximations

$l1$-Property	$u2$-Property
$1a.)\ Gr_*(X) \subseteq X$	$1b.)\ X \subseteq Gr_b^*(X)$
$2a.)\ (X \subseteq Y \longrightarrow Gr_*(X) \subseteq Gr_*(Y))$	$2b.)\ (X \subseteq Y \longrightarrow Gr_b^*(X) \subseteq Gr_b^*(Y))$
$3a.)\ Gr_*(\emptyset) = \emptyset$	$3b.)\ Gr_b^*(\emptyset) = \emptyset$
$4a.)\ Gr_*(S) = S$	$4b.)\ Gr_b^*(S) = S$
$5a.)\ Gr_*(Gr_*(X)) = Gr_*(X)$	$5b.)\ Gr_b^*(Gr_b^*(X)) = Gr_b^*(X)$
$6a.)\ Gr_*(X \cap Y) \subseteq Gr_*(X) \cap Gr_*(Y)$	$6b.)\ Gr_b^*(X \cap Y) \subseteq Gr_b^*(X) \cap Gr_b^*(Y)$
$7a.)\ Gr_*(X) \cup Gr_*(Y) \subseteq Gr_*(X \cup Y)$	$7b.)\ Gr_b^*(X) \cup Gr_b^*(Y) \subseteq Gr_b^*(X \cup Y)$
$8a.)\ Gr_*(X) \subseteq Gr_b^*(Gr_*(X))$	$8b.)\ Gr_*(Gr_b^*(X))\ \subseteq)\ Gr_b^*(X)$
$9a.)\ (Gr_*(X))^c = Gr_b^*(X^c)$	$9b.)\ (Gr_b^*(X))^c = Gr_*(X^c)$
$10A.)\ X \in Def_{Gr}(S) \longleftrightarrow X = Gr_*(X)$	
$10B.)\ X \in Cr_{Gr}(S) \longleftrightarrow Gr_*(X) = Gr_b^*(X)$	
$11A.)\ X, Y \in Def_{Gr}(S) \longrightarrow X \cup Y \in Def_{Gr}(S)$	(Implies equality in $7a$)
$11B.)\ X, Y \in Cr_{Gr}(S) \longrightarrow X \cap Y,\ X \cup Y \in Cr_{Gr}(S)$	(Implies equality in $6a$, $6b$, $7a$, $7b$, $8a$, $8b$)

6.3.1 Semantics for Bitten Rough Set Theory

The concept of granules to be used in the theory is essentially kept open. Many types of granules may not permit nice representation theory. Despite this, the semantics in [86] over roughly equivalent objects does well. This is due to the higher order approach used.

Definition 97 If S is a tolerance space , then over $\wp(S)$ let $A \sim B$ if and only if

$$Gr_*(A) = Gr_*(B) \text{ and } Gr_b^*(A) = Gr_b^*(B)$$

The following proposition and theorem basically say *the quotient structure (or the set of roughly equivalent objects) has very little structure* with respect to desirable properties of a partial algebra. They are clearly deficient from the rough perspective as proper conjunction and disjunction operations do not happen. But biting may actually make the partial operations total in many contexts.

Proposition 98 *$\sim$ is an equivalence on the power set $\wp(S)$. Moreover the following operations and relations on $\wp(S)| \sim$ are well-defined:*

- $L([A]) = [Gr_*(A)]$
- $\neg[A] = [A^c]$ *if defined*
- $\blacklozenge([A]) = [Gr_b^*(A)]$
- $[A] \leq [B]$ *if and only if, for any* $A \in [A]$ *and* $B \in [B]$ $Gr_*(A) \subseteq Gr_*(B)$ *and* $Gr_b^*(A) \subseteq Gr_b^*(B)$.
- $[A] \Cap [B] = [C]$ *if and only if* $[C]$ *is the infimum of* $[A]$ *and* $[B]$ *w.r.t* $\leq$. *It shall be taken to be undefined in other cases.*
- $[A] \Cup [B] = [C]$ *if and only if* $[C]$ *is the supremum of* $[A]$ *and* $[B]$ *w.r.t* $\leq$. *It shall be taken to be undefined in other cases.*

Moreover $\leq$ is a partial order on $\wp(S)| \sim$ that is partially compatible with L on the crisp elements.

Proof

- For any $A \in \wp(S)$ and any $B \in [A]$, $Gr_*(Gr_*(A) = Gr_*(A) = Gr_*(B)$ and $Gr_b^*(Gr_*(A)) = Gr_b^*(Gr_*(B))$. This proves that L is well-defined.
- For any $A \in \wp(S)$ and any $B, E \in [A]$, $(Gr_*(B))^c = Gr_b^*(B^c) = (Gr_*(E))^c = Gr_b^*(E^c)$ and $(Gr_b^*(B))^c = Gr_*(B^{c)} = (Gr_b^*(E))^c = Gr_*(E^{c)}$.

Rest of the statements can be verified along similar lines. □

Proposition 99 *In the above context,*

$$([A] \leq [B] \longrightarrow \neg[B] \leq \neg[A])$$

Theorem 100 *All of the following hold in $\wp(S)|\sim$ (the unary operators are assumed to bind more strongly than binary ones):*

1. $(\blacklozenge x \Cup \blacklozenge y = a, \blacklozenge(x \Cup y) = b \longrightarrow \blacklozenge x \Cup \blacklozenge y \leq \blacklozenge(x \Cup y))$
2. $x \Cap \blacklozenge x = x$
3. $\blacklozenge\blacklozenge x = \blacklozenge x$
4. $(\blacklozenge x \Cap \blacklozenge y = a, \blacklozenge(x \Cap y) = b \longrightarrow \blacklozenge(x \Cap y) \leq \blacklozenge x \Cap \blacklozenge y)$
5. $(Lx \Cup Ly = a, L(x \Cup y) = b \longrightarrow Lx \Cup Ly \leq L(x \Cup y))$
6. $Lx \Cap \blacklozenge Lx = Lx$
7. $\blacklozenge x \Cup L\blacklozenge x = \blacklozenge x$
8. $\neg\blacklozenge x = L\neg x$
9. $\neg Lx = \blacklozenge\neg x$

Proof

1. If $A \in x$ and $B \in y$, then $\blacklozenge x = [Gr_b^*(A)]$, $\blacklozenge y = [Gr_b^*(B)]$. Given the existence of the terms in the premise, it can be assumed that there exists $C \in [Gr_b^*(A)] \Cup [Gr_b^*(B)]$ and $E \in \blacklozenge(x \Cup y)$. $Gr_*(C) \subseteq Gr_*(E)$ and $Gr_b^*(C) \in Gr_b^*(E)$. So, given the existence of the terms in the premise, $\blacklozenge x \Cup \blacklozenge y \leq \blacklozenge(x \Cup y)$.
2. If $A \in x$, then $\blacklozenge x = [Gr_b^*(A)]$. As $A \subseteq Gr_b^*(A)$, so $x \Cap \blacklozenge x = x$.
3. If $A \in x$, then $\blacklozenge\blacklozenge x = \blacklozenge[Gr_b^*(A)] = [Gr_b^*(Gr_b^*(A))] = [Gr_b^*(A)] = \blacklozenge x$.
4. The proof of $(\blacklozenge x \Cap \blacklozenge y = a, \blacklozenge(x \Cap y) \longrightarrow \blacklozenge(x \Cap y) \leq \blacklozenge x \Cap \blacklozenge y)$ is similar to that of the first item.
5. Given the existence of the terms in the premise, if $A \in x$ and $B \in y$, then $Lx = L[A] = [Gr_*(A)]$ and $Ly = [Gr_*(B)]$. If $C \in Lx \Cup Ly$ and $E \in L(x \Cup y)$, then $Gr_*(C) \subseteq Gr_*(A) \cup Gr_*(B)$, $Gr_*(A) \cup Gr_B \subseteq Gr_*(E)$ and $Gr_b^*(C) \subseteq Gr_b^*(E)$. So, $Lx \Cup Ly \leq L(x \Cup y)$.
6. If $A \in x$, then $Lx = L[A] = [Gr_*(A)]$ and $\blacklozenge Lx = [Gr_b^* Gr_*(A)]$. But $Gr_*(A) \subseteq Gr_b^* Gr_*(A)$. So $Lx \Cap \blacklozenge Lx = Lx$.
7.
   - If $A \in x$, then $\blacklozenge x = \blacklozenge[A] = [Gr_b^*(A)]$ and $L\blacklozenge x = [Gr_* Gr_b^*(A)]$.
   - But $Gr_* Gr_b^*(A) \subseteq Gr_b^*(A)$.
   - So, $\blacklozenge x \Cup L\blacklozenge x = \blacklozenge x$.
8. If $A \in x$, then $\neg\blacklozenge x = \neg\blacklozenge[A] = \neg[Gr_b^*(A)]$. $\neg[Gr_b^*(A)] = [(Gr_b^*(A))^c], = [Gr_*(A^c)] = Lneg[A]$. So, $\neg\blacklozenge x = L\neg x$.
9. If $A \in x$, then $\neg Lx = \neg[Gr_*(A)] = [(Gr_*(A))^c]$.

   $$\text{But } [(Gr_*(A))^c] = [Gr_b^*(A^c)].$$

   This yields $\neg Lx = \blacklozenge\neg x$.

□

A semantics using the partial algebra over the associated quotient may be difficult because of axiomatisability issues. So a higher order approach that avoids introduction of extraneous properties is used. Eventually the constructed algebra ends up with three partial orders. In the following construction the use of a modified concept of filters simplifies the eventual representation theorem.

If $\wp(S)| \sim = K$, then let $K^* = \{f : f : K \mapsto I$ is isotone $\}$, I being the totally ordered two element set $\{0, 1\}$ under $0 < 1$. For any $A \subset K^*$, a subset F is an A-*ideal* if and only if

$$F = \bigcap_{x \in A} x^{-1}\{0\}.$$

Dually F is an A-*filter* if and only if

$$F = \bigcap_{x \in A} x^{-1}\{1\}.$$

All A-ideals are order ideals (w.r.t the induced order on K^*), but the converse need not hold. $A \subset K^*$ is said to be *full* if $\forall\, p \not\leq q\, \exists x \in A\, x(p) = 1,\ x(q) = 0$. A is said to be *separating* if for any disjoint ideal I and filter F, there exists a $x \in A$ such that $x_{|I} = 0$ and $x_{|F} = 1$

Lemma 101 *If A is a separating subset of K^* and $(\forall p, q \in K)(q \not\leq p \longrightarrow p \downarrow_A \cap q \uparrow_A = \emptyset)$, then A is full.*

If $p \in K$, then let $\mathcal{UP}(p) = \{x : x(p) = 1\}$ and $\mathcal{LO}(p) = \{x : x(p) = 0\}$, then the two closure operators C_1, C_2 can be defined via

$$C_1 = clos\{\mathcal{UP}(p)\}_{p \in K}$$

(a C_1-closed set is an intersection of elements of $\{\mathcal{UP}(p)\}_{p \in K}$) and

$$C_2 = clos\{\mathcal{LO}(p)\}_{p \in K}$$

Note that elements of $\mathcal{UP}(p)\}_{p \in K}$ are in fact $C_1 O_2$-sets (that is sets that are open with respect to the second closure system and closed with respect to the first). The set of such sets on a system (S, C_1, C_2), will be denoted by $C_1 O_2(S, C_1, C_2)$. The associated closure operators will be denoted by cl_1 and cl_2 respectively.

On any subset $A \subseteq K^*$, closure operators can be defined via $C_{iA}(X) = C_i \cap A$, with associated closure systems $\mathcal{UP}_A(p) = \mathcal{UP}(p) \cap A$ and $\mathcal{LO}_A(p) = \mathcal{LO}(p) \cap A$ respectively. It can be seen that, in the situation, $C_{1A} = clos\{\mathcal{UP}(p)\}_p \in P$ and $C_{2A} = clos\{\mathcal{LO}(p)\}_p \in P$.

Theorem 102 *If $A \subseteq K^*$ and $\sigma : K \mapsto C_1 O_2(A, C_{1A}, C_{2A})$ is a map defined by $\sigma(p) = \mathcal{UP}(p)$ then*

1. *σ is isotone*
2. *If A is full, then σ is injective*
3. *If A is separating, then σ is surjective.*

Proof This theorem and the following theorem are proved for an arbitrary partially ordered set K in [14]. □

Theorem 103 *If A is a full and separating subset of K^*, then*

$$K \cong C_1 O_2(A, C_{1A}, C_{2A})$$

Since K^* is a full and separating set, $K \cong C_1 O_2(K^*, C_1, C_2)$.
$K^* \setminus \{0, 1\}$ *is also a full and separating set.*

The following clarifies the connection with the more common way of using closure operators.

Proposition 104 *$A \in C_1 O_2(K^*, C_1, C_2)$ if and only if $A \subseteq K^*$ and $(\exists B, E \subseteq K^*)\, A = cl_1 B,\ A = K^* \setminus cl_2(E)$.*

K^* can be interpreted as the set of partitions of the set of roughly equivalent elements into an upper and lower region subject to the new order being a coarsening of the original order. The important thing is that this restricted global object is compatible with the natural global versions of the other operations and leads to a proper semantics. This is shown in what follows.

Definition 105 On K^*, the following global operations (relative those on K) can be defined:

- If $A \in C_1 O_2(K^*, C_1, C_2)$, then $\mathfrak{L}(A) = L(i(A))$, i being the canonical identity map from $C_1 O_2(K^*, C_1, C_2)$ onto $\wp K$.
- $A \vee B = cl_1(A \cup B)$ (if the right hand side is also open with respect to the second closure system), $\cup$ being the union operation over K^*
- $A \wedge B = cl_1(A \cap B)$ (if the right hand side is also open with respect to the second closure system), $\cap$ being the intersection operation over K^*
- If $A \in C_1 O_2(K^*, C_1, C_2)$ then $\Diamond A = \blacklozenge i(A)$
- If $A \in C_1 O_2(K^*, C_1, C_2)$ then $\sim A = \neg i(A)$
- $cl_1,\ cl_2$ can be taken as unary operators on K^*
- $1,\ \bot,\ \top$, shall be 0-ary operations with interpreted values corresponding to K, $\emptyset$ and K^* respectively
- If $A, B \in C_1 O_2(K^*, C_1, C_2)$ then $A \sqcap B, = i(A) \Cap i(B)$
- If $A, B \in C_1 O_2(K^*, C_1, C_2)$ then $A \sqcup B, = i(A) \Cup i(B)$

Proposition 106 *A partial operation is well defined if it is either uniquely defined or not ambiguously defined. In this sense all of the operations and partial operations are well-defined.*

Definition 107 An algebra of the form

$$\mathfrak{W} = \left\langle \underline{\wp(K^*)},\ \vee,\ \wedge,\ \sqcap,\ \sqcup,\ \cup,\ \cap,\ ^c,\ ,cl_1,\ cl_2,\ \sim,\ \mathfrak{L},\ \Diamond,\ \bot,\ 1,\ \top \right\rangle$$

of type $(2, 2, 2, 2, 1, 1, 1, 1, 1, 1, 0, 0, 0)$ in which the operations are as in the above definition will be called a *concrete bitten algebra*. $\cup$, $\cap$, ^{c} are the union, intersection and complementation operations respectively on $\wp(K^*)$. $\xi(x)$ will be used as an abbreviation for $cl_1 x = x,\ cl_2 x^c = x^c$. Further $\xi(x, y, \ldots)$ shall mean

$\xi(x), \xi(y)$ and so on. If S is the original tolerance space , then its associated concrete bitten algebra is denoted by $Bite(S)$.

Theorem 108 *A concrete bitten algebra $\mathfrak{W}$ satisfies all of the following:*

1. $\left\langle \underline{\wp(K^*)}, \cup, \cap, {}^c, \bot, \top \right\rangle$ *is a Boolean algebra. Note that after forming associated categories with the usual concept of morphisms, this can be realized through forgetful functors.*
2. $x \vee y \overset{\omega^*}{=} y \vee x$
3. $x \vee (y \vee z) \overset{\omega}{=} (x \vee y) \vee z$
4. $(\xi(x) \longrightarrow x \vee x = cl_1(x))$
5. $(x \vee y = z \longrightarrow cl_1 z = z, cl_2 z^c = z^c)$
6. $(x \vee x = y \longrightarrow cl_2(x^c) = x^c, y = cl_1(x) = x \wedge x)$
7. $cl_i(x) \cap x = x; cl_i cl_i(x) = cl_i(x); i = 1, 2$
8. $(x \cap y = x \longrightarrow cl_i(x) \cap cl_i(y) = cl_i(x)); i = 1, 2$
9. $x \wedge y \overset{\omega^*}{=} y \wedge x$
10. $x \wedge (y \wedge z) \overset{\omega}{=} (x \wedge y) \vee z$
11. $(x \wedge x = y \longrightarrow cl_2(x^c) = x^c, y = cl_1(x))$
12. $(cl_2((cl_1 x)^c) = (cl_1 x)^c \longrightarrow x \wedge x = cl_1(x))$
13. $(x \wedge y = z \longrightarrow cl_1 z = z, cl_2 z^c = z^c)$
14. $((x \wedge y) \vee x = z \longrightarrow z = cl_1(x))$
15. $(x \wedge y) \vee x \overset{\omega^*}{=} x \wedge (y \vee x)$
16. $\mathfrak{L}\bot = \bot; \mathfrak{L}1 = 1$
17. $(cl_1 x = x, cl_2(x^c) = x^c \longrightarrow x \vee \mathfrak{L}x = x, \mathfrak{L}\mathfrak{L}x = \mathfrak{L}x)$
18. $(x \sqcap y = z \longrightarrow \xi(x, y, z))$
19. $(x \sqcup y = z \longrightarrow \xi(x, y, z))$
20. $x \sqcap y \overset{\omega^*}{=} y \sqcap x$
21. $x \sqcap (y \sqcap z) \overset{\omega}{=} (x \sqcap y) \sqcap z$
22. $x \sqcup y \overset{\omega^*}{=} y \sqcup x$
23. $x \sqcup (y \sqcup z) \overset{\omega}{=} (x \sqcup y) \sqcup z$
24. $(\xi(x, y) \longrightarrow \Diamond(x \sqcup y) \cap (\Diamond x \sqcup \Diamond y) = \Diamond x \sqcup \Diamond y)$
25. $(\xi(x) \longrightarrow x \sqcap \Diamond x = x, \Diamond\Diamond x = \Diamond x)$
26. $(\xi(x, y) \longrightarrow \Diamond(x \sqcap y) \cap (\Diamond x \sqcap \Diamond y) = \Diamond(x \sqcup y))$
27. $(\xi(x, y) \longrightarrow \mathfrak{L}(x \sqcup y) \cap (\mathfrak{L}x \sqcup \mathfrak{L}y) = \mathfrak{L}x \sqcup \mathfrak{L}y)$
28. $(\xi(x) \longrightarrow \mathfrak{L}x \sqcap \Diamond\mathfrak{L}x = \mathfrak{L}x)$
29. $(\xi(x) \longrightarrow \Diamond x \sqcup \mathfrak{L}\Diamond x = \Diamond x)$
30. $(\xi x \longrightarrow \sim \Diamond x = \mathfrak{L} \sim x)$
31. $(\xi x \longrightarrow \sim \mathfrak{L}x = \Diamond \sim x)$
32. $(x \sqcap y = x \longrightarrow x \sqcup y = y)$

Proof

1. That $\left\langle \underline{\wp(K^*)}, \cup, \cap, {}^c, \bot, \top \right\rangle$ is a Boolean algebra can be proved by part of Stone's representation theorem.

2. For proving $x \vee y \stackrel{\omega^*}{=} y \vee x$, if $x \vee y$ is defined, then $cl_1(x \cup y)$ is open with respect to cl_2. So $cl_1(y \cup x$ is also open with respect to cl_2 and the two sides of the equality must be equal. Similarly for the reversed argument.
3. If $x \vee (y \vee z)$ and $(x \vee y) \vee z$ are defined, then they must equal $cl_1(x \cup cl_1(y \cup z))$ and $cl_1(cl_1(x \cup y)) \cup z)$ respectively. Further these and $cl_1(x \cup y)$, and $cl_1(y \cup z)$ must be open with respect to cl_2. But cl_i are topological closures. So $x \vee (y \vee z) \stackrel{\omega}{=} (x \vee y) \vee z$
4. $(\xi(x) \longrightarrow x \vee x = cl_1(x))$ can be derived directly.
5. If $(x \vee y = z$ then $z = cl_1(x \cup y)$ and it must be open with respect to the second closure system. So, $(x \vee y = z \longrightarrow cl_1 z = z,\ cl_2 z^c = z^c)$
6. If $x \vee x = y$, then $x \wedge x$ will also be equal to $cl_1(x)$. The rest of $(x \vee x = y \longrightarrow cl_2(x^c) = x^c,\ y = cl_1(x) = x \wedge x)$ follows from the previous observation.
7. $cl_i(x) \cap x = x; cl_i cl_i(x) = cl_i(x); i = 1, 2$ follows from definition
8. $(x \cap y = x \longrightarrow cl_i(x) \cap cl_i(y) = cl_i(x)); i = 1, 2$ expresses monotonicity
9. The proof of $x \wedge y \stackrel{\omega^*}{=} y \wedge x$ is similar to that of its dual.
10. The proof of $x \wedge (y \wedge z) \stackrel{\omega}{=} (x \wedge y) \vee z$ is similar but easier than that of its dual.
11. $(x \wedge x = y \longrightarrow cl_2(x^c) = x^c,\ y = cl_1(x))$ follows directly from definition.
12. $(cl_2(cl_1 x)^c) = (cl_1 x)^c \longrightarrow x \wedge x = cl_1(x))$ is also direct.
13. If $x \wedge y = z$, then z must necessarily be closed with respect to cl_1 and open with respect to cl_2. So $(x \wedge y = z \longrightarrow cl_1 z = z,\ cl_2 z^c = z^c)$
14. $(x \wedge y) = a$ (say) is certainly lesser than $cl_1(x)$, so $(x \wedge y) \vee x = cl_1(a \cup x) = cl_1(x)$. This proves $((x \wedge y) \vee x = z \longrightarrow z = cl_1(x))$
15. The argument of the previous conditional implication can be extended to prove $(x \wedge y) \vee x \stackrel{\omega^*}{=} x \wedge (y \vee x)$.
16. $\mathfrak{L}\bot = \bot$, and $\mathfrak{L}1 = 1$ follow from definition.
17. If $(cl_1 x = x,\ cl_2(x^c) = x^c$, then x is essentially in the main quotient structure of interest. So $\mathfrak{L}x$ will be defined and the rest of $(cl_1 x = x,\ cl_2(x^c) = x^c \longrightarrow x \vee \mathfrak{L}x = x,\ \mathfrak{L}\mathfrak{L}x = \mathfrak{L}x)$ follows.
18. If $(x \sqcap y = z$, then x, y are essentially in $C_1 O_2(K^*, C_1, C_2$, but then $x \sqcap y$ must be the infimum of x and y with respect to $\leq$. So z must also be in $C_1 O_2(K^*, C_1, C_2$ and $(x \sqcap y = z \longrightarrow \xi(x, y, z))$.
19. The proof of $(x \sqcup y = z \longrightarrow \xi(x, y, z))$ is similar to that of the above statement.
20. If either $x \sqcap y$ or $y \sqcap x$ is defined, then the other is and the two must be equal to their identification with $i(x) \Cap i(y)$. So $x \sqcap y \stackrel{\omega^*}{=} y \sqcap x$.
21. $x \sqcap (y \sqcap z) \stackrel{\omega}{=} (x \sqcap y) \sqcap z$
22. $x \sqcup y \stackrel{\omega^*}{=} y \sqcup x$ can be proved in the same way as its dual statement (with $\sqcap$).
23. The rest of the proof follows from the first theorem of this section.

□

The above definition of a concrete bitten algebra is well suited for easy abstraction.

Definition 109 An *abstract bitten algebra*, shall be a partial algebra of the form

$$A = \langle \underline{A}, \vee, \wedge, \sqcap, \sqcup, \cup, \cap, {}^c, , cl_1, cl_2, \sim, \mathfrak{L}, \Diamond, \bot, 1, \top \rangle$$

that satisfies all of the following:

1. $\langle \underline{A}, \cup, \cap, {}^c, \bot, \top \rangle$ is a Boolean algebra.
2. $x \vee y \stackrel{\omega}{=} y \vee x$
3. $x \vee (y \vee z) \stackrel{\omega}{=} (x \vee y) \vee z$
4. $(cl_2(cl_1 x)^c) = (cl_1 x)^c \longrightarrow x \vee x = cl_1(x))$
5. $(x \vee y = z \longrightarrow cl_1 z = z, cl_2 z^c = z^c)$
6. $(x \vee x = y \longrightarrow cl_2(x^c) = x^c, y = cl_1(x) = x \wedge x)$
7. $cl_i(x) \cap x = x; cl_i cl_i(x) = cl_i(x); i = 1, 2$
8. $(x \cap y = x \longrightarrow cl_i(x) \cap cl_i(y) = cl_i(x)); i = 1, 2$
9. $x \wedge y \stackrel{\omega}{=} y \wedge x$
10. $x \wedge (y \wedge z) \stackrel{\omega}{=} (x \wedge y) \vee z$
11. $(x \wedge x = y \longrightarrow cl_2(x^c) = x^c, y = cl_1(x))$
12. $(cl_2((cl_1 x)^c) = (cl_1 x)^c \longrightarrow x \wedge x = cl_1(x))$
13. $(x \wedge y = z \longrightarrow cl_1 z = z, cl_2 z^c = z^c)$
14. $((x \wedge y) \vee x = z \longrightarrow z = cl_1(x))$
15. $(x \wedge y) \vee x \stackrel{\omega}{=} x \wedge (y \vee x)$
16. $\mathfrak{L}\bot = \bot; \mathfrak{L}1 = 1$
17. $(cl_1 x = x, cl_2(x^c) = x^c \longrightarrow x \vee \mathfrak{L}x = x, \mathfrak{L}\mathfrak{L}x = \mathfrak{L}x)$
18. $(x \sqcap y = z \longrightarrow \xi(x, y, z))$
19. $(x \sqcup y = z \longrightarrow \xi(x, y, z))$
20. $x \sqcap y \stackrel{\omega}{=} y \sqcap x$
21. $x \sqcap (y \sqcap z) \stackrel{\omega}{=} (x \sqcap y) \sqcap z$
22. $x \sqcup y \stackrel{\omega}{=} y \sqcup x$
23. $x \sqcup (y \sqcup z) \stackrel{\omega}{=} (x \sqcup y) \sqcup z$
24. $(\xi(x, y) \longrightarrow \Diamond(x \sqcup y) \cap (\Diamond x \sqcup \Diamond y) = \Diamond x \sqcup \Diamond y)$
25. $(\xi(x, y) \longrightarrow \Diamond(x \sqcap y) \cap (\Diamond x \sqcap \Diamond y) = \Diamond(x \sqcup y))$
26. $(\xi(x, y) \longrightarrow \mathfrak{L}(x \sqcup y) \cap (\mathfrak{L}x \sqcup \mathfrak{L}y) = \mathfrak{L}x \sqcup \mathfrak{L}y)$
27. $(\xi(x) \longrightarrow \mathfrak{L}x \sqcap \Diamond\mathfrak{L}x = \mathfrak{L}x)$
28. $(\xi(x) \longrightarrow \Diamond x \sqcup \mathfrak{L}\Diamond x = \Diamond x)$
29. $(\xi x \longrightarrow \sim \Diamond x = \mathfrak{L} \sim x)$
30. $(\xi x \longrightarrow \sim \mathfrak{L}x = \Diamond \sim x)$

Definition 110 Let τ be a collection of subsets of K indexed by K, that satisfies

1. $(\forall x \in K)(\exists y \in \tau)\, x \in y$
2. $\bigcup \tau = K$
3. τ is an antichain with respect to inclusion

4. For a not necessarily disjoint partition $\mathcal{P}$ of K, $\{\cup_{x\in A}\{H_x : H_x \in \tau\}\}_{A\in\mathcal{P}}\} = \mathcal{B}$ satisfies:
   - $\mathcal{B}$ is an antichain with respect to the usual inclusion order.
   - If A is a subset of K not included in any element of $\mathcal{B}$, then there exists a two element subset of A with the same property.

   Then τ will be called an *ortho-normal cover* of K

Definition 111 Let τ be a collection of subsets of an algebra

$$K = \langle \underline{K}, f_1, f_2, \ldots f_l \rangle$$

indexed by K, that satisfies

1. $(\forall x \in K)(\exists y \in \tau)\, x \in y$
2. $\bigcup \tau = K$
3. τ is an antichain with respect to inclusion
4. For a not necessarily disjoint partition $\mathcal{P}$ of K, $\{\cup_{x\in A}\{H_x : H_x \in \tau\}\}_{A\in\mathcal{P}}\} = \mathcal{B}$ satisfies:
   - $\mathcal{B}$ is an antichain with respect to the usual inclusion order.
   - If A is a subset of K not included in any element of $\mathcal{B}$, then there exists a two element subset of A with the same property.
   - For any f_i of arity n and elements $B_1, \ldots, B_n \in \mathcal{B}$ there exists an element $B \in \mathcal{B}$ such that $f(B_1, \ldots, B_n) \subseteq B$

   Then τ will be called an *ortho-normal cover* of the algebra K for the tolerance determined by $\mathcal{B}$.

Note that $\mathcal{B}$ is a normal system of subsets of the algebra K and therefore determines a unique compatible tolerance on K (see [27]) and conversely. The same thing happens in case of the first definition for the set K.

Definition 112 Let the set of minimal elements in $\{\Diamond x : \xi x : \mathfrak{L}x \neq 0\}$ of an abstract bitten algebra S be H_0, then let $H = \{x : \Diamond x \in H_0\}$. If H determines an ortho-normal cover on a tolerance space $P = \langle \underline{P}, T \rangle$ with $Card(\underline{P}) = Card(H)$ then S will be said to be a *refined abstract bitten algebra*.

Theorem 113 *For each refined abstract bitten algebra S there exists a tolerance approximation space K, such that $Bite(K) \cong S$.*

Proof The proof has already been done above. The essential steps are:

1. The definition of a refined abstract bitten algebra S, ensures the existence of a related tolerance space P (say).
2. It can be checked that the tolerance space P and the tolerance space K (mentioned in the statement of the theorem) are isomorphic because of the representation theorem for tolerances.
3. Rest of the aspects have already been taken care of.

□

Theorem 113 is hardly constructive in any sense of the term and may prove to be difficult to apply in particular situations.

6.3.2 Discussion

In the above, a semantics for the logic of roughly similar objects is developed and it has a certain relationship with the original tolerance space . However the actual level of relationship that is desired between *such a semantics*, and an *original generalized approximation space along with the associated process* is still the subject to some judgment. This is definitely independent of logic-forming strategies like the Gentzen style algebraic [35, 36] or abstract algebraic approach that can be applied.

Any bitten semantics with no restrictions on the type of granules can be expected to fall short of a unique representation theorem (in the sense that given the semantics, a specification for obtaining the original tolerance space in a unique way is possible). In general, the process of forming approximations actually obscures the distribution of blocks and the latter is essential for a unique representation theorem because of [27].

When the set of granules used is the set of T-related elements, the required conditions for a unique representation theorem will necessarily include a constructive instance of the following process of formation of blocks from sets of T-related elements.

- Let $\mathcal{B}$ denote the set of all blocks of the tolerance space $S = \langle \underline{S}, T \rangle$ and let $\tau = \{[x]_T : x \in, \underline{S}\}$.
- Form the power set $\wp(\tau)$
- Let $\mu(\tau) = \{\cup(K) : K \in \wp(\tau), T_{|\cup(K)}$ is an equivalence$\}$. $T_{|\cup(K)}$ being the restriction of the tolerance to the set $\cup(K)$.
- $\mu(\tau)$ is partially ordered by the inclusion relation.
- $\mu_{max}(\tau)$, the set of maximal elements of $\mu(\tau)$, is the set of blocks of S. That is, $\mu_{max}(\tau) = \mathcal{B}$.

6.3.3 Illustrative Example

Let $S = \{x_1, x_2, x_3, x_4\}$ and let T be a tolerance T on it generated by

$$\{(x_1, x_2),\ (x_2, x_3)\}.$$

Denoting the statement that the granule generated by x_1 is $(x_1,\ x_2)$ by $(x_1 : x_2)$, let the granules be the set of predecessor neighborhoods:

$$Gr(S) = \{(x_1 : x_2),\ (x_2 : x_1, x_3),\ (x_3 : x_2),\ (x_4 :)\}$$

The different approximations are then as in Table 5.

Table 5 Example: bited approximation

$\wp(\mathbf{S})\| \sim$	Subset	**X**	$\mathbf{Gr}_*(\mathbf{X})$	$\mathbf{Gr}^*(\mathbf{X})$	$\mathbf{Gr}_*(\mathbf{X^c})$	$\mathbf{Gr}^*_\mathbf{b}(\mathbf{X})$
B_1	A_1	x_1	$\emptyset$	$x_2,\ x_1$	x_2, x_3, x_4	x_1
B_2	A_2	x_2	$\emptyset$	x_1, x_2, x_3	x_4	x_1, x_2, x_3
B_3	A_3	x_3	$\emptyset$	x_1, x_2, x_3	x_1, x_2, x_4	x_3
B_4	A_4	x_4	x_4	x_4	x_1, x_2, x_3	x_4
B_5	A_5	x_1, x_2	x_1, x_2	x_1, x_2, x_3	x_4	x_1, x_2, x_3
B_2	A_6	x_1, x_3	$\emptyset$	x_1, x_2, x_3	x_4	x_1, x_2, x_3
B_6	A_7	x_1, x_4	x_4	S	x_2, x_3	x_1, x_4
B_7	A_8	x_2, x_3	x_2, x_3	x_1, x_2, x_3	x_4	x_1, x_2, x_3
B_8	A_9	x_2, x_4	x_4	S	$\emptyset$	S
B_9	A_{10}	x_3, x_4	x_4	S	x_1, x_2	x_3, x_4
B_{10}	A_{11}	x_1, x_2, x_3	x_1, x_2, x_3	x_1, x_2, x_3	x_4	x_1, x_2, x_3
B_{11}	A_{12}	x_1, x_2, x_4	x_1, x_2, x_4	S	$\emptyset$	S
B_{12}	A_{13}	x_2, x_3, x_4	x_2, x_3, x_4	S	$\emptyset$	S
B_8	A_{14}	x_1, x_3, x_4	x_4	S	$\emptyset$	S
B_{14}	A_{15}	S	S	S	$\emptyset$	S
B_{13}	A_{16}	$\emptyset$	$\emptyset$	$\emptyset$	S	$\emptyset$

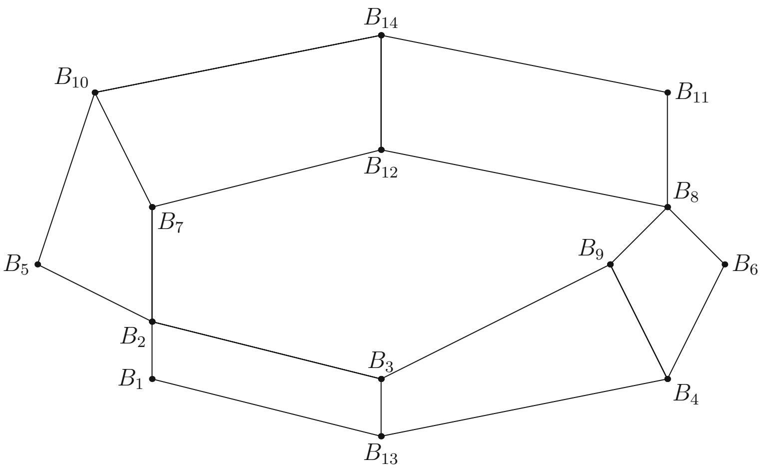

Fig. 4 Partial order on the quotient partial algebra

The first column in the table is for keeping track of the elements in the quotient $\wp(S)| \sim$ and can be used for checking the operations of Proposition 98. The order structure is given by the Hasse diagram (Fig. 4) following the table. More details of the construction are omitted because the next step will take some space. The braces on sets have been omitted.

6.4 Connections with AUAI Approximation Systems

In a $AUAI$ approximation system $\langle S, \mathcal{K}\rangle$ considered in Sect. 10, the collection $\mathcal{K}$ need not be the most appropriate concept of granule for the four different approximations of the theory. The implicit conditions on the possible concepts of a granule in the bitten approach are the following:

- The set of granules $\mathcal{S}$ is a cover of S, that is $\bigcup \mathcal{S} = S$.
- The form of the lower and bitten upper approximation are given as in the subsection on Bitten Approach.

Theorem 114 *Given a tolerance approximation space $\langle S, T\rangle$ with granulation $\mathcal{B}$ and approximations Gr_*, Gr_b^*, the $AUAI$ approximation system $\langle S, \mathcal{B}\rangle$ satisfies*

1. $(\forall X \in \wp(S))\, X^{l1} = X^l$
2. $(\forall X \in \wp(S))\, X^{u1} \subseteq X^u$
3. $(\forall X \in \wp(S))\, X^{ub} = X^u \cap X^{u2}$.

Proof

1. $X^{l1} = \bigcup\{A : A \subseteq X\,;\, A \in \mathcal{B}\} = X^l$
2. X^{u1} is the intersection of all unions of elements of $\mathcal{B}$, while X^u is the union of all elements of $\mathcal{B}$ that have non empty intersection with X. In general if $\mathcal{B}$ is a collection of pairwise disjoint sets then $X^{u1} = X^u$, else $X^{u1} \subseteq X^u$.
3. $X^{ub} = X^u \setminus Gr_*(X^c) = X^u \cap (Gr_*(X^c))^c$. But $\bigcap\{A_i^c : X \subseteq A_i^c\,;\, A_i \in \mathcal{B}\} = \bigcap\{A_i^c : X^c \subseteq A_i \in \mathcal{B}\} = (\bigcup\{A_i : X^c \subseteq A_i\})^c = (Gr_*(X^c))^c = X^{u2}$. So $X^{ub} = X^u \cap X^{u2}$ holds.

□

In the above theorem, the collection $\mathcal{K}$ of the $AUAI$ approximation system $\langle S, \mathcal{K}\rangle$ has been taken to coincide with $\mathcal{B}$. This need not be the case in general and many variations are possible on the point. In particular, the collection $\mathcal{K}$ can be selected subject to X^{ub} coinciding with X^{u2}.

Problems

Some important problems that originate from the previous sections are:

1. Under what general conditions will the operation of biting make the partial operation in Proposition 98 total?
2. Find simpler conditions under which an abstract bitten algebra becomes a refined abstract bitten algebra.
3. Describe the quasi equational classes (and variants) of refined abstract bitten algebras
4. Does a complete, atomic double Heyting algebra determine a unique BAS?
5. Which type of can reducts be computed with the help of these algebras?

6.5 Properties of Granules

In tolerance space of the form $\langle S, T\rangle$, all of the following types of granules with corresponding approximations have been used in the literature:

1. The collection of all subsets of the form $[x] = \{y : (x, y) \in T\}$ will be denoted by $\mathcal{T}$.
2. The collection of all blocks, the maximal subsets of S contained in T, will be denoted by $\mathcal{B}$. Blocks have been used as granules in [16, 83, 87, 160] and others.
3. The collection of all finite intersections of blocks will be denoted by $\mathcal{A}$.
4. The collection of all intersections of blocks will be denoted by $\mathcal{A}_\sigma$ [160].
5. Arbitrary collections of granules with choice functions operating on them [89].
6. The collection of all sets formed by intersection of sets in $\mathcal{T}$ will be denoted by $\mathcal{TT}$.

For convenience $H_0 = \emptyset$, $H_{n+1} = S$ will be assumed whenever the collection of granules $\mathcal{G}$ is finite and $\mathcal{G} = \{H_1, \ldots H_n\}$.

In a tolerance space $\langle S, T\rangle$, for a given collection of granules $\mathcal{G}$ definable approximations of a set $A \subseteq S$ include:

(i) $A^{l\mathcal{G}} = \bigcup\{H : H \subseteq A, H \in \mathcal{G}\}$,
(ii) $A^{u\mathcal{G}} = \bigcup\{H : H \cap A \neq \emptyset, H \in \mathcal{G}\}$,
(iii) $A^{l2\mathcal{G}} = \bigcup\{\cap_{i\in I} H_i^c : \cap_{i\in I} H_i^c \subseteq A, H \in \mathcal{G}\ I \subseteq \mathbf{N}(n+1)\}$,
(iv) $A^{u1\mathcal{G}} = \bigcap\{\cup_{i\in I} H_i : A \subseteq \cup_{i\in I} H_i, I \subseteq \mathbf{N}(n+1)\}$,
(v) $A^{u2\mathcal{G}} = \bigcap\{H_i^c : A \subseteq H_i^c, I \in \{0, 1, \ldots, n\}\}$.

But not all approximations fit it into these schemas in an obvious way. These include:

(i) $A^{l+} = \{y : \exists x (x, y) \in T, [x] \subseteq A\}$ [16],
(ii) $A^{u+} = \{x ; (\forall y)((x, y) \in T \longrightarrow [y]_T \cap A \neq \emptyset)\}$,
(iii) Generalized bitten upper approximation : $A^{ubg} = A^{ug} \setminus A^{clg}$—this is a direct generalization of the bitten approximation in [86, 157].

Theorem 115 *In the tolerance space context, with $\mathcal{T}$ being the set of granules and restricting to the approximations $l\mathcal{T}$, $u\mathcal{T}$, all of RA, MER, ST and weakenings thereof hold, but others do not hold in general.*

Proof RA follows from definition. For MER, if $A \subseteq [x]$ and $A^{l\mathcal{T}} = A^{u\mathcal{T}} = A$, then as $[x] \cap A \neq \emptyset$, so $[x] \subseteq A^{u\mathcal{T}} = A$. So $A = [x]$. Crispness fails because it is possible that $[x] \cap [y] \neq \emptyset$ for distinct x, y. □

Theorem 116 *If $\langle \underline{S}, T\rangle$ is a tolerance approximation space with granules $\mathcal{T}$ and the approximations $l\mathcal{T}$, $l+$, $u\mathcal{T}$, $u+$, then RA, NO, ACG do not hold, but SRA, SMER, SST, IMER, MER, US holds*

Proof RA does not hold due to $l+$, $u+$, ACG fails by the previous theorem. 'Sub' properties have been verified in the previous theorem, while the rest can be checked directly. □

Theorem 117 *In Bitten rough sets, (taking $\mathcal{G}$ to be the set of T-relateds and restricting to the lower, upper and bitten upper approximations alone),* SRA *does not hold for the bitten upper approximation if '$+$, $\cdot$' are interpreted as set union and intersection respectively.* MER, NO *do not hold in general, but* IMER, SCG, LS, LU, SRA *hold.*

7 Proto-Transitive Rough Sets

One of the reasons for including this long section is in the novel techniques used in getting to semantics and the abortive strategies invented for understanding rough objects in the context.

In forming semantics and logics of relation based rough sets, the absence of transitivity is known to be particularly problematic. Semantics of generalizations of transitivity are motivated by these and many other practical considerations. In [93, 95, 99], research into rough sets over generalized transitive relations like proto-transitive ones was initiated by the present author. In [95], approximation of proto-transitive relations by other relations was investigated and the relation with rough approximations was developed towards constructing semantics that can handle *fragments of structure*. It was also proved that difference of approximations induced by some approximate relations need not induce rough structures. Subsequently the characterization of rough objects has been improved and a theory of dependence for general rough sets developed in [97] by the present author. In [99] these have been used to internalize Nelson-algebra based approximate semantics of [95] and many other results have been proven by her. Most of the results have also been included in the present author's monograph [96] on proto transitive rough sets. This section includes an outline of these developments also because many new techniques are introduced in the study.

Proto-transitivity is one of the infinite number of possible generalizations of transitivity. These types of generalized relations happen often in application contexts. Failure to recognize them causes mathematical models to be inadequate or underspecified and tends to unduly complicate algorithms and approximate methods. From among the many possible alternatives that fall under *generalized transitivity*, *proto-transitivity* has been chosen because of application contexts, its simple set theoretic definition, connections with factor relations and consequent generative value among such relations. It has a special role in modeling knowledge as well.

Rough objects in a PRAX need not correspond to intervals of the form $]a, b[$ with the definite object b covering (in the ordered set of definite objects) the definite object a.

If R is a relation on a set S, then R can be approximated by a wide variety of partial/quasi-order relations in both classical and rough set perspective [59]. Though the methods are essentially equivalent for binary relations, the latter method is more general. When the relation R satisfies proto-transitivity, then many new properties emerge. This aspect is developed further in [95] and most of it is included.

When R is a quasi-order relation, then a semantics for the set of ordered pairs of lower and upper approximations $\{(A^l, A^u);\ A \subseteq S\}$ has recently been developed in [61, 63]. Though such a set of ordered pairs of lower and upper approximations are not rough objects in the PRAX context, the approximations can be used for an additional semantic approach to it. It is also shown that differences of consequent lower and upper approximations suggest partial structures for *measuring* structured deviation. It can be argued that the correspondence between point-wise and granular approximations of an entirely different kind is not justified. This is correct and in fact partial justifications are alone possible. The developed method should also be useful for studying correspondences between the different semantics [90, 92]. Because of this some space is devoted to the nature of transformation of granules by the relational approximation process.

A part of the investigations of the present author on the nature of possible concepts of *rough dependence* is also presented. Though the concept of independence is well studied in probability theory, the concept of dependence is rarely explored in any useful way. It has been shown to be very powerful in classical probability theory [39]—the formalism is valid over probability spaces, but its axiomatic potential is left unexplored. Connections between rough sets and probability theory have been explored from rough measure and information entropy viewpoint in a number of papers [8, 55, 128, 156, 168]. The nature of rough independence is also explored in [97] by the present author. Apart from problems relating to contamination, it is shown that the comparison by way of corresponding concepts of dependence fails in a very essential way.

Further, using the introduced concepts of rough dependence the approximate semantics is internalized instead of maintaining explicit dependences on correspondences. This allows for richer variants of the earlier semantics of rough objects.

This section is organized as follows: in the rest of this section the basics of proto-transitivity are introduced. In the following subsection, relevant approximations in PRAX are defined and their basic properties and those of definite elements are investigated. In the third subsection, abstract examples and three other extended examples are provided for justifying the study. In the following subsection, the algebraic structures that can be associated with the semantic properties of definite objects in a PRAX are characterized. The representation of rough objects is done from an interesting perspective in the following subsection. In the fifth subsection, atoms in the partially ordered set of rough object are described. This is followed by an algebraic semantics that relies on multiple types of aggregation and commonality operations.

7.1 Basic Concepts, Terminology

Definition 118 A binary relation R on a set S is said to be *weakly-transitive, transitive or proto-transitive* respectively on S if and only if S satisfies

- If whenever $Rxy,\ Ryz$ and $x \neq y \neq z$ holds, then Rxz. (i.e. $(R \circ R) \setminus \Delta_S \subseteq R$ (where $\circ$ is relation composition) , or
- whenever Rxy & Ryz holds then Rxz (i.e. $(R \circ R) \subseteq R$), or
- Whenever $Rxy,\ Ryz,\ Ryx,\ Rzy$ and $x \neq y \neq z$ holds, then Rxz follows, respectively. Proto-transitivity of R is equivalent to $R \cap R^{-1} = \tau(R)$ being weakly transitive.

The following simpler example will be used to illustrate many of the concepts and situations associated with in this section. For detailed motivations see Subsec.7.3 on motivation and examples.

Example 119 A simple real-life example of a proto-transitive, non transitive relation would be the relation $\mathbb{P}$, defined by $\mathbb{P}xy$ if and only if x thinks that y thinks that color of object O is a maroon.

The following simple example from databases will also be used as a persistent one (especially in the sections on approximation of relations) for illustrating a number of concepts. It has other attributes apart from the main one for illustrating more involved aspects.

Let $\mathfrak{J}$ be survey data in table form with column names being for Personal Data, Gender, Sexual Orientation, Other Personal Data, Responses to Statements and Opinions on sexist contexts expressed on a scale from -10 to $+10$ respectively with each row corresponding to a person. Let the assertions be

- Most men are blind towards patriarchal dynamics
- All religions are anti women
- The percentage of women in STEM research is low because of systemic discrimination
- Objectification of women leads to crimes against women
- 'Not All Men' arguments miss the point

Then an abridged version of the data can be represented as in Table 6

Let Rab if and only if person a agrees with b's opinions.

The predicate agrees with can be constructed empirically or from the data by a suitable heuristic like

- Form the intervals $\{[-10, -5], (-5, 0),\ [0, 1),\ [1, 8),\ [8, 10]\}$
- Let a agree with b in the following scenarios:
  - $b's$ scores are greater than a's for question-1 when score of a is in $[1, 8)$
  - $b's$ scores are greater than or equal a's for question-1 when score of a is in $[8, 10] \cup [0, 1]$

Table 6 Survey data

Id	Gender	Cis?	PD	Response-1	Response-2
1	Woman	Y	Data-1	10	10
2	Woman	Y	Data-2	5	−2
3	Woman	N	Data-3	10	10
4	Man	Y	Data-4	−7	−5
5	Man	Y	Data-5	0	0
6	Bigender	N	Data-6	7	8
7	Woman	Y	Data-7	5	4
8	Woman	Y	Data-8	4	6
9	Woman	Y	Data-9	10	9

  - $b's$ scores are less than or equal to a's for question-1 when score of a is in $[-10, -5] \cup (-5, 0)$.

- Response-2 can also be taken account to generate a more complex predicate.

R is proto-transitive. In general R is a proto-transitive, reflexive relation and this condition can be imposed to complete partial data as well (as a rationality condition). If a agrees with the opinions of b, then it will be said that a is an *ally* of b—if b is also an ally of a, then they are *comrades*. Finding optimal subsets of allies can be an interesting problem in many contexts especially given the fact that responses may have some vagueness in them.

Let $\alpha \subseteq \rho$ be two binary relations on S, then $\rho|\alpha$ will be the relation on $S|\rho$ defined via $(x, y) \in \rho|\alpha$ if and only if $(\exists b \in x, c \in y)(b, c) \in \rho$. The relation $Q|\tau(Q)$ for a relation Q will be denoted by $\sigma(Q)$.

The following are known:

Proposition 120

- *If Q is a quasi-order on S, then $Q|\tau(Q)$ is a partial order on $S|\tau(Q)$.*
- *If $R \in Ref(S)$, then $R \in p\tau(S)$ if and only if $\tau(R) \in EQ(S)$.*
- *In general,*

$$w\tau(S) \subseteq s\tau(S) \subseteq p\tau(S).$$

Proposition 121 *If $R \in p\tau(S) \cap Ref(S)$, then the following are equivalent:*

A1 $([a], [b]) \in R|\tau(R)$ *if and only if* $(a, b) \in R$.
A2 *R is semi-transitive.*

In [26], it is proved that

Theorem 122 *If $R \in Ref(S)$, then the following are equivalent:*

A3 *$R|\tau(R)$ is a pseudo order on $S|\tau(R)$ and A1 holds.*
A2 *R is semi-transitive.*

Note that *Weak transitivity* of [26] is *proto-transitivity* here. Successor, predecessor and related granules generated by elements alone will be considered and the precision based paradigm will be avoided.

7.2 The Approximate and Definite in PRAX

Definition 123 By a *Proto Approximation Space* S (PRAS for short), will be meant a pair of the form $\langle \underline{S}, R\rangle$ with $\underline{S}$ being a set and R being a proto-transitive relation on it. If R is also reflexive, then it will be called a *Reflexive Proto Approximation Space* (PRAX) for short). $\underline{S}$ may be infinite.

If S is a PRAX or a PRAS, then the *successor neighborhoods*, *inverted successor or predecessor neighborhoods* and *symmetrized successor neighborhoods* generated by an element $x \in S$ will respectively be denoted as in the following:

$$[x] = \{y;\ Ryx\}.$$

$$[x]_i = \{y;\ Rxy\}.$$

$$[x]_o = \{y;\ Ryx \,\&\, Rxy\}.$$

Taking these as granules, the associated granulations will be denoted by $\mathcal{G} = \{[x] : x \in S\}$, $\mathcal{G}_i$ and $\mathcal{G}_o$ respectively. In all that follows in this chapter S will be a PRAX unless indicated otherwise.

Definition 124 Definable approximations on S include ($A \subseteq S$):

$$A^u = \bigcup_{[x]\cap A\neq\emptyset} [x]. \quad \text{(Upper Proto)}$$

$$A^l = \bigcup_{[x]\subseteq A} [x]. \quad \text{(Lower Proto)}$$

$$A^{u_o} = \bigcup_{[x]_o\cap A\neq\emptyset} [x]_o. \quad \text{(Symmetrized Upper Proto)}$$

$$A^{l_o} = \bigcup_{[x]_o\subseteq A} [x]_o. \quad \text{(Symmetrized Lower Proto)}$$

$$A^{u+} = \{x : [x]\cap A \neq \emptyset\}. \quad \text{(Point-wise Upper)}$$

$$A^{l+} = \{x : [x] \subseteq A\}. \quad \text{(Point-wise Lower)}$$

Example 125 In the context of the example 119, $[x]$ is the *set of allies* x, while $[x]_o$ is the set of comrades of x. A^l is the *union of the set of all allies of at least one of the* *members of* A if they are all in A. A^u is the union of the set of all allies of persons having at least one ally in A. A^{l+} is the set of all those persons in A all of whose allies are within A. A^{u+} is the set of all those persons having allies in A.

Definition 126 If $A \subseteq S$ is an arbitrary subset of a PRAX or a PRAS S, then

$$A^{ux} = \bigcup_{[x]_o \cap A \neq \emptyset} [x].$$

$$A^{lx} = \bigcup_{[x]_o \subseteq A} [x].$$

$$A^{u*} = \bigcup \{[x] : [x] \cap A \neq \emptyset \& (\exists y \neq x)([x], [y]) \in \sigma(R),\ (x, y) \in R,\ [y] \subseteq A\}.$$

$$A^{l*} = \bigcup \{[x] : [x] \subseteq A\, \& (\exists y \neq x)(([x], [y]) \in \sigma(R), \& [y] \subseteq A)\}.$$

Proposition 127 *In a PRAX S and for a subset $A \subseteq S$, all of the following hold:*

- $(\forall x)\, [x]_o \subseteq [x]$
- *It is possible that $A^l \neq A^{l+}$ and in general, $A^l \parallel A^{lo}$ (that is they are not comparable set-theoretically).*

Proof The proof of the first two parts are easy. For the third part, the argument is chased up to a trivial counter example (see the following section).

$$\bigcup_{[x] \subseteq A} [x] \subseteq \bigcup_{[x]_o \subseteq A} [x] \supseteq \bigcup_{[x]_o \subseteq A} [x]_o$$

$$\bigcup_{[x]_o \subseteq A} [x]_o \supseteq \bigcup_{[x] \subseteq A} [x]_o \subseteq \bigcup_{[x] \subseteq A} [x].$$

□

Proposition 128 *For any subset A of S,*

$$A^{u_o} \subseteq A^u.$$

Proof Since $[x]_o \cap A \neq \emptyset$, therefore

$$A^{u_o} = \bigcup_{[x]_o \cap A \neq \emptyset} [x]_o \subseteq \bigcup_{[x] \cap A \neq \emptyset} [x]_o \subseteq A^{u_o} = \bigcup_{[x] \cap A \neq \emptyset} [x] = A^u.$$

□

Definition 129 If X is an approximation operator, then by a X-*definite element*, will be meant a subset A satisfying $A^X = A$. The set of all X-definite elements will be denoted by $\delta_X(S)$, while the set of X and Y-definite elements (Y being another approximation operator) will be denoted by $\delta_{XY}(S)$. In particular, we will speak of *lower proto-definite*, *upper proto definite* and *proto-definite* elements (those that are both lower and upper proto-definite).

Theorem 130 *In a PRAX S, the following hold:*

- $\delta_u(S) \subseteq \delta_{u_o}(S)$*, but* $\delta_{l_o}(S) = \delta_{u_o}(S)$ *and* $\delta_u(S)$ *is a complete sublattice of* $\wp(S)$ *with respect to inclusion.*
- $\delta_l(S) \parallel \delta_{l_o}(S)$ *in general. (*$\parallel$ *means* is not comparable.*)*
- *It is possible that* $\delta_u \not\subseteq \delta_{u_o}$.

Proof

- As R is reflexive, if A, B are upper proto definite, then $A \cup B$ and $A \cap B$ are both upper proto definite. So $\delta_u(S)$ is a complete sublattice of $\wp(S)$.
- If $A \in \delta_u$, then $(\forall x \in A)[x] \subseteq A$ and $(\forall x \in A^c)[x] \cap A = \emptyset$.
- So $(\forall x \in A^c)\,[x]_o \cap A = \emptyset$. But as $A \subseteq A^{u_o}$ is necessary, it follows that $A \in \delta_{u_o}$.

□

A^{u+}, A^{l+} have relatively been more commonly used in the literature and have also been the only kind of approximation studied in [60] for example (the inverse relation is also considered from the same perspective).

Definition 131 A subset $B \subseteq A^{l+}$ will be said to be *skeleton* of A if and only if

$$\bigcup_{x \in B} [x] = A^l,$$

and the set skeletons of A will be denoted by $\mathbf{sk}(A)$.

The skeleton of a set A is important because it relates all three classes of approximations.

Theorem 132 *In the context of the above definition, all of the following hold:*

- $\mathbf{sk}(A)$ *is partially ordered by inclusion with greatest element* A^{l+}.
- $\mathbf{sk}(A)$ *has a set of minimal elements* $\mathbf{sk}_m(S)$.
- $\mathbf{sk}(A) = \mathbf{sk}(A^l)$
- $\mathbf{sk}(A) = \mathbf{sk}(B) \leftrightarrow A^l = B^l \;\&\; A^{l+} = B^{l+}$.
- *If* $B \in \mathbf{sk}(A)$*, then* $A^l \subseteq B^u$.
- *If* $\cap \mathbf{sk}(A) = B$*, then* $A^{l_o} \cap \bigcup_{x \in B}[x] = \emptyset$.

Proof Much of the proof is implicit in other results proved earlier in this section.

- If $x \in A^l \setminus A^{l+}$, then $[x] \not\subseteq A^l$ and many subsets B of A^{l+} are in $\mathbf{sk}(A)$. If $B \subset K \subset A^{l+}$ and $B \in \mathbf{sk}(A)$, then $K \in \mathbf{sk}(A)$. Further minimal elements in the inclusion order (even if A is infinite) are induced by the inclusion of $\wp(S)$.
- has been proved above.
- More generally, if $A^l \subseteq B \subseteq A$, then $B^l = A^l$. So $\mathbf{sk}(A) = \mathbf{sk}(A^l)$.
- Follows from definition.
- If $B \in \mathbf{sk}(A)$, then $A^l = B^l \subseteq B^u$.

□

Theorem 133 *All of the following hold in PRAX or a PRAS S (∂ being the boundary operator):*

$$(\forall A \subseteq S)\, A^{cl+} = A^{u+c},\ A^{cu+} = A^{l+c} \quad \text{(Mutual Duality)}$$

$$u+ \textit{ is a monotone } \vee -\textit{morphism.} \quad (\vee\text{-Morphism})$$

$$l+ \textit{ is a monotone } \wedge -\textit{morphism.} \quad (\wedge\text{-Morphism})$$

$$\partial(A) = \partial(A^c) \quad \text{(Boundary operator)}$$

$$\Im(u+)\ \&\ \Im(l+) \textit{ are dually isomorphic lattices.} \quad \text{(Lattices)}$$

Further, $\Im(u+)$ is an interior system while $\Im(l+)$ is a closure system.

Proof This proof is known in the context of point-wise approximations

- For proving the mutual duality of the point-wise approximation operators, note that

$$x \in A^{cl+} \Rightarrow x \in A^c \,\&\, [x] \subseteq A^c$$

$$\text{So } A \subseteq [x]^c$$

$$\text{and } A \subseteq [x]^c \Rightarrow A^{u+}$$

$$\text{Therefore } [x]^{cu+c} \subseteq A^{u+c}.$$

$$\text{Again } z \in A^{u+c} \Rightarrow [z] \subseteq A^c$$

$$\text{This yields } z \in A^{cl+}$$

$$\text{and } A^{u+c} \subseteq A^{cl+}$$

- $\vee$-Morphism:
  - Suppose $x \in (A \cup B)^{u+}$, if and only if $[x] \cap (A \cup B) \neq \emptyset$
  - if and only if $[x] \cap A \neq \emptyset$ or $[x] \cap B \neq \emptyset$
  - if and only if $x \in A^{u+}$ or $x \in B^{u+}$,
  - So $(A \cup B)^{u+} = A^{u+} \cup B^{u+}$.
  - $u+$ is therefore a $\vee$-morphism.
  - The proof extends to arbitrary intersections and so the $\vee$-morphism is complete.

- $\wedge$-Morphism: The proof is similar to the above.
- Boundary: $\partial(A) = A^{u+} \setminus A^{l+} = A^{ccu+} \setminus A^{ccl+} = \partial(A^c)$, by the first part of this theorem.
- $\Im(u+)$ is an interior system and $\Im(l+)$ is a closure system because the operators are monotone increasing and $u+$ and $l+$ are $\vee$ and $\wedge$-morphisms respectively. If φ is a map defined by $\varphi(A^{u+}) = A^{cl+}$, then it is an isomorphism.

□

Theorem 134 *In a PRAX,* $(\forall A \in \wp(S))\, A^{l+} \subseteq A^{l},\ \ A^{u+} \subseteq A^{u}$ *and all of the following hold.*

$$(\forall A \in \wp(S))\, A^{ll} = A^{l} \,\&\, A^{u} \subseteq A^{uu}. \tag{Bi}$$

$$(\forall A, B \in \wp(S))\, A^{l} \cup B^{l} \subseteq (A \cup B)^{l}. \tag{l-Cup}$$

$$(\forall A, B \in \wp(S))\, (A \cap B)^{l} \subseteq A^{l} \cap B^{l}. \tag{l-Cap}$$

$$(\forall A, B \in \wp(S))\, (A \cup B)^{u} = A^{u} \cup B^{u}. \tag{u-Cup}$$

$$(\forall A, B \in \wp(S))\, (A \cap B)^{u} \subseteq A^{u} \cap B^{u}. \tag{u-Cap}$$

$$(\forall A \in \wp(S))\, A^{lc} \subseteq A^{cu}. \tag{Dual}$$

Proof

l-Cup For any $A, B \in \wp S$, $x \in (A \cup B)^{l}$

$$\begin{aligned}
&\Leftrightarrow (\exists y \in (A \cup B))\, x \in [y] \subseteq A \cup B.\\
&\Leftrightarrow (\exists y \in A)\, x \in [y] \subseteq A \cup B \text{ or } (\exists y \in B)\, x \in [y] \subseteq A \cup B.\\
&\Leftrightarrow (\exists y \in A)\, x \in [y] \subseteq A \text{ or } (\exists y \in A)\, x \in [y] \subseteq B\\
&\text{or } (\exists y \in B)\, x \in [y] \subseteq A\\
&\text{or} (\exists y \in B)\, x \in [y] \subseteq B\\
&-\text{this is implied by } x \in A^{l} \cup B^{l}.
\end{aligned}$$

l-Cap For any $A, B \in \wp S$, $x \in (A \cap B)^{l}$

$$\begin{aligned}
&\Leftrightarrow x \in A \cap B\\
&\Leftrightarrow (\exists y \in A \cap B)\, x \in [y] \subseteq A \cap B \,\&\, x \in A,\ x \in B\\
&\Leftrightarrow (\exists y \in A)\, x \in [y] \subseteq A \,\&\, (\exists y \in B)\, x \in [y] \subseteq B.
\end{aligned}$$

– Clearly the last statement implies $x \in A^{l} \& x \in B^{l}$, but the converse is not true in general.

u-Cup $x \in (A \cup B)^{u}$

$$\begin{aligned}
&\Leftrightarrow x \in \bigcup_{[y] \cap (A \cup B) \neq \emptyset} [y]\\
&\Leftrightarrow x \in \bigcup_{([y] \cap A) \cup ([y] \cap B) \neq \emptyset}
\end{aligned}$$

$$\Leftrightarrow \; x \in \bigcup_{[y]\cap A \neq \emptyset} [y] \text{ or } x \in \bigcup_{[y]\cap B \neq \emptyset} [y]$$

$$\Leftrightarrow \; x \in A^u \cup B^u.$$

u-Cap By monotonicity, $(A \cap B) \subseteq A^u$ and $(A \cap B) \subseteq B^u$, so $(A \cap B)^u \subseteq A^u \cap B^u$.

Dual If $z \in A^{lc}$, then $z \in [x]^c$ for all $[x] \subseteq A$ and either, $z \in A \setminus A^l$ or $z \in A^c$. If $z \in A^c$ then $z \in A^{cu}$. If $z \in A \setminus A^l$ and $z \neq A^{cu \setminus A^c}$ then $[z] \cap A^c = \emptyset$. But this contradicts $z \notin A^{cu} \setminus A^c$. So $(\forall A \in \wp(S))\, A^{lc} \subseteq A^{cu}$.

□

Theorem 135 *In a PRAX S, all of the following hold:*

$$(\forall A, B \in \wp(S))\, (A \cap B)^{l+} = A^{l+} \cap B^{l+}. \tag{19}$$

$$(\forall A, B \in \wp(S))\, A^{l+} \cup B^{l+} \subseteq (A \cup B)^{l+}. \tag{20}$$

$$(\forall A \in \wp(S))\, (A^{l+})^c = (A^c)^{u+} \;\&\; A^{l+} \subseteq A^{l_o}. \tag{21}$$

Proof

1. $x \in (A \cap B)^{l+}$

$$\Leftrightarrow \; [x] \subseteq A \cap B$$

$$\Leftrightarrow \; [x] \subseteq A \;\&\; [x] \subseteq B$$

$$\Leftrightarrow \; x \in A^{l+} \;\&\; x \in B^{l+}.$$

2. $x \in A^{l+} \cup B^{l+}$

$$\Leftrightarrow [x] \subseteq A^{l+} \text{ or } [x] \subseteq B^{l+}$$

$$\Leftrightarrow [x] \subseteq A \text{ or } [x] \subseteq B$$

$\Rightarrow [x] \subseteq A \cup B \;\Leftrightarrow\; x \in (A \cup B)^{l+}$.

3. $z \in A^{l+c}$

$$\Leftrightarrow \; z \notin A^{l+}$$

$$\Leftrightarrow \; [z] \nsubseteq A$$

$$\Leftrightarrow \; z \cap A^c \neq \emptyset.$$

Theorem 136

$$\textit{In a } \mathsf{PRAX}\ S,\ (\forall A \subseteq S)\, A^{l+} \subseteq A^l, \;\; A^{u+} \subseteq A^u.$$

Proof

- If $x \in A^{l+}$, then $[x] \subseteq A$ and so $[x] \subseteq A^l,\ x \in A^l$.
- If $x \in A^l$, then $(\exists y \in A)[y] \subseteq A,\ Rxy$. But it is possible that $[x] \nsubseteq A$, therefore it is possible that $x \notin A^{l+}$ and $A^l \nsubseteq A^{l+}$.
- If $x \in A^{u+}$, then $[x] \cap A \neq \emptyset$, so $x \in A^u$.
- So $A^{u+} \subseteq A^u$.
- Note that $x \in A^u$, if and only if $(\exists z \in S)\, x \in [z],\ [z] \cap A \neq \emptyset$, but this does not imply $x \in A^{u+}$.

□

Theorem 137 *In a PRAX S, all of the following hold:*

$$(\forall A \in \wp(S))\ A^{l+} \subseteq A^{l_o}. \tag{22}$$

$$(\forall A \in \wp(S))\ A^{u_o} \subseteq A^{u+}. \tag{23}$$

$$(\forall A \in \wp(S))\ A^{lc} \subseteq A^{cu}. \tag{24}$$

Proof

1. - If $x \in A^{l+}$, then $[x] \subseteq A$.
   - But as $[x]_o \subseteq [x]$, $A^{l+} \subseteq A^{l_o}$.
2. This follows easily from definitions.
3. - If $z \in A^{lc}$, then $z \in [x]^c$ for all $[x] \subseteq A$ and either, $z \in A \setminus A^l$ or $z \in A^c$.
   - If $z \in A^c$ then $z \in A^{cu}$.
   - If $z \in A \setminus A^l$ and $z \neq A^{cu \setminus A^c}$ then $[z] \cap A^c = \emptyset$.
   - But this contradicts $z \notin A^{cu} \setminus A^c$.
   - So $(\forall A \in \wp(S))\ A^{lc} \subseteq A^{cu}$.

□

From the above, the following relation (Fig. 5) between approximations in general can be deduced ($A^{u+} \longrightarrow A^u$ should be read as A^{u+} *is included in* A^u):

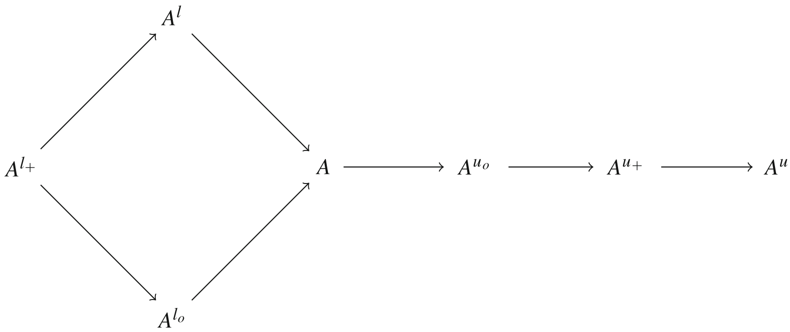

Fig. 5 Relationship between approximations

If a relation R is purely reflexive and not proto-transitive on a set S, then the relation $\tau(R) = R \cap R^{-1}$ will not be an equivalence and for a $A \subset S$, it is possible that $A^{u_o l} \subseteq A$ or $A^{u_o l} \parallel A$ or $A \subseteq A^{u_o l}$.

7.3 Motivation and Examples

Generalized transitive relations occur frequently in general information systems, but are often not recognized as such and there is hope for improved semantics and KI relative the situation for purely reflexive relation based rough sets. Not all of the definable approximations have been investigated in even closely related structures of general rough sets. Contamination-free semantics for the contexts are also not known. Finally these relate to RYS and variants. A proper characterization of roughly equal (requal) objects is also motivated by [91].

Abstract Example

Let $\S = \{a, b, c, e, f, g, h, l, n\}$ and let R be a binary relation on it defined via

$$R = \{(a, a), (l, l), (n, n), (n, h), (h, n), (l, n), (g, c), (c, g)$$
$$(g, l), (b, g), (g, b), (h, g), (a, b), (b, c), (h, a), (a, c)\}.$$

Then $\langle S, R \rangle$ is a PRAS.

If P is the reflexive closure of R (that is $P = R \cup \Delta_S$), then $\langle S, P \rangle$ is a PRAX. The successor neighborhoods associated with different elements of S are as follows ($\mathbf{E}$ is a variable taking values in S) (Table 7):

$$\text{If } A = \{a, h, f\},$$
$$\text{then } A^l = \{a, h, f\},$$
$$A^{l_o} = \{a, f\} \text{ and } A^{l_o} \subset A^l.$$
$$\text{If } F = \{l\},$$
$$\text{then } F^l = \emptyset,\ F^{l_o} = F$$
$$\text{and } F^l \subset F^{l_o}.$$

Table 7 Successor neighborhoods

E	a	b	c	g	e	f	h	l	n
$[E]$	$\{a, h\}$	$\{b, c, g\}$	$\{b, c, g\}$	$\{b, c, g, h\}$	$\{e\}$	$\{f\}$	$\{h, n\}$	$\{l, g\}$	$\{n, l, g, h\}$
$[E]_o$	$\{a\}$	$\{b, c, g\}$	$\{b, c, g\}$	$\{b, c, g\}$	$\{e\}$	$\{f\}$	$\{h, n\}$	$\{l\}$	$\{n, h\}$

Now let $Z = N \cup S \cup X$, where N is the set of naturals, X is the set of elements of the infinite sequences $\{x_i\}, \{y_j\}$. Let Q be a relation on Z such that

$$Q \cap S^2 = P, \tag{25}$$

$$Q \cap N^2 \text{ is some equivalence}, \tag{26}$$

$$(\forall i \in N)(i, x_{3i+1}),\ (x_{2i}, i),\ (x_i, x_{i+1}),\ (y_i, y_{i+1}) \in Q. \tag{27}$$

Q is then a proto-transitive relation. For any $i \in N$, let $P_i = \{y_k : k \neq 2j \& k < i\} \cup \{x_{2j} : 2j < i\}$—this will be used in later sections. The extension of the example to involve nets and densely ordered subsets is standard.

Caste Hierarchies and Interaction

The caste system and religion are among the deep-seated evils of Indian society that often cut across socio-economic classes and level of education. For the formulation of strategies aimed at large groups of people towards the elimination of such evils it would be fruitful to study interaction of people belonging to different castes and religions on different social fronts.

Most of these castes would have multiple subcaste hierarchies in addition. Social interactions are necessarily constrained by their type and untouchability perception. If x, y are two castes, then with respect to a possible social interaction α, people belonging to x will either regard people belonging to y as untouchable or otherwise. As the universality is so total, it is possible to write $\mathbb{U}_\alpha xy$ to mean that y is untouchable for x for the interaction α. Usually this is a asymmetric relation and y would be perceived as a *lower caste* by members of x and many others.

Other predicates will of course be involved in deciding on the possibility of the social interaction, but if $\mathbb{U}_\alpha xy$ then the interaction is forbidden relative x. If α is "context of possible marriage", then the complementary relation ($\mathbb{C}_\alpha$ say) is a reflexive proto-transitive relation. For various other modes of interaction similar relations may be found.

In devising remedial educational programmes targeted at mixed groups, it would be important to understand approximate perceptions of the group and the semantics of PRAX would be very relevant.

Compatibility Prediction Models

For predicting compatibility among individuals or objects the following model can be used. Specific examples include situations involving data from dating sites like OK-Cupid.

Let one woman be defined by a sequence of sets of features $a_1, \ldots, a_n$ at different temporal instants and another woman by $b_1, \ldots, b_n$. Let $\omega(a_i, b_i)$ be the set of features that are desired by a_i, but missing in b_i. Let ρ be an equivalence relation on a subset K of S – the set of all features, that determines the classical rough approximations l_ρ, u_ρ on $\wp(K)$.

Let $(a, b) \in R$ if and only if $(\omega(a_n, b_n))^{l_\rho}$ is *small* (for example, that can mean being an atom of $\wp(K)$). The predicate R is intended to convey *may like* *to be related*. In dating sites, this is understood in terms of profile matches: if a woman's profile matches another woman's and conversely and similarly with another woman's, then the other two woman are assumed to be mutually compatible.

Proposition 138 *R is a proto-transitive relation and $\langle \underline{S}, R\rangle$ is a* PRAS.

Proof Obviously R need not be reflexive or symmetric in general.

If $(a, b), (b, c), (b, a), (c, b) \in R$, then $(a, c), (c, a) \in R$ is a reasonable rule. □

The above is a concrete example of a PRAS that is suggestive of many more practical contexts.

Indeterminate Information Table Perspective
It is easy to derive PRAX from population census, medical, gender studies and other databases and these correspond to information systems. The connections are clarified through this example.

If the problem is to classify a specific population O, for a purpose based on scientific data on sex, gender continuum, sexual orientation and other factors, then our data base would be an indeterminate information system of the form

$$\mathfrak{I} = \langle O, At, \{V_a : a \in At\}, \{\varphi_a : a \in At\}\rangle,$$

where At is a set of attributes, V_a a set of possible values corresponding to the attribute a and $\varphi_a : O \longmapsto \wp(V_a)$ the valuation function. Sex is determined by many attributes corresponding to hormones, brain structure, karyotypes, brain configuration, anatomy, clinical sex etc. These six hormones have their associated free/bound values in the blood stream and the values vary widely over populations. The focus can be on a subset of attributes for which the inclusion/ordering of values (corresponding to any one of the attributes in the subset) of an object in another is relevant. For example, interest in patterns in sexual compatibility/relationships may be corresponded to such subsets. This relation is proto-transitive. Formally for a $B \subseteq At$, if we let $(x, y) \in \rho_B$ if and only if $(\exists a \in B)\varphi_a x \subseteq \varphi_a y$, then ρ_B is often proto-transitive via another predicate on B.

7.4 *Representation of Roughly Equal Elements*

The representation of roughly equal elements in terms of definite elements are well known in case of classical rough set theory. In case of more general spaces including tolerance spaces [91], most authors have been concerned with describing the interaction of rough approximations of different types and not of the interaction of roughly equal objects. Higher order approaches, developed by the present author as in [89] for bitten approximation spaces, permit constructs over sets of roughly equal objects. In the light of the contamination problem [89, 91], it would be an improvement to describe without higher order constructs. In this subsection the new method of representing roughly equal elements based on expanding concepts of definite elements in [93, 95, 99] is presented.

In the following theorem the fine structure of definite elements is described.

Theorem 139 *On the set of proto definite elements $\delta_{lu}(S)$ of a* PRAX *S, the following operations can be defined:*

$$x \wedge y \triangleq x \cap y. \tag{28}$$

$$x \vee y \triangleq x \cup y. \tag{29}$$

$$0 \triangleq \emptyset. \tag{30}$$

$$1 \triangleq S. \tag{31}$$

$$x^c \triangleq S \setminus x. \tag{32}$$

The resulting algebra $\delta_{proto}(S) = \langle \delta_{lu}(S), \vee, \wedge, c, 0, 1\rangle$ is a Boolean lattice.

Proof It is required to show that the operations are well defined. Suppose x, y are proto-definite elements, then

1.

$$(x \cap y)^u \subseteq x^u \cap y^u = x \cap y.$$

$$(x \cap y)^l = (x^u \cap y^u)^l = (x \cap y)^{ul} = (x \cap y)^u = x \cap y.$$

   Since $a^{ul} = a^u$ for any a.

2.

$$(x \cup y)^u = x \cup y = x^l \cup y^l \subseteq (x \cup y)^l.$$

3. $0 \triangleq \emptyset$ is obviously well defined.
4. Obvious.
5. Suppose $A \in \delta_{lu}(S)$, then $(\forall z \in A^c)\ [z] \cap A = \emptyset$ is essential, else $[z]$ would be in A^u. This means $[z] \subseteq A^c$ and so $A^c = A^{cl}$. If there exists a $a \in A$ such that $[a] \cap A^c \neq \emptyset$, then $[a] \subseteq A^u = A$. So $A^c \in \delta_{lu}(S)$.

□

Definition 140 On $\wp(S)$, the following relations can be defined:

$$A \preceq B \text{ if and only if } A^l \subseteq B^l \ \&\ A^u \subseteq B^u. \qquad \text{(Rough Inclusion)}$$

$$A \approx B \text{ if and only if } A \preceq B \ \&\ B \preceq A. \qquad \text{(Rough Equality)}$$

Proposition 141 *The relation $\preceq$ defined on $\wp(S)$ is a bounded partial order and $\approx$ is an equivalence. The quotient $\wp(S)| \approx$ will be said to be the set of* roughly equivalent objects.

Definition 142 A subset A of $\wp(S)$ will be said to a set of *roughly equal* elements if and only if

$$(\forall x, y \in A)\, x^l = y^l \,\&\, x^u = y^u.$$

It will be said to be *full* if no other subset properly including A has the property.

Relative the situation for a general RYS, the following result has already been proved.

Theorem 143 (Meta-Theorem) *In a PRAX S, full set of roughly equal elements is necessarily a union of intervals in $\wp(S)$.*

Definition 144 A non-empty set of non singleton subsets $\alpha = \{x : x \subseteq \wp(S)\}$ will be said to be a *upper broom* if and only if all of the following hold:

$$(\forall x, y \in \alpha)\, x^u = y^u.$$

$$(\forall x, y \in \alpha)\, x \parallel y.$$

$$\text{If } \alpha \subset \beta, \text{ then } \beta \text{ fails to satisfy at least one of the above two conditions.}$$

The set of upper brooms of S will be denoted by $\pitchfork(S)$.

Definition 145 A non-empty set of non singleton subsets $\alpha = \{x : x \subseteq \wp(S)\}$ will be said to be a *lower broom* if and only if all of the following hold:

$$(\forall x, y \in \alpha)\, x^l = y^l \neq x. \tag{33}$$

$$(\forall x, y \in \alpha)\, x \parallel y. \tag{34}$$

$$\text{If } \beta \subset \alpha \& Card(\beta) \geq 2, \text{ then } \beta \text{ fails to satisfy condition (1) or (2).} \tag{35}$$

The set of lower brooms of S will be denoted by $\psi(S)$.

Proposition 146 *If $x \in \delta_{lu}(S)$ then $\{x\} \notin \pitchfork(S)$ and $\{x\} \notin \psi(S)$.*

In the next definition, the concept of union of intervals in a partially ordered set is modified in a way for use with specific types of objects.

Definition 147 By a *bruinval*, will be meant a subset of $\wp(S)$ of one of the following forms:

- Bruinval-0: Intervals of the form (x, y), $[x, y)$, $[x, x]$, $(x, y]$ for $x, y \in \wp(S)$.
- Open Bruinvals: Sets of the form $[x, \alpha) = \{z : x \leq z < b \,\&\, b \in \alpha\}$, $(x, \alpha] = \{z : x < z \leq b \,\&\, b \in \alpha\}$ and $(x, \alpha) = \{z : x < z < b, b \in \alpha\}$ for $\alpha \in \wp(\wp(S))$.

- Closed Bruinvals: Sets of the form $[x, \alpha] = \{z : x \leq z \leq b \& b \in \alpha\}$ for $\alpha \in \wp(\wp(S))$.
- Closed Set Bruinvals: Sets of the form $[\alpha, \beta] = \{z : x \leq z \leq y \& x \in \alpha \& y \in \beta\}$ for $\alpha, \beta \in \wp(\wp(S))$
- Open Set Bruinvals: Sets of the form $(\alpha, \beta) = \{z : x < z < y, x \in \alpha \& y \in \beta\}$ for $\alpha, \beta \in \wp(\wp(S))$.
- Semi-Closed Set Bruinvals: Sets of the form $[[\alpha, \beta]]$ defined as follows: $\alpha = \alpha_1 \cup \alpha_2$, $\beta = \beta_1 \cup \beta_2$ and $[[\alpha, \beta]] = (\alpha_1, \beta_1) \cup [\alpha_2, \beta_2] \cup (\alpha_1, \beta_2] \cup [\alpha_2, \beta_1)$ for $\alpha, \beta \in \wp(\wp(S))$.

In the example of Sect. 2, the representation of the rough object (P_i^l, P_i^u) requires set bruinvals.

Proposition 148 *If S is a PRAX, then a set of the form $[x, y]$ with $x, y \in \delta_{lu}(S)$ will be a set of roughly equal subsets of S if and only if $x = y$.*

Proposition 149 *A bruinval-0 of the form (x, y) is a full set of roughly equal elements if*

- *$x, y \in \delta_{lu}(S)$,*
- *x is covered by y in the order on $\delta_{lu}(S)$.*

Proposition 150 *If $x, y \in \delta_{lu}(S)$ then sets of the form $[x, y)$, $(x, y]$ cannot be a non-empty set of roughly equal elements, while those of the form $[x, y]$ can be if and only if $x = y$.*

Proposition 151 *A bruinval-0 of the form $[x, y)$ is a full set of roughly equal elements if*

- *$x^l, y^u \in \delta_{lu}(S)$, $x^l = y^l$ and $x^u = y^u$,*
- *x^l is covered by y^u in $\delta_{lu}(S)$ and*
- *$x \setminus (x^l)$ and $y^u \setminus y$ are singletons*

Remark 152 In the above proposition the condition $x^l, y^u \in \delta_{lu}(S)$, is not necessary.

Theorem 153 *If a bruinval-0 of the form $[x, y]$ satisfies*

$$x^l = y^l = x \,\&\, x^u = y^u.$$

$$Card(y^u \setminus y) = 1.$$

then $[x, y]$ is a full set of roughly equal objects.

Proof Under the conditions, if $[x, y]$ is not a full set of roughly equal objects, then there must exist at least one set h such that $h^l = x$ and $h^u = y^u$ and $h \notin [x, y]$. But this contradicts the order constraint $x^l \leq h\, y^u$. Note that $y^u \notin [x, y]$ under the conditions. □

Theorem 154 *If a bruinval-0 of the form* $(x, y]$ *satisfies*

$$x^l = y^l = x \;\&\; (\forall z \in (x, y])\, z^u = y^u,$$

$$Card(y^u \setminus y) = 1.$$

then $(x, y]$ *is a full set of roughly equal objects, that does not intersect the full set* $[x, x^u]$.

Proof By monotonicity it follows that $(x, y]$ is a full set of roughly equal objects. then there must exist at least one set h such that $h^l = x$ and $h^u = y^u$ and $h \notin [x, y]$. But this contradicts the order constraint $x^l \leq h\, y^u$. Note that $y^u \notin [x, y]$ under the conditions. □

Theorem 155 *A bruinval-0 of the form* (x^l, x^u) *is not always a set of roughly equal elements, but will be so when* $x^{uu} = x^u$. *In the latter situation it will be full if* $[x^l, x^u)$ *is not full.*

The above theorems essentially show that the description of rough objects depends on too many types of sets and the order as well. Most of the considerations extend to other types of bruinvals as is shown below and remain amenable.

Theorem 156 *An open bruinval of the form* (x, α) *is a full set of roughly equal elements if and only if*

$$\alpha \in \pitchfork (S).$$

$$(\forall y \in \alpha)\, x^l = y^l, x^u = y^u$$

$$(\forall z)(x^l \subseteq z \subset x \longrightarrow z^u \subset x^u).$$

Proof It is clear that for any $y \in \alpha$, (x, y) is a convex interval and all elements in it have same upper and lower approximations. The third condition ensures that $[z, \alpha)$ is not a full set for any $z \in [x^l, x)$. □

Definition 157 An element $x \in \wp(S)$ will be said to be a *weak upper critical element relative* $z \subset x$ if and only if $(\forall y \in \wp(S))\, (z = y^l \,\&\, x \subset y \longrightarrow x^u \subset y^u)$.

An element $x \in \wp(S)$ will be said to be an *upper critical element relative* $z \subset x$ if and only if $(\forall v,\ y \in \wp(S))\, (z = y^l = v^l \,\&\, v \subset x \subset y \longrightarrow v^u = x^u \subset y^u)$. Note that the inclusion is strict.

An element a will be said to be *bi-critical relative* b if and only if $(\forall x,\ y \in \wp(S))(a \subset x \subseteq y \subset b \longrightarrow x^u = y^u \,\&\, x^l = y^l \,\&\, x^u \subset b^u \,\&\, a^l \subset x^l)$.

If x is an upper critical point relative z, then $[z, x)$ or (z, x) is a set of roughly equivalent elements.

Definition 158 An element $x \in \wp(S)$ will be said to be an *weak lower critical element relative* $z \supset x$ if and only if $(\forall y \in \wp(S))\, (z = y^u \,\&\, y \subset x \longrightarrow y^l \subset x^l)$.

An element $x \in \wp(S)$ will be said to be an *lower critical element relative* $z \supset x$ if and only if $(\forall y, v \in \wp(S))\,(z = y^u = v^u \,\&\, y \subset x \subset v \longrightarrow y^l \subset x^l = v^l)$.

An element $x \in \wp(S)$ will be said to be an *lower critical element* if and only if $(\forall y \in \wp(S))\,(y \subset x \longrightarrow y^l \subset x^l)$ An element that is both lower and upper critical will be said to be *critical*. The set of upper critical, lower critical and critical elements respectively will be denoted by $UC(S)$, $LC(S)$ and $CR(S)$.

Proposition 159 *In a PRAX, every upper definite subset is also upper critical, but the converse need not hold.*

The most important thing about the different lower and upper critical points is that they help in determining full sets of roughly equal elements by determining the boundaries of intervals in bruinvals of different types.

Types of Associated Sets

Because of reflexivity, it might appear that lower approximations in PRAX and classical rough sets are too similar at least in the perspective of lower definite objects. It is necessary to classify subsets of a PRAX S, to see the differences relative the behavior of lower approximations in classical rough sets. All this will be used in some of the semantics as well.

Definition 160 For each element $x \in \wp(S)$ the following sets can be associated:

$$F_0(x) = \{y : (\exists a \in x^c)\, Rya \,\&\, y \in x\} \quad \text{(Forward Looking)}$$

$$F_1(x) = \{y : (\exists a \in x^c)\, Rya \,\&\, Rzy \,\&\, z \in x\} \quad \text{(1-Forward Looking)}$$

$$\pi_0(x) = \{y : y \in x \,\&\, (\exists a \in x^c)\, Ray\} \quad \text{(Progressive)}$$

$$St(x) = \{y : [y] \subseteq x \,\&\, \neg(y \in F_0(x))\} \quad \text{(Stable)}$$

$$Sym(x) = \{y : y \in x \,\&\, (\forall z \in x)(Ryz \leftrightarrow Rzy)\} \quad \text{(Relsym)}$$

Forward looking set associated with a set x includes those elements not in x whose successor neighborhoods intersect x. Elements of the set may be said to be relatively forward looking. *Progressive* set of x includes those elements of x whose successor neighborhoods are not included in x. It is obvious that progressive elements are all elements of $x \setminus x^l$. Stable elements are those that are strongly within x and are not directly reachable in any sense from outside. $Sym(x)$ includes those elements in x which are symmetrically related to all other elements within x.

Even though all these are important these cannot be easily represented in the rough domain. Their approximations have the following properties:

Proposition 161 *In the above context, all of the following hold:*

$$(\pi_0(x))^l = \emptyset \,\&\, (\pi_0(x))^u \subseteq x^u \setminus x^l$$

$$(F_0(x))^u \subseteq x^u$$

$$St(x)^l \subseteq x^l \,\&\, F_0(x) = \emptyset \longrightarrow St(x) = x^{l+}$$

$$Sym(x)^u \subseteq x^u \,\&\, (Sym(x))^l \subseteq x^l.$$

Proof Proof is fairly direct. □

7.5 Atoms in the Poset of Roughly Equivalent Sets

Definition 162 For any two elements $x, y \in \wp(S)| \approx$, let

$$x \leq y \text{ if and only if } (\forall a \in x)(\forall b \in y) a^l \subseteq b^l \,\&\, a^u \subseteq b^u.$$

$\wp(S)| \approx$ will be denoted by H in what follows.

Proposition 163 *The relation $\leq$ defined on H is a bounded and directed partial order. The least element will be denoted by* 0 *($0 = \{\emptyset\}$) and the greatest by* 1 *($1 = \{S\}$).*

Definition 164 For any $a, b \in H$, let $UB(a, b) = \{x : a \leq x \,\&\, b \leq x\}$ and $LB(a, b) = \{x : x \leq a \,\&\, x \leq b\}$. By a *s-ideal* (strong ideal) of H, will be meant a subset K that satisfies all of

$$(\forall x \in H)(\forall a \in K)(x \leq a \longrightarrow x \in K),$$

$$(\forall a, b \in K)\, UB(a, b) \cap K \neq \emptyset.$$

An *atom* of H is any element that covers 0. The set of atoms of H will be denoted by $At(H)$.

Theorem 165 *Atoms of H will be of one of the following types:*

Type-0 *Elements of the form $(\emptyset, [x])$, that intersect no other set of roughly equivalent sets.*

Type-1 *Bruinvals of the form $(\emptyset, \alpha)$, that do not contain full sets of roughly equivalent sets.*

Type-2 *Bruinvals of the form (α, β), that do not contain full sets of roughly equivalent sets and are such that $(\forall x) x^l = \emptyset$.*

Proof It is obvious that a bruinval of the form (α, β) can be an atom only if α is the $\emptyset$. If not, then each element x of the bruinval $(\emptyset, \alpha)$ will satisfy $x^l = \emptyset \subset x^u$, thereby contradicting the assumption that (α, β) is an atom.

If $[x]$ intersects no other successor neighborhood, then

$$(\forall y \in (\emptyset, [x])) y^l = \emptyset \,\&\, x^u = [x]$$

and it will be a minimal set of roughly equal elements containing 0.

The other part can be verified based on the representation of possible sets of roughly equivalent elements. □

Theorem 166 *The partially ordered set H is atomic.*

Proof It is required to prove that any element x greater than 0 is either an atom or there exists an atom a such that $a \leq x$, that is

$$(\forall x)(\exists a \in At(H))(0 < x \longrightarrow a \leq x).$$

Suppose the bruinval (α, β) represents a non-atom, then it is necessary that

$$(\forall x \in \alpha)\, x^l \neq \emptyset \,\&\, x^u \subseteq S.$$

Suppose the neighborhoods included in x^u are $\{[y] \,:\, y \in B \subseteq S\}$. If all combinations of bruinvals of the form $(\emptyset, \gamma)$ formed from these neighborhoods are not atoms, then it is necessary that the upper approximation of every singleton subset of a set in γ properly contains another non-trivial upper approximation. This is impossible.

So H is atomic. □

7.6 Algebraic Semantics-1

If $A, B \in \wp(S)$ and $A \approx B$ then $A^u \approx B^u$ and $A^l \approx B^l$, but $\neg(A \approx A^u)$ in general. It has already been seen that $\leq$ is a partial order relation on $\wp(S)| \approx$. In this section elements of $\wp(S)| \approx$ would still be denoted by lower case Greek alphabets.

Theorem 167 *The following operations can be defined on $\wp(S)| \approx$ ($A, B \in \wp(S)$ and $[A], [B]$ are corresponding classes):*

$$L[A] \triangleq [A^l] \tag{36}$$

$$[A] \odot [B] \triangleq [\bigcup_{X \in [A],\, Y \in [B]} (X \cap Y)] \tag{37}$$

$$[A] \oplus [B] \triangleq [\bigcup_{X \in [A],\, Y \in [B]} (X \cup Y)] \tag{38}$$

$$U[A] \triangleq [A^u] \tag{39}$$

$$[A] \cdot [B] \triangleq \lambda(LB([A], [B])) \tag{40}$$

$$[A] \circledast [B] \triangleq \lambda(UB([A], [B])) \tag{41}$$

$$[A] + [B] \triangleq \{X \,:\, X^l = (A^l \cap B^l)^l \,\&\, X^u = A^u \cup B^u\} \tag{42}$$

$$[A] \times [B] \stackrel{\Delta}{=} \{X : X^l = A^l \cup B^l \,\&\, X^u = A^l \cup B^l \cup (A^u \cap B^u) \quad (43)$$

$$[A] \otimes [B] \stackrel{\Delta}{=} \{X : X^l = A^l \cup B^l \,\&\, X^u = A^u \cup B^u\}. \quad (44)$$

Proof If $A \approx B$ then $A^u \approx B^u$ and $A^l \approx B^l$, but $\neg(A \approx A^u)$ in general.

1. If $B \in [A]$, then $B^l = A^l$, $B^u = A^u$ and $L[A] = L[B] = [A^l]$.
2. $[A] \odot [B] \stackrel{\Delta}{=} [\bigcup_{X \in [A],\, Y \in [B]}(X \cap Y)]$ is obviously well defined as sets of the form $[A]$ are elements of partitions
3. Similar to the above.
4. If $B \in [A]$, then $B^u = A^u$ and so $[B^u] = [A^u]$.
5. $[A] \cdot [B] \stackrel{\Delta}{=} \lambda(LB([A], [B]))$.
6. $[A] \circledast [B] \stackrel{\Delta}{=} \lambda(UB([A], [B]))$.
7. $[A] + [B] \stackrel{\Delta}{=} \{X : X^l = A^l \cap B^l \,\&\, X^u = A^u \cup B^u\}$. As the definitions is in terms of A^l, B^l, A^u, B^u, so there is no issue.
8. Similar to above.
9. Similar to above.

□

$+$, $\times$ and $\otimes$ will be referred to as *pragmatic* aggregation, commonality and commonality operations as they are less ontologically committed to the classical domain and more dependent on the main rough domain of interest. $+$ and the other pragmatic operations cannot be compared by the $\leq$ relation and so do not confirm to intuitive understanding of the concepts of aggregation and commonality.

The following theorems summarize the essential properties of the defined operations:

Theorem 168

$$LL(\alpha) = L(\alpha). \quad \text{(L1)}$$

$$(\alpha \leq \beta \longrightarrow L(\alpha) \leq L(\beta)). \quad \text{(L2)}$$

$$(L(\alpha) = [\alpha] \longrightarrow \alpha = \{\alpha^l\}). \quad \text{(L3)}$$

$$(U(\alpha) \cap UU(\alpha) \neq \emptyset \longrightarrow U(\alpha) = UU(\alpha)). \quad \text{(U1)}$$

$$(UU(\alpha) = \emptyset \nrightarrow U(\alpha) = \emptyset). \quad \text{(U2)}$$

$$(\alpha \leq \beta \longrightarrow U(\alpha) \leq U(\beta). \quad \text{(U3)}$$

$$(U(\alpha) = \alpha \longrightarrow \alpha = \alpha^l = \alpha^u). \quad \text{(U4)}$$

$$UL(\alpha) \leq U(\alpha). \quad \text{(U5)}$$

$$LU(\alpha) = U(\alpha). \quad \text{(U6)}$$

Proof Let $\alpha \in \wp(S)| \approx$, then the pair of lower and upper approximations associated with it will be denoted by α_l and α_u respectively. By α^u and α^l is meant the result of global operations respectively on the set α (seen as an element of $\wp(S)$). These take singleton values and so there is no real need of the approximations α_l and α_u and the former will be used.

Proof of L1:

$$\alpha \in \wp(S)| \approx, \text{ so } \alpha = \{X\,;\, \alpha_l = X^l \,\&\, \alpha_u = X^u, \& X \in \wp(S)\}.$$
$$\alpha^l = \{X^l;\ X \in \alpha\} = \{\alpha_l\}$$
$$\text{So } [\alpha^l] = \{Y\,;\, Y^l = \alpha^l \,\&\, Y^u = \alpha^{lu}\}.$$
$$(L(\alpha))^l = \{Y^l\,;\, Y^l = \alpha^l \,\&\, Y^u = \alpha^{lu}\} = \{\alpha^l\}.$$
$$\text{This yields } LL(\alpha) = L(\alpha). \tag{L1}$$

Proof of U1:

$$\alpha^u = \{X^u\,;\, \alpha^l = X^l \,\&\, \alpha^u = X^u\} = \{\alpha^u\}.$$
$$U(\alpha) = [\alpha^u] = \{Y\,;\, Y^l = \alpha^u \,\&\, Y^u = \alpha^{uu}\}.$$
$$\text{So } U(\alpha)^u = \{\alpha^{uu}\}.$$
$$UU(\alpha) = [U(\alpha)^u] = [\alpha^{uu}] = \{Y\,;\, Y^l = \alpha^{uu} \,\&\, Y^u = \alpha^{uuu}\}.$$
$$\text{Since } \alpha \subseteq \alpha^u \subseteq \alpha^{uu} \subseteq \alpha^{uuu},$$
$$\text{therefore } (U(\alpha) \cap UU(\alpha) \neq \emptyset \longrightarrow U(\alpha) = UU(\alpha). \tag{U1}$$

The other parts can be proved from the above considerations. □

Theorem 169 *In the context of the above theorem, the following hold:*

$$\alpha \odot \beta = \beta \odot \alpha) \tag{CO1}$$
$$\alpha \leq \alpha \odot \alpha \tag{CO2}$$
$$\alpha \leq \alpha \odot \top \tag{CO3}$$
$$\alpha \odot \alpha = \alpha \odot (\alpha \odot \alpha) = \alpha \odot \top \tag{CO4}$$
$$\alpha \oplus \beta = \beta \oplus \alpha) \tag{AO1}$$
$$\alpha \leq \alpha \oplus \beta \tag{AO2}$$
$$\alpha \leq \alpha \oplus \bot \tag{AO3}$$
$$(\alpha \oplus \alpha) \oplus \alpha = \alpha \oplus \alpha \tag{AO4}$$
$$\textit{In general}, \alpha \oplus (\alpha \odot \beta) \neq \alpha. \tag{AC}$$

Proof

CO1 The definition of $\odot$ does not depend on the order in which the arguments occur as set theoretic intersection and union are commutative. To be precise $\bigcup_{X\in[A],\, Y\in[B]}(X \cap Y) = \bigcup_{X\in[A],\, Y\in[B]}(Y \cap X)$.

CO2 $\bigcup_{X\in[A],\, Y\in[A]}(X \cap Y) = \bigcup_{X\in[A]} X$. But because $X^l \cup Y^l \subseteq (X \cup Y)^l$ in general, so equality fails.

CO3 Follows from the last inequality.

CO4 In $[\alpha\odot(\alpha\odot\alpha)]$, any new elements that are not in $[\alpha\odot\alpha]$ cannot be introduced as the inequality in [CO2] is due to the lower approximation and all possible subsets have already been included.

AO1 The definition of $\oplus$ does not depend on the order in which the arguments occur as set theoretic union is commutative.

AO2 Even when $\beta = \alpha$, inequality can happen for reasons mentioned earlier.

Proof of [AO3, AO4, AC] are analogous or direct. □

The above result means that $\odot$ is an imperfect commonality relation. It is a proper commonality among a certain subset of elements of H.

Theorem 170 *In the context of the above theorem, the following properties of* $+, \times, \otimes$ *are provable:*

$$\alpha + \alpha = \alpha, \tag{+I}$$

$$\alpha + \beta = \beta + \alpha, \tag{+C}$$

$$\alpha \times \alpha = \alpha, \tag{cI}$$

$$\alpha \times \beta = \beta \times \alpha, \tag{cC}$$

$$\alpha \leq \beta \longrightarrow \alpha + \gamma \leq \beta + \gamma, \tag{+Is}$$

$$\alpha \leq \beta \longrightarrow \alpha \times \gamma \leq \beta \times \gamma, \tag{cIs}$$

$$\alpha \leq \beta \longrightarrow \alpha \leq \alpha \times \beta \leq \beta, \tag{+In}$$

$$\alpha + \beta \leq \alpha \oplus \beta, \tag{R1}$$

$$\alpha \times \beta \leq (\alpha \times \beta) \oplus \alpha. \tag{Mix1}$$

Proof Most of the proof is in Sect. 8, so they are not repeated. □

Definition 171 By a *Concrete Pre-PRAX Algebraic System* (CPPRAXA), will be meant a system of the form

$$\mathfrak{H} = \langle H, \leq, L, U, \oplus, \odot, +, \times, \otimes, \bot, \top \rangle ,$$

with all of the operations being as defined in this section.

Apparently the algebraic properties of the rough objects of l_o, u_o need to be involved for a representation theorem. The operations can be improved by related operations of the following section. Results concerning this will appear separately. Definable filters in general have reasonable properties.

Definition 172 Let K be an arbitrary subset of a CPPRAXA $\mathfrak{H}$. Consider the following statements:

$$(\forall x \in K)(\forall y \in \mathfrak{H})(x \leq y \Rightarrow y \in K). \tag{F1}$$

$$(\forall x, y \in K)\, x \oplus y, Lx \in K. \tag{F2}$$

$$(\forall a, b \in \mathfrak{H})(1 \neq a \oplus b \in K \;\Rightarrow\; a \in K \text{ or } b \in K). \tag{F3}$$

$$(\forall a, b \in \mathfrak{H})(1 \neq UB(a, b) \in K \;\Rightarrow\; a \in K \text{ or } b \in K). \tag{F4}$$

$$(\forall a, b \in K)\, LB(a, b) \cap K \neq \emptyset. \tag{F5}$$

- If K satisfies **F1** then it will be said to be an *order filter*. The set of such filters on $\mathfrak{H}$ will be denoted by $\mathfrak{O}_F(\mathfrak{H})$.
- If K satisfies **F1, F2** then it will be said to be a *filter*. The set of such filters on $\mathfrak{H}$ will be denoted by $\mathcal{F}(\mathfrak{H})$.
- If K satisfies **F1, F2, F3** then it will be said to be a *prime filter*. The set of such filters on $\mathfrak{H}$ will be denoted by $\mathcal{F}_P(\mathfrak{H})$.
- If K satisfies **F1, F4** then it will be said to be a *prime order filter*. The set of such filters on $\mathfrak{H}$ will be denoted by $\mathfrak{O}_{PF}(\mathfrak{H})$.
- If K satisfies **F1, F5** then it will be said to be an *strong order filter*. The set of such filters on $\mathfrak{H}$ will be denoted by $\mathfrak{O}_{SF}(\mathfrak{H})$.

Dual concepts of ideals of different kinds can be defined.

Proposition 173 *Filters of different kinds have the following properties:*

- *Every set of filters of a kind is ordered by inclusion.*
- *Every filter of a kind is contained in a maximal filter of the same kind.*
- $\mathfrak{O}_{SF}(\mathfrak{H})$ *is an algebraic lattice, with its compact elements being the finitely generated strong order filters in it.*

Definition 174 For $F, P \in \mathcal{F}(\mathfrak{H})$, the following operations can be defined:

$$F \wedge P \stackrel{\Delta}{=} F \cap P$$

$$F \vee P \stackrel{\Delta}{=} \langle F \cup P \rangle\,,$$

where $\langle F \cup P \rangle$ denotes the smallest filter containing $F \cup P$.

Theorem 175 $\langle \mathcal{F}(\mathfrak{H}),\, \vee,\, \wedge, \bot, \top \rangle$ *is an atomistic bounded lattice.*

8 Approximate Algebraic Semantics

In this section the approximate semantics of PRAX invented in [95, 99] is presented. Initially, the shortcomings of a direct approach are highlighted. Next ideas of approximation of relations by other relations are developed. Rough dependence is explained next and applied to form multiple approximate semantics. All of the details of the semantics have been omitted.

Definition 176 In a PRAX S, let

$$\mathcal{R}(S) = \{(A^l, A^u); A \in \wp(S)\}.$$

Then all of the following operations on $\mathcal{R}(S)$ can be defined:

$$(A^l, A^u) \vee (B^l, B^u) \triangleq (A^l \cup B^l, A^u \cup B^u). \qquad \text{(Aggregation)}$$

$$\text{If } (A^l \cap B^l, (A^u \cap B^u)) \in \mathcal{R}(S) \text{ then}$$

$$(A^l, A^u) \wedge (B^l, B^u) \triangleq (A^l \cap B^l, (A^u \cap B^u)). \qquad \text{(Commonality)}$$

$$\text{If } (A^{uc}, A^{lc}) \in \mathcal{R}(S) \text{ then}$$

$$\sim (A^l, A^u) \triangleq (A^{uc}, A^{lc}). \qquad \text{(Weak Complementation)}$$

$$\bot \triangleq (\emptyset, \emptyset). \ \top \triangleq (S, S). \qquad \text{(Bottom, Top)}$$

$$(A^l, A^u) \,\overline{\wedge}\, (B^l, B^u) \triangleq ((A^l \cap B^l)^l, (A^u \cap B^u)^l). \qquad \text{(Proper Commonality)}$$

Definition 177 In the context of the above definition, a partial algebra of the form $\mathfrak{R}(S) = \langle \mathcal{R}(S), \vee, \wedge, c, \bot, \top \rangle$ will be termed a *proto-vague algebra* and $\mathfrak{R}_f(S) = \langle \mathcal{R}(S), \vee, \wedge, \overline{\wedge}. c, \bot, \top \rangle$ will be termed a *full proto-vague algebra*.

More generally, if L, U are arbitrary rough lower and upper approximation operators over the PRAX, and if each occurrence of l is replaced by L and of u by U in the above definition then the resulting algebra of the above form will be called a *LU-proto-vague partial algebra*. Analogously, $l_o u_o$-proto-vague algebras and similar algebras can be defined.

Theorem 178 *A full proto-vague partial algebra* $\mathfrak{R}_f(S)$ *satisfies all of the following:*

1. $\vee, \overline{\wedge}$ *are total operations.*
2. $\vee$ *is a semi-lattice operation satisfying idempotency, commutativity and associativity.*
3. $\wedge$ *is a weak semi-lattice operation satisfying idempotency, weak strong commutativity and weak associativity. With* $\vee$ *it forms a weak distributive lattice.*
4. $\sim$ *is a weak strong idempotent partial operation;* $\sim\sim\sim \alpha \stackrel{\omega^*}{=} \sim \alpha$.

5. $\sim(\alpha \vee \beta) \overset{\omega}{=} \sim\alpha \wedge \sim\beta$ *(Weak De Morgan condition) holds.*
6. $\bar{\wedge}$ *is an idempotent, commutative and associative operation that forms a lattice with* $\vee$.
7. $\alpha \,\bar{\wedge}\, \bot = \alpha \wedge \bot = \bot$. $\alpha \vee \bot = \alpha$; $\alpha \,\bar{\wedge}\, \top = \alpha \wedge \top = \alpha$. $\alpha \vee \top = \top$.
8. $\sim(\alpha \wedge \beta) = (\sim\alpha \vee \sim\beta) \longrightarrow \sim(\alpha \,\bar{\wedge}\, \beta) = (\sim\alpha \vee \sim\beta)$.
9. $\alpha \vee (\beta \,\bar{\wedge}\, \gamma) \subseteq (\alpha \vee \beta) \,\bar{\wedge}\, (\alpha \vee \gamma)$, *but distributivity fails.*

Proof Let $\alpha = (X^l, X^u)$, $\beta = (Y^l, Y^u)$ and $\gamma = (Z^l, Z^u)$ for some $X, Y, Z \in \wp(S)$, then

1. $\alpha \vee \beta = (X^l \cup Y^l, X^u \cup Y^u)$ belongs to $\mathfrak{R}(S)$ because the components are unions of successor neighborhoods and $X^l \cup Y^l \subseteq X^u \cup Y^u$. The proof for $\wedge$ is similar.
2. $\alpha \vee (\beta \vee \gamma) = (X^l, X^u) \vee ((Y^l, Y^u) \vee (Z^l, Z^u)) = (X^l, X^u) \vee (Y^l \cup Z^l, Y^u \cup Z^u) = (X^l \cup Y^l \cup Z^l, X^u \cup Y^u \cup Z^u) = (\alpha \vee \beta) \vee \gamma$.
3. Weak absorptivity and weak distributivity are proved next.
   $(X^l \cap (X^l \cup Y^l)) = X^l$ and $(X^u \cap (X^u \cup Y^u)) = X^l$ hold in all situations. If $(X^l \cup (X^l \cap Y^l))$ is defined then it is equal to X^l and if $(X^u \cup (X^u \cup Y^u))$ is defined, then it is equal to X^u. So

$$\alpha \vee (\alpha \wedge \beta) \overset{\omega}{=} \alpha = \alpha \wedge (\alpha \vee \beta).$$

   For distributivity $(\alpha \vee (\beta \wedge \gamma) \overset{\omega}{=} (\alpha \vee \beta) \wedge (\alpha \vee \gamma)$ and $\alpha \wedge (\beta \vee \gamma) \overset{\omega}{=} (\alpha \wedge \beta) \vee (\alpha \wedge \gamma))$ again it is a matter of definability working in coherence with set-theoretic distributivity.
4. If $\sim\alpha$ is defined then $\sim\alpha = (X^{uc}, X^{lc})$ and

$$\sim\sim\alpha = \sim(X^{uc}, X^{lc}) = (X^{lcc}, X^{ucc}) = (X^l, X^u), \text{ by definition.}$$

   If $\sim\sim\alpha$ is defined, then $\sim\alpha$ is necessarily defined. So

$$\sim\sim\sim\alpha \overset{\omega^*}{=} \sim\alpha.$$

5. If $\sim(\alpha \vee \beta)$ and $\sim\alpha \wedge \sim\beta$ are defined then $\sim(\alpha \vee \beta) = \sim((X^l \cup Y^l), (X^u \cup Y^u)) = ((X^{uc} \cap Y^{uc}), (X^{lc} \cap Y^{lc})) \overset{\omega^*}{=} (X^{uc}, X^{lc}) \wedge (Y^{uc}, Y^{lc}) = \sim\alpha \wedge \sim\beta$. So $\sim(\alpha \vee \beta) \overset{\omega^*}{=} \sim\alpha \wedge \sim\beta$.
6. $\alpha \,\bar{\wedge}\, \beta = \beta \,\bar{\wedge}\, \alpha$ & $\alpha \,\bar{\wedge}\, \alpha = \alpha$ are obvious.
   $\alpha \,\bar{\wedge}\, (\beta \,\bar{\wedge}\, \gamma) = ((X^l \cap (Y^l \cap Z^l)^l)^l, (X^u \cap (Y^u \cap Z^u)^u)^u)$ The components are basically the unions of common granules among the three. No granule in the final evaluation is eliminated by choice of order of operations. So $\alpha \,\bar{\wedge}\, (\beta \,\bar{\wedge}\, \gamma) = (\alpha \,\bar{\wedge}\, \beta) \,\bar{\wedge}\, \gamma$.
   $\alpha \,\bar{\wedge}\, (\alpha \vee \beta) = ((X^l \cap (X^l \cup Y^l))^l, (X^u \cap (X^u \cup Y^u))^l) = \alpha$.
   Further, $\alpha \vee (\alpha \,\bar{\wedge}\, \beta) = ((X^l \cup (X^l \cap Y^l)^l), (X^u \cup (X^u \cap Y^u)^l)) = \alpha$. So $\vee, \bar{\wedge}$ are lattice operations.

7. • Since $\bot = (\emptyset, \emptyset)$, $\alpha \barwedge \bot = \alpha \wedge \bot = \bot$ and $\alpha \vee \bot = \alpha$ follow directly.
   • Since $\top = (S, S)$, $\alpha \barwedge \top = \alpha \wedge \top = \alpha$ and $\alpha \vee \top = \top$ follow directly.
8. Follows from the previous proofs.
9. • $\alpha \vee (\beta \barwedge \gamma) = ((X^l \cup (Y^l \cap Z^l)^l), (X^u \cup (Y^u \cap Z^u)^l))$. If $a \in S$ and $[a] \subseteq X^l \cup (Y^l \cap Z^l)^l$, and $[a] \subseteq (Y^l \cap Z^l)^l$, then $[a] \subseteq Y^l$ and $[a] \subseteq Z^l$. So $[a] \subseteq X^l \cup Y^l$ and $[a] \subseteq X^l \cup Z^l$.
   • If $[a] \subseteq X^l \cup (Y^l \cap Z^l)^l$ and if $[a] = P \cup Q$, with $P \subseteq X^l$, $Q \subseteq (Y^l \cap Z^l)^l$ then $[a] \subseteq X^l \cup Y^l$ and $[a] \subseteq X^l \cup Z^l$. This proves $\alpha \vee (\beta \barwedge \gamma) \subseteq (\alpha \vee \beta) \barwedge (\alpha \vee \gamma)$.
   • If $[a] \subseteq ((X^l \cup Y^l) \cap (X^l \cup Y^l))^l$ then $[a] \subseteq X^l \cup Y^l$ and $[a] \subseteq X^l \cup Z^l$. This means $[a] = P \cup Q$, with $P \subseteq X^l$, $Q \subseteq Y^l$ and $Q \subseteq Z^l$ and Q is contained in union of some other granules. So $Q \subseteq Y^l \cap Z^l$, but it cannot be ensured that $Q \subseteq (Y^l \cap Z^l)^l$ (required counterexamples are easy to construct). It follows that $((X^l \cup Y^l) \cap (X^l \cup Y^l))^l \nsubseteq X^l \cup (Y^l \cap Z^l)^l$.

□

The following theorem provides a condition for ensuring that $\sim \alpha$ is defined.

Theorem 179 *If $X^{uu} = X^u$, then $\sim (X^l, X^u) = (X^{uc}, X^{lc})$ but the converse is not necessarily true.*

Proof

- $\sim (X^l, X^u)$ is defined if and only if X^{uc} is a union of granules.
- If $X^{uu} = X^u$ then X^{uc} is a union of granules generated by *some* of the elements in X^{uc} , but the converse need not hold.
- So the result follows.

□

Let W be any quasi-order relation that approximates R, and let the granules $[x]_w$, $[x]_{wi}$ and l_w, u_w be lower and upper approximations defined by analogy with the definitions of l, u. If $R \subset W$, then $(\forall x \in S)\, [x] \subseteq [x]_w$ and $(A, B \in \wp(S)$. $A \parallel B$ in all that follows shall mean $A \nsubseteq B \,\&\, B \nsubseteq A$):

- If $A \subset B$ and $A^u = B^u$, then it is possible that $A^{u_w} \subset B^{u_w}$.
- If $A \subset B$ and $A^l = B^l$, then it is possible that $A^{l_w} \subset B^{l_w}$.
- If $A \subset B$ and $A^{u_w} = B^{u_w}$, then it is possible that $A^u \subset B^u$.
- If $A \subset B$ and $A^{l_w} = B^{l_w}$, then it is possible that $A^l \subset B^l$.
- If $A \parallel B$ and $A^l = B^l$, then it is possible that $A^{l_w} \parallel B^{l_w}$.
- If $A \parallel B$ and $A^{l_w} = B^{l_w}$, then it is possible that $A^l \parallel B^l$.
- If $A \parallel B$ and $A^u = B^u$, then it is possible that $A^{u_w} \parallel B^{u_w}$.
- If $A \parallel B$ and $A^{u_w} = B^{u_w}$, then it is possible that $A^u \parallel B^u$.
- If $A \subset B$, $A^l = B^l$ and $A^u = B^u$, then it is possible that $A^{u_w} \subset B^{u_w} \,\&\, A^{l_w} \subset B^{l_w}$.

The above properties mean that meaningful correspondences between vague partial algebras and Nelson algebras may be quite complex. Focusing on granular evolution alone, the following can be defined

$$(\forall x \in S)\, \varphi_o([x]) = \bigcup_{z \in [x]} [z]_w .$$

$$(\forall A \in \wp(S))\, \varphi(A^l) = \bigcup_{[x] \subseteq A^l} \varphi_o([x]).$$

$$(\forall A \in \wp(S))\, \varphi(A^u) = \bigcup_{[x] \subseteq A^u} \varphi_o([x]).$$

$\varphi(A^l \cup B^l) = \bigcup_{[x] \subseteq A^l \cup B^l}$.

If $[x] \subseteq A^l \cup B^l$

φ can be naturally extended by components to a map τ as per

$$\tau(A^l, A^u) = (\varphi(A^l), \varphi(A^u)).$$

Proposition 180 *If $R \subseteq R_w$ and R_w is transitive, then*

- *If $z \in [x]$ and $x \in [z]$, then $\varphi([z]) = \varphi([x])$.*
- *If $z \in [x]$, then $\varphi([z]) \subseteq \varphi([x])$.*
-

$$(\forall A \in \wp(S))\, \varphi(A^l) = \bigcup_{[x] \subseteq A^l} \varphi([x]) = \bigcup_{[x] \subseteq A^l} [x]_w$$

Proof

- $z \in [x]$ yields Rzx. So if Raz, then Rax and it is clear that $\varphi([z]) \subseteq \varphi([x])$. $Rbx \& Rzx \& Rxz$ implies $R_w bz$.
- This is the first part of the above.
- Follows from the above.

□

Definition 181 The following abbreviations will be used for handling different types of subsets of S:

$$\Gamma_u(S) = \{A^u;\ A \in \wp(S)\}. \quad \text{(Uppers)}$$

$$\Gamma_{uw}(S) = \{A^{u_w};\ A \in \wp(S)\}. \quad \text{(w-Uppers)}$$

$$\Gamma(S) = \{B;\ (\exists A \in \wp(S))\, B = A^l \text{ or } B = A^u\}. \quad \text{(lower definites)}$$

Note that $\delta_l(S)$ is the same as $\Gamma(S)$ and similarly for $\delta_{lw}(S)$.

τ has the following properties:

Proposition 182 *If $R \subseteq R_w$ and R_w is transitive, then*

$$\tau(\bot) = \bot_w.$$
$$\tau(\top) = \top_w.$$
$$(\forall \alpha, \beta \in \mathfrak{R}(S))\, \tau(\alpha \vee \beta) = \tau(\alpha) \vee \tau(\beta).$$
$$(\forall \alpha, \beta \in \mathfrak{R}(S))\, \tau(\alpha \wedge \beta) \overset{\omega}{=} \tau(\alpha) \wedge \tau(\beta).$$

Definition 183 For each $\alpha \in \mathfrak{R}_w(S)$, the set of ordered pairs $\tau^{\dashv}(\alpha)$ will be termed as a *co-rough object* of S, where

$$\tau^{\dashv}(\alpha) = \{\beta\,;\, \beta \in \mathfrak{R}(S)\,\&\,\tau(\beta) = \alpha\}.$$

The collection of all co-rough objects will be denoted by $\mathfrak{CR}(S)$.

This permits us to define a variety of closely related semantics of PRAX when $R \subseteq R_w$ and R_w is transitive. These include:

- The map $\tau : \mathfrak{R}_f(S) \longmapsto \mathfrak{R}_w(S)$. $\mathfrak{R}_w(S)$ being a Nelson algebra over an algebraic lattice.
- $\mathfrak{R}_f(S) \cup \mathfrak{CR}(S)$ along with induced operations yields another semantics of PRAX.
- $\mathfrak{R}(S) \cup \mathfrak{R}_w(S)$ enriched with algebraic and dependency operations described in 8.4.

8.1 Approximate Relations

If R is a binary relation on a set X, then let $R^o \overset{\partial}{=} R \cup \Delta_X$. The weak transitive closure of R will be denoted by $R^\#$. If $R^{(i)}$ is the i-times composition $\underbrace{R \circ R \ldots \circ R}_{\text{i-times}}$, then $R^\# = \bigcup R^{(i)}$. R is *acyclic* if and only if $(\forall x)\, \neg R^\# xx$. The relation $R^{\cdot}$ is defined by $R^{\cdot}ab$ if and only if $Rab\,\&\,\neg(R^\# ab\,\&\, R^\# ba)$.

Definition 184 If R is a relation on a set S, then the relations $R^{\curlywedge}$, R^{cyc} and R^h will be defined via

$$R^{\curlywedge}ab \text{ if and only if } [b]_{R^o} \subset [a]_{R^o}\,\&\,[a]_{iR^o} \subset [b]_{iR^o} \tag{45}$$
$$R^{cyc}ab \text{ if and only if } R^\# ab\,\&\,R^\# ba \tag{46}$$
$$R^h ab \text{ if and only if } R^{\curlywedge}ab\,\&\,R^{\cdot}ab. \tag{47}$$

In case of PRAX, $R^o = R$, so the definition of $R^{\curlywedge}$ would involve neighborhoods of the form $[a]$ and $[a]_i$ alone. $R^{\curlywedge} \subset R$ and $R^{\curlywedge}$ is a partial order.

Example 185 In our example 119, $R^{\#}ab$ happens when a is an ally of an ally of b. $R^{\curlywedge}ab$ happens if and only if every ally of b is an ally of a and if a is ally of c, then b is an ally of c—this can happen, for example, when b is a Marxist feminist and a is a socialist feminist. $R^{cyc}ab$ happens when *a is an ally of an ally of b and b is an ally of an ally of a*. $R^{\cdot}ab$ happens whenever a is an ally of b, but b is not an ally of anybody who is an ally of a.

Theorem 186 $R^h = \emptyset$.

Proof

$$R^h ab \Leftrightarrow R^{\curlywedge}ab \,\&\, R^{\cdot}ab$$
$$\Leftrightarrow \tau(R)ab \,\&\, (R \setminus \tau(R))ab$$
$$\text{But } \neg(\exists a)(R \setminus \tau(R))aa.$$

So $R^h = \emptyset$. □

Proposition 187 *All of the following hold in a PRAX S:*

$$R^{\cdot}ab \leftrightarrow (R \setminus \tau(R))ab \tag{48}$$
$$(\forall a, b)\neg(R^{\cdot}ab \,\&\, R^{\cdot}ba) \tag{49}$$
$$(\forall a, b, c)(R^{\cdot}ab \,\&\, R^{\cdot}bc \longrightarrow \neg R^{\cdot}ac). \tag{50}$$

Proof

- $R^{\cdot}ab \leftrightarrow Rab \,\&\, \neg(R^{\#}ab\, R^{\#}ba)$.
- But $\neg(R^{\#}ab\, R^{\#}ba)$ is possible only when both Rab and Rba hold.
- So $R^{\cdot}ab \leftrightarrow Rab \,\&\, \neg(\tau(R)ab) \leftrightarrow (R \setminus \tau(R))ab$.

□

Theorem 188

$$R^{\#\cdot} = R^{\#} \setminus \tau(R) \tag{51}$$
$$R^{\cdot\#} = (R \setminus \tau(R))^{\#} \tag{52}$$
$$(R \setminus \tau(R))^{\#} \subseteq R^{\#} \setminus \tau(R). \tag{53}$$

Proof

1.

$$R^{\#\cdot}ab \leftrightarrow R^{\#}ab \,\&\, \neg(R^{\#\#}ab \,\&\, R^{\#\#}ba)$$
$$\leftrightarrow R^{\#}ab \,\&\, \neg(R^{\#}ab \,\&\, R^{\#}ba)$$

$$\leftrightarrow R^{\#}ab \,\&\, \neg\tau(R)ab$$
$$\leftrightarrow (R^{\#} \setminus \tau(R))ab.$$

2.

$$R^{\cdot\#}ab \leftrightarrow (R^{\cdot})^{\#}ab$$
$$\leftrightarrow (R \setminus \tau(R))^{\#}ab.$$

3. Can be checked by a contradiction or a direct argument.

□

Possible properties that approximations of prototransitive relations may or should possess are considered next. If $<$ is a strict partial order on S and R is a relation, then consider the conditions :

$$(\forall a, b)(a < b \longrightarrow R^{\#}ab). \tag{PO1}$$

$$(\forall a, b)(a < b \longrightarrow \neg R^{\#}ba). \tag{PO2}$$

$$(\forall a, b)(R^{\leftthreetimes}ab \,\&\, R^{\cdot}ab \longrightarrow a < b. \tag{PO3}$$

$$\text{If } a \equiv_R b, \text{ then } a \equiv_< b. \tag{PO4}$$

$$(\forall a, b)(a < b \longrightarrow Rab). \tag{PO5}$$

As per [59], $<$ is said to be a *partial order approximation* POA (resp. *weak partial order approximation* WPOA) of R if and only if **PO1, PO2, PO3, PO4** (resp. **PO1, PO3, PO4**) hold. A POA $<$ is *inner approximation* IPOA of R if and only if **PO5** holds. **PO4** has a role beyond that of approximation and depends on both successor and predecessor neighborhoods. R^h, $R^{\cdot\leftthreetimes}$ are IPOA, while $R^{\cdot\#}$, $R^{\#\cdot}$ are POAs.

By a *lean quasi order approximation* $<$ of R, will be meant a quasi order satisfying **PO1** and **PO2**. The corresponding sets of such approximations of R will be denoted by $POA(R)$, $WPOA(R)$, $IPOA(R)$, $IWPOA(R)$ and $LQO(R)$

Theorem 189 *For any $A, B \in LQO(R)$, the operations $\&, \vee, \top$ can be defined via:*

$$(\forall x, y)(A\&B)xy \textit{ if and only if } (\forall x, y)Axy \,\&\, Bxy.$$

$$(A \vee B) = (A \cup B)^{\#},$$

$$\top = R^{\#}.$$

Proof

- If Aab then $R^{+}ab$ and if Bab then $R^{+}ab$.
- But if $(A\&B)ab$, then both Aab and Bab.
- So $R^{+}ab$.

Similarly it can be shown that $A \vee B \in LQO(R)$. It is always defined and contained within $R^{\#}$ as it is the transitive completion of $A \cup B$. $\top = R^{\#}$ as transitive closure is a closure operator. □

Theorem 190 *In a PRAX,* $R^{\cdot\#}\&R^{\#\cdot}xy \leftrightarrow (R \setminus \tau(R))^{\#}xy$.

8.2 Granules of Derived Relations

The behavior of approximations and rough objects corresponding to derived relations is investigated in this subsection.

Definition 191 The relation $R^{\#\cdot}$ will be termed the *trans ortho-completion* of R. The following granules will be associated with each $x \in S$:

$$[x]_{ot} = \{y\,;\ R^{\#\cdot}yx\} \tag{54}$$

$$[x]^{i}_{ot} = \{y\,;\ R^{\#\cdot}xy\} \tag{55}$$

$$[x]^{o}_{ot} = \{y\,;\ R^{\#\cdot}yx \,\&\, R^{\#\cdot}xy\}. \tag{56}$$

Let the corresponding approximations be $l_{ot},\ u_{ot}$ and so on.

Theorem 192 *In a PRAX S,* $(\forall x \in S)\,[x]^{o}_{ot} = \{x\}$.

Proof $R^{\#\cdot}xy \,\&\, R^{\#\cdot}yx$ means that the pair $(x,\ y)$ is in the transitive completion of R and not in $\tau(R)$. So $y \in [x]^{o}_{ot}$ if and only if

$$(\exists a,\ b)\ Rxa \,\&\, Ray \,\&\, (\neg Rax \vee \neg Rya) \,\&\, (Ryb \,\&\, Rbx) \,\&\, (\neg Rby \vee \neg Rxb).$$

If it is assumed that $x \neq y$, then each of the possibilities leads to a contradiction as is shown below. In the context of the above statement:

Case-1

- $Rxa \,\&\, Ray \,\&\, \neg Rax \,\&\, Rya \,\&\, Ryb \,\&\, Rbx \,\&\, \neg Rby \,\&\, Rxb$.
- This yields $R^{\#}xa \,\&\, R^{\#}bb \,\&\, R^{\#}ba \,\&\, R^{\#}ab$.
- So, $R^{\#}xb \,\&\, R^{\#}ya \,\&\, R^{\#}ax$ and this contradicts the original assumption.

Case-2

- $Rxa \,\&\, Ray \,\&\, Rax \,\&\, \neg Rya \,\&\, Ryb \,\&\, Rbx \,\&\, Rby \,\&\, \neg Rxb$.
- This yields the contradiction $R^{\#}ab$.

Case-3

- $Rxa \,\&\, Ray \,\&\, \neg Rax \,\&\, Rya \,\&\, Ryb \,\&\, Rbx \,\&\, Rby \,\&\, \neg Rxb$.
- This yields $R^{\#}ba \,\&\, R^{\#}ab \,\&\, R^{\#}aa \,\&\, R^{\#}bb$ and $R^{\#}yy\&R^{\#}xy\&R^{\#}yx\&Rya\&R^{\#}xa$.
- But such a $R^{\#}$ is not possible.

Somewhat similarly the other cases can be seen to lead to contradictions. □

By the *symmetric center* of a relation R, will be meant the set $K_R = \bigcup e_i(\tau(R)\setminus \Delta_S)$—basically the union of elements in either component of $\tau(R)$ minus the diagonal relation on S.

Proposition 193 $(\forall x)\, [x]\triangle[x]_{ot} \neq \emptyset$ *as*

$$x \notin K_R \longrightarrow [x] \subset [x]_{ot}$$

$$x \in K_R \longrightarrow [x] \nsubseteq [x]_{ot} \,\&\, \{x\} \subset [x] \cap [x]_{ot}.$$

Proof

$$\begin{aligned} z \in [x]_{ot} &\leftrightarrow R^{\#\cdot}zx \\ &\leftrightarrow R^{\#}zx \,\&\, \neg\tau(R)zx \\ &\leftrightarrow (Rzx \,\&\, \neg Rxz) \text{ or } (\neg Rzx \,\&\, \neg Rxz \,\&\, (R^{\#} \setminus R)zx). \end{aligned}$$

□

K_R can be used to partially categorize subsets of S based on intersection.

Proposition 194 $(R \setminus \tau(R))^{\#} \cup \tau(R)$ *is not necessarily a quasi order.*

Proof $(x, y) \in (R \setminus \tau(R))^{\#} \cup \tau(R)$ and $(x, y) \notin \tau(R)$ and $x \in K_R \,\&\, y \notin K_R$ and $\exists z \in K_R \,\&\, z \neq x \,\&\, Rzx$ do not disallow Rzy. So $(R \setminus \tau(R))^{\#} \cup \tau(R)$ is not necessarily a quasi-order. The missing part is left for the reader to complete. □

Proposition 195 $((R \setminus \tau(R))^{\#} \cup \tau(R))^{\#} = R^{\#}$.

Proof Clearly $R \subseteq ((R \setminus \tau(R))^{\#} \cup \tau(R))^{\#}$ and it can be directly checked that if $a \in ((R \setminus \tau(R))^{\#} \cup \tau(R))^{\#} \setminus R$ then $a \in R^{\#} \setminus R$ and conversely. □

8.3 Transitive Completion and Approximate Semantics

The interaction of the rough approximations in a PRAX and the rough approximations in the transitive completion can be expected to follow some order. *The definite or rough objects most closely related to the difference of lower approximations and those related to the difference of upper approximations can be expected to be related* *in a nice way*. It is shown that this *nice way* is not really a *rough way*. But the results proved remain relevant for the formulation of semantics that involves that of the transitive completion as in [61, 63]. A rough theoretical alternative is possible by simply starting from sets of the form $A^* = (A^l \setminus A^{l_\#}) \cup (A^{u_\#} \setminus A^u)$ and taking their lower ($l_\#$) and upper ($u_\#$) approximations—the resulting structure would be a partial algebra derived from a Nelson algebra over an algebraic lattice ([95]).

Proposition 196 *For an arbitrary proto-transitive reflexive relation R on a set S, (# subscripts will be used for neighborhoods, approximation operators and rough equalities of the weak transitive completion) all of the following hold:*

$$(\forall x \in S)\, [x]_R \subseteq [x]_{R^\#} \tag{Nbd}$$

$$(\forall A \subseteq S)\, A^l \subseteq A^{l_\#} \,\&\, A^u \subseteq A^{u_\#} \tag{App}$$

$$(\forall A \subseteq S)(\forall B \in [A]_\approx)(\forall C \in [A]_{\approx_\#})\, B^l \subseteq C^{l_\#} \,\&\, B^u \subseteq C^{u_\#} \tag{REq}$$

The reverse inclusions are false in general in the second assertion in a specific way. Note that the last condition induces a more general partial order $\preceq$ over $\wp(\wp(S))$ via $A \preceq B$ if and only if $(\forall C \in A)(\forall E \in B)\, C^l \subseteq E^{l_\#} \,\&\, C^u \subseteq E^{u_\#}$.

Proof The first of these is direct. For simplicity, the successor neighborhoods of x will be denoted by $[x]$ and $[x]_\#$ respectively. The possibility of tracking of the second assertion in the first part is also considered.

- If $z \in A^{l_\#}$ then $z \in A^l$ as $[x]_\# \subseteq A$ implies $[x] \subseteq A$.
- If $z \in A^l$ then $(\exists x)\, z \in [x] \subseteq A^l$.
- For this x, $z \in [x]_\#$, but it is possible that $[x]_\# \subseteq A$ or $[x]_\# \nsubseteq A$.
- If $[x]_\# \nsubseteq A$, and $(\exists b \notin A)\, R_\# ax \,\&\, Rab \,\&\, Rbx$ then a contradiction happens as Rbx means $b \in [x]$.
- If $[x]_\# \nsubseteq A$, and $(\exists b \in A)\, R_\# ax \,\&\, Rab \,\&\, Rbx$ all that is required is a $c \notin A \,\& Rcb$ that is compatible with $R_\# cx$ and $A^l \nsubseteq A^{l_\#}$.

□

Definition 197 By the *l-scedastic approximation* $\hat{l}$ and the *u-scedastic approximation* $\hat{u}$ of a subset $A \subseteq S$ will be meant the following approximations:

$$A^{\hat{l}} = (A^l \setminus A^{l_\#})^l, \quad A^{\hat{u}} = (A^{u_\#} \setminus A^u)^{u_\#}.$$

The above cross difference approximation is the best possible from closeness to properties of rough approximations.

Theorem 198 *For an arbitrary subset $A \subseteq S$ of a PRAX S,the following statements and diagram of inclusion ($\rightarrow$) hold (Fig. 6):*

- $A^{l_\# l} = A^{l_\#} = A^{l l_\#} = A^{l_\# l_\#}$
- *If $A^u \subset A^{u_\#}$ then $A^{u u_\#} \subseteq A^{u_\# u_\#}$.*

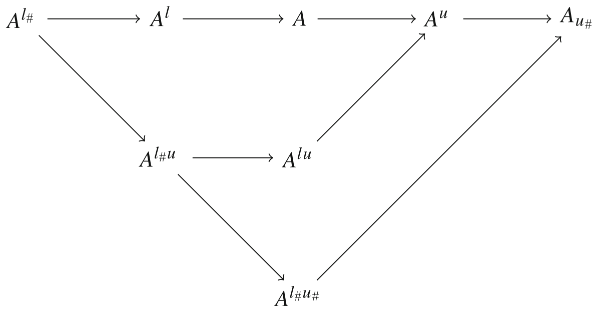

Fig. 6 Relation between approximate approximations

Proof It is clear that $A^l \subseteq A^u \subseteq A^{u\#}$. So $A^l \nsubseteq A^{u\#} \setminus A^u$.

$$\begin{aligned} x \in (A^l \setminus A^{l\#})^l &\Rightarrow (\exists y)\,[y]_\# \nsubseteq A \,\&\, x \in [y] \subset A \,\&\, x \in [y]_\# \\ &\Rightarrow x \in A^{u\#} \,\&\, x \in A^u \\ &\Rightarrow x \notin A^{u\#} \setminus A^u. \\ \text{But } [y]_\# \subset A^{u\#} &\ (\exists z)\, z \in A^{u\#} \,\&\, z \notin A^u \,\&\, z \in [y]_\#. \\ \text{So } [y]_\# \subset (A^{u\#} \setminus A^u)^{u\#} &\text{ and it is possible that } [y]_\# \nsubseteq (A^{u\#} \setminus A^u)^u. \end{aligned}$$

□

Theorem 199 *For an arbitrary subset $A \subseteq S$ of a PRAX S,*

$$(A^l \setminus A^{l\#})^l \nsubseteq (A^{u\#} \setminus A^u)^{u\#} \longrightarrow A^{u\#} = A^u.$$

$$A^{u\#} \neq A^u \longrightarrow A^l \setminus A^{l\#})^l \subseteq (A^{u\#} \setminus A^u)^{u\#}.$$

Proof

- Let $S = \{a, b, c, e, f\}$ and
- R be the transitive completion satisfying Rab, Rbc, Ref.
- If $B = \{a, b\}$, $B^{\hat{l}} = B$, but $B^{u\#} = \{a, b, c\} = B^u$.
- So $B^{\hat{u}} = \emptyset$.
- The second part follows from the proof of the above proposition under the restriction in the premise.

□

Theorem 200 *Key properties of the scedastic approximations follow:*

1. $(\forall B \in \wp(S))(B^{\hat{l}} = B \nrightarrow B^{\hat{u}} = B)$.
2. $(\forall B \in \wp(S))(B^{\hat{u}} = B \rightarrow B^{\hat{l}} = B)$.
3. $(\forall B \in \wp(S))\, B^{\hat{l}\hat{l}} = B^{\hat{l}}$.
4. $(\forall B \in \wp(S))\, B^{\hat{u}\hat{u}} \neq B^{\hat{u}}$.
5. *It is possible that* $(\exists B \in \wp(S)\, B^{\hat{u}\hat{u}} \subset B^{\hat{u}})$.

Proof

1. The counter example in the proof of the above theorem works for this statement.
2. $x \in B \leftrightarrow x \in (B^{u\#} \setminus B^{u})^{u\#} \leftrightarrow (\exists y \in B^{u\#})(\exists z \in B^{u\#} \setminus B^{u})\, x, z \in [y]_{\#} \,\&\, z \in B^{u\#} \,\&\, z \notin B^{u}$. But this situation requires that elements of the form z be related to x and so it is essential that $B^{u\#} = B^{u}$.
3. $B^{\hat{l}\hat{l}} = (B^{\hat{l}l} \setminus B^{\hat{l}l\#})^{l} = ((B^{l} \setminus B^{l\#})^{l} \setminus \emptyset)^{l} = B^{\hat{l}}$. The missing step is of proving $(B^{l} \setminus B^{l\#})^{ll\#} = \emptyset$.

4–5. The last two assertions shall be proven together by way of a counterexample and an essential pattern of deviation.

Let $S = \{a, b, c, e, f\}$ and R be a reflexive relation s.t. $Rab,\ Rbc,\ Ref$.
If $A = \{a, e\}$, then $A^{u\#} = \{a, b, c, e\}$ and $A^{u} = \{a, b, e\}$.
Therefore $A^{\hat{u}} = \{c\} \,\&\, A^{\hat{u}\hat{u}} = \emptyset \,\&\, A^{\hat{u}\hat{u}} \subset A^{\hat{u}}$.
In general if B is some subset, then $x \in B^{\hat{u}} = (A^{u\#} \setminus A^{u})^{u\#} \Rightarrow (\exists y \in A^{u\#})(\exists z)\, y \in [z]_{\#} \,\&\, y \notin A^{u} \,\&\, y \notin A \,\&\, z \in A \,\&\, y \notin [z] \,\&\, y \in [x]_{\#}$.

□

An interesting problem can be given A for which $A^{u\#} \neq A^{u}$, when does there exist a B such that

$$B^{l} = (A^{l} \setminus A^{l\#})^{l} = A^{\hat{l}} \,\&\, B^{u} = (A^{u\#} \setminus A^{u})^{u\#} = A^{\hat{u}}?$$

8.4 Rough Dependence

The concept of *rough dependence* was introduced in general rough set theory by the present author in [97]. By the term *rough dependence*, the present author seeks to capture the relation between two objects (crisp or rough) that have some representable rough objects in common. There is no process for similarity with the concept *mutual exclusivity* of probability theory and in rough evolution there temporality is not usually assumed. The present author would like to eventually analyze the extent to which ontology of not-necessarily-rough origin could be integrated in a seamless way. But in this chapter, basic concepts will be introduced, compared with probabilistic concepts and the semantic value of introduced functions and predicates will be considered.

Overall the following problems are basic and relevant for use in semantics:

- Which concepts of rough dependence provide for an adequate semantics of rough objects in the PRAX context?
- More generally how does this relation vary over other rough sets?
- Characterize the connection between granularity and rough dependence?

As mentioned earlier, *relation based RST* refers to rough theories originating from generalized approximation spaces of the form $U = \langle \underline{U}, R \rangle$, with $\underline{U}$ being a set and R being any binary relation on $\underline{U}$.

Definition 201 The $\tau\nu$-*infimal degree of dependence* $\beta_{i\tau\nu}$ of A on B will be defined as

$$\beta_{i\tau\nu}(A, B) = \inf_{\nu(S)} \oplus\{C : C \in \tau(S) \,\&\, \mathbf{P}CA \,\&\, \mathbf{P}CB\}.$$

Here the infimum means the largest $\nu(S)$ element contained in the aggregation.

The $\tau\nu$-*supremal degree of dependence* $\beta_{s\tau\nu}$ of A on B will be defined as

$$\beta_{s\tau\nu}(A, B) = \sup_{\nu(S)} \oplus\{C : C \in \tau(S) \,\&\, \mathbf{P}CA \,\&\, \mathbf{P}CB\}.$$

Here the supremum means the least $\nu(S)$ element containing the sets.

The definition extends to RYS [91] in a natural way.

Note that all of the definitions do not use real-valued rough measures and the cardinality of sets in accord with one of the principles of avoiding contamination. The ideas of dependence are more closely related to certain semantic operations in classical rough sets. But these were never seen to be of much interest. The connections with probability theories has been part of a number of papers including [127–129, 156, 170], however neither dependence nor independence have received sufficient attention. This is the case with other papers on entropy. It should be noted that the idea of independence in statistics is seen in relation to probabilistic approaches, but dependence has largely not been given much importance in applications.

The positive region of a set X is X^l, while the negative region is X^{uc} – this region is independent from x in the sense of attributes being distinct, but not in the sense of derivability or inference by way of rules. In considerations of dependence or independence of a set relative another, a basic question would also be about possible balance between the two meta principles of independence in the rough theory and relation to the granular concepts of independence.

Definition 202 Two elements x, y in a RBRST or CBRST S will be said to be *PN-independent* $I_{PN}(xy)$ if and only if

$$x^l \subseteq y^{uc} \,\&\, y^l \subseteq x^{uc}.$$

Two elements x, y in a RBRST or CBRST S will be said to be *PN-dependent* $\varsigma_{PN}(xy)$ if and only if

$$x^l \nsubseteq y^{uc} \,\&\, y^l \nsubseteq x^{uc}.$$

Theorem 203 *Over the RYS corresponding to classical rough sets, the following properties of dependence degrees hold when* $\tau(S) = \mathcal{G}(S)$*—the granulation of* S *and* $\nu(S) = \delta_l(S)$*—the set of lower definite elements. The subscripts* $\tau\nu$ *and braces in* $\beta_{i\tau\nu}(x, y)$ *are omitted in the following:*

1. $\beta_i xy = x^l \cap y^l = \beta_s xy$ *(subscripts* i, s *on* β *can therefore be omitted).*
2. $\beta xx = x^l$.
3. $\beta xy = \beta yx$.
4. $\beta(\beta xy)x = \beta xy$.
5. $\mathbf{P}(\beta xy)(\beta x(y \oplus z))$.
6. $(\mathbf{P}y^l z \longrightarrow \mathbf{P}(\beta xy)(\beta xz))$.
7. $\beta xy = \beta x^l y^l = \beta xy^l$.
8. $\beta 0x = 0;\ \beta x1 = x^l$.
9. $(\mathbf{P}xy \longrightarrow \beta xy = x^l)$.

This is proved in the next section.

Theorem 204 *For classical rough sets, a semantics over the classical semantic domain can be formulated with no reference to lower and upper approximation operators using the operations* $\cap, c, \beta$ *on the power-set of* S*,* S *being an approximation space.*

Proof It has already been shown that l is representable in terms of β. So the result follows. □

Dependence in PRAX

When $\nu(S) = \delta_l(S)$ and $\tau(S) = \mathcal{G}(S)$—the successor neighborhood granulation, then the situation in PRAX contexts is similar, but it would not be possible to define u from l and complementation. However when $\nu(S) = \delta_u(S)$, then the situation is very different.

Theorem 205 *Over the* RYS *corresponding to* PRAX *with* $\mathbf{P} = \subseteq$, $\oplus = \cup$ *and* $\odot = \cap$*, the following properties of dependence degrees hold when* $\tau(S) = \mathcal{S}$*—the granulation of* S *and* $\nu(S) = \delta_l(S)$*—the set of lower definite elements. In fact this holds in any reflexive* RBRST*. The subscripts* $\tau\nu$ *and braces in* $\beta_{i\tau\nu}(x, y)$ *are omitted in the following:*

1. $\beta_i xy = x^l \cap y^l = \beta_s xy$ *(subscripts* i, s *on* β *can therefore be omitted).*
2. $\beta xx = x^l; \beta xy = \beta yx$.
3. $(x \odot y = 0 \longrightarrow \beta_i xy = 0)$*, but the converse is false.*
4. $\beta(\beta xy)x = \beta xy$.
5. $\mathbf{P}(\beta xy)(\beta x(y \oplus z))$.
6. $(\mathbf{P}y^l z \longrightarrow \mathbf{P}(\beta xy)(\beta xz))$.
7. $\beta xy = \beta x^l y^l = \beta xy^l$.
8. $\beta 0x = 0;\ \beta x1 = x^l$.
9. $(\mathbf{P}xy \longrightarrow \beta xy = x^l)$.

Proof

1. $\beta_i xy$ is the union of the collection of successor neighborhoods generated by elements x and y that are included in both of them. So $\beta_i xy = x^l \cap y^l = \beta_s xy$.
2. $\beta xx = x^l; \beta xy = \beta yx$. is obvious

3. If $(x \odot y = 0$, then x and y have no elements in common and cannot have common successor neighborhoods. If $\beta_i xy = 0$, then x, y have no common successor neighborhoods, but can still have common elements. So the statement follows.
4. $\beta xy \subseteq x^l \subseteq x$ by the first statement. So $\beta(\beta xy)x = \beta xy$.
5. $\mathbf{P}(\beta xy)(\beta x(y \oplus z))$ follows by monotonicity.
6. If $\mathbf{P}y^l z$ is the same thing as $y^l \subseteq z$. $\beta xy = x^l \cap y^l$ and $\beta xz = x^l \cap z^l$ by the first statement. So $(\mathbf{P}y^l z \longrightarrow \mathbf{P}(\beta xy)(\beta xz))$ holds.
7. $\beta xy = \beta x^l y^l = \beta xy^l$ holds because l is an idempotent operation in a PRAX.
8. Rest of the statements are obvious.

□

Even though the properties are similar for reflexive RBRST when $\nu(S) = \delta_l(S)$ and $\tau(S) = \mathcal{G}(S)$, there are key differences that can be characterized in terms of special sets.

- $\beta xy = z$ if and only if $(\forall a \in z)(\exists b \in z)\, a \in [z] \subseteq x \cap y$.
- So a minimal $K_z \subseteq z$ satisfying $(\forall a \in z)(\exists b \in K_z)\, a \in [b] \subseteq x$ and $(\forall e \in K_z)\, [e] \subseteq x \cap y$ can be selected. Minimality being with respect to the inclusion order.
- Let $\mathcal{P}_z$ be the collection of all such K_z and let $\mathcal{B}_z$ be the subcollection of $\mathcal{P}_z$ satisfying the condition: if $K \in \mathcal{B}_z$ then $(\forall a \in K)(\forall b \in [a])(\exists J \in \mathcal{B}_z)\, b \in J$. $\mathcal{P}_z$ will be called the local basis and $\mathcal{B}_z$, the local super basis of z.

Proposition 206 *For classical rough sets* $(\forall z)\, \mathcal{B}_z = \mathcal{P}_z$ *and conversely.*

Theorem 207 *In the context of 205, if* $\nu(S) = \delta_u(S)$ *and* $\tau(S)$ *is as before, then all of the following hold (* βxy *is an abbreviation for* $\beta_i xy$*)*

1. $\mathbf{P}(\beta xy)(\beta_{i\delta_l(S)} xy)$,
2. $\mathbf{P}(\beta xx)(x^l)$; $\beta xy = \beta yx$.
3. $(x \odot y = 0 \longrightarrow \beta_i xy = 0)$, *but the converse is false.*
4. $\beta(\beta xy)x = \beta xy$.
5. $\mathbf{P}(\beta xy)(\beta x(y \oplus z))$.
6. $(\mathbf{P}y^l z \longrightarrow \mathbf{P}(\beta xy)(\beta xz))$.
7. $\beta xy = \beta x^l y^l$; $\mathbf{P}(\beta xy^l)(\beta x^u y^u)$.
8. $\beta 0x = 0$; $\mathbf{P}(\beta x1)(x^l)$.
9. $(\mathbf{P}xy \longrightarrow \mathbf{P}(\beta zx)(\beta zy))$
10. $(\beta xy)^l = \beta xy$.

Proof

1. By definition $\beta_{i\tau\nu}(A, B) = \inf_{\nu(S)} \oplus \{C : C \in \tau(S)\, \&\, \mathbf{P}CA\, \&\, \mathbf{P}CB\}$, so βxy is the greatest upper definite set contained in the union of common successor neighborhoods included in x and y. So it is necessarily a subset of $x^l \cap y^l$. In a PRAX, u is not idempotent and in general $x^u \subseteq x^{uu}$. So $\mathbf{P}(\beta xy)(\beta_{i\delta_l(S)} xy)$.
2. The statements $\mathbf{P}(\beta xx)(x^l)$ and $\beta xy = \beta yx$ follow from the above.

3. The proof is similar to that of third statement of 205.
4. In constructing $\beta(\beta xy)x$ from βxy, no effort is made to look for upper definite subsets strictly contained in the latter. So the property follows.
5. $\mathbf{P}(\beta xy)(\beta x(y \oplus z))$ follows by monotonicity.
6. Obvious from previous statements.
7. Note that $\beta x^u y^u$ is a subset of $x^u \cap y^u$ and in general contains βxy.
8. Is a special case of the first statement. 0 is the empty set and 1 is the top.
9. Follows by monotonicity.
10. Upper definite subsets are necessarily lower definite, so $(\beta xy)^l = \beta xy$.

□

The main properties of PN-dependence is as below:

Theorem 208 *In the context of 205, all of the following hold (the subscript 'PN' in ς_{PN} in the following):*

1. ςxx.
2. $(\varsigma xy \leftrightarrow \varsigma yx)$.
3. *In general, $\varsigma xy \,\&\, \varsigma zy$ does not imply ςxz. But $\neg\varsigma xz$ is more likely if a bit of frequentism is assumed.*
4. *In general, $\varsigma xy \nrightarrow \varsigma x^u y^u$ and $\varsigma x^u y^u \nrightarrow \varsigma xy$.*
5. $(x \cdot y = 0 \longrightarrow \neg\varsigma xy)$.
6. $(\mathbf{P}xy \longrightarrow \varsigma xy)$.

Theorem 209 *In the context of 205, if $\beta xy \neq 0$ then ςxy, but the converse need not hold. In classical rough sets, the converse holds as well.*

Proof If $\beta xy \neq 0$, then it follows that $x^l \cap y^l \neq \emptyset$ under the assumptions. If it is assumed that $x^l \subseteq y^{uc} \vee y^l \subseteq x^{uc}$, then in each of the three cases a contradiction happens. So the first part of the result follows.

In the classical case, if $x^l \subseteq y^{uc}$ is not empty, then it should be a union of successor neighborhoods and similarly for $y^l \subseteq x^{uc}$. These two parts should necessarily be common to x^l and y^l. So the converse holds for classical rough sets. The proof does not work for PRAX and the reasons for failure have been made clear. □

For a comparison of these concepts of dependence with those in probability theories, the reader is referred to the research papers [99, 101] by the present author.

8.5 Dependency Semantics of PRAX

Dependency based semantics are developed in at least two ways in this section. The *internalization based semantics* is essentially about adjoining predicates to the Nelson algebra corresponding to $\mathfrak{R}_w(S)$. The *cumulation based semantics* is essentially about cumulating both the semantics of $\mathfrak{R}(S)$, adjusting operations and adjoining predicates. Broader dependency based predicates are used in this case, but the value of the method is in fusion of the methodologies.

The central blocks of development of the cumulation based dependency semantics are the following:

- Take $\mathfrak{R}(S) \cup \mathfrak{R}_w(S)$ as the universal set of the intended partial/total algebraic system.
- Use a one point completion of τ to distinguish between elements of $\mathfrak{R}_w(S) \setminus \mathfrak{R}(S)$ and those in $\mathfrak{R}(S)$.
- Extend the idea of operational dependency to pairs of sets.
- Extend operations of aggregation, commonality and dual suitably.
- Interpret semantic dependence?

The first step is obvious, but involves elimination of other potential sets arising from the properties of the map τ.

One Point Completion

Since $R \subseteq R_w$ and R_w is transitive, so

Proposition 210

$$\alpha \in \mathfrak{R}(S) \cap \mathfrak{R}_w(S) \text{ if and only if } \tau(\alpha) = \alpha.$$

Adjoin an element 0 to $\mathfrak{R}(S) \cup \mathfrak{R}_w(S)$ to form $\mathfrak{R}^*(S)$ and extend τ (interpreted as a partial operation) to $\overline{\tau}$ as follows:

$$\overline{\tau}(\alpha) = \begin{cases} \tau(\alpha) & \text{if } \alpha \in \mathfrak{R}(S), \\ 0 & \text{if } \alpha \notin \mathfrak{R}(S). \end{cases}$$

Note that this operation suffices to distinguish between elements common to $\mathfrak{R}(S)$ and $\mathfrak{R}_w(S)$, and those exclusively in $\mathfrak{R}(S)$ and not in $\mathfrak{R}_w(S)$.

Dependency on Pairs

It is possible to consider all dependencies relative to the Nelson algebra or $\mathfrak{R}(S)$. In the proposed approach the former is considered first towards avoiding references to the latter.

Definition 211 By the *paired infimal degree of dependence* $\beta^{+}_{i\tau_1\tau_2\nu_1\nu_2}$ of α on β will be defined as

$$(\beta_{i\tau_1\nu_1}(e_1\alpha,\ e_1\beta),\ \beta_{i\tau_2\nu_2}(e_2\alpha,\ e_2\beta)).$$

Here the infimums involved are the largest $\nu_1(S)$ and $\nu_2(S)$ elements contained in the aggregation and the $e_j\alpha$ is the j-th component of α.

The following well defined specialization with $\tau_1(S) = \tau_2(S) = \mathcal{G}_w(S)$, $\nu_1 = \delta_{lw}(S)$ and $\nu_2 = \Gamma_{uw}(S)$ will also be of interest. For specializing the dependencies between a element in $\mathfrak{R}(S)$ and its image in $\mathfrak{R}_w(S)$, it suffices to define:

Definition 212 Under the above assumptions, by the *relative semantic dependence* $\varrho(\alpha)$ of $\alpha \in \mathfrak{R}(S)$, will be meant

$$\varrho(\alpha) = \beta_i^+(\alpha, \tau(\alpha)).$$

The idea of relative semantic dependence refers to elements in $\mathfrak{R}(S)$ and it can be reinterpreted as a relation on $\mathfrak{R}_w(S)$.

Internalization Based Semantics

Definition 213 By the *ϱ/σ-semantic dependences* $\varrho(\alpha)$, $\sigma(\alpha)$ of $\alpha \in \mathfrak{R}(S)$, will be meant $\varrho(\alpha) = \beta_i^+(\alpha, \tau(\alpha))$ and

$$\sigma(\alpha) = \beta_i^+(\alpha, ((\varphi(e_1\alpha) \setminus e_1\alpha)^l, (\varphi(e_2\alpha) \setminus e_1\alpha)^u))$$

respectively. Such relations are optional in the internalization process.

A relation Υ on $\mathfrak{R}_w(S)$ will be said to be a *relsem-relation* if and only if

$$\Upsilon\tau(\alpha)\gamma \leftrightarrow (\exists\beta \in \tau^{\dashv}\tau(\alpha))\, \gamma = \varrho(\beta).$$

Note that, $\tau(\alpha) = \tau(\beta)$ by definition of $\tau^{\dashv}$.

Through the above definitions the following internalized approximate definition has been arrived at:

Definition 214 By an *Approximate Proto Vague Semantics* of a PRAX S will be meant an algebraic system of the form

$$\mathfrak{P}(S) = \langle \mathfrak{R}_w(S),\ \Upsilon \vee, \wedge, c, \bot, \top \rangle,$$

with $\langle \mathfrak{R}_w(S),\ \vee_w, \wedge_w, c, \bot, \top \rangle$ being a Nelson algebra over an algebraic lattice and Υ being as above.

Theorem 215 *Υ has the following properties:*

$$\alpha = \tau(\alpha) \longrightarrow \Upsilon\alpha\alpha.$$

$$\Upsilon\alpha\gamma \longrightarrow \gamma \wedge_w \alpha = \gamma.$$

$$\Upsilon\alpha\gamma \,\&\, \Upsilon\gamma\alpha \longrightarrow \alpha = \gamma.$$

$$\Upsilon\bot\bot \,\&\, \Upsilon\top\top.$$

$$\Upsilon\alpha\gamma \,\&\, \Upsilon\beta\gamma \longrightarrow \Upsilon(\alpha \vee_w \beta)\gamma.$$

Proof

- If $\alpha = \tau(\alpha)$, then $\alpha = \varrho(\alpha) = \beta_i^+(\alpha, \tau(\alpha))$. So $\Upsilon\alpha\alpha$.
- If $\Upsilon\alpha\gamma$, then it follows from the definition of β_i^+, that the components of gamma are respectively included in those of α. So $\gamma \wedge \alpha = \gamma$.
- Follows from the previous.
- Proof is easy.
- From the premise we have $(\exists\mu \in \tau^{\dashv}\tau(\alpha))\, \gamma = \varrho(\mu)$ and $(\exists\nu \in \tau^{\dashv}\tau(\beta))\, \gamma = \varrho(\nu)$. This yields $(\exists\lambda \in \tau^{\dashv}\tau(\alpha \vee_w \beta))\, \gamma = \varrho(\lambda)$ as can be checked from the components.

□

$\Upsilon_{\tau(\alpha)} = \{\gamma\, ;\, \Upsilon\tau(\alpha)\gamma\}$ is the approximate reflection of the set of τ-equivalent elements in $\mathfrak{R}(S)$ identified by their dependence degree. In the approximate semantics aggregation and commonality are not lost track of as the above theorem shows. For a falls-down semantics, the natural candidates include the ones corresponding to largest equivalence or the largest semi-transitive contained in R. The latter will appear in a separate paper. For the former, the general technique (using $\sigma(\alpha)$) extends to PRAX as follows:

Definition 216

- Define a map from set of neighborhoods to l-definite elements $\int([x]_o) = \cup_{y\in[x]_o}[y]$ and extend it to images of l_o, u_o via,

$$\oint(A^{l_o}) = \cup_{[y]_o \subseteq A^{l_o}} \int([y]_o).$$

- Extend this to a map $\ltimes : \mathfrak{R}_o(S) \mapsto \mathfrak{R}(S)$ via $\ltimes(\alpha) = (\oint(e_1\alpha), \oint(e_2\alpha))$.
- Define $\Pi\alpha\nu$ on $\mathfrak{R}_o(S)$ if and only if $(\exists\gamma \in \ltimes^{\dashv} \ltimes(\alpha))\, \beta_i^+(\alpha, \gamma) = \nu$. Let $\Pi_\alpha = \{\nu\, ;\, \Pi\alpha\nu\}$.
- By a *Direct Falls Down* semantics of PRAX, will be meant an algebraic system of the form

$$\mathfrak{I}(S) = \langle \mathfrak{R}_o(S),\, \Pi, \vee, \wedge, c, \bot, \top \rangle ,$$

  with $\langle \mathfrak{R}_o(S),\, \vee_o, \wedge_o, \rightarrow, c, \bot, \top \rangle$ being a semi-simple Nelson algebra [124].
- The falls down semantics determines a cover $\mathfrak{I}^*(S) = \{\Pi_\alpha\, ;\, \alpha \in \mathfrak{R}_o(S)\}$

Theorem 217 *In the above context, all of the following hold:*

- $\Pi\alpha\alpha$.
- $(\Pi\alpha\mu \,\&\, \Pi\mu\alpha \longrightarrow \alpha = \mu)$.

- ($\Pi\alpha\gamma \longrightarrow \gamma \subseteq \alpha$). *The converse is false.*
- $\alpha \neq \bot \,\&\, \Pi\alpha\gamma \,\&\, \Pi\alpha\mu \longrightarrow \beta_i^+(\gamma, \mu) \neq \bot$.
- $\mu \in \Pi_\alpha \,\&\, \mu \subseteq \nu \subseteq \alpha \longrightarrow \nu \in \Pi_\alpha$.

The theorems mean that a purely order theoretic representation theorem is not possible for the falls down semantics, but other possibilities remain open.

Cumulation Based Semantics

The idea of cumulation is correctly a way of enhancing the original semantics based on proto-vagueness algebras with the Nelson algebraic semantics and the operational dependence. This is defined for a central problem relating to the underlying semantic domains.

Definition 218 By a *cumulative proto-vague algebra* will be meant a partial algebra of the form

$$\mathfrak{C}(S) = \left\langle \mathfrak{R}^*(S), \overline{tau}, \oplus, \odot, \otimes, \dagger, \bot, \top \right\rangle.$$

Problem:
When can the cumulation based semantics be deduced from (that is the extra operations can be defined from the original ones) within a full proto-vagueness algebra?

9 Connections with Non-monotonic Logic

The representation of rough objects in a PRAX has important connections with non-monotonic operators. This is considered in the present section

It has already been shown in the previous section that the representation of rough objects by definite objects is not possible in a relatively standard way in a PRAX. So it is important to look at possibilities based on other types of derived approximations. This problem was solved to an extent and connections with key properties of non-monotonic reasoning have been established in [99] by the present author.

Definition 219 If $x \in \wp(S)$, then

- Let $\Pi^o_\heartsuit(x) = \{y\,;\, x \subseteq y \,\&\, x^l = y^l \,\&\, y^u \subseteq x^{uu}\}$.
- Form the set of maximal elements $\Pi_\heartsuit(x)$ of $\Pi^o_\heartsuit(x)$ with respect to the inclusion order.
- Select a unique element $\chi(\Pi_\heartsuit(x))$ through a fixed choice function χ.
- Form $(\chi(\Pi_\heartsuit(x)))^u$.

- $x^{\heartsuit\chi} = (\chi(\Pi_{\heartsuit}(x)))^u$ will be said to be the *almost upper approximation* of x relative χ.
- $x^{\heartsuit\chi}$ will be abbreviated by $x^{\heartsuit}$ for fixed χ.

The choice function is be said to be *regular* if and only if $(\forall x, y)\,(x \subseteq y \,\&\, x^l = y^l \longrightarrow \chi(\Pi_{\heartsuit}(x)) = \chi(\Pi_{\heartsuit}(y)))$. Regularity will be assumed unless specified otherwise in what follows.

Definition 220 If $x \in \wp(S)$, then

- Let $\Pi^o_{\diamondsuit}(x) = \{y\,;\, x \subseteq y \,\&\, x^l = y^l\}$.
- Form the set of maximal elements $\Pi_{\diamondsuit}(x)$ of $\Pi^o_{\diamondsuit}(x)$ with respect to the inclusion order.
- Select a unique element $\chi(\Pi_{\diamondsuit}(x))$ through a fixed choice function χ.
- $x^{\diamondsuit\chi} = \chi(\Pi_{\diamondsuit}(x))$ will be said to be the *lower limiter* of x relative χ.
- $x^{\diamondsuit\chi}$ will be abbreviated by $x^{\diamondsuit}$ for fixed χ.

Definition 221 If $x \in \wp(S)$, then

- Let $\Pi^o_{\flat}(x) = \{y\,;\, y \subseteq x \,\&\, x^u = y^u\}$.
- Form the set of maximal elements $\Pi_{\flat}(x)$ of $\Pi^o_{\flat}(x)$ with respect to the inclusion order.
- Select a unique element $\xi(\Pi_{\flat}(x))$ through a fixed choice function ξ.
- $x^{\flat\xi} = \xi(\Pi_{\flat}(x))$ will be said to be the *upper limiter* of x relative χ.
- $x^{\flat\xi}$ will be abbreviated by $x^{\flat}$ for fixed ξ.

Proposition 222 *In the context of the above definition, the almost upper approximation satisfies all of the following:*

$$(\forall x)\, x \subseteq x^{\heartsuit} \quad \text{(Inclusion)}$$

$$(\forall x)\, x^{\heartsuit} \subseteq x^{\heartsuit\heartsuit} \quad \text{(Non-Idempotence)}$$

$$(\forall x\, y)\,(x \subseteq y \subseteq x^{\heartsuit} \longrightarrow x^{\heartsuit} \subseteq y^{\heartsuit}) \quad \text{(Cautious Monotony)}$$

$$(\forall x)\, x^u \subseteq x^{\heartsuit} \quad \text{(Supra Pseudo Classicality)}$$

$$S^{\heartsuit} = S \quad \text{(Top.)}$$

Proof

- Inclusion: Follows from the construction. If one element granules or successor neighborhoods are included in x, then these must be in the lower approximation. If a granule y is not included in x, but intersects it in f, then it is possible to include f in each of $\Pi_{\heartsuit}(x)$. So inclusion follows.
- Non-Idempotence: The reverse inclusion does not happen as $x^u \subseteq x^{uu}$.
- Cautious monotony: It is clear that monotony can fail in general because of the choice aspect, but if $x \subseteq y \subseteq x^{\heartsuit}$ holds, then $x^l \subseteq y^l$ and $y^{\heartsuit}$ has to be equal to $x^{\heartsuit}$ or include more granules because of regularity of the choice function.

- Supra Pseudo Classicality: The adjective *pseudo* is used because u is not a classical consequence operator. In the construction of $x^{\heartsuit}$, the selection is from super-sets of x^l that can generate maximal upper approximations. Upper approximation of the selected sets are done next. So that includes x^u in general. □

The conditions have been named in relation to the standard terminology used in non-monotonic reasoning. The upper approximation operator u is similar to classical consequence operator, but lacks idempotence. So the fourth property has been termed as *supra pseudo classicality* as opposed to *supra classicality*. This means the present domain of reasoning is more general than that of [80].

Theorem 223 *In the context of 222, the following additional properties hold:*

$$(\forall x)\, x^{\heartsuit} \subseteq x^{u\heartsuit} \qquad \text{(Sub Left Absorption)}$$

$$(\forall x)\, x^{\heartsuit} \subseteq x^{\heartsuit u} \qquad \text{(Sub Right Absorption)}$$

$$\boxdot(\forall x, y)\, (x^u = y^u \nrightarrow x^{\heartsuit} = y^{\heartsuit}) \qquad \text{(No Left Logical Equivalence)}$$

$$\boxdot(\forall x, y)\, (x^{\heartsuit} = y^{\heartsuit} \nrightarrow x^l = y^l) \qquad \text{(No Jump Equivalence)}$$

$$\boxdot(\forall x, y, z)\, (x \subseteq y^{\heartsuit} \,\&\, z \subseteq x^u \nrightarrow z \subseteq y^{\heartsuit}) \qquad \text{(No Weakening)}$$

$$\boxdot(\forall x, y)\, (x \subseteq y \subseteq x^u \nrightarrow x^{\heartsuit} = y^{\heartsuit}) \qquad \text{(No subclassical cumulativity)}$$

$$(\forall x, y)\, x^{\heartsuit} \cap y^{\heartsuit} \subseteq (x^u \cap y^u)^{\heartsuit} \qquad \text{(Distributivity)}$$

$$(\forall x, y, z)\, (x \cup z)^{\heartsuit} \cap (y \cup z)^{\heartsuit} \subseteq (z \cup (x^u \cap y^u))^{\heartsuit} \qquad \text{(Weak Distributivity)}$$

$$(\forall x, y, z)\, (x \cup y)^{\heartsuit} \cap (x \cup z)^{\heartsuit} \subseteq (x \cup (y \oplus z))^{\heartsuit}$$
(Disjunction in Antecedent)

$$(\forall x, y)\, (x \cup y)^{\heartsuit} \cap (x \cup y^c)^{\heartsuit} \subseteq x^{\heartsuit} \qquad \text{(Proof by Cases)}$$

$$\textit{If } y \subseteq (x \cup z)^{\heartsuit},\ \textit{then } x \implies y \subseteq z^{\heartsuit} \qquad \text{(Conditionalization.)}$$

Proof

Sub Left Absorption For any x, $x^{\heartsuit}$ is the upper approximation of a maximal subset y containing x such that $x^l = y^l$ and $x^{u\heartsuit}$ is the upper approximation of a maximal subset z containing x^u such that $x^{ul} = x^u = z^l$. Since, $x^l \subseteq x^{ul}$ and $x \subseteq x^u$, so $x^{\heartsuit} \subseteq x^{u\heartsuit}$ follows.

Sub Right Absorption Follows from the properties of u.

No Left Logical Equivalence Two subsets x, y can have unequal lower approximations and equal upper approximations and so the implication does not hold in general. $\boxdot$ should be treated as an abbreviation for *in general*.

No Jump Equivalence The reason is similar to that of the previous negative result.

No weakening In general if $x \subseteq y^{\heartsuit} \,\&\, z \subseteq x^u$, then it is possible that $x^u \subseteq y^{\heartsuit}$ or $y^{\heartsuit} \subseteq x^u$. So one cannot be sure about $z \subseteq y^{\heartsuit}$.

No Subclassical Cumulativity If $x \subseteq y \subseteq x^u$, then $x^l \subseteq y^l$ in general and so elements of $\Pi_{\heartsuit}(x)$ may be included in $\Pi_{\heartsuit}(y)$, the two may be unequal and it may not be possible to use a uniform choice function on them. So it need not happen that $x^{\heartsuit} = y^{\heartsuit}$.

Distributivity If $z \in x^{\heartsuit} \cap y^{\heartsuit}$, then $z \in (\chi(\Pi_{\heartsuit}(x)))^u$ and $z \in (\chi(\Pi_{\heartsuit}(y)))^u$. So if $z \in x^l$ and $z \in y^l$, then $z \in (x^u \cap y^u)^{\heartsuit}$. Since in general, $(a \cap b)^u \subseteq a^u \cap b^u$ and $(a^u \cap b^u)^l = (a^u \cap b^u)$, the required inclusion follows.

$$(\forall x, y, z)\,(x \cup z)^{\heartsuit} \cap (y \cup z)^{\heartsuit} \subseteq (z \cup (x^u \cap y^u))^{\heartsuit} \qquad \text{(Weak Distributivity)}$$

$$(\forall x, y, z)\,(x \cup y)^{\heartsuit} \cap (x \cup z)^{\heartsuit} \subseteq (x \cup (y \oplus z))^{\heartsuit} \qquad \text{(Disjunction in Antecedent)}$$

$$(\forall x, y)\,(x \cup y)^{\heartsuit} \cap (x \cup y^c)^{\heartsuit} \subseteq x^{\heartsuit} \qquad \text{(Proof by Cases)}$$

$$\text{If } y \subseteq (x \cup z)^{\heartsuit}, \text{ then } x \implies y \subseteq z^{\heartsuit} \qquad \text{(Conditionalization.)}$$

□

Proposition 224

$$(\forall x, y)(x^{\diamond} = y^{\diamond} \longrightarrow x^l = y^{l;})$$

$$(\forall x, y)(x^{\flat} = y^{\flat} \longrightarrow x^u = y^u.)$$

Discussion:
In non monotonic reasoning, if C is any consequence operator : $\wp(S) \longmapsto \wp(S)$, then the following named properties of crucial importance in semantics (in whatever sense, [80, 81]):

$$A \subseteq B \subseteq C(A) \longrightarrow C(B) \subseteq C(A) \qquad \text{(Cut)}$$

$$A \subseteq B \subseteq C(A) \longrightarrow C(B) = C(A) \qquad \text{(Cumulativity)}$$

$$x \subseteq y \subseteq x^u \longrightarrow x^{\heartsuit} = y^{\heartsuit} \qquad \text{(subclassical subcumulativity)}$$

Proposition 225 *In the context of the above definition, the lower limiter satisfies all of the following:*

$$(\forall x)\, x \subseteq x^{\diamond} \qquad \text{(Inclusion)}$$

$$(\forall x)\, x^{\diamond\diamond} = x^{\diamond} \qquad \text{(Idempotence)}$$

$$(\forall x\, y)\,(x \subseteq y \subseteq x^{\diamond} \longrightarrow x^{\diamond} = y^{\diamond}) \qquad \text{(Cumulativity)}$$

$$(\forall x)\, x^u \subseteq x^{\diamond} \qquad \text{(Upper Inclusion)}$$

$$S^{\diamond} = S \qquad \text{(Top)}$$

$$(\forall x\, y)\,(x \subseteq y \subseteq x^{\diamond} \longrightarrow x^{\diamond} = y^{\diamond}) \qquad \text{(Cumulativity)}$$

The above proposition means that the upper limiter corresponds to ways of reasoning in a stable way in the sense that the aggregation of conclusions does not affect inferential power or cut-like amplification.

A limited concrete representation theorem for operators like $\heartsuit$ in special cases and $\diamondsuit$ is proved next . The representation theorem is valid for similar operators in non-monotonic reasoning. It permits us to identify cover based formulations of PRAX.

Definition 226 A collection of sets $\mathcal{S}$ will be said to be a *closure system* of a type as per the following conditions:

$$(\forall \mathcal{H} \subseteq \mathcal{S})\ \cap \mathcal{H} \in \mathcal{S}. \quad \text{(Closure System)}$$

$$(\forall \mathcal{H} \subseteq \mathcal{S})\ (\cap \mathcal{H})^{u} \in \mathcal{S}. \quad \text{(U-Closure System)}$$

$$(\forall \mathcal{H} \subseteq \mathcal{S})\ (\cap \mathcal{H})^{l} \in \mathcal{S}. \quad \text{(L-Closure System)}$$

$$(\forall \mathcal{H} \subseteq \mathcal{S})\ (\cap \mathcal{H})^{l}, (\cap \mathcal{H})^{u} \in \mathcal{S}. \quad \text{(LU-Closure System)}$$

$$(\exists 0, \top \in \mathcal{S})(\forall X \in \mathcal{S})\ 0 \subseteq X \subseteq \top . \quad \text{(Bounded)}$$

Proposition 227 *In a PRAX S, the set* $\mathcal{U}(S) = \{x^{u};\ x \in \wp(S)\}$ *is not a bounded U-closure system.*

Proposition 228

$$(\forall x)\ x^{\heartsuit u} \subseteq x^{u \heartsuit}.$$

Proof Because $x^{l} \subseteq x^{u}$, an evaluation of possible granules involved in the construction of $x^{\heartsuit u}$ and $x^{u\heartsuit}$ proves the result. □

Theorem 229 *In a PRAX S, the set* $\heartsuit(S) = \{x^{\heartsuit};\ x \in \wp(S)\}$ *is a bounded LU-closure system if the choice operation is regular.*

Proof

- $x^{\heartsuit}$ is the upper approximation of a specific y containing x that is maximal subject to $x^{l} = y^{l}$.
- $x^{\heartsuit u}$ is the upper approximation of the upper approximation of a specific y containing x that is maximal subject to $x^{l} = y^{l}$ and its upper approximation.
- Clearly,

$$(\chi(\Pi_{\heartsuit}(x)) \cap \chi(\Pi_{\heartsuit}(y)))^{u} \subseteq (\chi(\Pi_{\heartsuit}(x)))^{u} \cap (\chi(\Pi_{\heartsuit}(y)))^{u}.$$

- The expression on the right of the inclusion is obviously a union of granules in the PRAX.

- From a constructive bottom-up perspective, let $p_1,\ p_2,\ \ldots\ p_s$ be a collection of subsets of $x \setminus x^l$ such that

$$\cup p_i \subseteq x \setminus x^l$$
$$(\exists z)\ p_i^u = [z]$$
$$\cup_{i \neq j}(p_i \cap p_j) \text{ is minimal on all such collections.}$$

- Now add subsets $k(p_i)$ of $x^{uu} \setminus x^c$ to x to form the required maximal subset.
- For the lower approximation part, it suffices to use the preservation of l by $\cap$.

□

Proposition 230 *For each $x \in \wp(S)$ let $x^{\curlyvee} = (x^{\heartsuit})^u$, then the following properties hold:*

$$(\forall x)\, x \subseteq x^{\curlyvee}.$$
$$(\forall x)\, x^{\curlyvee\curlyvee} = x^{\curlyvee}.$$

Proof $x^{\heartsuit u \heartsuit u} = x^{\heartsuit u}$. Because if a part of a class that retains the equality of lower approximations could be added, then that should be adjoinable in the construction of $x^{\heartsuit}$ as well. □

The following limited representation theorem can be useful for connections with covers.

Definition 231 Let X be a set and $C : \wp(X) \longmapsto \wp(X)$ a map satisfying all the following conditions:

$$(\forall A \in \wp(S))\, A \subseteq C(A) \quad \text{(Inclusion)}$$
$$(\forall A \in \wp(S))\, C(C(A)) = C(A) \quad \text{(Idempotence)}$$
$$(\forall A, B \in \wp(S))\, (A \subseteq B \subseteq C(A) \longrightarrow C(A) \subseteq C(B)) \quad \text{(Cautious Monotony,)}$$

then C will said to be a *cautious closure operator* (CCO) on X.

Definition 232 Let $H = \langle \underline{H}, \preceq \rangle$, be a partially ordered set over a set $\underline{H}$. A subset $\mathcal{K}$ of the set of order ideals $\mathcal{F}(H)$ of H will be said to be *relevant* for a subset $B \subseteq H$ (in symbols $\rho(\mathcal{K}, H)$) if and only if the following hold:

$$(\exists G \in \mathcal{K})(\forall P \in \mathcal{K})\, P \subseteq G.$$
$$(\forall P \in \mathcal{K})\, P \subseteq B.$$
$$\text{For any } \mathcal{L} \subseteq \mathcal{F}(H),\ \text{if } \mathcal{K} \subseteq \mathcal{L},\ \text{then}$$
$$(\exists \top \in \mathcal{L})(\forall Y \in \mathcal{L})\, Y \subseteq \top \neq H\ \&\ \cap \mathcal{L} = \cap \mathcal{K}.$$

Definition 233 In the context of Def.232, a map $\jmath : \wp(L) \longmapsto \wp(L)$ defined as below will be said to be *safe*

$$\jmath(Z) = \begin{cases} \cap\mathcal{K}, & \text{if all relevant collections for Z have same intersection.} \\ \cap\{\alpha : Z \subseteq \alpha \in \mathcal{F}(H)\}, & \text{else.} \end{cases}$$

Proposition 234 *A safe map $\jmath$ is a cautious closure operator.*

Proof The verification of idempotence and inclusion is direct.

- For $A, B \in \wp(L)$, if it is the case that $A \subseteq B \subseteq \jmath(A)$,
- then either $A \subseteq B \subseteq \jmath(B) \subseteq \jmath(A)$ or $A \subseteq B \subseteq \jmath(A) \subseteq \jmath(B)$ must be true.
- If the former inclusions hold, then it is necessary that $\jmath(A) = \jmath(B)$.
- If $\jmath(B)$ is defined as the intersection of order ideals and $\jmath(A)$ as that of relevant subcollections, then it is necessary that $\jmath(A) \subseteq \jmath(B)$. So cautious monotony holds. It can also be checked that monotonicity fails in this kind of situation.

□

Theorem 235 *On every Boolean ordered unary algebra of the form*

$$\mathcal{H} = \langle \wp(H), \subseteq, C \rangle ,$$

there exists a partial order $\leq$ on K such that $\langle \wp(K), \subseteq, \jmath \rangle$ is isomorphic to $\mathcal{H}$.

10 Cover-Based Rough Set Theory

Cover based rough sets can be traced to [127, 138, 139, 141]. Many approximations in the frameworks have been improved considerably in recent years in multiple directions [38, 91, 144, 174, 177, 180]. The neighborhood approach is also a cover based one with some connections to relations [165]. Associated dualities may also be found in the chapter on duality in this volume [107]. The concept of granules used in [174] is very restricted. This and the subsystem based approach considered in the same paper and earlier ones fall under the concept of granules used by present author [91].

A number of studies on point-wise cover based rough sets are known in a classical semantic domain. These have been surveyed to some extent in [38, 144, 174] and for different reasons. In [162], some of the non dual approximations are considered. All of these considerations have been relative to the classical semantic domain for assessment of abstract properties and connections with relation and neighborhood based rough sets. In [91], granular properties have been explored for six types of cover based approaches. Granular approximations have also been explored in [38, 174].

As far as algebraic semantics is concerned, the restriction to the classical semantic domain remains a problem. In other words, the semantics is not exclusively concerned with rough objects. Only in some cases, does cover based rough sets reduce to relation based rough sets in perfectly equivalent terms. Some connections happen through the algebraic models of modal logics associated. But these are in the classical semantic domain.

Much variation in the notation and terminology used in cover based rough sets can be found. This is also rectified for easy comprehension below and builds on earlier efforts of the present author. Superscript style of notation is strictly preferred for denoting approximations. 'l, u' stand for lower and upper approximations and anything else following those signify a type.

If $\mathcal{S}$ is a cover of the set $\underline{S}$, then the *neighborhood* of $x \in \underline{S}$ is defined via,

$$nbd(x) = \bigcap\{K : x \in K \in \mathcal{S}\} \quad \text{(Cover:Nbd)}$$

The *minimal description* of an element $x \in \underline{S}$ is defined to be the collection

$$\mathrm{md}(x) = \{A : x \in A \in \mathcal{S}, \forall B(x \in B \subseteq A \to A = B)\} \quad \text{(Cover:md)}$$

This is clearly the set of minimal subsets of $\mathcal{S}$ that contain the element x in question.

The *maximal description* of an element $x \in \underline{S}$ is defined to be the collection:

$$\mathrm{MD}(x) = \{A : x \in A \in \mathcal{S}, (\forall B \in \mathcal{S})(x \in B \to \sim (A \subset B))\} \quad \text{(Cover:MD)}$$

The *indiscernibility* (or friends) of an element $x \in \underline{S}$ is defined to be

$$Fr(x) = \bigcup\{K : x \in K \in \mathcal{S}\} \quad \text{(Cover:FR)}$$

An element $K \in \mathcal{S}$ is said to be *reducible* if and only if

$$(\forall x \in K)\, K \neq \mathrm{md}(x) \quad \text{(Cover:Red)}$$

The collection $\{nbd(x) : x \in S\}$ will be denoted by $\mathcal{N}$. The cover obtained by the removal of all reducible elements is called a *covering reduct*.

If every element K of a cover $\mathcal{S}$ contains an element $x \in S$ that satisfies

$$(\forall Z \in \mathcal{S})\, (x \in Z \longrightarrow K \subseteq Z)$$

then $\mathcal{S}$ and x are said to be a *representative* cover and element respectively.

A covering $\mathcal{S}$ is said to be *unary* if and only if $(\forall x \in S)\, \#(md(x)) = 1$. This condition is equivalent to $(\forall K_1, K_2 \in \mathcal{S})(\exists C_1, \ldots C_n \in \mathcal{S})\, K_1 \cap K_2 = \cup_1^n C_i$. For a proof, see [181] and the chapter on duality in this volume [107].

Definition 236 The *intersection closure* of $\mathcal{S}$, is denoted by $Cl_\cap(\mathcal{S})$, is the least subset of $\wp(S)$ that contains $\mathcal{S}$, $S, \emptyset$, and is closed under set intersection. The *union closure* of $\mathcal{S}$, is denoted by $Cl_\cup(\mathcal{S})$, is the least subset of $\wp(S)$ that contains $\mathcal{S}$, $S, \emptyset$, and is closed under set union. More generally if $\underline{\mathcal{H}}$ and $\overline{\mathcal{H}}$ are dual closure and closure systems contained in $\wp(S)$, then subsystem based approximations of a subset $X \subseteq S$ are defined as below:

$$X^{ls+} = \bigcup\{K : K \in \underline{\mathcal{H}} \,\&\, K \subseteq X\} \tag{s+-lower}$$

$$X^{us+} = \bigcap\{K : K \in \overline{\mathcal{H}} \,\&\, X \subseteq K\} \tag{s+-upper}$$

Theorem 237 ([172]) *The approximations satisfy*

- $(\forall X) X^{ls+} = ((X^c)^{us+})^c$
- $(\forall X) X^{ls+} \subseteq X \subseteq X^{us+}$
- *The subsystem based approximations are proper generalizations of classical rough sets.*

Proof If $\underline{\mathcal{H}} = \overline{\mathcal{H}}$, then the approximations coincide with those of classical rough sets. □

If $\mathcal{S}$ is any cover of S, then the *relation* $R_{\mathcal{S}}$ induced by $\mathcal{S}$ is defined by

$$R_{\mathcal{S}}ab \leftrightarrow b \in nbda \leftrightarrow b \in \cap md(a) \tag{57}$$

Conversely, if R is a binary relation on S, then the covering $\mathcal{C}(R)$ induced by R is taken to be $\{[x] : x \in S\}$.

The following result is proved in [181]

Theorem 238 *In the context of the above definition*

- $R_{\mathcal{S}}$ *is reflexive and transitive*
- $R_{\mathcal{S}}$ *is not uniquely determined by* $\mathcal{S}$.
- $\mathcal{C}(R_{\mathcal{S}}) = \mathcal{S}$

The proof may be found in the chapter on duality in this volume [107].

Example 239 This abstract example is intended to illustrate the computation of basic objects associated with covers on a set. Let $S = \{a, b, c, e, f, g, h, i, j\}$,

$$\mathcal{K} = \{K_1, K_2, K_3, K_4, K_5, K_6, K_7, K_8, K_9\},$$

$$K_1 = \{a, b\}, K_2 = \{a, c, e\}, K_3 = \{b, f\}, K_4 = \{j\}, K_5 = \{f, g, h\},$$

$$K_6 = \{i\}, K_7 = \{f, g, j, a\}, K_8 = \{f, g\}, K_9 = \{a, j\}.$$

The following table lists some of the popular granules or objects (Table 8):

Duality results relating to this section have been proved in the chapter on duality in this volume [107].

Table 8 Popular granules

Element: x	$Fr(x)$	$Md(x)$	$nbd(x)$
a	$S \setminus \{h, i\}$	$\{K_1, K_2, K_3\}$	$\{a\}$
b	$\{a, b, f\}$	$\{K_3\}$	$\{b\}$
c	$\{a, c, e\}$	$\{K_2\}$	$\{a, c, e\}$
e	$\{a, c, e\}$	$\{K_2\}$	$\{a, c, e\}$
f	$S \setminus \{c, e, i\}$	$\{K_3, K_8\}$	$\{f\}$
g	$\{a, f, g, h, j\}$	$\{K_8\}$	$\{f, g\}$
h	$\{f, g, h\}$	$\{K_5\}$	$\{f, g, h\}$
i	$\{i\}$	$\{K_6\}$	$\{i\}$
j	$\{a, f, g, j\}$	$\{K_4\}$	$\{j\}$

10.1 Suitable Granulations

In [91], it had been suggested by the present author that the best approach to granulation in cover based rough sets is to is to associate an initial set of granules and then refine them subsequently after suitable exploration. This approach can be useful in some practical contexts. Usually a refinement can be expected to be generable from the initial set through relatively simple set theoretic operations. In more abstract situations, the main problem would be of representation and the results in these situations would be the basis of possible abstract representation theorems. The theorems proved below throw light on the fine structure of granularity in the cover-based situations. The basic question that has been explored in this subsection *is the suitability of the current choice of granulation for the approximations that have happened.*

If $X \subseteq S$, then the covering lower approximation of type-1 ([12]) is defined by

$$X^{l1} = \bigcup\{K : K \in \mathcal{S} \,\&\, K \subseteq X\} \quad ()$$

Other key approximations of the cover based approach are [91]:

(i) $X^{u1+} = X^{l1} \cup \bigcup\{\mathrm{md}(x) : x \in X\}$ [12],
(ii) $X^{u2+} = \bigcup\{K : K \in \mathcal{S}, K \cap X \neq \emptyset\}$,
(iii) $X^{u3+} = \bigcup\{\mathrm{md}(x) : x \in X\}$,
(iv) $X^{u4+} = X^{l1} \cup \{K : K \cap (X \setminus X^{l1}) \neq \emptyset\}$,
(v) $X^{u5+} = X^{l1} \cup \bigcup\{nbd(x) : x \in X \setminus X^{l1}\}$,
(vi) $X^{u6+} = \{x : nbd(x) \cap X \neq \emptyset\}$,
(vii) $X^{l6+} = \{x : nbd(x) \subseteq X\}$.

Further, the approximation $X^{u1*} = X^{l1} \cup \bigcup\{\mathrm{md}(x) : x \in X \setminus X^{l1}\}$ is also of interest. The sixth type of lower and upper approximations [165, 181] of a set X are also written as $X_{\$}$ and $X^{\$}$. These are point-wise approximations that are not granular. It can be shown that:

Proposition 240 *In the above context,*

- *$md(x)$ is invariant under removal of reducible elements,*
- *$(nbd(x))^{l6+} = nbd(x)$,*
- *$nbd(x) \subseteq (nbd(x))^{6+}$.*

The approximation operators $u1+, \ldots, u5+$ (corresponding to first, ..., fifth approximation operators used in [181] and references therein) are considered with the lower approximation operator $l1$ in general. Some references for cover-based rough sets include [56, 73, 88, 139, 143, 165, 177, 179, 181]. The relation between cover-based rough sets and relation-based rough sets are considered in [88, 181] and in some detail in the chapter on duality in this volume [107]. For a cover to correspond to a tolerance, it is necessary and sufficient that the cover be normal—a more general version of this result can be found in [27]. When such reductions are possible, then good semantics in Meta-R perspective are possible. The main results of [88], due to the present author, provide a more complicated correspondence between covers and sets equipped with multiple relations or a relation with additional operations. The full scope of the results are still under investigation. So, in general, cover-based rough sets is more general than relation-based rough sets. From the point of view of expression, equivalent statements formulated in the former would be simpler than in the latter.

The following pairs of approximation operators have also been considered in the literature (the notation of [143] has been streamlined; $lp1$, $lm1$ corresponds to $\underline{P}_1$, $\underline{C}_1$ respectively and so on).

$$X^{lp1} = \{x : Fr(x) \subseteq X\},$$

$$X^{up1} = \bigcup\{K : K \in \mathcal{K},\ K \cap X \neq \emptyset\},$$

$$X^{lp2} = \bigcup\{Fr(x);\ Fr(x) \subseteq X\},$$

$$X^{up2} = \{z : (\forall y)(z \in Fr(y) \rightarrow Fr(y) \cap X \neq \emptyset)\},$$

$$X^{lp3} = X^{l1},$$

$$X^{up3} = \{y : \forall K \in \mathcal{K}(y \in K \rightarrow K \cap X \neq \emptyset)\},$$

$$X^{lp4}, X^{up4},$$

$$X^{lm1} = X^{l1} = X^{lp3},$$

$$X^{um1} = X^{u2},$$

$$X^{lm2} = X^{l6+},$$

$$X^{um2} = X^{u6+},$$

$$X^{lm3} = \{x;\ (\exists u)u \in nbd(x), nbd(u) \subseteq X\},$$

$$X^{um3} = \{x;\ (\forall u)(u \in nbd(x) \to nbd(u) \cap X \neq \emptyset)\},$$
$$X^{lm4} = \{x; (\forall x)(x \in nbd(u) \to nbd(u) \subseteq X)\},$$
$$X^{um4} = X^{u6+} = X^{um2},$$
$$X^{lm5} = \{x;\ (\forall u)(x \in nbd(u) \to u \in X)\},$$
$$X^{um5} = \bigcup\{nbd(x);\ x \in X\}.$$

Note that X^{lp4}, X^{up4} are the same as the classical approximations with respect to $\pi(\mathcal{K})$—the partition generated by the cover $\mathcal{K}$.

Theorem 241 *When the approximations are* $(lp1,\ up1)$ *and with the collection of granules being* $\{Fr(x)\}$*, all of* MER, URA, UMER *hold, but* ACG, NO, LS *do not hold necessarily*

Proof For an arbitrary subset X, $X \subseteq Fr(x)$ for some $x \in S$ and $X^{lp1} = X^{up1} = X$ would mean that $Fr(x) = X$ as $Fr(x) \subseteq X^{up1}$ would be essential. So UMER and the weaker condition MER holds. URA is obvious from the definitions. □

Theorem 242 *When the approximations are* $(lp2,\ up2)$ *and with the collection of granules being* $\{Fr(x)\}$*, all of* MER, LMER, RA, LCG *hold, but* ACG, NO, LS *do not hold necessarily*

Proof From the definition, it is clear that RA, LCG hold. If for an arbitrary subset X, $X \subseteq Fr(x)$ for some $x \in S$ and $X^{lp2} = X = X^{up2}$, then X is a union of some granules of the form $Fr(y)$. If $x \in X$, then it is obvious that $X = Fr(x)$.

If $x \in Fr(x) \setminus X$ and $F(x)$ is an element of the underlying cover $\mathcal{S}$, then again it would follow that $X = Fr(x)$. Finally $x \in Fr(x) \setminus X$ and $Fr(x)$ is a union of elements of the cover intersecting X, results in a contradiction. So MER follows.

If for an arbitrary subset X, $X \subseteq Fr(x)$ for some $x \in S$ and $X^{lp2} = X$ and $x \in Fr(x) \setminus X$, then a contradiction happens. So LMER holds. □

Theorem 243 *When the approximations are* $(lp3,\ up3)$ *and with the collection of granules being* $\mathcal{K}$*, all of* MER, RA, ST, LCG, LU *hold, but* ACG, NO *do not hold necessarily.*

Proof Both the lower and upper approximations of any subset of S is eventually a union of elements $\mathcal{K}$, so RA holds. Other properties follow from the definitions. Counter examples are easy. □

Theorem 244 *When the approximations are* $(lp4,\ up4)$ *and with the collection of granules being* $\pi(\mathcal{K})$*, all of* RA, ACG, MER, AS, FU, NO, PS *hold.*

Proof With the given choice of granules, this is like the classical case. □

Theorem 245 *When the approximations are* $(lp4,\ up4)$ *and with the collection of granules being* $\mathcal{K}$*, all of* WRA, ACG, AS *hold, while the rest may not.*

Proof WRA holds because elements of $\pi(\mathcal{K})$ can be represented set theoretically in terms of elements of $\mathcal{K}$. Lower and upper approximations of elements of $\mathcal{K}$ are simply unions of partitions of the elements induced by $\pi(\mathcal{K})$. □

Theorem 246 *When the approximations are* $(lm1,\ um1)$ *and with the collection of granules being* $\mathcal{K}$*, all of WRA, LS, LCG hold, but RA, ST, LMER do not hold necessarily. For WRA, complementation is necessary.*

Proof If $\mathcal{K}$ has an element properly included in another, then LMER will fail. If complementation is also permitted, then WRA will hold. Obviously RA does not hold. Note the contrast with the pair $(lp3, up3)$ with the same granulation. □

Theorem 247 *When the approximations are* $(lm2,\ um2)$ *and with the collection of granules being* $\mathcal{N}$*, all of LCG, LRA, ST, MER holds, but RA, ACG, LMER, NO do not hold necessarily.*

Proof If $y \in nbd(x)$ for any two $x,\ y \in S$, then $nbd(y) \subseteq nbd(x)$ and it is possible that $x \notin nbd(y)$, but it is necessary that $x \in nbd(x)$. So $(nbd(x))^{lm2} = nbd(x)$, but $(nbd(x))^{um2}$ need not equal $nbd(x)$. LRA will hold as the lower approximation will be a union of neighborhoods, but this need happen in case of the upper approximation. NO is obviously false. The upper approximation of a neighborhood can be a larger neighborhood of a different point. So ACG will not hold in general. MER can be easily checked. □

Theorem 248 *When the approximations are* $(l6+,\ u6+)$ *and with the collection of granules being* $\mathcal{N}$*, all of LCG, LRA, ST, MER holds, but RA, ACG, LMER, NO do not hold necessarily.*

Proof Same as above. □

Theorem 249 *When the approximations are* $(l1,\ u1+)$ *and with the collection of granules being* $\mathcal{K}$*, all of ACG, RA, FU, LS holds, but MER, LMER, NO do not hold necessarily.*

Proof RA holds as all approximations are unions of granules. For any granule K, $K^{l1} = K$ and so $K^{u1+} = K^{l} = K$. So ACG holds. If for two granules A, B, $A \subset B$, then $A^{l1} = A^{u1+} = A$, but $A \neq B$. So MER, $LMER$ cannot hold in general. □

Theorem 250 *When the approximations are* $(l1,\ u2+)$ *and with the collection of granules being* $\mathcal{K}$*, all of ACG, RA, FU, ST holds, but MER, LMER, NO do not hold necessarily.*

Proof RA holds as all approximations are unions of granules. For any granule K, $K^{l1} = K$ and so $K^{u1+} = K^{l} = K$. So ACG holds. If for two granules A, B, $A \subset B$, then $A^{l1} = A^{u1+} = A$, but $A \neq B$. So MER, LMER cannot hold in general. If a granule K is included in a subset X of S, then it will be included in the latter's lower approximation. If K intersects another subset, then the upper approximation of the set will include K. So ST holds. □

Theorem 251 *When the approximations are* $(l1, u3+)$ *and with the collection of granules being* $\mathcal{K}$, *all of* ACG, RA, FU, LS *holds, but* MER, LMER, NO *do not hold necessarily.*

Proof RA holds as all approximations are unions of granules. For any granule K, $K^{l1} = K$ and so $K^{u1+} = K^l = K$. So ACG holds. If for two granules A, B, $A \subset B$, then $A^{l1} = A^{u3+} = A$, but $A \neq B$. So MER, LMER cannot hold in general. The union of any two granules is a definite element, so FU holds. □

Theorem 252 *When the approximations are* $(l1, u4+)$ *and with the collection of granules being* $\mathcal{K}$, *all of* ACG, RA, FU, LS *holds, but* MER, LMER, NO *do not hold necessarily*

Proof RA holds as all approximations are unions of granules. For any granule K, $K^{l1} = K$ and so $K^{u1+} = K^l = K$. So ACG holds. If for two granules A, B, $A \subset B$, then $A^{l1} = A^{u3+} = A$, but $A \neq B$. So MER, LMER cannot hold in general. The union of any two granules is a definite element, so FU holds. □

Theorem 253 *When the approximations are* $(l1, u5+)$ *and with the collection of granules being* $\mathcal{K}$, *all of* ACG, RA, FU, LS *holds, but* MER, LMER, NO *do not hold necessarily.*

Proof RA holds as all approximations are unions of granules. For any granule K, $K^{l1} = K$ and so $K^{u1+} = K^l = K$. So ACG holds. If for two granules A, B, $A \subset B$, then $A^{l1} = A^{u3+} = A$, but $A \neq B$. So MER, LMER cannot hold in general. The union of any two granules is a definite element, so FU holds. □

In cover-based rough sets, different approximations are defined with the help of a determinate collection of subsets. These subsets satisfy the properties WRA, LS and FU and are therefore admissible granules. But they do not in general have many of the nicer properties of granules in relation to the approximations. However, at a later stage it may be possible to refine these and construct a better set of granules (see [88], for example) for the same approximations. Similar process of refinement can be used in other types of rough sets as well. For these reasons, the former are referred to as *initial granules* and the latter as relatively *refined granules*. It may happen that more closely related approximations may as well be formed by such process.

10.1.1 AUAI Approach

In [56], a theory of generalized rough sets based on covers of subsets of a given set $\underline{S}$ is considered. Let S be a set and $\mathcal{K} = \{K_i\}_1^n$ be a collection of subsets of it. If $X \subseteq S$, then consider the sets (with $K_0 = \emptyset$, $K_{n+1} = S$)

(i) $X^{l1} = \bigcup\{K_i \,:\, K_i \subseteq X,\ i \in \{0, 1, ..., n\}\}$

(ii) $X^{l2} = \bigcup\{\cap(S \setminus K_i) \,:\, \cap_I(S \setminus K_i) \subseteq X,\ I \subseteq \{1, ..., n+1\}\}$

(iii) $X^{u1} = \bigcap\{\cup_{i\in I} K_i \,:\, X, \subseteq \cup K_i,\ I \subseteq \{1, ..., n+1\}\}$

(iv) $X^{u2} = \bigcap\{S \setminus K_i \,:\, X, \subseteq S \setminus K_i,\ i \in \{0, ..., n\}\}$

The pair (X^{l1}, X^{u1}) is called a *AU-rough set* by union, while (X^{l2}, X^{u2}) a *AI-rough set* by intersection (in the notation of [56] these are

$$\left(\mathcal{F}_*^{\cup}(X),\ \mathcal{F}_{\cup}^*(X)\right) \text{ and } \left(\mathcal{F}_*^{\cap}(X),\ \mathcal{F}_{\cap}^*(X)\right)$$

respectively).

Theorem 254 *The following are true :*

(i) $X^{l1} \subseteq X \subseteq X^{u1}$
(ii) $X^{l2} \subseteq X \subseteq X^{u2}$
(iii) $\emptyset^{l1} = \emptyset^{l2} = \emptyset$
(iv) $(\cup\mathcal{K} = S \longrightarrow S^{u1} = S^{u2} = S)$
(v) $(\cup\mathcal{K} = S \longrightarrow \emptyset^{u2} = \emptyset,\ S^{l1} = S)$
(vi) $(\cap\mathcal{K} = \emptyset \longrightarrow \emptyset^{u1} = \emptyset,\ S^{l2} = S)$
(vii) $(X \cap Y)^{l1} \subseteq X^{l1} \cap Y^{l1}, (X \cap Y)^{l2} = X^{l2} \cap Y^{l2}$
(viii) $(X \cup Y)^{u1} = X^{u1} \cup Y^{u1}, X^{u2} \cup Y^{u2} \subseteq (X \cup Y)^{u2}$
(ix) $(X \subseteq Y \longrightarrow X^{l1} \subseteq Y^{l1},\ X^{l2} \subseteq Y^{l2})$
(x) *If* $\mathcal{K}$ *is pairwise disjoint then* $(X \cap Y)^{l1} = X^{l1} \cap Y^{l1}$, $(X \cup Y)^{u2} = X^{u2} \cup Y^{u2}$
(xi) $(X \subseteq Y \longrightarrow X^{u1} \subseteq Y^{u1},\ X^{u2} \subseteq Y^{u2})$
(xii) $X^{l1} \cup Y^{l1} \subseteq (X \cup Y)^{l1}$
(xiii) $X^{l2} \cup Y^{l2} \subseteq (X \cup Y)^{l2}$
(xiv) $(X \cap Y)^{u1} \subseteq X^{u1} \cap Y^{u1}$
(xv) $(X \cap Y)^{u2} \subseteq X^{u2} \cap Y^{u2}$
(xvi) $(S \setminus X)^{l1} = S \setminus X^{u2}$
(xvii) $(S \setminus X)^{l2} = S \setminus X^{u1}$
(xviii) $(S \setminus X)^{u1} = S \setminus X^{l2}$
(xix) $(S \setminus X)^{u2} = S \setminus X^{l1}$
(xx) $(X^{l1})^{l1} = X^{l1}, (X^{l2})^{l2} = X^{l2}$
(xxi) $(X^{u1})^{u1} = X^{u1}, (X^{u2})^{u2} = X^{u2}$
(xxii) $(X^{l1})^{u1} = X^{l1}, (X^{u2})^{l2} = X^{u2}$
(xxiii) $X^{l2} \subseteq (X^{l2})^{u2}, (X^{u1})^{l1} \subseteq X^{u1}$
(xxiv) $(\mathcal{K}_j^{\cap}(X))^{u2} = \mathcal{K}_j^{\cap}(X),\ j = 1, 2, ..., t_1$
(xxv) $(\mathcal{K}_j^{\cup}(X))^{l1} = \mathcal{K}_j^{\cup}(X),\ j = 1, 2, ..., t_2$

In this, $(\mathcal{K}_j^{\cup}(X))$ *is the minimal union for* j *being in the indicated range and* $(\mathcal{K}_j^{\cap}(X))$ *is the maximal intersection.*

Proof The properties are essentially set-theoretical as shown in the following proofs:

(i) $X^{l1} \subseteq X \subseteq X^{u1}$. Suppose X is non-empty. If $x \in X^{l1}$, then x is in some of those K_i) for which $K_i) \subseteq X$ holds. So $x \in X$ as well.

If $x \in X$, x is in all those unions of the form $\cup_{i \in I} K_i$ that include X for some $I \subseteq \{0, 1, \ldots, n\}$. So $x \in X^{u1}$.

(ii) If X is nonempty and $x \in X^{l2}$, then x is in some of the intersections of complements of K_is that are contained in X. So $X^{l2} \subseteq X$.

If $x \in X$, then if x is in a K_i that is contained in X, then there must exist a K_i for some $i \in \{0, 1 \ldots n\}$ such that its complement contains X. So also if x is in a K_i that is not contained in X. When x is in no K_i, then x is in every complement of K_is. So we have $X \subseteq X^{u2}$.

(iii) This follows from the definition.

(iv) If $\cup \mathcal{K} = S$ then $S^{u1} = \bigcap\{\cup K_i : S = \cup K_i\} = S$. $S^{u2} = \bigcap\{S \setminus K_i : S = S \setminus K_i, i \in \{0, 1, \ldots n\}\} = S$ in the situation too.

(v) If $\cup \mathcal{K} = S$, then $\emptyset^{u2} = \bigcap\{S \setminus K_i : i \in \{0, \ldots, n\} = \emptyset$, while $S^{l1} = \bigcup\{K_i : K_i \subseteq S\} = \cup \mathcal{K} = S$.

(vi) If $\cap \mathcal{K} = \emptyset$, then $\emptyset^{u1} = \bigcap\{K_i : i \in \{1, 2, \ldots, n\}\} = \emptyset$ and $S^{l2} = \bigcup\{\cap_I (S \setminus K_i) : I \subseteq \{1, 2, \ldots, n+1\}\} = S$.

(vii) If $x \in (X \cap Y)^{l1}$ then x is in some of the $K_i \subseteq (X \cap Y)$, but these will be subsets of both X and Y respectively. So x will be in $X^{l1} \cap Y^{l1}$.

For the second part it is clear that $X^{l2} \cap Y^{l2} \subseteq (X \cap Y)^{l2}$. Suppose that the reverse inclusion is false. Then there must exist $x \in (X \cap Y)^{l2}$ such that $x \notin X^{l2}$ and $x \notin Y^{l2}$. The latter means that there exist no intersection of complements of K_is that are included in X and Y, but are included in $X \cap Y$. This obvious contradiction proves that $(X \cap Y)^{l2} = X^{l2} \cap Y^{l2}$.

(viii) From the definitions it is clear that $X^{u1} \cup Y^{u1}$ will be a subset of $(X \cup Y)^{u1}$ as the latter is the intersections of the unions of K_is that contain $(X \cup Y)$. If $x \in (X \cup Y)^{u1}$ then x is in all of the unions of K_is that contain $(X \cup Y)$. Now each of these unions will contain X and Y respectively. If x is in X or Y, then there is nothing to prove. So suppose that x is in neither. Now if x is neither in X^{u1} and Y^2, then we will be able to form a collection of K_i that contains $X \cup Y$, contradicting our original assumption that $x \in (X \cup Y)^{u1}$.

(ix) Let $X \subset Y$ and let $K_i \subseteq X$, then K_i is also included in Y. The union of all such K_i's is the corresponding lower approximation. So $X^{l1} \subseteq Y^{l1}$.

Again if $X \subset Y$ and if $\cap_{i \in I} S \setminus K_i \subseteq X$, then it is a subset of Y as well, in fact some subsets of I may also have the property for Y and not for X. And as the unions of these intersections is the second lower approximation, so $X^{l2} \subseteq Y^{l2}$.

(x) $(X \cap Y)^{l1} \subseteq X^{l1} \cap Y^{l1}$ has already been proved. Let $x \in X^{l1} \cap Y^{l1}$. As the constituent K_i's in X^{l1} and Y^{l1} are respectively disjoint, suppose $x \in K_j$ for a fixed j. If x is not in $X \cap Y$, then an obvious contradiction to the existence of such a K_j becomes evident. Therefore $X^{l1} \cap Y^{l1} \subseteq (X \cap Y)^{l1}$ follows.

(xi) Let $X \subset Y$ and let $Y \subseteq \cup_{i \in I} K_i$ for some $I \subseteq \{1, 2, \ldots n+1\}$ then $X \subseteq \cup_{i \in I} K_i$. X^{u1} and Y^{u1} are formed by the intersection of such unions of K_i's. So $X^{u1} \subseteq Y^{u1}$.

Again if $X \subset Y$, and $Y \subseteq S \setminus K_i$ for some i, then X will also be a subset of the same. But there may exist some K_j for which, X is a subset of it's complement and Y is not so. So $X^{u2} \subseteq Y^{u2}$.

(xii) If $x \in X^{l1} \cup Y^{l1}$, then x is in at least one of the K_i's contained in X^{l1} or Y^{l1}. But that K_i must be contained in $X \cup Y$ and therefore in $(X \cup Y)^{l1}$ as well. So $X^{l1} \cup Y^{l1} \subseteq (X \cup Y)^{l1}$.

(xiii) If $x \in X^{l2} \cup Y^{l2}$, then x is in at least one of the intersections of the form $\cap_{i \in I}(S \setminus K_i)$ that is included in X or Y. Therefore x is in $X \cup Y$. But $\cap_{i \in I}(S \setminus K_i) \subseteq X \cup Y$. So $X^{l2} \cup Y^{l2} \subseteq (X \cup Y)^{l2}$.

(xiv) The intersection of the unions of K_is that contain X and Y respectively will intersect in a set containing $(X \cap Y)^{u1}$ (as a larger number of unions of K_is will contain $(X \cap Y)^{u1}$ and as their intersection is precisely $(X \cap Y)^{u1}$).

(xv) If $x \in (X \cap Y)^{u2}$, then x is present in all those $S \setminus K_i$ that contain $X \cap Y$. It is not present in any of those K_is that are included in $S \setminus (X \cap Y) = (S \setminus X) \cup (S \setminus Y)$. Suppose x is not in $(X^{u2} \cap Y^{u2}$, then in each of the three cases we have a contradiction to our original assumption. So $(X \cap Y)^{u2} \subseteq X^{u2} \cap Y^{u2}$.

(xvi) $(S \setminus X)^{l1} = \bigcup\{K_i : K_i \subseteq S \setminus X\}$. Now this is the same as $\bigcup\{K_i : X \subseteq S \setminus K_i\} = S \setminus X^{u2}$

(xvii) $(S \setminus X)^{l2} = S \setminus X^{u1}$ is proved in the same way as the above.

(xviii) $(S \setminus X)^{u1} = S \setminus X^{l2}$ is proved in the same way as the above.

(xix) $(S \setminus X)^{u2} = S \setminus X^{l1}$ is proved in the same way as the above.

The other statements are easy to prove through standard set-theoretic arguments.

□

Theorem 255 *In AUAI rough sets [56, 85], with the collection of granules being $\mathcal{K}$ and the approximation operators being (l_1, l_2, u_1 and u_2), WRA, LS, SCG, LU, IMER holds, but ACG, RA, SRA, MER do not hold in general.*

Proof WRA holds if the complement, union and intersection operations are used in the construction of terms in the WRA condition. ACG does not hold as the elements of $\mathcal{K}$ need not be crisp with respect to $l2$. Crispness holds with respect to $l1$, $u1$, so SCG holds. MER need not hold as it is violated when a granule is properly included in another. IMER holds as the pathology negates the premise. It can also be checked by a direct argument. □

From the definition of the approximations in AUAI and context, it should be clear that all pairings of lower and upper approximations are sensible generalizations of the classical context. In the following two of the four are considered. These pairs do not behave well with respect to duality, but are more similar with respect to representation in terms of granularity.

Theorem 256 *In AUAI rough sets, with the collection of granules being $\mathcal{K}$ and the approximation operators being (l_1, u_1), WRA, ACG, ST, LU holds, but MER, NO, FU, RA do not hold in general.*

Theorem 257 *In AUAI rough sets, with the collection of granules being $\mathcal{K}$ and the two approximation operators being (l_2, u_2), WRA, ST holds, but ACG, MER, RA, NO do not hold in general.*

Proof If $K \in \mathcal{K}$, $K \subseteq X$ (X being any subset of S) and $y \in K^{l2}$, then y must be in at least one intersection of the sets of the form $S \setminus K_i$ (for $i \in I_0$, say) and so it should be in each of these $S \setminus K_i \subseteq K \subseteq X$. This will ensure $y \in X^{l2}$. So lower stability holds. Upper stability can be checked in a similar way. □

10.1.2 Subsystem Based Approximations

Below a method of interpretation is proposed to ensure that the subsystem based approach falls under some level of granularity.

In the context of Def.236, the granulation $\mathcal{G}$ can be taken to be $\overline{\mathcal{H}} \cup \underline{\mathcal{H}}$ or simply $\underline{\mathcal{H}}$. In the latter case complementation operations will be required for representation of approximations, but are not needed in the former.

Theorem 258 *Under the assumptions, the granulation $\mathcal{G} = \underline{\mathcal{H}}$ satisfies all of WRA, LRA, LACG, LS, FU.*

10.2 The Algebras of CBRST

Very few explicit results that describe rough objects in the context of cover based rough sets are known. Results in the classical domain can be viewed as a semantics in the following ways:

- If a duality between general approximation spaces of the form $\langle \underline{S}, R\rangle$ and the covering approximation space $\langle \underline{S}, \mathcal{S}\rangle$ is known and the cover based approximations l, u are defined as point-wise approximations then the Boolean algebra with operators $\mathbb{B} = \left\langle \underline{\wp(S)}, l, u, \cup, \cap, ^c, 0, 1\right\rangle$ (satisfying a number of properties) can be interpreted as a semantics of the modal logic associated. This works because the relation R can be read as an accessibility relation between worlds and the general approximation space as a Kripke frame. This approach can be found in [143, 144]. It is necessary that the upper approximation be definable in terms of the lower approximation and complementation in the context for the interpretation to work.
- In general, a distributive set lattice with operators of the form

$$\mathbb{L} = \left\langle \underline{\wp(S)}, l, u, \cup, \cap, 0, 1\right\rangle$$

(satisfying a number of properties) is likely to be the semantic structure associated with a covering approximation space. A proper understanding of the context is required for claiming that L is not a Boolean algebra with operators. This semantic approach is not a satisfactory one for a number of reasons.

The last point refers to unexplored territories. The main question is: *Given a cover based rough set situation derived from an information system or otherwise, what properties and operations may be allowed for studying/constructing approximations?*

- The ideal based approach due to [1, 3, 64] has been generalized by the present author in [105] for point-wise relation based rough sets and should be generalizable to covering approximation provided the very definitions of approximations are changed. This is likely to work well for those granular approximations defined through subsystems and is an open research area.
- The antichain based approach due to the present author [98, 106] is bound to work for granular cover based rough sets. It applies to a distinct higher order semantic domain that consciously evaluates some implications of discernibility. The approach is described in a Sect. 12.
- Some cases of cover based rough sets that relate to tolerances have also been considered in [62] and have also been described in more detail in the chapter on irredundant coverings in this volume.
- Semantics through duality with general approximation spaces are possible in some cases. These are considered in the chapter on duality in this volume [107].
- Semantics by way of reduction to multiple nicer approximations—for example a similarity based approximations may be reduced to a number of classical approximations. This area is yet to see systematic development.

10.3 Quasi-Order Based Covers

In [68], a granular approach to a general approximation space $Q = \langle \underline{Q}, < \rangle$ with $<$ being a quasi-order is investigated. The granulation used is $\mathcal{Q} = \{[x]_i \,:\, x \in Q\}$—in the paper it is also referred to as a quasi-order generated cover and $\langle \underline{Q}, \mathcal{Q} \rangle$ as a *quasi-order generated covering approximation space* QOCAS. The reason for this terminology is in the well known fact that

Proposition 259

- *Any cover* $\mathcal{C}$ *on a set* $\underline{S}$ *generates a quasi-order on it and induces a QOCAS* $S = \langle \underline{S}, \{nbd(x) : x \in \underline{S}\} \rangle$.
- *The granulation is the set of order filters of* R.
- $a \in [x]_i$ *if and only if* $[a]_i \subseteq [x]_i$ *if and only if* $nbd(a) \subseteq nbd(x)$

Definition 260 On a QOAS S, the approximations of a subset $A \subseteq S$ are defined as below:

$$A^l = \bigcup\{[x]_i \,:\, [x]_i \subseteq A\} \quad \text{(qlower)}$$

$$A^{u*} = \bigcup\{[x]_i \,:\, x \in A\} \quad \text{(qupper)}$$

Theorem 261 *In a QOAS Q, the approximations satisfy top, bottom, idempotence and the following:*

$$(\forall x \in \wp(Q))\, x^{u*l} = x^{u*} \,\&\, x^{lu*} = x^l \quad \text{(Mix)}$$

$$(\forall x \in \wp(Q))\, x^l \subseteq x \subseteq x^{u*} \quad \text{(Inclusion+)}$$

$$(\forall x, a \in \wp(Q))\, (x \cap a)^l = x^l \cap a^l \,\&\, (x \cup a)^l \subseteq x^l \cup a^l \quad \text{(l-M.SA)}$$

$$(\forall x, a \in \wp(Q))\, (x \cap a)^{u*} \subseteq x^{u*} \cap a^{u*} \,\&\, (x \cup a)^{u*} = x^{u*} \cup a^{u*} \quad \text{(u*-SM.A)}$$

$$(\forall x \in \wp(Q))\, x^l = x \leftrightarrow x^{u*} = x \quad \text{(Definite+)}$$

$$(\forall x, a \in \wp(Q))\, x^{u*} \subseteq a \leftrightarrow x \subseteq a^l \quad \text{(Galois)}$$

Moreover $l, u*$ are topological interior and closure operators in the Alexandrov topologies τ and τ' associated with the quasi-order and dual quasi order respectively with neighborhood basis $\{[x]_i \,:\, x \in Q\}$. This permits the equivalent definition:

- $(\forall X \in \wp(Q))\, X^l = \bigcup\{H \,:\, H \in \tau \,\&\, H \subseteq X\}$
- $(\forall X \in \wp(Q))\, X^{u*} = \bigcap\{H \,:\, H \in \tau \,\&\, X \subseteq H\}$
- *The above result is a proper generalization of classical rough sets.*

Proof The proof is through standard set theoretic inclusion and membership arguments. The Galois connection on the ordered set $\wp(S)$ is proved by direct implications:

- $x^{u*} \subseteq a \rightarrow x^{u*l} \subseteq a^l \rightarrow x^{u*} \subseteq a^l \rightarrow x \subseteq x^{u*} \subseteq a^l$
- $x \subseteq a^l \rightarrow x^{u*} \subseteq a^{lu*} \rightarrow x^{u*} \subseteq a^l \rightarrow x^{u*} \subseteq a^l \subseteq a$

□

Proposition 262 *For a QOAS, $\Delta_R = \langle \underline{\delta_{lu*}(Q)}, \cap, \cup, ^+, 0, 1\rangle$ forms a Heyting algebra as it is the collection of open sets in the Alexandrov topology. ^{c} being the pseudo complementation operation.*

Theorem 263 *For a QOAS, $\Delta_R = \langle \underline{\delta_{lu*}(Q)}, \cap, \cup, ^+, 0, 1\rangle$ is a Boolean algebra if and only if $\{[x]_i \,:\, x \in Q\}$ is a partition of Q.*

Proof If $\{[x]_i \,:\, x \in Q\}$ is a partition of Q, then under the strong representation assumptions, it follows that Δ_R is a Boolean algebra.

For the converse, suppose $\{[x]_i \,:\, x \in Q\}$ is not partition of Q and Δ_R is a Boolean algebra

- So $(\exists z, a, b)[a]_i \neq [b]_i \,\&\, z \in [a]_i \cap [b]_i \neq \emptyset$
- So $[z]_i \subset [a]_i$ or $[z]_i \subset [b]_i$. Suppose the former
- Form the pseudo-complement ($^+$) and dual pseudo-complement ($^\circ$) of $[z]_i$
- $[z]_i^+ = \bigcup\{H \; H \in \Delta_R \,\&\, H \cap [z]_i = \emptyset\}$ and $[z]_i^\circ = \bigcap\{H \; H \in \Delta_R \,\&\, H \cup [z]_i = Q\}$.
- It can be checked that $a \notin [z]_i^+$ and $a \in [z]_i^\circ$ by a contradiction argument.
- But in a Boolean algebra, complementation coincides with both pseudo complementations.

□

The following representation theorem(s) is proved in [68]. This is considered in the chapter on duality and representation by the present author in [107].

Theorem 264

- *If L is a Heyting algebra, then there exists a QOAS Q such that L is embeddable in its Heyting algebra of definable sets.*
- *If L is a completely distributive lattice in which set of join irreducibles is also join dense, then there exists a QOAS Q such that L is embeddable in its lattice of definable sets.*

Not surprisingly, the relation between the sets $\mathbb{R} = \{(a, b) : a \subseteq b \,\&\, a, b \in \Delta_R\}$ and $\mathbb{RS} = \{(a^l, a^{u*}) : a \in \wp(Q)\}$ mimics that in classical rough sets due to the strong influence of transitivity and reflexivity of R.

Theorem 265 $\mathbb{R} = \mathbb{RS}$ *if and only if* $(\forall (a, b) \in \mathbb{R})\, \#(b \setminus a) \neq 1$.

Proof The proof, through set theoretic contradiction arguments and argument by cases, is a bit long and not very hard. □

The following weeding theorem is also proved in [68]. The converse construction has been added by the present author.

Theorem 266 *Let* $Q = \langle \underline{Q}, R\rangle$ *be a QOAS with at least one pair of definable sets* $a, b \in \Delta$ *such that* $a \subset b$ *and* $\#(b \setminus a) = 1$. *Then there exists a QOAS* $Q' = \langle \underline{Q'}, R'\rangle$ *with its lattice of definable sets* Δ' *being isomorphic to* Δ *and satisfying* $(\forall a, b \in \Delta')\, (a \subset b \longrightarrow \#(b \setminus a) > 1)$ *and conversely.*

Proof The basic idea of adding elements to remove the pathology works. Let

$$P = \{(a, b) : a \subset b \,\&\, a, b \in \Delta \,\&\, \#(b \setminus a) = 1\}$$

- Form $P' = \{c : \{c\} = b \setminus a \,\&\, a, b \in P\}$, a disjoint copy $P'' = \{x' : x \in P'\}$ and let $Q' = Q \cup P''$
- Define

$$[a']_i^{\prec} = \begin{cases} [a']_i \cup \{x' : x' \in P'', \; x \in [a'] \cap P'\} & \text{if } a' \in Q \\ [a]_i^{\prec} = [a]_i \cup \{x' : x' \in P'', \; x \in [a]_i \cap P'\} & \text{if } a' \in P'' \text{ for } a \in P' \end{cases}$$

- It can be checked that $\prec$ is a quasi order and $\langle \underline{Q}, \prec \rangle$ is a QOAS with granulation $\{[a']_i^{\prec} : a' \in Q'\}$.
- If Δ' is the set of definable elements of $\langle \underline{Q'}, \prec \rangle$ and let its set of join irreducible elements be $J' = \{[a']_i^{\prec} : a' \in Q'\}$
- Let $J = \{[a]_i : a \in Q\}$ be the set of join irreducibles of Δ, and let $(\forall x \in Q)\, \xi([x]_i) = [x]_i^{\prec}$
- ξ is an order isomorphism that can be extended to $\xi^* : \Delta \to \Delta'$ as below:

$$\xi^*(H) = \cup\{\xi([x]_i) : [x]_i \subseteq H\}$$

□

The reader is invited to fill in the gaps in the above proof.

Significance and Extensions
The significance of the result has not been discussed in [68]. The result basically says that it is possible to patch pathologies by adding elements instead of removing them—the intended meaning of the pathology being that a pathology is any instance of a crisp granule failing to upper approximate at least one non crisp object. In practice, this is a strange requirement to have. In fact no practical example may be found in the literature on the point. From the viewpoint of semantics, $\mathbb{RS}$ may not even form a lattice when such pathologies are present. In the present author's view the failure of ontology is more important than that of method.

10.3.1 QO-Algebraic Semantics

The algebraic structure on Δ extends to the collections $\mathbb{R}$:

Theorem 267 $\mathbb{R} = \langle \underline{\mathbb{R}}, \cup, \cap, \Rightarrow, \bot, \top \rangle$ *is a Heyting algebra in which the operations are defined as below for any* $(a, b),\ (c, e) \in \mathbb{R}$

$$(a, b) \cup (c, e) = (a \cup c, b \cup e) \qquad \text{(join)}$$

$$(a, b) \cap (c, e) = (a \cap c, b \cap e) \qquad \text{(meet)}$$

$$(a, b) \Rightarrow (c, e) = \bigcup\{(f, g);\ (f, g) \in \mathbb{R}\, \&\, (a, b) \cap (f, g) \subseteq (c, e)\} \qquad \text{(join)}$$

$$(\emptyset, \emptyset) = \bot \qquad \text{(bottom)}$$

$$(Q, Q) = \top \qquad \text{(top)}$$

Further, the pseudo-complement on the algebra can be represented by $\sim (a, b) = (b^+, b^+)$.

Proposition 268 *If a QOAS Q satisfies*

$$(\forall a, b \in \wp(Q))\, (a^l = b^l \,\&\, a^u = b^u \longrightarrow a = b) \qquad (58)$$

then $\langle \underline{\mathbb{RS}}, \vee, \wedge, \bot, \top \rangle$ *is a distributive lattice with the operations being defined as below:*

$$(a^l, a^{u*}) \vee (b^l, b^{u*}) = ((a \cup b)^l, (a \cup b)^{u*}) \quad \text{(join)}$$

$$(a^l, a^{u*}) \wedge (b^l, b^{u*}) = ((a \cap b)^l, (a \cap b)^{u*}) \quad \text{(meet)}$$

Example 269 Let $Q = \{a, b, c, e, f\}$ and its table of predecessor neighborhoods be as in Table 9

It can be checked that $(ce, ce) \vee (\emptyset, ef)$ is not defined. So $\mathbb{RS}$ is not a lattice in general (Table 10).

Abstract representation theorems[68] relating to these algebras have been considered in chapter on representation and duality in [107] of this volume by the present author.

11 General Choice Inclusive Approach

In this section a new choice inclusive approach towards the construction of approximations and semantics is invented by the present author. This is based on her earlier work in [87, 89]. Only an outline of the proposed theory is presented as the semantic part can be approached in many ways.

The main motivation for the approach has been the requirement that approximations should be representable in terms of pairwise independent (or disjoint) granules (this is referred to as the local clarity principle for approximations)

Table 9 Neighborhoods

S	a	b	c	e	f
$[x]_i$	$\{a, e, f\}$	$\{b, e, f\}$	$\{c, e\}$	$\{e\}$	$\{e, f\}$

Table 10 Approximations

A	(A^l, A^{u*})	A	(A^l, A^{u*})	A	(A^l, A^{u*})
$\emptyset$	$(\emptyset, \emptyset)$	a	$(\emptyset, aef)$	b	$(\emptyset, bef)$
c	$(\emptyset, ce)$	e	(e, e)	ab	$(\emptyset, abef)$
bc	$(\emptyset, bcef)$	ce	(ce, ce)	ef	(ef, ef)
ac	$(\emptyset, acef)$	ae	(e, aef)	af	$(\emptyset, aef)$
cf	$(\emptyset, cef)$	abc	$(\emptyset, Q)$	bce	$(ce, bcef)$
cef	(cef, cef)	abe	$(e, abef)$	abf	$(\emptyset, abef)$
ace	$(ce, acef)$	aef	(aef, aef)	bcf	$(\emptyset, bcef)$
bef	(bef, bef)	acf	$(\emptyset, acef)$	$abce$	(ce, Q)
$bcef$	$(bcef, bcef)$	$abcf$	$(\emptyset, Q)$	$abef$	$(abef, abef)$
$acef$	$(acef, acef)$	Q	(Q, Q)	–	–

and in case existing approximations do not satisfy the requirement then some methods of attaining such approximations is provided for. An example where such approximations matter is in the following:

Example 270 Six-year-old Jessica drew a shape on a deflated balloon and blew it up. . . .

Teacher: What makes it disappear?
Jessica: Because It's stretching. Because It's growing bigger, cause we're blowing air into it. Air.
Teacher: Does air make things grow bigger?
Jessica: Yes. Because It's stretching it inside and if you stretch it inside it grows bigger on the outside as well.

The above example is adapted from [130], where this is used to argue that children have access to powerful mathematical ideas from an early age. A reading of the paper would actually suggest that the authors are ascribing much stronger ideas to the children than is the case. But that is immaterial for the present application. Some of the many concepts that may be associated with Jessica's conception can be (semicolons and line breaks separate different concepts):

- Curved surfaces split space into an interior and exterior
- Curved surfaces partition 3-dimensional Euclidean space into an exterior and an interior space
- Surfaces are curved or flat; Surfaces are flat or non-flat
- Balloons have an inner layer and an outer layer
- Balloons can stretch on the inside and expand outside.
- Balloons can expand on the inside and the outside of the balloon surface
- Objects having area separate regions of space into inside regions and outside.
- Some objects can stretch; Objects that stretch may break
- If air stretches the inside of an enclosed space then it stretches the outside as well; Air can be blown

The third concept is part of the first, while the second is not usable as a granule as many concepts are part of it. The concept of *some objects can stretch depends on an understanding of* the concept of *objects that stretch may break*. Using such a parthood relation and subcollections of concepts on which the unclear concept being approximated depends, questions about integrations of concepts that are closest to the unclear concept may be asked. In doing so, it may be of interest to avoid repetition of subconcepts or conflicting information. In the above example concepts relating to curved surfaces seem to conflict reasoning of the form *balloons can stretch on the inside and expand outside* in the sense that the reasoner would not have thought about both at same time. Such a dialectics leads to choosing an appropriate subcollection of concepts and the leads to the concept of *primitive lower approximation* as a subcollection in [89].

Definition of aggregation can be very varied in the context, the most natural interpretation being the logical conjunction of the chosen granules. But in contexts involving meaning evolution, it would be more natural to interpret from the point of view of possible meanings. In [89], approximations that avoid conflicting information, overlap and are unions of non intersecting blocks have been proposed and studied by the present author.

Let $\mathbb{S} = \langle \underline{\mathbb{S}}, \mathcal{G}, l, u, \leq, \vee, \wedge, \bot, \top \rangle$ be a higher granular operator space.

Definition 271 A *choice function* on $\mathbb{S}$ is a map $\xi : \wp(\mathcal{G}) \mapsto \wp(\mathcal{G})$ that satisfies:

1. $(\forall x)\xi(x) \subseteq x$
2. $(\forall x)\xi(\xi(x)) = \xi(x)$
3. $\xi(\emptyset) = \emptyset$
4. $(\forall x, y)(x \subset y \longrightarrow \xi(x) \subseteq \xi(y))$

Definition 272 The above function can alternatively be replaced by a *higher choice function* $\xi^* : \wp(\wp(\mathcal{G})) \longmapsto \wp(\mathcal{G})$ (the intent being to suggest a choice among possible sets of granules in clearer terms) satisfying all of the following:

1. $(\forall x)\xi^*(x) \in x$
2. $(\forall x)\xi^*(\{\xi^*(x)\}) = \xi^*(x)$
3. $\xi^*(\{\emptyset\}) = \emptyset$
4. $(\forall x, y)(x \subset y \longrightarrow \xi^*(x) \subseteq \xi^*(y))$

Definition 273 For a set of granular axioms μ, a μ-*choice function* will be a choice function ξ for which $\xi(\mathcal{G})$ satisfies μ.

Definition 274 A choice function ξ will be said to satisfy the *local clarity principle* LCP if and only if

$$(\forall x \in \wp(\mathcal{G}))\,(a, b \in \xi(x)\,\&\, a \neq b \longrightarrow (\forall h \in \wp(S)\setminus\{\emptyset\})\neg\mathbf{P}ha\,\&\,\neg\mathbf{P}hb) \tag{59}$$

In case $\mathbf{P}$ is the $\subseteq$ relation, then the condition can be simplified to

$$(\forall x \in \wp(\mathcal{G}))\,(a, b \in \xi(x)\,\&\, a \neq b \longrightarrow a \cap b \neq \emptyset) \tag{60}$$

In the contexts of [89], a unique set of granules for lower approximation of a set is determined by additional conditions on possible choices. One of the strategies used for defining a granular lower approximation of a subset A in a tolerance space S is the following:

- Form the set of blocks $\{B_1, \ldots, B_n\}$ contained in A
- Let the collection of set of mutually pairwise disjoint blocks be $\mathcal{W}$. This is ordered by inclusion.
- Form the collection $\mathcal{W}_m$ of maximal elements of the collection.
- Compute $\xi^*(\mathcal{W}_m)$ or $\xi(\cup\mathcal{W}_m)$
- Define $A^l = \cup\xi^*(\mathcal{W}_m)$, for example.
- It is possible to define other lower approximations that satisfy LCP.

For defining upper approximations, a similar strategy can be used as follows:

- Form the set of blocks $\{B_1, \ldots, B_n\}$ intersecting A
- Let the collection of set of mutually pairwise disjoint blocks whose union includes A be $\mathcal{W}^+$. This is ordered by inclusion.
- Form the collection $\mathcal{W}_m^+$ of maximal elements of the collection.
- Compute $\xi^*(\mathcal{W}_m^+)$ or $\xi(\cup\mathcal{W}_m^+)$
- Define $A^u = \cup\xi^*(\mathcal{W}_m^+)$, for example.
- It is possible to define other upper approximations that satisfy LCP. To approximate more accurately, it makes sense to use the collection $\mathcal{W}_o^+$ of minimal elements of the collection $\mathcal{W}^+$.

Given the definitions of the lower and upper approximations, the semantics of different rough objects of interest can be approached in multiple ways. The semantics in [89] is a direct algebraic approach for tolerance spaces. Examples of the choice based approach [89] have been developed recently by others in [110]. Related software has also been developed by the authors.

11.1 Lambda Lattices

The appropriate semantic domain for the semantics of bitten rough sets in Sect. 6 may be considered by some to be less natural on subjective grounds. These include difficulty with reasoning within the power set of the set of possible order-compatible partitions of the set of rough objects. A simpler semantics that includes choice in aggregation and commonality operations is outlined in this sub section. Connections of this approach to that in [89] by the present author have not been investigated.

Choice functions are used in defining the rough operations of *combining sets* and *extracting the common part of two sets*.

Definition 275 For any $a, b \in \wp(S)|\sim$, let $UB(a, b)$ and $LB(a, b)$ be the set of minimal upper bounds and the set of maximal lower bounds of a and b (these are assumed to be nonempty for all pairs a, b). If $\lambda : \wp(\wp(S)|\sim) \longmapsto \wp(S)|\sim$ is a choice function, (by definition, it is such that $(a \leq b \longrightarrow \lambda(UB(\{a, b\})) = b,\ \lambda(LB(\{a, b\})) = a)$), then let

$$a + b = \lambda(UB(\{a, b\}))$$

$$a \cdot b = \lambda(LB(\{a, b\}))$$

$\mathfrak{S} = \langle \wp(S)|\sim, +\cdot, L, \blacklozenge, \neg \rangle$ is called the *simplified algebra of the bitten granular semantics* (SGBA)

Theorem 276 *A SGBA,* $\mathfrak{B} = \left\langle \underline{B}, +\cdot, L, \blacklozenge, \neg \right\rangle$ *satisfies all of the following:*

1. $\left\langle \underline{B}, +\cdot \right\rangle$ *is a* λ*-lattice*
2. $a + b = b + a;\ a \cdot b = b \cdot a$

3. $a + a = a;\ a \cdot a = a$
4. $a + (a \cdot b) = a;\ a \cdot (a + b) = a$
5. $a + (a + (b + c)) = a + (b + c);\ a \cdot (a \cdot (b \cdot c)) = a \cdot (b \cdot c)$
6. $a + La = a;\ a \cdot La = La$
7. $a + \blacklozenge a = \blacklozenge a;\ a \cdot \blacklozenge a = a$
8. $L(La) = La;\ \blacklozenge(\blacklozenge a) = \blacklozenge a$
9. $(a + b = a \longrightarrow La + Lb = La);\ (a \cdot b = a \longrightarrow La \cdot Lb = La)$
10. $(a + b = a \longrightarrow \blacklozenge a + \blacklozenge b = \blacklozenge a);\ (a \cdot b = a \longrightarrow \blacklozenge a \cdot \blacklozenge b = \blacklozenge a)$
11. $\neg(La) = \blacklozenge(\neg a);\ \neg(\blacklozenge a) = L(\neg a)$
12. $L0 = 0,\ L1 = 1;\ \blacklozenge 0 = 0,\ \blacklozenge 1 = 1$
13. $L\blacklozenge a + \blacklozenge a = \blacklozenge a;\ L\blacklozenge a \cdot \blacklozenge a = L\blacklozenge a$
14. $La + \blacklozenge La = \blacklozenge La;\ La \cdot \blacklozenge La = La$

Proof The proof consists in verification □

11.2 Representation Problem for SGBAs

Given a tolerance space S in a bitten rough semantic perspective, associating a single SGBA as its corresponding semantics amounts to modifying the original meaning by the introduction of artificial choice functions for the purpose of forming rough union and intersection-like operations. Either a justification of such preference is required or the option would be to accept all of the possible preferences. So the default semantics must be given by a set of SGBAs indexed by the set of all possible choice functions in the lambda lattice formation context. In this perspective the semantics can be explained directly and a sequent calculus associated (and with little additional representation theory).

Let $x \in S$, then $([x]_T)^l = [x]_T$, while $([x]_T)^u_b = ([x]_T)^u \setminus (S \setminus [x]_T)^l)$. It is obvious that elements with nonempty lower approximation that are minimal with respect to the rough order will be equivalent to elements of this type. Once the order relation on the set of roughly equivalent elements has been deduced, elements of this type can be found. This permits the reconstruction of the equivalence based partition of the power set of S.

If all possible choice functions involved are not known, then it is not possible in general to determine or construct the blocks of the tolerance T. But when will a knowledge of given subsets of choice functions permit us to determine the blocks of T? This is the problem of representation of SGBAs. It is also significant in a more general algebraic setting.

12 Anti Chains for Semantics

Any set of rough objects is quasi or partially orderable. The set of rough objects of various types and crisp objects of various types is also quasi- or partially ordered. Specifically in classical or Pawlak rough sets [124], the set of roughly equivalent sets has a quasi Boolean order on it while the set of rough and crisp objects is Boolean ordered. In the classical semantic domain or classical meta level, associated with general rough sets, the set of crisp and rough objects is quasi or partially orderable. Under minimal assumptions on the nature of these objects, many orders with rough ontology can be associated—these necessarily have to do with concepts of discernibility and variations thereof. Concepts of rough objects, in these contexts, depend additionally on approximation operators and granulations used. These were part of the motivations of the development of the concept of granular operator spaces by the present author in [98].

In quasi or partially ordered sets, sets of mutually incomparable elements are called *antichains* (for basics see[37, 54, 67]). The possibility of using antichains of rough objects for a possible semantics was mentioned in [95, 96, 99] by the present author and developed in [98, 106]. At one level, any rough set theory presented with approximations in operator form and well defined concepts of crisp and rough objects should suffice for formalization in the antic chain perspective—but then the semantics can be expected to miss key aspects of the relation between rough and crisp objects. In particular, the semantic approach is applicable to a large class of operator based rough sets including specific cases of RYS [91], other general approaches like [29, 58, 60] and all specific cases of relation based and cover based rough set approaches. In [29], negation like operators are assumed in general and these are not definable operations relative the order related operations/relation. A key problem in many of the latter types of approaches is of closure of possible aggregation and commonality operations [63, 99, 174, 178].

In the present section, the semantics of [98, 106] is presented in a compact way with additional examples. Connections with the concept of knowledge in the settings is also explored in some depth and related interpretations are offered. The basic framework of granular operator spaces used in [98] is generalized in this section as most of the mathematical parts carry over. The semantics of [98], as improved in the [106] by way of ternary terms, is very general, open ended, extendable and optimal for lateral studies. Most of it applies to *general granular operator spaces*, introduced in a separate paper by the present author. In the same framework, the machinery for isolation of deductive systems is developed by her and studied from a purely algebraic logic point of view. New results on representation of roughness related objects are also developed. Last but not least, the concept of knowledge as considered in [91, 94, 99, 127] is recast in very different terms for describing the knowledge associated with representation of data by maximal antichains. These representations are also examined for compatibility with triadic semiotics (that is not necessarily faithful to Peirce's ideas) for integration with ontology. Philosophical questions relating to perdurantism and endurantism are also solved in some directions. Illustrative examples that demonstrate applicability to *human reasoning contexts involving approximations but no reasonable data tables* have also been constructed in [106]. Parts of this are also presented.

In the next subsection, relevant background is presented. In the following subsection, the essential algebraic logic approach used is outlined. In the third subsection, possible operations on sets of maximal antichains derived from granular operator spaces are considered, AC-algebras are defined and their generation is studied. Representation of antichains derived from the context are also improved and earlier examples are refined. Ternary deduction terms in the context of the AC-algebra are explored next and various results are proved. The algebras of quasi-equivalential rough sets formed by related procedures is presented to illustrate key aspects of the semantics in the fifth and sixth subsections. The connections with epistemology and knowledge forms the following section. Further directions are provided in Sect. 9.

Background

Set framework with operators will be used as all considerations will require quasi orders in an essential way. The evolution of the operators need not be induced by a cover or a relation (corresponding to cover or relation based systems respectively), but these would be special cases. The generalization to some rough Y-systems RYS (see [91] for definitions), will of course be possible as a result.

Theorem 277 *Some known results relating to antichains and lattices are the following:*

1. *Let X be a partially ordered set with longest chains of length r, then X can be partitioned into k number of antichains implies $r \leq k$.*
2. *If X is a finite poset with k elements in its largest antichain, then a chain decomposition of X must contain at least k chains.*
3. *The poset $AC_m(X)$ of all maximum sized antichains of a poset X is a distributive lattice.*
4. *For every finite distributive lattice L and every chain decomposition C of J_L (the set of join irreducible elements of L), there is a poset X_C such that $L \cong AC_m(X_C)$.*

Proof The proof of the duality results can be found in the chapter on duality [107] in this volume. Proofs of the first three of the assertions can also be found in in [37, 69] for example. Many proofs of results related to Dilworth's theorems are known in the literature and some discussion can be found in [69] (pages 126–135).

1. To prove the first, start from a chain decomposition and recursively extract the minimal elements from it to form r number of antichains.
2. This is proved by induction on the size of X across many possibilities.
3. See [37, 69] for details.
4. In [67], the last connection between chain decompositions and representation by antichains reveals important gaps—there are other posets X that satisfy $L \cong AC_m(X)$. Further the restriction to posets is too strong and can be relaxed in many ways [153].

□

12.1 Parthood and Frameworks

Many of the philosophical issues relating to mereology take more specific forms in the context of rough sets in general and in the GSP framework. The axioms of parthood that can be seen to be not universally satisfied in all rough contexts include the following:

$$\mathbf{P}ab \& \mathbf{P}bc \longrightarrow \mathbf{P}ac \quad \text{(Transitivity)}$$

$$(\mathbf{P}ab \leftrightarrow \mathbf{P}ba) \longrightarrow a = b \quad \text{(Extensionality)}$$

$$(\mathbf{P}ab \& \mathbf{P}ba \longrightarrow a = b) \quad \text{(Antisymmetry)}$$

This affords many distinct concepts of *proper parthoods* $\mathbb{P}$:

$$\mathbb{P}ab \text{ if and only if } \mathbf{P}ab \,\&\, a \neq b \quad \text{(PP1)}$$

$$\mathbb{P}ab \text{ if and only if } \mathbf{P}ab \,\&\, \neg \mathbf{P}ba \quad \text{(PP2)}$$

$$\mathbb{P}ab \longrightarrow (\exists z)\mathbf{P}zb \,\&\, (\forall w)\neg(\mathbf{P}wa \,\&\, \mathbf{P}wz) \quad \text{(WS)}$$

PP1 does not follow from PP2 without antisymmetry and WS (weak supplementation) is a kind of proper parthood. All this affords a mereological approach with much variation to abstract rough sets.

12.2 Deductive Systems

In this section, key aspects of the approach to ternary deductive systems in [24, 25] are presented. These are intended as natural generalizations of the concepts of ideals and filters and classes of congruences that can serve as subsets or subalgebras closed under consequence operations or relations (also see [51]).

Definition 278 Let $\mathbb{S} = \langle S, \Sigma \rangle$ be an algebra, then the set of term functions over it will be denoted by $\mathbf{T}^{\Sigma}(\mathbb{S})$ and the set of r-ary term functions by $\mathbf{T}_r^{\Sigma}(\mathbb{S})$. Further let

$$g \in \mathbf{T}_1^{\Sigma}(\mathbb{S}),\ z \in S,\ \tau \subset \mathbf{T}_3^{\Sigma}(\mathbb{S}), \quad (0)$$

$$g(z) \in \Delta \subset S, \quad (1)$$

$$(\forall t \in \tau)(a \in \Delta \,\&\, t(a, b, z) \in \Delta \longrightarrow b \in \Delta), \quad (2)$$

$$(\forall t \in \tau)(b \in \Delta \longrightarrow t(g(z), b, z) \in \Delta), \quad (3)$$

then Δ is a $(g, z) - \tau$-*deductive system* of $\mathbb{S}$. If further for each k-ary operation $f \in \Sigma$ and ternary $p \in \tau$

$$(\forall a_i, b_i \in S)(\&_{i=1}^{k} p(a_i, b_i, z) \in \Delta \longrightarrow p(f(a_1, \ldots, a_k), f(b_1, \ldots, b_k), x) \in \Delta), \tag{61}$$

then Δ is said to be compatible.

τ is said to be a *g-difference system* for $\mathbb{S}$ if τ is finite and the condition

$$(\forall t \in \tau) t(a, b, c) = g(c) \text{ if and only if } a = b \text{ holds.} \tag{62}$$

A variety $\mathcal{V}$ of algebras is regular with respect to a unary term g if and only if for each $S \in \mathcal{V}$,

$$(\forall b \in S)(\forall \sigma, \rho \in con(S))([g(b)]_\sigma = [g(b)]_\rho \longrightarrow \sigma = \rho). \tag{63}$$

It should be noted that in the above τ is usually taken to be a finite subset and a variety has a g-difference system if and only if it is regular with respect to g.

Proposition 279 *In the above definition, it is provable that*

$$(\forall t \in \tau)(t(g(z), b, z) \in \Delta \longrightarrow b \in \Delta). \tag{64}$$

Definition 280 In the context of Def. 278, $\Theta_{Delta,z}$ shall be a relation induced on S by τ as per the following

$$(a, b) \in \Theta_{\Delta,z} \text{ if and only if } (\forall t \in \tau)\, t(a, b, z) \in \Delta. \tag{65}$$

Proposition 281 *In the context of Def.280,* $\Delta = [g(z)]_{\Theta_{\Delta,z}}$.

Proposition 282 *Let* $\tau \subset T_3^{\Sigma}(\mathbb{S})$ *with the algebra* $\mathbb{S} = \langle S, \Sigma \rangle$, $v \in T_1^{\Sigma}(\mathbb{S})$, $e \in S$, $K \subseteq S$ *and let* $\Theta_{K,e}$ *be induced by* τ. *If* $\Theta_{K,e}$ *is a reflexive and transitive relation such that* $K = [v(e)]_{Theta_{K,e}}$, *then* K *is a* (v, e)- τ*-deductive system of* $\mathbb{S}$.

Theorem 283 *Let* h *is a unary term of a variety* $\mathcal{V}$ *and* τ *a* h*-difference system for* $\mathcal{V}$. *If* $\mathbb{S} \in \mathcal{V} m$ $\Theta \in Con(\mathbb{S})$, $z \in S$ *and* $\Delta = [h(z)]_\Theta$, *then* $\Theta_{\Delta,z} = \Theta$ *and* Δ *is a compatible* (h, z)-τ*-deductive system of* $\mathbb{S}$.

The converse holds in the following sense:

Theorem 284 *If* h *is a unary term of a variety* $\mathcal{V}$, τ *is a* h*-difference system in it,* $\mathbb{S} \in \mathcal{V}$, $z \in S$ *and if* Δ *is a compatible* (h, z)-τ*-deductive system of* $\mathbb{S}$, *then* $\Theta_{\Delta,z} \in Con(\mathbb{S})$ *and* $\Delta = [g(z)]_{\Theta_{\Delta,z}}$.

When $\mathcal{V}$ is regular relative h, then $\mathcal{V}$ has a h-difference system relative τ and for each $\mathbb{S} \in \mathcal{V}$, $z \in S$ and $\Delta \subset S$, $\Delta = [h(z)]$ if and only if Δ is a (h, z)- τ-deductive system of $\mathbb{S}$.

In each case below, $\{t\}$ is a h-difference system ($x \oplus y = ((x \wedge y^*)^* \wedge (x^* \wedge y)^*)^*$):

$$h(z) = z \ \& \ t(a, b, c) = a - b + c \qquad \text{(Variety of Groups)}$$

$$h(z) = z \ \& \ t(a, b, c) = a \oplus b \oplus c \qquad \text{(Variety of Boolean Algebras)}$$

$$h(z) = z^{**} \ \& \ t(a, b, c) = (a + b) + c \qquad \text{(Variety of p-Semilattices)}$$

12.3 Anti Chains for Representation

In this section, the main algebraic semantics of [98] is summarized, extended to AC-algebras and relative properties are studied. It is also proved that the number of maximal antichains required to generate the AC-algebra is rather small.

Definition 285 $\mathbb{A}, \mathbb{B} \in \underline{S}| \approx$, will be said to be *simply independent* (in symbols $\Xi(\mathbb{A}, \mathbb{B})$)if and only if

$$\neg(\mathbb{A} \Subset \mathbb{B}) \text{ and } \neg(\mathbb{B} \Subset \mathbb{A}). \tag{66}$$

A subset $\alpha \subseteq \underline{S}| \approx$ will be said to be *simply independent* if and only if

$$(\forall \mathbb{A}, \mathbb{B} \in \alpha)\, \Xi(\mathbb{A}, \mathbb{B}) \vee (\mathbb{A} = \mathbb{B}). \tag{67}$$

The set of all simply independent subsets shall be denoted by $\mathcal{SY}(S)$.

A *maximal simply independent subset*, shall be a simply independent subset that is not properly contained in any other simply independent subset. The set of maximal simply independent subsets will be denoted by $\mathcal{SY}_m(S)$. On the set $\mathcal{SY}_m(S)$, $\ll$ will be the relation defined by

$$\alpha \ll \beta \text{ if and only if } (\forall \mathbb{A} \in \alpha)(\exists \mathbb{B} \in \beta)\, \mathbb{A} \Subset \mathbb{B}. \tag{68}$$

Theorem 286 *$\langle \mathcal{SY}_m(S), \ll \rangle$ is a distributive lattice.*

Analogous to the above, it is possible to define essentially the same order on the set of maximal antichains of $\underline{S}| \approx$ denoted by $\mathfrak{S}$ with the $\Subset$ order. This order will be denoted by $\lessdot$—this may also be seen to be induced by maximal ideals.

Theorem 287 *If $\alpha = \{\mathbb{A}_1, \mathbb{A}_2, \ldots, \mathbb{A}_n, \ldots\} \in \mathfrak{S}$, and if L is defined by*

$$L(\alpha) = \{\mathbb{B}_1, \mathbb{B}_2, \ldots, \mathbb{B}_n, \ldots\} \tag{69}$$

with $X \in \mathbb{B}_i$ if and only if $X^l = \mathbb{A}_i^{ll} = \mathbb{B}_i^l$ and $X^u = \mathbb{A}_i^{lu} = \mathbb{B}_i^u$, then L is a partial operation in general.

Proof The operation is partial because $L(\alpha)$ may not always be a maximal antichain. This can happen in general in which the properties $A^{ll} \subset A^l$ and/or

$A^{ul} \subset A$ hold for some elements. The former possibility is not possible by our assumptions, but the latter is scenario is permitted.

Specifically this can happen in bitten rough sets when the bitten upper approximation [86] operator is used in conjunction with the lower approximation. But many more examples are known in the literature (see [91]). □

Definition 288 Let $\chi(\alpha \cap \beta) = \{\xi;\ \xi \text{ is a maximal antichain } \& \ \alpha \cap \beta \subseteq \xi\}$ be the set of all possible extensions of $\alpha \cap \beta$. The function $\delta : \mathfrak{S}^2 \longmapsto \mathfrak{S}$ corresponding to *extension under cognitive dissonance* will be defined as per $\delta(\alpha, \beta) \in \chi(\alpha \cap \beta)$ and (LST means *maximal subject to*)

$$\delta(\alpha, \beta) = \begin{cases} \xi, & \text{if } \xi \cap \beta \text{ is a maximum subject to } \xi \neq \beta \text{ and } \xi \text{ is unique ,} \\ \xi, & \text{if } \xi \cap \beta \ \& \ \xi \cap \alpha \text{ are LST } \xi \neq \beta, \alpha \text{ and } \xi \text{ is unique ,} \\ \beta, & \text{if } \xi \cap \beta \ \& \ \xi \cap \alpha \text{ are LST } \& \ \xi \neq \beta, \alpha \text{ but } \xi \text{ is not unique ,} \\ \beta, & \text{if } \chi(\alpha \cap \beta) = \{\alpha, \beta\}. \end{cases} \tag{70}$$

Definition 289 In the context of the above definition, the function $\varrho : \mathfrak{S}^2 \longmapsto \mathfrak{S}$ corresponding to *radical extension* will be defined as per $\varrho(\alpha, \beta) \in \chi(\alpha \cap \beta)$ and (MST means *minimal subject to*)

$$\varrho(\alpha, \beta) = \begin{cases} \xi, & \text{if } \xi \cap \beta \text{ is a minimum under } \xi \neq \beta \text{ and } \xi \text{ is unique ,} \\ \xi, & \text{if } \xi \cap \beta \ \& \ \xi \cap \alpha \text{ are MST } \xi \neq \beta, \alpha \text{ and } \xi \text{ is unique ,} \\ \alpha, & \text{if } (\exists \xi)\, \xi \cap \beta \ \& \ \xi \cap \alpha \text{ are MST } \xi \neq \beta, \alpha \ \& \ \xi \text{ is not unique ,} \\ \alpha, & \text{if } \chi(\alpha \cap \beta) = \{\alpha, \beta\}. \end{cases} \tag{71}$$

Theorem 290 *The operations* ϱ, δ *satisfy all of the following:*

$$\varrho,\ \delta \text{ are groupoidal operations,} \tag{1}$$

$$\varrho(\alpha, \alpha) = \alpha \tag{2}$$

$$\delta(\alpha, \alpha) = \alpha \tag{3}$$

$$\delta(\alpha, \beta) \cap \beta \subseteq \delta(\delta(\alpha, \beta), \beta) \cap \beta \tag{4}$$

$$\delta(\delta(\alpha, \beta), \beta) = \delta(\alpha, \beta) \tag{5}$$

$$\varrho(\varrho(\alpha, \beta), \beta) \cap \beta \subseteq \varrho(\alpha, \beta) \cap \beta. \tag{6}$$

Proof

1. Obviously ϱ, δ are closed as the cases in their definition cover all possibilities. So they are groupoid operations. Associativity can be easily shown to fail through counterexamples.
2. Idempotence follows from definition.
3. Idempotence follows from definition.

For the rest, note that by definition, $\alpha \cap \beta \subseteq \delta(\alpha, \beta)$ holds. The intersection with β of $\delta(\alpha, \beta)$ is a subset of $\delta(\delta(\alpha, \beta), \beta) \cap \beta$ by recursion. □

In general, a number of possibilities (potential non-implications) like the following are satisfied by the algebra: $\alpha \lessdot \beta \,\&\, \alpha \lessdot \gamma \nrightarrow \alpha \lessdot \delta(\beta, \gamma)$. Given better properties of l and u, interesting operators can be induced on maximal antichains towards improving the properties of ϱ and δ. The key constraint hindering the definition of total l, u induced operations can be avoided in the following way:

Definition 291 In the context of Thm 287, operations $\Box, \Diamond$ can be defined as follows:

- Given $\alpha = \{\mathbb{A}_1, \mathbb{A}_2, \ldots, \mathbb{A}_n, \ldots\} \in \mathfrak{S}$, form the set $\gamma(\alpha) = \{\mathbb{A}_1^l, \mathbb{A}_2^l, \ldots, \mathbb{A}_n^l, \ldots\}$. If this is an antichain, then α would be said to be *lower pure*.
- Form the set of all relatively maximal antichains $\gamma_+(\alpha)$ from $\gamma(\alpha)$.
- Form all maximal antichains $\gamma_*(\alpha)$ containing elements of $\gamma_+(\alpha)$ and set $\Box(\alpha) = \wedge\gamma_*(\alpha)$
- For $\Diamond$, set $\pi(\alpha) = \{\mathbb{A}_1^u, \mathbb{A}_2^u, \ldots, \mathbb{A}_n^u, \ldots\}$. If this is an antichain, then α would be said to be *upper pure*.
- Form the set of all relatively maximal antichains $\pi_+(\alpha)$ from $\pi(\alpha)$
- Form all maximal antichains $\pi_*(\alpha)$ containing elements of $\pi_+(\alpha)$ and set $\Diamond(\alpha) = \vee\pi_*(\alpha)$

Theorem 292 *In the context of the above definition, the following hold:*

$$\alpha \lessdot \beta \longrightarrow \Box(\alpha) \lessdot \Box(\beta) \,\&\, \Diamond(\alpha) \lessdot \Diamond(\beta)$$

$$\Box(\alpha) \lessdot \alpha \lessdot \Diamond(\alpha), \;\; \Box(0) = 0 \,\&\, \Diamond(1) = 1$$

Based on the above properties, the following algebra can be defined.

Definition 293 By a *Concrete AC* algebra (AC -algebra) will be meant an algebra of the form $\langle \mathfrak{S}, \varrho, \delta, \vee, \wedge, \Box, \Diamond, 0, 1\rangle$ associated with a granular operator space S satisfying all of the following:

- $\langle \mathfrak{S}, \vee, \wedge\rangle$ is a bounded distributive lattice derived from a granular operator space as in the above.
- $\varrho, \delta, \Box, \Diamond$ are as defined above

The following concepts of ideals and filters are of interest as deductive systems in a natural sense and relate to ideas of rough consequence (detailed investigation will appear separately).

Definition 294 By a *LD-ideal* (resp. LD-filter)) K of an AC-algebra $\mathfrak{S}$ will be meant a lattice ideal (resp. filter) that satisfies:

$$(\forall \alpha \in K)\, \Box(\alpha), \Diamond(\alpha) \in K \tag{72}$$

By a *VE-ideal* (resp. VE-filter)) K of an AC-algebra $\mathfrak{S}$ will be meant a lattice ideal (resp. filter) that satisfies:

$$(\forall \xi \in \mathfrak{S})(\forall \alpha \in K)\, \varrho(\xi, \alpha), \delta(\xi, \alpha) \in K \tag{73}$$

Proposition 295 *Every VE filter is closed under* $\Diamond$

12.4 Generating AC-Algebras

Now it will be shown below that specific subsets of AC-algebras suffice to generate the algebra itself and that the axioms satisfied by the granulation affect the generation process and properties of AC-algebras and forgetful variants thereof.

An element $x \in \mathfrak{S}$ will be said to be *meet irreducible* (resp. join irreducible) if and only if $\wedge\{x_i\} = x \longrightarrow (\exists i)\, x_i = x$ (resp. $\vee\{x_i\} = x \longrightarrow (\exists i)\, x_i = x$). Let $W(S),\ J(S)$ be the set of meet and join irreducible elements of $\mathfrak{S}$ and let $l(\mathfrak{S})$ be the length of the distributive lattice.

Theorem 296 *All of the following hold:*

- $(\mathfrak{S}, \vee, \wedge, 0, 1)$ *is a isomorphic to the lattice of principal ideals of the poset of join irreducibles.*
- $l(\mathfrak{S}) = \#(J(S)) = \#(W(S))$
- $J(S)$ *is not necessarily the set of sets of maximal antichains of granules in general.*
- *When* $\mathcal{G}$ *satisfies mereological atomicity that is* $(\forall a \in \mathcal{G})(\forall b \in S)(\mathbf{P}ba,\ a^l = a^u = a \longrightarrow a = b)$*, and all approximations are unions of granules, then elements of* $J(S)$ *are proper subsets of* $\mathcal{G}$.
- *In the previous context,* $W(S)$ *must necessarily consist of two subsets of* S *that are definite and are not parts of each other.*

Proof

- The first assertion is a well known.
- Since the lattice is distributive and finite, its length must be equal to the number of elements in $J(S)$ and $W(S)$. For a proof see [108].
- Under the minimal assumptions on $\mathcal{G}$, it is possible for definite elements to be properly included in granules as in esoteric or prototransitive rough sets [85, 95]. These provide the required counterexamples.
- The rest of the assertions follows from the nature of maximal antichains and the constructive nature of approximations.

□

Theorem 297 *In the context of the previous theorem if* $R(\Diamond),\ R(\Box)$ *are the ranges of the operations* $\Diamond, \Box$ *respectively, then these have a induced lattice order on them. Denoting the associated lattice operations by* $\curlyvee, \curlywedge$ *on* $R(\Diamond)$*, it can be shown that*

- *$R(\Diamond)$ can be reconstructed from $J(R(\Diamond)) \cup W(R(\Diamond))$.*
- *$R(\Box)$ can be reconstructed from $J(R(\Box)) \cup W(R(\Box))$.*
- *When $\mathcal{G}$ satisfies mereological atomicity and absolute crispness (i.e. $(\forall x \in \mathcal{G})\, x^l = x^u = x$), then $R(\Diamond)$ are lattices which are constructible from two sets A, C (with $A = \{\mathcal{G} \cup \{g_1 \cup g_2\}^u \setminus \{g_1, g_2\};\ g_1, g_2 \in \mathcal{G}\}$ and C being the set of two element maximal antichains formed by sets that are upper approximations of other sets).*

Proof It is clear that $R(\Diamond)$ is a lattice in the induced order with $J(R(\Diamond))$ and $W(R(\Diamond))$ being the partially ordered sets of join and meet irreducible elements respectively. This holds because the lattice is finite.

The reconstruction of the lattice can be done through the following steps:

- Let $Z = J(R(\Diamond)) \cup W(R(\Diamond))$. This is a partially ordered set in the order induced from $R(\Diamond)$.
- For $b \in J(R(\Diamond))$ and $a \in W(R(\Diamond))$, let $b \prec a$ if and only if $a \neq b$ in $R(\Diamond)$.
- On the new poset Z with $\prec$, form sets including elements of $W(R(\Diamond))$ connected to it.
- The set of union of all such sets including empty set ordered by inclusion would be isomorphic to the original lattice. [108]
- Under additional assumptions on $\mathcal{G}$, the structure of Z can be described further.

When the granulation satisfies the properties of crispness and mereological atomicity, then $A = J(R(\Diamond))$ and $C = W(R(\Diamond))$. So the third part holds as well.

□

The results motivate this concept of purity: A maximal antichain will be said to *pure* if and only if it is both lower and upper pure.

12.5 Enhancing the Anti Chain Based Representation

An integration of the orders on sets of maximal antichains or antichains and the representation of rough objects and possible orders among them leads to interesting multiple orders on the resulting structure. A major problem is that of defining the orders or partials thereof in the first place among the various possibilities.

Definition 298 By the *rough interpretation of an antichain* will be meant the sequence of pairs obtained by substituting objects in the rough domain in place of objects in the classical perspective. Thus if $\alpha = \{a_1, a_2 \ldots, a_p\}$ is a antichain, then its rough interpretation would be ($\pi(a_i) = (a_i^l, a_i^u)$ for each i)

$$\underline{\alpha} = \{\pi(a_1), \pi(a_2), \ldots, \pi(a_p)\}. \tag{74}$$

Proposition 299 *It is possible that some rough objects are not representable by maximal antichains.*

Proof Suppose the objects represented by the pairs (a, b) and (e, f) are such that $a = e$ and $b \subset f$, then it is clear that any maximal antichain containing (e, f) cannot contain any element from $\{x : x^l = a \,\&\, x^u = b\}$. This situation can happen, for example, in the models of proto transitive rough sets. □

Definition 300 A set of maximal antichains V will be said to be *fluent* if and only if $(\forall x)(\exists \alpha \in V)(\exists (a, b) \in \alpha)\, x^l = a \,\&\, x^u = b$.

It will be said to be *well fluent* if and only if it is fluent and no proper subset of it is fluent.

A related problem is of finding conditions on $\mathfrak{G}$, that ensure that V is fluent.

12.6 Ternary Deduction Terms

Since AC-algebras are distributive lattices with additional operations, a natural strategy should be to consider terms similar to Boolean algebras and p-Semilattices. For isolating deductive systems in the sense of Sect. 12.2, a strategy can be through complementation-like operations. This motivates the following definition:

Definition 301 In a AC-algebra $\mathfrak{S}$, if an antichain $\alpha = (X_1, X_2, \ldots, X_n)$, then some possible general complements on the schema

$$\alpha^c = \mathfrak{H}(X_1^c, X_2^c, \ldots, X_n^c)$$

are as follows:

$$X_i^* = \{w;\ (\forall a \in X_i)\, \neg \mathbf{P}aw \,\&\, \neg \mathbf{P}wa\} \qquad \text{(Class A)}$$

$$X_i^\# = \{w;\ (\forall a \in X_i)\, \neg a^l = w^l \text{ or } \neg a^u = w^u\} \qquad \text{(Light)}$$

$$X_i^\flat = \{w;\ (\forall a \in X_i)\, \neg a^l = w^l \text{ or } \neg a^{uu} = w^{uu}\} \qquad \text{(UU)}$$

$\mathfrak{H}$ is intended to signify the maximal antichain containing the set if that is definable.

Note that under additional assumptions (similarity spaces), the light complementation is similar to the preclusivity operation in [19] for Quasi BZ-lattice or Heyting-Wajsburg semantics and variants.

The above operations on α are partial in general, but can be made total with the help of an additional order on α and the following procedure:

1. Let $\alpha = \{X_1, X_2, \ldots, X_n\}$ be a finite sequence,
2. Form α^c and split into longest ACs in sequence,
3. Form maximal ACs containing each AC in sequence
4. Join resulting maximal ACs.

Proposition 302 *Every general complement defined by the above procedure is well defined.*

Proof

- Suppose $\{X_1^c, X_2^c\}, \{X_3^c, \ldots X_n^c\}$ form antichains, but $\{X_1^c, X_2^c, X_3^c\}$ is not an antichain.
- Then form the maximal antichains $\eta_1, \ldots, \eta_p$ containing either of the two antichains.
- The join of this finite set of maximal antichains is uniquely defined. By induction, it follows that the operations are well defined.

□

12.6.1 Translations

As per the approach of Sect. 12.2, possible definitions of translations are as follows:

Definition 303 A *translation* in a AC-algebra $\mathfrak{S}$ is a $\sigma : \mathfrak{S} \longmapsto \mathfrak{S}$ that is defined in one of the following ways (for a fixed $a \in \mathfrak{S}$):

$$\sigma_\theta(x) = \theta(a, x) \; ; \theta \in \{\vee, \wedge, \varrho, \delta\}$$

$$\sigma_\mu(x) = \mu(x) \; ; \mu \in \{\Box, \Diamond\}$$

$$\sigma_t(x) = (x \oplus a) \oplus b \text{ for fixed a, b}$$

$$\sigma_{t+}(x) = (a \oplus b) \oplus x \text{ for fixed a, b}$$

Theorem 304

$$\sigma_\vee(0) = a = \sigma_\vee(a \; ; \sigma_\vee(1) = 1$$

$$Ran(\sigma_\vee) \textit{ is the principal filter generated by a}$$

$$Ran(\sigma_\wedge) \textit{ is the principal ideal generated by a}$$

$$x \lessdot w \longrightarrow \sigma_\vee(x) \lessdot \sigma_\vee(w) \,\&\, \sigma_\wedge(x) \lessdot \sigma_\wedge(w)$$

Proof

- Let $\mathbb{F}(a)$ be the principal lattice filter generated by a.
- If $a \lessdot w$, then $a \vee w = w = \sigma_\vee(w)$. So $w \in Ran(\sigma_\vee)$.
- $\sigma_\vee(x) \wedge \sigma_\vee(w) = (a \vee x) \wedge (a \vee w) = a \vee (x \wedge w) = \sigma_\vee(x \wedge w)$.
- So if $x, w \in Ran(\sigma_\vee)$, then $x \wedge w, x \vee w \in Ran(\sigma_\vee)$
- Similarly it is provable that $Ran(\sigma_\wedge)$ is the principal ideal generated by a.

□

12.6.2 Ternary Terms and Deductive Systems

Possible ternary terms that can cohere with the assumptions of the semantics include the following $t(a, b, z) = a \wedge b \wedge z$, $t(a, b, z) = a \oplus b \oplus z$ ($\oplus$ being as indicated earlier) and $t(a, b, z) = \Box(a \wedge b) \wedge z$. These have admissible deductive systems associated. Further under some conditions on granularity, the distributive lattice structure associated with $\mathfrak{S}$ becomes pseudo complemented.

Theorem 305 *If $t(a, b, z) = a \wedge b \wedge z$, $\tau = \{t\}$, $z \in H$, $h(x) = x$ $\sigma(x) = x \wedge z$ and if H is a ternary τ-deduction system at z, then it suffices that H be an filter.*

Proof All of the following must hold:

- If $a \in H$, $t(z, a, z) = a \wedge z \in H$
- If $t(a, b, z) \in H$, then $t(\sigma(a), \sigma(b), z) = t(a, b, z) \in H$
- If $a, t(a, b, z) \in H$ then $t(a, b, z) = (a \wedge z) \wedge b \in H$. But H is a filter, so $b \in H$.

□

Theorem 306 *If $t(a, b, z) = (a \vee (\Box b)) \wedge z$, $\tau = \{t\}$, $z \in H$, $h(x) = x$ $\sigma(x) = x \wedge z$ and if H is a ternary τ-deduction system at z, then it suffices that H be a principal LD-filter generated by z.*

Proof All of the following must hold:

- If $a \in H$, $t(z, a, z) = (z \vee (\Box a)) \wedge z \in H$ because $(z \vee (\Box a)) \in H$.
- If $t(a, b, z) \in H$, then $t(\sigma(a), \sigma(b), z) = t((a \wedge z), (b \wedge z), z) = ((a \wedge z) \vee \Box(b \wedge z)) \wedge z \in H$
- If $a, t(a, b, z) \in H$ then $t(a, b, z) = (a \vee \Box(b)) \wedge z = (a \wedge z) \vee (\Box(b) \wedge z) \in H$. But H is a LD-filter, so $a \vee \Box(b) \in H$. This implies $\Box(b) \in H$, which in turn yields $b \in H$.

□

In the above two theorems, the conditions on H can be weakened considerably. The converse questions are also of interest.

The existence of pseudo complements can also help in defining ternary terms that determine deductive systems (or subsets closed under consequence). In general, pseudo complementation $\circledast$ is a partial unary operation on $\mathfrak{S}$ that is defined by $x^{\circledast} = \max\{a\,;\ a \wedge x = 0\}$ (if the greatest element exists).

There is no one answer to the question of existence as it depends on the granularity assumptions of representation and stability of granules. The following result guarantees pseudo complementation (in the literature, there is no universal approach—it has always been the case that in some case they exist):

Theorem 307 *In the context of AC-algebras, if the granulation satisfies mereological atomicity and absolute crispness, then a pseudo complementation is definable.*

Proof Under the conditions on the granulation, it is possible to form the rough interpretation of each antichain. Moreover the granules can be moved in every case to construct the pseudo complement. The inductive steps in this proof have been omitted. □

12.7 Extended Abstract Example-1

The following example is intended to illustrate some aspects of the intricacies of the inverse problem situation where anti chains may be described. It is done within the relation based paradigm and the assumption that objects are completely determined by their properties.

Let $\S = \{a, b, c, e, f\}$ and let R be a binary relation on it defined via

$$R = \{(a, a), (b, b), (c, c), (a, b), \\ (c, e), (e, f), (e, c), (f, e), (e, b)\}$$

If the formula for successor neighborhoods is

$$[x] = \{z\,;\ Rzx\},$$

then the table for successor neighborhoods would be as given in Table 11.

Using the definitions

$$x^l = \bigcup_{[z]\subseteq x} [z] \ \&\ x^u = \bigcup_{[z]\cap x\neq\emptyset} [z],$$

the approximations and rough objects of Table 12 follow (strings of letters of the form *abe* are intended as abbreviation for the subset $\{a, b, e\}$ and $\lrcorner$ is for , among subsets).

Under the rough inclusion order, the bounded lattice of rough objects in Fig. 7 (arrows point towards smaller elements) is the result.

From this ordered structure, maximal antichains can be evaluated by standard algorithms or by a differential process of looking at elements, their order ideals (and order filters) and maximal antichains that they can possibly form. In the figure, for example, elements in the order ideals of 69 cannot form antichains with it. This computation is targeted at representation in terms of relatively exact objects. The direct computation that is likely to come first before representation in practice is presented after Table 13 in which some of the maximal antichains are computed.

Table 11 Successor neighborhoods

Objects **E**	a	b	c	e	f
Neighborhoods [**E**]	$\{a\}$	$\{a, b, e\}$	$\{c, e\}$	$\{c, f\}$	$\{e\}$

Table 12 Approximations and rough objects

Rough object **x**	$\mathbf{z}^{\mathbf{l}}$	$\mathbf{z}^{\mathbf{u}}$	RO identifier
$\{a \lrcorner b \lrcorner ab\}$	$\{a\}$	$\{abe\}$	{3}
$\{ae \lrcorner abe\}$	$\{a\}$	$\{abce\}$	{6}
$\{e \lrcorner be\}$	$\{e\}$	$\{abec\}$	{9}
$\{c\}$	$\{\emptyset\}$	$\{cef\}$	{15}
$\{f\}$	$\{\emptyset\}$	$\{cf\}$	{24}
$\{cf\}$	$\{cf\}$	$\{cef\}$	{27}
$\{bc \lrcorner bf\}$	$\{\emptyset\}$	$\{S\}$	{30}
$\{ac \lrcorner af \lrcorner abc \lrcorner abf\}$	$\{a\}$	$\{S\}$	{33}
$\{aef\}$	$\{ae\}$	$\{S\}$	{36}
$\{ef \lrcorner bef\}$	$\{e\}$	$\{S\}$	{42}
$\{ec \lrcorner bce\}$	$\{ec\}$	$\{S\}$	{45}
$\{bcf\}$	$\{fc\}$	$\{S\}$	{51}
$\{abef\}$	$\{abe\}$	$\{S\}$	{54}
$\{ace\}$	$\{ace\}$	$\{S\}$	{60}
$\{acf\}$	$\{acf\}$	$\{S\}$	{63}
$\{ecf \lrcorner bcef\}$	$\{cef\}$	$\{S\}$	{69}
$\{abcf\}$	$\{abcf\}$	$\{S\}$	{72}
$\{abce\}$	$\{abcf\}$	$\{S\}$	{78}

{60, 54, 69, 72} is a maximal antichain because no more elements can be added to the set without violating the incomparability assumption. Note that the singletons {0} and {1} are also maximal antichains by definition. A diagram of the associated distributive lattice will not be attempted because of the number of elements.

12.7.1 Comparative Computations

In practice, the above table corresponds to only one aspect of information obtained from information systems. The scope of the anti chain based is intended to be beyond that including the inverse problem [91]. The empirical aspect is explained in this part.

Antichains are formed from $\wp(S)$ or subsets of it with some implicit temporal order (because of the order in which elements are accessed). If the elements of $\wp(S)$ are accessed in lexicographic order, and the sequence is decomposed by rough object discernibility alone, then it would have the following form ($\lceil, \rceil$ being group boundaries):

$$\{\lceil\{a\}\rceil, \lceil\{b\}\,\{c\}, \{e\}, \{f\}\rceil, \lceil\{a, b\}, \{a, c\}, \{a, e\}\rceil,$$
$$\lceil\{a, f\}, \{b, c\}, \{b, e\}\rceil,$$
$$\lceil\{b, f\}, \{c, e\}, \{c, f\}, \{e, f\}, \{a, b, c\}, \{a, b, e\}\}\rceil, \ldots\}$$

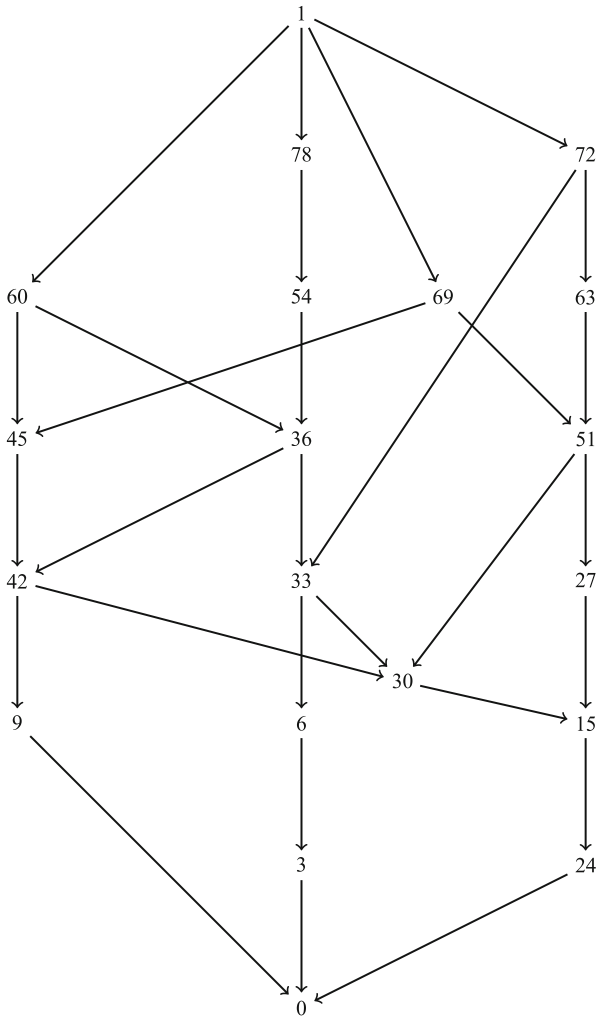

Fig. 7 Lattice of rough objects

If these are refined by rough inclusion, then a decomposition into antichains would have the following form ($\lceil, \rceil$ now serve as determiners of antichain boundaries)

$$\{\lceil\{a\}\rceil, \lceil\{b\}\,\{c\}, \{e\}\rceil, \lceil\{f\}, \{a, b\}\rceil, \lceil\{a, c\}\rceil, \lceil\{a, e\}\rceil,$$

$$\lceil\{a, f\}\rceil, \lceil\{b, c\}, \{b, e\}\rceil,$$

$$\lceil\{b, f\}\rceil, \lceil\{c, e\}, \{c, f\}\rceil, \lceil\{e, f\}, \{a, b, c\}\rceil, \lceil\{a, b, e\}\}\rceil, \ldots\}$$

Table 13 Maximal antichains

Rough object Z	Antichains including Z (differential)
78	{78, 69, 72}
60	{60, 54, 69, 72}, {60, 54, 69, 63}, {60, 54, 51}, {60, 54, 27}
54	{54, 45, 72}
72	{72, 45, 36}, {36, 69, 72}, {42, 72}, {9, 72}
69	{36, 69, 63}, {69, 33, 42}, {69, 6}, {69, 3}
42	{42, 33, 51}, {42, 33, 27}, {42, 6, 27}, {42, 6, 51}, {42, 63}
36	{36, 45, 63}, {36, 51, 45}, {36, 27, 45}
33	{45, 33, 51}, {45, 33, 27}
6	{9, 6, 15}, {9, 6, 27}, {9, 6, 51}, {9, 6, 24}
9	{9, 3, 15}, {9, 3, 24}, {9, 3, 27}, {9, 3, 51}, {9, 63}

Implicit in all this is that the agent can perceive

- rough approximations,
- rough inclusion,
- rough equality and

have good intuitive algorithms for arriving at maximal antichains. In the brute force approach, the agent would need as much as $\frac{2^{\#(\wp(S))}!}{2}$ orders for obtaining all maximal antichains. The number of computations can be sharply reduced by the table of rough objects and known algorithms in the absence of intuitive algorithms.

A reading of the above sequence of antichains in terms of approximations (the compact notation introduced earlier is used) is

$$\{\lceil(a, abe)\rceil, \lceil(a, abe), (\emptyset, cef), (e, abec)\rceil, \lceil(\emptyset, cf), (a, abe)\rceil, \lceil(a, S)\rceil, \lceil(a, abec)\rceil, \lceil(a, S)\rceil, \lceil(\emptyset, S), (e, abec)\rceil, \lceil(\emptyset, S)\rceil, \lceil(ec, S), (cf, cef)\rceil, \lceil(e, S), (a, S)\rceil, \lceil(a, abec)\rceil, \ldots\}$$

Relative the order structure this reads as

$$\{\lceil 3\rceil, \lceil 3, 15, 9\rceil, \lceil 24, 3\rceil, \lceil 33\rceil, \lceil 6\rceil, \lceil 33\rceil, \lceil 30, 9\rceil, \lceil 30\rceil, \lceil 45, 27\rceil, \lceil 42, 33\rceil, \lceil 6\rceil, \ldots\}$$

12.8 Example: Micro-Fossils and Descriptively Remote Sets

This is a somewhat extended version of the example mentioned by the present author in [98]. In the case study on numeric visual data including micro-fossils with the help of nearness and remoteness granules in [132], the difference between granules and approximations is very fluid as the precision level of the former can be varied. The data set consists of values of probe functions that extract pixel data from images of micro-fossils trapped inside other media like amethyst crystals.

The idea of remoteness granules is relative a fixed set of nearness granules formed from upper approximations—so the approach is about reasoning with sets of objects which in turn arise from tolerance relations on a set. In [132], antichains of rough objects are not used, but the computations can be extended to form maximal antichains at different levels of precision towards working out the best antichains from the point of view of classification.

Let X be an object space consisting of representation of some patterns and Φ : $X \longmapsto \mathbb{R}^n$ be a *probe function*, defined by

$$\Phi(x) = (\phi_1(x), \phi_2(x), \ldots, \phi_n(x)), \tag{75}$$

where $\phi_i(x)$ is intended as a measurement of the ith component in the feature space $\Im(\Phi)$. The concept of descriptive intersection of sets permits migration from classical ideas of proximity to ones based on descriptions. A subset $A \subseteq X$'s descriptive intersection with subset $B \subseteq X$ is defined by

$$A \cap_\Phi B = \{x \in A \cup B : \Phi(x) \in \Phi(A) \,\&\, \Phi(x) \in \Phi(B)\} \tag{76}$$

A is then *descriptively near* B if and only if their descriptive intersection is nonempty. Peter's version of proximity π_Φ is defined by

$$A\pi_\Phi B \leftrightarrow \Phi(A) \cap \Phi(B) \neq \emptyset \tag{77}$$

In [33], weaker implications for defining *descriptive nearness* are considered :

$$A \cap_\Phi B \neq \emptyset \rightarrow A\delta_\Phi B. \tag{78}$$

Specifically, if δ is a proximity on R^n, then a descriptive proximity δ_Φ is definable via

$$A\delta_\Phi B \leftrightarrow \Phi(A)\delta\Phi(B). \tag{79}$$

All these are again approachable from an anti-chain perspective.

12.9 Example: Beyond Data Tables

In this example subjective data is cast in terms of rough language for the purpose of understanding appropriate frameworks and solving context related problems.

Suppose agent X wants to complete a task and this task is likely to involve the use of a number of tools. X thinks tool-1 suffices for the task that a tool-2 is not suited for the purpose and that tool-3 is better suited than tool-1 for the same task. X also believes that tool-4 is as suitable as tool-1 for the task and that tool-5 provides more than what is necessary for the task. X thinks similarly about other tools but not much is known about the consistency of the information. X has a large repository of tools and limited knowledge about tools and their suitability for different purposes, and at the same time X might be knowing more about difficulty of tasks that in turn require better tools of different kinds.

Suppose also that similar heuristics are available about other similar tasks.

The reasoning of the agent in the situation can be recast in terms of lower, upper approximations and generalized equality and questions of interest include those relating to the agent's understanding of the features of tools, their appropriate usage contexts and whether the person thinks rationally.

To see this it should be noted that the key predicates in the context are as below:

- suffices for can be read as *includes potential lower approximation of* a right tool for the task.
- is not suited for can be read as *is neither a lower or upper approximation of* any of the right tools for the task.
- better suited than can be read as *potential rough inclusion* ,
- is as suitable as can be read as *potential rough equality* and
- provides more than what is necessary for is for *upper approximation* of a right tool for the task.

If *table rationality* is the process of reasoning by information tables and approximations, then when does X's reasoning become table rational?

This problem fits in easily with the antichain perspective, but not the information table approach because the latter requires extra information about properties.

12.10 Relation to Knowledge Interpretation

In Pawlak's concept of knowledge in classical RST [114, 127], if $\underline{S}$ is a set of attributes and P an indiscernibility relation on it, then sets of the form A^l and A^u represent clear and definite concepts (the semantic domain associated is the rough semantic domain). Extension of this to other kinds of rough sets have also been considered in [89, 94, 96, 99] by the present author. In [89], the concept of knowledge advanced by her is that of union of pairwise independent granules (in set context corresponding to empty intersection) correspond to clear concepts. This granular condition is desirable in other situations too, but may not correspond to the approximations of interest. In real life, clear concepts whose parts may not have clear relation between themselves are too common. If all of the granules are not definite objects, then analogous concepts of knowledge may be graded or typed based on the properties satisfied by them [96, 99]. *Then again the semantic domains in which these are being considered can be varied and so knowledge is relative*. Some examples of granular knowledge axioms are as follows:

1. Individual granules are atomic units of knowledge.
2. If collections of granules combine subject to a concept of mutual independence, then the result would be a concept of knowledge. The 'result' may be a single entity or a collection of granules depending on how one understands the concept of *fusion* in the underlying mereology. In set theoretic (ZF) setting the fusion operation reduces to set-theoretic union and so would result in a single entity.
3. Maximal collections of granules subject to a concept of mutual independence are admissible concepts of knowledge.
4. Parts common to subcollections of maximal collections are also knowledge.
5. All stable concepts of knowledge consistency should reduce to correspondences between granular components of knowledges. Two knowledges are *consistent* if and only if the granules generating the two have 'reasonable' correspondence.
6. Knowledge A is consistent with knowledge B if and only if the granules generating knowledge B are part of some of the granules generating A.

An antichain of rough objects is essentially a set of *some-sense mutually distinct rough concepts* relative that interpretation. Maximal antichains naturally correspond to represented knowledge that can be handled in a clear way in a context involving vagueness. The stress here should be on possible operations both within and over them. It is fairly clear that better the axioms satisfied by a concept of granular knowledge, better will be the nature of possible operations over sets of *some-sense mutually distinct rough concepts*.

From decision making perspectives, antichains of rough objects correspond to forming representative partitions of the whole and semantics relate to relation between different sets of representatives.

12.11 Knowledge Representation

In Subsection 12.5, the developed representation has the following features:

- Every object in a antichain is representable by a pair of objects (a, b) that are respectively of the form x^l and z^u.
- Some of these objects might be of the form (a, a) under the restriction $a = a^l = a^u$
- The above means that antichains can be written in terms of objects that are approximations of other objects or themselves.
- At another level, the concepts of rough objects mentioned in the background section suggest classification of the possible concepts of knowledge.
- The representation is perceivable in a rough semantic domain and this will be referred to as the *AC-representable rough domain* ACR.
- If *definable rough objects* are those rough objects representable in the form (a, b) with a, b being definite objects, then these together with definite objects may not correspond to maximal antichains in the classical semantic domain—the point is that some of the non crisp objects may fail to get represented under

the constraints. The semantic domain associated with the definable rough objects with the representation and crisp objects will be referred to as the *strict rough domain* (SRD).

The above motivates the following definition sequence

Definition 308 All of the following constitute the basic knowledge structure in the context of AC-semantics:

- A *Proper Knowledge Sequence* in ACR corresponds to the representation of any of the maximal antichains.
- An *Abstract Proper Knowledge Sequence* in ACR corresponds to the representation of possible maximal antichains. These may be realized in particular models.
- A *Knowledge Sequence* in SRD corresponds to the relatively maximal antichains formed by sequences of definable rough objects and definite objects.
- Definable rough objects
- Representation of rough and crisp objects

More complex objects formed by antichains are also of interest. The important thing about the idea of knowledge sequences is the explicit admission of temporality and the relation to all of the information available in the context. This is considered in [106] by the present author.

12.12 Problems

This research also motivates the following:

- Further study of specific rough sets from the perspective of antichains.
- Research into connections with the rough membership function based semantics of [28] and extensions by the present author in a forthcoming paper. This is justified by advances in concepts of so-called cut-sets in antichains.
- Research into computational aspects as the theory is well developed for antichains. The abstract example illustrates parts of this aspect in particular.
- Study of consequence and special ideals afforded by the semantics and
- Research into ontologies indicated by the triadic approach.

13 PQE Rough Sets

Entire semantics of various general rough set approaches can be recast in the antichain based perspective. For example, prototransitive rough sets [99] can be dealt with the same way. A finer characterization of the same will appear separately. Quasi equivalential rough sets were considered also as an example of the approach in [98]. Correctly, these should have been termed *pre-quasi equivalential rough sets* and this is done here and moreover the *quasi-equivalences* of [13] are considered.

One of the most interesting type of granulation $\mathcal{G}$ in relational rough sets is one that satisfies

$$(\forall x, y)\,(\phi(x) = \phi(y) \leftrightarrow Rxy \,\&\, Ryx), \tag{80}$$

where $\phi(x)$ is the granule generated by $x \in \underline{S}$. This granular axiom says that if x is left-similar to y and y is left-similar to x, then the elements left similar to either of x and y must be the same. R is being read as *left-similarity* because it is directional and has effect similar to tolerances on neighborhood granules.

A binary relation R on a set $\underline{S}$ will be said to be a *pre-quasi-equivalence* if and only if it satisfies:

$$(\forall a, b)\,(Rab \,\& Rba \longleftarrow [a] = [b]).$$

It is said to be a *quasi-equivalence* if and only if it satisfies:

$$(\forall a, b)\,([a] = [b] \leftrightarrow Rab \,\&\, Rba).$$

It is useful in algebras when it behaves as a good factor relation [13]. But the condition is of interest in rough sets by itself. In [98, 106], pre-quasi equivalences had been termed quasi equivalences. Here the notation, terminology and results have been updated.

Example 309 Every reflexive and transitive relation is a quasi-equivalence, but the converse need not hold.

In fact, the following is provable:

Proposition 310 *R is a quasi-equivalence on the set S if and only if all of the following hold:*

$$(\forall a) Raa \tag{Reflexive}$$

$$(\forall a, b, c)\; Rab \,\&\, Rba \,\&\, Rca \longrightarrow Rcb \tag{sy-transitive}$$

For example, it is possible to find quasi equivalences that do not satisfy other properties from contexts relating to numeric measures. Let S be a set such that Rxy if and only if $x \approx \kappa y \,\&\, y \approx \kappa' x \,\&\, \kappa, \kappa' \in (0.9, 1.1)$ for some interpretation of $\approx$ and $x, y \in S$.

Definition 311 By a *pre quasi equivalential approximation space* PQEAS will be meant a pair of the form $S = \langle \underline{S}, R \rangle$ with R being a pre quasi equivalence. For an arbitrary subset $A \in \wp(S)$, the following can be defined:

$$(\forall x \in \underline{S})\, [x] = \{y\,;\; y \in \underline{S} \,\&\, Ryx\}.$$

$$A^l = \bigcup\{[x]\,;\; [x] \subseteq A \,\&\, x \in \underline{S}\} \,\&\, A^u = \bigcup\{[x]\,;\; [x] \cap A \neq \emptyset \,\&\, x \in \underline{S}\}$$

$$A^{l_o} = \bigcup\{[x]\,;\,[x] \subseteq A \& x \in A\} \,\&\, A^{u_o} = \bigcup\{[x]\,;\,[x] \cap A \neq \emptyset \& x \in A\}$$

$$A^{L} = \{x\,;\,\emptyset \neq [x] \subseteq A \,\&\, x \in \underline{S}\} \,\&\, A^{U} = \{x\,;\,[x] \cap A \neq \emptyset \,\&\, x \in \underline{S}\}$$

$$A^{L_o} = \{x\,;\,[x] \subseteq A \,\&\, x \in A\} \,\&\, A^{U_o} = \{x\,;\,[x] \cap A \neq \emptyset \,\vee\, x \in A\}.$$

$$A^{L_1} = \{x\,;\,[x] \subseteq A \,\&\, x \in \underline{S}\} \,\&\, A^{U} = \{x\,;\,[x] \cap A \neq \emptyset \,\&\, x \in \underline{S}\}.$$

Note the requirement of non-emptiness of $[x]$ in the definition of A^L, but it is not necessary in that of A^{L_o}

Theorem 312 *The following properties hold when R is a PQE:*

1. *All of the approximations are distinct in general.*
2. $(\forall A \in \wp(S))\, A^{L_o} \subseteq A^{l_o} \subseteq A^{l} \subseteq A$ *and* $A^{L_o} \subseteq A^{L}$.
3. $(\forall A \in \wp(S))\, A^{l_o l} = A^{l_o} \,\&\, A^{l l_o} \subseteq A^{l_o} \,\&\, A^{l_o l_o} \subseteq A^{l_o}$
4. $(\forall A \in \wp(S))\, A^{u} = A^{ul} \subseteq A^{uu}$*, but it is possible that* $A \nsubseteq A^{u}$
5. *It is possible that* $A^{L} \nsubseteq A$ *and* $A \nsubseteq A^{U}$*, but* $(\forall A \in \wp(S))\, A^{L} \subseteq A^{U}$ *holds. In general* A^L *would not be comparable with* A^l *and similarly for* A^U *and* A^u.
6. $(\forall A \in \wp(S))\, A^{L_o L_o} \subseteq A^{L_o} \subseteq A \subseteq A^{U_o} \subseteq A^{U_o U_o}$. *Further* $A^{U} \subseteq A^{U_o}$.

Theorem 313 *The following additional properties properties hold when R is a QE*

1. $(\forall A \in \wp(S))\, A^{L_o} = A^{L} = A^{L_1} \subseteq A^{l_o} = A^{l} \subseteq A$
2. $(\forall A \in \wp(S))\, A^{ll} = A^{l} \,\&\, A^{l l_o} \subseteq A^{l_o} \,\&\, A^{l_o l_o} \subseteq A^{l_o}$
3. $(\forall A \in \wp(S))\, A^{u} = A^{ul} \subseteq A^{uu}$*, but it is possible that* $A \nsubseteq A^{u}$
4. *In general* A^L *would not be comparable with* A^l *and similarly for* A^U *and* A^u.
5. $(\forall A \in \wp(S))\, A^{L_o L_o} \subseteq A^{L_o} \subseteq A \subseteq A^{U_o} \subseteq A^{U_o U_o}$. *Further* $A^{U} \subseteq A^{U_o}$.

Clearly the operators l, u are granular approximations, but the latter is controversial as an upper approximation operator. The point-wise approximations L, U are problematic.

Example 314 (General)

$$\text{Let } \underline{S} = \{a, b, c, e, f, k, h, q\} \tag{81}$$

and let R be a binary relation on it defined via

$$R = \{(a, a), (b, a), (c, a), (f, a),$$
$$(k, k), (e, h), (f, c), (k, h)$$
$$(b, b), (c, b), (f, b), (a, b), (c, e), (e, q)\}.$$

The neighborhood granules $\mathcal{G}$ are then

$$[a] = \{a, b, c, f\} = [b], [c] = \{f\}, [e] = \{c\},$$
$$[k] = \{k\}, [h] = \{k, e\}, [f] = \emptyset \,\&\, [q] = \{e\}.$$

So R is a pre quasi-equivalence relation.

If $P = R \cup \Delta_S$, then P would be a quasi equivalence relation (Δ_S being the diagonal of S.)

If $A = \{a, k, q, f\}$, then

$$A^l = \{k, f\},\ A^u = \{a, b, c, f, k, e\},\ A^{uu} = \{a, b, c, f, k, e, h\}$$
$$A^{l_o} = \{k\},\ A^{u_o} = \{a, b, c, k, f\}.$$
$$A^L = \{k, f\},\ A^U = \{a, b, c, k, h, q\}.$$
$$A^{L_o} = \{q, k, f\},\ A^{U_o} = \{a, k, q, f, b, c, h\}.$$
$$A^{L_1} = \{k, c, f\},\ A^U = \{a, b, c, k, h, q\}.$$

$$\text{Note that } A^{L_1} \nsubseteq A \,\&\, A^{L_1} \nsubseteq A^U \,\&\, A \nsubseteq A^U. \tag{82}$$

13.1 Semantics of PQE-Rough Sets

In this section a semantics of pre quasi equivalential rough sets (PQE-rough sets), using antichains generated from rough objects, is developed. Interestingly the properties of the approximation operators of PQE-rough sets fall short of those required of granular operator space. Denoting the set of maximal antichains of rough objects by $\mathfrak{S}$ and carrying over the operations $\ll$, ϱ, δ, the following algebra can be defined.

Definition 315 A *maximal simply independent algebra Q* of pre quasi equivalential rough sets shall be an algebra of the form

$$Q = \langle \mathfrak{S}, \ll, \varrho, \delta \rangle \tag{83}$$

defined as in section 12.3 with the approximation operators being l, u uniformly in all constructions and definitions.

Theorem 316 *Maximal simply independent algebras are well defined.*

Proof None of the steps in the definition of the maximal antichains, or the operations ϱ or δ are problematic because of the properties of the operators l, u. □

The above theorem suggests that it would be better to try and define more specific operations to improve the uniqueness aspect of the semantics or at least the properties of ϱ, δ. It is clearly easier to work with antichains as opposed to maximal antichains as more number of suitable operations are closed over the set of antichains as opposed to those over the set of maximal antichains.

Definition 317 Let $\mathfrak{K}$ be the set of antichains of rough objects of S then the following operations $\mathfrak{L}$, $\mathfrak{U}$ and extensions of others can be defined:

- Let $\alpha = \{\mathbb{A}_1, \mathbb{A}_2, \ldots, \mathbb{A}_n, \ldots\} \in \mathfrak{K}$ with $\mathbb{A}_i$ being rough objects; the lower and upper approximation of any subset in $\mathbb{A}_i$ will be denoted by $\mathbb{A}_i^l$ and $\mathbb{A}_i^u$ respectively.
- Define $\mathfrak{L}(\alpha) = \{\mathbb{A}_1^l, \mathbb{A}_2^l, \ldots, \mathbb{A}_r^l, \ldots\}$ with duplicates being dropped
- Define $\mathfrak{U}(\alpha) = \{\mathbb{A}_1^u, \mathbb{A}_2^u, \ldots, \mathbb{A}_r^u, \ldots\}$ with duplicates being dropped
- Define

$$\mu(\alpha) = \begin{cases} \alpha & \text{if } \alpha \in \mathfrak{S} \\ \text{undefined,} & \text{else.} \end{cases} \tag{84}$$

- Partial operations ϱ^*, δ^* corresponding to ϱ, δ can also be defined as follows: Define

$$\varrho^*(\alpha, \beta) = \begin{cases} \varrho(\alpha, \beta) & \text{if } \alpha, \beta \in \mathfrak{S} \\ \text{undefined,} & \text{else.} \end{cases} \tag{85}$$

$$\delta^*(\alpha, \beta) = \begin{cases} \delta(\alpha, \beta) & \text{if } \alpha, \beta \in \mathfrak{S} \\ \text{undefined,} & \text{else.} \end{cases} \tag{86}$$

The resulting partial algebra $\mathfrak{K} = \langle \mathfrak{K}, \mu, \vee, \wedge, \varrho^*, \delta^*, \mathfrak{L}, \mathfrak{U}, 0 \rangle$ is said to be a *simply independent QE algebra*

Theorem 318 *Simply independent QE algebras are well defined and satisfy the following:*

- $\mathfrak{L}(\alpha) \vee \alpha = \alpha$.
- $\mathfrak{U}(\alpha) \vee \alpha = \mathfrak{U}(\alpha)$.

14 Concluding Remarks and Problems

In this research chapter, most if not all algebraic approaches to granular rough sets have been explained in some detail. This task required clarifications on the very nature of and formulation of usable concepts of granularity. The things that an algebraic approach must address are not fixed. The purpose of an algebraic approach may be to

- model reasoning with rough objects in a way that avoids contamination and a limited number of semantic operations,
- try and reduce models of reasoning to those of specific non classical logics,
- model reasoning and knowledge with rough objects, classical objects and hybrid ones
- aid in the computation of reducts
- explore the limits of definable models in a comparative way and
- model reasoning and decision making from an approximate perspective.

In this chapter, all of the above types of models have been considered except for those that explicitly seek to aid in the computation of reducts. In the present authors perspective, a large number of problems can be formulated within a model or semantics and without. Problems of the former class have been mentioned in the context of the sections and in related papers. Apart from problems that arise from algebraic or universal algebraic considerations, many problems require a choice of perspective for easier comprehensibility. It is important also to identify good perspectives. Some of the problems of interest are

- Development of rough sets with more complicated representation of approximations from granules. This may also help in reducing point-wise and other non granular approximations to granular ones satisfying weak representability.
- Admissible operations: Apart from definable operations, additional operations may be adjoinable to particular models to enhance and improve the semantics. Not many papers are known on this (see [92]).
- Rough methods of counting had been introduced and investigated in [91] by the present author. These can be used on all kinds of rough sets including ones that arise from granular considerations. Obviously the related algebras of [91] are apparently related to antichains. What are the connections between the two collections of algebras? How are they related to cylindric algebras and Boolean algebras with operators?

It is hoped that this research chapter would prove to be a useful reference for all granular approaches in rough sets.

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Three Lessons on the Topological and Algebraic Hidden Core of Rough Set Theory

Piero Pagliani

Abstract In what follows the reader will find an exposition of the basic, albeit not elementary, connections between Rough Set Theory and relation algebra, topology and algebraic logic.

Many algebraic aspects of Rough Set Theory, are known nowadays. Other are less known, although they are important, for instance because they unveil the "epistemological meaning" of some "unexplained" mathematical features of well-known algebraic structures.

We shall wrap everything in a simple exposition, illustrated by many examples, where just a few basic notions are required. Some new results will help the connection of the topics taken into account.

Important features in Rough Set Theory will be explained by means of notions connecting relation algebra, pre-topological and topological spaces, formal (pre) topological systems, algebraic logic and logic.

Relation algebra provides basic tools for the definition of approximations in general (that is, not confined to particular kind of relations). Indeed, these tools lead to pairs of operators fulfilling *Galois adjointness*, whose combinations, in turn, provide pre-topological and topological operators, which, in some cases, turn into approximation operators.

Once one has approximation operators, rough sets can be defined. In turn, rough set systems can be made into different logico-algebraic systems, such as Nelson algebras, three-valued Łukasiewicz algebras, Post algebras of order three, Heyting and co-Heyting algebras.

In addition, in the process of approximation, one has to deal with both exact and inexact pieces of information (definable and non-definable sets). Therefore, the concept of *local validity* comes into picture. It will be extensively discussed because it links the construction of Nelson algebras from Heyting ones with the notions of a *Grothendieck topology* and a *Lawvere-Tierney operator*.

As a side effect, we obtain an information-oriented explanation of the above logico-algebraic constructions which usually are given on the basis of pure formal motivations.

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The exposition will move from abstract levels (pointless) to concrete levels of analysis.

1 Introduction: Relations, Nearness and Granulation

From a general point of view, the approximation of a set A included in a universe U amounts to answering to the following questions:

(a) What elements of U are surely, or *necessarily*, in A?
(b) What elements of U are not in A but sufficiently near, or *possibly*, in A?
(c) What elements of U are surely outside A, that is neither necessarily nor possibly in A, that is, are *necessarily outside* A?

The difference with a sharp classification is crystal clear, since a sharp classification just provides a binary answer: either an element is inside A or it is outside of it. Otherwise stated, there is no notion of "possible in although not surely in". There is no indecision, no nuances: either "Yes" or "No".

However, classification on the basis of properties or attributes demands some more subtle answers. Suppose an item $u \in U$ is in A while another item $u' \in U$ is not in A from a classical set-theoretical point of view although it fulfils properties or attributes very close to those of u. It could be not correct to definitely exclude that u' belongs to the set A. Think of some dynamic situation, where all the patients in a hospital showing at least all the symptoms as u developed a particular disease α and a patient u' who shows *almost* all the symptoms as u but has not developed that disease. Probably it is not safe to rule out the possibility that u' will develop the disease, too. Therefore, we would say that u' belongs to a reasonable approximation of the set A of patients suffering from α. Such an approximation is *from above*, because it enlarges the actual set A: it is *possible* for u' to develop α, hence it can belong to A in the future.

The sentence "fulfilling almost all the properties as" defines a notion of *nearness*. From a mathematical point of view, "almost all" defines a *preorder*, that is, a binary relation R on U which is reflexive (uRu) and transitive (uRu' and $u'Ru''$ implies uRu''): the set A of symptoms showed by patient u' is included in the set of symptoms B of patient u. However, this preorder should be refined, because mathematically also the empty set $\emptyset$ is included in B, which is meaningless from a clinical point of view. Actually, the metric underlying the adverb "almost" depends on some particular knowledge and heuristic of the experts.

But in general R could be a binary relation denoting any sort of connection between items (or objects) which groups objects in *granules* of any kind. Indeed, there are cases in which it is difficult to recognise a "rule" behind the formation of the given granules of objects. In this case some relation R is in action, but this relation does not have any known "nice property".

If the set $\mathscr{G}(U)$ of the granules (subsets) of U is a covering, in many cases behind the granulation there is a *tolerance relations* (that is, reflexive and also symmetric: uRu' implies $u'Ru$)—see for instance [5, 18] and [36].

If $\mathscr{G}(U)$ forms a family of open subsets of a topological space, then we can restore from it a *preorder*, or a *partial order* (a preorder which is antisymmetric: uRu' and $u'Ru$ implies $u = u'$). In particular cases one obtains an *equivalence relation* (reflexive, transitive and symmetric) which is the original situation studied by Zdzisław Pawlak (see [40]).

As we shall see, also the reverse constructions hold, that is, from preorders, partial orders or equivalence relations to topological spaces with different features.

Algebraic structures induced by *rough set systems*, that is, the set of all rough sets, have been widely studied since inception. Considering only some early results, in [41] it was shown that classical rough sets form Stone algebras. In [25] rough sets were linked to Heyting algebras. Also [8] worked on this topic. In [28] rough set systems were proved to form semi-simple Nelson algebras, hence three-valued Łukasiewicz algebras. This result was improved in [3], in [4] and by other authors. Later, rough sets have been connected to other algebras of logic, such as Post algebras of order three, Chain-based Lattices, Heyting and bi-Heyting algebras (see [2, 32]). In [6] and [7] rough sets were embedded in the framework of Brouwer-Zadeh lattices and Heyting Wajsberg Algebras. More recently, interesting investigations about more general algebras linked to rough sets have been presented (see [46]).

Situations in which instead of topologies one has to deal with pre-topologies have been studied (see for instance [33] and [34]). Nonetheless, in a number of cases, preorders and partial orders occur (see for instance [16]). In these lessons we shall deal with this specific case.

From an abstract point of view topologies are *Heyting algebras*, which are particular structures which model *Intuitionistic Logic* in the same way *Boolean Algebras* model *Classical Logic*. Eventually, in this case rough set systems can be made into *Nelson algebras* which are built from Heyting algebras defined on the granulation.

A natural approach to rough sets is through relation algebra. We can cite as early works: [9, 10, 29] and [11].

It follows that we have to develop our exposition in three different framework, which will be connected each other:

$$\textit{Relation algebras} - \textit{Topology} - \textit{Algebraic logics}.$$

From now on, the sets we shall deal with are intended to be *finite*. This choice does no harm real-word application. Moreover, it avoids some complications which could disturb a beginner's comprehension. Many of the results are, nonetheless, applicable to the infinite case and, if required to avoid misinterpretation, we shall point out when this does not happen.

Since these are lessons a few results and proofs are really new, although much of the exposition is novel. We underline with references when a result is standard or well-known. Otherwise the theses or the proofs are new or already given in other publications by the author. Moreover, a number of elementary examples will be provided. These examples are connected each other to show how the topics intertwine.

Usually, in the meta-language, which is classical, we write, "$\exists$", "$\forall$", "$\Rightarrow$", "$\wedge$", "$\vee$" and "$\neg$" for "exists", "for all", "implies", "and", "or" and "non", respectively. However, in some cases to avoid confusion we use "&" instead of "$\wedge$".

2 Lesson 1: The Relational Framework

2.1 Binary Relations and Their Algebra

Let us formally define what we can do with binary relations.

Definition 1 Let U, U', U'' be three sets. In what follows, u^* is a dummy element of U and u'^* a dummy element of U' (that is, they represent any element of their domain):

1. A *binary relation* is a subset $R \subseteq U \times U'$ of ordered pairs $\langle u, u' \rangle$ of elements of U and U'.
2. $-R := \{\langle u, u' \rangle : \langle u, u' \rangle \notin R\}$ is the *complement* of R. If $R' \subseteq U \times U'$, then $R \cap R'$ and $R \cup R'$ are the usual set-theoretic operations.
3. $R^{\smile}$ denotes the *converse* of R: $R^{\smile} := \{\langle y, x \rangle : \langle x, y \rangle \in R\}$. Hence, for all $u \in U, u' \in U'$, $\langle u, u' \rangle \in R$ iff $\langle u', u \rangle \in R^{\smile}$.
4. If $Q \subseteq U' \times U''$, then $R \otimes Q := \{\langle u, u'' \rangle : \exists u' \in U'(\langle u, u' \rangle \in R \wedge \langle u', u'' \rangle \in Q)\}$—the *right composition* of R with Q. Converse is an involution with respect to composition: $R^{\smile\smile} = R$ and $(R \otimes Q)^{\smile} = Q^{\smile} \otimes R^{\smile}$.
5. If $A \subseteq U$, then $A_R^{\rightarrow} := \{\langle a, u' \rangle : a \in A \wedge u' \in U'\}$ is called the *right cylinder* of A with respect to R. It is the relational embedding of a subset A of U in $U \times U'$. If $B \subseteq U'$, then $B_R^{\leftarrow} := \{\langle u, b \rangle : b \in B \wedge u \in U\}$ is the *left cylinder* of B with respect to R. It is the relational embedding of a subset B of U' in $U \times U'$.

   We set $A_{R^{\smile}}^{\leftarrow} := (A_R^{\rightarrow})^{\smile}$, the *left cylinder* of A with respect to $R^{\smile}$ to have the relational embedding of A in $U' \times U$ and $B_{R^{\smile}}^{\rightarrow} := (R_R^{\leftarrow})^{\smile}$, the *right cylinder* of B with respect to $R^{\smile}$, which is the relational embedding of B in $U' \times U$.

   A cylinder represents a set in the language of relations. In any ordered pair $\langle x, y \rangle$ of a cylinder, the element y is any element of the codomain of the relation. This is the result and meaning of a cylindrification, that now we formally motivate.
6. The operation $R^{\smile} \otimes A_R^{\rightarrow} = \{\langle u', u'^* \rangle : \exists u(\langle u, u' \rangle \in R \wedge \langle u, u'^* \rangle \in A_R^{\rightarrow})\}$ is a right cylinder of type $U' \times U'$. Since u'^* is a dummy element of U the operation can be rephrased in terms of relations and sets, as it will be formally proved in Lemma 5:

$$R(A) := \{u' : \exists u(\langle u, u' \rangle \in R \wedge u \in A)\}. \tag{1}$$

   $R(A)$ is called the *left Peirce product* of R and A, R-*neighbourhood* of A, or the *filter* of A, if R is an order relation, in which case we denote it by $\uparrow A$ or $\uparrow_R A$ if necessary.

Similarly, the operation $R \otimes B^{\rightarrow}_{R^{\smile}} = \{\langle u, u^* \rangle : \exists u'(\langle u, u' \rangle \in R \wedge \langle u', u^* \rangle \in B^{\rightarrow}_{R^{\smile}})\}$ can be rephrased in terms of relations and sets as:

$$R^{\smile}(B) := \{u : \exists u'(\langle u, u' \rangle \in R \wedge u' \in B)\} \tag{2}$$

$R^{\smile}(B)$ is called the *left Peirce product* of $R^{\smile}$ and B (the right Peirce product of R and B), the $R^{\smile}$-*neighbourhood* of B, or the R-*ideal* of B, denoted also by $\downarrow B$ or $\downarrow_R B$ if we need to specify the relation. For any $u' \in U'$, $u \in U$:

$$(A^{\rightarrow}_{R})^{\smile}(u') = (A^{\leftarrow}_{R^{\smile}})(u') = A, \quad (B^{\rightarrow}_{R^{\smile}})^{\smile}(u) = B^{\leftarrow}_{R}(u) = B. \tag{3}$$

7. Given two relations $R \subseteq U \times U'$ and $Z \subseteq U \times U''$ the relation

$$R \longrightarrow Z = \{\langle u', u'' \rangle : \forall u(\langle u, u' \rangle \in R \Rightarrow \langle u, u'' \rangle \in Z)\} \tag{4}$$

is called the *right residual* of R and Z. $R \longrightarrow Z$ is the largest relation K such that $R \otimes K \subseteq Z$:

$$R \otimes K \subseteq Z \ \textit{ iff } \ K \subseteq R \longrightarrow Z. \tag{5}$$

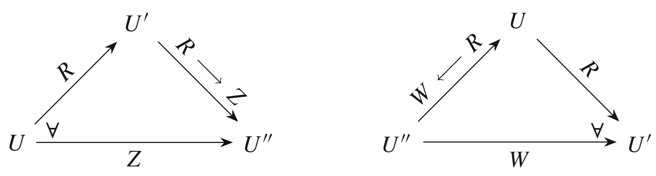
If $R \subseteq U \times U'$ and $W \subseteq U'' \times U'$ the relation

$$W \longleftarrow R = \{\langle u'', u \rangle : \forall u'(\langle u, u' \rangle \in R \Rightarrow \langle u'', u' \rangle \in W)\} \tag{6}$$

is called the *left residual* of R and W. $W \longleftarrow R$ is the largest relation K such that $K \otimes R \subseteq W$:

$$K \otimes R \subseteq W \ \textit{ iff } \ K \subseteq W \longleftarrow R. \tag{7}$$

The above operations can be depicted as follows:



For any set $A \subseteq U$ and $B \subseteq U'$, one has: $R \longrightarrow A^{\rightarrow}_{R} = \{\langle u', u'^* \rangle : \forall u(\langle u, u' \rangle \in R \Rightarrow \langle u, u'^* \rangle \in A^{\rightarrow}_{R})\}$ and $R^{\smile} \longrightarrow B^{\rightarrow}_{R^{\smile}} = \{\langle u, u^* \rangle : \forall u'(\langle u', u \rangle \in R^{\smile} \Rightarrow \langle u', u^* \rangle \in B^{\rightarrow}_{R^{\smile}})\}$. Since the elements decorated with $*$ are generic, one can get rid of the cylindrification and rephrase the operations in terms of relations and sets as follows:

$$R \longrightarrow A = \{u' : \forall u(\langle u, u' \rangle \in R \Rightarrow u \in A)\} \tag{8}$$

$$R^{\smile} \longrightarrow B = \{u : \forall u'(\langle u, u' \rangle \in R \Rightarrow u' \in B)\} \tag{9}$$

The above operations are fundamental to study approximations by means of relations.

Lemma 2 *Given* $R \subseteq U \times U'$, $Z \subseteq U \times U''$ *and* $W \subseteq U'' \times U'$*:*

$$R \longrightarrow Z = -(R^{\smile} \otimes -Z); \tag{10}$$

$$W \longleftarrow R = -(-W \otimes R^{\smile}). \tag{11}$$

Proof

$$\begin{aligned}
-(R^{\smile} \otimes -Z) &= -\{\langle u', u''\rangle : \exists u(\langle u', u\rangle \in R^{\smile} \wedge \langle u, u''\rangle \notin Z)\} \\
&= -\{\langle u', u''\rangle : \exists u\neg(\langle u', u\rangle \notin R^{\smile} \vee \langle u, u''\rangle \in Z)\} \\
&= \{\langle u', u''\rangle : \neg\exists u\neg(\langle u', u\rangle \notin R^{\smile} \vee \langle u, u''\rangle \in Z)\} \\
&= \{\langle u', u''\rangle : \forall u(\langle u', u\rangle \notin R^{\smile} \vee \langle u, u''\rangle \in Z)\} \\
&= \{\langle u', u''\rangle : \forall u(\langle u, u'\rangle \notin R \vee \langle u, u''\rangle \in Z)\} \\
&= \{\langle u', u''\rangle : \forall u(\langle u, u'\rangle \in R \Rightarrow \langle u, u''\rangle \in Z)\} \\
&= R \longrightarrow Z
\end{aligned}$$

The other proof comes from symmetry. □

The above equations parallel the logical equivalence $\alpha \Longrightarrow \beta \equiv \neg(\alpha \wedge \neg\beta)$).

Definition 3 The structure $\langle U, U', R\rangle$, with $R \subseteq U \times U'$ will be called a *relational system*. If a is in relation R with b we write $\langle a, b\rangle \in R$. Especially if R is an order relation we also use the notation aRb. If $R \subseteq U \times U$ we shall write $\langle U, R\rangle$ instead of $\langle U, U, R\rangle$.

Example 4 A relation $R \subseteq U \times U'$ will be usually represented by means of a Boolean matrix with rows labelled by the elements of U and columns by the elements of U'. If $\langle x, y\rangle \in R$ then the entry of row x, column y is 1. It is 0 otherwise. The operation $\longrightarrow$ has a higher precedence than the others. Thus, for instance, $R \otimes Q \longrightarrow Z$ means $R \otimes (Q \longrightarrow Z)$.
$U = \{a, b, c, d\}$, $U' = \{\alpha, \beta, \gamma\}$, $U'' = \{\varkappa, \lambda, \mu\}$, $R \subseteq U \times U'$, $Q \subseteq U' \times U''$.

R	α	β	γ
a	1	0	1
b	1	1	0
c	0	1	1
d	1	0	0

$-R$	α	β	γ
a	0	1	0
b	0	0	1
c	1	0	0
d	0	1	1

$R^{\smile}$	a	b	c	d
α	1	1	0	1
β	0	1	1	0
γ	1	0	1	0

Q	$\varkappa$	λ	μ
α	1	1	0
β	0	1	0
γ	1	0	1

$R \otimes Q$	$\varkappa$	λ	μ
a	1	1	1
b	1	1	0
c	1	1	1
d	1	1	0

Indeed, for instance, $\langle a, \alpha\rangle \in R$ and $\langle \alpha, \lambda\rangle \in Q$, thus $\langle a, \lambda\rangle \in R \otimes Q$. Analogously, $\langle c, \gamma\rangle \in R$ and $\langle \gamma, \mu\rangle \in Q$, thus $\langle c, \mu\rangle \in R \otimes Q$. On the contrary, there is no intermediate element of U' linking d and μ. And so on.

Z	η	δ	ϵ	ζ	ι
a	1	0	0	1	1
b	1	1	1	1	0
c	0	1	1	1	0
d	1	0	1	1	0

$R \longrightarrow Z$	η	δ	ϵ	ζ	ι
α	1	0	0	1	0
β	0	1	1	1	0
γ	0	0	0	1	0

$R \otimes R \longrightarrow Z$	η	δ	ϵ	ζ	ι
a	1	0	0	1	0
b	1	1	1	1	0
c	0	1	1	1	0
d	1	0	0	1	0

Therefore, $R \otimes R \longrightarrow Z \subsetneq Z$. For instance, $\langle a, \iota \rangle \notin R \otimes R \longrightarrow Z$, while $\langle a, \iota \rangle \in Z$. Since $\langle a, \alpha \rangle, \langle a, \gamma \rangle \in R$, in order to have $\langle \alpha, \iota \rangle \in R \otimes R \longrightarrow Z$ we should have either $\langle \alpha, \iota \rangle$ or $\langle \gamma, \iota \rangle$ in $R \longrightarrow Z$. In the former case also $\langle b, \iota \rangle \in R \otimes R \longrightarrow Z$, because $\langle b, \alpha \rangle \in R$, too. But $\langle b, \iota \rangle \notin Z$. In the latter case also $\langle c, \iota \rangle \in R \otimes R \longrightarrow Z$ because $\langle c, \gamma \rangle \in R$, but $\langle c, \iota \rangle \notin Z$.

Let $A = \{a, b, c\}$. Then the right cylindrification of A is

$A^{\rightarrow}_{R}$	α	β	γ
a	1	1	1
b	1	1	1
c	1	1	1
d	0	0	0

Notice that for any $u' \in U'$, $(A^{\rightarrow}_{R})^{\smile}(u') = A$. Moreover,

$R \longrightarrow A^{\rightarrow}_{R}$	α	β	γ
α	0	0	0
β	1	1	1
γ	1	1	1

Thus, $R \longrightarrow A = \{\beta, \gamma\}$.

Let $B = \{\alpha, \gamma\}$. The right cylindrification of B is

$B^{\rightarrow}_{R^{\smile}}$	a	b	c	d
α	1	1	1	1
β	0	0	0	0
γ	1	1	1	1

Then one obtains

$R^{\smile} \longrightarrow B^{\rightarrow}_{R^{\smile}}$	a	b	c	d
a	1	1	1	1
b	0	0	0	0
c	0	0	0	0
d	1	1	1	1

. Thus, $R^{\smile} \longrightarrow B = \{a, d\}$.

Lemma 5 *Given $R \subseteq U \times U'$, $Z \subseteq U \times U''$, $W \subseteq U' \times U''$, $A \subseteq U$ and $B \subseteq U'$:*

$$R \longrightarrow Z = \{\langle u', u'' \rangle : R^{\smile}(u') \subseteq Z^{\smile}(u'')\} \tag{12}$$

$$R \longrightarrow A = \{u' : R^{\smile}(u') \subseteq A\} \tag{13}$$

$$R^{\smile} \longrightarrow W = \{\langle u, u'' \rangle : R(u) \subseteq W^{\smile}(u'')\} \tag{14}$$

$$R^{\smile} \longrightarrow B = \{u : R(u) \subseteq B\} \tag{15}$$

$$R \otimes W = \{\langle u, u'' \rangle : R(u) \cap W^{\smile}(u'') \neq \emptyset\} \tag{16}$$

$$R^{\smile}(B) = \{u : R(u) \cap B \neq \emptyset\} \tag{17}$$

$$R^{\smile} \otimes Z = \{\langle u', u'' \rangle : R^{\smile}(u') \cap Z^{\smile}(u'') \neq \emptyset\} \tag{18}$$

$$R(A) = \{u' : R^{\smile}(u') \cap A \neq \emptyset\} \tag{19}$$

Proof (12), (14), (16) and (18) come straightforwardly from the definitions. We just prove a couple of other points.
(14)⇒(15): Let $B \subseteq U'$ and $B^{\rightarrow}_{R^{\smile}}$ its right cylinder. Then from (14) $R^{\smile} \longrightarrow B^{\rightarrow}_{R^{\smile}} = \{\langle u, u^{*}\rangle : R(U) \subseteq (B^{\rightarrow}_{R^{\smile}})^{\smile}(u^{*})\}$. But from (3) $(B^{\rightarrow}_{R^{\smile}})^{\smile}(u^{*}) = B$. Since u^{*} is a dummy element, we can dispense with it and the cylindrification of B and obtain (15).
(18)⇒(19): Let $A^{\rightarrow}_{R}$ be the right cylinder of a set $A \subseteq U$. Then from (18) $R^{\smile} \otimes A^{\rightarrow}_{R} = \{\langle u', u'^{*}\rangle : R^{\smile}(u') \cap (A^{\rightarrow}_{R})^{\smile}(u'^{*}) \neq \emptyset\}$. But from (3) $(A^{\rightarrow}_{R})^{\smile}(u'^{*}) = A$. Thus $R^{\smile} \otimes A^{\rightarrow}_{R} = \{\langle u', u'^{*}\rangle : R^{\smile}(u') \cap A \neq \emptyset\}$. Since u'^{*} is a dummy element, we can dispense with it and the cylindrification of A and obtain (19). □

3 Lesson 2: The Topological Framework

3.1 Galois Adjunctions and Their Operators

Pre-topologies and topologies are definable from a particular mathematical notion called a *Galois adjunction*. It is not the usual way to introduce topologies but it is an effective one.

Definition 6 Let $\mathbf{O} = \langle U, R\rangle$ be an ordered set and $\mathbf{L} = \langle U, \vee, \wedge, 0, 1\rangle$ a bounded lattice such that for any $a, b \in U$, aRb iff $a \wedge b = b$ ($a \vee b = b$). Let $\varphi : \mathbf{O} \longmapsto \mathbf{O}$ and $\theta : \mathbf{L} \longmapsto \mathbf{L}$ be two operators. Then, given any $a, b \in U$:

- φ is a *projection* if it is (a) monotonic: aRb implies $\langle \varphi(a), \varphi(b)\rangle \in R$ and (b) idempotent: $\varphi(\varphi(a)) = \varphi(a)$.
- a projection operator φ is a *closure* if it is increasing: $\langle a, \varphi(a)\rangle \in R$.
- A projection operator φ is an *interior* if it is decreasing: $\langle \varphi(a), a\rangle \in R$.
- θ is a *modal* or *possibility* operator if it is (a) normal: $\theta(0) = 0$ and (b) additive: $\theta(a \vee b) = \theta(a) \vee \theta(b)$.
- θ is a *co-modal* or *necessity* operator if it is (a) co-normal: $\theta(1) = 1$ and (b) multiplicative: $\theta(a \wedge b) = \theta(a) \wedge \theta(b)$.
- A closure operator θ on a lattice is *topological* if it is modal.
- An interior operator θ on a lattice is *topological* if it is co-modal.

Now we investigate two pairs of operators which are defined by means of a binary relation R. In the definitions of these operators (as well as of many mathematical operators) some logic combinations recur, namely the pairs $\langle \Rightarrow, \wedge\rangle$, $\langle \exists, \wedge\rangle$, $\langle \forall, \Rightarrow\rangle$, $\langle \exists, \forall\rangle$ and $\langle \forall, \exists\rangle$. These combinations enjoy particular mathematical properties which are inherited by the operators they define and which are introduced in the next definition.

Definition 7 (Galois Adjunctions) Let $\mathbf{O}$ and $\mathbf{O}'$ be two pre-ordered sets (possibly lattices) with order $\leq$, resp. $\leq'$ and $\sigma : \mathbf{O} \longmapsto \mathbf{O}'$ and $\iota : \mathbf{O}' \longmapsto \mathbf{O}$ be two maps such that for all $p \in \mathbf{O}$ and $p' \in \mathbf{O}'$

$$\iota(p') \leq p \ \textit{ iff } \ p' \leq' \sigma(p) \tag{20}$$

then σ is called the *upper adjoint* of ι and ι is called the *lower adjoint* of σ. This fact is denoted by $\mathbf{O}' \dashv^{\iota,\sigma} \mathbf{O}$ and we say that the pair $\langle \iota, \sigma \rangle$ forms a *Galois adjunction* or an *axiality*.

Remarks 3.1 The contravariant version of (20), i.e. $\iota(p') \leq p$ *iff* $p' \geq' \sigma(p)$ is called a *Galois connection* and $\langle \iota, \sigma \rangle$ a *polarity*. Galois connections from binary relations were basically introduced in [26] and applied in Formal Concept Analysis (FCA) in [49]. FCA and polarities are not in the scope of the chapter. Galois adjunctions, that is, the covariant form we are dealing with, have been introduced in classical Rough Set Theory in [17] with the name "dual Galois connections". Independently and in the present general setting, which is derived from Intuitionistic Formal Spaces (see [44] and [45]), they were applied to approximation theory in [37].

Adjoint operators enjoy interesting properties:

Facts 3.1

1. *σ preserves all existing* infs *(i.e. it is multiplicative), thus it is monotonic.*
2. *ι preserves all existing* sups *(i.e. it is additive), thus it is monotonic.*
3. $\sigma(a) \vee' \sigma(b) \leq' \sigma(a \vee b)$; $\iota(a') \wedge \iota(b') \leq \iota(a' \wedge' b')$.
4. *$\sigma\iota$ is a closure operator on $\mathbf{O}'$, $\iota\sigma$ is an interior operator on $\mathbf{O}$;*
5. $\sigma\iota(a' \wedge' b') \leq \sigma\iota(a') \wedge' \sigma\iota(b')$, $\sigma\iota(a' \vee' b') \geq \sigma\iota(a') \vee' \sigma\iota(b')$;
6. $\iota\sigma(a \vee b) \geq \iota\sigma(a) \vee \iota\sigma(b)$, $\iota\sigma(a \wedge b) \leq \iota\sigma(a) \wedge \iota\sigma(b)$;
7. $\iota\sigma\iota = \iota$; $\sigma\iota\sigma = \sigma$.

For the proofs of the above Facts, see for instance [37, 38] or [39].

3.2 Galois Adjunction from Relations

Definition 8 Let $R \subseteq U \times U'$ be a binary relation, $A \subseteq U$, $B \subseteq U'$. Then we define the following operators:

1. $\langle i \rangle : \wp(U) \longmapsto \wp(U')$; $\langle i \rangle(A) = R(A)$
   – the *intensional possibility* of A. It is also denoted by $\langle R^{\smile} \rangle(A)$.
2. $\langle e \rangle : \wp(U') \longmapsto \wp(U)$; $\langle e \rangle(B) = R^{\smile}(B)$
   – the *extensional possibility* of B. It is also denoted by $\langle R \rangle(B)$.
3. $[i] : \wp(U) \longmapsto \wp(U')$; $[i](A) = R \longrightarrow A$
   – the *intensional necessity* of A. It is also denoted by $[R^{\smile}](A)$.
4. $[e] : \wp(U') \longmapsto \wp(U)$; $[e](B) = R^{\smile} \longrightarrow B$
   – the *extensional necessity* of B. It is also denoted by $[R](B)$.
5. $int : \wp(U) \longmapsto \wp(U)$; $int(A) = \langle e \rangle [i](A)$—the *interior* of A.

6. $cl : \wp(U) \longmapsto \wp(U)$; $cl(A) = [e]\langle i\rangle(A)$—the *closure* of A.
7. $\mathscr{C} : \wp(U') \longmapsto \wp(U')$; $\mathscr{C}(B) = \langle i\rangle[e](B)$—the *co-interior* of B.
8. $\mathscr{A} : \wp(U') \longmapsto \wp(U')$; $\mathscr{A}(B) = [i]\langle e\rangle(B)$—the *co-closure* of B.

The above notation and terms have the following motivations. In a relational system $\langle U, U', R\rangle$, U can be interpreted as a set of items or objects and U' as a set of properties, so that $\langle u, u'\rangle \in R$ means that object u enjoys property u'. According to this interpretation, if $u' \in \langle i\rangle(A)$, then any element of A has the possibility to enjoy u'. On the other hand, if $u' \in [i](A)$ then in order to enjoy u' it is necessary to be a member of A, although this is not a sufficient condition, since there can be elements of A which does not enjoy u' (to put it another way, at most all the elements of A enjoy u'). A symmetric interpretation holds for $\langle e\rangle(B)$ and $[e](B)$. The terms "necessity" and "possibility" are also associated with some models for modal logic. A Kripke model is a relational system $\langle U, R\rangle$ equipped with a forcing relation $\Vdash$ between members of U and formulas, such that:

$$u \Vdash \Box\alpha \quad iff \quad \forall u'(\langle u, u'\rangle \in R \Rightarrow u' \Vdash \alpha)$$

$$u \Vdash \Diamond\alpha \quad iff \quad \exists u'(\langle u, u'\rangle \in R \wedge u' \Vdash \alpha)$$

where $\Box$ is the necessity modality and $\Diamond$ the possibility. If $[\![\alpha]\!] = \{x : x \Vdash \alpha\}$ is the domain of validity of α, then $u \Vdash \Box\alpha$ iff $u \in [e]([\![\alpha]\!])$, while $u \Vdash \Diamond\alpha$ iff $u \in \langle e\rangle([\![\alpha]\!])$. Therefore, from (15) one has $[e]([\![\alpha]\!]) = ([\![\Box\alpha]\!])$ and from (17), $\langle e\rangle([\![\alpha]\!]) = ([\![\Diamond\alpha]\!])$. In turn, $[i]$ and $\langle i\rangle$ model the modality operators with respect to the reverse relation $R^{\smile}$. For this reason we equate the symbols in the following pairs: $([e], [R])$, $(\langle e\rangle, \langle R\rangle)$, $([i], [R^{\smile}])$ and $(\langle i\rangle, \langle R^{\smile}\rangle)$.

Notation We call the operators $\langle\bullet\rangle$ and $[\bullet]$ *constructors*. If $X = \{x\}$, for any operator op of the above definition, we shall usually write $op(x)$ instead of $op(\{x\})$. If needed we write op_R to specify the relation from which an operator op is defined. A relation $R \subseteq U \times U'$ is called *serial* if $R(u) \neq \emptyset$, for any $u \in U$.
From now on, if not otherwise stated given a relation $R \subseteq U \times U'$, A will denote a subset of the domain U and B a subset of the codomain U'.

Through the notion of a Peirce product one arrives at the notion of a *granule*:

Definition 9 Let $R \subseteq U \times U$ be a binary relation, $u \in U$, $A \subseteq U$. The set $R(u)$ (i.e. $\langle i\rangle(\{u\})$) is called the R-*granule* of u and $R(A) = \bigcup\{R(a) : a \in A\}$ is called the R-*granule* of A.

Remarks 3.2 The above definition of $R(A)$ is consistent with (1) of Definition 1 because the operation $R(-)$ is additive. This can be easily proved from the very definition of R-*neighbourhoods* based on the quantifier $\exists$. However, we shall see below that there is another more general proof.

A set of granules of U is called a *granulation*. More in general, granules are subsets of U, so that they are not necessarily generated by some binary relation. For instance, any covering is a granulation although only particular covering are induced by binary relations (more precisely, particular tolerance relations—see the Introduction). Anyway, in what follows we shall deal with preorders and equivalence relations. These kinds of binary relations induce particular features in the above operators, which will be essential in the algebraic analysis of rough set systems.

We list some straightforward consequences of the above definitions and Lemma 5:

$$[i](A) = \{u' : R^{\smile}(u') \subseteq A\} \tag{21}$$

$$[e](B) = \{u : R(u) \subseteq B\} \tag{22}$$

$$\langle i\rangle(A) = R(A) = \{u' : R^{\smile}(u') \cap A \neq \emptyset\} \tag{23}$$

$$\langle e\rangle(B) = R^{\smile}(B) = \{u : R(u) \cap B \neq \emptyset\} \tag{24}$$

$$int(A) = \bigcup\{R^{\smile}(u') : u' \in [i](A)\} = \bigcup\{R^{\smile}(u') : R^{\smile}(u') \subseteq A\} \tag{25}$$

$$cl(A) = \{u : R(u) \subseteq R(A)\} = \{u : \forall u'(u \in R^{\smile}(u') \Rightarrow R^{\smile}(u') \cap A \neq \emptyset)\} \tag{26}$$

$$\mathscr{C}(B) = \bigcup\{R(u) : u \in [e](B)\} = \bigcup\{R(u) : R(u) \subseteq B\} \tag{27}$$

$$\mathscr{A}(B) = \{u' : R^{\smile}(u') \subseteq R^{\smile}(B)\} = \{u' : \forall u(u' \in R(u) \Rightarrow R(u) \cap B \neq \emptyset)\} \tag{28}$$

The following duality properties are easily obtained by means of the logical equivalences $\neg\exists \equiv \forall\neg$ and $\neg(\alpha \wedge \neg\beta) \equiv \alpha \Longrightarrow \beta$:

Lemma 10 $\langle e\rangle(B) = -[e](-B)$; $\langle i\rangle(A) = -[i](-A)$

Moreover, the operators acting on opposite directions fulfil the following adjointness properties (see [37] and [39]):

Theorem 11 *Let* $\mathbf{P} = \langle U, U', R\rangle$ *be a relational system. Then for* $\mathbf{U}' = \langle\wp(U'), \subseteq\rangle$ *and* $\mathbf{U} = \langle\wp(U), \subseteq\rangle$*:*

$$1.\ \mathbf{U}' \dashv^{\langle e\rangle,[i]} \mathbf{U}, \quad 2.\ \mathbf{U} \dashv^{\langle i\rangle,[e]} \mathbf{U}' \tag{29}$$

Proof Let $A \subseteq U$, $B \subseteq U'$. Then (1): $\langle e\rangle(B) \subseteq A$ iff for all $y \in B$, $\langle e\rangle(y) \subseteq A$, iff for all $y \in B$ if xRy then $x \in A$ iff for all $y \in B$, $y \in [i](A)$, iff $B \subseteq [i](A)$. (2): By symmetry. □

Remarks 3.3 One can verify that all the above operators are isotonic. Moreover, $\exists$ and $\forall$ are, from the position of the sub-formula "$y \in B$" and "$x \in A$" in their definitions, respectively lower and upper adjoints to the pre-image $f^{-1} : \wp(Y) \longmapsto \wp(X)$ of a function $f : X \longmapsto Y$. That is, for all $A \subseteq X$, $B \subseteq Y$ one has $\exists_f(A) \subseteq B$ iff $A \subseteq f^{-1}(B)$ and $B \subseteq \forall_f(A)$ iff $f^{-1}(B) \subseteq A$, where $\exists_f(A) = \{b \in B : \exists a(f(a) = b \wedge a \in A)\}$ and $\forall_f(A) = \{b \in B : \forall a(f(a) = b \Rightarrow a \in A)\}$.

Finally, the operators $\langle\bullet\rangle$ has the logical structure $\exists\wedge$, while the operators $[\bullet]$ has the structure $\forall \Rightarrow$ and we shall see that $\wedge$ is lower adjoint to $\Rightarrow$. Therefore, since "e" (i.e. R-based) and "i" (i.e. $R^{\smile}$-based) operators act in opposite directions, the preceding result comes as no surprise.[1]

Remarks 3.4 From (5) it follows that $\otimes$ is lower adjoint to $\longrightarrow$ with respect to the ordered set $\langle \mathscr{R}(U, U'), \subseteq\rangle$, where $\mathscr{R}(U, U') = \{R : R \subseteq U \times U'\}$. Therefore, $\otimes$ is additive and from point 6 of Definition 1, $R(\ )$ is additive, too. From this observation and Definition 8 one obtains another proof of Theorem 11.

Corollary 12 *Let $\langle U, U', R\rangle$ be a relational system. Then for any X, Y belonging to the due domain:*

1. $[i](U) = U'$; $[e](U') = U$; $[\bullet](\emptyset) = \emptyset$ *if the relation is serial.*
2. $\langle\bullet\rangle(\emptyset) = \emptyset$; $\langle e\rangle(U') = U$ *if R is serial;* $\langle i\rangle(U) = U'$ *if $R^{\smile}$ is serial.*
3. $\langle\bullet\rangle(X \cup Y) = \langle\bullet\rangle(X) \cup \langle\bullet\rangle(Y)$.
4. $[\bullet](X \cap Y) = [\bullet](X) \cap [\bullet](Y)$.
5. $\langle\bullet\rangle(X \cap Y) \subseteq \langle\bullet\rangle(X) \cap \langle\bullet\rangle(Y)$.
6. $[\bullet](X \cup Y) \supseteq [\bullet](X) \cup [\bullet](Y)$.
7. $int(X) \subseteq X \subseteq cl(X)$.
8. $\mathscr{C}(Y) \subseteq Y \subseteq \mathscr{A}(Y)$.

Proof (1) and (2) are trivial. (3) Because $\langle\bullet\rangle$ constructors are lower adjoints. (4) Because $[\bullet]$ constructors are upper adjoints. (5) and (6) can be proved in many a way which are worth mentioning: (a) Straightforwardly from point 3 of Facts 3.1. (b) Using the distributive properties of quantifiers. For instance one has $\forall x A(x) \vee \forall x B(x) \Rightarrow \forall x(A(x) \vee B(x))$, but not the opposite. Incidentally, this proves that $\forall$ cannot have an upper adjoint, otherwise it should be additive. (c) $A \subseteq X$ or $A \subseteq Y$ implies $A \subseteq X \cup Y$ but not the other way around. Also (7) can be proved in many a way: (a) straightforwardly from 5 of Facts 3.1; (b) from (25) and (26) one trivially obtains $int(X) \subseteq X$ and, respectively, $X \subseteq cl(X)$; (c) by adjointness $\langle e\rangle([i](X)) \subseteq X$ iff $[i](X) \subseteq [i](X)$ and $X \subseteq [e](\langle i\rangle(X))$ iff $\langle i\rangle(X) \subseteq \langle i\rangle(X)$; but the rightmost inequalities are tautologies. (8) is obtained by symmetry. □

Thus, if $U = U'$ and R and $R^{\smile}$ are serial, $[\bullet]$ and $\langle\bullet\rangle$ are co-modal, respectively modal, operators on $\langle \wp(U), \subseteq\rangle$, but in general $\langle\bullet\rangle$ are not increasing and $[\bullet]$ are not decreasing. Therefore they are not interiors, respectively, closures.

Vice-versa, $\mathscr{A}$ and cl are closure operators, while $\mathscr{C}$ and int are interior operators on $\langle \wp(U'), \subseteq\rangle$, respectively $\langle \wp(U), \subseteq\rangle$. However, they are not modal, respectively co-modal. Indeed, as like as topological interior operators, int and $\mathscr{C}$ are not additive, because the internal constructors $[\bullet]$ are not, but they are not multiplicative either, because the external constructors $\langle\bullet\rangle$ are not. Symmetrically, cl and $\mathscr{A}$ are neither additive nor multiplicative. We call them *pretopological*.

[1]Often, a lower adjoint is called "left adjoint" and an upper adjoint is called "right adjoint". We avoid the terms "right" and "left" because they could make confusion with the position of the arguments of the operations on binary relations.

However, it is worth noticing that $u \in [e](B)$ iff $R(u) \subseteq B$, that is, if there exists an R-neighbourhood of u included in B, so that $[e](B)$ is similar to the topological definition of an open set. In turn, $u \in \langle e \rangle(B)$ iff $R(u) \cap B \neq \emptyset$, that is, if all the R-neighbourhoods of u have non void intersection with B, since $R(u)$ is the least R-neighbourhood of u. Therefore, we are close to the definition of topological operators. We achieve the goal if the properties of $[\bullet]$ and $\langle \bullet \rangle$ join those of $\mathscr{C}$ and int, respectively $\mathscr{A}$ and cl.

We sum up the previous results in the following table:

Modal constructors	Pre-topological operators
$[e](B) = \{u : R(u) \subseteq B\}$	$\mathscr{C}(B) = \bigcup\{R(u) : R(u) \subseteq B\}$
$[i](A) = \{u' : R^{\smile}(u') \subseteq A\}$	$int(A) = \bigcup\{R^{\smile}(u') : R^{\smile}(u') \subseteq A\}$
$\langle e \rangle(B) = \{u : u \in R^{\smile}(B)\}$	$\mathscr{A}(B) = \{u' : R^{\smile}(u') \subseteq R^{\smile}(B)\}$
$\langle i \rangle(A) = \{u' : u' \in R(A)\}$	$cl(A) = \{u : R(u) \subseteq R(A)\}$

Example 13 $U = \{a, b, c, d\}, U' = \{\alpha, \beta, \gamma\}$

R	α	β	γ
a	1	0	1
b	1	1	0
c	0	1	1
d	1	0	0

$[i](\{a\}) = \emptyset, [i](\{a, b, c\}) = \{\beta, \gamma\}, \langle i \rangle(\{a\}) = \{\alpha, \gamma\},$
$[e](\{\alpha\}) = \{d\}, [e](\{\alpha, \beta\}) = \{b, d\}, \langle e \rangle(\alpha) = \{a, b, d\},$
$int(\{c, d\}) = \emptyset, int(\{a, c, d\}) = \{a, c\},$
$cl(\{a\}) = \{a, d\}, cl(\{d\}) = \{d\},$
$\mathscr{A}(\{\alpha\}) = \{\alpha\}, \mathscr{A}(\{\alpha, \beta\}) = \{\alpha, \beta, \gamma\},$
$\mathscr{C}(\{\alpha\}) = \{\alpha\}, \mathscr{C}(\{\alpha, \beta\}) = \{\alpha, \beta\}.$

Given $R \subseteq U \times U'$, for any operator $op \in \{[e], [i], \langle e \rangle, \langle i \rangle, cl, int, \mathscr{C}, \mathscr{A}\}$ we set $\mathbf{S}_{op}(D) = \{(op(X) : X \in dom(op)\}$, where D is U or U' according to the operator. Then we can define the following lattices:

Definition 14 Let $\langle U, U', R \rangle$ be a relational system. Then:

1. $\mathbf{L}_{\langle i \rangle}(U) = \langle \mathbf{S}_{\langle i \rangle}(U), \wedge, \cup, \emptyset, U' \rangle$, where $\bigwedge_{i \in I} X_i = \mathscr{C}(\bigcap_{i \in I} X_i)$
2. $\mathbf{L}_{[i]}(U) = \langle \mathbf{S}_{[i]}(U), \cap, \vee, \emptyset, U' \rangle$, where $\bigvee_{i \in I} X_i = \mathscr{A}(\bigcup_{i \in I} X_i)$
3. $\mathbf{L}_{\langle e \rangle}(U') = \langle \mathbf{S}_{\langle e \rangle}(U'), \wedge, \cup, \emptyset, U \rangle$, where $\bigwedge_{i \in I} X_i = int(\bigcap_{i \in I} X_i)$
4. $\mathbf{L}_{[e]}(U') = \langle \mathbf{S}_{[e]}(U'), \cap, \vee, \emptyset, U \rangle$, where $\bigvee_{i \in I} X_i = cl(\bigcup_{i \in I} X_i)$
5. $\mathbf{L}_{int}(U) = \langle \mathbf{S}_{int}(U), \wedge, \cup, \emptyset, U \rangle$, where $\bigwedge_{i \in I} X_i = int(\bigcap_{i \in I} X_i)$
6. $\mathbf{L}_{cl}(U) = \langle \mathbf{S}_{cl}(U), \cap, \vee, \emptyset, U \rangle$, where $\bigvee_{i \in I} X_i = cl(\bigcup_{i \in I} X_i)$
7. $\mathbf{L}_{\mathscr{A}}(U') = \langle \mathbf{S}_{\mathscr{A}}(U'), \cap, \vee, \emptyset, U' \rangle$, where $\bigvee_{i \in I} X_i = \mathscr{A}(\bigcup_{i \in I} X_i)$
8. $\mathbf{L}_{\mathscr{C}}(U') = \langle \mathbf{S}_{\mathscr{C}}(U'), \wedge, \cup, \emptyset, U' \rangle$, where $\bigwedge_{i \in I} X_i = \mathscr{C}(\bigcap_{i \in I} X_i)$

Proposition 15 *The structures* $\mathbf{L}_{op}(D)$ *of Definition 14 are complete lattices.*

Proof The proof for $\mathbf{L}_{int}(U)$, $\mathbf{L}_{cl}(U)$, $\mathbf{L}_{\mathscr{A}}(U)$ and $\mathbf{L}_{\mathscr{C}}(U)$ can be found in section 1.4 of [39]. As for $\mathbf{L}_{\langle i \rangle}(U)$ we have to prove that given $\langle i \rangle(X)$ and $\langle i \rangle(Y)$, $\mathscr{C}(\langle i \rangle(X) \cap \langle i \rangle(Y)) = inf\{\langle i \rangle(X), \langle i \rangle(Y)\}$. That is: (a) $\mathscr{C}(\langle i \rangle(X) \cap \langle i \rangle(Y)) \subseteq \langle i \rangle(X)$ and $\mathscr{C}(\langle i \rangle(X) \cap \langle i \rangle(Y)) \subseteq \langle i \rangle(Y)$, and (b) if $\langle i \rangle(Z) \subseteq \langle i \rangle(X)$ and $\langle i \rangle(Z) \subseteq \langle i \rangle(Y)$, then $\langle i \rangle(Z) \subseteq \mathscr{C}(\langle i \rangle(X) \cap \langle i \rangle(Y))$. But (a) is obvious because $C(\langle i \rangle(X) \cap \langle i \rangle(Y)) \subseteq \langle i \rangle(X) \cap \langle i \rangle(Y)$ and $\langle i \rangle(X) \cap \langle i \rangle(Y)$ is included both in $\langle i \rangle(X)$ and $\langle i \rangle(Y)$. As to

(b) $C(\langle i\rangle(X) \cap \langle i\rangle(Y)) = \langle i\rangle[e](\langle i\rangle(X) \cap \langle i\rangle(Y)) = \langle i\rangle([e]\langle i\rangle(X) \cap [e]\langle i\rangle(Y))$. Suppose $\langle i\rangle(Z) \subseteq \langle i\rangle(X)$ and $\langle i\rangle(Z) \subseteq \langle i\rangle(Y)$. Then for adjunction, $Z \subseteq [e]\langle i\rangle(X)$ and $Z \subseteq [e]\langle i\rangle(Y)$, so that $Z \subseteq ([e]\langle i\rangle(X) \cap [e]\langle i\rangle(Y)$. Therefore, by isotonicity $\langle i\rangle(Z) \subseteq \langle i\rangle([e]\langle i\rangle(X) \cap [e]\langle i\rangle(Y))$. The proof for $\mathbf{L}_{[i]}$ comes from duality and by symmetry we obtain the proof for $\mathbf{L}_{[e]}$ and $\mathbf{L}_{\langle e\rangle}$. □

From the definitions above it follows that the lattice order of $\mathbf{L}_{op}(D)$ is inherited from $\langle \mathbf{S}_{op}(D), \subseteq\rangle$.

Lemma 16 *Let* $\mathbf{P} = \langle U, U', R\rangle$ *be a relational system. Then for all* $A \subseteq U$, $B \subseteq U'$:
$A \in \mathbf{S}_{int}(U)$ *iff* $A = \langle e\rangle(B')$, $A \in \mathbf{S}_{cl}(U)$ *iff* $A = [e](B')$, *for some* $B' \subseteq U'$
$B \in \mathbf{S}_{\mathscr{A}}(U')$ *iff* $B = [i](A')$, $B \in \mathbf{S}_{\mathscr{C}}(U')$ *iff* $B = \langle i\rangle(A')$, *for some* $A' \subseteq U$

Proof If $A = \langle e\rangle(B')$ then $A = \langle e\rangle[i]\langle e\rangle(B')$, from point 7 of Facts 3.1. Therefore, by definition of int, $A = int(\langle e\rangle(B')) = int(A)$. Vice-versa, if $A = int(A)$, then $A = \langle e\rangle[i](A)$. Hence, $A = \langle e\rangle(B')$ for $B' = [i](A)$. The other cases are proved in the same way, by exploiting the appropriate equations of point 7 of Facts 3.1. □

Corollary 17 (See [39]) *Let* $\mathbf{P} = \langle U, U', R\rangle$ *be a relational system. Then,*

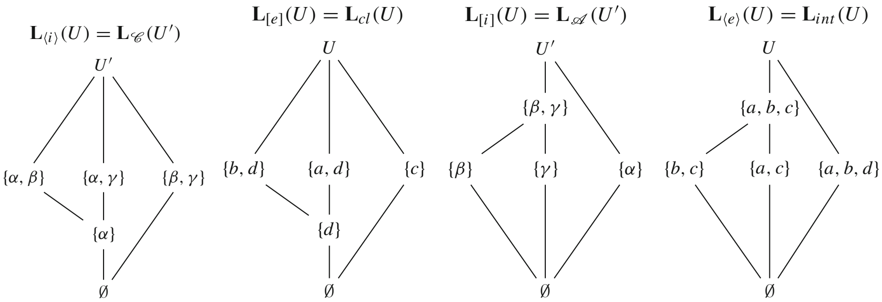
1. $\langle e\rangle$ *is an isomorphism between* $\mathbf{L}_{\mathscr{A}}(U')$ *and* $\mathbf{L}_{int}(U)$*;*
2. $[i]$ *is an isomorphism between* $\mathbf{L}_{int}(U)$ *and* $\mathbf{L}_{\mathscr{A}}(U')$*;*
3. $[e]$ *is an isomorphism between* $\mathbf{L}_{\mathscr{C}}(U')$ *and* $\mathbf{L}_{cl}(U)$*;*
4. $\langle i\rangle$ *is an isomorphism between* $\mathbf{L}_{cl}(U)$ *and* $\mathbf{L}_{\mathscr{C}}(U')$.
5. *the set-theoretic complementation is an anti-isomorphism between* $\mathbf{L}_{cl}(U)$ *and* $\mathbf{L}_{int}(U)$ *and between* $\mathbf{L}_{\mathscr{C}}(U')$ *and* $\mathbf{L}_{\mathscr{A}}(U')$.

Proof Let us notice that the proof for an operator requires the proof for its adjoint operator. Then, let us prove points (1) and (2) together. First, let us prove bijection for $\langle e\rangle$ and $[i]$. From Lemma 16 the codomain of $\langle e\rangle$ is $\mathbf{S}_{int}(U)$ and the codomain of $[i]$ is $\mathbf{S}_{\mathscr{A}}(U')$. Moreover, for all $A \in \mathbf{S}_{int}(U)$, $A = \langle e\rangle[i](A)$ and for all $B \in \mathbf{S}_{\mathscr{A}}(U')$, $B = [i]\langle e\rangle(B)$. From the adjunction properties we have:
(i) $\langle e\rangle$ is surjective onto $\mathbf{S}_{int}(U)$ and (ii) $[i]$ is injective from $\mathbf{S}_{int}(U)$.
(iii) $\langle e\rangle$ is injective from $\mathbf{S}_{\mathscr{A}}(U')$ and (iv) $[i]$ is surjective onto $\mathbf{S}_{\mathscr{A}}(U')$.
Moreover, if $[i]$ is restricted to $\mathbf{S}_{int}(U)$, then its codomain is the set $H = \{B : B = [i](A) \wedge A \in \mathbf{S}_{int}(U)\}$. Clearly, $H \subseteq \mathbf{S}_{\mathscr{A}}(U')$. In turn, if $\langle e\rangle$ is restricted to $\mathbf{S}_{\mathscr{A}}(U')$, then its codomain is the set $K = \{A : A = \langle e\rangle(B) \wedge B \in \mathbf{S}_{\mathscr{A}}(U')\}$. Clearly $K \subseteq \mathbf{S}_{int}(U)$. Therefore, (i) and (iii) give that $\langle e\rangle$ is bijective if restricted to $\mathbf{S}_{\mathscr{A}}(U')$, while (ii) and (iv) give that $[i]$ is a bijection whenever restricted to $\mathbf{S}_{int}(U)$. Now it is to show that $\langle e\rangle$ and $[i]$ preserve joins and meets. For $\langle e\rangle$ we proceed as follows: (v) $\langle e\rangle(\bigvee_{i\in I}(\mathscr{A}(Y_i))) := \langle e\rangle(\mathscr{A}(\bigcup_{i\in I}(\mathscr{A}(Y_i))))$. But $\langle e\rangle\mathscr{A} = \langle e\rangle$, from point 7 of Facts 3.1. Moreover, $\langle e\rangle$ distributes over unions. Hence the right side of (v) equals to $\bigcup_{i\in I}\langle e\rangle(\mathscr{A}(Y_i))$. But in view of Theorem 15, the union of extensional open subsets is open and from Lemma 16 $\langle e\rangle(\mathscr{A}(Y_i))$ belongs to $\mathbf{S}_{int}(U)$ indeed, so that the right side of (v) turns into $int(\bigcup_{i\in I}\langle e\rangle(\mathscr{A}(Y_i))) = \bigcup_{i\in I}\langle e\rangle(\mathscr{A}(Y_i))$. (vi)

$\langle e \rangle(\bigcap_{i\in I} \mathscr{A}(Y_i)) = \langle e \rangle(\bigcap_{i\in I} [i]\langle e \rangle(Y_i))$. Since $[i]$ distributes over intersections, the right side of (vi) turns into $\langle e \rangle[i](\bigcap_{i\in I} \langle e \rangle(Y_i)) = int(\bigcap_{i\in I} \langle e \rangle(Y_i))$. But $\langle e \rangle = \langle e \rangle \mathscr{A}$, so that the last term is exactly $\bigwedge_{i\in I} \langle e \rangle(\mathscr{A}(Y_i))$. Since $[i]$ is the inverse of $\langle e \rangle$, *qua* isomorphism, we have that $[i]$ preserves meets and joins, too.
As to (3) and (4) the results come by symmetry. Finally, (5) is trivial. □

Corollary 18 *For any binary relation R, $\mathbf{L}_{[e]}(U) = \mathbf{L}_{cl}(U)$, $\mathbf{L}_{\langle e \rangle}(U) = \mathbf{L}_{int}(U)$, $\mathbf{L}_{\langle i \rangle}(U) = \mathbf{L}_{\mathscr{C}}(U')$, $\mathbf{L}_{[i]}(U) = \mathbf{L}_{\mathscr{A}}(U')$.*

Example 19 Example 13 continued.



Thus, so far we have seen how binary relations induce modal and pretopological operators. However, if $U = U'$ and R is a preorder the operators and constructors gain the topological properties. To prove that, first we show that if R is a preorder then int and $[i]$ coincide. At this point, since int is an interior operator and $[i]$ is comodal, we immediately obtain that int (aka $[i]$) is topological (see Definition 6). By duality the same can be proved of cl (aka $\langle i \rangle$) and by symmetry for $\mathscr{C}$ (i.e. $[e]$) and $\mathscr{A}$ (i.e. $\langle e \rangle$).

However, we shall complete the proof in a more specific manner, this time with a focus on the opposite direction: it will be proved that if R is a preorder, then $\mathscr{C}$ (thus $[e]$) is the interior operator of a particular topology induced by R. Therefore, now we enter the topological framework.

3.3 Topologies and Relations

Definition 20 Let U be a set. Then:

- Let $\Omega(U)$ be a distributive lattice of subsets of U which is bounded by U and $\emptyset$ and is closed under infinite unions and finite intersections. Then $\Omega(U)$ is called a *topology* on U, its elements *open sets* and $\tau(U) = \langle U, \Omega(U) \rangle$ a *topological space*.
- $\mathbb{I}(X) = \bigcup\{A \in \Omega(U) : A \subseteq X\}$ is called the *interior* of X and $\mathbb{I}$ the *interior operator* of $\tau(U)$.

- $\mathbb{C}(X) = \{x : \forall A \in \Omega(U)(x \in A \Rightarrow A \cap X \neq \emptyset)\}$ is called the *closure* of X and $\mathbb{C}$ the *closure operator* of $\tau(U)$. We put $\Gamma(U) = \{\mathbb{C}(X) : X \subseteq U\}$—the set of closed sets of $\tau(U)$.

Facts 3.2 *From the above definitions it follows that:*

- $\Omega(U) = \{X : X \subseteq U \wedge \mathbb{I}(X) = X\} = \{\mathbb{I}(X) : X \subseteq U\}$.
- $\mathbb{I}(X) = -\mathbb{C}(-X)$ *and* $\mathbb{C}(X) = -\mathbb{I}(-X)$, *any* $X \subseteq U$.
- $\Gamma(U) = \{-A : A \in \Omega(U)\}$ *and* $\Omega(U) = \{-A : A \in \Gamma(U)\}$.
- *The inner logical structure of the operator* $\mathbb{I}$ *is* $(\forall \Rightarrow)$. *Indeed* $x \in \mathbb{I}(X)$ *iff there exists* $A \in \Omega(U)$ *containing* x *such that* $\forall y(y \in A \Rightarrow y \in B)$.
- *The inner logical structure of the operator* $\mathbb{C}$ *is* $(\exists \wedge)$. *Indeed,* $x \in \mathbb{C}(X)$ *iff for all* $A \in \Omega(U)$ *containing* x, $\exists z(x \in A \wedge z \in X)$.

Let now R be a binary relation on a set U, which is assumed to be at most countable. From now on we set $\mathbf{P} := \langle U, R\rangle$. If R is a preorder, then the family of granules $B_R(U) = \{R(u) : u \in U\}$ is a basis of a topology on U (that is, any open set of the topology is given by the union of a family, possibly empty, of elements of $B_R(U)$). This topology is called an *Alexandrov topology*. In this kind of topologies, $R(A)$ is an open set, for any $A \subseteq U$ because the operator $R(-)$ is additive, i.e. $R(A) \cup R(B) = R(A \cup B)$. We denote with $\Omega_R(U)$ the family $\{R(A) : A \subseteq U\}$ of open subsets of the topology and by $\mathbb{I}_R$ and $\mathbb{C}_R$ its interior and, respectively, closure operators.

In Alexandrov spaces the intersection of any family of open sets is open and each point has a least open neighbourhood (indeed the basis $B_R(U)$ provides these least open neighbourhoods). Moreover, in any topological space, a *specialisation preorder* $\preceq$ can be defined as follows: $x \preceq y$ iff for all open set O if $x \in O$ then $y \in O$. An Alexandrov topology $\Omega_R(U)$ is such that its specialisation preorder and R coincide.

Remarks 3.5 The definition of a specialisation preorder can be rephrased using the interior operator $\mathbb{I}$: $x \preceq y$ iff for all $A \subseteq U, x \in \mathbb{I}(A)$ implies $y \in \mathbb{I}(A)$. Indeed, given a set X it can be proved that a preorder can be defined by means of any monadic operator $\odot$ on $\wp(X)$ as follows:

$$x \preceq_{\odot} y \text{ iff } \forall A \subseteq X, x \in \odot(A) \Longrightarrow y \in \odot(A).$$

The relation $\preceq_{\odot}$ is a preorder: clearly it is reflexive because by substituting x for y we obtain a tautology, and it is transitive, because implication is transitive and the universal quantifier distributes over implications. Thus we can call $\preceq_{\odot}$ the *specialisation preorder induced by* $\odot$.

If we denote by $\mathbb{I}_R$ the interior operator of an Alexandrov topology Ω_R induced by a preorder R, we have $R =\preceq_{\mathbb{I}_R}$ and $\Omega_R = \Omega_{\preceq_{\mathbb{I}_R}}$. However, there can be Alexandrov topologies $\Omega_R(U)$ induced by bases $B_R(U)$ such that R is not a preorder, so that $R \neq\preceq_{\mathbb{I}_R}$ but $\Omega_R = \Omega_{\preceq_{\mathbb{I}_R}}$, all the same. We shall illustrate this delicate issue in order to avoid some traps. Moreover, to our knowledge this topic has not been treated before.

Lemma 21 *Let* $\langle U, R\rangle$ *be a relational space. Then* $\forall u \in U, u \in [i](R^{\smile}(u))$.

Proof Trivially, $u \in [i](R^{\smile}(u))$ iff $R^{\smile}(u) \subseteq R^{\smile}(u)$. □

Theorem 22 *Let $\langle U, R\rangle$ be a relational space. Then for all $A \subseteq U, int(A) = [i](A)$ if and only if R is a preorder.*

Proof

A) If $\exists A \subseteq U$ such that $int(A) \neq [i](A)$ then R is not a preorder (either reflexivity or transitivity fail). Proof. The antecedent holds in two cases: (i) $\exists x \in [i](A), x \notin int(A)$; (ii) $\exists x \in int(A), x \notin [i](A)$. In case (i) from (25) one has that $\forall y \in [i](A), x \notin R^{\smile}(y)$. In particular, $x \notin R^{\smile}(x)$, so that reflexivity fails. In case (ii) $\exists y \in [i](A)$ such that $x \in R^{\smile}(y)$. Therefore, since $y \in [i](A)$, from (21) x must belong to A. Moreover, it must exists $z \notin A, \langle z, x\rangle \in R$, otherwise $x \in [i](A)$. Since $\langle x, y\rangle \in R$, if R were transitive, $\langle z, y\rangle \in R$, so that $y \notin [i](A)$. Contradiction.
B) If R is not a preorder, then $\exists A \subseteq U, int(A) \neq [i](A)$. Proof. (i) Take $A = R^{\smile}(x)$. From Lemma 21, $x \in [i](R^{\smile}(x))$. Suppose R is not reflexive with $\langle x, x\rangle \notin R$. Thus $x \notin R^{\smile}(x)$. Hence, it cannot exists an y such that $x \in R^{\smile}(y)$ and $R^{\smile}(y) \subseteq R^{\smile}(x)$. So, $x \notin int(R^{\smile}(x))$. (ii) Suppose transitivity fails, with $\langle x, y\rangle, \langle y, z\rangle \in R, \langle x, z\rangle \notin R$. From Lemma 21, $z \in [i](R^{\smile}(z))$, but $y \notin [i](R^{\smile}(z))$, because $x \in R^{\smile}(y)$ while $x \notin R^{\smile}(z)$ so that $R^{\smile}(y) \nsubseteq R^{\smile}(z)$. On the contrary, $y \in R^{\smile}(z)$ and $R^{\smile}(z) \subseteq R^{\smile}(z)$. Therefore, $y \in int(R^{\smile}(z))$. We conclude that $int(R^{\smile}(z)) \neq [i](R^{\smile}(z))$.

□

We write $op = op'$ if for all elements X of their domain $op(X) = op'(X)$.

Corollary 23 *In a relational space $\langle U, R\rangle$, the following are equivalent: (i) R is a preorder, (ii) $\mathscr{C} = [e]$, (iii) $int = [i]$, (iv) $cl = \langle i\rangle$, (v) $\mathscr{A} = \langle e\rangle$.*

Proof (i)⇔ (iii) is Theorem 22, (ii)⇔ (iv) by duality and the other equivalences by symmetry. □

Corollary 24 *Given a relational space $\langle U, R\rangle$, if R is a preorder, then $int, [i], \mathscr{C}$ and $[e]$ are topological interior operators; $cl, \langle i\rangle, \mathscr{A}, \langle e\rangle$ are topological closure operators.*

The converse of Corollary 24 holds just partially:

Corollary 25 *Let $\langle U, R\rangle$ be a SRS. If $[\bullet]$ and $\langle\bullet\rangle$ are topological interior, respectively closure, operators, then R is a preorder.*

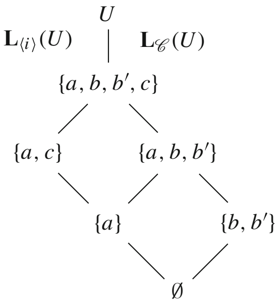
The proof follows from Corollary 23. However, the converse of Corollary 24 does not hold for *int*, *cl*, $\mathscr{A}$ and $\mathscr{C}$ as the following example illustrates and Theorem 30 will prove:

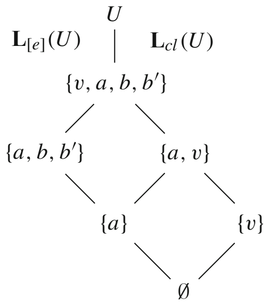
Example 26

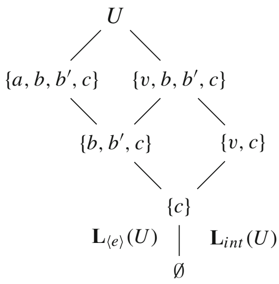
$$U = \{v, a, b, b', c\}, \text{ and } B_R(U) = \{\{a\}, \{b, b'\}, \{a, c\}, U\}$$

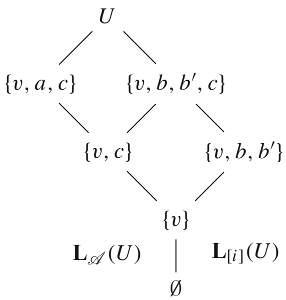
R	v	a	b	b'	c
v	0	0	1	1	0
a	0	1	0	0	0
b	0	1	0	0	1
b'	0	1	0	0	1
c	1	1	1	1	1

R is neither reflexive (e.g. $\langle v, v\rangle \notin R$) nor transitive (e.g. $\langle b, c\rangle \in R$ and $\langle c, v\rangle \in R$, but $\langle b, v\rangle \notin R$). Therefore, it is not a preorder. Indeed, from the lattices below one verifies that the equalities of Corollary 23 do not hold. However, in these lattices $inf = \cap$ and $sup = \cup$. Therefore, they are bounded distributive, hence topologies.









Albeit obvious, it is worth pointing out that if $\mathbf{L}_{op} = \mathbf{L}_{op'}$ the equality is related to the entire lattice which the operators output, not to the operators. For instance $\langle i\rangle(\{v\}) = \{b, b'\} \neq \emptyset = \mathscr{C}(\{v\})$, $\langle e\rangle(\{v\}) = \{c\} \neq \emptyset = int(\{v\})$, $[e](\{v, a\}) = \{a\} \neq \{v, a\} = cl(\{v\})$ and $[i](\{v, b', c\}) = \{v, b, b'\} \neq \{v, b, b', c\} = \mathscr{A}(\{v, b, b'\})$.

Remarks 3.6 Corollary 24 amends point (iv) of Corollary 1 of [35] and point (ii) of Facts 3 of [36], which state also the converse implication, erroneously. However, one can state that if $\{R(A) : A \subseteq U\}$ is a topology, then $\langle U, R\rangle$ is a renaming of the elements of a preorder $\langle U', R\rangle$. To see this, we need some results about the duality between topologies $\Omega_R(U)$ from preorders R and the specialisation preorder $\preceq_{\mathbb{I}_R}$.

Lemma 27 *If $R \subseteq X \times X$ is transitive, then $\forall x, y \in X$, $\langle x, y\rangle \in R$ implies $R(y) \subseteq R(x)$. If R is reflexive, then $R(y) \subseteq R(x)$ implies $\langle x, y\rangle \in R$.*

Proof Suppose $\langle x, y\rangle \in R$ and $z \in R(y)$. Then $\langle y, z\rangle \in R$ and by transitivity $\langle x, z\rangle \in R$ so that $z \in R(x)$. Thus, $R(y) \subseteq R(x)$. Vice-versa, if $R(y) \subseteq R(x)$ then for all a, $\langle y, a\rangle \in R$ implies $\langle x, a\rangle \in R$. In particular $\langle y, y\rangle \in R$ by reflexivity. Hence $\langle x, y\rangle \in R$. □

Theorem 28 *Let $\langle U, R\rangle$ be a relational system such that R is preorder. Then the specialization preorder induced by $[i]$ coincides with $R^{\smile}$ and that induced by $[e]$ coincides with R.*

Proof If $x \preceq_{[i]} y$ then for all $A \subseteq X$, $x \in [i](A)$ implies $y \in [i](A)$. Therefore, $R^{\smile}(x) \subseteq A$ implies $R^{\smile}(y) \subseteq A$, all A. In particular, $R^{\smile}(x) \subseteq R^{\smile}(x)$ implies $R^{\smile}(y) \subseteq R^{\smile}(x)$. But the antecedent is true, so the consequence must be true, too, so that $R^{\smile}(y) \subseteq R^{\smile}(x)$. Since R is reflexive, so is $R^{\smile}$ and from Lemma 27, $\langle x, y\rangle \in R^{\smile}$. The opposite implication is proved analogously by transitivity. The thesis for $[e]$ and R is a trivial consequence. □

Corollary 29 *Let $\mathscr{C}_R$ be the operator induced by a preorder $\langle U, R\rangle$. Then $\mathscr{C}_R$ is the interior operator $\mathbb{I}_R$ of the Alexandrov topology induced by R.*

Proof If R is a preorder then from Corollary 23, $\mathscr{C}_R = [e]$. Therefore, from Theorem 28, the specialisation preorder induced by $\mathscr{C}_R$ coincides with R which, in turn, coincides with the specialisation preorder of the Alexandrov topology induced by R. □

Obviously, if R is symmetric (as in equivalence relations), then $R = R^{\smile}$, with all the simplifications due to this fact which operates for standard Rough Set Theory. Now we prove that $\mathscr{C}_R$ of Example 26 is a topological interior operator, that is, multiplicative. The proof is based on the following fact:

Theorem 30 *Let $\mathbf{L} = \langle L, \wedge, \vee\rangle$ be a lattice and $\odot$ an interior operator on L such that $\odot(a) \wedge \odot(b) \geq \odot(a \wedge b)$ and $\mathbf{L}_{\odot} = \{\odot(x) : x \in L\}$ is a sublattice of $\mathbf{L}$. Then $\odot$ is multiplicative.*

Proof Since $\mathbf{L}_{\odot}$ is a sublattice of $\mathbf{L}$, for all $x, y \in L$, $\odot(x) \wedge \odot(y) = \odot(z)$ for some $z \in L$. Since $\odot(x) \leq x$ and $\odot(y) \leq y$, $\odot(x) \wedge \odot(y) \leq x \wedge y$. Therefore, $\odot(z) \leq x \wedge y$ so that from isotonicity and idempotency of $\odot$ we obtain $\odot(z) = \odot\odot(z) \leq \odot(x \wedge y)$. To prove multiplicativity of $\odot$ we then just need $\odot(z) \geq \odot(x \wedge y)$, which is given by hypothesis. □

$\mathbf{L}_{\mathscr{C}_R}(U)$ is a sublattice of $\wp(U)$, therefore, in $\mathbf{L}_{\mathscr{C}_R}(U)$, $inf = \cap$ and $\mathscr{C}_R$ fulfils the hypotheses of the theorem. So we obtain that for any $X, Y \subseteq U$, $\mathscr{C}_R(X) \cap \mathscr{C}_R(Y) = \mathscr{C}_R(X \cap Y)$.[2]

Let $R \subseteq U \times U$ be such that $\mathscr{C}_R$ is a topological interior operator. Then $\mathbf{L}_{\mathscr{C}_R}$ is a distributive lattice, hence a topology $\Omega_{\mathscr{C}_R}(U)$. Let $\preceq_{\mathscr{C}_R}$ be the specialisation preorder induced by $\mathscr{C}_R$. It is possible to prove that the interior operator $\mathbb{I}_{\mathscr{C}_{\preceq_R}}$ and $\mathscr{C}_R$ coincide and that there is a transformation from R to $\preceq_{\mathscr{C}_R}$. Moreover, this transformation is given by the operation $\longrightarrow$. However, this topic and its mathematical connections are still under investigation.

Example 31 The specialisation preorder $\preceq_{\mathscr{C}}$ induced by the lattice $\mathbf{L}_{\mathscr{C}}(U)$ of Example 26 is the one given in Example 34 below.

[2] A direct proof of point 6 of Facts 3.1 runs as follows. Let $x \in \mathscr{C}_R(X \cap Y)$. Therefore, $\exists y, x \in R(y)$ and $R(y) \subseteq X \cap Y$. It follows that $R(y) \subseteq X$ and $R(y) \subseteq Y$, so that $x \in \mathscr{C}_R(X)$ and $x \in \mathscr{C}_R(Y)$ which amounts to $x \in \mathscr{C}_R(X) \cap \mathscr{C}_R(Y)$. Therefore, $\mathscr{C}_R(X \cap Y) \subseteq \mathscr{C}_R(X) \cap \mathscr{C}_R(Y)$.

3.4 Approximation and Topology

In view of the notions introduced so far, the above three questions can be given precise mathematical answers. Let $A \subseteq U$, $R \subseteq U \times U$ a "nearness relation" of any kind, and $x \in U$:

1. x is in necessarily in A if all the elements R-near x are in A. Since u is R-near x if $u \in R(x)$, we obtain that x is necessarily in A if $R(x) \subseteq A$. Let us set:

$$(lR)(A) = \{x : R(x) \subseteq A\} \tag{30}$$

   It is easy to see that for any $x \in U$, $\{R(X) : x \in R(X)\}$ is a neighbourhood system, so that if R is a preorder $x \in (lR)(A)$ if there is an open set of $\Omega_R(U)$ containing x and included in A.
2. x is possibly in A if there is some element R-near x which is in A. Thus, x is possibly in A if $R(x) \cap A \neq \emptyset$. Let us set:

$$(uR)(A) = \{x : R(x) \cap A \neq \emptyset\} \tag{31}$$

   If R is a preorder, $R(x)$ is the least open set containing x, so that the previous condition implies that all open sets containing x has non void intersections with A.
3. Finally, x is necessarily outside A if $x \in -(uR)(A)$, that is, if $R(x) \subseteq -A$, i.e. $x \in (lR)(-A)$.

Theorem 32 *Given a relational space $\langle U, R\rangle$,*

$$(i)\ (lR)(A) = [e]_R(A).\ (ii)\ (uR)(A) = \langle e\rangle_R(A). \tag{32}$$

Proof (i) From (22) and (30). (ii) From (24) and (31). □

Therefore all the previous results about the extensional constructors apply to the approximation operators.

Definition 33 Given a relational space (U, R) and $A \subseteq U$:

1. $(lR)(A)$ is called the *lower approximation* of A.
2. $(uR)(A)$ is called the *upper approximation* of A.
3. $\langle U, (lR)\rangle$ is called an *approximation space* and is denoted with $\mathbf{AS}(U/R)$.

In view of Lemma 10, $\langle U, (lR)\rangle$ is enough to define an approximation space. If R is a preorder we identify $\mathbf{AS}(U/R)$ with the Alexandrov topological space $\langle U, \Omega_R(U)\rangle$, and we have the following correspondences:

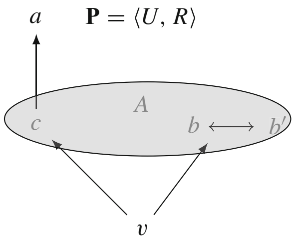
- $(lR)(A)$ is the *interior* $\mathbb{I}_R(A)$ of A,
- $(uR)(A)$ is the *closure* $\mathbb{C}_R(A)$ of A,
- $(bR)(A) = (uR)(A) \cap -(lR)(A)$ is the *boundary* $\mathbb{B}_R(A)$ of A,
- $(eR)(A) = -(uR)(A) = (lR)(-A)$ is the *exterior* of A, denoted by $\mathbb{E}_R(A)$.

The usual topological transformations *via* the complement hold trivially:
$-(lR)(A) = (uR)(-A)$, hence $-(uR)(A) = (lR)(-A)$.

The reader is invited to pay attention that, for instance, $(bR)(A)$ is a notion which applies to any R, while $\mathbb{B}_R(A)$ works just if R is a preorder, and so on.

Example 34 $U = \{v, a, b, b', c\}$

R	v	a	b	b'	c
v	1	1	1	1	1
a	0	1	0	0	0
b	0	0	1	1	0
b'	0	0	1	1	0
c	0	1	0	0	1



$R(a) = \{a\}$
$R(c) = \{a, c\}$
$R(v) = \{v, a, b, b', c\}$
$R(b) = R(b') = \{b, b'\}$
$A = \{b, b', c\}$
$(lR)(A) = \{b, b'\}$
$(uR)(A) = \{v, b, b', c\}$

Remarks 3.7 It is worth noticing that, provided R is a preorder:

$$(lR)(A) = \{x : R(x) \subseteq A\} \tag{33}$$

$$= \bigcup\{R(x) : R(x) \subseteq A\} = \bigcup\{R(X) : R(X) \subseteq A\} \tag{34}$$

$$= \bigcup\{O \in \Omega_R(U) : O \subseteq A\}. \tag{35}$$

Formula (33) can be used to define lower approximations on the basis of any binary relation R. However, this formula does not guarantee a proper *lower* approximation, that is, less than or equal to the set to be approximated and, dually, R does not guarantee a proper upper approximation. For instance, if R is not reflexive and $x \notin R(x)$, then $x \in (lR)(R(x))$, trivially, so that $(lR)(R(x)) \not\subseteq R(x)$. Even worst, if $R(x) = \emptyset$, then for any set A, x belongs to $(lR)(A)$, according to (33), while it does not belong to $(uR)(A)$. An odd situation: x necessarily belongs to A but not possibly.

Formula (34) serves the same purpose and by definition the resulting lower approximation is proper. But (33) coincides with (34) only if R is at least a preorder. Indeed, if $a \in (lR)(A)$ then $R(a) \subseteq A$. But by reflexivity of R, $a \in R(A)$. It follows that $a \in \bigcup\{R(x) : R(x) \subseteq A\}$ and we can conclude $(lR)(A) \subseteq \bigcup\{R(x) : R(x) \subseteq A\}$. Conversely, assume $a \in \bigcup\{R(x) : R(x) \subseteq A\}$. Then for some $b \in U$, $a \in R(b)$ and $R(b) \subseteq A$. By transitivity, $R(a) \subseteq R(b) \subseteq A$, so that $a \in (lR)(A)$ and we conclude $\bigcup\{R(x) : R(x) \subseteq A\} \subseteq (lR)(A)$.

If R is an equivalence relation then $\langle U, \Omega_R(U)\rangle$ is a Pawlak approximation space, $R(x)$ is an equivalence class, $B_R(U)$ is a partition and any element of $\Omega_R(U)$ is the union of equivalence classes so that its complement is a union of equivalence classes, too. As a consequence, $\langle U, \Omega_R(U)\rangle$ is a 0-dimensional topological space, i.e. any open set is closed and vice-versa: they are *clopen*. In this case the upper approximation can be defined in this way: $(uR)(A) = \bigcap\{O \in \Omega_R(U) : A \subseteq O\}$.

If R is a preorder, then by setting $[\![\Box\alpha]\!] = (lR)([\![\alpha]\!])$ and $[\![\Diamond\alpha]\!] = (uR)([\![\alpha]\!])$ the approximation space $\langle U, (lR), (uR)\rangle$ is a model of the modal logic $S4$ (or, more precisely, if U is finite, S4.1). If R is an equivalence relation, the modelled logic is S5.

Geometrically we can depict this fact by embedding $\Omega_R(U)$ into the powerset $\wp(U)$ which with the intersection, union and complement operators provides the ambient Boolean algebra. Notice that one can generalise this approach by defining a modal space as a pair $\langle \mathbf{L}, \mathbf{L}'\rangle$ such that $\mathbf{L}'$ is embeddable in $\mathbf{L}$ and setting for any $a \in \mathbf{L}$, $\Box(a) = \bigsqcup\{x : x \in \mathbf{L}' \;\&\; a \wedge x = x\}$ and $\Diamond(a) = \bigsqcap\{x : x \in \mathbf{L}' \;\&\; a \vee x = a\}$, where $\sqcap$, $\sqcup$ give the lattice order of $\mathbf{L}'$, while $\wedge$, $\vee$ give the order of $\mathbf{L}$, provided the two orders are linked by some coherence property (see [12, 14]).

4 Lesson 3: The Algebraic Framework

If (U, R) is a preorder then the lattice $\langle \Omega_R(U), \cap, \cup, \emptyset, U\rangle$ can be made into a Heyting algebra. If R is an equivalence relation, it is a Boolean algebra. Therefore, we have to explore these notions, from a general point of view.

4.1 Heyting Algebras

Definition 35

- A structure $\mathbf{H} = \langle X, \wedge, \vee, \neg, \Longrightarrow, 1, 0\rangle$ is a *Heyting algebra* if $\langle X, \wedge, \vee, 1, 0\rangle$ is a bounded lattice, $\neg a = a \Longrightarrow 0$, and the following holds, for any $x, a, b \in X$:

$$x \wedge a \leq b \;\; iff \;\; x \leq a \Longrightarrow b \tag{36}$$

  The operation $x \Longrightarrow y$ is called the *relative pseudo-complementation* of x with respect to y and $\neg x$ is called the *pseudo-complementation* of x.
- A Heyting algebra such that for any element x, $\neg\neg x = x$ (or equivalently $x \vee \neg x = 1$), is called a *Boolean algebra*.

The relative pseudo-complement $x \Longrightarrow y$ is the largest element of $\mathbf{H}$ (more precisely, of the carrier X) whose meet with the antecedent x is less than or equal to y. In other terms, $x \Longrightarrow y$ is what x needs to reach y. The relation (36) which defines the relative pseudo-complementation may be re-written by parametrizing the operation with the shared argument a, as follows:

$$\wedge_a(x) \leq b \;\; iff \;\; x \leq\Longrightarrow_a (b)$$

From (20) it immediately appears that in a Heyting algebra $\wedge$ is lower adjoint to $\Longrightarrow$ and $\Longrightarrow$ is upper adjoint to $\wedge$. Therefore, $\wedge$ is additive, so that due to the very properties of adjointness Heyting algebras are distributive lattices: for all a, b and c, $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

Notice that in Heyting algebras $\neg(a \vee b) = \neg a \wedge \neg b$ but $\neg(a \wedge b) \geq \neg a \vee \neg b$, witness the killing case $\neg(a \wedge \neg a)$. In Boolean algebras also the second De Morgan law holds, because $\neg\neg a = a$.

The following standard results will be useful in Sect. 6.5:

Lemma 36 (Cf.[1] and [42]) *In any Heyting algebra* **H**, (1) $b \leq a \Longrightarrow b$. (2) $\Longrightarrow$ *is monotonic (i.e. order preserving) in the second argument, and antitonic (i.e. order reversing) in the first, that is* $a \leq b$ *implies* $c \Longrightarrow a \leq c \Longrightarrow b$, *and* $b \Longrightarrow c \leq a \Longrightarrow c$, *any* c. (3). $a \leq \neg\neg a$. (4) $a \Longrightarrow b \leq \neg b \Longrightarrow \neg a$. (5) $\neg$ *is antitonic.* (6) $\neg\neg$ *is monotonic.*

Also the following results are standard, but a glance to their proofs is worthwhile, to see how adjunction work.

Theorem 37 *In any Heyting algebra*

1. $\neg\neg$ *preserves* $\Longrightarrow$ *and finite meets.*
2. $\neg(a \Longrightarrow b) = \neg\neg a \wedge \neg b$.

Proof (1) The proof for $\Longrightarrow$ will be given in a footnote of Theorem 87. As for meet, since $\neg\neg$ is monotonic, $\neg\neg(x \wedge y) \leq \neg\neg x$ and $\neg\neg(x \wedge y) \leq \neg\neg y$, therefore $\neg\neg(x \wedge y) \leq \neg\neg x \wedge \neg\neg y$. On the other hand, $\neg\neg x \wedge \neg\neg y \wedge \neg(x \wedge y) \leq 0$. From adjunction one obtains $\neg\neg x \wedge \neg\neg y \leq \neg(x \wedge y) \Longrightarrow 0 = \neg\neg(x \wedge y)$. (2) Since $\Longrightarrow$ is monotonic in the second argument, $\neg a = a \Longrightarrow 0 \leq a \Longrightarrow b$. But $\neg$ is antitonic so that $\neg(a \Longrightarrow b) \leq \neg\neg a$. Moreover, $\Longrightarrow$ is antitonic in the first argument so that $b = 1 \Longrightarrow b \leq a \Longrightarrow b$ and, hence, $\neg(a \Longrightarrow b) \leq \neg b$ and we conclude that $\neg(a \Longrightarrow b) \leq \neg\neg a \wedge \neg b$. On the other hand, since $a \Longrightarrow b \leq \neg b \Longrightarrow \neg a$ and $\neg b \wedge (\neg b \Longrightarrow \neg a) \leq \neg a$, one obtains that $\neg\neg a \wedge \neg b \wedge (a \Longrightarrow b) \leq \neg\neg a \wedge \neg a \leq 0$ so that by adjunction $\neg\neg a \wedge \neg b \leq (a \Longrightarrow b) \Longrightarrow 0 = \neg(a \Longrightarrow b)$. □

4.2 Heyting Algebras from Topological Spaces

We now define Heyting algebras using the family of open subsets of a topological space. Indeed, abstract Heyting algebras can be considered the *pointless* companion of the properties of "concrete" topologies, that is open sets populated by points. In this respect, Heyting algebra are part of Algebraic Geometry. In a sense, Heyting algebras are obtained by "zooming-out" topological spaces, while, in turn, topological spaces are obtained by "zooming-in" Heyting algebras. The duality theorem provides such a zooming-in, which will be discussed in the next section.

Definition 38 Let $\langle U, \Omega(U)\rangle$ be a topological space, $A, B \in \Omega(U)$:

$$1 := U \tag{37}$$

$$0 := \emptyset \tag{38}$$

$$A \wedge B := A \cap B \tag{39}$$

$$A \vee B := A \cup B \tag{40}$$

$$A \Longrightarrow B := \mathbb{I}(-A \cup B) \tag{41}$$

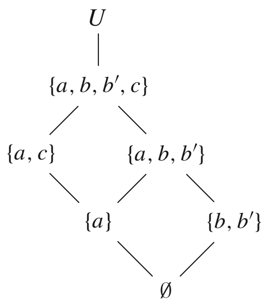
$$\neg A := A \Longrightarrow \emptyset = \mathbb{I}(-A) = -\mathbb{C}(A) \tag{42}$$

Theorem 39 $\Omega(U)$ *equipped with the above operations, is a Heyting algebra.*

The proof is folklore in mathematical logic (indeed, it is key to the very duality theorem for Heyting algebras). It is easy to verify that $X \Longrightarrow Y = \bigcup\{Z : Z \cap X \subseteq Y\}$, so that $\neg X = \bigcup\{Z : Z \cap X = \emptyset\}$.

If $\langle U, R\rangle$ is a pre-ordered space, by $\Omega_R(U)$ we denote three objects: (i) the set of all order filters (or up-sets) of $\langle U, R\rangle$, (ii) the Alexandrov topology of the topological space $\tau(U) = \langle U, \Omega_R(U)\rangle$, (iii) the Heyting algebra with the operations of Definition 38.

Example 40 Example 34 continued. The relational space (U, R) induces the following topological space $\langle U, \Omega_R(U)\rangle$, a.k.a. Heyting algebra:



Verify: $\{a, c\} \Longrightarrow \{a\} = \{a, b, b'\}$.
$\neg\{a\} = \{b, b'\}$, $\neg\{b, b'\} = \{a, c\}$, $\{a\} \cup \neg\{a\} = \{a\} \cup \{b, b'\} = \{a, b, b'\} \subsetneq U$.
The operators $\neg$ and $(lR)(-)$ are different though formally both correspond to $\mathbb{I}-$. Indeed, $\neg$ applies to elements of the algebra and not to generic subsets of U, as (lR) does.
$\neg\mathbb{I}(\{b, b', c\}) = \mathbb{I} - \mathbb{I}(\{b, b', c\}) = \mathbb{I} - \{b, b'\} = \{a, c\}$ $\mathbb{I}(-\{b, b', c\}) = \mathbb{I}(\{a, v\}) = \{a\}$. Therefore, $\neg(lR)(A) \neq (lR)(-A)$. Indeed, $\neg(lR)(A) = \mathbb{I}\mathbb{C}(-A) \neq \mathbb{I}(-A) = (lR)(-A)$. It is immediate to verify that the specialisation preorder $\preceq$ is R itself. For instance $c \preceq a$ because a is in all the open sets containing c, but not the converse.

4.3 Duality

The relationship between finite Heyting algebras $\mathbf{H}$ and preorders is expressed in terms of *duality*. We say that an element x of a (at most countable) Heyting algebra $\mathbf{H}$ is *co-prime*, if $x = \bigvee X$ implies $x \in X$. In other words, x is not the union of elements different from it. Let $J(\mathbf{H})$ be the set of co-prime elements of $\mathbf{H}$ and $\mathbf{J}(\mathbf{H}) = (J(\mathbf{H}), \sqsubseteq)$ where $\sqsubseteq$ is the reverse of the order $\leq$ of $\mathbf{H}$. Let $\Omega_{\sqsubseteq}(\mathbf{J}(\mathbf{H})) = \{\uparrow_{\sqsubseteq} X : X \subseteq J(\mathbf{H})\}$ be the set of all the order filters of $\mathbf{J}(\mathbf{H})$ and $\mathbf{H}(\mathbf{J}(\mathbf{H})) = \langle \Omega_{\sqsubseteq}(\mathbf{J}(\mathbf{H})), \subseteq \rangle$. Then $\mathbf{H}$ and $\mathbf{H}(\mathbf{J}(\mathbf{H}))$ are lattice isomorphic. The isomorphism φ is given by: $\varphi(a) = \{p \in J(\mathbf{H}) : p \leq a\}$ (pay attention that $\leq$ is the order in $\mathbf{H}$). Moreover, $\mathbf{J}(\mathbf{H}(\mathbf{J}(\mathbf{H})))$ and $\mathbf{J}(\mathbf{H})$ are order isomorphic.

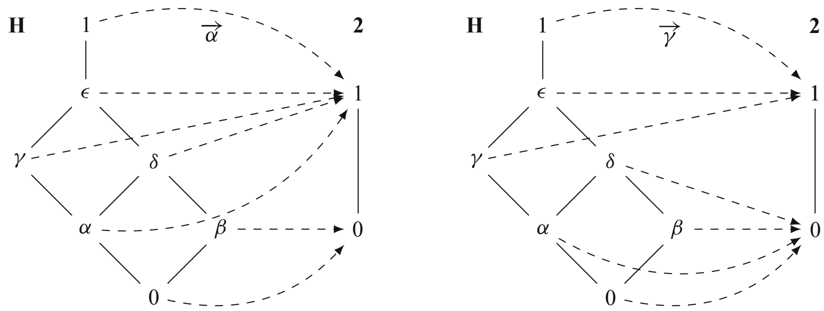
Notice that $\mathbf{H}$ is used both as an algebraic structure and a constructor of algebraic structures.

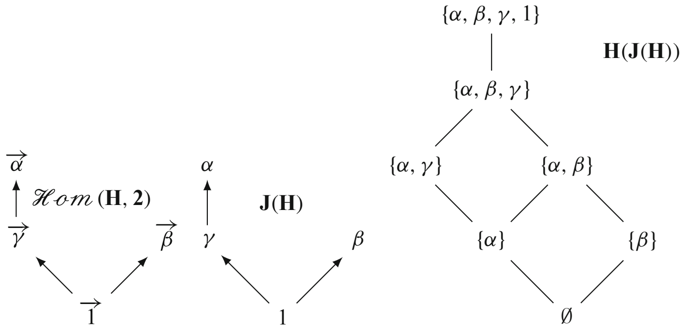
The elements of $J(\mathbf{H})$ may be thought of as "abstract points". If the elements of $\mathbf{H}$ are thought of as "properties", then abstract points are bundles of properties (which means that points have their properties as proxies). This interpretation is supported mathematically. In fact, let $\mathbf{2} = \langle \{0, 1\}, 0 \leq 1 \rangle$ be the so called Sierpiński frame. Let $\psi : \mathbf{H} \longmapsto \mathbf{2}$ be a lattice homomorphism. If the elements of $\mathbf{H}$ are seen as a properties, then the *true kernel* $\psi^{-1}(\mathbf{2})$, which is $\{x : \psi(x) = 1\}$, is a principal filter $\uparrow_{\leq} p = \{x : p \leq x\}$ in $\mathbf{H}$ and, intuitively, gathers together the "virtual points" which fulfil the property p, for some p. If a point a fulfils a property p, we write $p \models a$. The element p of $\mathbf{H}$ which generates the filter is the least "virtual point" fulfilling that property. Otherwise stated, p is both a property and the representative of the "virtual points" which fulfil p itself. Therefore we can denote this homomorphism ψ with $\overrightarrow{p}$. Under this respect, the isomorphism φ gains a straightforward interpretation:

$$\varphi(a) = \{p \in J(\mathbf{H}) : a \in \uparrow_{\leq} p\} = \{p \in J(\mathbf{H}) : a \in \overrightarrow{p}^{-1}(1)\} = \{p \in J(\mathbf{H}) : p \models a\}$$

The set $\mathcal{Hom}(\mathbf{H}, \mathbf{2})$ of all the homomorphism from $\mathbf{H}$ to $\mathbf{2}$ is, thus, the set of properties representing the virtual points which fulfil them. There is a bijection between $J(\mathbf{H})$ and $\mathcal{Hom}(\mathbf{H}, \mathbf{2})$. Therefore we consider $J(\mathbf{H})$ to be the set of *abstract points* defined by $\mathbf{H}$. Now the reverse order $\sqsubseteq$ is understood: $x \sqsubseteq y$ in $\mathbf{J}(\mathbf{H})$ if and only if $\uparrow_{\leq} x \subseteq \uparrow_{\leq} y$ and from Lemma 27, if $y \leq x$ in $\mathbf{H}$, then $\uparrow_{\leq} x \subseteq \uparrow_{\leq} y$. Actually, this is the order on $\mathcal{Hom}(\mathbf{H}, \mathbf{2})$.

Example 41 In order to appreciate the construction of abstract points through duality, let us start with an abstract version $\mathbf{H}$ of the Heyting algebra of Example 40.





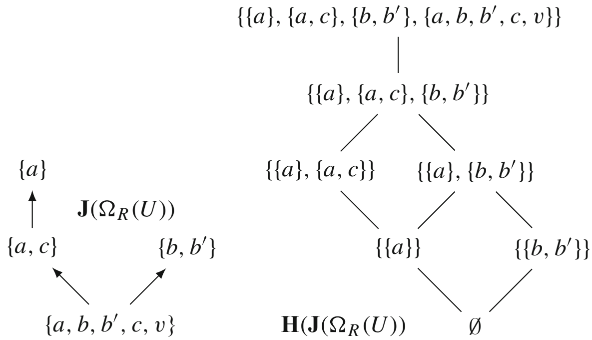
Notice, for instance, that there is no homomorphism $\overrightarrow{\delta}$ because if all the elements of $\uparrow \delta$ were mapped on 1 and the others to 0, then $\overrightarrow{\delta}(\alpha) = 0$, $\overrightarrow{\delta}(\beta) = 0$ so that $\overrightarrow{\delta}(\alpha \vee \beta) = \overrightarrow{\delta}(\delta) = 1 \neq \overrightarrow{\delta}(\alpha) \vee \overrightarrow{\delta}(\beta) = 0$. And the same happens to all the non co-prime elements of $\mathbf{H}$. The properties fulfilled by δ are: $\varphi(\delta) = \{p \in J(\mathbf{H}) : p \leq \delta)\} = \{\alpha, \beta\} = \uparrow_{\sqsubseteq} \{\alpha, \beta\}$.

If $\mathbf{H}$ is a lattice of set (that is, a topological space), it might be the case that the abstract points are fewer than the original points. Indeed, if two points p, p' cannot be separated by means of an open set (i.e. by means of a "personal" property), the homomorphism makes them collapse onto the same abstract point. In other words, from a topological point of view $\mathbf{H}$ and $\mathbf{H}(\mathbf{J}(\mathbf{H}))$ are isomorphic but not homeomorphic. We shall see later that the collapse of such "redundant" points is called T_0*-ification* (or soberification) of the space.[3]

This is what can be seen in our example, indeed.

[3]In the infinite case there can occur a dual situation: there are not enough points to separate properties. In this case the dual operation of T_0-ification is called *spatialisation*.

Example 42 Example 13 continued.



It is evident that $\mathbf{H}(\mathbf{J}(\Omega_R(U))$ and $\Omega_R(U)$ are isomorphic but the isomorphism φ makes the two "twin" points b and b' collapse onto the single abstract point $\{b, b'\}$. In fact, it maps the two-points element $\{b, b'\}$ onto the singleton $\{\{b, b'\}\}$. In turn, $\mathbf{J}(\Omega_R(U))$ is order isomorphic to $(U/_{\equiv}, \leq)$, where $\equiv$ is defined as $p \equiv p'$ iff pRp' and $p'Rp$ and $\leq$ is the order inherited by the equivalence classes from R: $[x]_{\equiv} \leq [y]_{\equiv}$ if and only if xRy (equivalently, one can set $p \equiv p'$ iff $p \preccurlyeq p'$ and $p' \preccurlyeq p$, where $\preccurlyeq$ is the specialisation preorder of the topological space $\langle U, \Omega_R(U)\rangle$).

5 Rough Sets and the Algebras of Rough Set Systems

A *rough set* is an equivalence class on the powerset $\wp(U)$ modulo the equivalence of the two approximations $(lR)(\,)$ and $(uR)(\,)$:

Definition 43 Let $\mathbf{AS}(U/R)$ be an approximation space, with R any binary relation on U. Two sets $A, B \in \wp(U)$ are said to be *rough equal*, denoted $A \approx B$, if $(lR)(A) = (lR)(B)$ and $(uR)(A) = (uR)(B)$. A *rough set* is an equivalence class modulo $\approx$. The rough set of a set A is denoted as $[A]_{\approx}$.

Since the two approximations uniquely define a rough set, given any subset A of U, $[A]_{\approx}$ can be represented in the following ways (see Definition 33):

Definition 44 Let $\mathbf{AS}(U/R)$ be an approximation space and $A \subseteq U$.

1. $\langle (lR)(A), (uR)(A)\rangle$—*increasing representation*—$Icr(A)$
2. $\langle (uR)(A), (lR)(A)\rangle$—*decreasing representation*—$Dcr(A)$
3. $\langle (lR)(A), (eR)(A)\rangle$—*disjoint representation*—$Dsj(A)$
4. $\langle (lR)(A), (bR)(A)\rangle$—*boundary representation*—$Bdr(A)$

Let us focus our attention on the decreasing and disjoint representations:

Definition 45 Let $\mathbf{AS}(U/R)$ be an approximation space. Then we set:

$$Dsj(\mathbf{AS}(U/R)) = \{\langle (lR)(A), (eR)(A)\rangle : A \subseteq U\} \tag{43}$$

$$Dcr(\mathbf{AS}(U/R)) = \{\langle (uR)(A), (lR)(A)\rangle : A \subseteq U\} \tag{44}$$

The above representations are interchangeable. For instance, the following functions link decreasing represented and disjoint represented rough sets:

$$\rho : Dsj(A) \longmapsto Dcr(A);\ \rho(\langle a_1, a_2\rangle) = \langle -a_2, a_1\rangle \tag{45}$$

$$\rho^{-1} : Dcr(A) \longmapsto Dsj(A);\ \rho^{-1}(\langle a_1, a_2\rangle) = \langle a_2, -a_1\rangle \tag{46}$$

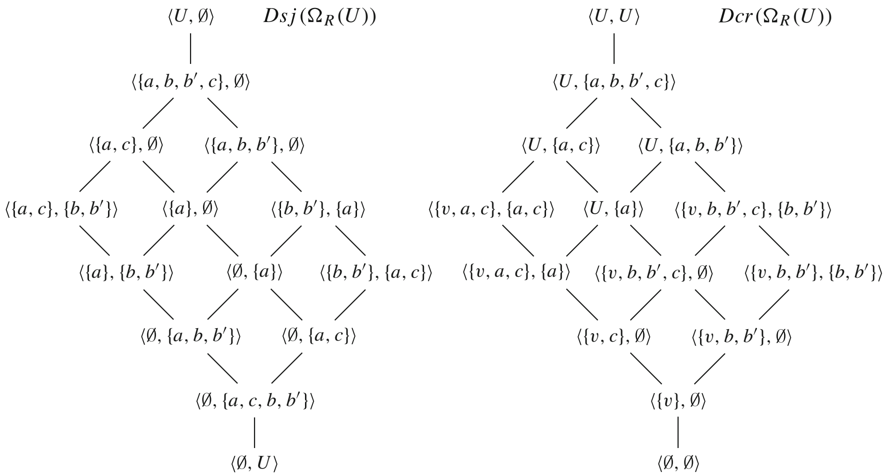
The justification of these functions are straightforward. For instance, ρ operates as follows given $a_1 = \mathbb{I}_R(X)$ and $a_2 = -\mathbb{C}_R(X)$ for some $X \subseteq U$: $\rho(\langle \mathbb{I}_R(X), -\mathbb{C}_R(X)\rangle = \langle -{-\mathbb{C}_R(X)}, \mathbb{I}_R(X)\rangle = \langle \mathbb{C}_R(X), \mathbb{I}_R(X)\rangle$, which is the decreasing representation of the rough set of X.

Notation In view of the above discussion, if R is a preorder from now on the approximation space $\mathbf{AS}(U/R)$ will be considered a topological space and identified with its topology $\Omega_R(U)$. The system of rough sets from this space in disjoint representation will be denote by $Dsj(\Omega_R(U))$ and in decreasing representation by $Dcr(\Omega_R(U))$.

Example 46 Continued from Example 34 where $A = \{b, b', c\}$:

- $Icr(A) = \langle\{b, b'\}, \{b, b', c, v\}\rangle$ • $Dcr(A) = \langle\{b, b', c, v\}, \{b, b'\}\rangle$
- $Dsj(A) = \langle\{b, b'\}, \{a\}\rangle$ • $Bdr(A) = \langle\{b, b'\}, \{c, v\}\rangle$

Below we depict the entire rough set system, in disjoint and in decreasing representation induced by the approximation space $\mathbf{AS}(U/R)$ which is depicted in Example 40 and we identify with the topology $\Omega_R(U)$.



The following table shows the Dsj image of $\wp(U)$. If only b is in a set X then the corresponding set with b' is omitted (for instance, $\langle\{c, v, b\}$ stays also for $\{c, v, b'\}$). First $Dsj(\emptyset) = \langle\emptyset, U\rangle$, $Dsj(U) = \langle U, \emptyset\rangle$. Then:

X	$\{v\}$	$\{c\}, \{c, v\}$	$\{a\}, \{a, v\}$	$\{a, c\}, \{a, c, v\}$
$Dsj(X)$	$\langle\emptyset, \{a, b, b', c\}\rangle$	$\langle\emptyset, \{a, b, b'\}\rangle$	$\langle\{a\}, \{b, b'\}\rangle$	$\langle\{a, c\}, \{b, b'\}\rangle$
X	$\{b\}, \{b, v\}$	$\{c, v, b\}, \{c, b\}$	$\{a, b\}, \{a, v, b\}$	$\{a, c, v, b\}, \{a, c, b\}$
$Dsj(X)$	$\langle\emptyset, \{a, c\}\rangle$	$\langle\emptyset, \{a\}\rangle$	$\langle\{a\}, \emptyset\rangle$	$\langle\{a, c\}, \emptyset\rangle$
X	$\{b, b'\}, \{b, b', v\}$	$\{b, b', c\}, \{b, b', v, c\}$	$\{a, b, b'\}, \{a, b, b', v\}$	$\{a, b, b', c\}$
$Dsj(X)$	$\langle\{b, b'\}, \{a, c\}\rangle$	$\langle\{b, b'\}, \{a\}\rangle$	$\langle\{a, b, b'\}, \emptyset\rangle$	$\langle\{a, b, b', c\}, \emptyset\rangle$

The reader can verify that $Dcr(\Omega_R(U)) = \rho(Dsj(\Omega_R(U)))$.

For instance, $\rho(\langle\{a\}, \{b, b'\}\rangle = \langle -\{b, b'\}, \{a\}\rangle = \langle\{v, a, c\}, \{a\}\rangle$.

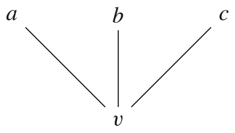
Any ordered pair of $Dcr(\Omega_R(U))$ has a closed set of $\langle U, \Omega_R(U)\rangle$ as first element and an open set as second, which is included in the closed set. Closed sets are order-ideals in (U, R), while open sets are order-filters. Notice that any open set of $\langle U, \Omega_R(U)\rangle$ is a closed set in $\langle U, \Omega_{R^{\smile}}(U)\rangle$, and vice-versa.

Now we have to pay attention to a basic fact. The ordered pair $\langle\{v, a, c\}, \emptyset\rangle$ is made of the above ingredient, namely, decreasing elements of $\Omega_R(U)$. Still it is not the representation of any rough set. Indeed, assume $\{v, a, c\} = (uR)(X)$ and $\emptyset = (lR)(X)$, for some subset X of U. Since $a \in (uR)(X)$, $R(a) \cap X \neq \emptyset$. But $R(a) = \{a\}$, so that $a \in X$. Therefore, $R(a) \subseteq X$ and we conclude that $a \in (lR)(X)$, which contradicts the assumption $(lR)(X) = \emptyset$.

In turn, the ordered pairs of $Dsj(\Omega_R(U))$ are made of disjoint open sets. However, the disjoint ordered pair $\langle\{b, b'\}, \emptyset\}\rangle$ is made of these ingredients, but it is not a rough set. Indeed, if the second element, $\emptyset$, is the complement of the upper approximation (i.e. closure) of a set X, then this upper approximation is U, so that a belongs to it. Hence, from the previous reasoning one obtains that a should be in the lower approximation of X, too, which is not the case.

Pay attention that this problem does not occur because the lower approximation, in the first case, or the complement of the upper approximation, in the second case, are empty sets. Consider the following example:

Example 47 $U = \{v, a, b, c\}$ and R is the partial order below:



$\langle\{a\}, \{c\}\rangle$ is a pair of disjoint open sets, but if $\{a\} = \mathbb{I}_R(X)$ then $R(b) \nsubseteq X$ and since $R(b) = \{b\}$, $b \notin X$. So, $b \in -X$. In consequence $R(b) \subseteq -X$ and $b \in \mathbb{I}_R(-X) = -\mathbb{C}_R(X)$, which, therefore, cannot be $\{c\}$.

Actually, the next sections are focused on obtaining $Dsj(\Omega_R(U))$ from the lattice of all the ordered pairs of disjoint elements of a topological approximation space $\Omega_R(U)$, which now we formally define:

Definition 48 Let $\Omega_R(U)$ be an approximation space with R a preorder.
$Dsj(U/R) = \{\langle a_1, a_2\rangle : a_1, a_2 \in \Omega_R(U)\ \&\ a_1 \cap a_2 = \emptyset\}$

If we want, instead, a lattice of decreasing elements, we have to decide "elements of what?". Since the first element, in decreasing representation, is the closure of a set X, it cannot belong to $\Omega_R(U)$. For instance $\langle\{a, c\}\{a\}\rangle$ is a pair of decreasing elements of $\Omega_R(U)$ but it does not represent any rough set. Actually, the first element of a decreasing representation of rough sets is the complement of some open set of $\Omega_R(U)$ which is an open set in $\Omega_{R^\smile}(U)$. If R is an equivalence relation then $R = R^\smile$, so we do not notice the difference. But if R is a preorder or a partial order then we must take care with it.

In particular we have to take care of the definition of the operations which manipulate ordered pairs of decreasing elements, because in some cases they transform elements of the topology $\Omega_R(U)$ into elements of the opposite topology $\Omega_{R^\smile}(U)$. We shall see this interesting point at due time. By now, our analysis will focus on the disjoint representation which make it possible to operate just on elements of a single structure.

Now we have to face another problem.

As we have seen, if we take the set of all ordered pairs of disjoint elements of $\Omega_R(U)$, which we denote by $Dsj(U/R)$, respectively of all the ordered pairs of decreasing elements in $\Omega_{R^\smile}(U) \times \Omega_R(U)$, denoted $Dcrj(U/R)$), we have elements which do not represent any rough set.

From the above discussion, we need a way to exclude from $Dsj(U/R)$ the ordered pairs which do not fulfil the following condition:

$$X_1 \cup X_2 \supseteq S \tag{47}$$

where S is the set of all isolated points: $S \bigcup\{x : R(x) = \{x\}\}$.

In Example 94 below one can see an illustration of what we have to do, with some mathematical means.

From $Dcr(U/R)$ we have to exclude the ordered pairs which do not fulfil the condition $X_1 \cap S = X_2 \cap S$.

To end this section, we sum up the issue. If $R(X) = \{x\}$ then topologically x is an *isolated point*. Isolated points cannot belong to the boundary of any set, for the reason illustrated in Example 47, which formally runs as follows: if $x \in X$, then $R(x) \subseteq X$ so that $x \in \mathbb{I}_R(X)$. If $x \notin X$, then $x \in -X$ so that $R(x) \subseteq -X$ and, in consequence, $x \in \mathbb{I}_R(-X)$. In the first case $x \in \mathbb{C}_R(U)$ but $x \notin -\mathbb{I}_R(X)$. In the second case $x \in -\mathbb{I}_R(X)$ but $x \notin \mathbb{C}_R(X)$. In both cases $x \notin \mathbb{C}_R(X) \cap -\mathbb{I}_R(X) = \mathbb{B}_R(X)$.

6 Rough Set Systems, Grothendieck Topologies and Lawvere-Tierney Operators

The fact that any isolated point must belong either to the positive part $(lR)(X)$, or to the negative part $-(uR)(X)$ of a rough set, is a sort of Excluded Middle localized on S. Indeed, in general given $x \in U$ and $X \subseteq U$, the assignment of x to X is given by a three-valued characteristic function:

$$\chi_x(X) = \begin{cases} 1 & \text{if } x \in (lR)(X) \\ 1/2 & \text{if } x \in (uR)(X) \cap -(lR)(X) \\ 0 & \text{if } x \in -(uR)(X) \end{cases} \tag{48}$$

But if $x \in S$, then χ takes just value 0 or 1.

Intuitively, Classical Logic is *locally valid* on S.

Local validity is a notion wide studied in Algebraic Geometry which provides a powerful tool, *Grothendieck topology*, which we shall apply in this Section.

6.1 Grothendieck Topologies and Local Validity

Definition 49 (Grothendieck Topology) Let $\mathbf{P} = \langle U, R \rangle$ be a preorder. We recall that $\Omega_R(U) = \{R(X) : X \subseteq U\}$ is the set of all order filters over $\mathbf{P}$. A *Grothendieck topology* on the preorder $\mathbf{P}$ is a map $J : U \longmapsto \wp(\Omega_R(U))$; $J_{[x]} \subseteq \Omega_R(R(x))$ such that:

GT1. $R(x) \in J_{[x]}, \forall x \in U$,

GT2. $R(x') \cap S \in J_{[x']}, \forall x' \geq x, \forall S \in J_{[x]}$.

GT3. $\forall x \in U, \forall S \in J_{[x]}, \forall S' \subseteq R(x)$ such that $S' \in \Omega_R(U)$, if $\forall x' \in S$, $R(x') \cap S' \in J_{[x']}$, then $S' \in J_{[x]}$.

If a filter S belongs to $J_{[x]}$, we say that S *covers* x. $J_{[x]}$ is called the *open-cover system* of x. $\mathbf{G} = \{J_{[x]} : x \in U\}$ and $\langle \mathbf{P}, \mathbf{G} \rangle$ is called an *ordered site*.

From a "granular" point of view, Grothendieck topologies formalize the notion "*To be locally true*" in the following sense: a property P is locally true at point x in a granulated space S if every granule G such that $x \in G$ contains a granule G' such that $x \in G'$ and G has property P, that is, the validity set $[\![P]\!]$ is included in G'.

If S is a topological space, as in the case we are dealing with, then one substitutes "open neighbourhood" for "granule" and obtains that a topological space S has property P locally valid at a point x, if the set of P-neighbourhoods of x (i.e. the set of neighbourhoods of x included in $[\![P]\!]$) is *cofinal* in the neighbourhood filter of x.

By definition, a Grothendieck open-cover of x is a set of open neighbourhoods of x in the given topology.

Example 50 In our standard example $\Omega_R(U)$, suppose $[\![P]\!] = \{a\}$, then $c \not\models P$. But the granules (i.e. open neighbourhoods) containing c are $\{a, c\}$, $\{a, c, b, b'\}$, and U an all of them contain $\{a\}$. So P is locally valid at c. On the contrary, for instance, b has three granules containing $\{a\}$ (they are $\{a, b, b'\}$, $\{a, c, b,'\}$ and U), but there is an open neighbourhood of b not containing $\{a\}$ and it is $\{b, b'\}$. So P is not locally valid at b (or at b', of course).

As much as the family of open (closed) sets of a topology induces an interior (closure) operator, an ordered site induces a particular operator. This operator is partially an interior and partially a closure operator of a usual topology (and we shall see the reason why).

6.2 Lawvere-Tierney Operators

Any Grothendieck topology induces on $\Omega_R(U)$ a closure operator $J : J(A) = \{x : A \cap R(x) \in J_{[x]}\}$. In other terms, if $A = [\![P]\!]$ then $J(A)$ is the set of points in which P is locally valid. J is a *Lawvere-Tierney operator* which we define at pointless level:

Definition 51 (Lawvere-Tierney Operators) Given a Heyting algebra $\mathbf{H}$, $J : \mathbf{H} \longmapsto \mathbf{H}$ is a *Lawvere-Tierney operator* if the following hold:

- $x \leq J(x)$—inflation,
- $J(J(x)) = J(x)$—idempotence,
- $J(x \wedge y) = J(x) \wedge J(y)$—multiplicativity.

From multiplicativity we obtain *monotonicity*: if $x \leq y$ then $J(x) \leq J(y)$.

The above properties has the following intuitive motivations:

In the first place, since x is more specialised than y if it enjoys more properties than y, that is, it belongs to open sets from which y is excluded, but not the other way around, conversely we say that a property P is stronger than Q if its domain of validity is "more specialised" than that of Q. So we have: (i) If the property P is stronger than property Q, then P is locally stronger than Q. (ii) The domain of validity of P is stronger than the domain of local validity of P. (iii) The domain of local validity of the domain of local validity of a property P equals the domain of local validity of P itself. (iv) The domain of local validity of $(P \wedge Q)$ is the *inf* of the domains of local validity of P and Q.

We have seen that given a Grothendieck topology we can produce a Lawvere-Tierney operator connected to it. Now, symmetrically, we restore a Grothendieck topology from a Lawvere-Tierney operator on $\Omega_R(U)$. The following result can be found in [15] or [23].

Lemma 52 *Given a preorder* $\mathbf{P} = \langle U, R\rangle$ *and a Lawvere-Tierney operator* J *on* $\Omega_R(U)$*, the family*

$$\{J_{[x]} : J_{[x]} = \{R(x) \cap X : x \in J(X) \ \&\ x \in U\} \tag{49}$$

is a Grothendieck topology.

Definition 53 (Local Validity in an Ordered Site) A *forcing relation* $\models$ between elements of $\mathbf{P}$ and the set $\mathscr{I}$ of formulas of propositional Intuitionistic logic is a relation $\models \subseteq U \times \mathscr{I}$, such that for any formula $\alpha \in \mathscr{I}$, $p \in U$:

$$If\ \ p \models \alpha\ then\ \forall p'(p' \in R(p) \Rightarrow p' \models \alpha). \tag{50}$$

which means that if $p \in [\![\alpha]\!]$ then $p \in [e]_{\models}([\![\alpha]\!])$. Clearly, for any α, $[\![\alpha]\!]$ belongs to $\Omega_R(U)$.

Given an ordered site $\langle \mathbf{P}, \mathbf{G}\rangle$, we say that a formula α is *locally valid* at point $p \in \mathbf{P}$, in symbols $p \models \langle l\rangle(\alpha)$, if $R(p) \cap [\![\alpha]\!]$ covers p in the topology $\mathbf{G}$, that is, if $\{p' \geq p : p' \models \alpha\}$ belongs to the open-cover system of p in $\mathbf{G}$.

Example 54 The following is a Grothendieck topology which we name $\mathbf{G}^\delta$ for reasons that will be clear soon:

x	a	b	b'	c	v
$J^\delta_{[x]}$	$\{\{a\}\}$	$\{\{b, b'\}\}$	$\{\{b, b'\}\}$	$\{\{a\}, \{a, c\}\}$	$\{\{a, b, b'\}, \{a, b, b'c\}, U\}$

Let us compute $J^\delta(\{a\})$, where J^δ is the Lawvere-Tierney operator induced by $\mathbf{G}^\delta$: $R(a) \cap \{a\} = \{a\} \in J^\delta_{[a]}$, $R(c) \cap \{a\} = \{a\} \in J^\delta_{[c]}$. For no other x, $R(x) \cap \{a\} \in J^\delta_{[x]}$. So $J^\delta(\{a\}) = \{a, c\}$. The other cases follow suit. Therefore, if $[\![P]\!] = \{a\}$ then P is locally valid at c even if $c \notin [\![P]\!]$, as anticipated in Example 50.

X	$\emptyset$	$\{a\}$	$\{b, b'\}$	$\{a, b, b'\}$	$\{a, c\}$	$\{a, b, b', c\}$	U
$J^\delta(X)$	$\emptyset$	$\{a, c\}$	$\{b, b'\}$	U	$\{a, c\}$	U	U

Vice-versa, given J^δ we can compute $\mathbf{G}^\delta$. For instance $J^\delta_{[c]}$ is obtained as $c \in J^\delta(\{a\})$, $J^\delta(\{a, c\})$, $J^\delta(\{a, b, b'\})$, $J^\delta(\{a, b, b', c\})$ and $J^\delta(U)$. Therefore:

$$\begin{aligned} J^\delta_{[c]} &= \{R(c) \cap \{a\}, R(c) \cap \{a, c\}, R(c) \cap \{a, b, b'\}, R(c) \cap \{a, b, b', c\}, R(c) \cap U\} \\ &= \{\{a, c\} \cap \{a\}, \{a, c\} \cap \{a, c\}, \{a, c\} \cap \{a, b, b'\}, \{a, c\} \cap \{a, b, b', c\}, \{a, c\} \cap U\} \\ &= \{\{a\}, \{a, c\}\}. \end{aligned}$$

Notice that for any x, $J^\delta_{[x]} = \{Y : J^\delta(Y) = J^\delta(X) \ \&\ Y \subseteq X\}$.

6.3 Congruence

For the comfort of the reader we recall some definitions and results.

Definition 55 (Congruence) Let $\mathbf{L} = \langle U, \varphi \rangle$ be a set equipped with an n-ary operation φ and $\equiv$ an equivalence relation on U. Then $\equiv$ is called a *congruence* if for all $a_1 \ldots a_n, b_1 \ldots b_n \in U$, $a_1 \in [b_1]_{\equiv}, \ldots, a_n \in [b_n]_{\equiv}$ implies $\varphi(a_1, \ldots, a_n) \in [\varphi(b_1, \ldots, b_n)]_{\equiv}$. If this is the case, we say that $\equiv$ is *compatible* with φ.

Therefore, given a Heyting algebra $\mathbf{H}$ and an equivalence relation $\equiv$ on $\mathbf{H}$ we say that $\equiv$ is a $\wedge$-*congruence* if it is compatible with $\wedge$, that is if $a_1 \in [b_1]_{\equiv}$, $a_2 \in [b_2]_{\equiv}$ implies $a_1 \wedge a_2 \in [b_1 \wedge b_2]_{\equiv}$. Similarly we define the notion of $\vee$-*congruence* and $\Longrightarrow$-*congruence*. If all the three compatibilities are satisfied, we say that $\equiv$ is a *Heyting algebra-congruence*. Remember that $0 = \bigvee \emptyset$ and $1 = \bigwedge \emptyset$.

Lemma 56 *Let* $\mathbf{L} = \langle U, \varphi \rangle$ *and* $\mathbf{L}' = \langle U', \varphi \rangle$ *be two sets equipped with the same n-ary operation* φ. *Let* f *be an homomorphism between* $\mathbf{L}$ *and* $\mathbf{L}'$. *Set for all* $a, b \in \mathbf{L}$, $a \equiv_f b \iff f(a) = f(b)$. *Then* $\equiv_f$ *is a congruence on* $\mathbf{L}$.

Proof The standard and straightforward proof is the following: Since $\equiv_f$ is defined by means of an equality, then it is an equivalence. Now assume $a_1 \in [b_1]_{\equiv_f}, \ldots, a_n \in [b_n]_{\equiv_f}$. Hence, since f preserves φ, $f(\varphi(a_1, \ldots, a_n)) = \varphi(f(a_1), \ldots, f(a_n))$. But by assumption $f(a_i) = f(b_i)$. Therefore, $f(\varphi(a_1, \ldots, a_n)) = \varphi(f(b_1), \ldots, f(b_n)) = f(\varphi(b_1, \ldots, b_n))$, so that $\varphi(a_1, \ldots, a_n) \in [\varphi(b_1, \ldots, b_n)]_{\equiv_f}$. □

The congruence $\equiv_f$ is called the *kernel* of f.

Definition 57 (Quotient Structure) Let $\mathbf{L} = \langle U, \varphi \rangle$ be a set with an n-ary operation φ and $\equiv$ an equivalence relation on $\mathbf{L}$. Then:

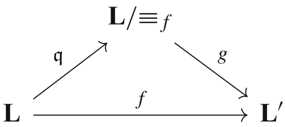
1. $U/\equiv := \{[a]_{\equiv} : a \in U\}$ is called the *quotient set* of U.
2. For all $a, b \in U$, define $\varphi_{\equiv}([a]_{\equiv}, [b]_{\equiv}) := [\varphi(a, b)]_{\equiv}$. Then $\mathbf{L}/\equiv := \langle U/\equiv, \varphi_{\equiv} \rangle$ is called the *quotient structure* of $\mathbf{L}$.

If there is no risk of confusion, we write φ also for $\varphi_{\equiv}$, thus, for instance, $\wedge$ instead of $\wedge_{\equiv}$.

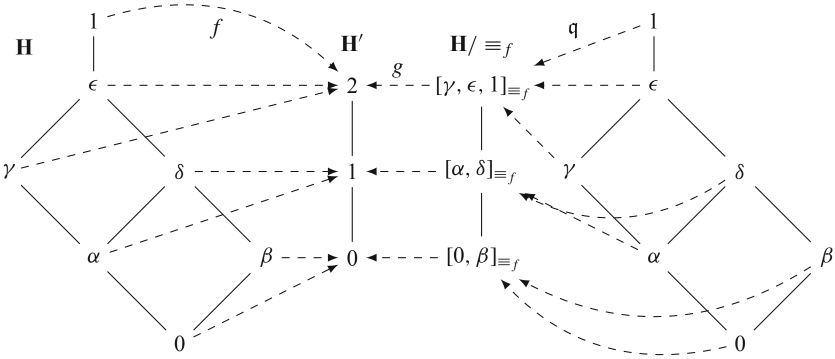
Lemma 58 *If* $\mathbf{L}$ *is a lattice and* $\equiv$ *a congruence on* $\mathbf{L}$, *then* $\mathbf{L}/\equiv$ *is a lattice and the map* $\mathfrak{q} : \mathbf{L} \longmapsto \mathbf{L}/\equiv$; $\mathfrak{q}(a) = [a]_{\equiv}$ *is a homomorphism. The map* $\mathfrak{q}$ *is called the* natural quotient map.

Theorem 59 (Fundamental Homomorphism Theorem for Lattices) *Let* $\mathbf{L}$ *and* $\mathbf{L}'$ *be lattices and* f *an homomorphism of* $\mathbf{L}$ *onto* $\mathbf{L}'$. *Then the map* $g : \mathbf{L}/\equiv_f \longmapsto \mathbf{L}'$ *given by* $g([a]_{\equiv_f}) = f(a)$ *is independent of the representative a, that is, for all* $a, b \in \mathbf{L}$, $[a]_{\equiv f} = [b]_{\equiv_f}$ *implies* $g([a]_{\equiv_f}) = g([b]_{\equiv_f})$.
Moreover g *is an isomorphism between* $\mathbf{L}/\equiv_f$ *and* $\mathbf{L}'$.

Finally, if $\mathfrak{q}$ *is the natural quotient map, then* $\equiv_{\mathfrak{q}}$ *and* $\equiv_f$ *coincide and the following diagram commutes, that is,* $g(\mathfrak{q}) = f$*:*



Example 60 Consider the abstract Heyting algebra of Example 41.



6.4 Lawvere-Tierney Operators and Heyting Algebra Congruences

Lemma 61 *Let* **H** *be a Heyting algebra and* $a \in \mathbf{H}$. *Let us set the following operator on* **H***:*

$$J^a(x) = a \Longrightarrow x \tag{51}$$

Then J^a *is a Lawvere-Tierney operator.*

Proof This is a standard result in Algebraic Geometry (see [23]). However we provide a simple proof which exhibits how adjointness properties may be used. Indeed, in any Heyting algebra **H**, from the adjointness property, $y \wedge a \leq x$ iff $y \leq a \Longrightarrow x$, for any y. But $x \wedge a \leq x$ and one obtains $x \leq a \Longrightarrow x$, hence J^a is increasing. Idempotence follows from the following equations: $a \Longrightarrow (a \Longrightarrow x) = (a \wedge a) \Longrightarrow x = a \Longrightarrow x$. This is an application of the Curry property of $\wedge$ and $\Longrightarrow$ which can be obtained from adjointness: $x \leq y \Longrightarrow (w \Longrightarrow z)$ iff $x \wedge y \leq w \Longrightarrow z$ iff $x \wedge (y \wedge w) \leq z$, that is, $x \leq (y \wedge w) \Longrightarrow z$. Finally, multiplicativity derives again from the multiplicativity of upper adjoints and $\Longrightarrow$ is upper adjoint. □

Lemma 62 *Let* $\mathbf{H}$ *be a Heyting algebra and* $\mathscr{F}$ *a filter on* $\mathbf{H}$. *Let us define for all* $a, b \in \mathbf{H}$, $a \equiv_{\mathscr{F}} b$ *iff* $\exists f \in \mathscr{F}$ *such that* $a \wedge f = b \wedge f$. *Then* $\equiv_{\mathscr{F}}$ *is a Heyting algebra congruence.*

Proof Suppose $x \in [y]_{\equiv_{\mathscr{F}}}$ and $z \in [w]_{\equiv_{\mathscr{F}}}$. Then for some $f \in \mathscr{F}$, $x \wedge f = y \wedge f$ and $z \wedge f = w \wedge f$. Now, $(x \vee z) \wedge f = (x \wedge f) \vee (z \wedge f) = (y \wedge f) \vee (w \wedge f) = (y \vee w) \wedge f$. Hence, $x \vee z \in [y \vee w]_{\equiv_{\mathscr{F}}}$. Dually one obtains $\wedge$-compatibility. We have to prove the case of $\Longrightarrow$, that is, $(x \Longrightarrow z) \wedge f = (y \Longrightarrow w) \wedge f$. Now, let $q \wedge x \leq z$ Then $q \wedge f \wedge x \leq z \wedge f$. From the congruence assumptions we obtain $q \wedge f \wedge y \leq w \wedge f$ and, *a fortiori*, $q \wedge f \wedge y \leq w$, which means that $q \wedge f \leq y \Longrightarrow w \wedge f$. Since q is arbitrary among the elements a such that $a \wedge x \leq z$, in particular we can set $q = x \Longrightarrow z$, obtaining $x \Longrightarrow z \wedge f \leq y \Longrightarrow w \wedge f$. With a symmetric reasoning we obtain $y \Longrightarrow w \wedge f \leq x \Longrightarrow z \wedge f$ and conclude $x \Longrightarrow z \wedge f = y \Longrightarrow w \wedge f$. □

In particular, if $\mathscr{F}$ is a principal filter generated by the element p, one has that $a \wedge f = b \wedge f$ for some $f \in \mathscr{F}$ if and only if $a \wedge p = b \wedge p$. However, if we set $\hat{\mathscr{F}}(a) = a \wedge p$, we indeed have that $a \equiv_p b \iff \hat{\mathscr{F}}(a) = \hat{\mathscr{F}}(b)$ is a congruence, but $\hat{\mathscr{F}}$ is not a Lawvere-Tierney operator, because $\hat{\mathscr{F}}(a) \leq a$.

Anyway, there is a Lawvere-Tierney operator J such that $\equiv_J$ coincides with $\equiv_{\hat{\mathscr{F}}}$, and it is J^p:

Theorem 63 *Given a Heyting algebra* $\mathbf{H}$ *and* $p \in \mathbf{H}$, *set* $a \equiv_p b$ *if and only if* $a \wedge p = b \wedge p$ *and let* $a \equiv_{J^p} b$ *iff* $J^p(a) = J^p(b)$. *Then* $\equiv_p$ *coincides with* $\equiv_{J^p}$.

Proof A proof is given in Proposition 7.4.2 of [39]. A straightforward proof is the following, anyway. The lower adjoint of J^p is $\wedge_p$. From this the result is just an application of the adjointness relation (cf. (4.1)). □

Therefore, given an Heyting algebra $\mathbf{H}$ and an element $a \in \mathbf{H}$, $\equiv_{J^a}$ is a congruence on $\mathbf{H}$. Here we report a more general result about the relation between congruence and Lawvere-Tierney operators on a Heyting algebra (see [15]):

Theorem 64 *Let* $\equiv$ *be a congruence on a Heyting algebra* $\mathbf{H}$. *Define:*

$$J_{\equiv} : \mathbf{H} \longmapsto \mathbf{H};\ J_{\equiv}(p) = \bigvee\{x : x \equiv p\} \tag{52}$$

Then $J_{\equiv}$ *is a Lawvere-Tierney operator and* $x \equiv b$ *iff* $J_{\equiv}(x) = J_{\equiv}(y)$, *that is,* $\equiv$ *and* $\equiv_{J_{\equiv}}$ *coincide.*

Proof We just prove the first part. From (52) $p \leq J_{\equiv}(p)$ and $J_{\equiv}(J_{\equiv}(p)) = J_{\equiv}(p)$ since $J_{\equiv}(p)$ is a maximal element. From $p \equiv J_{\equiv}(p)$ and $q \equiv J_{\equiv}(q)$ one sees that $(p \vee q) \equiv J_{\equiv}(p) \vee J_{\equiv}(q)$. Therefore, again for (52), $J_{\equiv}(p) \vee J_{\equiv}(q) \leq J_{\equiv}(p \vee q)$. Suppose now $p \leq q$. Then $p \vee q = q$ so that $J_{\equiv}(p) \vee J_{\equiv}(q) \leq J_{\equiv}(p \vee q) = J_{\equiv}(q)$. Thus, $J_{\equiv}$ is monotone. Similarly, $J_{\equiv}(p) \wedge J_{\equiv}(q) \leq J_{\equiv}(p \wedge q)$. But from $p \wedge q \leq p$ and $p \wedge q \leq q$ monotonicity gives $J_{\equiv}(p \wedge q) \leq J_{\equiv}(p)$ and $J_{\equiv}(p \wedge q) \leq J_{\equiv}(q)$ so that $J_{\equiv}(p \wedge q) \leq J_{\equiv}(p) \wedge J_{\equiv}(q)$, from which multiplicativity follows. □

Actually, there is a one-one correspondence between Heyting algebra congruences $\equiv$ and Lawvere-Tierney operators which can be found in [15].

In particular we obtain:

Theorem 65 *Let* **H** *be a Heyting algebra and* J *a Lawvere-Tierney operator on* **H**. *Let us set for all* $a, b \in \mathbf{H}$,

$$a \equiv_J b \iff J(a) = J(b) \tag{53}$$

Then $\equiv_J$ *is congruence on* **H**.

Later in this chapter we provide a proof for a particular important case of the above theorem that we are going to introduce.

6.5 Dense Elements of a Heyting Algebra

If not otherwise stated, in this section **H** will denote a Heyting algebra and a, b, $a_1, b_1, x, y, \ldots$ will denote elements of **H**.

Definition 66 An element $x \in \mathbf{H}$ is said to be *dense* if $\neg x = 0$. It is called *regular* if $\neg\neg x = x$.

The following is immediate:

Theorem 67 *An element* δ *is dense iff* $\neg\neg\delta = 1$ *iff for all* $x \in \mathbf{H}$, $x \wedge \delta \neq 0$.

Theorem 68 *If* $\mathscr{D}$ *is the filter of all dense elements of* **H**, *then for all* $a, b \in \mathbf{H}$, $a \equiv_{\mathscr{D}} b$ *iff* $\neg a = \neg b$ *iff* $\neg\neg a = \neg\neg b$.

The relation $\equiv_{\mathscr{D}}$ is called *Glivenko congruence*. The proof is a standard result in Geometric Logic (see [23]). However, we give an algebraic proof in the case $\mathscr{D}$ is a principal filter generated by an element δ which, therefore, is the least dense element of **H**.[4]

Lemma 69 *Let* δ *be the least dense element of* **H**. *Then for all a,* $\delta \wedge a = \delta \wedge \neg\neg a$.

Proof $a \vee \neg a \leq \neg\neg a \vee \neg a$ and both terms are dense elements in **H**. Therefore, $\delta \leq a \vee \neg a \leq \neg\neg a \vee \neg a$. Hence, $\delta \wedge (a \vee \neg a) = \delta = \delta \wedge (\neg\neg a \vee \neg a)$, which means $(\delta \wedge a) \vee (\delta \wedge \neg a) = \delta = (\delta \wedge \neg\neg a) \vee (\delta \wedge \neg a)$. Since $\delta \wedge \neg a$ is disjoint from the other terms of the disjunctions, one obtains $\delta \wedge a = \delta \wedge \neg\neg a$. □

Corollary 70 *Let* δ *be the least dense element of* **H**, *then for all* a, b, $\delta \wedge a = \delta \wedge b$ *if and only if* $\neg a = \neg b$.

[4] If **H** is finite then it has the least dense element. An infinite Heyting algebra, instead, may lack this element.

Proof $\neg a = \neg b$ if and only if $\neg\neg a = \neg\neg b$. Therefore, from Lemma 69 one obtains: $\delta \wedge a = \delta \wedge b$ iff $\delta \wedge b = \delta \wedge \neg\neg a$ iff $\delta \wedge a = \delta \wedge \neg\neg b$ iff $\delta \wedge \neg\neg b = \delta \wedge \neg\neg a$ iff $\delta \wedge \neg b = \delta \wedge \neg a$. From these equations $\neg a = \neg b$. □

Corollary 71 *If δ is the least dense element of* $\mathbf{H}$*, then $a \equiv_{J^\delta} b$ if and only if $\neg\neg a = \neg\neg b$.*

Lemma 72 *Let $a \leq x \leq \neg\neg a$. Then $\neg a = \neg x$ and, thus, $\neg\neg a = \neg\neg x$.*

Proof From the hypothesis one has $\neg\neg\neg a \leq \neg x \leq \neg a$, that is, $\neg a \leq \neg x \leq \neg a$. Therefore, $\neg a = \neg x$. □

Lemma 73 *Let $a \leq b$ and for some y, $b \wedge y \leq a$. Then $a \wedge y = b \wedge y$.*

Proof From $a \leq b$ and $b \wedge y \leq a$ one obtains, respectively, $a \wedge y \leq b \wedge y$ and $b \wedge y \wedge y \leq a \wedge y$, that is, $b \wedge y \leq a \wedge y$. Therefore, $a \wedge y \leq b \wedge y \leq a \wedge y$, so that $a \wedge y = b \wedge y$. □

Lemma 74 *Let $a \leq x \leq \neg\neg a$. Then $x \Longrightarrow a$ is dense.*

Proof From Lemma 72 $\neg\neg a = \neg\neg x$. Moreover, from Theorem 37, $\neg(x \Longrightarrow a) = \neg\neg x \wedge \neg a = \neg\neg a \wedge \neg a = 0$. We conclude that $x \Longrightarrow a$ is dense. □

Lemma 75 *Let δ be dense. Then $\neg(\delta \Longrightarrow a) = \neg a$.*

Proof $\neg(\delta \Longrightarrow a) = \neg\neg\delta \wedge \neg a = 1 \wedge \neg a = \neg a$ □

Corollary 76 *Let δ be dense. Then $\delta \Longrightarrow a \leq \neg\neg a$.*

Proof $\delta \Longrightarrow a \leq \neg\neg(\delta \Longrightarrow a) = \neg(\neg(\delta \Longrightarrow a)) = \neg\neg a$. □

Theorem 77 *Let* $\mathbf{H}$ *be a Heyting algebra with least dense element.*

1. Let δ be a dense element. Then $\delta \Longrightarrow a = a$ or $\delta \Longrightarrow a = \neg\neg a$.
2. Let δ be the least dense element of $\mathbf{H}$*. Then $\delta \Longrightarrow a = \neg\neg a$.*
3. Let δ be the least dense element of $\mathbf{H}$*. Then $\delta \Longrightarrow a = \neg a \Longrightarrow \neg\delta$.*

Proof (1) We have two cases: C1: $\neg\neg a \wedge \delta \leq a$ and C2: $\neg\neg a \wedge \delta \gneq a$. Case C1. By adjointness, from $\neg\neg a \wedge \delta \leq a$ one obtains $\neg\neg a \leq \delta \Longrightarrow a$ and from $\delta \Longrightarrow a \leq \neg\neg a$, provided by Lemma 76, $\delta \Longrightarrow a = \neg\neg a$. This result encompasses also the case $a = 0$, because $\neg\neg 0 = 0$. Case C2. Let $x = \delta \Longrightarrow a$. Therefore, from adjunction $x \wedge \delta \leq a$. From Lemma 6.18, $x \leq \neg\neg a$. It cannot be $x = \neg\neg a$, because from assumption $\neg\neg a \wedge \delta \gneq a$, so that both $x \wedge \delta \leq a$ and $x \wedge \delta \gneq a$ ought to be true. Since $\delta \gneq a$, this contradiction occurs for every x such that $a \lneq x \leq \neg\neg a$. It follows that x must be a itself.
(2) If in the proof of (1) one uses the fact that $\delta \wedge a = \delta \wedge \neg\neg a$ because δ is the least dense element, one obtains the result. Alternatively, from Theorem 63 and Lemma 69, $\delta \Longrightarrow a = \delta \Longrightarrow \neg\neg a \geq \neg\neg a$. So, from Lemma 76 we obtain the proof. Another proof exploits adjunction: from Lemma 69 $\neg\neg a \wedge \delta = a \wedge \delta \leq a$ so that by adjunction $\neg\neg a \leq \delta \Longrightarrow a$. But from (1) $\delta \Longrightarrow a \leq \neg\neg a$. (3) is a corollary of (2) because $\neg a \Longrightarrow \neg\delta = \neg a \Longrightarrow 0 = \neg\neg a$. □

Therefore, the Lawvere-Tierney operator J^{δ}, with δ least dense element of a Heyting algebra **H** can be re-written as $\neg\neg$. Now we prove the specialisation to J^{δ} of Theorem 65.

Theorem 78 *The relation* $a \equiv_{\neg\neg} b$ *iff* $\neg\neg a = \neg\neg b$ *is a congruence on* **H**.

Proof $a \in [x]_{\neg\neg}$ and $b \in [y]_{\neg\neg}$ iff $\neg\neg a = \neg\neg x$ and $\neg\neg b = \neg\neg y$ which implies $\neg\neg a \wedge \neg\neg b = \neg\neg x \wedge \neg\neg y$. But in view of Theorem 37.(1), $\neg\neg$ preserves meets so that one obtains $\neg\neg(a \wedge b) = \neg\neg(x \wedge y)$ and can conclude that $a \wedge b \in [x \wedge y]_{\neg\neg}$. Regarding disjunction, $\neg\neg(a \vee b) = \neg\neg(x \vee y)$ iff $\neg(\neg a \wedge \neg b) = \neg(\neg x \wedge \neg y)$. But from the hypothesis of congruence, $\neg a = \neg x$ and $\neg b = \neg y$, therefore, the latter equation is true. A similar proof holds for $\Longrightarrow$: from the hypothesis $\neg\neg a \Longrightarrow \neg\neg b = \neg\neg x \Longrightarrow \neg\neg y$. Since $\neg\neg$ preserves $\Longrightarrow$ we obtain $\neg\neg(a \Longrightarrow b) = \neg\neg(x \Longrightarrow y)$ and we conclude that $a \Longrightarrow b \in [x \Longrightarrow y]_{\neg\neg}$. From this and the fact that $\neg\neg 0 = 0$ (or because $\neg\neg\neg a = \neg a$) one obtains the congruence for $\neg$. □

Clearly the same result can be obtained from Theorem 63 plus Lemma 62.

So far we have seen the properties of Lawvere-Tierney operators on an abstract Heyting algebra, and in particular the operator J^{δ}. Now we discuss the same properties in terms of Heyting algebras of an Alexandrov topology $\Omega_R(U)$. In this way we zoom-in the abstract structures, populate them with points and see how the above manipulations act on them.

In this kind of spaces there is a fundamental example of Grothendieck topology and conjugate Lawvere-Tierney operator, which will be key to our construction: the so-called *dense topology* and its corresponding *local operator*.

Definition 79 Let $\tau(U) = \langle U, \Omega(U)\rangle$ be a topological space on a set U and $X \in \Omega(U)$. Then:

$$X \textit{ is called } \text{dense} \text{ in } \tau(U) \textit{ if } \mathbb{C}(X) = U. \tag{54}$$

$$X \textit{ is called } \text{regular} \text{ in } \tau(U) \textit{ if } \mathbb{I}\mathbb{C}(X) = X. \tag{55}$$

Facts 6.1 *The abstract (pointless) notions of Definition 66 and the concrete (with points) ones coincide. Indeed, for any open set X of a topology $\Omega(U)$, $\mathbb{C}(X) = U$ iff $-\mathbb{C}(X) = \emptyset$ iff $\mathbb{I}(-X) = \emptyset$ iff $\neg X = 0$. Moreover, $\mathbb{I}\mathbb{C}(X) = \mathbb{I}\mathbb{C}(- - X) = \mathbb{I}(-\mathbb{I}(-X)) = \neg\neg X$.*

Definition 80 (Dense Topology) Given a finite topological space $\tau(U)$ on a set U the *dense topology* is obtained by tacking the least dense element δ and using the Lawvere-Tierney operator J^{δ} on the Heyting algebra $\Omega(U)$.

From the very definition (41) of $\Longrightarrow$ the following is straightforward:

Theorem 81 *Given a topological space $\tau(U)$, for any $X \in \Omega(U)$, $J^{\delta}(X) = \mathbb{I}(-\delta \cup X)$.*

Theorem 82 *For any $X \in \Omega(U)$, $J^{\delta}(X) = \mathbb{IC}(X)$.*

Proof From Theorem 77.(2). □

As before, we call $\mathbf{G}^{\delta}$ the Grothendieck topology induced by the operator J^{δ}. Therefore, given any open set $O \in \Omega_R(U)$, in the ordered site $\langle \mathbf{G}^{\delta}, \mathbf{P} \rangle$ a point p is covered by O iff $p \in \neg\neg O$ (i.e. $p \in \mathbb{IC}(O)$). Clearly, if $p \in O$ then $p \in \mathbb{IC}(O)$, but the interesting fact is when $p \notin O$.

The following result is well-known and it is the translation at the point-level of the above theorems:

Lemma 83 *Let $\langle \mathbf{G}^{\delta}, \mathbf{P} \rangle$ be the ordered site induced by the Lawvere-Tierney operator J^{δ}. Let α be any Intuitionistic formula. Let us set for any $p \in U$, $p \models \langle l \rangle(\alpha)$ iff $[\![\alpha]\!]$ covers p in the ordered site. Then $p \models \langle l \rangle(\alpha)$ iff $\forall p'(p \preceq p' \Rightarrow \exists p''(p' \preceq p'' \wedge p'' \models \alpha))$.*

So this is the origin of the Grothendieck topology of Example 54, which the reader can use an example of what we have just said.

In general, an upper adjoint is just multiplicative. When it is also additive we are in a particular situation that will be analysed later during the discussion about standard rough set systems. By now we have an operator J^{δ} which does not preserve disjunctions. However, the image of J^{δ}, actually of any Lawvere-Tierney operator, can be made into a Heyting algebra. This structure is very important to Rough Set Systems.

6.6 The Boolean Algebra of the Regular Elements of a Heyting Algebra

Given an operator φ on a lattice $\mathbf{L}$, let us denote with $\mathscr{F}_{\varphi}(\mathbf{L})$ the set of its fixed points: $\mathscr{F}_{\varphi}(\mathbf{L}) := \{x : x \in \mathbf{L} \wedge \varphi(x) = x\}$. If φ is idempotent, then $\mathscr{F}_{\varphi}(\mathbf{L})$, is just the image of φ.

So, we have seen that given an Heyting algebra $\mathbf{H}$ the image $\mathscr{F}_{J^{\delta}}(\mathbf{H})$ of the operator J^{δ} inherits from $\mathbf{H}$ the operations $\wedge$ and $\Longrightarrow$, but not $\vee$.

However on $\mathscr{F}_{J^{\delta}}(\mathbf{H})$ one can set a disjunction $\sqcup$ and obtain a Boolean algebra. We prove it in a general manner. The starting point is a classical result:

Lemma 84 *[Tarski] Let $\mathbf{L}$ be a complete lattice and φ a multiplicative and monotone operator on $\mathbf{L}$. Then the set of fixed points of φ, $\mathscr{F}_{\varphi}(\mathbf{L})$, is a complete lattice.*

Proof (See [15]) Let $a, b \in \mathscr{F}_\varphi(\mathbf{L})$. Let $\mathscr{F}_\varphi^{a,b} := \{x : \varphi(x) \leq x \;\&\; a, b \leq x\}$ and set $a \sqcup b := \bigwedge \mathscr{F}_\varphi^{a,b}$. Since $\mathbf{L}$ is complete, such inf exists in $\mathbf{L}$. We have to show that it belongs to $\mathscr{F}_\varphi(\mathbf{L})$. Since $a, b \leq a \sqcup b$ by monotonicity $\varphi(a), \varphi(b) \leq \varphi(a \sqcup b)$ which means $a, b \leq \varphi(a \sqcup b)$. Similarly it is proved that if $x \in \mathscr{F}_\varphi^{a,b}$, then $\varphi(x) \in \mathscr{F}_\varphi^{a,b}$, too. But by definition, if $x \in \mathscr{F}_\varphi^{a,b}$ then $a \sqcup b \leq x$. It follows that for all such x, $a \sqcup b \leq x$, hence $\varphi(a \sqcup b) \leq \varphi(x) \leq x$, so that by definition $\varphi(a \sqcup b) \leq a \sqcup b$ and one concludes that $a \sqcup b \in \mathscr{F}_\varphi^{a,b}$. Hence $\varphi(a \sqcup b) \in \mathscr{F}_\varphi^{a,b}$ and from this, $a \sqcup b \leq \varphi(a \sqcup b)$. Therefore, $\varphi(a \sqcup b) = a \sqcup b$, so $a \sqcup b \in \mathscr{F}_\varphi(\mathbf{L})$ and, moreover, for all $x, y \in \mathscr{F}_\varphi(\mathbf{L})$, $a, b \leq x$ implies $a \sqcup b \leq x$. □

Now we show that in the case of the operator J^δ, alias $\neg\neg$, the above operation $\sqcup$ is the double negation of the disjunction $\vee$ of $\mathbf{H}$:

Lemma 85 *Let the lattice* $\mathbf{L}$ *of Theorem 84 be a Heyting algebra* $\mathbf{H}$ *and* φ *be* $\neg\neg$. *Then for all* $a, b \in \mathscr{F}_{\neg\neg}(\mathbf{H})$, $a \sqcup b = \neg\neg(a \vee b)$.

Proof Since $p \leq \neg\neg p$ the requirement $\varphi(p) \leq p$ turns into $\neg\neg p = p$. Therefore, $\mathscr{F}_{\neg\neg}^{a,b} = \{x : \neg\neg(x) = x \;\&\; a, b \leq x\}$. Since $\neg\neg(a \vee b)$ is a fixed point, it belongs to $\mathscr{F}_{\neg\neg}$ and we have just to show that for all y such that $\neg\neg y = y$ and $a, b \leq y$, $\neg\neg(a \vee b) \leq y$. Now, if $a, b \leq y$, then $a \vee b \leq y$. By monotonicity, $\neg\neg(a \vee b) \leq \neg\neg y = y$. □

As a corollary of Theorem 84, it is easy to show that $\Uparrow \varphi = \{x : x \leq \varphi(x)\}$ and $\Downarrow_\varphi = \{x : \varphi(x) \leq x\}$ are complete lattices. In the case of operator $\neg\neg$, $x \leq \neg\neg x$, all $x \in \mathbf{H}$. Therefore $\Uparrow_\varphi = \mathbf{H}$ and $\Downarrow_\varphi = \mathscr{F}_{\neg\neg}(\mathbf{H})$, because $x \in \Downarrow_{\neg\neg}$ iff $\neg\neg x \leq x \leq \neg\neg x$.

From Theorem 84, it can be proved that if $\mathbf{H}$ is a Heyting algebra, then for any Lawvere-Tierney operator J on $\mathbf{H}$, $\mathscr{F}_J(\mathbf{H})$ is a Heyting algebra, too (remember that J is idempotent, so that for all x, $J(x)$ is a fixed point):

Theorem 86 (See [15]) *Given a Heyting algebra* $\mathbf{H}$, *for any Lawvere-Tierney operator* J *on* $\mathbf{H}$, *the set* $\mathscr{F}_J(\mathbf{H}) = \{J(x) : x \in \mathbf{H}\}$ *forms a Heyting algebra.*

Proof Since J is multiplicative, $\mathscr{F}_J(\mathbf{H})$ is closed under $\wedge$ because if $a = J(a)$ and $b = J(b)$, $a \wedge b = J(a) \wedge J(b) = J(a \wedge b)$. Define on $\mathscr{F}_J(\mathbf{H})$ a disjunction $\sqcup$ as above. We have to show the distributive property for elements of $\mathscr{F}_J(\mathbf{H})$: $p \wedge (a \sqcup b) = (p \wedge a) \sqcup (p \wedge b)$. Actually, since in one sense it works, we have to show that $p \wedge (a \sqcup b) \leq (p \wedge b) \sqcup (p \wedge b)$. Trivially, $p \wedge a \leq ((p \wedge a) \sqcup (p \wedge b))$ and $p \wedge b \leq ((p \wedge a) \sqcup (p \wedge b))$. Hence, by the adjunction relation, $a \leq p \Longrightarrow ((p \wedge a) \sqcup (p \wedge b))$ and $b \leq p \Longrightarrow ((p \wedge a) \sqcup (p \wedge b))$. Again by applying adjunction, $p \wedge (p \Longrightarrow ((p \wedge a) \sqcup (p \wedge b))) \leq (p \wedge a) \sqcup (p \wedge b)$. By monotonicity and multiplicativity of J, $J(p) \wedge J((p \Longrightarrow ((p \wedge a) \sqcup (p \wedge b)))) \leq J((p \wedge a) \sqcup (p \wedge b))$. One more time by adjunction $J((p \Longrightarrow ((p \wedge a) \sqcup (p \wedge b)))) \leq J(p) \Longrightarrow J((p \wedge a) \sqcup (p \wedge b)) = p \Longrightarrow ((p \wedge a) \sqcup (p \wedge b))$. The equation holds because p, a and b are fixed points of

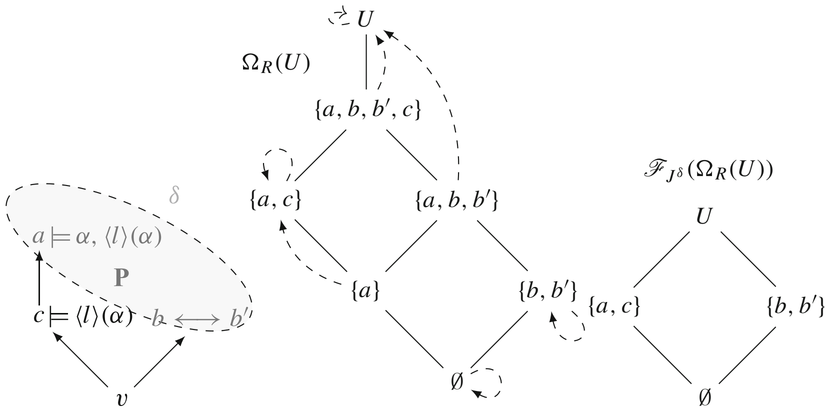
J which is multiplicative. Therefore, from the proof of Tarski's theorem one obtains $a \sqcup b \leq p \Longrightarrow ((p \wedge a) \sqcup (p \wedge b))$ and finally, again adjunction gives the thesis: $p \wedge (a \sqcup b) \leq (p \wedge b) \sqcup (p \wedge b)$. □

But $\mathscr{F}_{J^\delta}(\mathbf{H})$ is not only a Heyting algebra. In fact, one has $a \equiv_{J^\delta} b$ iff $\neg a = \neg b$ and, therefore, from Definition 35, $\mathscr{F}_{J^\delta}(\mathbf{H})$ is a Boolean algebra[5]:

Theorem 87 *$\mathscr{F}_{J^\delta}(\mathbf{H})$ with the operation $\wedge, \sqcup, \neg$ forms a Boolean algebra.*[6]

If X is an element of the Heyting algebra $\Omega_R(U)$ one has that the families $J^X_{[x]}$ are congruence classes of $\equiv_{J^X}$ and if $A \in J^X_{[x]}$ then $J^X(A) = \bigcup J^X_{[x]}$. In other terms, $J^X(A)$ is the top element of the $\equiv_{J^X}$ congruence class of A.

Example 88 Consider the preorder of Example 13 and the corresponding Heyting algebra (aka Alexandrov topology) of Example 40. The elements $\{a, b, b'\}$, $\{a, b, b', c\}$ and U are dense and $\{a, b, b'\}$ is the least dense element δ of the algebra.



[5]One obtains a Boolean algebra from a Heyting one by applying another Lawvere-Tierney operator, namely $B_x(p) = (p \Longrightarrow x) \Longrightarrow x$. The congruence relation is $a \equiv b$ iff $a \Longrightarrow x = b \Longrightarrow x$. If $x = 0$, $a \equiv b$ iff $\neg a = \neg b$. By definition a is a fixed point of B_x if $(a \Longrightarrow x) \Longrightarrow x \leq a$, so that if $x = 0$, a is a fixed point if $\neg\neg a \leq a$, hence if $\neg\neg a = a$.

[6]Moreover, by means of $\sqcup$ we have a proof that $\neg\neg$ preserves $\Longrightarrow$:

$$\neg\neg(a \Longrightarrow b) = \neg(\neg\neg a \wedge \neg b) = \neg(\neg\neg a \wedge \neg\neg\neg b)$$
$$= \neg\neg(\neg a \vee \neg\neg b) = \neg a \sqcup (\neg\neg b)$$
$$= \neg(\neg\neg a) \sqcup \neg\neg b = \neg\neg a \Longrightarrow \neg\neg b$$

The last equation is legal because it is calculated in the Boolean algebra $\mathscr{F}_{J^\delta}(\mathbf{H})$ of the regular elements of $\mathbf{H}$.

The arrows represent the action of the operator J^{δ}. The elements connected by arrows form a congruence class of $\equiv_{J^{\delta}}$.

$$\begin{aligned} J^{\delta}(\{a\}) \sqcup J^{\delta}(\{b, b'\}) &= J^{\delta}(J^{\delta}(\{a\}) \cup J^{\delta}(\{b, b'\})) \\ &= J^{\delta}(\{a, c\} \cup \{b, b'\}) = J^{\delta}(\{a, b, b', c\}) \\ &= U = J^{\delta}(\{a, b, b'\}) = J^{\delta}(\{a\} \cup \{b, b'\}). \end{aligned}$$

Let us see some instance of operations of the Heyting algebra $\Omega_R(U)$: $\{a, b, b'\} \Longrightarrow \{a\} = \{a, c\}$ and $\neg\{a\} = \{b, b'\}$, so that $\neg\neg\{a\} = \{a, c\}$. $\mathscr{F}_{J^{\delta}}(\Omega_R(U))$ is a Boolean algebra in which $\{a, c\} \sqcup \{b, b'\} = J^{\delta}(\{a, c\} \cup \{b, b'\}) = \neg\neg\{a, b, b', c\} = U$.

Notice, also, that $\{a, c\}$ and $\{b, b'\}$ are regular elements. However they are not complemented in $\Omega_R(U)$ because their union is $\{a, b, b', c\}$, not the top element U. On the contrary, U and $\emptyset$ are both regular and complemented. If we drop v from $\mathbf{P}$, in $\Omega_R(U)$ the aforementioned elements are complemented. Furthermore, in this case *all* the regular elements are complemented. Also this is a particular situation which will be discussed during the analysis of standard rough set systems.

We have seen that if $[\![\alpha]\!] = \{a\}$, then $c \models \langle l \rangle(\alpha)$, although $c \not\models \alpha$. Obviously, $a \models \langle l \rangle(\alpha)$. On the contrary, $v \not\models \langle l \rangle(\alpha)$ because $R(v) \cap \{a\} = \{a\} \notin J^{\delta}_{[v]}$. In fact, for instance, $b \in R(v)$ but $[\![\alpha]\!] \not\subseteq R(b)$. Notice that $v \models \langle l \rangle(\beta)$ iff $\delta \subseteq [\![\beta]\!]$.

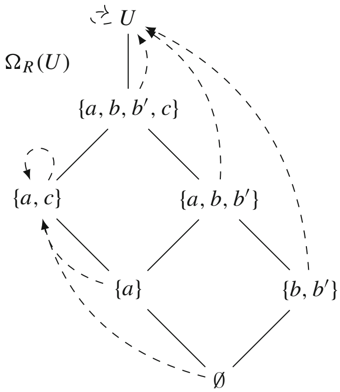
Finally, from the very definition of $J^{\delta}_{[c]}$ one can trivially verify that $[\![\alpha]\!]$ is locally valid at c in the intuitive sense discussed at the beginning of the section: $R(c) \cap [\![\alpha]\!] = \{a, c\} \cap \{a\} = \{a\}$ and $\{a\} \in J^{\delta}_{[c]}$. In Example 54 we have proved it by computing $J^{\delta}(\{a\})$.

Definition 89 Let $\mathbf{H}$ be a Heyting algebra and $\equiv$ a congruence on it. If $\mathbf{H}/_{\equiv}$ is a Boolean algebra, then $\equiv$ is called a *Boolean congruence*.

If $A \subseteq \delta$, then $\equiv_{J^A}$ is a Boolean congruence. However if $A \subsetneq \delta$, then a paradoxical situation is obtained. In fact, if $[\![\alpha]\!] = A$, then it is not dense so that there exists an x such that $x \models \neg\alpha$. Hence $x \in \neg[\![\alpha]\!] = [\![\alpha]\!] \Longrightarrow \emptyset = J^A(\emptyset)$. In consequence, $\emptyset \in J^A_{[x]}$. But since $x \models \neg\alpha$, then $R(x) \subseteq [\![\neg\alpha]\!]$ so that $R(x) \cap [\![\alpha]\!] = \emptyset$, which is a member of $J^A_{[x]}$, and one concludes that $x \models \langle l \rangle(\alpha)$. That is, $\neg\alpha$ is valid at x and nonetheless α is locally valid at x itself, as well.

Example 90 Consider the operator $J^{\{b, b'\}}$.

X	$J^{\{b,b'\}}(X)$
$\emptyset$	$\{a,c\}$
$\{a\}$	$\{a,c\}$
$\{a,c\}$	$\{a,c\}$
$\{b,b'\}$	U
$\{a,b,b'\}$	U
$\{a,b,b',c\}$	U
U	U



$\mathbf{G}^{\{b,b'\}}$

x	$J_{[x]}^{\{b,b'\}}$
a	$\{\emptyset,\{a\}\}$
b	$\{\{b,b'\}\}$
b'	$\{\{b,b'\}\}$
c	$\{\emptyset,\{a\},\{a,c\}\}$
v	$\{\{b,b'\},\{a,b,b'\},\{a,b,b',c\}\}$

Therefore, if $[\![\alpha]\!] = \{b, b'\}$ then $a \models \langle l \rangle(\alpha)$ and $c \models \langle l \rangle(\alpha)$ because $R(a) \cap [\![\alpha]\!] = \emptyset$ and $\emptyset \in J_{[a]}^{\{b,b'\}}$, and the same holds for c. Thus, we have the paradoxical situation that $a \models \neg\alpha$, $c \models \neg\alpha$ and both a and c force $\langle l \rangle(\alpha)$.

In view of the above discussion, if a is a non-dense element of a Heyting algebra, that is, $\neg a \neq 0$, then we call J^a and its related Grothendieck topology *paradoxical Lawvere-Tierney operator* and, respectively, *paradoxical Grothendieck topology*.

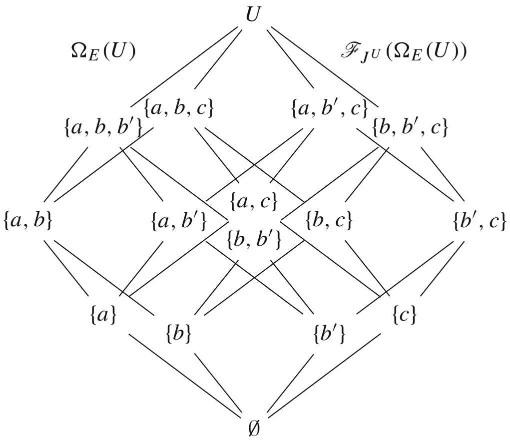
6.7 Grothendieck Topologies and Rough Set Systems

Now we see how all the above machinery applies to Rough Set Systems.

Let us start with standard rough sets. Therefore, we are given a set U and an equivalence relation $E \subseteq U \times U$. We know that we have to use the set of all singletons S as the parameter for the Lawvere-Tierney operator J^X. What does it happen to the conjugate Grothendieck topology $\mathbf{G}^S$?

The approximation space $\Omega_E(U)$ is a Heyting algebra which, in particular, is a Boolean algebra. In this algebra the only dense element is the top element U. If the set of singletons in $\Omega_R(U)$ coincides with U, then for any $O \in \Omega_R(U))$, $J^U(O) = O$, so that $\mathbf{G}^U = \{\{O\} : O \in \Omega_E(U)\}$. Since U is the least (actually only) dense element of the algebra, we know that $\mathscr{F}_{J^U}(\Omega_E(U))$ is a Boolean algebra, but in this case the disjunction is $\cup$ itself. In fact, $\mathscr{F}_{J^S}(\Omega_E(U))$ equals $\Omega_E(U)$ for the very reason that for any O, $J^U(O) = O$ and from the previous results we know that $J^U(O) = \neg\neg O$ because U is the least dense element of $\Omega_E(U)$. In turn, $\Omega_E(U)$ equals the Boolean algebra $\mathbf{B}(U) = \langle \wp(U), \cap, \cup, -, \emptyset, U \rangle$ because for any $x \in U$, $E(x) = \{x\}$. This is consistent with the fact that in the Boolean algebra $\mathbf{B}(U)$, $\neg\neg O = O$, any O because in this algebra the negation $\neg$ is the set-theoretic complement.

Example 91 Let $U = \{a, b, b', c\}$ and $E = \{\langle x, x \rangle : x \in U\}$. Then:



Since for all $X \in \Omega_R(U)$ (i.e. $X \in \wp(U)$) $J^U(X) = X$, one has for instance: $J^U_{[b]} = \{\{b\} \cap \{b\}, \{b\} \cap \{a, b\} \ldots \{b\} \cap \{a, b, b'\}, \ldots\} = \{\{b\}\}$. Therefore, $\mathbf{G}^U = \{\{\{a\}\}, \{\{b\}\}, \{\{b'\}\}, \{\{c\}\}\}$ and $\mathscr{F}_{J^U}(\Omega_E(U)) = \Omega_E(U)$. Thus, for any α interpreted on $\mathbf{E} = \langle U, E \rangle$, for any $X \in U$, one has $x \models \langle l \rangle(\alpha)$ iff $x \models \alpha$. That is, local and global validity coincide.

Things drastically change if the set S of singletons in $\Omega_R(U)$ is strictly less than U. Usually, in the rough set community it is assumed that there are no singleton classes in U/E. If in many real-work applications this assumption may be acceptable, in a broader framework it is questionable because, actually, also the real word is not made of just incomplete information but of a melange of complete and incomplete information. In the complete part Classical Logic is locally valid while it can be assumed that the global logic is three-valued.

Now we frame this issue in the case of Rough Set Systems induced by preorders because it has a particular logic importance, it is a broader point of view and because the issue of singleton granules cannot be avoided if the items we are dealing with are connected by means of a preorder or, even worst, a partial order $\mathbf{P}$. In fact, if in $\mathbf{P}$ there is at least a non infinite chain, the greatest point x of this chain is an isolated point because $R(x) = \{x\}$.

In preorders we can have both maximal and pre-maximal points, that is, points x such that if $x \leq y$ then $y \leq x$. If x is a pre-maximal point, then the cardinality of $R(x)$ may be greater than 1. For instance, in our Example b and b' are pre-maximal and $R(b) = R(b') = \{b, b'\}$ and both b and b' can be in the boundary of some set. For instance, $b, b' \in \mathbb{B}(\{a, b\})$.

From the point of view of an information order, however, the larger an information the smaller the set of items it filters. This is the so-called "Loi de Port Royal" (Low of Port Royal: "L'*extension*, ou *étendue* varie in proportion inverse de l'*intension comprehénsion*"). Thus, extension and intention are contravariant. If we intend "strictly contravariant", that is, if we are interested not in items but in information increases, items which are not separable by means of granules play exactly the same role with respect of our knowledge. If this is the case, one would instead use topological spaces where points are not redundant with respect to the available properties, a.k.a. open sets. These spaces, as we have already mentioned, are called *sober*, or, which is the same in the finite case, T_0 topological spaces:

Definition 92 (T_0-Spaces) A topological space $\tau(U)$ is said to be T_0 if for each two points $x, y \in U$, if $x \neq y$ there exists an open set O such that $x \in O$ and $y \notin O$.

Therefore, a space is T_0 if any two different points are separable by means of an open set. In our interpretation: any two different items are separable by means of some property. That is, there is at least a property which is enjoyed by one item but not by the other.

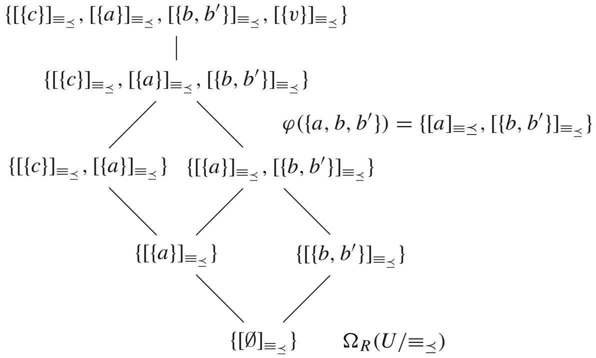
T_0 spaces are obtained from any space $\tau(U)$ with specialisation preorder $\preceq$, by taking the quotient set $U/\equiv_{\preceq}$, where $x \equiv_{\preceq} y$ if and only if $x \preceq y$ and $y \preceq x$, and setting for all $O \in \Omega(U)$, $\varphi(O) = \{[x]_{\equiv_{\preceq}} : x \in O\}$. The topology $\Omega(U/\equiv_{\preceq}) = \{\varphi(O) : O \in \Omega(U)\}$ is called the T_0-*ification* of $\Omega(U)$ and φ is an isomorphism between it and $\Omega(U)$. Notice that φ is the extension to $\wp(U)$ of the natural map $\mathfrak{q}$ of Lemma 58: $\varphi(O) = \{\mathfrak{q}(x) : x \in O\}$.

If $\mathbf{P} = \langle U, \preceq\rangle$ is a preorder (in particular the specialisation preorder of a topological space $\tau(U)$), then on $U/\equiv_{\preceq}$ we can define the relation $[x]_{\equiv_{\preceq}} \sqsubseteq [y]_{\equiv_{\preceq}} \iff x \preceq y$. It is immediate to verify that $\sqsubseteq$ is a partial order. Indeed, reflexivity and transitivity are inherited from $\preceq$ and if $[x]_{\equiv_{\preceq}} \sqsubseteq [y]_{\equiv_{\preceq}}$ and $[y]_{\equiv_{\preceq}} \sqsubseteq [x]_{\equiv_{\preceq}}$ then $x \preceq y$ and $y \preceq x$ so that $x \equiv_{\preceq} y$ and, in conclusion, $[x]_{\equiv_{\preceq}} = [y]_{\equiv_{\preceq}}$.

Let $\tau(U)$ be an Alexandrov space with specialisation preorder $\preceq$. Let $\mathbf{P}/\equiv_{\preceq} := \langle U/\equiv_{\preceq}, \sqsubseteq\rangle$. Then $\Omega_{\sqsubseteq}(\mathbf{P}/\equiv_{\preceq}) = \Omega(U/\equiv_{\preceq})$ and it is isomorphic to $\Omega(U)$. The isomorphism from $\Omega(U)$ to $\Omega(U/\equiv_{\preceq})$ is φ. However, the two isomorphic topologies are not homeomorphic, because if $x \neq y$ but $x \equiv y$, then $\mathfrak{q}(x) = \mathfrak{q}(y)$, so that an open set $\varphi(O)$ of $\Omega(U/\equiv_{\preceq})$ can have less points than O.

As it is clear, one can chose a representative of $[x]$ and obtain a partial order $\mathbf{P}'$ on the new set $U' \subseteq U$. It is not difficult to verify through $\mathfrak{q}$ that $\Omega(\mathbf{P}')$ and $\Omega_{\sqsubseteq}(\mathbf{P}/\equiv_{\preceq})$ are homeomorphic.

Example 93 Consider again our standard point-level example on $U = \{v, a, b, b', c\}$. The specialisation preorder $\preceq$ is R (in fact $\Omega_R(U)$ is an Alexandrov topology). Therefore, the T_0-ification of $\Omega(U)$ is:



It is clear that duality produces a T_0-ification. Anyway, if we are interested in the granules, then T_0-ification is not an appropriate move. In what follows we show what happens algebraically in case the space is not T_0-ficated and when it is. We shall see that the algebraic structure of Rough Set Systems induced by the first case will be sensibly transformed, as well as the logical properties of the systems (that is, the logic they model).

Moreover, we also show the algebraic and logic difference between the case **H** is a generic Heyting algebra and the case **H** is a Boolean algebra. The latter case in the classical one.

First, we present the topic from the point-level perspective of Rough Set Theory. Then we develop it at the abstract level. After that, we zoom-in again to achieve the intermediate, or hybrid, level of algebras of concrete open sets.

7 Algebras of Rough Set Systems

Given a granulation $\Omega_R(U)$ induced by a preorder $\mathbf{P} = \langle U, R\rangle$, the issue concerning singletons is how to filter the set $Dsj(U/R)$ of all the ordered pairs of disjoint elements of $\Omega_R(U)$ in order to have just elements actually representing the pairs $\langle (lR)(X), -(uR)(X)\rangle$ for some $X \subseteq U$.

As it is already clear, the solution is obtained by applying the Lawvere-Tierney operator J^S, where S is the union of all isolated points of $\Omega_R(U)$ intended as a topology. That is, $S = \{x : R(x) = \{x\}\}$.

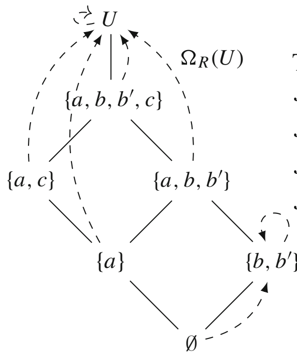
7.1 Defining the Set of Rough Sets

Remember that $\Omega_R(U)$ is a Heyting algebra and S is an element of this algebra. So for any element O of the algebra (any open set) we can define the operator $J^S(O) = S \Longrightarrow O$.

What is the mechanism that makes J^S act as a filter which is able to discern true from apparent rough sets? The answer, as we are going to see, is that J^S forces any element of S to be in the first or in the second element of a rough set when it is represented by an ordered pair of disjoint elements $\langle (lR)(X), -(uR)(X)\rangle$. But what is the mathematical and logical significance of J^S?

Let us consider again our example:

Example 94 Consider the preorder **P** of Example 34 and the Heyting algebra or topology or approximation space $\Omega_R(U)$. The set of all singletons is $S = \{a\}$. The action of the operator J^S on $\Omega_R(U)$ is the following:



The Grothendieck topology $\mathbf{G}^S$

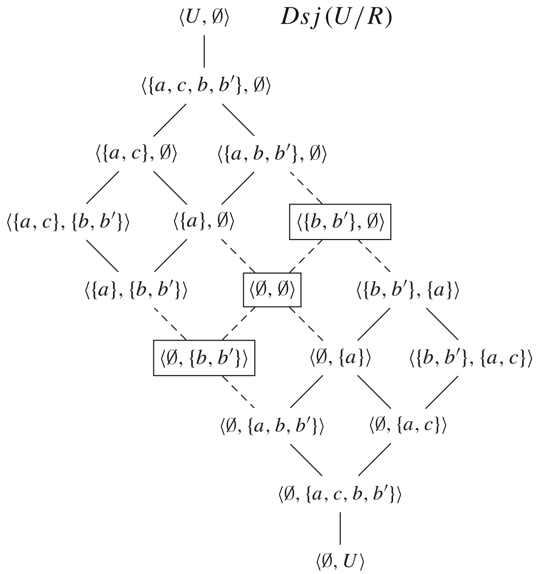
$J^S_{[a]} = \{\{a\}\}$

$J^S_{[b]} = J^S_{[b']} = \{\emptyset, \{b, b'\}\}$

$J^S_{[c]} = \{\{a\}, \{a, c\}\}$

$J^S_{[v]} = \{\{a\}, \{a, c\}, \{a, b, b'\}, \{a, c, b, b'\}, U\}$

This topology gives us some information. For instance, in no ordered pair the open set $\{b, b'\}$ can have $\emptyset$ as partner even if $\neg S = \{b, b'\}$. In fact the presence of $\emptyset$ in $J^S_{[b]}$ and $J^S_{[b']}$ tells us that a partner must be found between S and $\neg\neg S$, in our case $\{a\}$ and $\{a, c\}$. Actually, the only elements which can have $\emptyset$ as a partner in a disjoint pair, are the members of $\uparrow S$. Therefore, the following filtration is applied by J^S on $Dsj(U/R)$:



The framed elements are discharged by the filtration J^S. Indeed, $\emptyset \cup \{b, b'\} \not\supseteq S$. Therefore, the ordered pairs $\langle \emptyset, \{b, b'\} \rangle$ and $\langle \{b, b'\}, \emptyset \rangle$ are filtered out. *A fortiori* $\langle \emptyset, \emptyset \rangle$ is discharged. The rough set system $Dsj(\Omega_R(U))$ is then the lattice without the framed elements depicted in Example 34.

Remarks 7.1 The Grothendieck topology above is paradoxical, like the one of Example 90. In a sense, the very filtration rules out the paradoxical situations.

Now we transform $Dsj(U/R)$ into $Dsj(\Omega_R(U))$ step by step, explaining the inner mathematical and logical mechanisms of the transformation. At the end of the process we will find that $Dsj(\Omega_R(U))$ is a particular structure called *Nelson algebra* that we define at the pointless level:

Definition 95 (Nelson Algebras) A lattice $\mathbf{N} = \langle A, \wedge, \vee, \sim, \longrightarrow, 0, 1 \rangle$ is a *Nelson algebra* if:

1. $\langle A, \wedge, \vee, \sim, 0, 1 \rangle$ is a *de Morgan lattice*, that is, a distributive lattice such that for all a, b, $\sim\sim a = a$ and $\sim (a \vee b) = \sim a \wedge \sim b$. Therefore, $\sim (a \wedge b) = \sim a \vee \sim b$ and $a \leq b$ iff $\sim b \leq \sim a$.
2. $a \wedge \sim a \leq b \vee \sim b$, so that it is also a *Kleene algebra*.
3. The operation $\longrightarrow$ fulfils the following adjunction property:

$$a \wedge c \leq \sim a \vee b \iff c \leq a \longrightarrow b \tag{56}$$

Nelson algebras are a bit tricky to one who is just familiar with Classical or Intuitionistic logic, that is, Boolean or Heyting algebras. Indeed we have that $\sim\sim a = a$ but nonetheless, $\sim a \vee a \leq 1$. This suggests that $\sim$ is not a pseudo-complementation, otherwise the lattice would be a Boolean algebra. In turn, $\longrightarrow$ is not a relative pseudo-complementation. Indeed the adjunction property fulfilled by $\longrightarrow$ is (56) and we have that $\sim a \leq a \longrightarrow 0$. So we can define two other negations and an additional implication. This operations will play a key role in the connection of Nelson algebras and rough set systems.

$$\lrcorner a := a \longrightarrow 0 \quad (57)$$

$$a \supset b := \sim \lrcorner \sim a \vee b \vee (\lrcorner a \wedge \lrcorner \sim b) \quad (58)$$

$$\neg a := a \supset 0 = \sim \lrcorner \sim a \quad (59)$$

The negation $\sim$ is called *strong negation* because one has $\sim a \leq \lrcorner a$, and $\lrcorner$ is intended to be an intuitionistic negation (it is the implication of the 0-element). We shall see, that $\lrcorner$ is far from replicating a pseudo-complementation, because $a \wedge \lrcorner a$ is not 0 and $\lrcorner\lrcorner$ is not able to grasp classical tautologies. In a particular case, that we shall discuss below, $\lrcorner$ is a dual pseudo-complementation, this is the reason of the symbol, while $\neg$ turns into a real pseudo-complementation. Anyway, all finite Nelson lattices are, also, Heyting algebras (but there are infinite Nelson lattices which are not Heyting algebras). What we will show is that if a Nelson algebra is semi-simple, then the relative pseudo-complementation is definable by means of the very operations of the Nelson algebra itself.

Definition 96 A Nelson algebra is called *semi-simple* if $a \vee \lrcorner a = 1$, any a.

The link between this abstract structure and the concrete structure $Dsj(\Omega_R(U))$ is given by the duality theory of Nelson algebras. Therefore the first step is the construction of the dual space of a Nelson algebra from an abstract point of view.

So, let $\mathbf{N}$ be a Nelson algebra. We recall that we assume that $\mathbf{N}$ is finite and that our meta-theory is Classical Logic. Define on $J(\mathbf{N})$ the following endomorphism:

$$f(x) = min_{\leq_N}(J(\mathbf{N}) \cap -\{\sim b : b \in \uparrow_{\leq_N} x\}) \quad (60)$$

where $min_{\leq_N}$ and $\uparrow_{\leq_N}$ refer to the lattice order $\leq_N$ of $\mathbf{N}$, which is a partial order. The restriction to $J(\mathbf{N})$ of this order will be denoted by $\leq$.

It can be proved that f is a linear involutive anti-order isomorphism in $\mathbf{J}(\mathbf{N}) = \langle J(\mathbf{N}), \leq \rangle$, that is, $x \leq y$ implies $f(y) \leq f(x)$, $x \leq f(x)$ or $f(x) \leq x$ and $f(f(x)) = x$. Moreover, the following interpolation property holds:

$$\text{if } a \geq f(a), b \geq f(a), a \geq f(b), b \geq f(b)$$

$$\text{then } \exists c \in J(\mathbf{N}) \text{ such that } c \leq a, c \leq b, f(a) \leq c, f(b) \leq c.$$

That is, there is an intermediate element which prevents f and the order of $J(\mathbf{N})$ from crossing. If $\mathbf{N}$ were a Kleene algebra, then the interpolation property could fail. The space $\mathscr{N}(\mathbf{J}(\mathbf{N})) := \langle J(\mathbf{N}), \leq, f \rangle$ is called a *Nelson space*.

Actually, a Nelson space is any preorder $\mathscr{N}(U) = \langle U, \leq, f \rangle$ where f is an endomorphism with the properties above. A Nelson algebra is restored from a Nelson space $\mathscr{N}(U)$ by defining the following operations on $\Omega_{\leq}(U)$:

- $1 := U, 0 := \emptyset$
- $A \vee B := A \cup B, A \wedge B := A \cap B$
- $A \longrightarrow B = -\mathbb{C}_{\leq}(A \cap f(A) \cap -B)$
- $\sim A := U \cap -f(A)$

One has that $\mathbf{N}(\mathscr{N}(U)) := \langle \Omega_{\leq}(U), \vee, \wedge, \longrightarrow, \sim, 0, 1\rangle$ is a Nelson algebra. In particular $\mathbf{N}(\mathscr{N}(\mathbf{J}(\mathbf{N})))$ is isomorphic to $\mathbf{N}$.

In the process, we see also that $\lrcorner A := -\mathbb{C}_{\leq}(A \cap f(A))$—i.e. $A \longrightarrow 0$.

All this is very useful, but if we start from an approximation space $\Omega_R(U)$, how to define the involution f? What is the relation between f and the granulation provided by $\Omega_R(U)$?

In order to answer, let us observe, in the first place, that a Nelson space can be split into two parts linked by the involution f.

Notation In order to avoid confusions later, the universe of a Nelson space will be denoted by U^*.

Definition 97 Given a Nelson space $\mathscr{N}(U^*) = \langle U^*, \leq, f\rangle$, define

$$U^+ := \{x \in U^* : x \leq f(x)\}, \text{ with an order } \leq^+ \text{ inherited from } \mathscr{N}(U^*)$$

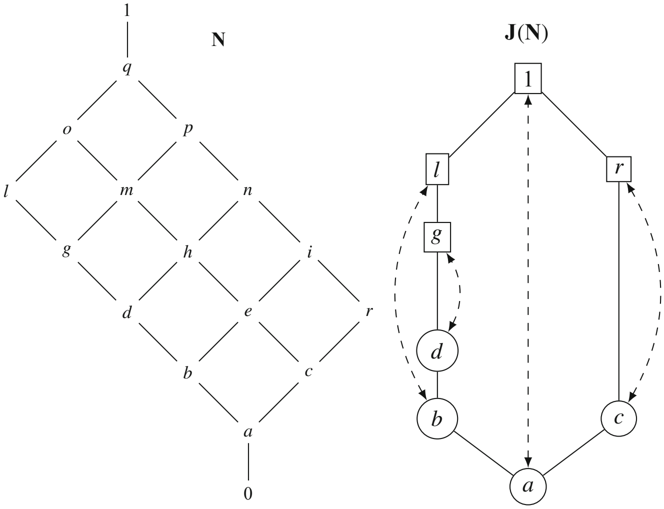
$$U^- := \{x \in U^* : f(x) \leq x\}, \text{ with an order } \leq^- \text{ inherited from } \mathscr{N}(U^*)$$

Let $k_f : \Omega_{\leq}(U^*) \longmapsto \Omega_{\leq^+}(U^+) \times \Omega_{\leq^+}(U^+) := \langle U^+ \cap X, U^+ \cap -f(X)\rangle$.
Define the following operations where inside the ordered pairs the operations are those of the Heyting algebra $\Omega_{\leq^+}(U^+)$ (i.e. $\wedge = \cap$, $\vee = \cup$ and so on):

- $1 := \langle U^+, \emptyset\rangle$, $0 = \langle \emptyset, U^+\rangle$
- $\langle X_1, X_2\rangle \vee \langle Y_1, Y_2\rangle := \langle X_1 \vee Y_1, X_2 \wedge Y_2\rangle$
- $\langle X_1, X_2\rangle \wedge \langle Y_1, X_2\rangle := \langle X_1 \wedge Y_1, X_2 \vee Y_2\rangle$
- $\sim \langle X_1, X_2\rangle := \langle X_2, X_1\rangle$
- $\langle X_1, X_2\rangle \longrightarrow \langle Y_1, Y_2\rangle := \langle X_1 \Longrightarrow Y_1, X_1 \wedge Y_2\rangle$
- $\lrcorner\langle X_1, X_2\rangle := \langle \neg X_1, X_1\rangle$
- $\neg\langle X_1, X_2\rangle := \langle X_2, \neg X_2\rangle$

It is possible to show that $\mathbf{N}_{k_f}(\mathscr{N}(U^*)) := \langle k_f(\Omega_{\leq}(U^*)), \wedge, \vee, \sim, \longrightarrow, \lrcorner, 0, 1\rangle$ is a Nelson algebra. Just notice, that any $X \in \Omega_{\leq}(U^*)$ is an up-set (i.e. a $\leq$ filter). Therefore, if $x \in X \cap U^+$ then for each $y \in U^+$, if $x \leq y$ then $y \in X$ and, thus, $y \in X \cap U^+$. That is, $X \cap U^+$ is an up-set in U^+ (a $\leq^+$ filter), hence belongs to $\Omega_{\leq^+}(U^+)$. Similarly, since X is an up-set in U^*, $f(X)$ is a down-set in U^* (a $\leq$ ideal) because f is order reversing. In consequence, $-f(X)$ is an up-set in U^* and, again, $-f(X) \cap U^+$ is an $\leq^+$ filter.

Example 98 Although we shall prove only later that $Dsj(U/R)$ is a Nelson algebra, in order to follow the construction of the dual space let us assume it is a Nelson algebra and consider a pointless version of it.



	0	a	b	c	d	e	r	g	h	i	l	m	n	o	p	q	1
$\sim$	1	q	p	o	n	m	l	i	h	g	r	e	d	c	b	a	0

Notice that the strong negation of x is symmetric to x. $J(\mathbf{N}) = \{a, b, c, d, g, l, r, 1\}$. Let us calculate the involution f:
$f(a) = min_{\leq_N} - \{\sim x : x \in \uparrow_{\leq_N} a\} = min_{\leq_N} - \downarrow_{\leq_N} (\sim a) = min_{\leq_N} - \downarrow_{\leq_N} q = min_{\leq_N}\{1\} = 1$; $f(b) = min_{\leq_N} - \downarrow_{\leq_N} (\sim b) = min_{\leq_N} - \downarrow_{\leq_N} p = min_{\leq_N}\{l, o, q, 1\} = l$. The reader now has the mechanism. Then:
$f(d) = min_{\leq_N} - \downarrow_{\leq_N} (\sim d) = min_{\leq_N} - \downarrow_{\leq_N} n = min_{\leq_N} \uparrow_{\leq_N} g = g$. $f(g) = min_{\leq_N} - \downarrow_{\leq_N} i = min_{\leq_N} \uparrow_{\leq_N} d = d$; $f(l) = min_{\leq_N} \uparrow_{\leq_N} b = b$. $f(c) = r$ and $f(r) = c$.
The circled elements form the subset U^+ while the boxed ones form U^-.
It is easy to verify, for instance, that given the up-set $\{c, r, 1, l\}$, $f(\{c, r, 1, l\}) = \{c, r, a, b\}$ which is a down-set. Therefore, $-f(\{c, r, 1, l\}) = -\{c, r, a, b\} = \{d, g, l, 1\}$ is an up-set.

Now we know what our goal is. So the first step is to transform our approximation space (Heyting algebra, Alexandrov topology) $\Omega_R(U)$ into the space previously denoted as $\Omega_{\leq^+}(U^+)$. Then we have to define from it the duplicate space $\Omega_{\leq^-}(U^-)$, an order $\leq$ glueing the two spaces into a preorder on $U^* = U^+ \cup U^-$ and, finally, an involution f on U^* fulfilling the properties discussed above, with respect to $\leq$. The resulting space $\mathcal{N}(U^*)$ is a Nelson space. After that, we are eventually in position to transform $\Omega_R(U)$ into the Nelson algebra $\mathbf{N}_{k_f}(\mathcal{N}(U^*))$ and prove that its domain (or carrier) is $Dsj(\Omega_R(U))$. Since from now to the end of the section the universe U of our approximation space $\Omega_R(U)$ plays the role of U^+, let us use the last symbol (with our usual R which turns into $\leq^+$). Thus, now the approximation space $\Omega_R(U)$ is called $\Omega_{\leq^+}(U^+)$.

A particular attention is due to the involution f, because it is strictly connected with the singleton issue. Indeed, look at the role of f in the definition of $\mathbf{N}_{k_f}(\Omega_{\leq}(U^*))$. It actually decides which elements not belonging to the open set $X_1 = U^+ \cap X$ are allowed to belong to X_2. In fact, suppose that $X \in \Omega_{\leq}(U^*)$ and $U^+ \cap X = \mathbb{I}_{\leq^+}(A)$ for some subset A of U^+ (aka U). We know that $\mathbb{I}_{\leq^+}(A)$ is an up-set in U^+ and $\mathbb{C}_{\leq^+}(A)$ a down set in U^+ therefore a down-set in U^*. Vice-versa, if X is a down-set in U^*, it is a down set also in U^+. But we have seen that actually $f(X)$ is a down-set in U^*. Indeed, $f(X) \cap U^+$ "is" the closure $\mathbb{C}_{\leq^+}(A)$. In consequence, $-f(X) \cap U^+$ is the complement of the closure of A, that is, $-\mathbb{C}_{\leq^+}(A)$ (to see that, we need only to prove $-(f(X) \cap U^+) = -f(X) \cap U^+$).

Now, how can f rule out from $-f(X)$ the elements c of A which are isolated in $\Omega_{\leq^+}(U^+)$ and are not already in X? Or, which is the same, how can it put c in $f(X)$? How does f work?

We need a formal definition of a notion we have already seen:

Definition 99 Let $\mathbf{P} = \langle P, \leq \rangle$ be a preorder. An element $x \in P$ is said to be *pre-maximal* if $\forall y \in P(x \leq y \Rightarrow y \leq x)$. It is called *maximal* if $\forall y \in P(x \leq y \Rightarrow y = x)$.

Clearly, if $\leq$ is a partial order then the two notions coincide. Notice that in the case $\mathbf{P}$ is the dual space $\mathbf{J}(\mathbf{A})$ of a Nelson algebra or a Heyting algebra $\mathbf{A}$, then the order of $\mathbf{J}(\mathbf{A})$, $x \leq y \iff \uparrow(y) \subseteq \uparrow(x)$, is a partial order because it is induced by the subset relation $\subseteq$ which is an extensional relation, so that if $X \subseteq Y$ and $Y \subseteq X$ then $X = Y$.

Now, c is isolated in $\langle U^+, \leq^+ \rangle$ if $\uparrow_{\leq^+} c = \{c\}$. And this happens if and only if c is a maximal element in $\leq^+$.

We have to pay attention to a fact that maybe evaded the reader: U^+ and U^- must have the same cardinality (otherwise f is not an anti-isomorphism), however it is not required they are disjoint. In Example 98 they are disjoint, but their definitions do not imply disjunction. So, let us investigate their, possibly not empty, intersection.

Theorem 100 *Let $\mathcal{N}(U^*) = \langle U^*, \leq, f \rangle$ be a Nelson space. Let $B = U^+ \cap U^-$. Then:*

1. *$c \in B$ if and only if $f(c) \in B$.*
2. *If $\leq$ is a partial order, then $c \in B$ if and only if $f(c) = c$.*
3. *If $c \in B$ then c is maximal or pre-maximal in $\langle U^+, \leq^+ \rangle$ (maximal if $\leq^+$ is a partial order).*

Proof (1) If $c \in B$ then $c \in U^+$ and $c \in U^-$, so that $f(c) \leq c \leq f(c)$. From this it is immediate to see that $f(c) \in B$, too: $c \leq f(c)$ gives $f(f(c)) \leq f(c)$ and $f(c) \leq c$ gives $f(c) \leq f(f(c))$. The reverse is obvious. (2) comes trivially from (1). (3) Suppose now that for $x \in U^+$, $c \leq x$, so that $f(x) \leq f(c)$. Since $x \in U^+$, $x \leq f(x)$ and we immediately obtain the following relation: $x \leq f(x) \leq f(c) \leq c \leq x \leq f(x)$. Therefore, $f(c) \leq f(x)$, so that $x \leq c$. Hence c is pre-maximal or maximal (therefore, maximal if $\leq$ is a partial order). □

Remarks 7.2 The reverse implication of Theorem 100.(3) does not hold: if c is pre-maximal (maximal), then not necessarily $c \in U^+ \cap U^-$. This is important, because it means that it depends on the application to decide if a maximal element c is to be set to $f(c) = c$, which, indeed, is the case of rough set systems, as we are going to see.

We have enough material to proceed with our construction. Given an approximation space $\Omega_{\leq^+}(U^+)$ take a copy $\Omega_{\leq^-}(U^-)$, where $\leq^-$ is the reversed order of $\leq^+$ (i.e. $\leq^{+\smile} =\leq^-$). If $\leq^+$ is a preorder (partial order), then $\leq^-$ is a preorder (partial order), too. The elements of U^- will be decorated by an apex "−", while the corresponding elements in U^+ will be decorated by "+". As we have just seen, it is possible for some element to have both decorations (if it belongs to $U^+ \cap U^-$).

Define now a relation $\varphi \subseteq U^+ \times U^-$ as follows: $\varphi(x^+) = \{x^-\}$. Therefore, $\varphi^\smile(x^-) = \{x^+\}$ and for x^{+-}, $\varphi(x^{+-}) = \varphi^\smile(x^{+-}) = \{x^{+-}\}$. Clearly, φ is an order anti-isomorphism between U^+ and U^-. The new relation φ enables the definition of the final order $\leq$ on $U^* = U^+ \cup U^-$. The intermediate step is the definition of an order connecting U^+ and U^-. It is defined passing through φ in the composition $\leq^+ \otimes\varphi\otimes \leq^-$. It is then possible to prove that $\leq:=\leq^+ \cup \leq^- \cup(\leq^+ \otimes\varphi\otimes \leq^-)$ is the required order on $U^* = U^+ \cup U^-$: it preserves both $\leq^+$ and $\leq^-$ and glues them together in a minimal way. Finally, we define the involution f on U:

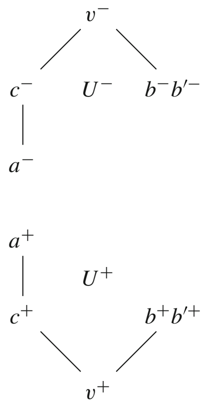
$$f(x) = \begin{cases} x^- & \text{if } x = x^+ \\ x & \text{if } x = x^{+-} \\ x^+ & \text{if } x = x^- \end{cases} \tag{61}$$

It is a little bit long but not difficult to prove that f satisfies the required properties so that the space $\mathscr{N}(U^*) = \langle U^*, \leq, f\rangle$ is a Nelson space. In consequence $\mathbf{N}_{k_f}(\mathscr{N}(\Omega_\leq(U^*)))$ is a Nelson algebra of ordered pairs of disjoint elements of $\Omega_{R^+}(U^+)$.

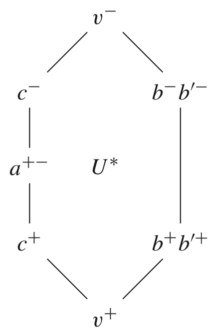
In synthesis, we have transformed a Heyting algebra into a Nelson one applying a particular filtration which operates as follows. In the first place, let us focus our attention to the involution f. From the above discussion, it is evident that if $x \in U^+$ then $f(\uparrow_{\leq^+} x) \cap U^+ = \mathbb{C}_{\leq^+}(\{x\})$. Therefore, if x is an isolated point in $\Omega^{+}_{\leq}(U^+)$ (i.e. $\uparrow_{\leq^+} x = \{x\}$) and $x \notin X$ but $x \in f(X)$, for some $X \in \Omega_\leq(U)$, then we do have x neither in $U^+ \cap X$ nor in $U^+ \cap -f(X)$. Suppose that for some $A \subseteq U^+$, $X \cap U^+ = \mathbb{I}_{\leq^+}(A)$ so that $U^+ \cap -f(X) = -\mathbb{C}_{\leq^+}(A)$. Therefore x would belong to the boundary of A, which is impossible for isolated points.

But suppose $f(x) = x$. Then $x \notin X \cap U^+$ if and only if $f(x) \notin X$. In consequence, $x \in -f(X)$ and, finally x is in $U^+ \cap -f(X)$, that is, $x \in -\mathbb{C}_{\leq^+}(A)$. It follows that if S is the set of isolated points of $\Omega_{\leq^+}$ and we put $f(c) = c$ for all $c \in S$, then there is a one-one correspondence between the set $\{-\mathbb{C}_{\leq^+}(A) : A \in \wp(U^+)\}$ and the set $\{X^+ \cap -f(X) : X \in \Omega_\leq(U^*)\}$, that is, the set $\{X_2 : \langle X_1, X_2\rangle \in k_f(\Omega_\leq(U^*))\}$. This is the tricky part, because $\Omega_{\leq^+}(U^+)$ is by definition $\{\mathbb{I}_{\leq^+}(A) : A \in \wp(U^+)\}$. Otherwise stated, in $\langle X_1, X_2\rangle$ the element X_1 is not affected by the choice of the subset of maximal elements.

Example 101 In this example we use our familiar preorder $\mathbf{P} = \langle U, R\rangle$. But to follow the construction comfortably, we rename its elements by setting U^+ instead of U and $\leq^+$ instead of R. Therefore, now our approximation space is called $\Omega_{\leq^+}(U^+)$.



We are not obliged to force any point into the intersection of U^+ and U^-, but since we want to represent rough sets, we must put in $U^+ \cap U^-$ all the isolated points, that is, all the maximal points of U^+. In our case a^+. Therefore we obtain the following lattice, where the elements are linked by the preorder $\leq^+ \cup \leq^- \cup(\leq^+ \otimes\varphi\otimes \leq^-)$ which is shown below.



$\leq^+$	v^+	c^+	b^+	b'^+	a^+
v^+	1	1	1	1	1
c^+	0	1	0	0	1
b^+	0	0	1	1	0
b'^+	0	0	1	1	0
a^+	0	0	0	0	1

$\leq^-$	v^-	c^-	b^-	b'^-	a^-
v^-	1	0	0	0	0
c^-	1	1	0	0	0
b^-	1	0	1	1	0
b'^-	1	0	1	1	0
a^-	1	1	0	0	1

φ	v	c^-	b^-	b'^-	a^-
v^+	1	0	0	0	0
c^+	0	1	0	0	0
b^+	0	0	1	0	0
b'^+	0	0	0	1	0
a^+	0	0	0	0	1

$\leq^+ \otimes\varphi\otimes \leq^-$	v^-	c^-	b^-	b'^-	a^-
v^+	1	1	1	1	1
c^+	0	1	1	0	1
b^+	1	0	1	1	0
b'^+	1	0	1	1	0
a^+	1	1	0	0	1

and finally

$\leq$	v^+	c^+	b^+	b'^+	a^{+-}	v^-	c^-	b^-	b'^-
v^+	1	1	1	1	1	1	1	1	1
c^+	0	1	0	0	1	1	1	0	0
b^+	0	0	1	1	0	1	0	1	1
b'^+	0	0	1	1	0	1	0	1	1
a^{+-}	0	0	0	0	1	1	1	0	0
v^-	0	0	0	0	0	1	0	0	0
c^-	0	0	0	0	0	1	1	0	0
b^-	0	0	0	0	0	1	0	1	1
b'^-	0	0	0	0	0	1	0	1	1

It is clear that the codomain of $\leq^+ \otimes\varphi$ is just a renaming of the codomain of $\leq^+$, from U^+ to U^-, so that $\leq^+ \otimes\varphi\otimes \leq^-$ amounts to $\leq^+ \otimes{\leq^+}^{\smile}$. So we obtain:

The above pre-ordered space together with the involution f of (61) is our Nelson space $\mathscr{N}(U^*) = \langle U^*, \leq, f\rangle$. Now let us apply to $\mathscr{N}(U^*)$ the transformation k_f. We compute some instances.

$$k_f(\{b^+, b'^+, b^-, b'^-, v^-\}) = \langle U^+ \cap \{b^+, b'^+, b^-, b'^-, v^-\}, U^+ \cap -f(\{b^+, b'^+, b^-, b'^-, v^-\})\rangle$$
$$= \langle\{b^+, b'^+\}, U^+ \cap -(\{b^+, b'^+, b^-, b'^-, v^+\})\rangle$$
$$= \langle\{b^+, b'^+\}, U^+ \cap \{v^-, c^+, a^{+-}, c^-\}\rangle = \langle\{b^+, b'^+\}, \{c^+, a^{+-}\}\rangle.$$

$k_f(\{b^+, b'^+, b^-, b'^-, v^-, c^-\}) = \langle\{b^+, b'^+\}, \{a^{+-}\}\rangle$.
Since $f(a^{+-}) = a^{+-}$, for any $X \in \Omega_{\leq}(U^+)$ it is not possible to exclude a^{+-} from both X_1 and X_2 in $k_f(X) = \langle X_1, X_2\rangle$.
Notice, indeed, that in the construction of the present space we have applied the Grothendieck topology of Example 94. On the contrary, consider the Nelson space $\mathscr{N}(U^*)$ of Example 98 and set $X = \{c, r, 1, g, l\}$. Then $k_f(X) = \langle\{c\}, \emptyset\rangle$ which corresponds in the present space to $\langle\{b^+, b'^+\}, \emptyset\rangle$. So, a (i.e. a^+) is excluded both from the interior and the complement of the closure of X.

Observe now that from Theorem 100 we can put $f(c) = c$ for all $c \in S$ for the very reason that S is the set of isolated, hence maximal, points of $\Omega_{\leq^+}(U^+)$. Since $\uparrow S$ contains all the dense elements of $\Omega_{\leq^+}(U^+)$, it induces a Boolean congruence.
Indeed, the above construction is an instance of the following general result at the pointless level which shows how to transform Heyting algebras into Nelson ones.

Theorem 102 (Sendlewski) *Let* $\mathbf{H}$ *be a Heyting algebra and* $\equiv$ *a Boolean congruence on* $\mathbf{H}$*. Then:*

$$N_{\equiv}(\mathbf{H}) = \{\langle a_1, a_2\rangle : a_1 \wedge a_2 = 0 \text{ and } a_1 \vee a_2 \equiv 1\} \tag{62}$$

equipped with the abstract version of the operations of Definition 97 is a Nelson algebra $\mathbf{N}_{\equiv}(\mathbf{H})$*. If* $a \cong b \iff a \longrightarrow b \wedge b \longrightarrow a$*, then* $\cong$ *is a congruence with respect to all the operations of* $\mathbf{N}_{\equiv}(\mathbf{H})$ *but the strong negation* $\sim$ *and* $\mathbf{N}_{\equiv}(\mathbf{H})/\cong$ *is isomorphic to* $\mathbf{H}$*. Moreover,* $\mathbf{N}_{k_f}(\mathscr{N}(\mathbf{J}(\mathbf{N}_{\equiv}(\mathbf{H})))) = \mathbf{N}_{\equiv}(\mathbf{H})$*. Finally, all the Nelson algebras* $\mathbf{N}$ *such that* $\mathbf{N}/\cong$ *is isomorphic to* $\mathbf{H}$ *are isomorphic to* $\mathbf{N}_{\equiv}(\mathbf{H})$ *for some Boolean congruence* $\equiv$*.*

Remarks 7.3 With "abstract version" of the operations we mean, for instance, $\sim \langle a_1, a_2\rangle = \langle a_2, a_1\rangle$ or $\langle a_1, a_2\rangle \longrightarrow \langle b_1, b_2\rangle = \langle a_1 \Longrightarrow b_1, a_1 \wedge b_2\rangle$.

The congruence $\cong$ takes into account just the first elements of the ordered pairs. But these are the elements of $\mathbf{H}$. Therefore it is immediate that $\mathbf{N}_{\equiv}(\mathbf{H})/\cong$ is isomorphic to $\mathbf{H}$, provided $\equiv$ is a Boolean congruence. If we look at this result from the point of view of Nelson spaces, we see that there is a one-one correspondence between the subsets of the set M of the maximal elements of the dual space of $\mathbf{H}$ and the Boolean congruences of **H**. The least Boolean congruence corresponds to M itself and to our operator J^{δ} discussed above because M is the least dense element of **H**. This congruence will play a key role that we see after showing how Boolean congruences, that is, subsets of maximal elements, are connected to rough set systems from the point of view of the transformation $N_{\equiv}$.

The dual space of a Heyting algebra is a partial order. So, any pre-maximal element is maximal. But if we start with a preordered system we have to take into account the subsets of pre-maximal and maximal elements. We have adapted the dual construction of Theorem 102, due to [47], to the finite case and preorders. In the process, we have found Theorem 100. Moreover, we have linked the Boolean congruence $\equiv$ to Lawvere-Tierney operators and Grothendieck topologies. This link enables us to understand the importance of $\equiv$ in the construction of rough set systems.

In fact, the rough set companion of Theorem 102, expressed in terms of Lawvere-Tierney operators, is:

Theorem 103 *Let $\Omega_R(U)$ be an approximation space and S the set of its isolated elements. Let us set:*

$$N_{\equiv_{J^S}}(\Omega_R(U)) = \{\langle X_1, X_2\rangle \in \Omega_R(U) : X_1 \cap X_2 = \emptyset \text{ and } X_1 \cup X_2 \equiv_{J^S} U\}$$

Then $Dsj(\Omega_R(U)) = N_{\equiv_{J^S}}(\Omega_R(U))$.

Proof The proof is immediate: In the first place, S is a subset of maximal elements of $\langle U, R\rangle$. So $\equiv_{J^S}$ is a Boolean congruence.

Moreover, if $X_1 \cup X_2 \equiv_{J^S} U$ then $S \Longrightarrow (X_1 \cup X_2) = S \Longrightarrow U = U$. It follows that $S \subseteq (X_1 \cup X_2)$.

Let $X_1 = \mathbb{I}_R(A)$ and $X_2 = -\mathbb{C}_R(A)$ for some $A \subseteq U$. Then for each $c \in S$ either $c \in \mathbb{I}_R(A)$ or $c \in -\mathbb{C}_R(A)$. □

Consider Example 101. It is easy to verify that $k_f(\Omega_{\leq}(U^*)) = N_{\equiv_{J^{\{a\}}}}(\Omega_R(U))$ and that they coincide with the lattice $Dsj(\Omega_R(U))$ of Example 46 once we get rid of the decoration $^+$.

We can see the above construction from a different point of view: it provides an information-like interpretation of the filtration clause "$\equiv 1$" which appears not only in the definition of Nelson algebras, but also of Stone algebras and Łukasiewicz algebras. Three-valued Łukasiewicz algebras will be linked to rough set systems in the next Section. Now we have to conclude the story about rough set systems from preorders and partial orders, with a particular and interesting case.

7.2 Rough Set Systems Based on Partial Orders and Effective Lattices

7.2.1 Constructive Logic with Strong Negation: CLSN

The Hilbert-style axioms of Nelson Logic, called *Constructive Logic with Strong Negation*—CLSN—are essentially the axioms for Nelson algebras. There is also an equational definition which can be found in [42]. A Natural Deduction-style set of rules can be found in [39]. Kripke models for CLSN were introduced by Thomason (see [48]). They are partial orders, that for a familiar reason we denote with $\langle U^+, \leq^+\rangle$, equipped with a standard forcing relation $\models$ for positive formulas, that is, if $p \models \alpha$ then for all $p' \geq p$, $p' \models \alpha$. In Kripke models for Intuitionistic logic, the forcing clause for the intuitionistic negation $\neg$ is defined as $p \models \neg\alpha$ if for all $p' \geq p$, $p' \not\models \alpha$. In Thomason's models the strong negation $\sim$ of CLSN is defined as like as a positive formula: $p \models\sim \alpha$ implies that for all $p' \geq p$, $p' \models\sim \alpha$. It is only required that if $p \models \alpha$ then $p \not\models\sim \alpha$.

Therefore, in the Intuitionistic case, $[\![\neg\alpha]\!]$ can be calculated from $[\![\alpha]\!]$. In particular, given an Heyting algebra $\Omega_R(U)$, $[\![\neg\alpha]\!] = -\mathbb{C}_R([\![\alpha]\!])$. On the contrary, there is not a function sending $[\![\alpha]\!]$ to $[\![\sim \alpha]\!]$. Actually, there can be α and β such that $[\![\alpha]\!] = [\![\beta]\!]$ but $[\![\sim \alpha]\!] \neq [\![\sim \beta]\!]$, a situation which does not occur for $\neg$. In a sense, $\sim \alpha$ is an "explicit negation" not an "implicit" one as $\neg\alpha$; one has to positively state where α is false.

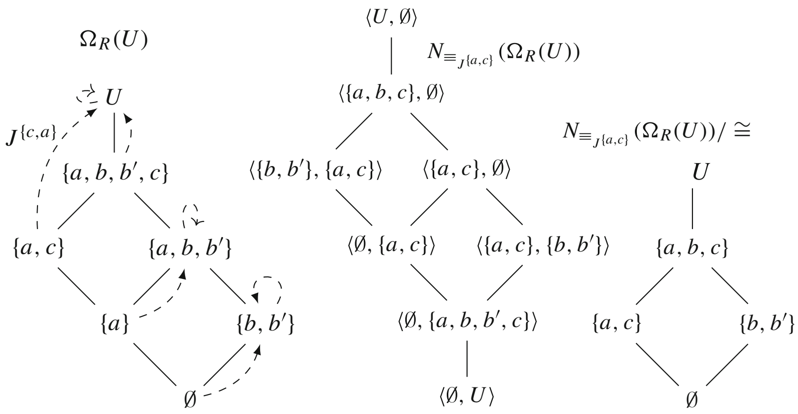
Therefore, it is natural to represent the evaluation of a CLSN formula α by means of an ordered pair of disjoint elements $\kappa(\alpha) = \langle [\![\alpha]\!], [\![\sim \alpha]\!]\rangle$.

Clearly, there are states p such that $p \not\models \alpha$ and $p \not\models\sim \alpha$ because it is not required the existence of maximal states m such that either $m \models \alpha$ or $m \models\sim \alpha$. This behaviour is different from the one of intuitionistic negation, because by its very definition if α is not forced by some state above p, then $p \models \neg\alpha$. Otherwise stated, in models for CLSN it is not required that *eventually* all formulas are decidable.

This behaviour is mirrored by the dual construction of Nelson algebras. In fact, if there is a maximal p which is required to decide every formula α, than p must be either in $[\![\alpha]\!]$ or in $[\![\sim \alpha]\!]$. Therefore, in the dual construction one must put $f(p) = p$ so that p must be maximal. But from the Remarks 7.2 this is not mandatory for maximal elements, as Example 98 shows.

Pay attention that both in the case of pre-orders and partial orders if p is not maximal or pre-maximal and still we put $f(p) = p$, then in view of Theorem 100.(3), f cannot be an involutive anti order isomorphism and for $X = \{p : f(p) = p\}$, the filter $\uparrow X$ does not contain all the dense elements of $\Omega_{\leq^+}(U^+)$ so that $\mathbf{N}_{\equiv_{JX}}(\Omega_{\leq^+}(U^+))/\cong$ is isomorphic to another Heyting algebra $\mathbf{H}' \neq \Omega_{\leq^+}(U^+)$.

Example 104 If we set $X = \{a, c\}$, then J^X gives:



The relations between **H** and **H**$'$ are not in the scope of our chapter. Therefore, we focus our attention on *maximal* and *pre-maximal* states.

From a logical point of view, since formula are evaluated on order filters, the distinction between partial and pre orders, hence between maximal and pre-maximal states, is not very important. On the contrary, it is relevant from the point of view of rough sets. In fact, in order to be approximated through a relation R on U, a subset A of U is "evaluated" on the points of $\Omega_R(U)$, so that, as we are going to see, there is a difference if R is a partial order or a preorder. More precisely, the difference concerns the existence of maximal states. From now on, therefore, we consider partial orders or preorders bounded by maximal states.

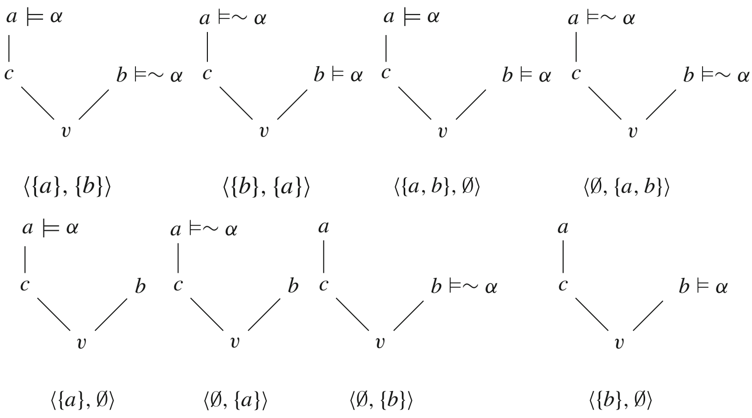
We have two extreme cases and an intermediate one: (1) No maximal states decide every formula. (2) All the maximal states decide every formula. (3) Some maximal states but not all decide all formulas.

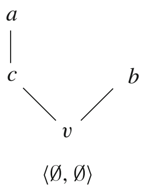
In the first case, the filtering congruence relation is $\equiv_{J^\emptyset}$, so it is required that $\emptyset \subseteq X_1 \cup X_2$ which is a relation always fulfilled. In particular in the resulting Nelson lattice the pair $\langle\emptyset, \emptyset\rangle$ appears, which represents a state of "complete absence of information".

Notice that $\sim\langle\emptyset, \emptyset\rangle = \langle\emptyset, \emptyset\rangle$. An element a such that $\sim a = a$ is called *central*. Central elements are fixed elements of the negation, thus. In Nelson algebras there can be only one central element.

The intermediate case is the generic one discussed so far. The lattice of Example 98 (a.k.a. $Dsj(U/R)$) illustrates the first extreme case. The other intermediate case will be discussed in the next section.

Example 105 Let us drop from our standard preorder **P** the element b'. Then R turns into a partial order Q on a set $W = \{a, b, c, v\}$. Suppose we are given just one CLSN formula α to be evaluated on $\langle W, Q\rangle$. Then the situations at the maximal state a, are the following (below any diagram the corresponding $\kappa(\alpha)$ is displayed):





In the last case, $\kappa(\alpha)$ is the central element $\langle\emptyset,\emptyset\rangle$.

But if the parameter of the operator J^X is the set of maximal elements M, then only the first four cases are admitted. Notice again that ordered pairs are not admitted not because they have an empty component. In fact $\langle M,\emptyset\rangle$ and $\langle\emptyset, M\rangle$ are admitted (for another counterexample see the next section).

Before analysing the second and fundamental extreme case, we display some interesting relations between the three negations. We need an easy but useful lemma:

Lemma 106 *Let $\langle a_1, a_2\rangle$ be a pair of disjoint elements of a Heyting algebra. Then $a_1 \le \neg a_2$ and $a_2 \le \neg a_1$.*

Proof Immediate from adjointness: $a_1 \wedge a_2 \le 0$ if and only if $a_1 \le a_2 \Longrightarrow 0 = \neg a_2$ if and only if $a_2 \le a_1 \Longrightarrow 0 = \neg a_1$. □

Moreover, it is not difficult to verify that for any two elements $a = \langle a_1, a_2\rangle$ and $b = \langle b_1, b_2\rangle$ of any Nelson algebra $\mathbf{N}_{\equiv_{J^X}}(\mathbf{H})$ of ordered pairs of disjoint elements of a Heyting algebra $\mathbf{H}$ (hence with $\equiv_{J^X}$ not necessarily a Boolean congruence) the following hold:

$$a \le b \text{ if and only if } a_1 \le_{\mathbf{H}} b_1 \text{ and } b_2 \le_{\mathbf{H}} a_2. \tag{63}$$

$$(i)\ \neg a \le\sim a \le \lrcorner a,\ \ (ii) \text{ if } a \le b \text{ then } \sim a \le\sim b,\ \neg a \le \neg b,\ \lrcorner a \le \lrcorner b \tag{64}$$

$$(i)\ \neg\neg\neg a \ge \neg a,\ \ (ii)\ \sim\sim\sim a =\sim a,\ \ (iii)\ \lrcorner\lrcorner\lrcorner \le \lrcorner a \tag{65}$$

$$(i) \lrcorner\lrcorner a \geq \sim\lrcorner a = \neg \sim a = \neg\lrcorner a \leq a, \; (ii)\; \neg\neg a \leq \sim \neg a = \lrcorner \sim a = \lrcorner\neg a \geq a \tag{66}$$

$$(i)\; \lrcorner\lrcorner a \geq a \text{ only if } a_2 = \neg a_1, \;\; (ii)\; \neg\neg a \geq a \text{ only if } \neg\neg a_2 = a_2 \tag{67}$$

By easy inspection using Lemma 106. For instance, (66).(ii) is immediate from (i): $\sim \neg a = \sim \neg \sim\sim a = \sim\sim\lrcorner \sim a = \lrcorner \sim a$. As to (67) $a = \langle a_1, a_2\rangle$ and from Lemma 106 $\lrcorner\lrcorner a = \langle \neg\neg a_1, \neg a_1\rangle$. Clearly $\neg\neg a_1 \geq_{\mathbf{H}} a_1$ and $\neg a_1 \geq_{\mathbf{H}} a_2$, so that the first pair is above the second only if $a_2 = \neg a_1$. In turn, $\neg\neg a = \langle \neg a_2, \neg\neg a_2\rangle$ and $\neg a_2 \geq_{\mathbf{H}} a_1$, $\neg\neg a_2 \geq_{\mathbf{H}} a_2$, so that now we need $\neg\neg a_2 = a_2$ (we recall that $\langle a_1, a_2\rangle \leq \langle b_1, b_2\rangle$ iff $a_1 \leq_{\mathbf{H}} b_1$ and $b_2 \leq_{\mathbf{H}} a_2$).

Example 107 Consider the following examples from the lattice $Dsj(\Omega(U))$ of Example 46: $\lrcorner\lrcorner\langle\{a\}, \emptyset\rangle = \langle\{a, c\}, \{b, b'\}\rangle$ which is incomparable with $\langle\{a\}, \emptyset\rangle$ because $\{a\} \subseteq \{a, c\}$ but also $\emptyset \subseteq \{b, b'\}$.

Similarly, $\neg\neg\langle\emptyset, \{a\}\rangle = \langle\{b, b'\}, \{a, c\}\rangle$ which is incomparable with $\langle\emptyset, \{a\}\rangle$. $\lrcorner\lrcorner\langle\emptyset, \{a\}\rangle = \langle\emptyset, U\rangle \leq \langle\emptyset, \{a\}\rangle$ and $\neg\neg\langle\{b, b'\}, \{a\}\rangle = \langle\{b, b'\}, \{a, c\}\rangle \leq \langle\{b, b'\}, \{a\}\rangle$. On the contrary, $\lrcorner\lrcorner\langle\{a\}, \{b, b'\}\rangle = \langle\{a, c\}, \{b, b'\}\rangle \geq \langle\{a\}, \{b, b'\}\rangle$ and $\neg\neg\langle\emptyset, \{a, c\}\rangle = \langle\{b, b'\}, \{a, c\}\rangle \geq \langle\emptyset, \{a, c\}\rangle$.

7.3 Effective Lattices, the Logic E_0 and Rough Set Systems

Now we have to focus on the particular case in which the preorder, or the partial order, is bounded by a set M of maximal states. From a rough set perspective, this means that the set S of isolated elements coincides with M. What are the logical consequences of this situation? Otherwise stated, what does it happen if in a model every state has a state above it which decides every formula?

In view of Theorem 83, in such model for any formula α, $[\![\alpha \vee \sim \alpha]\!]$ is locally valid everywhere. This is the main feature which makes CLSN transform into a new logic called E_0. This logic (also called *Effective Logic* 0) was introduced for studying program synthesis and program specification (see [24]) and its algebraic models were studied in [27]. This logic was presented by means of rules of Natural Deduction by adding to the rules for CLSN the following schemas:

$$(T1)\quad \dfrac{\begin{array}{cc}[\alpha] & [\alpha] \\ \vdots & \vdots \\ [\beta] & [\sim \beta]\end{array}}{\sim \mathbf{T}(\alpha)} \qquad\qquad (T2)\quad \dfrac{\begin{array}{cc}[\sim \alpha] & [\sim \alpha] \\ \vdots & \vdots \\ [\beta] & [\sim \beta]\end{array}}{\mathbf{T}(\alpha)}$$

Now, let us translate the two rules in the operation of a Nelson algebra $\mathbf{N}_{\equiv}(\mathbf{H})$ of ordered pairs of disjoint elements of a Heyting algebra $\mathbf{H}$. In what follows we put $a = \kappa(\alpha)$, $b = \kappa(\beta)$, $a = \langle a_1, a_2\rangle$, $b = \langle b_1, b_2\rangle$, and so on. Remember that $a \longrightarrow b = 1$ iff $a_1 \leq_{\mathbf{H}} b_1$. Then, since the first element of $\kappa(b \wedge \sim b)$ is $b_1 \wedge b_2 = 0$, we have:

$$(i)\ (a \longrightarrow 0) \longrightarrow \mathbf{T}(\sim a);\ (ii)\ (\sim a \longrightarrow 0) \longrightarrow \mathbf{T}(a) \tag{68}$$

Moreover, from the two rules it is possible to derive

$$\mathbf{T}(\sim \alpha) \equiv \sim \mathbf{T}(\alpha). \tag{69}$$

Let us set $\mathbf{T}(a) = \langle T_1, T_2\rangle$ and from the above ingredients discover what T_1 and T_2 must be.

From (68).(i) and $\mathbf{T}(\sim \alpha) \equiv \sim \mathbf{T}(\alpha)$ one has $\lrcorner a \longrightarrow \sim \mathbf{T}(a)$. From (68).(ii) one obtains $\lrcorner \sim a \longrightarrow \mathbf{T}(a)$. Since $\lrcorner a = \langle \neg a_2, a_2\rangle$ and $\lrcorner a = \langle \neg a_1, a_1\rangle$ it must be:

$$(i)\ \neg a_2 \leq_{\mathbf{H}} T_1,\ \ (ii)\ \neg a_1 \leq_{\mathbf{H}} T_2,\ \text{ so } (iii)\ \neg T_1 \leq_{\mathbf{H}} \neg\neg a_2,\ \ (iv)\ \neg T_2 \leq_{\mathbf{H}} \neg\neg a_1 \tag{70}$$

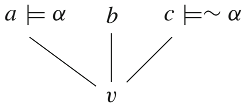
From Lemma 106 we have:

$$(i) \neg\neg a_1 \leq_{\mathbf{H}} \neg a_2,\ \ \ (ii)\ T_1 \leq_{\mathbf{H}} \neg T_2,\ \ \ (iii)\ T_2 \leq_{\mathbf{H}} \neg T_1 \tag{71}$$

From (70).(ii) and (71).(i) $\neg\neg a_1 \leq_{\mathbf{H}} \neg a_2 \leq_{\mathbf{H}} T_1$. From this, (70).(iv) and (71).(ii) $\neg\neg a_1 \leq_{\mathbf{H}} T_1 \leq_{\mathbf{H}} \neg T_2 \leq_{\mathbf{H}} \neg\neg a_1$. In consequence: $\neg\neg a_1 = T_1 = \neg T_2$. Therefore, $\neg T_1 = \neg a_1$. From this and (71).(iii) $T_2 \leq_{\mathbf{H}} \neg a_1$ and from (70).(ii) $T_2 \leq_{\mathbf{H}} \neg a_1 \leq_{\mathbf{H}} T_2$, so that $T_2 = \neg a_1$. We conclude that:

$$\mathbf{T}(a) = \langle \neg\neg a_1, \neg a_1\rangle \text{ for any } a \text{ of a Nelson algebra modelling } E_0. \tag{72}$$

Now, in a *generic* Nelson algebra we do have $\lrcorner\lrcorner a = \langle \neg\neg a_1, \neg a_1\rangle$, but we cannot set $\mathbf{T}(a) = \lrcorner\lrcorner a$ because of (69). In fact, $\sim \lrcorner\lrcorner a = \neg\neg \sim a = \langle \neg a_1, \neg\neg a_1\rangle$. Therefore it should be $\neg a_1 = \neg\neg a_2$ and $\neg a_2 = \neg\neg a_1$, all a. However, usually in Nelson algebras these equations do not hold. Look at the following model:



$\kappa(\alpha) = \langle \{a\}, \{c\}\rangle$, $\sim \lrcorner\lrcorner(\kappa(\alpha)) = \langle \neg\{a\}, \neg\neg\{a\}\rangle = \langle \{b, c\}, \{a\}\rangle$, while $\lrcorner\lrcorner \sim \kappa(\alpha) = \langle \neg\neg\{c\}, \neg\{c\}\rangle = \langle \{c\}, \{a, b\}\rangle$.

The reader has immediately recognised the problem: b does not decide α or $\sim \alpha$.

At this point we must find the conditions for a Nelson algebra to make $\neg a_1 = \neg\neg a_2$, which, obviously, is the same as $\neg\neg a_1 = \neg a_2$.

Lemma 108 *Let* $\mathbf{L}$ *be a distributive lattice,* $a, b, c \in \mathbf{L}$*,* $a \wedge b = 0 = a \wedge c$*. Then* $a \vee b = a \vee c$ *if and only if* $b = c$*.*

Proof If $a \vee b = a \vee c$ then $(a \vee c) \wedge b = (a \vee b) \wedge b = b$. Therefore, $b = (a \wedge b) \vee (c \wedge b) = 0 \vee (c \wedge b)$ so that $b \leq c$. Similarly one proves that $c \leq b$. The converse implication is more than trivial. □

Theorem 109 *Let* $\mathbf{N}_{\equiv}(\mathbf{H})$ *be a Nelson algebra of ordered pairs of disjoint elements of a Heyting algebra* $\mathbf{H}$*. Then for any* $\langle a_1, a_2 \rangle \in N_{\equiv}(\mathbf{H})$*,* $\neg a_1 = \neg\neg a_2$ *if and only if* $\equiv$ *is* $\equiv_{J\delta}$*, where* δ *is the least dense element of* $\mathbf{H}$*.*

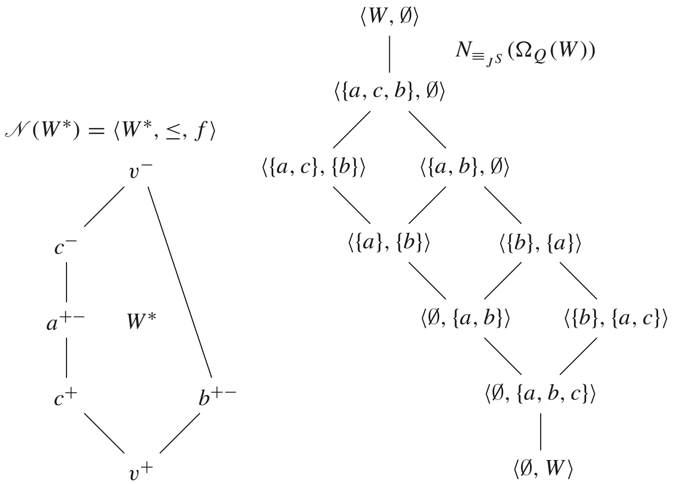
Proof If $\langle a_1, a_2 \rangle \in N_{\equiv_{J\delta}}(\mathbf{H})$ then $[\langle a_1 \vee a_2 \rangle]_{\neg\neg} = [1]_{\neg\neg}$. It follows that $[\langle a_1 \vee a_2 \rangle]_{\neg\neg} = [\langle a_1 \vee \neg a_1 \rangle]_{\neg\neg}$. Since $a_1 \wedge a_2 = 0$ and $a_1 \wedge \neg a_1 = 0$, then $[a_1]_{\neg\neg} \wedge [a_2]_{\neg\neg} = [0]_{\neg\neg}$ and $[a_1]_{\neg\neg} \wedge \neg[a_1]_{\neg\neg} = [0]_{\neg\neg}$. Moreover, $[a_1 \vee a_2]_{\neg\neg} = [a_1]_{\neg\neg} \sqcup [a_2]_{\neg\neg} = [a_1 \vee \neg a_1]_{\neg\neg} = [a_1]_{\neg\neg} \sqcup [\neg a_1]_{\neg\neg}$. Therefore, from Lemma 108 $[a_2]_{\neg\neg} = [\neg a_1]_{\neg\neg}$ and we conclude $\neg\neg a_2 = \neg\neg\neg a_1 = \neg a_1$. The example above proves the converse (more precisely, it proves the contrapositive of the converse implication, that is, if $\equiv$ is not the minimal Boolean congruence, then—see Lemma 106—$\neg a_1 \geq_{\mathbf{H}} \neg\neg a_2$). □

In [30] the above result is provided by a parallel algebraic and proof-theoretic derivation.

If an approximation space $\Omega_R(U)$ is induced by an order R upper bounded by maximal elements (i.e. with no infinite ascending chains), like any finite partial order, then the set of maximal elements of R is the set of isolated points, which for the philosophy of Rough Set Theory, are completely describable items, that is, items which are describable with no ambiguity by the given properties. In consequence $Dsj(\Omega_R)(U) = N_{\equiv_{J\delta}}(\Omega_R)(U)$. In this case the *intrinsic logic* (in the sense of [21]) of the rough set system is E_0, not CLSN (see [19]):

Theorem 110 *Let* $\Omega_R(U)$ *be an approximation space such that* R *is an order with no infinite ascending chains, and* S *the set of its maximal elements. Then* $N_{\equiv_{JS}}(\Omega_R(U)) = Dsj(\Omega_R(U))$*.*

Example 111 Consider the partial order of Example 105. The set of maximal elements is $S = \{a, b\}$. Below we show the Nelson space built on $\langle W, Q \rangle$ with the usual decorations "+" and "−" and the resulting Nelson lattice of ordered pair of elements of $\Omega_Q(W)$ without the decoration "+".



Pay attention that if we apply the above construction to our preorder $\langle U, R\rangle$ enlarging S to a set S' containing also the pre-maximal states, then $N_{\equiv_{JS'}}(\Omega_R(U))$ is, indeed, an effective lattice, but it is a sublattice of $Dsj(\Omega_R(U)$. For instance, $Dsj(\{a, b\}) = \langle\{a\}, \emptyset\rangle$ which is not an element of $N_{\equiv_{JS'}}(\Omega_R(U))$. In fact in this lattice $R(b)$ and $R(b')$, that is, $\{b, b'\}$, must be included either in the first or in the second element of any ordered pair. Practically, $N_{\equiv_{JS'}}(\Omega_R(U))$ is the above lattice with b' added in any element containing b. Hence it is a lattice quite different from $Dsj(\Omega_R(U))$ which is shown in Example 46.

7.4 Algebraic Logic from Equivalence Relations

Originally, Rough Set Theory was based on equivalence relations (see [40]). Also in this case the intrinsic logic of the resulting rough set system changes drastically according to the filter J^X.

Lemma 112 *An approximation space $\Omega_R(U)$ with R an equivalence relation and equipped with the Heyting algebra operations of Definition 38 is a Boolean algebra.*

Proof A Heyting algebra such that $\neg\neg a = a$ is a Boolean algebra. The topological space $\Omega_R(U)$ is made of clopen (closed and open subsets). Hence, if $A \in \Omega_R(U)$ then $-A \in \Omega_R(U)$. In consequence for any $A, B \in \Omega_R(U)$, $\mathbb{I}(-A \cup B) = -A \cup B$, so that $\neg A = -A$. Therefore $\neg\neg A = - - A = A$. $\square$

Lemma 113 *If* **B** *is a Boolean algebra, then for any* $a \in \mathbf{B}$, $\equiv_{J^a}$ *is a Boolean congruence.*

Proof Trivial: the only dense element of a Boolean algebra is 1 and since $a \leq 1$, any a, $\uparrow a$ contains all dense elements (actually the only one). □

Lemma 114 *Let* **B** *be a Boolean algebra, then for any congruence* $\equiv_{J^a}$, $\mathbf{N}_{\equiv_{J^a}}(\mathbf{B})$ *is a semi-simple Nelson algebra.*

Proof For any $a \in N_{\equiv_{J^a}}(\mathbf{B})$, $a \vee \lrcorner a = \langle a_1, a_2\rangle \vee \langle \neg a_1, a_1\rangle = \langle a_1 \vee \neg a_1, a_2 \wedge a_1\rangle = \langle 1, 0\rangle = 1$. □

Semi-simple Nelson algebras are the same mathematical objects as three-valued Łukasiewicz algebras. We explain this correspondence by showing interesting facts connected to rough sets.

In semi-simple Nelson algebras the operation $\neg$ is a pseudo-complementation: $\neg a$, that is, $\langle a_2, \neg a_2\rangle$, is the maximal element $\langle x_1, x_2\rangle$ such that $x_1 \wedge a_1 = 0$ and $x_2 \vee a_2 = 1$. This may be surprising, because the complement of a_1 in the underlying Boolean algebra is its pseudo-complement $\neg a_1$, not a_2. But we have to consider that in any Nelson algebra of ordered pairs of a Heyting algebra **H**, $a \leq b$ if and only if $a_1 \leq_{\mathbf{H}} b_1$ and $b_2 \leq_{\mathbf{H}} a_2$. Therefore, the orders of the first and the second elements are contravariant. It follows that $\langle x_1, x_2\rangle$ is the pseudo-complement of $\langle a_1, a_2\rangle$ if x_2 is the minimal element z such that $z \vee a_2 = 1$ and x_1 is the maximal element w such that $w \wedge a_1 = 0$ and $w \wedge x_2 = 0$. But, $\neg a_2$ is the minimal complement of a_2 to 1 (it is its supplement, indeed—see Definition 123). And a_2 is the maximal element disjoint from $\neg a_2$.

However, we give a proof, by introducing another kind of implication, which is fundamental to understand the intrinsic logic of rough set systems from equivalence relations.

Theorem 115 *In a Nelson algebra* $\mathbf{N}_{\equiv}(\mathbf{B})$ *with* **B** *a Boolean algebra, the operation* $\supset$ *defined in* (58) *is a relative-pseudocomplementation*

The proof can be found in [32] or in [39].[7] As a corollary, since $\neg a = a \supset 0$, one obtains that $\neg a$ is the pseudo-complement of a.

Excursus

Now we have enough material to discuss a point we have left open: how to define the Nelson operations on a decreasing representation $Dcr(\Omega_R(U))$? If R is an equivalence relation the answer is straightforward, as we shall see. But if R is a preorder some difficulties arise. We recall that any element $\langle A_1, A_2\rangle$ of the decreasing representation belongs to $\Omega_{R^\smile}(U) \times \Omega_R(U)$ and that $\Omega_{R^\smile}(U)$ is a co-Heyting algebra with respect to $\Omega_R(U)$.

It is a tricky point. For meet and join there is no problem: $a \wedge b = \langle A_1 \cap B_1, A_2 \cap B_2\rangle$ and analogously for $\vee$. For the strong negation just a little effort gives:

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[7]The first proof was presented in [28]. In that paper there is a misprint: $\neg(\neg B_2 \cup A_2)$ instead of $(\neg B_2 \cup A_2)$.

$\sim \langle A_1, A_2\rangle = \langle -A_2, -A_1\rangle$. Now we apply the map ρ of (45) which transforms a disjoint representation into a decreasing one. Thus pay attention that before applying ρ, $\sim \langle A_1, A_2\rangle = \langle A_2, A_1\rangle$ while after applying ρ, $\sim \langle A_1, A_2\rangle = \langle -A_2, -A_1\rangle$. Let us then verify that $\rho(\sim a) =\sim \rho(a)$: $\rho(\sim a) = \rho(\langle A_2, A_1\rangle) = \langle -A_1, A_2\rangle = \langle -A_1, --A_2\rangle =\sim \langle -A_2, A_1\rangle =\sim \rho(a)$. It is more difficult to define the implication. In disjoint representation $a \longrightarrow b = \langle A_1 \Longrightarrow B_1, A_1 \cap B_2\rangle$ so $\rho(\langle A_1 \Longrightarrow B_1, A_1 \cap B_2\rangle) = \langle -(A_1 \cap B_2), A_1 \Longrightarrow B_1\rangle = \langle -A_1 \cup -B_2, A_1 \Longrightarrow B_1\rangle$. Now, $\rho(a) = \langle -A_2, A_1\rangle$, $\rho(b) = \langle -B_2, B_1\rangle$. It follows that given $a = \langle A_1, A_2\rangle$ and $b = \langle B_1, B_2\rangle$ *in decreasing representation* $a \longrightarrow b = \langle -A_2 \cup B_1, A_2 \Longrightarrow B_2\rangle$. Notice that $A_2 \Longrightarrow B_2 = \mathbb{I}_R(-A_2 \cup B_2)$. Therefore, $\lrcorner a = a \longrightarrow \langle 0, 0\rangle = \langle -A_2, \neg A_2\rangle$. One more time, notice that the first element is a set-theoretic complementation while the second is the interior of a set theoretic complementation. Finally, the definition of $\neg$ is interesting, and tricky: in disjoint representation $\neg a = \langle a_2, \neg a_2\rangle$. Hence $\rho(\neg a) = \langle -\neg A_2, A_2\rangle = \langle -\mathbb{I}_R(-A_2), A_2\rangle$. In consequence, for a in decreasing representation $\neg a = \langle -\mathbb{I}_R(A_1), A_1\rangle = \langle \lrcorner A_1, -A_1\rangle$, where $\lrcorner$ is the pseudo-complementation of the opposite Heyting algebra $\Omega_{R^{\smile}}(U)$. In fact, $-\mathbb{I}_R(X) = -\mathbb{C}_{R^{\smile}}(X) = \mathbb{I}_{R^{\smile}}(-X) = \lrcorner X$. Rephrased with the constructors introduced in the first lesson, if a is in decreasing representation, then $\neg a = \langle [i]_R(-A_1), A_1\rangle = \langle -\langle i\rangle_R(-A_1), A_1\rangle$, while $\lrcorner a = \langle -A_2, [e]_R(A_2)\rangle$.

At an abstract level, in view of Theorem 11, these operations require a Boolean algebra equipped with a topological modal operator M and a topological co-modal operator L such that $\langle M, L\rangle$ is an axiality (an adjoint pair). Alternatively, we need a bi-Heyting algebra providing $\neg$ and $\lrcorner$.

But in the case R is an equivalence relation, $R = R^{\smile}$ and $\Omega_R(U)$ is a topology of clopen sets. Then things are much easier and the operations are smoother: in the first place, there are no chains of alternate topological operators, that is, $\mathbb{C}_R\mathbb{I}_R = \mathbb{I}_R$ and $\mathbb{I}_R\mathbb{C}_R = \mathbb{C}_R$. This fact simplifies a lot, since any X_i has the form $\mathbb{I}_R(Y)$ for some $Y \subseteq U$. Furthermore, $\lrcorner a = \langle \neg A_2, \neg A_2\rangle$, $\neg a = \langle \neg A_1, \neg A_1\rangle$ and $a \longrightarrow b = \langle A_2 \Longrightarrow B_1, A_2 \Longrightarrow B_2\rangle$, where $\neg X_i$ is the set-theoretic complement of X_i and $X_i \Longrightarrow Y_j = -X_i \cup Y_j$.

7.5 Rough Set Interpretation of Tautologies and Contradictions

The following hold in *any* Nelson algebra $\mathbf{N}_{\equiv_{Jx}}(\mathbf{H})$:

$$1 \geq a \vee \lrcorner a \geq a \vee \neg a = a \vee \sim a \geq \langle x, 0\rangle \tag{73}$$

$$0 \leq a \wedge \neg a \leq a \wedge \lrcorner a = a \wedge \sim a \leq \langle 0, x\rangle \tag{74}$$

(73) and (74) are easily proved: $a \vee \lrcorner a = \langle a_1 \vee \neg a_1, 0\rangle \geq \langle a_1 \vee a_2\rangle = a \vee \neg a$. Symmetrically, $a \wedge \neg a = \langle 0, a_2 \vee \neg a_2\rangle \leq \langle 0, a_1 \vee a_2\rangle = a \wedge \lrcorner a$. But $a_1 \vee a_2 \geq x$, by definition of the domain of the algebra. Therefore, $a \vee \neg a \geq \langle x, 0\rangle$ and $a \wedge \lrcorner a \leq \langle 0, x\rangle$.

If $\mathbf{N}_{\equiv_{Jx}}(\mathbf{H})$ is semi-simple:

$$(i)\ \neg\neg\neg a = \neg a,\ (ii)\ \lrcorner\lrcorner\lrcorner a = \lrcorner a \tag{75}$$

$$(i)\ \lrcorner\lrcorner a = \sim\lrcorner a = \neg \sim a \leq a,\ \ (ii) \neg\neg a = \sim \neg a = \lrcorner \sim a \geq a \tag{76}$$

$$(i)\ a \vee \lrcorner a = \langle 1, 0\rangle;\ \ (ii)\ a \wedge \neg a = \langle 0, 1\rangle \tag{77}$$

(75), (76) and (77) come from (65), (66) and (67) because $\neg\neg a_1 = a_1$ and $\neg\neg a_2 = a_2$. (77) comes trivially from the fact that $a_1 \vee \neg a_1 = a_2 \vee \neg a_2 = 1$, so that $a \vee \lrcorner a = \langle a_1 \vee \neg a_1, 0\rangle = \langle 1, 0\rangle$ and $a \wedge \neg a = \langle 0, a_2 \vee \neg a_2\rangle = \langle 0, 1\rangle$.

Suppose $\Omega_R(U)$ is a topology and $a = Dsj(X) = \langle \mathbb{I}_R(X), -\mathbb{C}_R(X)\rangle$ for $X \subseteq U$. Then:

$$a \vee \sim a = a \vee \neg a = \langle \mathbb{I}_R(X) \cup -\mathbb{C}_R(X), \emptyset\rangle = \langle -\mathbb{B}_R(X), \emptyset\rangle \tag{78}$$

$$a \vee \lrcorner a = \langle \mathbb{I}_R(X) \cup \mathbb{I}_R - \mathbb{I}_R(X), \emptyset\rangle = \langle \mathbb{I}_R(X) \cup \mathbb{I}_R\mathbb{C}_R(-X), \emptyset\rangle \tag{79}$$

Opposite relations hold for the contradictions:

$$a \wedge \sim a = a \wedge \lrcorner a = \langle \emptyset, -\mathbb{B}_R(X)\rangle \tag{80}$$

$$a \wedge \neg a = \langle \emptyset, -\mathbb{C}_R(X) \cup \mathbb{I}_R - -\mathbb{C}_R(X)\rangle = \langle \emptyset, -\mathbb{C}_R(X) \cup \mathbb{I}_R\mathbb{C}_R(X)\rangle \tag{81}$$

Therefore, we note that contradictions are not equivalent to 0 and tautologies are not equivalent to 1 because of the presence of an "indecision area", that is, the boundary of a set.

In the case $\Omega_R(U)$ is a Boolean algebra, so that $Dsj(X)$ is an element of a semi-simple Nelson algebras, since the open sets of $\Omega_R(U)$ are clopen, $\mathbb{I}_R\mathbb{C}_R(-X) = \mathbb{C}_R(-X) = -\mathbb{I}_R(X)$, so that $a \vee \lrcorner a = \langle U, \emptyset\rangle$ and $a \wedge \neg a = \langle \emptyset, U\rangle$, as anticipated by (77).

What about, then, $Dsj(\mathbb{B}_R(X))$ itself? It is $\langle \mathbb{I}_R(\mathbb{B}_R(X)), -\mathbb{C}_R(\mathbb{B}_R(X))\rangle$. But since $\mathbb{B}_R(X)$ is the intersection of two closed sets, that is, $\mathbb{C}_R(X)$ and $-\mathbb{I}_R(X)$, it is itself a closed set. In consequence,

$$Dsj(\mathbb{B}_R(X)) = \langle \mathbb{I}_R\mathbb{B}_R(X), -\mathbb{B}_R(X)\rangle = \sim\lrcorner(a \wedge \sim a) = \sim\lrcorner(a \wedge \lrcorner a) \tag{82}$$

In the semi-simple case, since $\mathbb{I}_R\mathbb{B}_R(X) = \mathbb{B}_R(X)$, $Dsj(\mathbb{B}_R(X)) = \langle \mathbb{B}_R(X), -\mathbb{B}_R(X)\rangle$. Therefore, $\mathbb{B}_R(X)$ is an exact set.

7.6 Double Negations, Modalities and Approximations

The following are immediate in any Nelson algebra:

$$(i)\ \neg\neg a = \langle \neg a_2, \neg\neg a_2 \rangle,\ \ (ii)\ \lrcorner\lrcorner a = \langle \neg\neg a_1, \neg a_1 \rangle\ (iii)\ \lrcorner\lrcorner a \leq \neg\neg a \tag{83}$$

In what follows, $\mathbb{I}$ and $\mathbb{C}$ stand for $\mathbb{I}_R$ and, respectively, $\mathbb{C}_R$.

Theorem 116 *Let $\Omega_R(X)$ be a topology. Then for all $X \subseteq U$:*

$$Dsj((lR)(X)) = \langle \mathbb{I}(X), -\mathbb{C}\mathbb{I}(X) \rangle,\ \ Dsj((uR)(X)) = \langle \mathbb{I}\mathbb{C}(X), -\mathbb{C}(X) \rangle \tag{84}$$

$$\neg\neg Dsj(X) = \langle \mathbb{I}\mathbb{C}(X), \mathbb{I}\mathbb{C}\mathbb{I}(-X) \rangle,\ \ \lrcorner\lrcorner Dsj(X) = \langle \mathbb{I}\mathbb{C}\mathbb{I}(X), \mathbb{I}\mathbb{C}(-X) \rangle \tag{85}$$

$$\lrcorner\lrcorner Dsj((lR)(X)) = \lrcorner\lrcorner Dsj(X),\ \ \lrcorner\lrcorner Dsj((uR)(X)) = \neg\neg Dsj(X) \tag{86}$$

$$\neg\neg Dsj((lR)(X)) = \lrcorner\lrcorner Dsj(X),\ \ \neg\neg Dsj((uR)(X)) = \neg\neg Dsj(X) \tag{87}$$

$$Dsj((lR)(X)) \leq \lrcorner\lrcorner Dsj((lR)(X)) = \lrcorner\lrcorner Dsj(X) \leq \tag{88}$$

$$\leq \neg\neg Dsj(X) = \neg\neg Dsj((uR)(X)) \leq Dsj((uR)(X)) \tag{89}$$

Proof First, $\neg - X = -\mathbb{C}(-X) = \mathbb{I}(X)$, $\neg\neg - X = \mathbb{I} - \mathbb{I}(- - X) = \mathbb{I} - \mathbb{I}(X)$, and so on. Second, $\mathbb{I}\mathbb{C}\mathbb{I}(X) \subseteq \mathbb{I}\mathbb{C}(X)$. In view of these equations and dis-equations the proofs are just easy calculations. For instance, (85) is proved as follows:

$$\begin{aligned}\neg\neg Dsj(X) &= \neg\neg\langle \mathbb{I}(X), -\mathbb{C}(X) \rangle = \langle \neg - \mathbb{C}(X), \neg\neg - \mathbb{C}(X) \rangle = \\ &= \langle \mathbb{I} - -\mathbb{C}(X), \mathbb{I} - \mathbb{I} - -\mathbb{C}(X) \rangle = \langle \mathbb{I}\mathbb{C}(X), \mathbb{I} - \mathbb{I}\mathbb{C}(X) \rangle = \\ &= \langle \mathbb{I}\mathbb{C}(X), \mathbb{I}\mathbb{C}\mathbb{I}(-X) \rangle\end{aligned}$$

□

Theorem 117 *Let $\Omega_R(U)$ be a Boolean algebra, then for all $X \subseteq U$,*

$$Dsj((lR)(X)) = \lrcorner\lrcorner Dsj(X),\ \ Dsj((uR)(X)) = \neg\neg Dsj(X) \tag{90}$$

In other terms, the following diagrams commute:

$$\begin{array}{ccc} X & \xrightarrow{Dsj} & \cdot \\ {\scriptstyle (lR)}\downarrow & & \downarrow{\scriptstyle \lrcorner\lrcorner} \\ \cdot & \xrightarrow[Dsj]{} & \cdot \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{Dsj} & \cdot \\ {\scriptstyle (uR)}\downarrow & & \downarrow{\scriptstyle \neg\neg} \\ \cdot & \xrightarrow[Dsj]{} & \cdot \end{array}$$

Proof (90) is based on the fact that $\Omega_R(U)$ is made of clopen sets, so that for all $X \subseteq U$, $\mathbb{I}\mathbb{C}(X) = \mathbb{C}(X)$ and $\mathbb{C}\mathbb{I}(X) = \mathbb{I}(X)$. It follows from (85) and (84) that $\lrcorner\lrcorner Dsj(X) = \langle \mathbb{I}\mathbb{C}\mathbb{I}(X), \mathbb{I}\mathbb{C}(-X) \rangle = \langle \mathbb{I}(X), \mathbb{I}\mathbb{C}(-X) \rangle = Dsj((lR)(X))$. The other equation comes from duality. □

Actually, $\lrcorner\lrcorner Dsj(X) = Dsj((lR)(X)) = \langle \mathbb{I}(X), -\mathbb{I}(X)\rangle$ follows from $\mathbb{IC}(-X) = \mathbb{C}(-X) = -\mathbb{I}(X)$. Similarly, $\neg\neg Dsj(X) = Dsj((uR)(X)) = \langle \mathbb{C}(X), -\mathbb{C}(X)\rangle$. Indeed, if **B** is a Boolean algebra, then in the semi-simple Nelson algebra $\mathbf{N}_{\equiv}(\mathbf{B})$, $\neg\neg a = \langle \neg a_2, a_2\rangle$ and $\lrcorner\lrcorner a = \langle a_1, \neg a_1\rangle$, any a.

We know that $\neg\neg(Dsj(\Omega_R(U))$ is a Boolean algebra if equipped with $\cap, \sqcup$ and $\neg$, where $\neg$ is the set-theoretic complementation. In case R is an equivalence relation, we prove something special:

Theorem 118 *Let* **N** *be a semi-simple Nelson algebra and* $\neg\neg(\mathbf{N})$ *the lattice of regular elements of* **N**. *Then* $\sqcup$ *coincides with* $\vee$.

In Theorem 117 it was actually proved that in semi-simple Nelson algebras $\neg\neg$ is a topological closure operator, hence it is additive. Another proof is the following: if **N** is semi-simple, then $\neg$ is a pseudo-complementation. It follows that $\neg\neg(\neg\neg a \vee \neg\neg b) = \neg(\neg\neg\neg a \wedge \neg\neg\neg b) = \neg(\neg a \wedge \neg b)$. In Sect. 7.8 we shall prove that $\neg(x \wedge y) = \neg x \vee \neg y$. Hence, $\neg(\neg a \wedge \neg b) = \neg\neg a \vee \neg\neg b$.

In a rough set perspective, one can prove by easy calculation that

$$\neg\neg Dsj(X \cup Y) = \neg\neg Dsj(X) \cup \neg\neg Dsj(Y).$$

$$\text{Therefore, } \neg\neg(\neg\neg Dsj(X) \cup \neg\neg Dsj(Y)) = \neg\neg\neg\neg Dsj(X \cup Y) =$$

$$\neg\neg Dsj(X \cup Y) = \neg\neg Dsj(X) \cup \neg\neg Dsj(Y).$$

Notice, on the contrary, that $Dsj(X \cup Y) \geq Dsj(X) \cup Dsj(Y)$, because $\mathbb{I}_R(X \cup Y) \supseteq \mathbb{I}_R(X) \cup \mathbb{I}_R(Y)$.

It is natural to ask what is the element x of this algebra such that $\neg\neg a = J^x(a)$ for any element a.

Theorem 119 *Let* **B** *be a Boolean algebra. Then for any* $a \in \mathbf{B}$, *the least dense element of* $\mathbf{N}_{\equiv_{Ja}}(\mathbf{B})$ *is* $\langle a, 0\rangle$.

Proof An element $\langle x_1, x_2\rangle$ is dense if $\neg\langle x_1, x_2\rangle = \langle x_2, \neg x_2\rangle = 0$. Hence $x_2 = 0$. But because of the filtration clause, $x_1 \vee 0 \geq_{\mathbf{B}} a$. In consequence $x_1 \geq_{\mathbf{B}} a$. □

Corollary 120 *In any semi-simple Nelson algebra* $\mathbf{N}_{\equiv_{Ja}}(\mathbf{B})$, $J^{\langle a,0\rangle}(x) = \neg\neg x$, *any* x.

Proof From Theorem 77.(1). □

So we have seen that the set $J^{\langle a,0\rangle}(N_{\equiv_{Ja}}(\mathbf{B}))$ forms a subalgebra of $N_{\equiv_{Ja}}(\mathbf{B})$. All the elements of this subalgebra are regular and complemented. The set of complemented elements of an algebra is called *center* of the algebra.

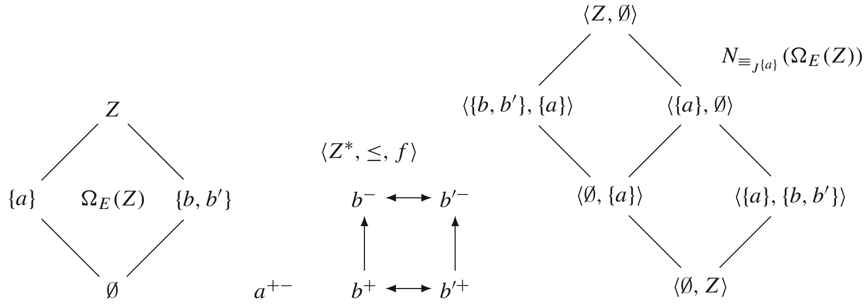
Theorem 121 *Given a semi-simple Nelson algebra, for any element a, if* $a = \neg\neg a$ *then all the three negations are complementations of* a.

Proof From (66) and (76) $\neg\neg\neg a =\lrcorner\neg\neg a =\sim \neg\neg a$. Therefore, from (77) all the three negations are complementations of $\neg\neg a$. □

In view of (76).(i), the above result does not hold for generic elements.

From the above discussion and Definition 6, $\sim\lrcorner = \lrcorner\lrcorner = \neg\lrcorner$ is a necessity operator $\mathbb{L}$ and $\sim\neg = \neg\neg = \lrcorner\neg$ is a possibility operator $\mathbb{M}$.

Example 122 Consider the usual preorder **P** without the elements v and c. Then we obtain an equivalence relation $E = \{\langle a, a\rangle, \langle b, b\rangle, \langle b', b'\rangle, \langle b, b'\rangle, \langle b', b\rangle\}$ on a set $Z = \{a, b, b'\}$. The only isolated element is a. Below we depict the Boolean algebra $\Omega_E(Z)$, the resulting Nelson space and the rough set system $Rsj(\Omega_E(Z))$ as the lattice $N_{\equiv_{J\{a\}}}(\Omega_E(Z))$ without decorations "+":



Notice that $\langle Z^*, \leq, f\rangle$ is not obtained from the dual space of $\Omega_E(Z)$ but from $\langle Z, E\rangle$. In fact, as we have seen, the dual space of $\Omega_E(Z)$ is a T_0-ification in which b and b' collapse into a single point $\{b, b'\}$.

$Dsj(\{a\}) = \langle\{a\}, \{b, b'\}\rangle$. $Dsj(\{a, b\}) = \langle\{a\}, \emptyset\rangle$. Notice that if X is an exact set, that is, $X = (lR)(X) = (uR)(X)$, then $Dsj(X)$ is a regular element of the algebra. The exact sets are $\{a\}$, $\{b, b'\}$, Z and $\emptyset$. Their Dsj-images lay in the center of $N_{\equiv_{J\{a\}}}(\Omega_E(Z))$. Let us verify some cases: $\neg\langle\{a\}, \emptyset\rangle = \langle\emptyset, Z\rangle$. $\langle\{a\}, \emptyset\rangle$ it the least dense element. $\neg\langle\emptyset, \{a\}\rangle = \langle\{a\}, \{b, b'\}\rangle$. $\neg\langle\emptyset, \{a\}\rangle \wedge \langle\emptyset, \{a\}\rangle = \langle\emptyset, Z\rangle$. $\neg\langle\emptyset, \{a\}\rangle \vee \langle\emptyset, \{a\}\rangle = \langle\{a\}, \emptyset\rangle$. $\lrcorner\langle\emptyset, \{a\}\rangle = \langle Z, \emptyset\rangle$. $\lrcorner\langle\emptyset, \{a\}\rangle \wedge \neg\langle\emptyset, \{a\}\rangle\wedge = \langle\emptyset, \{a\}\rangle$. $\neg\neg\langle\{a\}, \emptyset\rangle = \langle Z, \emptyset\rangle$. $\lrcorner\lrcorner\langle\{a\}, \emptyset\rangle = \langle\{a\}, \{b, b'\}\rangle$.

7.7 *Negations and Dual Pseudo-Complementation*

Given a semi-simple Nelson algebra $\mathbf{N}_{\equiv}(\mathbf{B})$, from a Boolean algebra **B**, we know that $a \supset b$ is the pseudo-complement of a relative to b and $\neg a$ the pseudo-complement of a. If we reverse the order of $\supset$ we obtain another operation:

Definition 123 (Pseudo-Supplementation) Let **L** be a bounded lattice, if for all $a, b, x \in \mathbf{L}$ the following holds:

$$a \vee x \geq b \text{ if and only if } x \geq a \subset b \tag{91}$$

then $a \subset b$ is called the *psudosupplement of* a relative to b. The element $a \subset 1$ is called *pseudo-supplement* of a.

Therefore, $a \subset b$ is the least element x such that $a \vee x \geq b$. In other terms, $\subset$ is lower adjoint to $\vee$. In consequence $\vee$ is multiplicative, so that **L** is distributive.

Theorem 124 *Let* **N** *be a semi-simple Nelson algebra, then*

$$a \subset b = \sim (\sim a \supset \sim b) \tag{92}$$

Let us verify that for any a in **N**, $a \subset 1 = \lrcorner a$: $\sim (\sim a \supset \sim b) = \sim (\sim \lrcorner \sim\sim a \vee \sim b \vee (\lrcorner \sim a \wedge \lrcorner \sim\sim b)) = \sim (\sim \lrcorner a \vee \sim b \vee (\lrcorner \sim a \wedge \lrcorner b))$. Therefore, $a \subset 1 = \sim (\sim \lrcorner a \vee 0 \vee (\lrcorner \sim a \wedge 0)) = \sim (\sim \lrcorner a) = \lrcorner a$.

This justifies why $\lrcorner$ and $\neg$ have dual properties, for instance with respect to the De Morgan laws, as we shall see in the next section.

Definition 125 (Co-Heyting Algebras) A bounded distributive lattice **L** such that $a \subset b$ is defined for all $a, b \in$ **L**, is called a *co-Heyting algebra*.

Definition 126 (Bi-Heyting Algebras) A bounded distributive lattice **L** such that it is both a Heyting and a co-Heyting algebra is called a *bi-Heyting algebra*.

Notice that the system of all closed subsets of a topological space is a co-Heyting algebra in which given two closed sets X and Y, $X \subset Y = \mathbb{C}(Y \cap -X)$ and $\lrcorner X = \mathbb{C}(-X) = -\mathbb{I}(X)$. This justifies the definition of the operations for the decreasing representation $Dcr(\Omega_R(U))$ that were provided in the excursus before Sect. 7.5.

Given a co-Heyting algebra, in [22] William Lawvere defined a boundary operation $\delta(a) := a \wedge \lrcorner a$ and pointed out that this operation fulfils the following rule $\delta(a \wedge b) = (\delta(a) \wedge b) \vee (a \wedge \delta(b))$. This rule is consistent with our spatial intuition, if we think of two overlapping sets. Moreover, it is consistent with the Leibniz rule for differentiation of a product: $\frac{d}{dx}(a \cdot b) = \frac{da}{dx} \cdot b + a \cdot \frac{db}{dx}$.

But in Sect. 7.5 we have seen, indeed, that given a semi-simple Nelson algebra $\mathbf{N}_{\equiv_{JS}}(\Omega_R(U))$, with R an equivalence relation, $Dsj(X) \wedge \lrcorner Dsj(X)$ "represents" the boundary of X (see [31]).

We now prove that in a semi-simple Nelson algebra $\mathbf{N}_{\equiv}(\mathbf{B})$ the Leibniz rule holds for $\delta(x) = x \wedge \lrcorner x$:

$$\begin{aligned}
(a \wedge b) \wedge \lrcorner (a \wedge b) &= \langle a_1 \wedge b_1, a_2 \vee b_2 \rangle \wedge \langle \neg(a_1 \wedge b_1), a_1 \wedge b_1 \rangle \\
&= \langle 0, a_2 \vee b_2 \vee (a_1 \wedge b_1) \rangle = \langle 0, (a_2 \vee b_2 \vee a_1) \wedge (a_2 \vee b_2 \vee b_1) \rangle \\
&= \langle 0, a_2 \vee b_2 \vee a_1 \rangle \vee \langle 0, a_2 \vee b_2 \vee b_1 \rangle \\
&= \langle a_1 \wedge \neg a_1 \wedge b_1, a_2 \vee b_2 \vee a_1 \rangle \vee \langle a_1 \wedge b_1 \wedge \neg b_1, a_2 \vee b_2 \vee b_1 \rangle \\
&= (\langle a_1, a_2 \rangle \wedge \langle \neg a_1, a_1 \rangle \wedge \langle b_1, b_2 \rangle) \vee (\langle a_1, a_2 \rangle \wedge \langle b_1, b_2 \rangle \wedge \langle \neg b_1, b_1 \rangle) \\
&= ((a \wedge \lrcorner a) \wedge b) \vee (a \wedge (b \wedge \lrcorner b))
\end{aligned}$$

We conclude the section with these straightforward results on rough set systems:

Theorem 127 *Let* **B** *be a Boolean algebra and* $\equiv$ *a congruence on* **B**. *Then:*

$$\mathbf{H}(\mathbf{B}) := \langle N_{\equiv}(\mathbf{B}), \vee, \wedge, \neg, \supset, 0, 1\rangle \text{ is a Heyting algebra.}$$

$$\mathbf{CH}(\mathbf{B}) := \langle N_{\equiv}(\mathbf{B}), \vee, \wedge, \lrcorner, \subset, 0, 1\rangle \text{ is a co-Heyting algebra.}$$

$$\mathbf{BH}(\mathbf{B}) := \langle N_{\equiv}(\mathbf{B}), \vee, \wedge, \neg, \lrcorner, \supset, \subset, 0, 1\rangle \text{ is a bi-Heyting algebra.}$$

Corollary 128 *Let* $\Omega_R(U)$ *be an approximation space with* R *an equivalence relation. then* $\mathbf{H}(\Omega_R(U))$ *is a Heyting algebra,* $\mathbf{CH}(\Omega_R(U))$ *is a co-Heyting algebra, and* $\mathbf{BH}(\Omega_R(U))$ *is a bi-Heyting algebra.*

7.8 Negations and De Morgan Laws

By definition, both De Morgan laws hold for the strong negation $\sim$. On the contrary, by means of some calculation we obtain:

$$\neg(a \wedge b) = \neg a \vee \neg b, \tag{93}$$

$$\neg(a \vee b) = \langle a_2 \wedge b_2, \neg(a_2 \wedge b_2)\rangle \le \langle a_2 \wedge b_2, \neg a_2 \vee \neg b_2\rangle = \neg a \wedge \neg b \tag{94}$$

$$\lrcorner(a \vee b) = \lrcorner a \wedge \lrcorner b \tag{95}$$

$$\lrcorner(a \wedge b) = \langle \neg(a_1 \wedge b_1), a_1 \wedge b_1\rangle \ge \langle \neg a_1 \vee \neg b_1, a_1 \wedge b_1\rangle \ge \lrcorner a \vee \lrcorner b. \tag{96}$$

The same equalities and disequalities hold for double negated elements, too. For instance, $\neg(\neg\neg a \wedge \neg\neg b) = \neg\neg\neg a \vee \neg\neg\neg b$.

Also, $\neg(\lrcorner\lrcorner a \vee \lrcorner\lrcorner b) \le \neg\lrcorner\lrcorner a \wedge \neg\lrcorner\lrcorner b$ and so on, while $\neg(\lrcorner\lrcorner a \wedge \lrcorner\lrcorner b) = \neg\lrcorner\lrcorner a \vee \neg\lrcorner\lrcorner b$.

Pay attention that the above relations hold for the operations $\neg$ and $\lrcorner$ in a generic Nelson algebra **N**. On the contrary, if $\neg$ is the pseudo-complementation of **N** *qua* Heyting algebra (for instance if **N** is a finite Nelson lattice), then $\neg(a \wedge b) \ge \neg a \vee \neg b$, while $\neg(a \vee b) = \neg a \wedge \neg b$. Symmetrically, if $\lrcorner$ is the dual-pseudocomplementation in the co-Heyting algebra $\mathbf{N}^{op}$ then $\lrcorner(a \vee b) \le \lrcorner a \wedge \lrcorner b$ and $\lrcorner(a \wedge b) = \lrcorner a \vee \lrcorner b$.

If the Nelson algebra is semi-simple, things change sensibly. In fact, in this case the Nelson operator $\neg$ is really a pseudo complementation and $\lrcorner$ a co-pseudocomplementation. Clearly, we expect that the De Morgan law for Heyting algebras and co-Heyting algebras hold:

$$\neg(a \vee b) = \neg a \wedge \neg b \text{ and } \lrcorner(a \wedge b) = \lrcorner a \vee \lrcorner b.$$

It holds because the same is valid in the underlying Boolean algebra.

The law (93) suggests that semi-simple Nelson algebras can be made into Heyting algebras with very peculiar properties, because that law does not hold in general for pseudo-complementation. Symmetrically for their co-Heyting algebras. Indeed, in [43] it was proved that if (93) and (95) hold then $\lrcorner\neg$ and $\neg\lrcorner$ are idempotent operators and $\lrcorner\neg a$ is the least complemented element above a, while $\neg\lrcorner a$ is the largest complemented element below a.

In fact we have seen in Theorem 117 that in semi-simple Nelson algebras, $\lrcorner\neg = \neg\neg =\sim \neg$ and $\neg\lrcorner =\lrcorner\lrcorner =\sim\lrcorner$ correspond to topological modal (possibility) and, respectively, topological co-modal (necessity) operators which project an element a onto the sublattice of regular elements which is, also, the sublattice of complemented elements of the algebra.

Actually, another reason why the above De Morgan laws hold in semi-simple Nelson algebras is a general result by Johnstone (see [20]): in a Heyting algebra **H**, the De Morgan law (93) is equivalent to the fact that the set of regular elements form a sublattice of **H**. And in Theorem 118 it was proved that this is the case for semi-simple Nelson algebras, indeed.

7.9 *Changing Information and Changing Logic*

We have mentioned that semi-simple Nelson algebras are equivalent to three-valued Łukasiewicz algebras. Now we enter some details.

Definition 129 (Łukasiewicz Algebra) A *three-valued Łukasiewicz algebra* is a distributive lattice $\langle A, \vee, \wedge, \sim, 0, 1\rangle$ with two additive and multiplicative unary operations φ_1, φ_2 satisfying:

$$\varphi_1(x) \geq \varphi_2(x),\ \ \varphi_i(x) \vee \sim \varphi_i(x) = 1,\ \ \varphi_i(x) \wedge \sim \varphi_i(x) = 0,\ \ \varphi_i(\sim x) = \sim \varphi_i(x)$$

$$\varphi_i(\varphi_j(x)) = \varphi_j(x),\ \ x \vee \varphi_1 = \varphi_1(x),\ \ x \wedge \varphi_2 = \varphi_2(x),\ \ \varphi_i(0) = 0,\ \ \varphi_i(1) = 1$$

$$\sim x \wedge \varphi_2(x) = 0,\ \ \sim x \vee \varphi_1(x) = 1,\ \ y \wedge (x \vee \sim \varphi_1(x) \vee \varphi_2(y)) = y.$$

It is possible to prove (see [39] or [32]):

Theorem 130 *Let* **B** *be a Boolean algebra and* $\equiv$ *a congruence on* **B**. *Then:* $\mathbf{L}(\mathbf{B}) := \langle N_{\equiv}(\mathbf{B}), \vee, \wedge, \sim, \varphi_1, \varphi_2, 0, 1\rangle$ *is a three-valued Łukasiewicz algebra, where* $\varphi_1 = \lrcorner\neg = \sim\neg = \neg\neg$ *and* $\varphi_2 = \neg\lrcorner = \sim\lrcorner = \lrcorner\lrcorner$.

Corollary 131 *Let* $\Omega_R(U)$ *be an approximation space with* R *an equivalence relation. Then,* $\mathbf{L}(\Omega_R(U))$ *is a three-valued Łukasiewicz algebra.*

See Example 122.

It is interesting to note that our relative pseudo-complementation $\supset$ of Theorem 115 coincides with the so-called *Moisil residuation* $\sqsupset$ which is definable in three-valued Łukasiewicz algebras: $a \sqsupset b := b \vee \sim \varphi_1(a) \vee (\sim \varphi_2(a) \wedge \varphi_1(b))$.

Semi-simple Nelson algebras or three-valued Łukasiewicz algebras represent the case in which the filtration congruence $\equiv$ is generic, that is, $\equiv$ is $\equiv_{J^a}$ for $0 \leq a \leq 1$. What happens in the extreme cases, that is, when $a = 1$ and $a = 0$?

We have seen that $\mathbf{N}_{\equiv_{J^1}}(\Omega_R(U))$ is a Boolean algebra isomorphic to $\Omega_R(U)$. If, instead, $a = 0$, then we obtain a Post algebra of order three.

Definition 132 (Post Algebra) A *Post algebra of order three* is a Heyting algebra $\langle A, \vee, \wedge, \Longrightarrow, \neg, 0, 1\rangle$ equipped with a three chain element $0 = e_0 \leq e_1 \leq e_2 = 1$ and two unary multiplicative and additive operators D_1, D_2 such that, for $1 \leq i, j \leq 2$:

$$D_1(x) \vee \neg D_1(x) = 1, \;\; D_i(\neg x) = \neg D_i(x), \;\; D_i(D_j(x)) = D_j(x)$$

$$x = (D_1(x) \wedge e_1) \vee (D_2(x) \wedge e_2) \text{ - monotonic representation of } x$$

$$D_i(x \Longrightarrow y) = (D_1(x) \Longrightarrow D_1(y)) \wedge (D_2(x) \Longrightarrow D_2(y))$$

$$D_i(e_j) = \begin{cases} 1 & \text{for } 1 \leq i \leq j \leq 2 \\ 0 & \text{for } 2 \geq i \gneq j \geq 0 \end{cases}$$

Let then $\mathbf{P} = \langle A, \vee, \wedge, \Longrightarrow, \neg, e_0, e_1, e_2, D_1, D_2, 0, 1\rangle$. Since $D_i(x) \vee \neg D_i(x) = (D_1(D_i(x)) \vee \neg D_1(D_i(x)) = 1$, it is evident that for any x, $D_i(x)$ belongs to the centre of $\mathbf{P}$.

Post algebras of order three are special cases of three-valued Łukasiewicz algebras. In fact, if a Łukasiewicz algebra $\mathbf{L} = \langle A, \vee, \wedge, \sim, \varphi_1, \varphi_2, 0, 1\rangle$ has a chain $0 \leq \delta \leq 1$, by setting $D_1 = \varphi_1$, $D_2 = \varphi_2$, $\neg x =\sim D_1(x)$, one obtains a Post algebra of order three (see [32]).

Notice that, indeed, $\sim D_1(x) =\sim \varphi_1(x) =\sim\sim \neg x = \neg x$.

Now we exhibit a Post algebra of ordered pairs of disjoint elements (see [32]). We have just noticed that $D_i(x)$ is complemented. Moreover, $D_1(x) =\sim \neg(x)$. In view of (65) and Theorem 121, this suggests that the underlying algebra is Boolean. Therefore, let $\mathbf{B}$ be a Boolean algebra and $\equiv$ be the largest congruence on $\mathbf{B}$. Let $N_{\equiv}(\mathbf{B})$ be a set of ordered pairs of disjoint elements of $\mathbf{B}$. We have already seen that since $\equiv$ is the largest congruence on $\mathbf{B}$, $1 \equiv 0$ so that any pair of disjoint elements of $\mathbf{B}$ is admitted by the filtration rule $a_1 \vee a_2 \equiv 1$, hence also $\langle 0, 0\rangle$ is. We know that $N_{\equiv}(\mathbf{B})$ can be made into a three-valued Łukasiewicz algebra and how to transform it into a Post algebra of order three. We only need a chain of values. Obviously, $e_0 = \langle 0, 1\rangle$ and $e_2 = \langle 1, 0\rangle$. We claim that e_1 is $\langle 0, 0\rangle$. Clearly, $\langle 0, 1\rangle \leq \langle 0, 0\rangle \leq \langle 1, 0\rangle$. Moreover $D_1(\langle 0, 0\rangle) =\sim \neg\langle 0, 0\rangle =\sim \langle 0, 1\rangle = 1$ and $D_2(\langle 0, 0\rangle) =\sim\lrcorner\langle 0, 0\rangle =\sim \langle 1, 0\rangle = 0$.

Now, the largest congruence on **B** is $\equiv_{J0}$. So, we obtain:

Theorem 133 *Let* **B** *be a Boolean algebra, then*

$$\mathbf{P}(\mathbf{B}) := \langle N_{\equiv_{J0}}(\mathbf{B}), \vee, \wedge, \neg, \supset, \sim\neg, D_1, D_2, e_0, e_1, e_2\rangle$$

is a Post algebra of order three, if $D_1 = \sim\neg$*,* $D_2 = \sim\lrcorner$*,* $e_0 = \langle\emptyset, U\rangle$*,* $e_1 = \langle\emptyset, \emptyset\rangle$ *and* $e_2 = \langle U, \emptyset\rangle$*.*

From the rough set perspective, given an approximation space $\Omega_R(U)$ with R an equivalence relation, we obtain a Post algebra of order three if and only if there are no isolated elements:

Theorem 134 *Let R be an equivalence relation on a set U such that there are no isolated elements, then* $N_{?\equiv_{J0}}(\Omega_R(U)) = Dsj(\Omega_R(U))$ *and*

$$\mathbf{P}(\Omega_R(U)) := \langle N_{\equiv_{J\emptyset}}(\Omega_R(U)), \vee, \wedge, \neg, \supset, \sim\neg, D_1, D_2, e_0, e_1, e_2\rangle$$

is a Post algebra of order three.

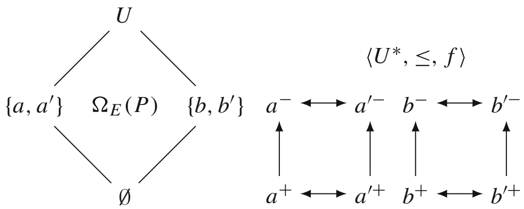
We have noticed that the intermediate value of the above Post algebra is $\langle 0, 0\rangle$ which is the least dense element of the algebra. And we have also noticed that given a Boolean algebra **B** and $a \in \mathbf{B}$, $\langle a, 0\rangle$ is the least dense element in the lattice $\langle N_{\equiv_{Ja}}(\mathbf{B}), \vee, \wedge, \neg, \mathbf{0}, \mathbf{1}\rangle$, therefore also in $\mathbf{L}(\mathbf{B})$ we can set a chain $0 = \langle 0, 1\rangle = e_0 \le e_1 = \langle a, 0\rangle \le e_2 = \langle 1, 0\rangle = 1$.

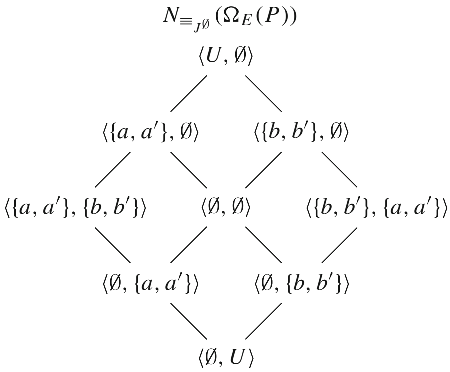
However, $D_1(e_1) = \varphi_1(e_1) = \sim\neg\langle a, 0\rangle = \sim\langle 0, 1\rangle = 1$, which is consistent with Post algebras, but $D_2(e_1) = \sim\lrcorner e_1 = \sim\langle\neg a, a\rangle = \langle a, \neg a\rangle \ge \langle 0, 1\rangle = 0$, while in Post algebras $D_2(e_1) = D_2(e_0) = 0$.

Actually, if one assumes $\langle a, 0\rangle$ as intermediate value, one obtains another kind of lattices, called *chain-based lattices*, namely P_2*-lattices* which are generalisations of Post algebras (see [13]).

Example 135 Let $P = \{a, a', b, b'\}$ and E be the equivalence relation depicted below together with the Boolean algebra $\Omega_E(P)$, the resulting Nelson space and $Dsj(\Omega_E(P))$ as the Post algebra of order three $N_{\equiv_{J\emptyset}}(\Omega_E(P))$ without decorations '+'. For instance, $Dsj(\{a, b\} = \langle\emptyset, \emptyset\rangle$, $Dsj(\{a, a', b\}) = \langle\{a, a'\}, \emptyset\rangle$, $Dsj(\{a, a'\}) = \langle\{a, a'\}, \{b, b'\}\rangle$.

E	a	a′	b	b′
a	1	1	0	0
a′	1	1	0	0
b	0	0	1	1
b′	0	0	1	1

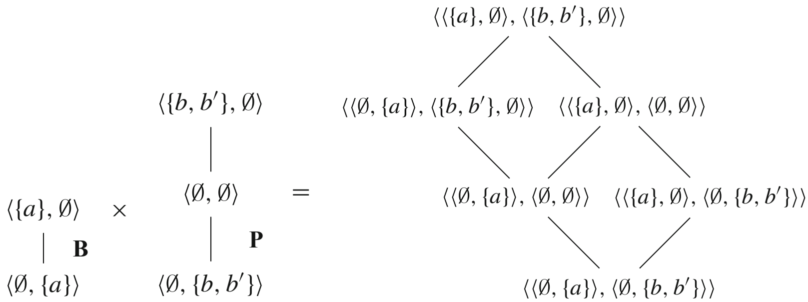




8 Conclusions

In many respects, the logic of rough sets is still to be defined. In the case of classic rough sets based on equivalence relations, we have seen that their logic depends on the geometry of isolated points. In other terms, it depends on the set of items completely describable by the given properties, that is, items which are singled out by the properties, or information, we have. If all the items can be isolated by the properties, then we obtain a Boolean algebra. This is no surprise if we think of Classic Logic as the logic of perfect information: either α or $\neg\alpha$. If some pieces of information are complete and other are incomplete, and we gather the completely describable items into a set S, then we obtain a semi-simple Nelson algebra, i.e. a three-valued Łukasiewicz algebra, in which the pair $\langle S, \emptyset\rangle$ is the least dense element and a *local top element*, in the sense that classical tautologies takes values between $\langle S, \emptyset\rangle$ and the absolute top element $\langle U, \emptyset\rangle$. Vice-versa, all classical contradictions are between $\sim \langle S, \emptyset\rangle = \langle \emptyset, S\rangle$ and the absolute bottom element $\langle \emptyset, U\rangle$, so that $\langle \emptyset, S\rangle$ is a *local bottom element*. If no items are completely described, then $S = \emptyset$ and rough set systems turns into Post algebras of order three, where the local top and bottom elements fuse into a state $\langle \emptyset, \emptyset\rangle$ of *complete indecision* or *totally uninformed* situation. Since it is often assumed that there are no isolated points, or completely described items, then the logic of rough sets should be the one modelled by Post algebras of order three, not three-valued Łukasiewicz logic or connected mathematical objects (regular Stone algebras and the like).

However, in our opinion, the real world is a melange of perfect and imperfect information. Then three-valued Łukasiewicz logic, or Constructive Logic with Strong Negation plus an axiom for $a \vee \lrcorner a = 1$ approximate the intrinsic logic of rough sets. But they are not able to account for the double nature of perfect and imperfect information which is implicit in these algebraic models as it is shown by the fact that any such algebra is the product of a Post algebra of order three, modelling the imperfect part, and a Boolean algebra modelling the perfect part. Look at the Łukasiewicz algebra of Example 122. It is the product of the Boolean algebra **B** whose (in this case only) atom is $\{a\}$ and a Post algebra **P** with elements built on the indiscernible elements b and b':



Notice that the product of the least (only) dense element of **B**, that is, the top element $\langle\{a\}, \emptyset\rangle$ and the least dense element of **P**, which is the intermediate value $\langle\emptyset, \emptyset\rangle$, gives the least dense element of the resulting three-valued Łukasiewicz algebra.

Finally, if an approximation space is induced by a partial or pre-order bounded by maximal states, then the intrinsic logic of the rough set system is E_0. In particular all the usual approximation spaces induced by a finite partial order are of this kind.

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Irredundant Coverings, Tolerances, and Related Algebras

Jouni Järvinen and Sándor Radeleczki

Abstract This chapter deals with rough approximations defined by tolerance relations that represent similarities between the elements of a given universe of discourse. We consider especially tolerances induced by irredundant coverings of the universe U. This is natural in view of Pawlak's original theory of rough sets defined by equivalence relations: any equivalence E on U is induced by the partition U/E of U into equivalence classes, and U/E is a special irredundant covering of U in which the blocks are disjoint. Here equivalence classes are replaced by tolerance blocks which are maximal sets in which all elements are similar to each other. The blocks of a tolerance R on U always form a covering of U which induces R, but this covering is not necessarily irredundant and its blocks may intersect. In this chapter we consider the semantics of tolerances in rough sets, and in particular the algebraic structures formed by the rough approximations and rough sets defined by different types of tolerances.

1 Tolerances, Information Systems, and Rough Approximations

In this section, we show that tolerances can be used for representing information about objects. We also consider incomplete information systems and tolerances determined by them. We define rough approximations and study their properties. In particular, we concentrate on the structures of the ordered sets of lower and upper approximations and show that they form ortholattices.

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1.1 Knowledge Representation and Tolerances

Knowledge about objects may be represented as binary relations. For instance, if we classify all human beings into disjoint sets based on their place of birth, then this classification determines a binary relation R by setting $x \, R \, y$ whenever x and y are born in the same place. This relation is *reflexive*, that is, $x \, R \, x$ for all human beings x. It is also *symmetric*: if $x \, R \, y$, then x and y were born in the same place and $y \, R \, x$ holds. The relation is *transitive*, because if $x \, R \, y$ and $y \, R \, z$, then x, y, and z are all born in the same place and also $x \, R \, z$ is true.

Originally Z. Pawlak defined rough approximations in terms of *equivalence relations* [16], which are reflexive, symmetric, and transitive relations. Pawlak considered equivalences as *indistinguishability relations*: two objects are equivalent if we cannot distinguish them by using the given information. Reflexivity is a natural property of indistinguishability, because each object is indistinguishable from itself. We may also assume that indistinguishability is symmetric: if x is indistinguishable from y, then y is indistinguishable from x. Transitivity is the most controversial property of indistinguishability: we may have a finite sequence of objects $x_1, x_2, \ldots, x_n$ such that each two consecutive objects x_i and x_{i+1} are indistinguishable, but x_1 and x_n are very different from each other. The reason for this is that the difference between x_i and x_{i+1} is so small that it cannot be perceived, but if we go far enough in the chain of indistinguishable objects, we have a clear difference. For instance, if we compare photographs of a person's face, then the photographs taken on consecutive days should not differ much from each other. However, the pictures that are taken with separation of 10 years certainly look different.

In this chapter, we concentrate on cases in which the information about objects is given by a relation which is reflexive and symmetric, but not necessarily transitive. Such a relation is called a *tolerance relation*. The term tolerance relation was introduced in the context of visual perception theory by Zeeman [22], motivated by the fact that indistinguishability of "points" in the visual world is limited by the discreteness of retinal receptors. We view tolerances as *similarity* relations.

Tolerances correspond to simple graphs. A *simple graph* is an undirected graph that has no loops (edges connected at both ends to the same vertex) and no multiple edges. Any tolerance R on U determines a graph $\mathcal{G} = (U, R)$, where U is interpreted as the set of vertices and R as the set of edges. There is a line connecting x and y if and only if $x \, R \, y$. Because each point is R-related to itself, loops connecting a point to itself are not drawn.

Example 1 Assume that $U = \{1, 2, 3, 4, 5, 6\}$. Let R be a tolerance depicted by the graph $\mathcal{G} = (U, R)$ of Fig. 1.

Now, for example, $(1, 4), (4, 1) \in R$ and $(2, 5), (5, 2) \in R$, because there is an edge connecting the points 1 and 4, and the points 2 and 5. The elements 1 and 2 are not R-related, because there is no edge connecting them.

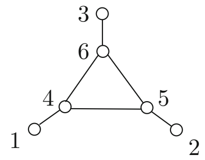


Fig. 1 A graph $\mathscr{G} = (U, R)$

Information systems were introduced by Pawlak in [15]. An *information system* is a triple $\mathscr{S} = (U, A, \{V_a\}_{a \in A})$, where U is a nonempty set of *objects*, A is a nonempty set of *attributes*, and $\{V_a\}_{a \in A}$ is an A-indexed family of sets of *attribute values*. Each attribute $a \in A$ is a function $a \colon U \to V_a$. Usually the sets U, A, and V_a are assumed to be finite, which is often a natural assumption. However, in general we do not assume anything about the cardinalities of these sets.

Let $\mathscr{S} = (U, A, \{V_a\}_{a \in A})$ be an information system. An *indistinguishability relation* can be defined for any $B \subseteq A$ by setting

$$IND_B = \{(x, y) \in U \times U \mid a(x) = a(y) \text{ for all } a \in B\}.$$

This means that two objects are B-indistinguishable if and only if their values for all the attributes in B are equal. It is obvious that IND_B is an equivalence for any $B \subseteq A$.

In real-world situations, some attribute values for some objects may be undefined or unknown. Data may be missing for several reasons, but they do not concern us. In [12] these *null values* are marked by $*$. This kind of information systems are called *incomplete information systems*. For each $B \subseteq A$, the following relation is defined:

$$SIM_B = \{(x, y) \in U \times U \mid (\forall a \in B)\, a(x) = a(y) \text{ or } a(x) = * \text{ or } a(y) = *\}.$$

For any $B \subseteq A$, SIM_B is a tolerance on U such that $IND_B \subseteq SIM_B$. For each attribute $a \in A$, let us denote $SIM_{\{a\}}$ simply by SIM_a. It is clear that

$$SIM_B = \bigcap_{a \in B} SIM_a.$$

Example 2 An information system $\mathscr{S}$ in which the sets U and A are finite can be represented by a table. The rows of the table are labelled by the objects and the columns by the attributes of the system $\mathscr{S}$. In the intersection of the row labelled by an object x and the column labelled by an attribute a we find the value $a(x)$.

Let us consider an information system $\mathscr{S} = (U, A, \{V_a\}_{a \in A})$, where the object set $U = \{1, 2, 3, 4, 5\}$ consists of five persons called 1, 2, 3, 4 and 5, respectively. The attribute set A has the attributes Age, Eyes, and Height. Let the values of attributes be defined as in Table 1.

The tolerance SIM_A defined in the information system $\mathscr{S}$ is depicted in Fig. 2.

Table 1 A simple incomplete information system

	Age	Eyes	Height
1	Young	*	*
2	Middle-aged	Brown	Tall
3	Young	Blue	Short
4	*	Brown	Tall
5	Young	Brown	Tall

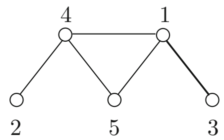


Fig. 2 The tolerance SIM_A on $\{1, 2, 3, 4, 5\}$

1.2 Rough Approximations Defined by Tolerances

We begin by defining the rough set approximations based on an arbitrary tolerance R on a universe U. For any $x \in U$, we denote

$$R(x) = \{y \in U \mid x \; R \; y\}.$$

The set $R(x)$ is called the *R-neighbourhood* of x. It consists of the elements that are similar to x in view of the knowledge R. Consider any $X \subseteq U$ and let X^c denote the complement $U \setminus X = \{x \in U \mid x \notin X\}$ in U. For any $x \in U$, we have three possibilities:

(N1) $R(x) \subseteq X$: These elements x are certainly in X in view of the knowledge R, because all elements that are similar to x are in X.

(N2) $R(x) \cap X = \emptyset$, that is, $R(x) \subseteq X^c$: These are the elements x which certainly are not in X, because their R-neighbourhood is totally outside X.

(N3) $R(x) \cap X \neq \emptyset$ and $R(x) \cap X^c \neq \emptyset$: These elements x are such that their belonging to X cannot be decided by the means of the knowledge R; both in X and outside X there are elements which are similar to x.

Example 3 The three kinds of elements with respect to set $X \subseteq U$ are depicted in Fig. 3. Element x belongs certainly to X, element y is certainly not in X, and z is such that its belonging to X cannot be decided in view of the knowledge R.

Next we define the rough approximation operators. Let R be a tolerance on a set U. The *upper approximation* of a set $X \subseteq U$ is

$$X^{\blacktriangle} = \{x \in U \mid R(x) \cap X \neq \emptyset\}$$

and the *lower approximation* of X is

$$X^{\blacktriangledown} = \{x \in U \mid R(x) \subseteq X\}.$$

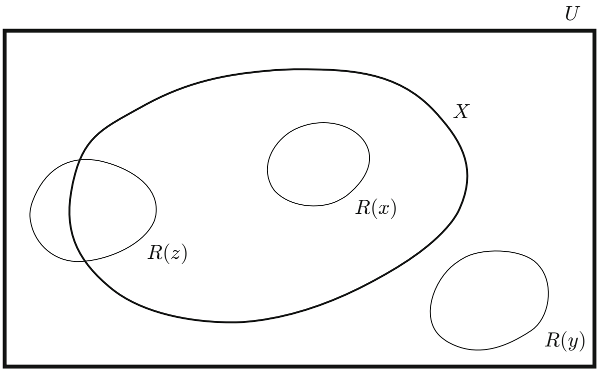


Fig. 3 Three kinds of elements with respect to set $X \subseteq U$

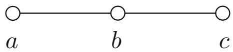


Fig. 4 The graph of a tolerance R is a 3-element chain

Thus the lower approximation $X^{\blacktriangledown}$ consists of the elements of type (N1) and the upper approximation $X^{\blacktriangle}$ of contains the objects of type (N1) and (N3). The *boundary* $B(X) := X^{\blacktriangle} \setminus X^{\blacktriangledown}$ is the actual area of uncertainty; it consists of the elements of type (N3). Note that if $x \in B(X)$, then $|R(x)| \geq 2$. The set $X^{\blacktriangle c}$ contains the objects of type (N2).

Example 4 Let $U = \{a, b, c\}$ and let R be the tolerance on U defined in Fig. 4. The lower and upper approximations of subsets of U are given in Table 2.

In the following are listed the basic properties of rough approximations defined by tolerances. Notice that $X^{\blacktriangle\blacktriangledown}$ denotes $(X^{\blacktriangle})^{\blacktriangledown}$ and a similar convention is used in this chapter for combinations of different mappings. The proofs of these claims are easy to verify, and they can also be found in [6, 7].

Table 2 All lower and upper approximations

X	$X^{\blacktriangledown}$	$X^{\blacktriangle}$
$\emptyset$	$\emptyset$	$\emptyset$
$\{a\}$	$\emptyset$	$\{a, b\}$
$\{b\}$	$\emptyset$	U
$\{c\}$	$\emptyset$	$\{b, c\}$
$\{a, b\}$	$\{a\}$	U
$\{a, c\}$	$\emptyset$	U
$\{b, c\}$	$\{c\}$	U
U	U	U

Proposition 5 *Let R is a tolerance on U and $X, Y \subseteq U$.*

(a) $\emptyset^{\blacktriangledown} = \emptyset^{\blacktriangle} = \emptyset$ *and* $U^{\blacktriangledown} = U^{\blacktriangle} = U$*;*
(b) $X^{\blacktriangledown} \subseteq X \subseteq X^{\blacktriangle}$*;*
(c) $X^{\blacktriangledown\blacktriangle} \subseteq X \subseteq X^{\blacktriangle\blacktriangledown}$*;*
(d) $X \subseteq Y$ *implies* $X^{\blacktriangledown} \subseteq Y^{\blacktriangledown}$ *and* $X^{\blacktriangle} \subseteq Y^{\blacktriangle}$*;*
(e) $(X \cup Y)^{\blacktriangle} = X^{\blacktriangle} \cup Y^{\blacktriangle}$ *and* $(X \cap Y)^{\blacktriangledown} = X^{\blacktriangledown} \cap Y^{\blacktriangledown}$*;*
(f) $X^{\blacktriangledown c} = X^{c\blacktriangle}$ *and* $X^{\blacktriangle c} = X^{c\blacktriangledown}$*;*
(g) $B(X) = B(X^c)$*;*
(h) $X^{\blacktriangle\blacktriangledown\blacktriangle} = X^{\blacktriangle}$ *and* $X^{\blacktriangledown\blacktriangle\blacktriangledown} = X^{\blacktriangledown}$.

Remark 6 Let us make some observations concerning Proposition 5. Item (a) says that an element belongs neither certainly nor possibly to the empty set and that every element belongs possibly and certainly to the whole universe.

Statement (b) says that if an element belongs certainly to X in view of the knowledge R, it must be in X. Further, if an element belongs to X, it belongs also possibly to X in the view of knowledge R. This property follows from the relation R being reflexive.

In (c), $X \subseteq X^{\blacktriangle\blacktriangledown}$ says that if $x \in X$, then $R(x) \subseteq X^{\blacktriangle}$. This means that if x belongs to X, then the elements R-related to x are possibly in X. Similarly, if $x \in X^{\blacktriangledown\blacktriangle}$, then x is R-related to some element in $X^{\blacktriangledown}$. But $X^{\blacktriangledown}$ consists of such elements that all element R-related to them are in X. Therefore, also x must be in X. Note that this condition holds since R is symmetric.

Assertion (d) says simply that if all elements of X are in Y, then all elements which are certainly (resp. possibly) in X are also certainly (resp. possibly) in Y.

By (e), the set of elements which are possibly in the union of X and Y equals the set of elements which are possibly in X or possibly in Y. Similarly, the elements which certainly are in the intersection of X and Y are those elements which are certainly in X and certainly in Y. Note that $X^{\blacktriangledown} \cup Y^{\blacktriangledown} \neq (X \cup Y)^{\blacktriangledown}$ may hold. For instance, consider the tolerance of Example 4. If $X = \{a\}$ and $Y = \{b\}$, then $X^{\blacktriangledown} = \emptyset$ and $Y^{\blacktriangledown} = \emptyset$, but $(X \cup Y)^{\blacktriangledown} = \{a, b\}^{\blacktriangledown} = \{a\}$. Additionally, $X^{\blacktriangle} \cap Y^{\blacktriangle} \neq (X \cap Y)^{\blacktriangle}$ for these particular sets X and Y.

Claim (f) says that the operators $^{\blacktriangle}$ and $^{\blacktriangledown}$ are mutually dual. Statement (g) says that if we cannot decide whether an element is in X, we cannot decide whether it is in the set-complement X^c of X either. This natural property follows from (f).

The equalities in (h) are consequences of (c) and (d). Indeed, since $X \subseteq X^{\blacktriangle\blacktriangledown}$, we have $X^{\blacktriangle} \subseteq X^{\blacktriangle\blacktriangledown\blacktriangle}$ by (d). For $X^{\blacktriangle}$, (c) gives $X^{\blacktriangle} \supseteq (X^{\blacktriangle})^{\blacktriangledown\blacktriangle}$. The other part behaves in a similar manner.

Let us denote by $\wp(U)^{\blacktriangledown}$ the set of all lower approximations and by $\wp(U)^{\blacktriangle}$ the set of all upper approximations, that is,

$$\wp(U)^{\blacktriangledown} = \{X^{\blacktriangledown} \mid X \subseteq U\} \qquad \text{and} \qquad \wp(U)^{\blacktriangle} = \{X^{\blacktriangle} \mid X \subseteq U\}.$$

Next we present some lattice-theoretical properties of $\wp(U)^{\blacktriangledown}$ and $\wp(U)^{\blacktriangle}$. For that we need to recall some definitions from the literature [2, 4].

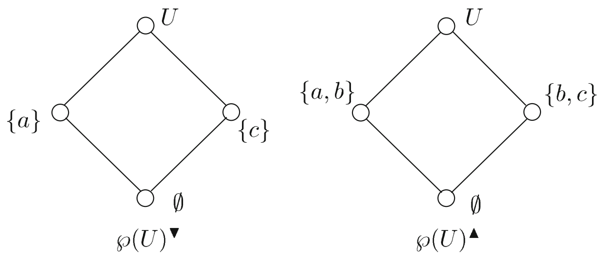


Fig. 5 The ordered sets $\wp(U)^{\blacktriangledown}$ and $\wp(U)^{\blacktriangle}$

An *order* (or a *partial order*) on a set P is a binary relation $\leq$ such that, for all $a, b, c \in P$, (1) $a \leq a$, (2) $a \leq b$ and $b \leq a$ imply $a = b$, (3) $a \leq b$ and $b \leq c$ imply $a \leq c$, that is, the relation $\leq$ is reflexive, antisymmetric, and transitive. A set P equipped with an order relation $\leq$ is called an *ordered set* (or a *partially ordered set*). We usually denote an ordered set $(P, \leq)$ simply by P.

Let a and b be elements of an ordered set P. We say that a is *covered by* b (or that b *covers* a), and write $a \prec b$, if $a < b$ and $a \leq c < b$ implies $a = c$. Every finite ordered set can be drawn by using its covering relation $\prec$. The *Hasse diagram* of an ordered set P represents the elements of P with circles, and the circles representing two elements a and b are connected by a line if $a \prec b$ or $b \prec a$. If a is covered by b, the circle representing a is below the circle representing b.

It is clear that for any tolerance R on U, the lower approximations and upper approximations form ordered sets with respect to the set-inclusion order, that is, $(\wp(U)^{\blacktriangledown}, \subseteq)$ and $(\wp(U)^{\blacktriangle}, \subseteq)$ are ordered sets. We usually denote these ordered sets simply by $\wp(U)^{\blacktriangledown}$ and $\wp(U)^{\blacktriangle}$.

Example 7 Let R be the tolerance on Example 4. The Hasse diagrams of $\wp(U)^{\blacktriangledown}$ and $\wp(U)^{\blacktriangle}$ are presented in Fig. 5.

If P and Q are ordered sets, then a mapping $\varphi\colon P \to Q$ is an *order-embedding*, if $a \leq b$ in P if and only if $\varphi(a) \leq \varphi(b)$ in Q. Note that an order-embedding is always an injection, because if $\varphi(a) = \varphi(b)$, then $\varphi(a) \leq \varphi(b)$ and $\varphi(a) \geq \varphi(b)$, which imply $a \leq b$ and $a \geq b$, that is, $a = b$. An *order-isomorphism* is a surjective order-embedding. When there exists an order-isomorphism from P to Q, we say that P and Q are *order-isomorphic* and write $P \cong Q$. Since any order-isomorphism $\varphi\colon P \to Q$ is a bijection, φ has an inverse mapping φ^{-1} which also is an order-isomorphism $\varphi^{-1}\colon Q \to P$.

Proposition 8 *If R is a tolerance on U, then $\varphi\colon X^{\blacktriangledown} \mapsto X^{\blacktriangledown\blacktriangle}$ defines an order-isomorphism between $\wp(U)^{\blacktriangledown}$ and $\wp(U)^{\blacktriangle}$.*

Proof Let $X, Y \subseteq U$. If $X^{\blacktriangledown} \subseteq Y^{\blacktriangledown}$, then $\varphi(X^{\blacktriangledown}) = X^{\blacktriangledown\blacktriangle} \subseteq Y^{\blacktriangledown\blacktriangle} = \varphi(Y^{\blacktriangledown})$. Conversely, if $\varphi(X^{\blacktriangledown}) = X^{\blacktriangledown\blacktriangle} \subseteq Y^{\blacktriangledown\blacktriangle} = \varphi(Y^{\blacktriangledown})$, then $X^{\blacktriangledown} = X^{\blacktriangledown\blacktriangle\blacktriangledown} \subseteq Y^{\blacktriangledown\blacktriangle\blacktriangledown} = Y^{\blacktriangledown}$. Thus, φ is an order-embedding.

If $Z^\blacktriangle \in \wp(U)^\blacktriangle$, then $Z^{\blacktriangle\blacktriangledown} \in \wp(U)^\blacktriangledown$ and $\varphi(Z^{\blacktriangle\blacktriangledown}) = Z^{\blacktriangle\blacktriangledown\blacktriangle} = Z^\blacktriangle$. This means that φ is also surjective. □

Let us note that if R is an equivalence on U, then $X^{\blacktriangle\blacktriangledown} = X^\blacktriangle$ and $X^{\blacktriangledown\blacktriangle} = X^\blacktriangledown$ for all $X \subseteq U$. This implies that $\wp(U)^\blacktriangledown = \wp(U)^\blacktriangle$.

A family $\mathscr{L}$ of subsets of U is a *closure system* on U if $\mathscr{L}$ is closed under arbitrary intersections. In particular, $\mathscr{L}$ always contains $U = \bigcap \emptyset$. A map $\mathscr{C} \colon \wp(U) \to \wp(U)$ is a *closure operator* on U if for any $X, Y \subseteq U$:

(1) $X \subseteq \mathscr{C}(X)$ (extensive)
(2) $\mathscr{C}(\mathscr{C}(X)) = \mathscr{C}(X)$ (idempotent)
(3) $X \subseteq Y$ implies $\mathscr{C}(X) \subseteq \mathscr{C}(Y)$ (order-preserving)

A subset X of U is *closed* (with respect to $\mathscr{C}$) if $\mathscr{C}(X) = X$.

A closure system $\mathscr{L}$ on U defines a closure operator $\mathscr{C}_\mathscr{L}$ on U by the rule

$$\mathscr{C}_\mathscr{L}(B) = \bigcap \{L \in \mathscr{L} \mid B \subseteq L\}.$$

Conversely, if $\mathscr{C}$ is a closure operator on U, then the family

$$\mathscr{L}_\mathscr{C} = \{B \subseteq U \mid \mathscr{C}(B) = B\}$$

of $\mathscr{C}$-closed subsets of U is a closure system. The relationship between closure systems and closure operators is bijective.

Lemma 9 *Let R be a tolerance on U. The family of sets $\wp(U)^\blacktriangledown$ is a closure system on U.*

Proof Consider $\{X^\blacktriangledown \mid X \in \mathscr{H}\} \subseteq \wp(U)^\blacktriangledown$ for some $\mathscr{H} \subseteq \wp(U)$. We show that

$$\bigcap_{X \in \mathscr{H}} X^\blacktriangledown = \left(\bigcap \mathscr{H}\right)^\blacktriangledown,$$

which means that $\bigcap \{X^\blacktriangledown \mid X \in \mathscr{H}\}$ belongs to $\wp(U)^\blacktriangledown$. For every $X \in \mathscr{H}$, $\left(\bigcap \mathscr{H}\right)^\blacktriangledown \subseteq X^\blacktriangledown$ because $\bigcap \mathscr{H} \subseteq X$. Therefore,

$$\left(\bigcap \mathscr{H}\right)^\blacktriangledown \subseteq \bigcap_{X \in \mathscr{H}} X^\blacktriangledown.$$

On the other hand, if $x \in \bigcap \{X^\blacktriangledown \mid X \in \mathscr{H}\}$, then $x \in X^\blacktriangledown$ and $R(x) \subseteq X$ for all $X \in \mathscr{H}$. Thus, $R(x) \subseteq \bigcap \mathscr{H}$ and $x \in \left(\bigcap \mathscr{H}\right)^\blacktriangledown$. This completes the proof. □

Let R be a tolerance on U. We define a map $\Diamond$ ("diamond") on $\wp(U)$ by setting

$$\Diamond X := X^{\blacktriangle\blacktriangledown} \tag{1}$$

for all $X \subseteq U$.

Lemma 10 *The mapping $\Diamond$ is the closure operator corresponding to the closure system $\wp(U)^{\blacktriangledown}$.*

Proof This follows from Proposition 5. Indeed, for all $X \subseteq U$, $X \subseteq X^{\blacktriangle\blacktriangledown} = \Diamond X$ and $\Diamond\Diamond X = X^{\blacktriangle\blacktriangledown\blacktriangle\blacktriangledown} = X^{\blacktriangle\blacktriangledown} = \Diamond X$. If $X \subseteq Y$, then $X^{\blacktriangle} \subseteq Y^{\blacktriangle}$ and $\Diamond X = X^{\blacktriangle\blacktriangledown} \subseteq Y^{\blacktriangle\blacktriangledown} = \Diamond Y$. Thus, $\Diamond$ is a closure operator on U. It is clear that $\Diamond X \in \wp(U)^{\blacktriangledown}$, and that if $X^{\blacktriangledown} \in \wp(U)^{\blacktriangledown}$, then $\Diamond X^{\blacktriangledown} = X^{\blacktriangledown\blacktriangle\blacktriangledown} = X^{\blacktriangledown}$. □

Note that since the closure system corresponding $\Diamond$ is $\wp(U)^{\blacktriangledown}$, we have

$$\Diamond X = \bigcap\{A \in \wp(U)^{\blacktriangledown} \mid X \subseteq A\},$$

for any $X \subseteq U$. Therefore, $\Diamond X$ is the least set in $\wp(U)^{\blacktriangledown}$ which contains X. Note also that $X^{\blacktriangle} = (\Diamond X)^{\blacktriangle}$ for all $X \subseteq U$ and $\wp(U)^{\blacktriangledown} = \{\Diamond X \mid X \subseteq U\}$.

A map $\mathscr{I} \colon \wp(U) \to \wp(U)$ is called an *interior operator* on U if for any $X, Y \subseteq U$:

(1) $\mathscr{I}(X) \subseteq X$ (contractive)
(2) $\mathscr{I}(\mathscr{I}(X)) = \mathscr{I}(X)$ (idempotent)
(3) $X \subseteq Y$ implies $\mathscr{I}(X) \subseteq \mathscr{I}(Y)$ (order-preserving)

It is known (see e.g. [13]) that each closure operator $\mathscr{C} \colon \wp(U) \to \wp(U)$ defines an interior operator $\mathscr{I}_{\mathscr{C}} \colon \wp(U) \to \wp(U)$ by the rule $\mathscr{I}_{\mathscr{C}}(X) = \mathscr{C}(X^c)^c$. A family $\mathscr{N}$ of subsets of A is said to be an *interior system* if $\mathscr{N}$ is closed under arbitrary unions. Note that interior systems always contain $\emptyset = \bigcup\emptyset$. Also the relationship between interior systems and interior operators is bijective.

Because for any tolerance R on U the mapping $\Diamond$ is a closure operator on U, it defines an interior operator $\Box$ ("box") by the rule

$$\Box X := (\Diamond(X^c))^c = X^{c\blacktriangle\blacktriangledown c} = X^{\blacktriangledown\blacktriangle}. \tag{2}$$

This gives that $\Box(X^c) = (\Diamond X)^c$ and $\Diamond(X^c) = (\Box X)^c$, that is, the operators $\Box$ and $\Diamond$ are mutually dual.

The interior system corresponding to the interior operator $\Box$ is $\wp(U)^{\blacktriangle}$, and

$$\Box X = \bigcup\{A \in \wp(U)^{\blacktriangle} \mid A \subseteq X\}.$$

Now $\Box X$ is the greatest set in $\wp(U)^{\blacktriangle}$ contained in X. Note also that $X^{\blacktriangledown} = (\Box X)^{\blacktriangledown}$ for all $X \subseteq U$ and $\wp(U)^{\blacktriangle} = \{\Box X \mid X \subseteq U\}$. In addition,

$$(\Diamond X)^{\blacktriangle} = \Box(X^{\blacktriangle}) \qquad \text{and} \qquad (\Box X)^{\blacktriangledown} = \Diamond(X^{\blacktriangledown}).$$

An ordered set $(P, \leq)$ is a *complete lattice* if each subset $S \subseteq P$ has a greatest lower bound $\bigwedge S$ and a least upper bound $\bigvee S$.

It is known that if $\mathscr{L}$ is a closure system on U, then the ordered set $(\mathscr{L}, \subseteq)$ is a complete lattice in which for a subset $\mathscr{H}$ of $\mathscr{L}$,

$$\bigwedge \mathscr{H} = \bigcap \mathscr{H} \qquad \text{and} \qquad \bigvee \mathscr{H} = \mathscr{C}_{\mathscr{L}}(\bigcup \mathscr{H}).$$

Because for a tolerance R on U, $\wp(U)^{\blacktriangledown}$ is a closure system, the ordered set $(\wp(U)^{\blacktriangledown}, \subseteq)$ is a complete lattice in which

$$\bigwedge \mathscr{H} = \bigcap \mathscr{H} \qquad \text{and} \qquad \bigvee \mathscr{H} = \Diamond(\bigcup \mathscr{H}) \tag{3}$$

for all $\mathscr{H} \subseteq \wp(U)^{\blacktriangledown}$. Similarly, the ordered set $(\wp(U)^{\blacktriangle}, \subseteq)$ is a complete lattice in which

$$\bigwedge \mathscr{H} = \Box(\bigcap \mathscr{H}) \qquad \text{and} \qquad \bigvee \mathscr{H} = \bigcup \mathscr{H} \tag{4}$$

for all $\mathscr{H} \subseteq \wp(U)^{\blacktriangle}$.

We say that an ordered set $(P, \leq)$ is *bounded* if it has a least element, denoted usually by 0, and a greatest element, denoted by 1. It is obvious and well-known that any complete lattice is bounded. The complete lattices $\wp(U)^{\blacktriangledown}$ and $\wp(U)^{\blacktriangle}$ are bounded in such a way that $\emptyset$ is their smallest element and U is the greatest element.

A mapping $x \mapsto x^{\perp}$ on a bounded lattice L is called an *orthocomplementation*, and $x^{\perp}$ an *orthocomplement* of x, if the following conditions hold for all $x, y \in L$:

(O1) $x \leq y$ implies $y^{\perp} \leq x^{\perp}$ (order-reversing)
(O2) $x^{\perp\perp} = x$ (involution)
(O3) $x \vee x^{\perp} = 1$ and $x \wedge x^{\perp} = 0$ (complement)

An *ortholattice* is a bounded lattice equipped with an orthocomplementation. Note that orthocomplementations are not always unique.

Remark 11 For an ordered set $(P, \leq)$, a mapping $\varphi\colon P \to P$ satisfying (O1) and (O2) is called a *polarity*. Such a polarity φ is an order-isomorphism from $(P, \leq)$ to its dual $(P, \geq)$. This means that P is *anti-isomorphic* to itself. Hence, the Hasse diagram of P looks the same when it is turned upside-down.

Proposition 12 *Let R be a tolerance on U.*

(a) *The map $X \mapsto X^{c\blacktriangledown}$ is an orthocomplementation in $\wp(U)^{\blacktriangledown}$.*
(b) *The map $X \mapsto X^{c\blacktriangle}$ is an orthocomplementation in $\wp(U)^{\blacktriangle}$.*

Proof We show that $X^{\perp} = X^{c\blacktriangledown}$ is an orthocomplement of $X \in \wp(U)^{\blacktriangledown}$ which proves (a). Claim (b) may be proved similarly. Suppose $X, Y \in \wp(U)^{\blacktriangledown}$. It is clear that $X^{\perp}$ belongs to $\wp(U)^{\blacktriangledown}$, so the mapping is well defined.

(O1) If $X \subseteq Y$, then $Y^c \subseteq X^c$ and $Y^{\perp} = Y^{c\blacktriangledown} \subseteq X^{c\blacktriangledown} = X^{\perp}$.
(O2) $X^{\perp\perp} = X^{c\blacktriangledown c\blacktriangledown} = X^{cc\blacktriangle\blacktriangledown} = X^{\blacktriangle\blacktriangledown}$. Because $X \in \wp(U)^{\blacktriangledown}$, we have $X = A^{\blacktriangledown}$ for some $A \subseteq U$. Thus, $X^{\perp\perp} = X^{\blacktriangle\blacktriangledown} = A^{\blacktriangledown\blacktriangle\blacktriangledown} = A^{\blacktriangledown} = X$.

(O3) By straightforward computation,

$$X \wedge X^{\perp} = X \cap X^{c\blacktriangledown} \subseteq X \cap X^{c} = \emptyset$$

and

$$X \vee X^{\perp} = (X \cup X^{c\blacktriangledown})^{\blacktriangle\blacktriangledown} = (X^{\blacktriangle} \cup X^{\blacktriangle c \blacktriangle})^{\blacktriangledown} \supseteq (X^{\blacktriangle} \cup X^{\blacktriangle c})^{\blacktriangledown} = U^{\blacktriangledown} = U.$$

□

2 Tolerances Induced by an Irredundant Covering

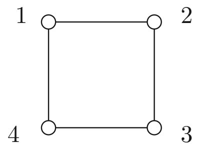
In this section, we first consider blocks of a tolerance. A block of a tolerance can be seen as a counterpart of an equivalence class of an equivalence relation. First we recall the notion of a covering [18]. Each covering induces a tolerance, and in this section tolerances induced by irredundant coverings assume a special role. We characterize the tolerances induced by an irredundant covering and we also give an algorithm which checks whether a tolerance is induced by an irredundant covering. Moreover, in case R is such a tolerance, the algorithm also returns that unique irredundant covering. Section 2.3 is devoted to rough approximation operators defined by irredundant coverings. There are many ways to define approximations operators in terms of a covering. We show that if $\mathscr{H}$ is an irredundant covering, many of these operators can be expressed by using rough approximation operators defined by the tolerance induced by $\mathscr{H}$. The last subsection deals with tolerances and formal concept analysis. Many results presented in this section appear already in our previous works [8–10].

2.1 Blocks of Tolerances and Set Coverings

We begin by considering blocks of tolerances. Blocks of tolerances can be seen as generalizations of equivalence classes, because each block of a tolerance R on U is a maximal set within which all elements are R-related. On the other hand, the R-neighbourhood $R(x)$ of $x \in U$ is not necessarily a block of R. In fact, we shall see that those $R(x)$-neighbourhoods that are blocks play a special role.

Let R be a tolerance on U. A nonempty subset X of U is an *R-preblock* if $X \times X \subseteq R$. Note that if B is an R-preblock, then $B \subseteq R(x)$ for all $x \in B$. An *R-block* is an R-preblock that is maximal with respect to the inclusion relation. Each tolerance R is completely determined by its blocks, that is, $a\,R\,b$ if and only if there exists a block B such that $a, b \in B$. Blocks are "clusters" of similar objects, because each object in a block B is R-related with all other elements in B, and no element outside B is R-related to all elements of B.

Fig. 6 Tolerance R on $\{1, 2, 3, 4\}$



Example 13 Let R be a tolerance on $U = \{1, 2, 3, 4\}$ depicted by the graph $\mathscr{G} = (U, R)$ in Fig. 6. A nonempty set $X \subseteq U$ is a preblock if and only if all points in X are connected by an edge of $\mathscr{G}$. Blocks are the maximal preblocks, and the blocks of R are $\{1, 2\}$, $\{2, 3\}$, $\{3, 4\}$, $\{1, 4\}$.

Next we present two lemmas related to R-neighbourhoods of objects.

Lemma 14 *Let R be a tolerance on U and $x \in U$. The following are equivalent:*

(a) *$R(x)$ is a preblock;*
(b) *$R(x)$ is a block.*

Proof It is clear that (b) implies (a). Let $R(x)$ be a preblock. Suppose that $R(x) \subseteq X$ for some preblock X. If $y \in X$, then $x \, R \, y$ because $x \in R(x) \subseteq X$ and X is a preblock. Therefore, $y \in R(x)$. Hence, $R(x) = X$ and $R(x)$ is a block. □

By the above lemma, we can also write that for all $x \in U$,

$$R(x) \text{ is a block} \iff R(x) \subseteq R(y) \text{ for all } y \in R(x). \tag{5}$$

As we have noted, if B is a block, then $B \subseteq R(y)$ for all $y \in B$, so "$\Longrightarrow$" follows from this. On the other hand, if $R(x) \subseteq R(y)$ for all $y \in R(x)$, then $a, b \in R(x)$ implies that $a \in R(b)$ and $b \in R(a)$. Therefore, all elements in $R(x)$ are related and by Lemma 14 $R(x)$ is a block.

Lemma 15 *A tolerance R on U is an equivalence if and only if $R(x)$ is a block for each $x \in U$.*

Proof If R is an equivalence, then each equivalence class $R(x)$ is a block.

On the other hand, suppose that there is $x \in U$ such that $R(x)$ is not a block. By Lemma 14 this means that there are $a, b \in R(x)$ which are not related. Now we have that $a \, R \, x$ and $x \, R \, b$, but $(a, b) \notin R$. Thus, the relation R is not transitive. □

Example 16 Let us consider the tolerance R on $U = \{a, b, c, d, e\}$ depicted in Fig. 7. By Lemma 14, $R(a) = R(b) = \{a, b, e\}$ and $R(c) = R(d) = \{c, d, e\}$ are blocks, because all their elements are R-related. The neighbourhood $R(e) = U$ is not a block, because, for instance, a and c are not related. By Lemma 15, the tolerance R is not an equivalence, because $R(e)$ is not a block.

A *partition* π on U is a collection of nonempty subsets of U such that every element x of U belongs to exactly one member of π. For any equivalence E on U, the set of all equivalence classes $U/E = \{[x]_E \mid x \in U\}$ forms a partition of U.

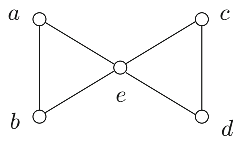


Fig. 7 Tolerance R on $\{a, b, c, d, e\}$

On the other hand, each partition π on U defines an equivalence E on U by setting $x \mathrel{E} y$ if there is a set $X \in \pi$ such that $x, y \in X$.

Tolerances do not in general determine partitions, but for tolerances the counterparts of partitions are set coverings. A collection $\mathscr{H}$ of nonempty subsets of U is called a *covering* of U if $\bigcup \mathscr{H} = U$. A covering $\mathscr{H}$ is *irredundant* if $\mathscr{H} \setminus \{X\}$ is not a covering for any $X \in \mathscr{H}$.

Example 17 The family $\mathscr{H} = \{\{a\}, \{a, b\}, \{b, c, d\}, \{c, d\}\}$ forms a covering of $U = \{a, b, c, d\}$, but this covering is not irredundant, because an irredundant covering cannot contain sets B and C such that $B \subseteq C$. The subfamilies $\{\{a\}, \{b, c, d\}\}$, $\{\{a, b\}, \{b, c, d\}\}$, and $\{\{a, b\}, \{c, d\}\}$ of $\mathscr{H}$ are irredundant coverings.

The family $\mathscr{K} = \{\{a, b\}, \{b, c\}, \{c, d\}, \{a, d\}\}$ also forms a covering of U. It is not irredundant even $B \nsubseteq C$ for all $B, C \in \mathscr{K}$. The subfamilies $\{\{a, b\}, \{c, d\}\}$ and $\{\{a, d\}, \{b, c\}\}$ of $\mathscr{K}$ are irredundant coverings.

It is clear that the blocks of a tolerance R on U form a covering of U, because for each $x \in U$, $(x, x) \in R$ means that there must be a block containing x. On the other hand, each covering $\mathscr{H}$ of U defines a tolerance $\bigcup\{X \times X \mid X \in \mathscr{H}\}$, called the *tolerance induced* by $\mathscr{H}$. If R is a tolerance induced by a covering $\mathscr{H}$, then

$$R(x) = \bigcup\{B \in \mathscr{H} \mid x \in B\}.$$

A natural problem is, how to recognize the tolerances that are induced by an irredundant covering. We consider this problem next.

Proposition 18 *Let R be a tolerance induced by an irredundant covering $\mathscr{H}$ of U. Then $\mathscr{H} = \{R(x) \mid R(x)$ is a block$\}$.*

Proof Let $B \in \mathscr{H}$. Because $\mathscr{H}$ is an irredundant covering, there is an element $x \in B$ such that $x \notin \bigcup(\mathscr{H} \setminus \{B\})$. Since R is induced by $\mathscr{H}$, $x \mathrel{R} y$ for all y in B. Therefore, $B \subseteq R(x)$. On the other hand, if $a \in R(x)$, then there is a set $C \in \mathscr{H}$ such that $x, a \in C$. But because x belongs only to B, we have $C = B$ and $a \in B$. Thus, also $R(x) \subseteq B$ and hence $R(x) = B$. Additionally, because $\mathscr{H}$ induces R, we have that $a \mathrel{R} b$ for all $a, b \in R(x) = B$. By Lemma 14, this means that $R(x)$ is a block. We have now proved that $\mathscr{H} \subseteq \{R(x) \mid R(x)$ is a block$\}$.

We need to show that also $\{R(x) \mid R(x)$ is a block$\} \subseteq \mathscr{H}$. Assume that $R(x)$ is a block. Because $\mathscr{H}$ is a covering, there is $B \in \mathscr{H}$ such that $x \in B$. If $a \in B$, then because $\mathscr{H}$ induces R, $a \in R(x)$ and thus $B \subseteq R(x)$. Since $\mathscr{H}$ is irredundant covering inducing R, we have by the beginning of the proof that $\mathscr{H} \subseteq \{R(x) \mid R(x)$ is a block$\}$. This means that there is an element $y \in U$ such that $B = R(y)$

and $R(y)$ is a block. Because $R(x)$ and $R(y)$ are blocks, $R(y) = B \subseteq R(x)$ gives $B = R(x)$ and $R(x) \in \mathscr{H}$. □

Proposition 18 says that if R is a tolerance induced by an irredundant covering, then this covering is unique and contains exactly the $R(x)$-neighbourhoods that are blocks. This means that we may simply speak about tolerances induced by an irredundant covering without specifying the covering in question.

Lemma 19 *Let R be a tolerance on U. If $R(x)$ and $R(y)$ are distinct blocks, then $x \notin R(y)$.*

Proof If $x \in R(y)$, then $R(y) \subseteq R(x)$ by (5) since $R(y)$ is a block. But $x \in R(y)$ means that also $y \in R(x)$. Since $R(x)$ is a block, we have $R(x) \subseteq R(y)$. Hence $R(x) = R(y)$. □

Note that Lemma 19 means that if $\{R(x) \mid R(x) \text{ is a block}\}$ is a covering, then it is irredundant. In fact, we can write the following characterization.

Theorem 20 *Let R be a tolerance on U. The following are equivalent:*

(a) *R is a tolerance induced by an irredundant covering;*
(b) *$\{R(x) \mid R(x)$ is a block$\}$ induces R.*

Proof That (a) implies (b) is clear by Proposition 18. On the other hand, if $\mathscr{H} = \{R(x) \mid R(x) \text{ is a block}\}$ induces R, then $\mathscr{H}$ is a covering, because for all $y \in U$, $y\,R\,y$ requires that there is an $R(x) \in \mathscr{H}$ such that $y \in R(x)$. By Lemma 19, the covering $\mathscr{H}$ is irredundant, because if $R(x)$ is a block, then x cannot belong to any other $R(y)$ that is a block. Thus, (b) implies (a). □

Example 21 It is possible that the family $\mathscr{H} = \{R(x) \mid R(x) \text{ is a block}\}$ is an irredundant covering, but does not induce R. In this case, R is not a tolerance induced by an irredundant covering. For instance, consider the tolerance of Example 1. Now

$$\mathscr{H} = \{R(x) \mid R(x) \text{ is a block}\} = \{R(1), R(2), R(3)\} = \{\{1, 4\}, \{2, 5\}, \{3, 6\}\}.$$

The family $\mathscr{H}$ is an irredundant covering of $U = \{1, 2, 3, 4, 5, 6\}$, but $\mathscr{H}$ does not induce R. For example, $(4, 5) \in R$, but there is no block in $\mathscr{H}$ containing 4 and 5.

Example 22 Let us consider the tolerance on $U = \{a, b, c, d, e, f, g\}$ given in Fig. 8. This tolerance appears in [18, Figure 3.5]. The neighbourhoods that are blocks are $R(a) = \{a, b, d, e\}$, $R(c) = \{b, c, d, f\}$, and $R(g) = \{d, e, f, g\}$. All edges of the graph are inside these blocks, which means that $\{R(a), R(c), R(g)\}$ induces R. By Theorem 20, R is a tolerance induced by an irredundant covering.

Notice that there may exist blocks which are not R-neighbourhoods of any object. For instance, $\{b, d, e, f\}$ is such a block.

Remark 23 A nonempty set $X \subseteq U$ is an R-preblock if and only if it is a *clique* of the graph $\mathscr{G} = (U, R)$, that is, all pairs of vertices in X are connected by an edge of $\mathscr{G}$. A block of R is thus a maximal clique.

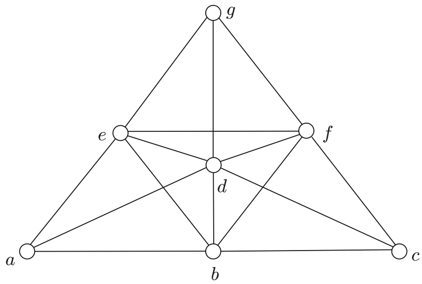


Fig. 8 Tolerance R induced by an irredundant covering

In computer science, the *clique problem* is the decision problem whether a clique of a given size k exists in a graph. A brute-force method for solving this problem is to list all sets of k vertices and to check each one to see whether it forms a clique. For a graph with n vertices, the running time of this algorithm is $\Omega(k^2\binom{n}{k})$. In fact, it is known that the clique problem is NP-complete and therefore an efficient algorithm for the clique problem is unlikely to exist [1].

Fortunately, our problem is simpler. We do not need to find all maximal cliques of a graph. We are only interested in the question whether $R(x)$ is a clique for some $x \in U$, because such an $R(x)$ is necessarily a maximal clique by Lemma 14. Our Algorithm 31 solves this in $O(|R|)$ steps, where $|R|$ denotes the cardinality of R.

We end this subsection by considering how tolerances induced by an irredundant covering may arise from incomplete information systems.

Let $\mathscr{S} = (U, A, \{V_a\}_{a \in A})$ be an incomplete information system, where the null values are marked by $*$. For each $B \subseteq A$, the tolerance SIM_B is defined as earlier, that is,

$$SIM_B = \{(x, y) \in U \times U \mid (\forall a \in B)\, a(x) = a(y) \text{ or } a(x) = * \text{ or } a(y) = *\}.$$

We define the set $compl_B$ of *B-complete elements* by

$$compl_B = \{x \in U \mid a(x) \neq * \text{ for all } a \in B\}.$$

Our next lemma shows that the neighbourhoods of complete elements are blocks.

Lemma 24 *Let $\mathscr{S} = (U, A, \{V_a\}_{a \in A})$ be an incomplete information system and $B \subseteq A$. For any $c \in compl_B$, the neighbourhood $SIM_B(c)$ is a block.*

Proof Assume that $x, y \in SIM_B(c)$. Then $a(x) = a(c)$ for all $a \in B$ such that $a(x) \neq *$. Similarly, $a(y) = a(c)$ for all $a \in B$ such that $a(y) \neq *$. This means that $a(x) = a(y)$ and $(x, y) \in SIM_a$ for all $a \in B$ such that $a(x) \neq *$ and $a(y) \neq *$. On the other hand, if $a(x) = *$ or $a(y) = *$, then $(x, y) \in SIM_a$. Thus, $(x, y) \in SIM_a$ for all $a \in B$, which means that $(x, y) \in SIM_B$. Hence $SIM_B(c)$ is a block, according to Lemma 14. □

Table 3 The information system $\mathscr{S}$

	a	b
1	*	w_1
2	v_1	w_1
3	v_2	w_2
4	v_2	*

In what follows, we present a condition under which for each $B \subseteq A$, the tolerance SIM_B is induced by an irredundant covering. Let us begin with an example.

Example 25 Let $\mathscr{S} = (U, A, \{V_a\}_{a \in A})$ be the incomplete information system defined in Table 3.

The elements 2 and 3 are A-complete and their SIM_A-neighbourhoods are $\{1, 2\}$ and $\{3, 4\}$. The tolerance induced by this covering is an equivalence which differs from the tolerance SIM_A, because the objects 1 and 4 are SIM_A-related. This means that the covering $\{\{1, 2\}, \{3, 4\}\}$ does not induce SIM_A.

Let $\mathscr{S} = (U, A, \{V_a\}_{a \in A})$ be an incomplete information system and $B \subseteq A$. We introduce the following condition:

$$(x, y) \in SIM_B \iff (\exists c \in compl_B)\, x, y \in SIM_B(c) \qquad (\star)$$

Proposition 26 *Let $\mathscr{S} = (U, A, \{V_a\}_{a \in A})$ be an incomplete information system and let $B \subseteq A$ be such that $(\star)$ is satisfied. Then $\mathscr{H}_B = \{SIM_B(c) \mid c \in compl_B\}$ is an irredundant covering and it induces SIM_B.*

Proof Let $B \subseteq A$. Because $(x, x) \in SIM_B$ for all $x \in U$, by $(\star)$ there must be an element $c \in compl_B$ such that $x \in SIM_B(c)$. Therefore, $\mathscr{H}_B$ is a covering.

The covering $\mathscr{H}_B$ is clearly irredundant, because each B-complete element c can belong only to $SIM_B(c)$. By condition $(\star)$, $\mathscr{H}_B$ induces SIM_B □

Example 27 Let us consider the incomplete information system of Example 2. The A-complete elements are 2, 3, 5. Now $SIM_A(2) = \{2, 4\}$, $SIM_A(3) = \{1, 3\}$, and $SIM_A(5) = \{1, 4, 5\}$. Because every edge of the graph belongs to these blocks, the irredundant covering

$$\{SIM_A(2), SIM_A(3), SIM_A(5)\}$$

induces SIM_A.

Remark 28 For any B-complete element c, $R(c)$ is a cluster in which all elements are similar to each other with respect to the B-attributes. The element c can be seen as a "prototype element" for this cluster because an object $x \in U$ belongs to this cluster if and only if x is B-similar to c. Such prototype elements c are called "medoids" in cluster analysis. If condition $(\star)$ holds for some $B \subseteq A$, then this means that two objects x and y are SIM_B-related if and only if there is a B-complete prototype object c such that x and y are SIM_B-related to this c.

2.2 Algorithms

Next we present an algorithm which decides whether a given tolerance is induced by an irredundant covering. If that is the case, then the method also produces the unique irredundant covering.

Since inputs to the algorithms have to be finite, we assume here that U is finite. We suppose that the relation R is given as the collection $\{R(x) \mid x \in U\}$ of all R-neighbourhoods. More precisely, we use an *adjacency list representation* of (U, R) consisting of an array *Adj* of $|U|$ lists, one for each element in U. For each $x \in U$, the adjacency list $Adj[x]$ contains the members of $R(x)$ (see [1], for example). We also assume that the elements in each adjacency list are ordered according to some given linear order of U. Each adjacency list is implemented as a linked list, where each list element a has a reference $a.next$ to the next element in the list. The end of the list is marked by NULL. The "value" identifying the element stored in the list element a is in $a.key$. For each pair $(x, y) \in R$, the adjacency list $Adj[x]$ contains a list element a such that $a.key = y$. Thus the sum of the lengths of all adjacency lists is $|R|$. Because the relation R is reflexive, $|U| \leq |R|$, and so the amount of memory the adjacency-list representation requires is $O(|R|)$.

Example 29 The adjacency-list representation of the relation R of Example 2 is depicted in Fig. 9. The *next*-references are marked by an arrow and the NULL-value is indicated by a diagonal slash.

Our first algorithm checks whether $R(x) \subseteq R(y)$ for some $x, y \in U$.

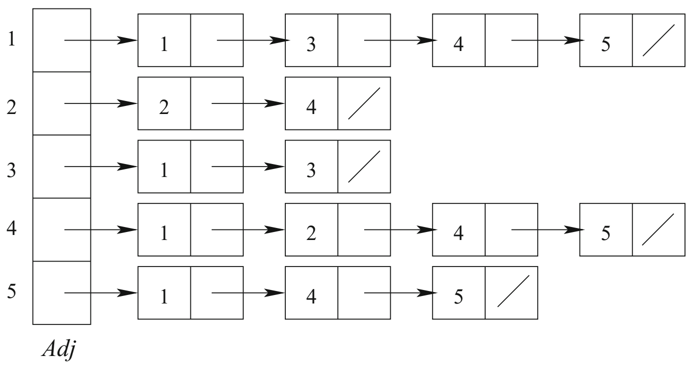


Fig. 9 An adjacency list representation

Algorithm 30 (Inclusion)

Input: The adjacency list representation *Adj* of (U, R) and two elements $x, y \in U$.
Output: "yes" if $R(x) \subseteq R(y)$; "no" otherwise.

(1) Let a be the first element in the list $Adj[x]$ and let b be the first element in $Adj[y]$.
(2) While $a \neq$ NULL and $b \neq$ NULL, repeat the following:
   (a) if $a.key < b.key$, then output "no" and halt.
   (b) if $a.key = b.key$, then $a \leftarrow a.next$ and $b \leftarrow b.next$.
   (c) if $a.key > b.key$, then $b \leftarrow b.next$.
(3) If $a =$ NULL, then output "yes"; otherwise output "no".

The time-complexity of Algorithm 30 is $O(|Adj[y]|)$. This is because in the worst case $b \leftarrow b.next$ is executed for each element b in $Adj[y]$.

Next we present an algorithm which decides whether $R(x)$ is a block for a given $x \in U$. Using Algorithm 30 it checks whether $R(x) \subseteq R(y)$ holds for each $y \in R(x)$. If this is true, then $R(x)$ is a block by (5).

Algorithm 31 (Block)

Input: The adjacency list representation *Adj* of (U, R) and an element $x \in U$.
Output: "yes" if $R(x)$ is a block; "no" otherwise.

1. For each $y \in R(x) \setminus \{x\}$, test using Algorithm 30 whether $R(x) \subseteq R(y)$. If no, output "no" and halt.
2. If all elements $y \in R(x) \setminus \{x\}$ are checked without halting, output "yes" and halt.

The running time of Algorithm 31 is $O(|R|)$, because when $R(x)$ is a block, we need to check $R(x) \subseteq R(y)$ for all $y \in R(x) \setminus \{x\}$. Each such test takes $O(|Adj[y])|$ time. The sum of the lengths of all adjacency lists is $|R|$, from which we get the upper bound.

Our next algorithm decides whether the tolerance R is induced by an irredundant covering. As far as we know, this is the first algorithm solving this problem.

Algorithm 32 (Irredundant Covering)

Input: The adjacency list representation *Adj* of (U, R) and the set U.
Output: "yes" if R is induced by an irredundant covering and a set C such that $\{R(x) \mid x \in C\}$ is the irredundant covering inducing R; "no" otherwise.

1. Divide U into $C = \{x \in U \mid R(x) \text{ is a block}\}$ and $D = U \setminus C$.
2. For all $d \in D$ and all $e \in R(d)$, find out whether $\{d, e\} \subseteq R(c)$ for some $c \in C$. If such an element c cannot be found for some (d, e)-pair, output "no" and halt.
3. If all (d, e)-pairs are tested without halting, output "yes" and the set C.

The correctness of the algorithm follows from Theorem 20: if R is induced by an irredundant covering $\mathscr{H}$, then $\mathscr{H}$ must be $\{R(x) \mid R(x) \text{ is a block}\}$. Therefore, the algorithm needs to check whether $\{R(x) \mid R(x) \text{ is a block}\}$ induces R. The algorithm first produces the set C, which can be done in $O(|U| \cdot |R|)$ time (step 1). This is because there are $O(|U|)$ elements in C, and checking whether $R(c)$ is a block for some $c \in C$ takes $O(|R|)$ steps by using Algorithm 31.

Next the algorithm decides whether $\{R(c) \mid c \in C\}$ induces R. This is done by considering the remaining R-neighbourhoods $\{R(d) \mid d \in D\}$. It suffices to check that for any $e \in R(d)$ (meaning that $d\,R\,e$), there is an element $c \in C$ such that $d, e \in R(c)$. Since there are at most $|R|$ this kind of (d, e)-pairs, and the sum of the lengths of the adjacency lists of the elements in C at most $|R|$, the time complexity of step 2 is $O(|R|^2)$. The total running time of Algorithm 32 is therefore $O(|U|\cdot|R|) + O(|R|^2) = O(|R|^2)$.

2.3 Rough Approximations Defined by Tolerances Induced by Irredundant Coverings

In Sect. 1, we showed that $\wp(U)^{\blacktriangledown}$ and $\wp(U)^{\blacktriangle}$ are complete order-isomorphic ortholattices. In this section our aim is to study the properties of these complete lattices defined by tolerances induced by irredundant coverings.

A *distributive lattice* is a lattice L satisfying the *distributive laws*:

(D1) $(\forall x, y, z \in L)\ x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge y)$;
(D2) $(\forall x, y, z \in L)\ x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

It is known that a lattice satisfies (D1) if and only if it satisfies (D2). Therefore, checking that a lattice is distributive requires only checking the validity of either (D1) or (D2).

Example 33 In general, $\wp(U)^{\blacktriangledown}$ and $\wp(U)^{\blacktriangle}$ are not distributive. For instance, consider a tolerance R of Example 13. The lattices of lower and upper approximations are given in Fig. 10. Note that for the sake of simplicity, we sometimes denote sets by sequences of their elements. For example, $\{1, 2, 3\}$ is denoted by 123. The lattice

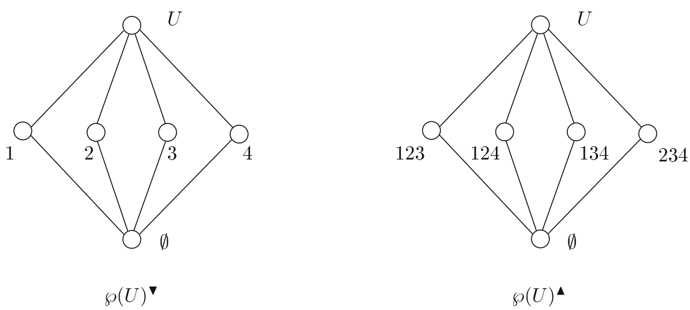


Fig. 10 $\wp(U)^{\blacktriangledown}$ and $\wp(U)^{\blacktriangle}$ are not distributive

$\wp(U)^{\blacktriangledown}$ is not distributive, because

$$(\{1\} \vee \{2\}) \wedge \{3\} = U \wedge \{3\} = \{3\}$$

and

$$(\{1\} \wedge \{3\}) \vee (\{2\} \wedge \{3\}) = \emptyset \vee \emptyset = \emptyset.$$

Because $\wp(U)^{\blacktriangle} \cong \wp(U)^{\blacktriangledown}$, $\wp(U)^{\blacktriangle}$ is not distributive neither.

Let L be a bounded lattice with least element 0 and greatest element 1. An element $b \in L$ is a *complement* of an element $a \in L$ if

$$a \vee b = 1 \qquad \text{and} \qquad a \wedge b = 0.$$

A *complemented lattice* is a bounded lattice in which every element has a complement. Notice that the orthocomplement of an element is a complement in the sense of the above definition. Complements need not be unique and if a lattice is not distributive, then an element may have several complements. For instance, in $\wp(U)^{\blacktriangledown}$ of Example 33, the element $\{1\}$ has the complements $\{2\}$, $\{3\}$, and $\{4\}$.

A *Boolean lattice* is a complemented distributive lattice. It is known that in a distributive lattice any element can have at most one complement, so in Boolean lattices the complement of any element is unique. The complement of x is denoted by x'. The complement operation in a Boolean lattice has the following properties:

(B1) $0' = 1$ and $1' = 0$;
(B2) $a'' = a$;
(B3) $(a \vee b)' = a' \wedge b'$ and $(a \wedge b)' = a' \vee b'$;
(B4) $a \wedge b = 0$ if and only if $a \leq b'$;
(B5) $a \leq b$ implies $b' \leq a'$.

Let L be a lattice with a least element 0. Then $a \in L$ is called an *atom* of L, if $0 \prec a$. The set of atoms of L is denoted by $\mathscr{A}(L)$. A lattice L with zero is *atomistic* if every element $x \in L$ is a join of atoms.

Let L be an atomistic lattice. For any $x \in L$, let us denote

$$A(x) = \{a \in \mathscr{A}(L) \mid a \leq x\}.$$

Then clearly $x = \bigvee A(x)$, and for any $y \in L$, $x \leq y$ if and only if $A(x) \subseteq A(y)$. Hence $A(x) = A(y)$ if and only if $x = y$. The facts

$$A(x \wedge y) = A(x) \cap A(y) \quad \text{and} \quad A(x \vee y) \supseteq A(x) \cup A(y) \tag{6}$$

follow directly from the definitions of joins and meets.

Next we prove that $\wp(U)^{\blacktriangledown}$ and $\wp(U)^{\blacktriangle}$ are Boolean lattices when R is a tolerance induced by an irredundant covering. For that we will need the following two lemmas.

Lemma 34 *Let R be a tolerance on U.*

(a) *Any atom of $\wp(U)^{\blacktriangle}$ has the form $R(x)$ for some $x \in U$.*
(b) *If $R(x)$ is a block of R, then $R(x)$ is an atom of $\wp(U)^{\blacktriangle}$.*
(c) *If R is induced by an irredundant covering of U, then $\{R(x) \mid R(x)$ is a block$\}$ is the set of the atoms of the lattice $\wp(U)^{\blacktriangle}$.*

Proof

(a) Atoms of $\wp(U)^{\blacktriangle}$ have to be of the form $R(x)$, because the map $^{\blacktriangle}$ is order-preserving and $R(x) = \{x\}^{\blacktriangle}$.
(b) Suppose that $R(x)$ is a block and $R(y) \subseteq R(x)$. Because $y \in R(x)$ and $R(x)$ is a block, we have $R(x) \subseteq R(y)$ by (5). Thus, $R(y) = R(x)$ and $R(x)$ is an atom.
(c) Suppose that $R(y)$ is an atom of $\wp(U)^{\blacktriangle}$. Because $\{R(x) \mid R(x)$ is a block$\}$ is a covering, there exists a block $R(x)$ such that $y \in R(x)$. Since $R(x)$ is a block, $\emptyset \subset R(x) \subseteq R(y)$ by (5). Because $R(y)$ is an atom of $\wp(U)^{\blacktriangle}$, we get $R(y) = R(x)$, and this means that $R(y)$ is a block. □

In a lattice L with a least element 0, an element x^* is a *pseudocomplement* of an element $x \in L$ if, for any $z \in L$, $x \wedge z = 0$ if and only if $z \leq x^*$. Obviously, an element can have at most one pseudocomplement. The lattice L itself is called *pseudocomplemented*, if every element of L has a pseudocomplement. Every pseudocomplemented lattice is necessarily bounded, having 0^* as the greatest element.

Lemma 35 *Any complete atomistic pseudocomplemented lattice L is a Boolean lattice.*

Proof First, we show that $A(x \vee y) = A(x) \cup A(y)$ for any $x, y \in L$. By (6), it suffices to prove that $A(x \vee y) \subseteq A(x) \cup A(y)$. Take any $a \in \mathscr{A}(L)$ with $a \leq x \vee y$. We show that $a \notin A(x) \cup A(y)$ is not possible. Indeed, if $a \notin A(x)$ and $a \notin A(y)$, then $a \wedge x = 0$ and $a \wedge y = 0$. This gives $x, y \leq a^*$ and $x \vee y \leq a^*$. We get $a = a \wedge (x \vee y) \leq a \wedge a^* = 0$, a contradiction. Hence $a \in A(x) \cup A(y)$, which proves $A(x \vee y) = A(x) \cup A(y)$.

Next, we show that L satisfies identity (D1). Take any $x, y, z \in L$ and observe that

$$\begin{aligned} A(x \wedge (y \vee z)) &= A(x) \cap A(y \vee z) = A(x) \cap (A(y) \cup A(z)) \\ &= (A(x) \cap A(y)) \cup (A(x) \cap A(z)) = A(x \wedge y) \cup A(x \wedge z) \\ &= A((x \wedge y) \vee (x \wedge z)). \end{aligned}$$

Thus $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, proving that (D1) holds and L is distributive.

To prove that L is complemented, take any $x \in L$. Then, for any $a \in \mathscr{A}(L)$,

$$a \notin A(x) \iff a \nleq x \iff a \wedge x = 0 \iff a \leq x^* \iff a \in A(x^*).$$

This yields $A(x^*) = \mathscr{A}(L) \setminus A(x)$, whence we get

$$A(x \vee x^*) = A(x) \cup A(x^*) = \mathscr{A}(L) = A(1).$$

This implies $x \vee x^* = 1$. Since $x \wedge x^* = 0$, we obtain that x^* is the complement of x in L. Therefore, L is a Boolean lattice. □

Proposition 36 *If R is a tolerance induced by an irredundant covering, then $\wp(U)^{\blacktriangle}$ is atomistic and pseudocomplemented.*

Proof Since in the lattice $\wp(U)^{\blacktriangle}$ the joins coincide with unions, in view of Lemma 34(c), $\wp(U)^{\blacktriangle}$ is atomistic if for any $A \in \wp(U)^{\blacktriangle}$,

$$\bigcup\{R(x) \subseteq A \mid R(x) \text{ is a block}\} = A. \tag{7}$$

As the left side of (7) is included in A, we have to show only the converse inclusion. Because $A \in \wp(U)^{\blacktriangle}$, $A = X^{\blacktriangle}$ for some $X \subseteq U$. Let $a \in A$. Then $a\,R\,b$ for some $b \in X$. Because R is induced by an irredundant covering, by Proposition 18, there is $x \in U$ such that $a, b \in R(x)$ and $R(x)$ is a block. Because $R(x)$ is a block, we have $R(x) \subseteq R(c)$ for all $c \in R(x)$. In particular, $b \in R(c) \cap X$, $R(c) \cap X \neq \emptyset$ and $c \in X^{\blacktriangle}$ for all $c \in R(x)$. Thus, $R(x) \subseteq X^{\blacktriangle} = A$ and $a \in \bigcup\{R(x) \subseteq A \mid R(x) \text{ is a block}\}$. This proves (7) and hence $\wp(U)^{\blacktriangle}$ is atomistic.

Now let $B, C \in \wp(U)^{\blacktriangle}$ be such that $B \wedge C = \emptyset$. In order to prove that $\wp(U)^{\blacktriangle}$ is pseudocomplemented, we show that $C \subseteq B^{\perp}$. Note that we have already proved that $B \wedge B^{\perp} = \emptyset$ in Proposition 12. Since $\wp(U)^{\blacktriangle}$ is atomistic, to prove $C \subseteq B^{\perp}$, it is enough to show that each atom $R(x) \subseteq C$ satisfies $R(x) \subseteq B^{\perp} = B^{c\blacktriangle}$.

Assume by contradiction that there exists an element $x \in U$ such that $R(x)$ is a block and $R(x) \subseteq C$, but $R(x) \nsubseteq B^{c\blacktriangle} = B^{\blacktriangledown c}$. This means that $R(x) \cap B^{\blacktriangledown} \neq \emptyset$. Hence there is an element $y \in B^{\blacktriangledown}$ with $y \in R(x)$. Then $y\,R\,x$ and $R(y) \subseteq B$. Since $R(x)$ is a block, we get $R(x) \subseteq R(y) \subseteq B$ by (5). Because $R(x) \in \wp(U)^{\blacktriangle}$, $R(x) \subseteq B, C$ yields $B \wedge C \neq \emptyset$, a contradiction. This proves that $B^{\perp}$ is the pseudocomplement of any $B \in \wp(U)^{\blacktriangle}$. □

We can now write the following conclusion of Lemmas 34 and 35, and Proposition 36. Notice that since ${}^{\blacktriangledown}$ is an isomorphism from $\wp(U)^{\blacktriangle}$ to $\wp(U)^{\blacktriangledown}$, the atoms of $\wp(U)^{\blacktriangledown}$ are the ${}^{\blacktriangledown}$-images of the atoms of $\wp(U)^{\blacktriangle}$.

Corollary 37 *Let R be a tolerance induced by an irredundant covering of U.*

(a) *$\wp(U)^{\blacktriangle}$ is an atomistic Boolean lattice such that for all $X \in \wp(U)^{\blacktriangle}$, $X' = X^{c\blacktriangle}$. The set of atoms is $\{R(x) \mid R(x)$ is a block$\}$.*
(b) *$\wp(U)^{\blacktriangledown}$ is an atomistic Boolean lattice such that for all $X \in \wp(U)^{\blacktriangledown}$, $X' = X^{c\blacktriangledown}$. The set of atoms is $\{R(x)^{\blacktriangledown} \mid R(x)$ is a block$\}$.*

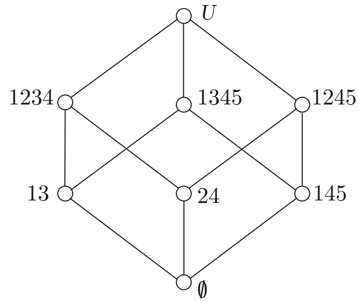


Fig. 11 The Boolean lattice $\wp(U)^{\blacktriangle}$

Example 38 Let us consider the tolerance R of Fig. 2 on $U = \{1, 2, 3, 4, 5, 6\}$. The Boolean lattice $\wp(U)^{\blacktriangle}$ is given in Fig. 11. The elements $R(2) = \{2, 4\}$, $R(3) = \{1, 3\}$, $R(5) = \{1, 4, 5\}$ are its atoms.

Note that for tolerances induced by an irredundant covering, the sets in $\wp(U)^{\blacktriangle}$ are in a sense "definable". Let us recall that each $R(x)$ being a block is completely determined by the element x. As we noted, these "prototype elements" are called medoids in cluster analysis. Because $\mathscr{A}(\wp(U)^{\blacktriangle}) = \{R(x) \mid R(x) \text{ is a block}\}$, each $X^{\blacktriangle}$ is a union of some $R(x)$-neighbourhoods that are blocks. This means that $X^{\blacktriangle}$ can be defined just by listing the appropriate "prototype elements". For instance, in Example 38, the set $\{1, 2, 4, 5\}$ is "defined" by the elements 2 and 5: an element of U belongs to $\{1, 2, 4, 5\}$ if and only if it is R-related to 2 or 5.

Next we show how approximations defined by tolerances and approximations defined by coverings are related when we consider irredundant coverings and tolerances induced by them. For an equivalence relation E, lower and upper approximations for $X \subseteq U$ can be also written in the form

$$X^{\blacktriangledown} = \bigcup \{[x]_E \mid [x]_E \subseteq X\}$$

and

$$X^{\blacktriangle} = \bigcup \{[x]_E \mid [x]_E \cap X \neq \emptyset\},$$

respectively. Because there is a one-to-one correspondence between equivalences and partitions, we could as well define the rough approximations of $X \subseteq U$ in terms of a partition π on U by

$$X^{\blacktriangledown} = \bigcup \{B \in \pi \mid B \subseteq X\}$$

and

$$X^{\blacktriangle} = \bigcup \{B \in \pi \mid B \cap X \neq \emptyset\}.$$

In [21], W. Żakowski presented generalizations of these definitions by replacing the partition π of U by a covering $\mathscr{H}$ of U. His operators do not form a dual pair, but Pomykała [17] associated with coverings several pairs of mutually dual approximation operators. We recall here a couple of them.

Let $\mathscr{H}$ be a covering of U. For each $x \in U$, the set

$$N(x) = \bigcup\{B \in \mathscr{H} \mid x \in B\}$$

is called the *$\mathscr{H}$-neighbourhood* x. For any $X \subseteq U$, we define

$$\begin{aligned} X^{\blacktriangleleft} &= \{x \in U \mid N(x) \subseteq X\}, \\ X^{\blacktriangleright} &= \bigcup\{B \in \mathscr{H} \mid B \cap X \neq \emptyset\}, \\ X^{\triangleleft} &= \bigcup\{B \in \mathscr{H} \mid B \subseteq X\}, \text{ and} \\ X^{\triangleright} &= \{x \in U \mid B \cap X \neq \emptyset \text{ for all } B \in \mathscr{H} \text{ with } x \in B\}. \end{aligned}$$

As noted by Pomykała [17], these operators form dual pairs, that is, $X^{\blacktriangleright c} = X^{c \blacktriangleleft}$ and $X^{\triangleright c} = X^{c \triangleleft}$ for all $X \subseteq U$. The operators $^{\blacktriangleright}$ and $^{\triangleleft}$ are the ones defined by Żakowski.

Let us see how these operators relate to the rough set operators $^{\blacktriangle}$ and $^{\blacktriangledown}$ when $\mathscr{H}$ is a covering of U and R is induced by it. Note that now the $\mathscr{H}$-neighbourhoods and the R-neighbourhoods are equal for all $x \in U$, that is, for any $x \in U$, $N(x) = R(x)$. This means that $X^{\blacktriangledown} = X^{\blacktriangleleft}$ for every $X \subseteq U$. Because also the operators $^{\blacktriangle}$ and $^{\blacktriangledown}$ are dual, we can write the following proposition.

Proposition 39 *If $\mathscr{H}$ is a covering of U and R is induced by $\mathscr{H}$, then*

$$X^{\blacktriangle} = X^{\blacktriangleright} \quad \textit{and} \quad X^{\blacktriangledown} = X^{\blacktriangleleft}$$

for any $X \subseteq U$.

In particular, by Proposition 39 we can write for every $X \subseteq U$,

$$X^{\blacktriangle} = \bigcup\{B \in \mathscr{H} \mid X \cap B \neq \emptyset\}.$$

In (2), the interior operation $\square$ on U is defined by $\square X = X^{\blacktriangledown\blacktriangle}$. The corresponding interior system is $\wp(U)^{\blacktriangle}$ and we noted that

$$\square X = \bigcup\{A \in \wp(U)^{\blacktriangle} \mid A \subseteq X\}.$$

Assume now that the covering $\mathscr{H}$ is irredundant. If R is induced by $\mathscr{H}$, then we have $\mathscr{H} = \{R(x) \mid R(x) \text{ is a block}\}$.

Lemma 40 *If $\mathscr{H}$ is an irredundant covering of U and R is induced by $\mathscr{H}$, then*

$$\bigcup\{A \in \wp(U)^{\blacktriangle} \mid A \subseteq X\} = \bigcup\{B \in \mathscr{H} \mid B \subseteq X\}$$

for every $X \subseteq U$.

Proof Since for any $B \in \mathscr{H}$, $B = R(x)$ for some $x \in U$, we have $B = \{x\}^{\blacktriangle} \in \wp(U)^{\blacktriangle}$. Hence, $\bigcup\{B \in \mathscr{H} \mid B \subseteq X\}$ is included in $\bigcup\{A \in \wp(U)^{\blacktriangle} \mid A \subseteq X\}$.

Conversely, if $x \in \bigcup\{A \in \wp(U)^{\blacktriangle} \mid A \subseteq X\}$, then $x \in A$ for some $A \in \wp(U)^{\blacktriangle}$, that is, there is a set $Y \subseteq U$ such that $x \in Y^{\blacktriangle} = A$. Since $Y^{\blacktriangle} = \bigcup\{B \in \mathscr{H} \mid Y \cap B \neq \emptyset\}$, this means that $x \in B$ for some $B \in \mathscr{H}$ with $Y \cap B \neq \emptyset$. Then $B \subseteq Y^{\blacktriangle} = A \subseteq X$, which implies $x \in \bigcup\{B \in \mathscr{H} \mid B \subseteq X\}$. Therefore also $\bigcup\{A \in \wp(U)^{\blacktriangle} \mid A \subseteq X\}$ is included in $\bigcup\{B \in \mathscr{H} \mid B \subseteq X\}$. This completes our proof. □

Lemma 40 means that for every $X \subseteq U$,

$$\Box X = \bigcup\{B \in \mathscr{H} \mid B \subseteq X\}.$$

Hence, we can write the following proposition (for the definition of $\Diamond$ see (1)).

Proposition 41 *If $\mathscr{H}$ is an irredundant covering of U and R is induced by $\mathscr{H}$, then*

$$X^{\triangleleft} = \Box X \qquad \textit{and} \qquad X^{\triangleright} = \Diamond X$$

for any $X \subseteq U$.

2.4 *Complement Formal Contexts Based on Tolerances*

A *formal context* is a triple $\mathscr{K} = (G, M, I)$, where G is a set of *objects*, M is a set of *attributes*, and $I \subseteq G \times M$ is a binary relation called *incidence relation*. The notations $(g, m) \in I$ and $g\,I\,m$ both express that an object g is in relation I with an attribute m, and we read it as "the object g has the attribute m". The basic definitions and results concerning formal concept analysis can be found in [2, 3], for example. By defining for all subsets $A \subseteq G$ and $B \subseteq M$,

$$A^I = \{m \in M \mid g\,I\,m \text{ for all } g \in A\}$$

and

$$B^I = \{g \in G \mid g\,I\,m \text{ for all } m \in B\},$$

Table 4 A simple formal context

	Angular	Right angles	Equilateral	Central symmetry
Triangle	×			
Square	×	×	×	×
Circle				×
Rectangle	×	×		×
Rhombus	×		×	×

we establish a connection between the powerset lattices $\wp(G)$ and $\wp(M)$. In fact, for any subsets $A, A_1, A_2 \subseteq G$ and $B, B_1, B_2 \subseteq M$ the following hold:

(1) $A_1 \subseteq A_2$ implies $A_2^I \subseteq A_1^I$, and $B_1 \subseteq B_2$ implies $B_2^I \subseteq B_1^I$;
(2) $A \subseteq A^{II}$ and $B \subseteq B^{II}$;
(3) $A^I = A^{III}$ and $B^I = B^{III}$.

By these properties, the map $A \mapsto A^{II}$ is a closure operator on G and $B \mapsto B^{II}$ is a closure operator on M.

A small context usually is represented by a table, similar to an information system. The table rows are labelled by objects and the columns are labelled by attributes. A cross (×) in row g and column m means $g\,I\,m$, that is, the object g has the attribute m. In a sense, contexts are like 2-valued information systems, where the values are "cross" and "no cross".

Example 42 A formal context describing some geometrical shapes is given in Table 4.

A formal concept of the context (G, M, I) is a pair $(A, B) \in \wp(G) \times \wp(M)$ with $A^I = B$ and $B^I = A$. The set A is called the *extent* and B the *intent* of the concept (A, B). Hence any concept has the form (A^{II}, A^I) for some $A \subseteq G$, and A is a concept extent if and only if $A^{II} = A$. Similarly, for any $B \subseteq M$, (B^I, B^{II}) is a concept and B is a concept intent if and only if $B^{II} = B$. The set of all concepts of the context (G, M, I) is denoted by $\mathfrak{B}(G, M, I)$.

Let (G, M, I) be a formal context. For any concepts (A_1, B_1) and (A_2, B_2) in $\mathfrak{B}(G, M, I)$, we set $(A_1, B_1) \leq (A_2, B_2)$ if $A_1 \subseteq A_2$. Note that $A_1 \subseteq A_2$ implies that $B_1 = A_1^I \supseteq A_2^I = B_2$ and $A_1^I \supseteq A_2^I$ implies $A_1 = A_1^{II} \subseteq A_2^{II} = A_2$. Therefore,

$$(A_1, B_1) \leq (A_2, B_2) \iff A_1 \subseteq A_2 \iff B_1 \supseteq B_2.$$

It is known [3, Theorem 3] that $\mathfrak{B}(G, M, I)$ forms a complete lattice such that for $\{(A_j, B_j) \mid j \in J\} \subseteq \mathfrak{B}(G, M, I)$,

$$\bigwedge_{j \in J} (A_j, B_j) = \Big(\bigcap_{j \in J} A_j, \big(\bigcup_{j \in J} B_j\big)^{II} \Big), \text{ and } \bigvee_{j \in J} (A_j, B_j) = \Big(\big(\bigcup_{j \in J} A_j\big)^{II}, \bigcap_{j \in J} B_j \Big).$$

This lattice is called *concept lattice*.

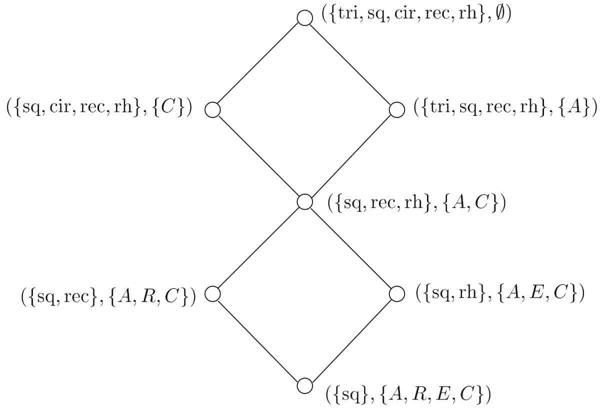


Fig. 12 The concept lattice $\mathfrak{B}(G, M, I)$

Example 43 The concept lattice of the context of Example 42 is given in Fig. 12. We use shorthand for the geometric shapes: "tr" (triangle), "sq" (square), "cir" (circle), "rec" (rectangle), and "rh" (rhombus). The attributes are denoted simply by capital letters, that is, A denotes "angular", R denotes "right angles", E denotes "equilateral" and C denotes "central symmetry".

If $(P, \leq)$ is a partially ordered set, then it is known that the concept lattice $\mathfrak{B}(P, P, \leq)$ is the smallest complete lattice into which $(P, \leq)$ can be order-embedded [3, Theorem 4]. If R is a symmetric binary relation on U, then (U, U, R) forms a formal context in which the concepts are the maximal pairs $(A, B) \in \wp(U) \times \wp(U)$ such that every element of A is R-related to every element of B. Thus, if $(A, B) \in \mathfrak{B}(U, U, R)$, then also $(B, A) \in \mathfrak{B}(U, U, R)$. The map

$$\sim : \mathfrak{B}(U, U, R) \to \mathfrak{B}(U, U, R),\ (A, B) \mapsto (B, A)$$

is a polarity on $\mathfrak{B}(U, U, R)$. It is easy to see that if the relation R is irreflexive, then the extent and the intent of each concept must be disjoint and we have

$$(A, B) \wedge (B, A) = (\emptyset, M) \text{ and } (A, B) \vee (B, A) = (G, \emptyset).$$

This means that the map $\sim$ is an orthocomplementation and the concept lattice $\mathfrak{B}(U, U, R)$ is an orthocomplemented lattice.

Now for a tolerance R on U, we consider the context (U, U, R^c), where $R^c = \{(a, b) \in U^2 \mid (a, b) \notin R\}$. Following the terminology by Yao [20], (U, U, R^c) is called a *complement formal context*. Then R^c is an irreflexive and symmetric relation and in view of the previous observations, $\mathfrak{B}(U, U, R^c)$ is an orthocomplemented complete lattice, where the orthocomplement of an element $(A, B) \in \mathfrak{B}(U, U, R^c)$ is just (B, A).

Let us now consider the complement formal context (U, U, R^c) in more detail. For any $X \subseteq U$, we obtain

$$\begin{aligned} X^I &= \{x \in U \mid y\, R^c\, x \text{ for all } y \in X\} \\ &= \{x \in U \mid (y, x) \notin R \text{ for all } y \in X\} \\ &= \{x \in U \mid (x, y) \notin R \text{ for all } y \in X\} \\ &= \{x \in U \mid R(x) \cap X = \emptyset\} \\ &= X^{\blacktriangle c} = X^{c\blacktriangledown}. \end{aligned}$$

Thus, $X^{\blacktriangle} = X^{Ic}$ and $X^{\blacktriangledown} = X^{cI}$. From here we get that

$$X^{II} = X^{\blacktriangle c \blacktriangle c} = X^{\blacktriangle\blacktriangledown} = \Diamond X.$$

Since the orthocomplement of $\Diamond X$ in $\wp(U)^{\blacktriangledown}$ equals $(\Diamond X)^{\perp} = X^{\blacktriangle\blacktriangledown c\blacktriangledown} = X^{\blacktriangle c} = X^I$, the concept lattice of the complement context $\mathscr{K} = (U, U, R^c)$ has the form

$$\mathfrak{B}(\mathscr{K}) = \{(X^{\blacktriangle\blacktriangledown}, X^{c\blacktriangledown}) \mid X \subseteq U\} = \{(\Diamond X, (\Diamond X)^{\perp}) \mid X \subseteq U\}.$$

On the other hand, for each $A \in \wp(U)^{\blacktriangledown}$, we have $A = \Diamond A = A^{\blacktriangle\blacktriangledown}$, and so $(A, A^{\perp})$ belongs to $\mathfrak{B}(\mathscr{K})$. Hence,

$$\mathfrak{B}(\mathscr{K}) = \{(A, A^{\perp}) \mid A \in \wp(U)^{\blacktriangledown}\}.$$

We can write the following proposition.

Proposition 44 *Let R be a tolerance on a set U and let $\mathscr{K}$ be the complement formal context (U, U, R^c).*

(a) $\mathfrak{B}(\mathscr{K})$ *is isomorphic to* $(\wp(U)^{\blacktriangle}, \subseteq)$ *and* $(\wp(U)^{\blacktriangledown}, \subseteq)$.
(b) $\mathfrak{B}(\mathscr{K})$ *is a complete sublattice of the direct product of* $(\wp(U)^{\blacktriangledown}, \subseteq)$ *and* $(\wp(U)^{\blacktriangledown}, \supseteq)$.

Proof

(a) It is obvious that the map $A \mapsto (A, A^{\perp})$ is an isomorphism between $\wp(U)^{\blacktriangledown}$ and $\mathfrak{B}(\mathscr{K})$. In Proposition 8 we proved that $\wp(U)^{\blacktriangledown}$ and $\wp(U)^{\blacktriangle}$ are isomorphic.
(b) Clearly, $\mathfrak{B}(\mathscr{K}) \subseteq \wp(U)^{\blacktriangledown} \times \wp(U)^{\blacktriangledown}$. Let $\{(A_j, B_j)\}_{j \in J} \subseteq \mathfrak{B}(\mathscr{K})\}$. The join $\bigvee_{j \in J}(A_j, B_j)$ in $\mathfrak{B}(\mathscr{K})\}$ equals $((\bigcup_{j\in J} A_j)^{II}, \bigcap_{j \in J} B_j)$. Because $\bigcap_{j\in J} B_j$ is the meet operation in $(\wp(U)^{\blacktriangledown}, \subseteq)$, it is the join operation in $(\wp(U)^{\blacktriangledown}, \supseteq)$. Moreover, $(\bigcup_{j\in J} A_j)^{II} = \Diamond(\bigcup_{j \in J} A_j)$ is the join in $(\wp(U)^{\blacktriangledown}, \subseteq)$. Therefore, the join of any $\{(A_j, B_j)\}_{j\in J}$ in $\mathfrak{B}(\mathscr{K})$ coincides with join in the direct product of $(\wp(U)^{\blacktriangledown}, \subseteq)$ and $(\wp(U)^{\blacktriangledown}, \supseteq)$. An analogous argument is valid for meets. These facts mean that $\mathfrak{B}(\mathscr{K})$ is a complete sublattice of the direct product of the complete lattices $(\wp(U)^{\blacktriangledown}, \subseteq)$ and $(\wp(U)^{\blacktriangledown}, \supseteq)$. □

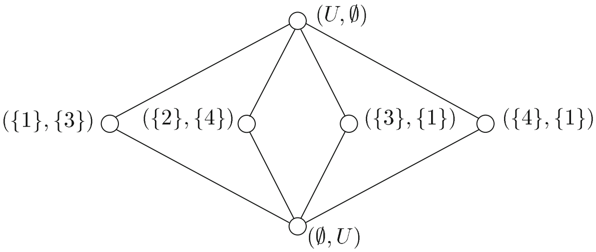


Fig. 13 The concept lattice $\mathfrak{B}(U, U, R^c)$

Example 45 Let us consider the tolerance R defined in Example 13 on the universe $U = \{1, 2, 3, 4\}$. Then, $R^c = \{(1, 3), (3, 1), (2, 4), (4, 2)\}$ and the concept lattice of the complement formal context (U, U, R^c) is depicted in Fig. 13. Obviously, the concept lattice $\mathfrak{B}(U, U, R^c)$ is isomorphic to the ortholattices $\wp(U)^{\blacktriangle}$ and $\wp(U)^{\blacktriangledown}$ depicted on Fig. 10.

By Corollary 37 we obtain the following result concerning tolerances induced by irredundant coverings.

Corollary 46 *Let R be a tolerance induced by an irredundant covering of U. Then $\wp(U)^{\blacktriangle}$, $\wp(U)^{\blacktriangledown}$ and $\mathfrak{B}(U, U, R^c)$ are isomorphic complete atomistic Boolean lattices.*

It is worth to mention that if L is a complete ortholattice, then there exists a context $\mathscr{K} = (U, U, I)$, where I is an irreflexive and symmetric binary relation on U, such that $L \cong B(\mathscr{K})$ [3, p. 54]. If we set $R = I^c$, then R is a tolerance on U, and by Proposition 44, the rough approximations lattices $(\wp(U)^{\blacktriangle}, \subseteq)$ and $(\wp(U)^{\blacktriangledown}, \subseteq)$ are isomorphic to $\mathfrak{B}(\mathscr{K})$. Therefore, we get the following representation theorem for complete ortholattices in terms of rough approximations.

Proposition 47 *A complete lattice L is an ortholattice if and only if there exist a set U and a tolerance R on U such that $L \cong \wp(U)^{\blacktriangledown} \cong \wp(U)^{\blacktriangle}$.*

3 Rough Set Systems Determined by Tolerances

In this section we consider the rough sets defined by a tolerance R and the order-theoretical properties of the collection *RS* of them. Section 3.1 is devoted to rough sets defined by tolerances in general. We show that even these structures do not necessarily form lattices, they have a polarity operation, which is an order-isomorphism between $(RS, \leq)$ and its dual $(RS, \geq)$. In Sect. 3.2 we study rough sets defined by tolerances induced by irredundant coverings. We show that *RS* forms a Kleene algebra and a double pseudocomplemented lattice. As a double pseudocomplemented lattice, *RS* is determination trivial. Viewed as a pseudocomplemented Kleene algebra, *RS* is normal. This chapter ends by Sect. 3.3, where we show that in case the tolerance is induced by an irredundant covering, the relation-based and covering-based rough sets systems are isomorphic.

3.1 The General Case

Originally Pawlak [16, p. 351] defined a rough set as an equivalence class of sets which look the same in view of the knowledge restricted by the given indistinguishability relation, that is, as a class of sets having the same lower approximation and the same upper approximation. This concept generalizes in a natural way to similarity relations.

Let R be a tolerance on U. A relation $\equiv$ is defined on $\wp(U)$ by

$$X \equiv Y \iff X^{\blacktriangledown} = Y^{\blacktriangledown} \text{ and } X^{\blacktriangle} = Y^{\blacktriangle}.$$

The equivalence classes of $\equiv$ are called *rough sets*. Each element in a given rough set looks the same, when observed through the knowledge given by the tolerance R. Namely, if $X \equiv Y$, then exactly the same elements belong certainly or possibly to X and Y.

The order-theoretical study of rough sets was initiated by T.B. Iwiński in [5]. In his approach rough sets on U are the pairs $(X^{\blacktriangledown}, X^{\blacktriangle})$, where $X \subseteq U$. This is justified because if $\mathscr{C} \subseteq \wp(U)$ is a rough set as defined before, that is, $\mathscr{C}$ is an equivalence class of $\equiv$, then $\mathscr{C}$ is uniquely determined by the pair $(X^{\blacktriangledown}, X^{\blacktriangle})$, where X is any member of $\mathscr{C}$: a set $Y \subseteq U$ belongs to $\mathscr{C}$ if and only if $(Y^{\blacktriangledown}, Y^{\blacktriangle}) = (X^{\blacktriangledown}, X^{\blacktriangle})$. Therefore, we call

$$RS = \{(X^{\blacktriangledown}, X^{\blacktriangle}) \mid X \subseteq U\}$$

the *set of rough sets*. The set *RS* is ordered by the componentwise inclusion:

$$(X^{\blacktriangledown}, X^{\blacktriangle}) \leq (Y^{\blacktriangledown}, Y^{\blacktriangle}) \iff X^{\blacktriangledown} \subseteq Y^{\blacktriangledown} \text{ and } X^{\blacktriangle} \subseteq Y^{\blacktriangle}.$$

Example 48 Let $U = \{1, 2, 3, 4, 5\}$ and let R be the tolerance on U depicted in Fig. 14. The lower and upper approximations defined by R are presented in Table 5.

The Hasse diagram of *RS* is given in Fig. 15. In the figure, sets are denoted simply by sequences of letters, that is, 124 denotes the set $\{1, 2, 4\}$. Now *RS* is not a lattice because, for instance, the elements $(1, 123)$ and $(\emptyset, 1234)$ do not have a

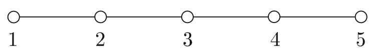


Fig. 14 The graph of a tolerance R

Table 5 Approximations based on the tolerance R

X	$(X^{\blacktriangledown}, X^{\blacktriangle})$	X	$(X^{\blacktriangledown}, X^{\blacktriangle})$
∅	(∅, ∅)	{1, 2, 3}	(12, 1234)
{1}	(∅, 12)	{1, 2, 4}	(1, U)
{2}	(∅, 123)	{1, 2, 5}	(1, U)
{3}	(∅, 234)	{1, 3, 4}	(∅, U)
{4}	(∅, 345)	{1, 3, 5}	(∅, U)
{5}	(∅, 45)	{1, 4, 5}	(5, U)
{1, 2}	(1, 123)	{2, 3, 4}	(3, U)
{1, 3}	(∅, 1234)	{2, 3, 5}	(∅, U)
{1, 4}	(∅, U)	{2, 4, 5}	(5, U)
{1, 5}	(∅, 1245)	{3, 4, 5}	(45, 2345)
{2, 3}	(∅, 1234)	{1, 2, 3, 4}	(123, U)
{2, 4}	(∅, U)	{1, 2, 3, 5}	(12, U)
{2, 5}	(∅, U)	{1, 2, 4, 5}	(15, U)
{3, 4}	(∅, 2345)	{1, 3, 4, 5}	(45, U)
{3, 5}	(∅, 2345)	{2, 3, 4, 5}	(345, U)
{4, 5}	(5, 345)	U	(U, U)

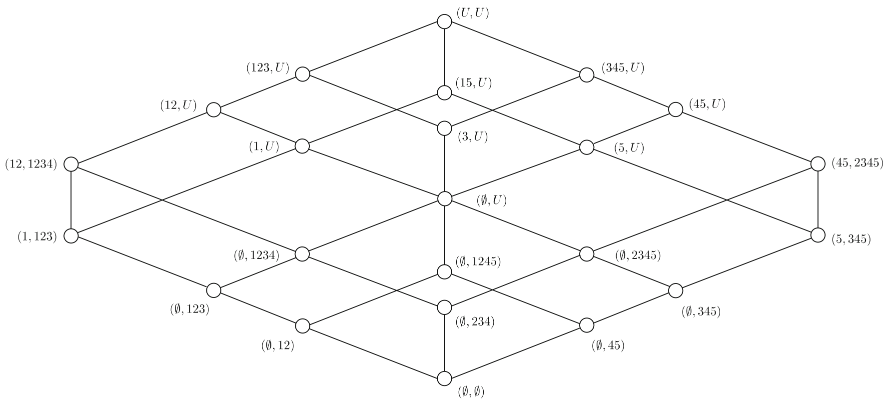


Fig. 15 The ordered set *RS* not forming a lattice

join, because (12, 1234) and (1, U) are the minimal upper bounds for (1, 123) and (∅, 1234), but there is no smallest upper bound. Similarly, we may observe that the elements (12, 1234) and (1, U), do not have a meet.

We end this subsection by some basic observations on the structure of *RS*. Because $(\emptyset^{\blacktriangledown}, \emptyset^{\blacktriangle}) = (\emptyset, \emptyset)$ and $(U^{\blacktriangledown}, U^{\blacktriangle}) = (U, U)$, the ordered set *RS* is always bounded. The pair (∅, ∅) is the least element and (U, U) is the greatest element.

Let us consider the mapping

$$\sim\colon RS \to RS,\ (X^{\blacktriangledown}, X^{\blacktriangle}) \mapsto (X^{\blacktriangle c}, X^{\blacktriangledown c}).$$

It is easy to see that $\sim(X^{\blacktriangledown}, X^{\blacktriangle}) = (X^{c\blacktriangledown}, X^{c\blacktriangle})$, which means that $\sim(X^{\blacktriangledown}, X^{\blacktriangle})$ belongs to *RS*, and that the map $\sim$ is well defined. Clearly, $\sim\sim(X^{\blacktriangledown}, X^{\blacktriangle}) = (X^{\blacktriangledown}, X^{\blacktriangle})$. Furthermore, $(X^{\blacktriangledown}, X^{\blacktriangle}) \leq (Y^{\blacktriangledown}, Y^{\blacktriangle})$ implies $\sim(Y^{\blacktriangledown}, Y^{\blacktriangle}) = (Y^{\blacktriangle c}, Y^{\blacktriangledown c}) \leq (X^{\blacktriangle c}, X^{\blacktriangledown c}) = \sim(X^{\blacktriangledown}, X^{\blacktriangle})$. This means that $\sim$ is a polarity and hence *RS* is anti-isomorphic to itself, that is, *RS* looks the same when turned upside-down (see e.g. Fig. 15).

Because $(\wp(U)^{\blacktriangledown}, \subseteq)$ and $(\wp(U)^{\blacktriangle}, \subseteq)$ are complete lattices, their direct product

$$\wp(U)^{\blacktriangledown} \times \wp(U)^{\blacktriangle} = \{(A, B) \mid A \in \wp(U)^{\blacktriangledown} \text{ and } B \in \wp(U)^{\blacktriangle}\}$$

ordered coordinatewise by $\subseteq$ is a complete lattice in which

$$\bigwedge_{i \in I}(A_i, B_i) = \Big(\bigcap_{i \in I} A_i, \Box\big(\bigcap_{i \in I} B_i\big)\Big) \tag{8}$$

and

$$\bigvee_{i \in I}(A_i, B_i) = \Big(\Diamond\big(\bigcup_{i \in I} A_i,\big), \bigcup_{i \in I} B_i\Big) \tag{9}$$

for all $(A_i, B_i)_{i \in I} \subseteq \wp(U)^{\blacktriangledown} \times \wp(U)^{\blacktriangle}$.

We have proved in [8] that for any tolerance R on U, *RS* is a complete lattice if and only if it is a complete sublattice of the direct product $\wp(U)^{\blacktriangledown} \times \wp(U)^{\blacktriangle}$. This means that whenever *RS* is a complete lattice, we know how the joins and meets are defined. Namely, if $\mathcal{H} \subseteq \wp(U)$, then in *RS*,

$$\bigwedge_{X \in \mathcal{H}} (X^{\blacktriangledown}, X^{\blacktriangle}) = \Big(\bigcap_{X \in \mathcal{H}} X^{\blacktriangledown}, \Box\big(\bigcap_{X \in \mathcal{H}} X^{\blacktriangle}\big)\Big) \tag{10}$$

and

$$\bigvee_{X \in \mathcal{H}} (X^{\blacktriangledown}, X^{\blacktriangle}) = \Big(\Diamond\big(\bigcup_{X \in \mathcal{H}} X^{\blacktriangledown}\big), \bigcup_{X \in \mathcal{H}} X^{\blacktriangle}\Big). \tag{11}$$

Let us emphasize that showing that *RS* is a complete lattice is not a simple task, because it needs to show that for any $\mathcal{H} \subseteq \wp(U)$, there are sets $A, B \subseteq U$ such that

$$A^{\blacktriangledown} = \bigcap_{X \in \mathcal{H}} X^{\blacktriangledown} \quad \text{and} \quad A^{\blacktriangle} = \Diamond\big(\bigcap_{X \in \mathcal{H}} X^{\blacktriangle}\big) \tag{12}$$

and

$$B^{\blacktriangledown} = \Diamond\big(\bigcup_{X\in\mathcal{H}} X^{\blacktriangledown}\big) \quad \text{and} \quad B^{\blacktriangle} = \bigcup_{X\in\mathcal{H}} X^{\blacktriangle}. \tag{13}$$

3.2 Rough Sets Defined by Tolerances Induced by an Irredundant Covering

In this section we recall some results which can be found in [8, 10]. We omit the proof of Proposition 49 because it is rather long and technical, and the interested reader may find it in [8].

Proposition 49 *Let R be a tolerance on U induced by an irredundant covering. Then RS is a complete lattice such that for all $\mathcal{H} \subseteq \wp(U)$,*

$$\bigwedge_{X\in\mathcal{H}} (X^{\blacktriangledown}, X^{\blacktriangle}) = \Big(\bigcap_{X\in\mathcal{H}} X^{\blacktriangledown}, \Box\big(\bigcap_{X\in\mathcal{H}} X^{\blacktriangle}\big)\Big)$$

and

$$\bigvee_{X\in\mathcal{H}} (X^{\blacktriangledown}, X^{\blacktriangle}) = \Big(\Diamond\big(\bigcup_{X\in\mathcal{H}} X^{\blacktriangledown}\big), \bigcup_{X\in\mathcal{H}} X^{\blacktriangle}\Big).$$

□

Notice that if R is a tolerance induced by an irredundant covering, then $\wp(U)^{\blacktriangledown}$ and $\wp(U)^{\blacktriangle}$ are distributive lattices, and hence their direct product

$$\wp(U)^{\blacktriangledown} \times \wp(U)^{\blacktriangle} = \{(A, B) \mid A \in \wp(U)^{\blacktriangledown} \text{ and } B \in \wp(U)^{\blacktriangle}\}$$

is a distributive lattice in which the operations are defined coordinatewise. As a sublattice of the distributive lattice $\wp(U)^{\blacktriangledown} \times \wp(U)^{\blacktriangle}$, also RS is distributive.

Example 50 Let R be the tolerance of Fig. 2 on $U = \{1, 2, 3, 4, 5\}$. As we have noted, R is a tolerance induced by an irredundant covering, so RS forms a lattice by Proposition 49. The rough approximations of subsets of U are given in Table 6, and the rough set lattice RS is given in Fig. 16.

For a detailed study of the structure of RS, we will need some further notions. A *De Morgan algebra* is a structure $(L, \vee, \wedge, \sim, 0, 1)$ such that $(L, \vee, \wedge, 0, 1)$ is a bounded distributive lattice and the operation $\sim$ satisfies the following equations:

(DM1) $\sim(x \wedge y) = \sim x \vee \sim y$;
(DM2) $\sim(x \vee y) = \sim x \wedge \sim y$;
(DM3) $\sim\sim x = x$.

Table 6 Approximations based on the tolerance R of Example 50

X	$(X^{\blacktriangledown}, X^{\blacktriangle})$	X	$(X^{\blacktriangledown}, X^{\blacktriangle})$
$\emptyset$	$(\emptyset, \emptyset)$	$\{1, 2, 3\}$	$(3, U)$
$\{1\}$	$(\emptyset, 1345)$	$\{1, 2, 4\}$	$(2, U)$
$\{2\}$	$(\emptyset, 24)$	$\{1, 2, 5\}$	$(\emptyset, U)$
$\{3\}$	$(\emptyset, 13)$	$\{1, 3, 4\}$	$(3, U)$
$\{4\}$	$(\emptyset, 1245)$	$\{1, 3, 5\}$	$(3, 1345)$
$\{5\}$	$(\emptyset, 145)$	$\{1, 4, 5\}$	$(5, U)$
$\{1, 2\}$	$(\emptyset, U)$	$\{2, 3, 4\}$	$(2, U)$
$\{1, 3\}$	$(3, 1345)$	$\{2, 3, 5\}$	$(\emptyset, U)$
$\{1, 4\}$	$(\emptyset, U)$	$\{2, 4, 5\}$	$(2, 1245)$
$\{1, 5\}$	$(\emptyset, 1345)$	$\{3, 4, 5\}$	$(\emptyset, U)$
$\{2, 3\}$	$(\emptyset, 1234)$	$\{1, 2, 3, 4\}$	$(23, U)$
$\{2, 4\}$	$(2, 1245)$	$\{1, 2, 3, 5\}$	$(3, U)$
$\{2, 5\}$	$(\emptyset, 1245)$	$\{1, 2, 4, 5\}$	$(245, U)$
$\{3, 4\}$	$(\emptyset, U)$	$\{1, 3, 4, 5\}$	$(135, U)$
$\{3, 5\}$	$(\emptyset, 1345)$	$\{2, 3, 4, 5\}$	$(2, U)$
$\{4, 5\}$	$(\emptyset, 1245)$	U	(U, U)

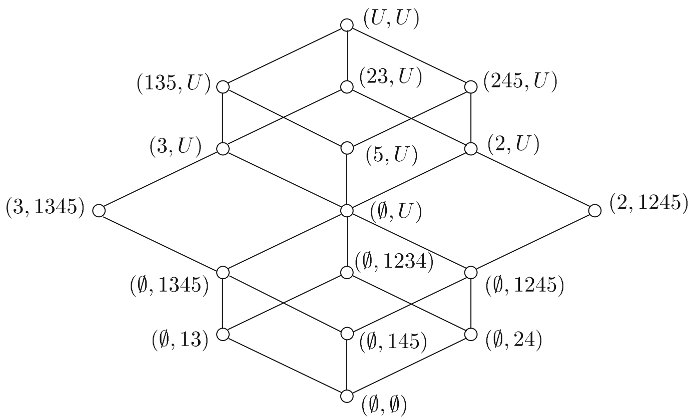


Fig. 16 The lattice *RS*

Remark 51 Notice that under (DM3), (DM1) and (DM2) are equivalent. For example, if (DM1) holds, then

$$\sim(x \vee y) = \sim(\sim\sim x \vee \sim\sim y) = \sim\sim(\sim x \wedge \sim y) = \sim x \wedge \sim y,$$

that is, (DM2) holds. Similarly, (DM2) implies (DM1).

It should be also noted that in a De Morgan algebra $(L, \vee, \wedge, \sim, 0, 1)$, the map $\sim$ is a polarity on L, because (DM3) is part of the definition of an polarity, and $x \leq y$ implies $\sim x = \sim(x \wedge y) = \sim x \vee \sim y$, which is equivalent to $\sim y \leq \sim x$.

On the other hand, if $\sim$ is a polarity on a distributive lattice $(L, \leq)$, then $x, y \leq x \vee y$ implies that $\sim(x \vee y)$ is a lower bound of $\sim x$ and $\sim y$. Assume that z is a lower bound of $\sim x$ and $\sim y$. Because $\sim$ is an involution, $\sim z \geq x$ and $\sim z \geq y$, and therefore $x \vee y \leq \sim z$ and $z \leq \sim(x \vee y)$. Hence, $\sim(x \vee y)$ is the greatest lower bound of $\sim x$ and $\sim y$, that is, $\sim(x \vee y) = \sim x \wedge \sim y$. If L is also bounded by 0 and 1, then $(L, \vee, \wedge, \sim, 0, 1)$ is a De Morgan algebra.

Proposition 52 *If R is a tolerance induced by an irredundant covering, then*

$$(RS, \vee, \wedge, \sim, (\emptyset, \emptyset), (U, U))$$

is a De Morgan algebra.

Proof If R is a tolerance induced by an irredundant covering, then RS is a distributive lattice by Proposition 49. We have already noted that RS is bounded by $(\emptyset, \emptyset)$ and (U, U). In Sect. 3.1 we showed that $\sim\colon (X^{\blacktriangledown}, X^{\blacktriangle}) \mapsto (X^{\blacktriangle c}, X^{\blacktriangledown c})$ is a polarity. □

It is quite obvious that even when RS is defined by a tolerance induced by an irredundant covering, it is not in general a Boolean lattice. For instance, in Example 50, the only elements that have complements are $(\emptyset, \emptyset)$ and (U, U). Therefore, the equalities $x \wedge \sim x = 0$ or $y \vee \sim y = 1$ do not hold, in general. If a De Morgan algebra satisfies the inequality

$$x \wedge \sim x \leq y \vee \sim y, \tag{K}$$

it is called a *Kleene algebra*.

Proposition 53 *If R is a tolerance induced by an irredundant covering, then*

$$(RS, \vee, \wedge, \sim, (\emptyset, \emptyset), (U, U))$$

is a Kleene algebra.

Proof Let $X, Y \subseteq U$. Then,

$$(X^{\blacktriangledown}, X^{\blacktriangle}) \wedge \sim(X^{\blacktriangledown}, X^{\blacktriangle}) = (X^{\blacktriangledown} \cap X^{c\blacktriangledown}, \Box(X^{\blacktriangle} \cap X^{c\blacktriangle})) = (\emptyset, \Box(X^{\blacktriangle} \cap X^{c\blacktriangle}))$$

and

$$(Y^{\blacktriangledown}, Y^{\blacktriangle}) \vee \sim(Y^{\blacktriangledown}, Y^{\blacktriangle}) = (\Diamond(Y^{\blacktriangledown} \cup Y^{c\blacktriangledown}), Y^{\blacktriangle} \cup Y^{c\blacktriangle}) = (\Diamond(Y^{\blacktriangledown} \cup Y^{c\blacktriangledown}), U).$$

From these equations we see directly that

$$(X^{\blacktriangledown}, X^{\blacktriangle}) \wedge \sim(X^{\blacktriangledown}, X^{\blacktriangle}) \leq (Y^{\blacktriangledown}, Y^{\blacktriangle}) \vee \sim(Y^{\blacktriangledown}, Y^{\blacktriangle}).$$

□

In Sect. 2.3 we already considered pseudocomplements. Let us recall that in a lattice L with a least element 0, an element denoted by x^* is the *pseudocomplement* of an element $x \in L$, if for any $z \in L$, $x \wedge z = 0$ if and only if $z \leq x^*$, and that L is said to be *pseudocomplemented* if every element has a pseudocomplement. Analogously, a *dual pseudocomplement* of $x \in L$ is an element x^+ such that $x \vee z = 1$ if and only if $z \geq x^+$. If L is such that each element has a pseudocomplement and a dual pseudocomplement, then L is called a *double pseudocomplemented lattice*.

In the following we list some properties of pseudocomplements. Let L be a double pseudocomplemented lattice and $a, b \in L$. Then

(1) $a \leq b$ implies $b^* \leq a^*$ and $b^+ \leq a^+$,
(2) $a^{++} \leq a \leq a^{**}$,
(3) $a^* = a^{***}$ and $a^+ = a^{+++}$.

The next proposition shows that if R is induced by an irredundant covering, then RS is a double pseudocomplemented lattice.

Proposition 54 *Let R be a tolerance induced by an irredundant covering. Then RS is a double pseudocomplemented lattice in which*

$$(A, B)^* = (B^{c\blacktriangledown}, B^{c\blacktriangle}) \qquad \textit{and} \qquad (A, B)^+ = (A^{c\blacktriangledown}, A^{c\blacktriangle})$$

for any $(A, B) \in RS$.

Proof This proof uses many of the properties listed in Proposition 5. Let $(A, B) \in RS$. First, we show that $(A, B) \wedge (B^{c\blacktriangledown}, B^{c\blacktriangle}) = (A \cap B^{c\blacktriangledown}, (B \cap B^{c\blacktriangle})^{\blacktriangledown\blacktriangle})$ equals $(\emptyset, \emptyset)$. It suffices to show that the right component $(B \cap B^{c\blacktriangle})^{\blacktriangledown\blacktriangle}$ is $\emptyset$, because then necessarily the left component $A \cap B^{c\blacktriangledown}$ is also empty. Indeed,

$$(B \cap B^{c\blacktriangle})^{\blacktriangledown\blacktriangle} = (B^{\blacktriangledown} \cap B^{c\blacktriangle\blacktriangledown})^{\blacktriangle} = (B^{\blacktriangledown} \cap B^{\blacktriangledown\blacktriangle c})^{\blacktriangle} \subseteq (B^{\blacktriangledown\blacktriangle} \cap B^{\blacktriangledown\blacktriangle c})^{\blacktriangle} = \emptyset^{\blacktriangle} = \emptyset.$$

On the other hand, if $(A, B) \wedge (X, Y) = \emptyset$ for some $(X, Y) \in RS$, then $B \wedge Y = \emptyset$ in the corresponding Boolean lattice $\wp(U)^{\blacktriangle}$. This gives $Y \subseteq B^{c\blacktriangle}$, since $B^{c\blacktriangle}$ is the complement of B in the Boolean lattice $\wp(U)^{\blacktriangle}$ by Corollary 37. To show that $X \subseteq B^{c\blacktriangledown}$ requires more work. Because $(X, Y) \in RS$, $X = Z^{\blacktriangledown}$ and $Y = Z^{\blacktriangle}$ for some $Z \subseteq U$. We have $X^{\blacktriangle} = Z^{\blacktriangledown\blacktriangle} \subseteq Z \subseteq Z^{\blacktriangle\blacktriangledown} = Y^{\blacktriangledown}$. This implies $X^{\blacktriangle\blacktriangle} \subseteq Y^{\blacktriangledown\blacktriangle} \subseteq Y \subseteq B^{c\blacktriangle}$ and further $X^{\blacktriangle} \subseteq (X^{\blacktriangle})^{\blacktriangle\blacktriangledown} \subseteq B^{c\blacktriangle\blacktriangledown}$. Now $B \in \wp(U)^{\blacktriangle}$ means that $B = C^{\blacktriangle}$ for some $C \subseteq U$. We have

$$B^{c\blacktriangle\blacktriangledown} = B^{\blacktriangledown\blacktriangle c} = C^{\blacktriangle\blacktriangledown\blacktriangle c} = C^{\blacktriangle c} = B^c.$$

We get by the above that

$$X \subseteq X^{\blacktriangle\blacktriangledown} \subseteq B^{c\blacktriangle\blacktriangledown\blacktriangledown} = B^{c\blacktriangledown}.$$

We have now shown that $(X, Y) \leq (B^{c\blacktriangledown}, B^{c\blacktriangle})$ which completes the proof.

The claim concerning $(A, B)^+$ can be proved similarly. □

Let L be a double pseudocomplemented lattice. We say that L is *determination-trivial* if for all $x, y \in L$,

$$x^* = y^* \text{ and } x^+ = y^+ \text{ imply } x = y. \tag{M}$$

Proposition 55 *If R is a tolerance induced by an irredundant covering, then the double pseudocomplemented lattice RS is determination trivial.*

Proof If $(A, B)^* = (C, D)^*$, then $B^{\blacktriangledown c} = B^{c\blacktriangle} = D^{c\blacktriangle} = D^{\blacktriangledown c}$. So $B^{\blacktriangledown} = D^{\blacktriangledown}$ and $B^{\blacktriangledown\blacktriangle} = D^{\blacktriangledown\blacktriangle}$. Because $B, D \in \wp(U)^{\blacktriangle}$, $B = B^{\blacktriangledown\blacktriangle} = D^{\blacktriangledown\blacktriangle} = D$. Similarly, $(A, B)^+ = (C, D)^+$ implies $A = C$. We have proved that $(A, B) = (C, D)$. □

A *pseudocomplemented De Morgan algebra* is an algebra $(L, \vee, \wedge, \sim, ^*, 0, 1)$ such that $(L, \vee, \wedge, \sim, 0, 1)$ is a De Morgan algebra and $^* \colon L \to L$ is a pseudocomplement operation of L. Every pseudocomplemented De Morgan algebra forms a double pseudocomplemented lattice in which the pseudocomplements determine each other by:

$$x^* = \sim(\sim x)^+ \quad \text{and} \quad x^+ = \sim(\sim x)^*. \tag{14}$$

A pseudocomplemented De Morgan algebra $(L, \vee, \wedge, \sim, ^*, 0, 1)$ is *normal* (see [14]), if for all $x \in L$,

$$x^* \le \sim x. \tag{N}$$

Note that if $(L, \vee, \wedge, \sim, ^*, 0, 1)$ is a normal pseudocomplemented De Morgan algebra, then for every $x \in L$ and $y = \sim x$, we have $\sim(\sim y)^+ = y^* \le \sim y$. Hence $(\sim y)^+ \ge y$ and so $x^+ \ge \sim x$. Thus,

$$x^* \le \sim x \le x^+.$$

It is known [11, 19] that in any distributive double pseudocomplemented lattice, condition (M) is equivalent to condition

$$x \wedge x^+ \le y \vee y^*. \tag{D}$$

We say that a pseudocomplemented Kleene algebra is normal if the underlying pseudocomplemented De Morgan algebra is normal.

Proposition 56 *If R is a tolerance induced by an irredundant covering, then the pseudocomplemented De Morgan algebra $(RS, \vee, \wedge, \sim, ^*, (\emptyset, \emptyset), (U, U))$ is normal.*

Proof Let $(A, B) \in RS$. Then by Proposition 54,

$$(A, B)^* = (B^{c\blacktriangledown}, B^{c\blacktriangle}).$$

We also have that

$$\sim (A, B) = (B^c, A^c).$$

Trivially, $B^{c\blacktriangledown} \subseteq B^c$. Since $(A, B) \in RS$, $A = X^\blacktriangledown$ and $B = X^\blacktriangle$ for some $X \subseteq U$. We have

$$A^\blacktriangle = X^{\blacktriangledown\blacktriangle} \subseteq X \subseteq X^{\blacktriangle\blacktriangledown} = B^\blacktriangledown.$$

This, implies

$$B^{c\blacktriangle} = B^{\blacktriangledown c} \subseteq A^{\blacktriangle c} = A^{c\blacktriangledown} \subseteq A^c.$$

We have now proved that $(A, B)^* \leq {\sim}(A, B)$. □

3.3 Covering-Based Rough Set Systems

We end this chapter by showing that certain rough set lattices based on irredundant coverings are isomorphic to relation-based rough sets lattices. Therefore, they have all the properties listed in Sect. 3.2.

In Sect. 2.3, we defined the operators $^\blacktriangleright$, $^\blacktriangleleft$, $^\triangleright$, and $^\triangleleft$ in terms of a covering $\mathscr{H}$ of U. Let us first consider the operators $^\blacktriangleright$ and $^\triangleleft$ introduced by Żakowski. We denote by RS_0 the set of all rough sets defined by these operators, that is:

$$RS_0 = \{(X^\triangleleft, X^\blacktriangleright) \mid X \subseteq U\}.$$

We order RS_0, similarly as RS, by coordinatewise inclusion, that is to say,

$$(X^\triangleleft, X^\blacktriangleright) \leq (Y^\triangleleft, Y^\blacktriangleright) \iff X^\triangleleft \subseteq Y^\triangleleft \text{ and } X^\blacktriangleright \subseteq Y^\blacktriangleright.$$

Proposition 57 *If R is a tolerance induced by an irredundant covering $\mathscr{H}$, then*

$$RS \cong RS_0.$$

Proof We prove that the map $\varphi\colon (X^\blacktriangledown, X^\blacktriangle) \mapsto (X^\triangleleft, X^\blacktriangleright)$ is the required order-isomorphism. Suppose $(X^\blacktriangledown, X^\blacktriangle) \leq (Y^\blacktriangledown, Y^\blacktriangle)$. By Proposition 39, $X^\blacktriangleright = X^\blacktriangle \subseteq Y^\blacktriangle = Y^\blacktriangleright$. Proposition 41 gives that

$$X^\triangleleft = \Box X = X^{\blacktriangledown\blacktriangle} \subseteq Y^{\blacktriangledown\blacktriangle} = \Box Y = Y^\triangleleft.$$

Thus, $(X^\triangleleft, X^\blacktriangleright) \leq (Y^\triangleleft, Y^\blacktriangleright)$.

On the other hand, assume that $(X^{\lhd}, X^{\blacktriangleright}) \leq (Y^{\lhd}, Y^{\blacktriangleright})$. Then again trivially, $X^{\blacktriangle} = X^{\blacktriangleright} \subseteq Y^{\blacktriangleright} = Y^{\blacktriangle}$. Because

$$X^{\blacktriangledown\blacktriangle} = \Box X = X^{\lhd} \subseteq Y^{\lhd} = \Box Y = Y^{\blacktriangledown\blacktriangle},$$

we have $X^{\blacktriangledown} = X^{\blacktriangledown\blacktriangle\blacktriangledown} \subseteq Y^{\blacktriangledown\blacktriangle\blacktriangledown} = Y^{\blacktriangledown}$. Therefore, also $(X^{\blacktriangledown}, X^{\blacktriangle}) \leq (Y^{\blacktriangledown}, Y^{\blacktriangle})$ and the map φ is an order-embedding.

It is obvious that φ is onto RS_0. So, the map φ is an order-isomorphism. □

By definition, $^{\blacktriangleright}$ and $^{\blacktriangleleft}$ form a pair of dual operators. We denote the rough set system defined by them by RS_1, that is,

$$RS_1 = \{(X^{\blacktriangleleft}, X^{\blacktriangleright}) \mid X \subseteq U\}.$$

In Proposition 39 we showed that if R is a tolerance induced by a covering $\mathscr{H}$ of U, then $X^{\blacktriangle} = X^{\blacktriangleright}$ and $X^{\blacktriangledown} = X^{\blacktriangleleft}$ for every $X \subseteq U$. Therefore, we can write the following proposition.

Proposition 58 *If R is a tolerance induced by a covering $\mathscr{H}$, then*

$$RS = RS_1.$$

We denote the rough set system defined by the dual operators $^{\rhd}$ and $^{\lhd}$ by RS_2, that is,

$$RS_2 = \{(X^{\lhd}, X^{\rhd}) \mid X \subseteq U\}.$$

Again, we may write an isomorphism theorem for RS_2.

Proposition 59 *If R is a tolerance induced by an irredundant covering $\mathscr{H}$, then*

$$RS \cong RS_2.$$

Proof We noted in Proposition 41 that if $\mathscr{H}$ is an irredundant covering of U and R is induced by $\mathscr{H}$, then for any $X \subseteq U$, $X^{\lhd} = \Box X$ and $X^{\rhd} = \Diamond X$. Because $\Diamond X = X^{\blacktriangle\blacktriangledown}$ and $\Box X = X^{\blacktriangledown\blacktriangle}$, it suffices to show that the map

$$\varphi\colon (X^{\blacktriangledown}, X^{\blacktriangle}) \mapsto (X^{\blacktriangledown\blacktriangle}, X^{\blacktriangle\blacktriangledown})$$

is an order-isomorphism.

If $(X^{\blacktriangledown}, X^{\blacktriangle}) \leq (Y^{\blacktriangledown}, Y^{\blacktriangle})$, then $(X^{\blacktriangledown\blacktriangle}, X^{\blacktriangle\blacktriangledown}) \leq (Y^{\blacktriangledown\blacktriangle}, Y^{\blacktriangle\blacktriangledown})$. Similarly, $(X^{\blacktriangledown\blacktriangle}, X^{\blacktriangle\blacktriangledown}) \leq (Y^{\blacktriangledown\blacktriangle}, Y^{\blacktriangle\blacktriangledown})$ gives $X^{\blacktriangledown} = X^{\blacktriangledown\blacktriangle\blacktriangledown} \subseteq Y^{\blacktriangledown\blacktriangle\blacktriangledown} = Y^{\blacktriangledown}$ and $X^{\blacktriangle} = X^{\blacktriangle\blacktriangledown\blacktriangle} = Y^{\blacktriangle\blacktriangledown\blacktriangle} = X^{\blacktriangle}$. This means that the map φ is an order-embedding. The map φ is clearly onto. □

The next theorem summarizes the results presented in this section.

Table 7 Different approximation operators

X	$X^{\blacktriangledown}$	$X^{\blacktriangle}$	$X^{\lhd}$	$X^{\blacktriangleright}$	$\Box X$	$\Diamond X$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{a\}$	$\emptyset$	$\{a, b\}$	$\emptyset$	$\{a, b\}$	$\emptyset$	$\{a\}$
$\{b\}$	$\emptyset$	U	$\emptyset$	U	$\emptyset$	U
$\{c\}$	$\emptyset$	$\{b, c\}$	$\emptyset$	$\{b, c\}$	$\emptyset$	$\{c\}$
$\{a, b\}$	$\{a\}$	U	$\{a, b\}$	U	$\{a, b\}$	U
$\{a, c\}$	$\emptyset$	U	$\emptyset$	U	$\emptyset$	U
$\{b, c\}$	$\{c\}$	U	$\{b, c\}$	U	$\{b, c\}$	U
U	U	U	U	U	U	U

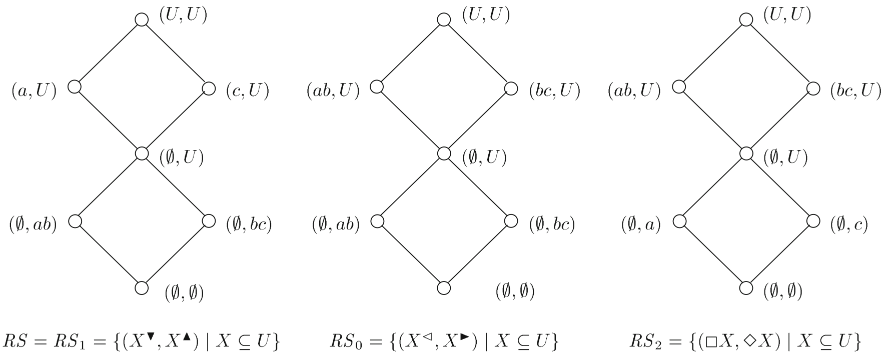


Fig. 17 The lattices $RS = RS_1$, RS_0, and RS_2

Theorem 60 *If R is tolerance induced by an irredundant covering $\mathscr{H}$, then*

$$RS = RS_1 \cong RS_0 \cong RS_2.$$

The following example demonstrates that, although the lattices $RS = RS_1$, RS_0, and RS_2 always are isomorphic, their elements may be different, and that different definitions may assign different rough approximations to a given set.

Example 61 Let us consider the tolerance R on $\{a, b, c\}$ of Example 4. The relation R was defined by $R(a) = \{a, b\}$, $R(b) = U$, and $R(c) = \{b, c\}$. The family of sets $\{R(a), R(c)\} = \{\{a, b\}, \{b, c\}\}$ is an irredundant covering inducing R. The different approximation operators are presented in Table 7.

The isomorphic rough sets lattices $RS = RS_1$, RS_0, and RS_2 are depicted in Fig. 17.

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Algebraic Representation, Dualities and Beyond

A. Mani

Abstract In this research chapter, dualities and representations of various kinds associated with the semantics of rough sets are explained, critically reviewed, new proofs have been proposed, open problems are specified and new directions are suggested. Some recent duality results in the literature are also adapted for use in rough contexts. New results are also proved on granular connections between generalized rough and L-fuzzy sets by the present author. Philosophical aspects of the concepts have also been considered by her in this research chapter.

1 Introduction

If one wants to restrict oneself to Mathematical problems within Mathematics alone from a reductionist perspective (first order or otherwise), then *representation problems* are the problems of constructing isomorphisms from a given class of algebraic systems into a subclass of the same class. *Duality problems*, in the same perspective are those of constructing adjoint equivalences between classes of algebraic or topological algebraic systems of possibly different types. Relative to this perspective, the problems considered in this chapter are representation, duality and somewhat-related problems. In the present author's perspective, this chapter is also about duality, representation and problems beyond these—her concept of representation differs from the one mentioned.

Many results of the mentioned kind are known in the study of rough sets. But a comprehensive or representative overview is not known. This research chapter is intended to fulfill this need from a critical perspective.

The problem of handling concepts of representation and duality in the contexts of rough sets is not a straightforward one. Even if one restricts oneself to full dualities or a specific sense of category-theoretic duality like adjunctions, a number of issues

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stand out. It is also that the mathematical formalisms used for handling rough sets have ontological commitments that become more jarring when categorical formalisms with its additional ontological commitments are initiated. These are explained to an extent by the present author in [98] in this volume. The *inverse problems* considered by her in [86, 94] is not fully included by duality in a mathematical sense, but are definitely so in a philosophical sense (at least as of this writing). In this research chapter, some of the available representation results, dualities and issues in formalization will be looked into. Some other representation and duality are also considered in other chapters [12, 61, 109] in this volume and some optimization has been at play.

Consider the following definitions:

Let $\mathbb{S}$ and $\mathbb{Z}$ be two classes of algebraic systems of the same type τ. If every algebraic system $S \in \mathbb{S}$ is isomorphic to a $Z \in \mathbb{Z}$, then $\mathbb{S}$ is said to be *represented in* $\mathbb{Z}$. In some cases, $\mathbb{Z}$ is a subclass of $\mathbb{S}$.

Let $\mathbb{S}$ and $\mathbb{Z}$ be two classes of algebras of type τ_1 and τ_2. If every algebraic system $S \in \mathbb{S}$ is isomorphic to an algebra Z^* derived (in the algebraic sense) from an algebra $Z \in \mathbb{Z}$, then $\mathbb{S}$ can be said to be *represented over* $\mathbb{Z}$.

Both definitions can be generalized to include partial operations. It can also be argued that both types of representation are essentially the same.

Mathematicians typically understand representation in a dyadic or triadic scheme of things. The latter happens because of the contexts in which they happen. For example, the problem of *representing lattices having some property* π *as the lattices of congruences of an algebra of type* τ *satisfying some property* ξ has been solved in a number of cases [51]. This can be read as a problem of representing a class within a subclass—but the formalism presumes first order objective conditions that do not exist.

Fundamental to these considerations is the concept of interpretation. Across ideas of representation and duality, it gets modified as below:

- If S is an algebraic system of type τ, then every endomorphism $\varphi \in E(S)$ *re-interprets* S *within itself.*
- If S, K are algebraic systems of signature Σ and Σ' respectively of type τ and τ' with τ being part of τ', then every forgetful Σ-morphism $\varphi \in Mor(K, S)$ *interprets* K *in* S.
- In particular, morphisms between algebraic systems of the same type interpret one in the other and isomorphisms are essentially equivalences that express bounds on interpretation by way of algebraic methods and properties.
- Typically, in duality theory, a category $\mathbf{C}$ is interpreted in another category $\mathbf{K}$ by way of interpretations of $\mathbf{C}_{obj}$ in $\mathbf{K}_{obj}$ and $\mathbf{C}_{mor}$ in $\mathbf{K}_{mor}$. That is, the interpretation spans both objects and morphisms.
- Representations of Algebraic Systems may be classified into abstract and concrete representations—these require a triadic understanding of representation for proper classification.
- There are many types of dualities in category theory and the ones of interest for rough sets go beyond what is considered as standard mathematics [86]. Rough

sets can, for example, be interpreted from the perspective of semisets. This in turn means connections with other types of non set-theoretic mereology. In this chapter, only some key mathematical results will be considered.

Some examples for representations are

- Every group is isomorphic to a group of permutations [15].
- Every Boolean ring is isomorphic to a Boolean algebra.
- Every Boolean algebra is isomorphic to a field of sets [136]. More specifically, Stone's theorem states that every Boolean algebra B can be extended to a complete and atomic Boolean algebra, and that this embedding is an isomorphism if the algebra B itself is atomic.
- Every Boolean algebra with operators is a subalgebra of a powerset algebra where the operators are obtained from relations on the base set [64]. This can be seen as a proper generalization of the previous result.
- Every regular double Stone algebra is isomorphic to a subalgebra of the algebra of rough sets for some approximation space $\langle \underline{S}, R \rangle$ [24].

The third, fourth and fifth of these also correspond to dualities. The last is a duality result between an algebra and a relational system of different types. In this regard it should be noted that fully fledged dualities involve two categories and functors between them with special properties:

- The category of Boolean algebras with Boolean homomorphisms is dual to the category of totally disconnected compact Hausdorff spaces and continuous mappings—an extension of Stone's Representation Theorem, see e.g. [71, Chapter 3] for details.
- The category of distributive lattice is dual to the category of compact totally order disconnected topological spaces and continuous increasing maps [120].

Many more representation and duality theorems can be found in the literature. Often, the representation or duality will involve a class (or category) of topological spaces. If one considers only the discrete topology then a duality will be called *discrete duality*, see [105] for a comprehensive overview.

Discrete dualities have the following form: If $\mathcal{A}$ is a class of algebras of a type τ and $\mathcal{F}$ a class of algebraic systems of type τ', then a discrete duality between the two classes is the tuple $\langle \mathcal{A}, \mathcal{F}, \mathfrak{f}, \mathfrak{g} \rangle$ such that

- $\mathfrak{f} : \mathcal{A} \longmapsto \mathcal{F}$ and $\mathfrak{g} : \mathcal{F} \longmapsto \mathcal{A}$ are maps,
- each $S \in \mathcal{A}$ is embeddable in $\mathfrak{g}\mathfrak{f}(S)$, and
- each $F \in \mathcal{F}$ is embeddable in $\mathfrak{f}\mathfrak{g}(F)$.

Some discrete dualities, of relevance in rough contexts, are also proved in this chapter.

Definition 1 A result or theory that concerns *representation or duality* will be referred to as a *red* result or theory respectively.

1.1 More on Duality

The concept of duality, in general, is not a crisp one and many competing definitions that capture aspects of philosophical motivations are known (see [101] for example). From a category-theoretical perspective, it can be argued that the loosest general notion of "duality" is that of adjunction or dual adjunction, as in pairs of adjoint functors (Lambek 81). Here one may omit any concretizations via functors to Set, or even for that matter any explicit mention of opposite categories, and just work at the level of abstract categories themselves.

Nevertheless, many adjunctions come packaged in *dual pairs*. A famous slogan from Categories for the Working Mathematician is that *all concepts are Kan extensions*, and in that light the dual pairs are instances of the general dual pair (*right Kan extension, left Kan extension*) which are formal duals in the axiomatic sense described earlier. Via the many incarnations of universal constructions in category theory, dualities may concern

- limit and colimits
- end and coends
- dependent sum and dependent products
- existential quantification and universal quantifications
- terminal and initial objects

When the adjoint functors are monads and hence modalities, then adjointness between them has been argued to specifically express the concept of duality of opposites.

Again, adjunctions and specifically dual adjunctions (*Galois connections*) may be thought of as generalized dualities, more general than "perfect duality" which involves equivalences between categories (*Galois correspondences* or adjoint equivalences). However, it should also be noted that any such adjunction (or dual adjunction) restricts to a maximal (dual) equivalence between subcategories, by considering objects where the appropriate units and counits are isomorphisms. This generalizes the manner by which any Galois connection induces a Galois correspondence (where in this special case, one need only take the images of the partially ordered set maps which constitute the connection). Any adjunction (or dual adjunction) between two categories is a duality context.

Furthermore, although an adjoint equivalence is more structured notion than a mere equivalence, the property of *being adjoint equivalent* is no stronger than *being equivalent*. This is because every equivalence may be refined to an adjoint equivalence by modifying one of the natural isomorphisms involved.

1.1.1 Syntactic Dualities

A number of logics have been found to be relevant in the context of reasoning with rough objects in the rough, classical and some other semantic domains. All related

Table 1 Dual operators

Operator	Meaning	Dual operator	Meaning
$\top$	Truth	$\perp$	Falsity
$\wedge$	Conjunction	$\vee$	Disjunction
$\Rightarrow$	Implication	$\setminus$	Without
$\Leftrightarrow$	Equivalence	$\oplus$	XOR
$\neg$	Negation	$\ominus$	B-negation
$\forall$	Universal Q	$\exists$	Existential Q
$\Box$	Necessitation	$\Diamond$	Possibility

syntactic and semantic dualities are of natural interest and results may be found all over this volume. For an abstract view of syntax-semantics duality and recent developments, the reader is referred to [2, 4, 28].

In logic, a De Morgan duality or a syntactic duality is a duality between logical operators of intuitionistic logic and dual-intuitionistic paraconsistent logic that extends to other logics. In classical logic and linear logic, it can be read as a self-duality. Modal operators are also accommodated in such dualities. As the dualities operate at the level of the respective languages (the language $\mathcal{L}$ underlying a logic forms a partial algebra in general), they can be read as algebraic dualities. De Morgan dualities in category theory are category-theoretic generalizations of these dualities of logic (Table 1).

Note that in the table *Quantification* is abbreviated by the letter 'Q'. Further $\neg p = p \Rightarrow \perp$ and $\ominus p = \top \setminus p$ may be definable operators in many logics including intuitionist logic.

All of the operators are used (or are definable) in classical logic. The two negations $\neg$ and $\ominus$ coincide in classical logic. The first two operators in each column are used in both intuitionist and dual-intuitionist propositional logic. The last two in each column are used in both predicate logic and many modal logics (respectively).

The first two rows of the intuitionist duality are also valid for bounded lattices, including sub object lattices in coherent categories. These generalize to duality between limits and colimits in category theory in particular. in fact, all duality in category theory can be perceived as a generalization of De Morgan duality. This aspect will not be considered in this chapter as the focus is not on logics. For more details the reader is referred to [12, 98, 109, 110, 147] and related work.

1.2 Representation and Duality (Red) Results in Rough Sets

Various types of semantic dualities and representations have been developed by different authors for rough sets over the last quarter of a century. Some references are [5, 10, 24, 32, 35, 36, 46, 57, 60, 62, 80–83, 85–87, 92, 97, 104, 106–108, 110, 116–118]. The abbreviation Red will be used for *representation and duality* throughout this chapter.

Table 2 Medical diagnostics

Nom	Temp.	Body Pain	Skin	H.ache	State
A	Set1	Medium	1	No	(F0, 0.5)
B	Set2	None	1	Yes	(F0, 0.3)
C	Set1	Mild	NA	No	(Test,0.8)
E	Medium	Medium	1	Yes	(F1, 0.9)
F	High	None	1	Yes	(Test, 0.4)
G	Set1	High	NA	Yes	(F1,0.8)

Often, rough approximations and related properties are studied over information tables. These information tables can also be seen from a formal perspective. Apart from formalisms in fixed language [70], it is possible to relate them to cylindric algebras [23, 24] and to model them using CIS algebras [66]. A duality result relating to cylindric algebras is actually proved in [24]. But the cylindric algebra does not faithfully model information tables—information is lost in the process of defining models.

Given an information table and the process of obtaining relation based general approximation spaces or cover based approximation systems, it is usually not possible to construct the reverse process as some information is lost in the construction of the latter. To see this, consider the following information table (Table 2) with the last column referring to grades of illness.

If the attributes { Temp., Body Pain } are used to approximate after replacing None with Mild and taking Set1 is Medium and Set2 is High, then the resulting equivalence does not have any access to these values after the construction of the relation in question. Typically the valuation data used in the construction of approximation spaces is not present in the approximation space and cannot be recovered from it. In [23], the quality of the information loss is improved. Better models of information tables are also known.

From the available literature on rough sets, it can be said that the main dualities and representations relating to rough sets concern

- General approximation spaces and covering approximation spaces,
- Information Systems in logical or mathematical formalism,
- Derived algebraic systems that serve as semantics of reasoning with objects in some semantic domain. The semantic domains fall into the following subcategories (these are determined by nature of objects of interest in the semantics [98]):
  - rough semantic domain,
  - classical semantic domain,
  - hybrid and higher order semantic domains.

The collection of rough objects (or roughly equal objects) have order structures of different kind associated, many order theoretic dualities are of natural interest in rough sets. Because of this, a part of this chapter is devoted to relatively more fundamental duality and representation results of an ordered structures and another part concerns actual dualities and representation theorems of semantics.

1.3 Foundational Issues in Representation

In the present author's opinion, the distinction between abstract and concrete representation contexts require a triadic understanding of the context involving the representation, the represented and the representation process. This is argued for in this subsection. This subsection is not absolutely essential for an understanding of the rest of the research chapter and the reader may skip to the next subsection.

Ideas of concrete and abstract representation are also popular in the study of specific algebras. Many authors in the study of algebraic approaches to rough sets often speak of abstract and concrete representation results. In all cases of representation, it does not happen that the type (and fundamental operations and predicates) of the original class and the class used for representation are the same. This aspect of usage is reflected in some results in this chapter.

The distinction between concrete and abstract representation is not a perfectly unambiguous one—especially if one omits major steps involved in their formalism. The concepts of *concrete* and *abstract* representation theorems refer to concrete and abstract models that may respectively be constructed from natural numbers or sets with set-theoretic operations that can be abstracted from properties observed in more specific contexts (or apparent subclasses). In fact, the very subject of *abstract algebra* is committed to this distinction. In the perspective of category theory, the connections are not easily abstractable in a uniform way—concrete isomorphisms are not particularly helpful in deciding between the two.

The concepts of *concrete* and *abstract* representation theorems are also of a relative nature in rough sets. General rough sets may or may not concern information tables. There are many different ways of obtaining structures like general approximation spaces or covers or operators from information structures. The appropriate process depends on the data set, context, computational considerations and related heuristics of a subjective nature. If approximation spaces are to be derived from a complete information table with a finite set of attributes, then only a finite number of these can be derived. For every process a determinate number of related structures can be derived. Some methods of handling related collections of semantics are available.

From a purely topological algebraic representation or canonical duality perspective, representation results may also be

- of a concrete nature (of concrete representation theorems) or
- of a relatively abstract nature (of abstract representation theorems),
- natural (as natural as algebraic naturalists think they are),
- purely algebraic (or set-theoretical),
- constructive or
- approximate.

The concept of abstract representation theorems in mathematics often refers to models abstracted from mathematical models that may have no direct relation to real processes. Any mathematical model constructed from a model of arithmetic or of sets and set operations can also be designated as the only admissible concrete models. Absence of reference to such models, subject to a few minimal conditions being satisfied, may also seen to be the defining conditions of abstract representation theorems. Any abstract representation theorem in the context is typically arrived at through the following steps:

- Let $K = \langle \underline{K}, \Sigma, \nu \rangle$ be an algebraic system of signature Σ (ν being the interpretation of Σ on the set $\underline{K}$) and type τ and let it satisfy a number of properties Π that seek to capture some phenomenon in a context (and so is designated as a concrete algebraic system). Let $\mathcal{K}$, the collection of all such algebraic systems of the type satisfying Π, be closed under isomorphism.
- Let $S = \langle \underline{S}, \Sigma_o, \nu_o \rangle$ be a derived algebraic system of signature Σ_o (ν_o being the interpretation of Σ_o on the set $\underline{S}$) and type τ_o that satisfies a subset of properties $\Xi \subset \Pi$. Here *derived* is in the universal algebraic sense of *having derived operations (or terms) and predicates or universal algebraic construction involving class operators* [42, 79]. Terms are the [49]. Let $\mathcal{S}$, the collection of all such algebraic systems of the type satisfying Ξ, be closed under isomorphism.
- An *abstract representation theorem* is a provable statement of the form Every algebraic system of the form $S \in \mathcal{S}$ is derived from an algebraic system of the form $K \in \mathcal{K}$ by a construction C and an application of the construction C on any $K \in \mathcal{K}$ yields an algebraic system $S \in \mathcal{S}$.

For example, it is known that [114], *A finite lattice is isomorphic to the congruence lattice of a finite algebra if and only if it is isomorphic to an interval in the lattice of subgroups of a finite group*. In this result, the finite lattice is an abstract algebra, while the concrete algebra is a class of algebraic lattices derived by a higher order class operator on a finite group. This is an example of a complicated concrete representation theorem.

In the well known result [121] *every finite lattice L is isomorphic to a sublattice of the congruence lattice of some finite algebra S*, L is an abstract algebra as it satisfies no additional conditions, while S is also an algebra whose construction is actually determined by the lattice. So this is an example of an abstract representation theorem.

From the above discussion it should be clear that the concepts of *abstract* and *concrete* are relative to the algebraic machinery allowed in a context. In other words, the concepts are triadic in nature.

In Ch. 2 of the book on universal algebra [63], a problem of the form *Is every monoid of transformations of a non empty set U equal (not just isomorphic) to the monoid of all endomorphisms of some structure?* is referred to as a *concrete representation problem*. The problem has a negative solution. There are other usages of the term in the book, and the basic understanding is closer to what has been stated above.

In terms of formal languages and structures, abstract and concrete representation contexts can be written in the following form: Suppose that Π is a set of statements in a logical language $\mathcal{L}$, and $\mathcal{K}$ be the class of $\mathcal{L}$-models satisfying Π. If $\mathcal{L}' \subseteq \mathcal{L}$, Ξ is a set of statements of $\mathcal{L}'$ and $\Xi \subseteq \Pi$, we let $\mathcal{S}$ be the class of $\mathcal{L}'$-models of Ξ. Historically the former formalism has been the standard approach in universal algebra, while the structure based formalism has been used for concrete and abstract model theory [65].

Explicit designation of representation theorems as *abstract* also depend on the context. It may also happen that $\Sigma = \Sigma_o$. A theorem is a *concrete representation theorem* just in case it fits into the following schema:

- Let $K = \langle \underline{K}, \Sigma, \nu \rangle$ be an algebraic system of signature Σ and type τ and let it satisfy a number of properties Π. Let $\mathcal{K}$, the collection of all such algebraic systems of the type satisfying Π. Further let K be constructed from sets and set-theoretic operations or the set of real numbers with related operations.
- Let $S = \langle \underline{S}, \Sigma_o, \nu \rangle$ be a derived algebraic system of signature Σ_o and type τ_o that satisfies a subset of properties $\Xi \subset \Pi$. Let $\mathcal{S}$, the collection of all such algebraic systems of the type satisfying Ξ.
- A *concrete representation theorem* is a provable statement of the form Every algebraic system of the form $S \in \mathcal{S}$ is derived from an algebraic system of the form $K \in \mathcal{K}$.

The first of the statements is open ended and additional conditions of mathematical or logical naturality may be used in practice. For example, The irrational number $\sqrt{2}$ can be regarded as more natural than other irrational numbers because it is algebraic and can be constructed through ruler and compass methods.

1.4 Chapter Organization

This chapter is organized as follows:

- In this introduction, a number of philosophical issues concerning the concepts of representation and duality are also discussed.
- In the following section, some of the essential background is considered. This includes concepts of rough sets, topology, closure operators and discrete dualities.
- In the third section, Galois connections and correspondences are also considered.
- Concrete representation theorems are proved apparently in the next section.
- Three representation theorems are proved for classical rough sets in the fifth section.
- In the sixth section, apart from basic results concerning relations and covers, order theoretic representations, representation of quasi-order based CAS and dualities for Tarski algebras are proved. Tarski algebras have actually been adapted to the rough context in the last subsection.
- Discrete dualities are proved for double Stone algebras in the next section. The proofs have been reworked.
- Some very interesting duality results on preference relations have been reworked for rough sets contexts in the next section.

- In the ninth section, distributive lattices with Galois connections are studied. Specific connections with Heyting algebras with Galois connections are also pointed out.
- In the following section, other representation and duality results that have been left out are mentioned. Canonical extensions are explained in a subsection.
- In the next two subsections connections between rough and Fuzzy sets are studied.
- Results on L-fuzzy sets and quasi-orders are part of the next section.
- New representation results between specific classes of fuzzy sets and rough sets are proved in the next section by the present author.
- In the fourteenth section, representation results of antichain based semantics are explored in detail.

2 Background

Information tables (also referred to as information systems in the rough set literature) are basically representations of structured data tables. When columns for decision are also included, then they are referred to as *decision tables*. Often rough sets arise from *information tables* and decision tables. As mentioned in [22, 98], the term *information system* is used in other broader senses in the literature on computer science.

An *information table* $\mathcal{I}$, is a relational system of the form

$$\mathcal{I} = \langle \mathfrak{O},\ \mathbb{A},\ \{V_a : a \in \mathbb{A}\},\ \{f_a : a \in \mathbb{A}\}\rangle$$

with $\mathfrak{O}$, $\mathbb{A}$ and V_a being respectively sets of *objects*, *attributes* and *values* respectively. $f_a : \mathfrak{O} \longmapsto \wp(V_a)$ being the valuation map associated with attribute $a \in \mathbb{A}$. Values may also be denoted by the binary function $\nu : \mathbb{A} \times \mathfrak{O} \longmapsto \wp V$ defined by for any $a \in \mathbb{A}$ and $x \in \mathfrak{O}$, $\nu(a, x) = f_a(x)$.

An information table is *deterministic* (or complete) if

$$(\forall a \in \mathbb{A})(\forall x \in \mathfrak{O})\, f_a(x) \text{ is a singleton.}$$

It is said to be *indeterministic* (or incomplete) if it is not deterministic that is

$$(\exists a \in \mathbb{A})(\exists x \in \mathfrak{O})\, f_a(x) \text{ is not a singleton.}$$

Relations may be derived from information tables by way of conditions of the following form: For $x,\ w \in \mathfrak{O}$ and $B \subseteq \mathbb{A}$, $(x,\ w) \in \sigma$ if and only if

$(\forall a \in B)\, \nu(a,\ x) = \nu(a,\ w)$.
$(\forall a \in B)\, \nu(a,\ x) \cap \nu(a,\ w) \neq \emptyset$.
$(\forall a \in B)\, \nu(a,\ x) \subseteq \nu(a,\ w)$.

$(\forall a \in B)(\exists z \in B)\, \nu(a, x) \cup \nu(a, w) = \nu(a, z).$
$(\forall a, b \in B)(\nu(a, x) \cap \nu(b, x) \neq \emptyset \longrightarrow \nu(a, w) \cap \nu(b, w) \neq \emptyset).$
$(\forall a \in B)(\exists b \in B)\, \nu(a, x) = \nu(b, w).$
$(\forall a, b \in B)(\nu(a, x) = \nu(b, w) \longrightarrow \nu(a, w) = \nu(b, x)).$
$(\forall a, b \in B)(\nu(a, x) \subset \nu(a, w) \,\&\, \nu(b, x) \subset \nu(b, w) \longrightarrow \nu(a, x) \cap \nu(b, x) = \nu(a, w) \cap \nu(b, w)).$
$(\forall a, b \in B)\, (\nu(a, x) \cap \nu(b, x) \subseteq \nu(a, w))$ or $(\nu(a, x) \cap \nu(b, x) \subseteq \nu(b, w)).$
$(\forall a, b \in B)\, (\nu(a, x) \cap \nu(b, w) \subseteq \nu(a, w))$ or $(\nu(a, w) \cap \nu(b, x) \subseteq \nu(b, w)).$

For a reference of relations that may be obtained from information tables also see [92, 103, 111, 129, 130].

Definition 2 The following definitions relate to quasi-orders on sets. If $QO(S)$ is the set of all quasi-orders on a set S, then

- $<\in QO(S)$ is said to be a *up-directed* if and only if

$$(\forall a, b)(\exists c)\, a < c \,\&\, b < c.$$

  $x \parallel y$ will be an abbreviation for $x \nless y \,\&\, y \nless x$.
- $<$ is *well founded* if and only if every nonempty subset of S has a minimal element. The latter condition is equivalent to S having no infinite descending chains

$$\ldots < x_n < \ldots < x_o.$$

- A well founded quasi-order $<$ is *well quasi-ordered* (WQO) if it has no infinite anti-chains.
- A quasi-ordered set S is *total* if the quasi-order $S^2 \subseteq <$*pre-well* if the quasi-order on it is total and well-founded.
- A subset A of the quasi-ordered set $S = \langle \underline{S}, < \rangle$ is an *o-ideal* (or *initial segment*) if and only if

$$x < y \,\&\, y \in A \longrightarrow x \in A.$$

  The o-ideal generated by a subset X will be the o-ideal

$$X \downarrow = \{y\, ;\; y \in S \,\&\, (\exists x \in X)\, y < x\}.$$

  An o-ideal is *principal* if it is generated by a singleton $\{x\}$ and in this case is denoted by $x \downarrow$. The set of all o-ideals (resp principal, finitely generated) on S will be denoted by $\mathcal{I}(S)$ (resp. $\mathcal{I}_p(S)$, $\mathcal{I}_f(S)$). An o-ideal A will be said to be an *ideal* if it is up-directed.
- A subset A of the quasi-ordered set will be said to be an *o-filter* (or *final segment*) if and only if

$$x < y \,\&\, x \in A \longrightarrow y \in A.$$

The o-filter generated by a subset X will be the o-filter

$$X\uparrow = \{y;\ y \in S\ \&\ (\exists x \in X)\, x < y\}.$$

An o-filter is *principal* if it is generated by a singleton $\{x\}$ and in this case is denoted by $x\uparrow$. The set of all o-filters (resp principal, finitely generated) on S will be denoted by $\mathcal{F}(S)$ (resp. $\mathcal{F}_p(S)$, $\mathcal{F}_f(S)$). An o-filter A will be said to be a *filter* if it is down-directed. A subset A is said to be a *double o-ideal* if and only if it is both an o-filter and an o-ideal. The set of double o-ideals of S will be denoted by $\mathcal{FI}(S)$.

- In a quasi-ordered set S, $x \in S$ is an atom if and only if

$$(\forall y)(y < x \longrightarrow y = x).$$

A lattice S is *atomistic* if and only if every element is the join of atoms below it. This definition extends to join-semi-lattices naturally.

- In a lattice, *ideals* are o-ideals that are closed under $\vee$ (filters are defined dually). If a lattice ideal is also a filter, then it is said to be a *double ideal*. *Principal double ideals* would be those principal ideals that are also filters.

A lattice L is *weakly atomic* if, given $x < y$ in L, there exist $a, b \in L$ such that $x \leq a \prec b \leq y$, where $\prec$ denotes the covering relation of L. All algebraic lattices are weakly atomic

A complete lattice L satisfies the *join-infinite distributive law* (JID) if and only if for any $S \subseteq L$ and $x \in L$,

$$(\forall K \subseteq L)(\forall x \in L)\, x \wedge \left(\bigvee K\right) = \bigvee\{x \wedge b : b \in K\}. \tag{JID}$$

The dual condition is the *meet-infinite distributive law* (MID).

It is well known that a complete lattice is a Heyting algebra if and only if it satisfies (JID). Also, a complete lattice is a Heyting-Brouwer algebra if and only if it satisfies both (JID) and (MID). In particular as any distributive algebraic lattice satisfies JID and so they are Heyting algebras.

An element a of a complete lattice L is called *completely join-irreducible* if $(\forall K \subseteq L)\,(a = \bigvee K \longrightarrow a \in K)$. $\mathcal{J}$ will be used to denote the set of completely join-irreducible elements of L.

A complete lattice L is *spatial* if and only if

$$(\forall a \in L)\, a = \bigvee\{b \in \mathcal{J} : b \leq a\}$$

Topology

An *Alexandrov topology* τ on a set S is a topology that satisfies

$$(\forall \sigma \subseteq \tau) \bigcap \sigma \in \tau.$$

For each $x \in S$, $N_\tau(x) = \cap\{X\,;\, x \in X \in \tau\}$ is the neighborhood generated by x. $\mathcal{A}(S)$ will denote the set of all Alexandrov topologies on S.

In an arbitrary topological space $\langle S, \tau\rangle$, the T_o axiom is for any two points $x, y \in S$, there exists a neighborhood U satisfying $x \in U \,\&\, y \notin U$ or conversely. The T_1 axiom is for any two points $x, y \in S$, there exist neighborhoods U, V satisfying $x \in U \,\&\, y \in V \,\&\, y \notin U \,\&\, x \notin V$. Any Alexandroff space satisfying the T_1 axiom is known to be discrete.

If an Alexandrov space satisfies the T_o axiom, then it has nicer topological properties and the topology will be referred to as a T_o Alexandrov topology. In particular in such a space it is provable that

$$N_\tau(a) = N_\tau(b) \longleftrightarrow x = y.$$

In a topological space is $\langle S, \tau\rangle$ a *path* from a point a to b is a continuous function $p : [0, 1] \longmapsto S$ that satisfies $p(0) = a$ and $p(1) = b$. S is said to be *path connected* if and only if for every $a, b \in S$, there exists a path p from a to b.

If $\varphi : X \longmapsto L$ is a L-fuzzy set, and a topological property π is induced on X through some process $\boldsymbol{\mu}$, then φ can be said to be a π *L- fuzzy set*. This concept is an optional abbreviation.

A *simple chain* from a to b in a topological space S is a sequence of open sets $\{A_i\}_{i=0}^n$ satisfying

$$\begin{aligned}
(\forall i)\, A_i \cap A_{i+1} &\neq \emptyset,\\
(\forall i, j)\, (|i - j| > 1 \longrightarrow A_i \cap A_j &= \emptyset),\\
(\forall i \neq 0)\, a &\in A_0 \setminus A_i,\\
(\forall i \neq n)\, b &\in A_n \setminus A_i.
\end{aligned}$$

S is said to be *chain connected* if given any open cover $\mathcal{A}$ of S, every pair of distinct points has a chain consisting of elements of $\mathcal{A}$ from one to the other. In a Alexandrov space that satisfies the T_0 axiom, the property of chain connectedness reduces to the following condition: For every $a, b \in S$, there exist $\{x_i\}_{i=0}^n$ in S such that

$$a = x_0,\ b = x_n \,\&\, N(x_i) \cap N(x_j) \neq \emptyset \text{ whenever } |i - j| \leq 1.$$

In any topological space, the following implication holds:

$$\text{Path Connectedness} \implies \text{Connectedness} \implies \text{Chain Connectedness.}$$

Alexandroff topologies have also been referred to as *principal topologies* or *fixed topologies*. They have been found to be useful in few papers in rough sets and soft computing. Some references are [3, 72, 132, 134].

From [135], it can be deduced that

Theorem 3 *The lattice of all Alexandroff topologies on a set is complemented.*

The following result is due to [124].

Proposition 4 *Let $<$ be a quasi-order on a set X and $\tau_<$ the Alexandrov topology on X induced by $<$. If*

$$A \to B = \{a \in X : (\forall a \leq b)\,(b \in A \to b \in B)\},$$
$$A \leftarrow B = \{a \in X : (\exists b \leq a)\, b \notin A \,\&\, b \in B\},$$

then $(\tau_<, \cup, \cap, \to, \leftarrow, \emptyset, X)$ is a Heyting-Brouwer algebra.

2.1 Closure and Related Operators

Closure, interior operators and variants thereof have a central role to play in duality theory in general and rough sets. These are reviewed in this section and some advanced theorems are proved. Let $S = \langle \underline{S}, \leq \rangle$ be a partially ordered set.

Definition 5 A mapping $F : S \to S$ is called a *closure operator* if

$$(\forall x)\, x \leq F(x) \qquad \text{(Inclusion)}$$
$$(\forall x, w)\, x \leq w \longrightarrow F(x) \leq F(w) \qquad \text{(Monotonicity)}$$
$$(\forall x)\, F(F(x)) = F(x) \qquad \text{(Idempotence)}$$

Inclusion is also referred to as *extensivity* and monotonicity as *isotonicity* or *increasing* respectively. An element $x \in S$ is *closed*, if $F(x) = x$.

If the partial-order on S is a lattice order then the lattice operations $\vee$ and $\wedge$ will be assumed. Consider the following conditions:

$$(\forall x)\, F(x) \leq x \qquad \text{(Contraction)}$$
$$(\forall x, w)\, F(x \vee w) = F(x) \vee F(w) \qquad \text{(Additivity)}$$
$$(\forall x, w)\, F(x \wedge w) = F(x) \wedge F(w) \qquad \text{(Multiplicativity)}$$
$$(\forall x, w)\, F(x) \vee F(w) \leq F(x \vee w) \qquad \text{(Sub-Additivity)}$$
$$(\forall x, w)\, F(x \wedge w) \leq F(x) \wedge F(w) \qquad \text{(Sub-Multiplicativity)}$$
$$(\forall x, w)\, (F(x) = F(w) \longrightarrow F(x \wedge w) = F(x) = F(w)) \qquad \text{(UniqueGen)}$$
$$(\forall x_i)\, F(\bigvee x_i) = \bigvee F(x_i) \qquad \text{(Complete additivity)}$$

$$(\forall x_i)\ F(\bigwedge x_i)\ =\ \bigwedge F(x_i)$$
(Complete multiplicativity)

$$F(0) = 0 \qquad \text{(Bottom)}$$

$$F(1) = 1 \qquad \text{(Top)}$$

Note that for complete additivity and multiplicativity to hold, the operations need to be defined in the first place.

$$(\neg(a \leq x)\ \&\ \neg(b \leq x)\ \& b \leq F(x \vee a)\ \&\ \neg(a = b)\ \longrightarrow\ a \leq F(x \vee b)) \qquad \text{(Exch)}$$

$$(\neg(a \leq x)\ \&\ \neg(b \leq x)\ \& b \leq F(x \vee a)\ \&\ \neg(a = b)\ \longrightarrow\ \neg(a \leq F(x \vee b)))$$
(AExch)

The following associations/definitions are well known:

- *Pre-topological operators* are operators that satisfy inclusion, additivity and bottom.
- Closure operators that satisfy bottom, and additivity are called *topological or Kuratowski* closure operators.
- Topological closure operators that satisfy Infinite Join on a complete lattice are *Alexandrov* closure operators.
- An *kernel* or *interior* operator is an operator that satisfies idempotence, monotonicity and non-extensivity.
- Any closure operator C on a lattice S induces a partial-order $\leq_C$ defined by

$$a \leq_C b\ \leftrightarrow\ C(a) \wedge b \leq a \leq C(b)$$

- Any closure operator C on a lattice S induces a quasi-order $\lhd$ defined by

$$a \lhd b\ \leftrightarrow\ C(a) \leq C(b)$$

Even if the lattice is a Boolean algebra, $\lhd$ need not be a partial-order. To see this let B be the four element BA with atoms a, b, and set

$$C(x) = \begin{cases} 0, & \text{if } x = 0, \\ 1, & \text{otherwise.} \end{cases} \qquad (1)$$

This yields $a \lhd b\ \&\ b \lhd a$.

If C is a closure operator on a collection $\langle S, \subseteq\rangle$ of sets partially ordered by set inclusion then

- C is said to be *finitary*, if for any $s \subseteq S$ and $b \in C(s)$, then there exists a finite $e \subseteq s$ such that $b \in C(e)$.
- A *generator* of a closed subset Z by a closure operator C is any minimal subset G such that $C(G) = Z$. C is said to be *uniquely generated* if and only if every closed subset has a unique generator.
- The exchange property is common for algebraic closure operators in linear algebra, matroids and projective geometries, while the anti-exchange property can be encountered in the study of convex geometry [100] and anti-matroids.

The reader should note that not every closed set has a generator, for example, if B is an atomless Boolean algebra, and C is defined as in (1), then 1 is closed without a generator.

An element a of a complete lattice L is *compact* if and only if if for any $X \subseteq L$, $a \leq \bigvee X$ then there exists a finite subset $B \subseteq X$ such that $a \leq \bigvee B$. An *algebraic lattice* is a complete lattice L in which every element is a join (possibly infinite) of compact elements.

Definition 6 A collection $\mathcal{C}$ of subsets of a non-empty set S is said to be a *closure system* on S if $\mathcal{C}$ is closed under arbitrary intersections, that is,

$$(\forall Z \subseteq \mathcal{C}) \bigcap Z \in \mathcal{C}.$$

By definition, $\bigcap \emptyset = S$, so $S \in \mathcal{C}$.

Proposition 7 ([9]) *A closure system $\mathcal{C}$ forms a complete lattice with respect to $\subseteq$, where for all $X \subseteq \mathcal{C}$,*

$$\bigwedge X = \bigcap X, \quad \bigvee X = \bigcap \{Z : Z \in \mathcal{C} \,\&\, \bigcup X \subseteq Z\}. \tag{2}$$

Theorem 8 ([9])

1. *Let F be a closure operator on S. Then, the collection $\mathcal{C}_F$ of closed elements is a closure system.*
2. *If $\mathcal{C}$ is a closure system on S, then the mapping $F_{\mathcal{C}}$ defined by*

$$F_{\mathcal{C}}(X) = \bigcap \{Y \in \mathcal{C} : X \subseteq Y\}$$

   is a closure operator.
3. *If F is a closure operator, then $F = F_{\mathcal{C}_F}$.*
4. *If $\mathcal{C}$ is a closure system on S, then $\mathcal{C} = \mathcal{C}_{F_{\mathcal{C}}}$.*

If a closure system $\mathcal{C}$ is also closed under the union of subcollections that are (upward) directed under set inclusion, then it is called an *algebraic closure system* (or an algebraic closed-set system). Equivalently if $\mathcal{C}$ is *inductive* (closed under union of nonempty chains) then it is an algebraic closure system.

Theorem 9

- *An algebraic closure system $\mathcal{C}$ forms an algebraic lattice under the set-theoretic inclusion order and the operations of* (2). *All algebraic lattices are necessarily complete.*
- *If $\mathcal{C}$ is an algebraic closure system over a non-empty set A, and if B is a non-empty subset of A, then $\{F \cap B : F \in \mathcal{C}\}$ is an algebraic closure system over B.*
- *If $h : B \longmapsto A$ is a map, then $h^{-1}(\mathcal{C}) = \{h^{-1}(F) : F \in \mathcal{C}\}$ is an algebraic closure system over B.*

Theorem 10 *If S is an algebraic lattice, then*

1. *The set of compact elements $K(S)$ is a join semilattice with least element in the induced order;*
2. *S is order-isomorphic to the lattice of order ideals of the compact elements, that is $S \cong \mathcal{I}(K(S))$. The associated map is : $x \mapsto x \downarrow \cap K(S)$ ($x \downarrow$ being the principal order ideal generated by x).*

In a join-semilattice L with least element 0, an *ideal* is any subset of K of L that is an order ideal and is closed under $\vee$. For each $x \in L$, $x \downarrow$ is the principal ideal generated by x. The next representation result is due to [50] and the first part is due to [69]. The proofs can be found in [49].

Theorem 11 *If L is a join-semilattice with least element, then*

1. *The collection $\mathcal{I}(L)$ of all ideals of L is an algebraic lattice (the meet corresponds to set intersection)*
2. *The compact elements of $\mathcal{I}(L)$ are the principal ideals.*

Moreover, for every algebraic lattice P, there exists a join-semilattice Q such that $\mathcal{I}(Q)$ is isomorphic to P.

Closure operators can be directly related to ideas of consistency under minimal additional assumptions.

Let F be a closure operator on a set S. Call a set $X \subseteq S$ *F-consistent* if $F(X) \neq S$, and *F-inconsistent* otherwise. Maximal F-consistent sets are said to be *F-complete*. F is called *logically compact* (l-compact) if every F-inconsistent set includes a finite F-inconsistent subset. The terminology is motivated by the behavior of the consequence operator in algebraic logic [28].

Theorem 12 ([28])

1. *F is logically compact if and only if the collection of all F-consistent sets is inductive or finitary.*
2. *If F is logically compact, then every F-consistent set is contained in a F-complete set.*

3. *A closure operator F on S is finitary and logically compact if and only if $\mathcal{F} \setminus \{S\}$ is inductive.*

The next theorem is a negative result, due to the present author, for upper approximations in granular rough sets. The theorem is a variation of the result in Boolean algebras [29, 40]. The second part can be read as an abstract representation theorem and is relevant for matroidal approaches to rough sets [78, 138, 144].

Theorem 13 *In any lattice S, if a closure operator C is uniquely generated, then C satisfies the anti-exchange property. Every complemented lattice generated by a closure operator C that is unique generated is isomorphic to a complemented lattice generated by a closure operator F that satisfies anti-exchange property and conversely.*

Proof

- Assuming Unique Generation, let $\neg(a \leq x) \,\&\, \neg(b \leq x)$ and $b \leq C(a \vee x)$.
- Suppose also that $a \leq C(b \vee x)$,
- then by Unique Generation, it is necessary that $C(a \vee x) = C(b \vee x) = C(x)$.
- This contradicts $\neg(a \leq x)$.
- Therefore Unique Generation $\Longrightarrow$ AntiExchange.

For the converse in a complemented lattice,

- Suppose $C(x) = C(w)$ and let z be a minimal element satisfying $C(x) = C(z)$.
- If $a \leq z$, then $C(a) \leq C(x)$.
- If $\neg(a \leq w)$, let $k \leq w$ be a minimal element satisfying $C((z \wedge a') \vee k) = C(x)$.
- Let $p \leq k$ and $h = (z \wedge a') \vee (k \wedge p')$, then $C(h) \leq C(x)$
- All this yields, $\neg(p \leq x)$, $\neg(a \leq x)$, but $a \leq C(h \vee p) = C(x)$ and $p \leq C(h \vee a) = C(x)$.
- This contradicts Anti exchange, so it is essential that $z \leq w$ and $C(x) = C(w) = C(x \wedge w)$

□

Proposition 14 *In classical rough sets, the upper approximation operator satisfies all of idempotence, inclusion, monotonicity, exchange, submultiplicativity and join. The properties of unique generation and anti-exchange do not hold.*

Proof If X is an approximation space, then u is a closure operator on the Boolean algebra $S = \left\langle \underline{\wp(X)}, \cup, \cap, ^c, 0, 1 \right\rangle$ then the properties of idempotence, inclusion, monotonicity, submultiplicativity and join are well known [110].

- For proving exchange, note that
- for any distinct $A, B, C \in \wp(X)$, if $\neg(A \subset C)$, $\neg(B \subset C)$ and $B \subset (A \cup C)^u$,
- then for any element $x \in B$, there exists a class $[z]$ with $z \in A \cup C$ such that $x \in [z]$.
- It follows that $z \in [x]$ must hold as well.
- Therefore, $A \subseteq (B \cup C)^u$ holds.

□

The last two results relate to the closure operators of matroids in the study of rough sets with matroids. Some aspects are mentioned below.

Definition 15 A *matroid* is a pair of the form $H = (S, \mathcal{H})$, with S being a finite set and $\mathcal{H}$ being a set of subsets of S that satisfies:

- $\mathcal{H}$ is an order ideal with respect to set inclusion,
- $\emptyset \in \mathcal{H}$ and
- For any $A, B \in \mathcal{H}$, if $\#(A) < \#(B)$ ($\#(A)$ being the cardinality of A), then $(\exists a \in B \setminus A)\, A \cup \{a\} \in \mathcal{H}$.

On matroids a rank function r_H is a map $r_H : \wp(S) \longmapsto N$ that satisfies

$$\text{For } X \in \wp(S)\; r_H(X) = \max\{\#(K) : K \subseteq X \,\&\, K \in \mathcal{H}\}$$

Closure operators cl_H on H are defined as a map $cl_H : \wp(S) \longmapsto \wp(S)$ satisfying

$$(\forall X \in \wp(S))\; cl_H(X) = \{a : a \in S \,\&\, r_H(X) = r_H(X \cup \{a\})\}$$

The main directions of studies on the connections between covering based rough sets and matroids have been along the following directions:

- Study of matroidal properties of matroids derived from given covers (see for example [78, 144]), and
- Study of structures like graphs and matrices formed through matroidal constructions (see for example [138]).

The theorems on exchange and anti-exchange properties considered concern the closure operator on matroids.

2.2 *Discrete Dualities for Rough Sets*

A *frame* may be seen as another name for a general approximation space. Then these are simply relational systems of the form $S = \langle \underline{S}, \sigma \rangle$ (with $\underline{S}$ being a set and σ a binary relation over it). This view is reductionist because frames have been used in literature on modal logic with many ontological assumptions (see [47] for example). Over such frames, pointwise approximations are definable and have a number of nice properties that are modal logic friendly (see the subsection on pointwise approximations in [98] by the present author for a summary). For frame semantics of modal logic, the reader is referred to books like [127]. Often it happens that frames have additional structure and in almost all of these sub-cases they are relational systems with one or more quasi-orders or meaningful relations. In general, however general approximations may not have much to do with modal understanding of things.

In simplified terms any discrete duality is between a class $\mathcal{A}$ of algebras and a class $\mathcal{K}$ of frames and may be seen as a tuple $\langle \mathcal{A}, \mathcal{K}, \mathfrak{C}, \mathfrak{Cf} \rangle$ satisfying all of the following:

- Every $S \in \mathcal{A}$, there exists a $\mathfrak{Cf}(S) \in \mathcal{K}$ called the *canonical frame of* S.
- Every $K \in \mathcal{K}$, there exists a $\mathfrak{C}(K) \in \mathfrak{A}$ called the *complex algebra* of K.
- For every $S \in \mathfrak{A}$, there exists an embedding $\iota_S : S \longmapsto \mathfrak{C}(\mathfrak{Cf}(S))$.
- For every $K \in \mathfrak{K}$, there exists an embedding $\kappa_K : K \longmapsto \mathfrak{Cf}(\mathfrak{C}(S))$.

A discrete duality has the following properties:

- The topology associated in the duality is the discrete one.
- It preserves truth in the sense that the concept of truth associated with the class of algebras $\mathcal{S}$ corresponds to the truth associated with its frame semantics corresponding to the class of all frames $\mathbb{B}$ associated. A special case of this is when the latter class is the class of Kripke frames. A relatively minimalist case for lattices is considered in [104].
- Instead of isomorphisms, two-way embeddings are seen to be sufficient for the purpose.

In rough sets, many discrete dualities have been found to be useful and some have been specifically developed for application in rough semantics. Results relating to Heyting algebras, double Stone algebras [35] and rough relation algebras [36] are known.

3 Basic Results on Order Structures

In this section some basic results on order structures are presented. Often collections of all definite objects in rough sets form a distributive lattice. Sometimes the definite objects are taken to unions of basic definite granules. For a number of cases, the reader is referred to [98] in this volume. Because of these reasons some stress has been laid on related concepts.

A *lattice ideal* K of a lattice $L = (L, \vee, \wedge)$ is a subset of L that satisfies the following ($\leq$ is assumed to the definable lattice order on L):

$$(\forall a \in L)(\forall b \in K)(a \leq b \longrightarrow a \in K) \quad \text{(o-Ideal)}$$

$$(\forall a, b \in K)\, a \vee b \in K \quad \text{(Join Closure)}$$

The set of all order ideals and lattice ideals of a lattice L will respectively be denoted by $\mathcal{I}_o(L)$ and $\mathcal{I}(L)$ respectively.

An ideal P in a lattice L is *prime* if and only if $(\forall a, b)(a \wedge b \in P \longrightarrow a \in P \text{ or } b \in P)$. $Spec(L)$ shall denote the set of all prime ideals. Maximal lattice filters are the same as ultrafilters. In Boolean algebras, any filter F that satisfies $(\forall a) a \in F \text{ or } a^c \in F$ is an ultra filter. *Chains* are subsets of a partially ordered set in which any two elements are comparable, while *antichains* are subsets of a partially ordered set in which no two distinct elements are comparable. Singletons are both chains and antichains.

A filter (resp. ideal) of a lattice is *irreducible* provided it is proper and not the intersection of two proper filters (resp. proper ideals). All maximal filters (maximal ideals) are irreducible. All prime filters (prime ideals) are irreducible.

The next theorem is easy to prove, has many generalizations and extensions and is very important in algebraic semantics of a wide variety of logics.

Theorem 16 *If A and B are two disjoint subsets of a lattice L whose union is L (in other words if A and B are complementary subsets of L), then A is a prime ideal if and only if B is a prime filter.*

In a lattice, L, an element $s \in L$ is *standard* if and only if

$$(\forall a, b \in L)\, a \wedge (s \vee b) = (a \wedge s) \vee (a \wedge b) \tag{Standard}$$

An ideal K of L is said to be *standard* if and only if

$$(\forall A, B \in \mathcal{I}(L))\, A \wedge (K \vee B) = (A \wedge K) \vee (A \wedge B) \tag{Standard Ideal}$$

The set of standard ideals will be denoted by $\mathcal{I}_S(S)$.

3.1 Galois Connections and Correspondences

Let $P = \langle \underline{P}, \leq \rangle$ and $Q = \langle \underline{Q}, \preceq \rangle$ be two quasi-ordered relational systems. A pair of maps (f, h) with $f : P \longmapsto Q$ and $h : Q \longmapsto P$ is said to be *Galois connection* [102] (or a *residuated-residual pair*) between P and Q if and only if

$$(\forall p \in P)(\forall q \in Q)(f(p) \preceq q \leftrightarrow p \leq h(q)) \tag{3}$$

h is also called the *adjoint* (or *residual*) and f is called the *co-adjoint* (or *residuated* map) in the context. It is possible to define these concepts through the following proposition:

Proposition 17 *A pair of maps (f, h) with $f : P \longmapsto Q$ and $h : Q \longmapsto P$ is a* Galois connection *if and only if*

- *f, h are isotone (order-preserving maps, and)*
- $$(\forall p \in P)(\forall q \in Q)(p \leq h(f(p))\, \&\, q \preceq h(f(q))) \tag{4}$$

In the definition above, if f and h are partial maps instead and the condition (3) holds, then (f, h) is referred to as a *partial Galois connection*.

Theorem 18 *If (f, h) is a Galois connection between the complete lattices P and Q, then all of the following hold:*

1. $(\forall a \in P)(\forall b \in Q)\, f(h(f(a))) = f(a)\,\&\, h(f(h(b))) = h(b)$
2. *f is a complete $\vee$-morphism and h is a complete $\wedge$-morphism.*
3. *$h \circ f$ and $h \circ f$ are closure and interior operators on P and Q respectively.*
4. *The maps $f(a) \mapsto h(f(a))$ and $h(b) \mapsto f(h(b))$ are mutually inverse order-isomorphisms : $\Im(f) \longmapsto \Im(h)$*
5. *f and h are inter-definable by the equations:*

$$f(a) = \bigwedge\{b : b \in Q\,\&\, a \leq h(b)\}\ \&\ h(b) = \bigvee\{a : a \in P\,\&\, f(a) \leq b\} \tag{5}$$

Galois connections on residuated lattices have been studied in [119] for example. Importantly Galois connections between lattices have connections with compatible tolerances.

Proposition 19 *If P and Q are lattices and $f : P \longmapsto Q$ is a map, then*

- *f is $\wedge$-morphism and K is a filter of Q, then $f^{-1}(K)$ is a filter of P.*
- *f is $\vee$-morphism and J is an ideal of Q, then $f^{-1}(J)$ is an ideal of P.*

Proof Using the same operation symbols in both lattices, if $a, b \in f^{-1}(K)$, then $f(a \wedge b) = f(a) \wedge f(b) \in K$ and $f(a), f(b) \in K$. So $f(a) \wedge f(b) \in K$. It follows that $a \wedge b \in f^{-1}(K)$.

If $x \in f^{-1}(K)$ and $w \in P$ and $x \leq w$. f is a order preserving map necessarily. So $f(x) \leq f(w)$. As $f(x) \in K$ and K is a filter, therefore $f(w) \in K$. It follows that $w \in f^{-1}(K)$. □

If $R \subseteq P \times Q$, then define the maps $\zeta, \xi : \wp(Q) \longmapsto \wp(P)$ via

$$(\forall H \in \wp(Q))\, \zeta(H) = \{x : x \in P\,\&\,(\exists a \in H)\, Rxa\}$$

$$(\forall H \in \wp(Q))\, \xi(H) = \{x : x \in P\,\&\,(Rxa \Rightarrow a \in H)\}$$

ζ is $\vee$-preserving and ξ is $\wedge$-preserving.

Theorem 20 *If L is a Boolean algebra,*

- *F a proper filter and K an ideal of L satisfying $F \cap K = \emptyset$, then there is a maximal filter F^* such that $F \subseteq F^*$ and $F^* \cap K = \emptyset$*
- *If L is complete, then L is atomic if and only if it is atomistic.*
- *$a \in L$ is an atom if and only if $a\uparrow$ is a maximal filter.*
- *Any Power set Boolean algebra is complete and atomic.*
- Stone Representation: *If $\mathfrak{F}_{\max}(L)$ is the set of maximal filters, then the function:*

$$\nu : L \longmapsto \wp(\mathfrak{F}_{\max}(L))$$

defined by $\nu(x) = \{U : U \in \mathcal{F}_{\max}(L);\ x \in L\}$ *is an injective morphism of lattices.*

Proposition 21 *In any lattice L,*

$$(\forall a, b, c)\,(a \wedge b) \vee (a \wedge c) \ \leq\ a \wedge (b \vee c)$$

$$(\forall a, b, c)\,a \vee (b \wedge c) \ \leq\ (a \vee b) \wedge (a \vee c)$$

Definition 22 A lattice L has *prime filter property* PFP if and only if given a proper filter K and an ideal J of L such that $K \cap J = \emptyset$ there is a prime filter P of L such that $K \subseteq P$ and $P \cap J = \emptyset$.

Theorem 23 *A lattice has PFP if and only if it is distributive.*

Proof Suppose that a lattice has the PFP, but is not distributive, then

$$(\exists a, b, c)\,(a \vee b) \wedge (a \vee c) \nleq a \vee (b \wedge c)$$

because in all lattices the RHS $\leq$ LHS.

Let $x = (a \vee b) \wedge (a \vee c)$, $w = a \vee (b \wedge c)$, $K = x\uparrow$ and $J = w\downarrow$. By the condition, K and J are proper and $K \cap J = \emptyset$. So by assumption, there exists a prime filter P satisfying $x \in K \subseteq P$. Consequently, $a \vee b,\ a \vee c \in K$.

If now $a \in P$, then as $a \leq a \vee (b \wedge c) = w$, $w \in P$—a contradiction. Again if $a \notin P$, then as $a \vee b \in P$, (P being prime) it is necessary that $b \in P$. Using the same argument c, $b \wedge c \in P$. So $w \in P$—again a contradiction.

So it follows that if L has PFP then L must necessarily be distributive.

For the converse, starting from a proper filter K and ideal J of L that satisfy $K \cap J = \emptyset$, form the collection

$$\mathcal{K} = \{H : H \in \mathcal{F}(L), \emptyset \subset H \subset L,\ \& K \subseteq H \,\&\, H \cap J = \emptyset\}$$

under the inclusion ordering.

Each union of a chain of filters in $\mathcal{K}$ must be disjoint from J and contains K. So every chain has an upper bound in $\mathcal{K}$. By Zorn's lemma, $\mathcal{K}$ has a maximal element P.

P must be prime, otherwise a contradiction follows. To see this let for some $a, b \in L$, $a \vee b \in P$ with $a, b \notin P$. Form the filter P_a generated by $P \cup \{a\}$. By the maximality of P in $\mathcal{K}$, it is necessary that $P_a \cap J \neq \emptyset$. So there exists a $g \in P$ such that $g \wedge a \in J$. Similarly there is a $h \in P$ with $h \wedge b \in J$.

So $(h \wedge a) \vee (g \wedge b) \in J$. By distributivity,

$$p = (g \vee h) \wedge (h \vee b) \wedge (g \vee a) \wedge (a \vee b) \in J$$

Since filters are increasing and $h \leq h \vee g$, $h \leq h \vee b$ and $g \leq g \vee a$, and $g, h \in P$ therefore $h \vee g, h \vee b, g \vee a \in P$. So $p \in P$ and this contradicts $P \cap J = \emptyset$.

□

A subset H of a partially ordered set P is join dense in it if and only if

$$(\forall x \in P)(\exists H_x \subseteq H)\, x = \bigvee H_x$$

If P is a complete lattice, then the defining condition is equivalent to

$$(\forall x \in P)\, x = \bigvee \{a \,:\, a \in H \,\&\, a \leq x\}.$$

The *spectrum* $Spec(a)$ of an element a is the set of join-irreducible elements below it:

$$Spec(a) = \{x \,:\, x \in JI(L) \,\&\, x \leq a\} = a\downarrow \cap JI(L)$$

Let S be a set and $\mathcal{H} \subseteq \wp(S)$ be a complete ring of sets. For any $x \in S$, if $nbd(x) = \bigcap\{X \,;\, x \in X \in \mathcal{H}\}$, then $nbd(x) \in \mathcal{H}$ and $\mathcal{N} = \{nbd(x)\,;\, x \in S\}$ is the set of join-irreducible elements of $\mathcal{H}$. It is also the smallest join-dense sets in $\mathcal{H}$.

Proposition 24 *In a distributive lattice L, $a \in L$ is join-irreducible if and only if the principal filter generated by a is prime.*

Theorem 25 *For every finite distributive lattice L, there exists a partially ordered set P unique upto isomorphism such that $L \cong \mathcal{I}(P)$. Associated classes of nonempty distributive lattices and partially ordered sets correspond bijectively. Also, a lattice is distributive if and only it is isomorphic to a ring of sets.*

Proof The proofs are simplified by the provable statement that the map $\varphi : a \longmapsto Spec(a)$ is an isomorphism between L and $JI(L)\downarrow$. To see this, note that every element of L is the join of nonzero join-irreducible elements because L is finite:

$$a = \bigvee Spec(a).$$

So φ is injective. Also

- Since $Spec(a) \cap Spec(b) = Spec(a \wedge b)$,
- So $\varphi(a \wedge b) = \varphi(a) \wedge \varphi(b)$
- and $\varphi(a \vee b) = \varphi(a) \vee \varphi(b)$.

because if $x \in Spec(a \vee b)$, then $x = x \wedge (a \vee b) = (x \wedge a) \vee (x \wedge b)$. Since $x \in JI(L)$, either $x = x \wedge a$ or $x = x \wedge b$. That is, $x \in Spec(a) \cup Spec(b)$. So $Spec(a \vee b) = Spec(a) \cup Spec(b)$.

To show that $(\exists a \in L)\, \varphi(a) = A$ if $A \in JI(L)\downarrow$, let $a = \bigvee A$. Then $A \subseteq Spec(a)$. If $x \in Spec(a)$, then

$$x = x \wedge a = x \wedge \bigvee A = \bigvee\{x \wedge z \,:\, z \in A\}.$$

By join-irreducibility, $x = x \wedge z$ for some $z \in A$ as A is a down set.

A lattice is distributive if and only it is isomorphic to a ring of sets because of Stone's theorem [137] and because any ring of sets is a distributive lattice. □

The collection $Spec(L)$ of all prime ideals of a distributive lattice L does not characterize it. But it does so when endowed with an additional topology.

Definition 26 If $\psi : L \longmapsto Spec(L)$ is a map defined by

$$(\forall x \in L)\, \psi(x) = \{F : F \in Spec(L)\,\&\, x \in Spec(L)\},$$

then $Spec(L)$ with sub base $\mathcal{P}(L) = \{\psi(x) : x \in L\}$ forms a topological space called the *Stone space* of L.

Theorem 27 *Every distributive lattice* $L = \langle \underline{L}, \vee, \wedge \rangle$ *is homeomorphic to the Stone space* $St(L) = \langle \underline{\mathcal{P}(L)}, \cup, \cap \rangle$ *with the* T_o *topology. If* L *has a top, then the Stone space is compact.*

Definition 28 A *Priestley Space* is a compact totally order-disconnected topological space. That is a structure of the form $\langle X, \leq, \tau_X \rangle$ in which all of the following hold:

- $\langle X, \leq \rangle$ is a partially ordered set,
- $\langle X, \tau_X \rangle$ is a topological space in which for all $a, b \in X$ satisfying $a \not\leq b$, there exists a clopen increasing set K for which $a \in K$ and $b \notin K$ holds and
- the topology is compact.

The set of clopen increasing sets is denoted by $\Delta(X)$.

Theorem 29 *Given a bounded distributive lattice* L*,*

- *the algebra* $\left\langle \underline{\mathcal{S}(Spec(L))}, \cup, \cap, \emptyset, Spec(L) \right\rangle$ *over the set of all* $\subseteq$*-order filters of* $Spec(L)$ *is also a bounded lattice,*
- *The function* $\alpha : L \longmapsto \mathcal{S}(Spec(L))$ *defined by* $\alpha(x) = \{F : F \in Spec(L) : x \in F\}$ *is an injective lattice morphism (embedding)*
- $\langle Spec(L), \subseteq, \tau_{Spec(L)} \rangle$ *is a Priestley space.*
- $L \cong \Delta(Spec(L))$ *and the subbase of the topology is*

$$\alpha(L) \cup \{Spec(L) \setminus \alpha(x) : \alpha(x) \in \alpha(L)\}$$

- *If* $\mathcal{O}(Spec(L))$ *is the set of all open increasing subsets of* $Spec(L)$*, then the map* $\varphi : \mathcal{I}(L) \longmapsto \mathcal{O}(Spec(L))$ *defined by*

$$(\forall H \in \mathcal{I}(L))\, \varphi(H) = \{T : T \in Spec(L)\,\&\, T \cap H \neq \emptyset\}$$

  is a lattice isomorphism.

Theorem 30 *If* $\langle X, \leq, \tau_X \rangle$ *is a Priestley space, then the map* $\xi_X : X \longmapsto Spec(\Delta(X))$ *defined by*

$$(\forall x)\, \xi_X(x) = \{U : x \in U \in \Delta(X)\}$$

is a homeomorphism and order isomorphism.

4 Concrete Representation Theorems

In this section, concrete representation for lattices or partially ordered sets with additional operations are considered.

4.1 Quasi-Boolean Algebras

By a *De Morgan lattice* or a *quasi-Boolean algebra* (ΔML) will be meant an algebra of the form $L = \langle \underline{L}, \vee, \wedge, c, 0, 1 \rangle$ with $\vee, \wedge$ being distributive lattice operations and c satisfying

$$x^{cc} = x \quad \text{(Complement-1)}$$

$$(x \leq a \leftrightarrow a^c \leq x^c) \quad \text{(Complement-2)}$$

The De Morgan properties (including $(x \vee a)^c = x^c \wedge a^c$) follow from the above.

Proposition 31 *In a quasi-Boolean algebra* $S = \langle \underline{S}, \cup, \cap, \sim, 0, 1 \rangle$, *the operation* $^\circ$ *defined by*

$$(\forall A \in \wp(S))\, A^\circ = \{\sim x : x \in A\}$$

on $\wp(S)$ *satisfies*

$$(S \setminus A)^\circ = S \setminus (A^\circ),\ \&\, A^{\circ\circ} = A. \tag{6}$$

Let X be a non-empty set with $h : X \longmapsto X$ being an involution (that is satisfies $(\forall x \in X)\, h(h(x)) = x$). Involutions are necessarily bijective. On $A \in \wp(X)$, if

$$\sim A = X \setminus h(A),$$

a *quasi-field of sets* is the algebra $\langle \underline{Q(X)}, \cup, \cap, \sim, X \rangle$, with the $\underline{Q(X)} \subseteq \wp(X)$ being any nonempty subcollection of subsets of X closed under the induced operations.

Theorem 32 *Every quasi Boolean algebra is isomorphic to a quasi-field of some open subsets of a topological, compact T_o-space.*

Proof Let $L = \langle \underline{L}, \cup, \cap, \sim, 0, 1 \rangle$ be a quasi-Boolean algebra and $Spec(L)$ be the set of prime filters of the lattice. For each $a \in A$, let $\psi(a) = \{F : F \in Spec(L) \,\&\, a \in F\}$. By the representation of distributive lattices, $\{\psi(a)\, a \in L\}$ is a sub base for $Spec(L)$ with compact, T_o topology.

ψ is injective and satisfies:

$$\psi(a \cup b) = \psi(a) \cup \psi(b)$$

$$\psi(a \cap b) = \psi(a) \cap \psi(b)$$

$$\psi(1) = Spec(L)$$

Let $\alpha : Spec(L) \longmapsto Spec(L)$, be a map defined by

$$(\forall F \in Spec(L))\, \alpha(F) = L \setminus F^{\circ} \tag{7}$$

By definition, for each $F \in Spec(L)$, F° is a prime ideal and $L \setminus F^{\circ}$ is a prime filter. So α is a self map : $Spec(L) \longmapsto Spec(L)$.

Further,

$$\alpha(\alpha(F)) = L \setminus ((\alpha(F))^{\circ} = L \setminus (L \setminus F^{\circ})^{\circ} = L \setminus (L \setminus F^{\circ\circ}) = F$$

- Clearly, $F \in \alpha(\psi(a))$ if and only there exists a prime filter $G \in \psi(a)$ subject to $F = \alpha(G)$.
- But $G \in \psi(a)$ if and only if $a \in G$ if and only if $\sim a \in G^{\circ}$ if and only if $\sim a \notin L \setminus (G^{\circ})$.
- So, $F \in \alpha(\psi(a))$ if and only if there exists a prime filter G subject to $\sim a \notin L \setminus (G^{\circ}) = F$.
- This yields $F \in \alpha(\psi(a))$ if and only if $\sim a \notin F$
- So $F \in \sim \psi(a) = Spec(L) \setminus \alpha(\psi(a))$ if and only if $\sim a \in F$, if and only if $F \in \psi(\sim a)$.
- This proves $\sim \psi(a) = \psi(\sim a)$.

So $Q(Spec(L)) = \{\psi(a) : a \in L\}$ is a quasi-field of open sets of $Spec(L)$ and the map ψ is an injection from L onto $Q(Spec(L))$. □

Example 33

$$\text{Let } \underline{S} = \{\top, a, b, \bot\} \,\&\, \mathfrak{U}_o = \langle \underline{S}, \cup, \cap, \sim, \top \rangle$$

$$\text{defined by } (\forall x, w)\, x + x = x \,\&\, x + w = w + x \text{ for } + \in \{\cap, \cup\}$$

$$\top \cup x = \top \,\&\, \top \cap x = x = \bot \cup x \,\&\, \bot \cap x = \bot$$

$$\sim \top = \bot \,\&\, \sim \bot = \top \,\&\, \sim a = a \,\&\, \sim b = b$$

$$\text{Let } \underline{B} = \{\top, \bot\} \,\&\, \mathfrak{B}_o = \langle \underline{B}, \cap, \cup, \sim, \top \rangle$$

$$\text{Let } \underline{C} = \{\top, a, \bot\} \,\&\, \mathfrak{C}_o = \langle \underline{C}, \cap, \cup, \sim, \top \rangle$$

It is easy to check that $\mathfrak{U}_o$ is a quasi-Boolean algebra and all its subalgebras upto isomorphism are $\mathfrak{B}_o$ and $\mathfrak{C}_o$.

Theorem 34

- *The class of all quasi-Boolean algebras forms a variety $\mathcal{V}_{qba}$ that is a generated by the subdirectly irreducible algebra $\mathfrak{U}_o$.*
- *Every quasi-Boolean algebra is isomorphic to a subalgebra of a product of the form $\Pi_{i \in I} \mathfrak{U}_i$ with $\mathfrak{U}_i = \mathfrak{U}_o$ for $i \in I$—a directed set.*
- *$\mathfrak{U}_o$ is functionally free relative to $\mathcal{V}_{qba}$. That is any two terms are identically equal in $\mathcal{V}_{qba}$ if and only if they are identically equal in $\mathfrak{U}_o$. Further, the three-element quasi-Boolean algebra $\mathfrak{C}_o$ is functionally free for the class $\mathcal{K}_o$ of all quasi-Boolean algebras satisfying the Kleene condition $a \wedge \sim a \leq b \vee \sim b$.*

Proof It is easy to verify that $\mathcal{V}_{qba}$ is closed under **HSP** (that is $\mathbf{HSP}\mathcal{V}_{qba} = \mathcal{V}_{qba}$).

By the previous representation theorem, it suffices to deal with quasi-Boolean algebras formed by quasi-fields of subsets alone. Let

$$\mathcal{Q}(X) = \langle \underline{Q(X)}, \cup, \cap, \sim, X \rangle$$

be a quasi-field of subsets of a set X and let $h : X \longmapsto X$ be an involution defining $\sim$:

$$(\forall Z \in Q(X)) \ \sim Z = X \setminus h(Z). \tag{8}$$

Let

$$I = \{(x, h(x)) : \ x \in X\} \tag{9}$$

and define a map $h_i : Q(X) \longmapsto \mathfrak{U}_o$ via

$$h_i(Z) = \begin{cases} \bot & \text{if } \{x, h(x)\} \cap Z = \emptyset, \\ a & \text{if } \{x, h(x)\} \cap Z = \{x\} \neq \{h(x)\}, \\ b & \text{if } \{x, h(x)\} \cap Z = \{h(x)\} \neq \{x\}, \\ \top & \text{if } \{x, h(x)\} \cap Z = \{x, h(x)\}. \end{cases} \tag{10}$$

Since $h_i : Q(X) \longmapsto \mathfrak{U}_o$ is a morphism for each i, the map $h : Q(X) \longmapsto \Pi_{i \in I} \mathfrak{U}_o$ defined by $(\forall Z \in Q(X))\, h(Z) = (h_i(Z))_{i \in I}$ is a morphism. It is easy to verify that it is a monomorphism.

The reader is invited to supply the missing parts of the proof. □

Next a relational representation theorem will be proved. It can be claimed that it is relatively more concrete than the previous representation.

- Let S be a set and $T \in Tol(S)$.
- Let $Q_o(S, T) = \{K : K \subseteq T\}$
- Define $\sim R = T \setminus R^{-1}$ for any $R \subseteq S^2$
- Define $\underline{Q(S, T)} = \{K : K \in Q_o(S, T) \,\&\, \sim K \in Q_o(S, T)\}$
- $Q(S, T) = \langle \underline{Q(S, T}, \cup, \cap, \sim, T \rangle$ is a *quasi-Boolean algebra of relations*.

Theorem 35 *Every quasi-Boolean algebra is isomorphic to a quasi-Boolean algebra of relations.*

Proof The proof is not too difficult. □

4.2 Red Results of Neighborhood Systems

A *neighborhood operator* n on a set $\underline{S}$ is any map of the form $n : \underline{S} \longmapsto \wp \underline{S}$.

Proposition 36 *Every neighborhood operator* n *induces a global map* $N : \wp(S) \longmapsto \wp(S)$ *that satisfies*

$$(\forall X \in \wp(S) \setminus \{\emptyset\})\; N(X) = \bigcup_{x \in X} n(x) \text{ and } N(\emptyset) = \emptyset \tag{11}$$

The following properties of neighborhood operators have important connections with relations and help in transforming results on approximations to cover based rough contexts and vice versa.

$$(\forall a)(\exists b)\, b \in n(a) \tag{Serial}$$

$$(\forall a)(\exists b)\, a \in n(b) \tag{Cover}$$

$$(\forall a)\, a \in n(a) \tag{Reflexive}$$

$$(\forall a, b)\, (a \in n(b) \longrightarrow b \in n(a)) \tag{Symmetric}$$

$$(\forall a, b, c)\, (b \in n(a) \,\&\, c \in n(b) \longrightarrow c \in n(a)) \tag{Transitive}$$

$$(\forall a, b, c)\, (b \in n(a) \,\&\, c \in n(a) \longrightarrow c \in n(b)) \tag{Euclidean}$$

Let the set of all binary relations on a set S be denoted by $\mathcal{R}(S)$. The following operations can be defined on it (for any $P, Q \in \mathcal{R}(S)$ and for any $a, b \in S$):

- $(P \cup Q)ab$ if and only if Pab or Qab.
- $(P \cap Q)ab$ if and only if $Pab \,\&\, Qab$.
- $(P^c)ab$ if and only if $\neg Pab$.
- $(P \circ Q)ab$ if and only if $(\exists c)\; Pac \,\&\, Qcb$

- $(P^{-1})ab$ if and only if Pba
- $\bot = \emptyset$, $\Delta = \{(a, a) : a \in S\}$ and $\top = S^2$

The *relation algebra* formed on S is then the algebra

$$\Re = \left\langle \underline{\mathcal{R}(S)}, \cup, \cap, ^c, \bot, \top, \circ, ^{-1}, \Delta \right\rangle$$

of type (2, 2, 1, 0, 0, 2, 1, 0) that satisfies

- $\left\langle \underline{\mathcal{R}(S)}, \cup, \cap, ^c, \bot, \top \right\rangle$ is a Boolean algebra.
- $\left\langle \underline{\mathcal{R}(S)}, \circ, ^{-1}, \Delta \right\rangle$ is an involuted monoid. Δ being the diagonal relation.
- $(\forall a, b, c)\, ((a \circ b) \cap c = 0 \leftrightarrow (a^{-1} \circ c) \cap c = 0 \leftrightarrow (c \circ b^{-1}) \cap a = 0)$

Recall that the different neighborhoods generated by a relation are as below

$$[x] = \{a;\ Rax\} \quad \text{(Successor)}$$

$$[x]_i = \{a;\ Rxa\} \quad \text{(Predecessor)}$$

$$[x]_o = \{a;\ Rax\ \&\ Rxa\} \quad \text{(Multiplicative)}$$

$$[x]_\vee = \{a;\ Rax \vee Rxa\} \quad \text{(Additive)}$$

Relations will be adjoined as superscripts whenever multiple relations are under consideration.

Proposition 37

- *If* $P \subseteq Q$, *then* $(\forall x)\, [x]_P \subseteq [x]_Q$.
- $[x]_o^R = [x]^{R \cap R^{-1}}$
- $[x]_\vee^R = [x]^{R \cup R^{-1}}$

Theorem 38 *If a binary relation R has a property then its associated neighborhood operators also possess related properties. This is tabulated below (Sym, Se, i.Se, Re, Tr and Eu stand respectively for symmetry, serial, inverse serial, reflexive, transitive and Euclidean respectively)* (Table 3)*:*

Theorem 39 *The collection of all neighborhoods* $\mathcal{N} = \{n(x) : x \in \underline{S}\}$ *of* $\underline{S}$ *will form a cover if and only if* $(\forall x)(\exists y) x \in n(y)$ *(anti-seriality).*

Table 3 Relation and neighborhoods

R	$[x]$	$[x]_i$	$[x]_o$	$[x]_\vee$
Any			Sym	Sym
Se.	Se.	iSe.		Se.& i.Se.
Re.	Re.	Re.	Re.	Re.
Sym	Sym	Sym	Sym	Sym
Tr.	Tr.	Tr.	Tr.	Tr.
Eu.	Eu.		Eu.	

So in particular a reflexive relation on $\underline{S}$ is sufficient to generate a cover on it. Of course, the converse association does not necessarily happen in a unique way.

5 Red Results of Classical Rough Sets

For rough sets over approximation spaces, a number of representation, duality and canonical duality results are known. In the literature some partial surveys [6] are also known. All of the following semantic approaches have red results associated:

- Regular Double Stone Algebras[24, 35]
- Pre-Rough Algebras [5]
- Rough Algebras [5]
- Semi-simple Nelson Algebras [106]
- 3-Valued Lukasiewicz Algebras[55]
- Cylindric Algebras[23]
- BZ lattices and Variants [12, 13]
- Ortho-pair approach [14]
- Super Rough algebras[81]
- Post Algebras [106]
- Stone Algebras[46]
- AntiChain based approach[89, 95]

Some of these are considered in this section.

5.1 TQBA and Related Algebras

In this subsection dualities associated with topological quasi Boolean algebras (TQBA) including rough algebras are examined. For the basic theory, notation and related references refer to Chapter "Algebraic Methods for Granular Rough Sets" in this volume [98]. An important difference with [5] is the absence of the definable rough implication and order in these considerations as they are not relevant for the essential duality. It should be mentioned that semantics have been strongly influenced by a number of other papers including [10, 46, 57].

- $X = \langle \underline{X}, R \rangle$ is an approximation space.
- $\wp(X) = \left\langle \underline{\wp(X)}, l, u, \cup, \cap, \sim, 0, 1 \right\rangle$ is the topological Boolean algebra associated with X
- $\Re(X) = \left\langle \underline{\wp(X)|\approx}, L, \sqcup, \sqcap, \neg, \bot, \top \right\rangle$ is the topological quasi Boolean algebra on roughly equal elements.
- If $B = \langle \underline{B}, \vee, \wedge, \sim, 0, 1 \rangle$ is a Boolean algebra, then $TQ(B)$ is the topological quasi Boolean algebra generated by ordered pairs.

- $\delta(X)$ is the Boolean algebra of definites over the approximation space.

Theorem 40 *Every pre-rough algebra of the form* $S = \langle \underline{S}, \sqcap, \sqcup, \neg, L, \bot, \top \rangle$ *is isomorphic to a subalgebra of* $TQ(L(S))$.

Proof

- $L(S) = \{Lx : x \in S\}$ is a Boolean algebra and $TQ(L(S))$ is a topological quasi-Boolean algebra.
- The set $\{(Lx, \Diamond x) : x \in S\}$ generates a topological quasi-Boolean algebra that is a subalgebra of $TQ(L(S))$. S is epimorphic to the generated topological quasi-Boolean algebra.
- Injectivity follows from the property $La \leq Lb \,\&\, \Diamond a \leq \Diamond b \longrightarrow a \leq b$.

□

Theorem 41 *For every approximation space X, there exists a unique rough algebra S (upto isomorphism) associated with it and conversely every rough algebra is a subalgebra of a rough algebra generated by an approximation space.*

Proof

- Let $S = \langle \underline{S}, \sqcap, \sqcup, \neg, L, \bot, \top \rangle$ be a rough algebra. By definition it is completely distributive.
- The set $L(S)$ (the image of L) with induced operations is a complete Boolean subalgebra of S and is isomorphic to a complete field of sets $K = \langle \underline{K}, \cap, \cup, ^c, \emptyset, \underline{K} \rangle$
- Since K is atomic, let $X = \bigcup At(K)$. The atoms induce an equivalence relation R on X and $X = \langle X, R \rangle$ becomes an approximation space.
- Clearly $K = \delta(X)$ and the isomorphism of $L(S)$ and K induces the isomorphism of $TQ(L(S))$ and $TQ(\delta(X))$.
- It should be noted that in all these considerations singleton definite sets do not matter as any atomic Boolean algebra is isomorphic to an atomic Boolean algebra of sets without singleton atoms.

□

A few topological quasi-Boolean algebras that are not rough algebras or rough pre-algebras have been studied in the literature [128]. But no duality results have been proved in the study.

5.2 Super Rough Dualities

Super rough dualities refer to dualities in the context of a higher order approach to rough sets due to the present author [81]. Prerequisites can be found in this chapter [98]. The essential duality is presented here.

As mentioned before, a higher order similarity relation between represented rough objects is used in the semantics to arrive at super rough algebras.

Theorem 42 *For every super rough algebra S, there exists an approximation space X such that the super rough set algebra generated by X is isomorphic to S.*

Proof

- As S is a long lattice so there exists a partially ordered set P such that the lattice of convex subsets $Co(P)$ generated by it is isomorphic to a sublattice of it.
- The finiteness part ensures that an isomorphic copy of $Co(P)$ is obtainable.
- The convex structure ensures better expression in terms of *total* operations as opposed to partial ones and is always available and helps in simplifying the proof.
- By the fundamental characterization of tolerances by blocks (Theorem 46), it is possible to reconstruct a lattice F along with a compatible tolerance T on it from the set of fixed points of the map L_T.
- L_T is definable by a set of conditional equations and F is also constructible as the set of 'singletons' in S. These singletons are definable via the covering property with respect to the empty set.
- Again note that in any partially ordered set all singletons are convex subsets. This allows the definition of the operations $\sqcup$, $\sqcap$, L, $\neg$ and the distinguished elements on the desired prerough algebra.

Now the representation theorem for rough algebras Theorem 41 allows the existence of the approximation space X. Checking that the super rough set algebra generated by the approximation space is isomorphic to S is by a direct contradiction argument. □

5.3 Nelson Algebras

All Nelson algebras are quasi-Boolean algebras and semantics using these have been considered in two chapters in this volume [98, 109]. These are constructed as follows:

- Let S be an approximation space.
- Form the collection of $\underline{N} = \{(x^l, x^{uc}) : x \in \wp(S)\}$,
- For $anya, b \in \wp(S)$, define $(a^l, a^{uc}) \wedge (b^l, b^{uc}) = (a^l \cap b^l, a^{uc} \cup b^{uc})$
- For $anya, b \in \wp(S)$, define $(a^l, a^{uc}) \vee (b^l, b^{uc}) = (a^l \cup b^l, a^{uc} \cap b^{uc})$
- For $anya, b \in \wp(S)$, define $(a^l, a^{uc}) \Rightarrow (b^l, b^{uc}) = (a^{lc} \cup b^l, a^l \cap b^{uc})$
- For $anya \in \wp(S)$, define $\neg(a^l, a^{uc}) = (a^{uc}, a^l)$ & $\sim (a^l, a^{uc}) = (a^{lc}, a^l)$
- Define $0 = (\emptyset, S)$ and $1 = (S, \emptyset)$.

The following abstract representation theorem holds:

Theorem 43 *The algebra $N = \langle \underline{N}, \vee, \wedge, \Rightarrow, \sim, \neg, 0, 1 \rangle$ is a semi-simple Nelson algebra as it also satisfies*

$$(\forall a \in N)\, a \vee \sim a = 1 \qquad \text{(Nelson-SS)}$$

Further, any finite semi-simple Nelson algebra is isomorphic to a semi-simple Nelson derived by the above construction from an approximation space.

It should also be noted that in the above construction if the term functions

$$^* = \neg \sim\sim \;\; \&^+ = \sim$$

are used as fundamental operations instead of $\neg, \sim$, then

Theorem 44 *The algebra $L = \langle \underline{N}, \vee, \wedge, ^*, ^+, 0, 1\rangle$ is a regular double Stone algebra as it also satisfies all of the following apart from being a pseudo-complemented bounded distributive lattice that satisfies*

$$(\forall a, b)\, (b \leq a^* \leftrightarrow b \wedge a = 0) \qquad \text{(pseudo complement)}$$

$$(\forall a)\, a^* \vee a^{**} = 1 \qquad \text{(Stone Id)}$$

$$(\forall a, b)\, (a^+ \leq b \leftrightarrow b \vee a = 1) \qquad \text{(dual pc)}$$

$$(\forall a)\, a^+ \wedge a^{++} = 0 \qquad \text{(dual Stone Id)}$$

$$(\forall a, b)\, (a^* = b^* \,\&\, a^+ = b^+ \longrightarrow a = b) \qquad \text{(regularity)}$$

This result is due to [23].

6 Red Results of Cover Based Rough Sets

In this section, the red results considered are of the following forms:

- Representation between General Approximation Spaces and Covers
- Order Theoretic Dualities

The duality result in the subsection on Tarski algebras is a new adaptation due to the present author. The result has not been used in the context of covering approximation spaces previously.

A Basic Problem that can substantially reduce efforts used in logico-algebraic approaches to general rough sets is the following: Given a general approximation space $S = \langle \underline{S}, R\rangle$, does there exist a unique covering approximation space of the form $S_{cov} = \langle \underline{S}, \mathcal{C}\rangle$ from which R can be reconstructed in a unique way and conversely?

Variations of the question are also of interest from the point of view of multiple general approximation spaces that seek to model multi agent systems.

6.1 Basic Results

Theorem 45

- *Every partition* $\mathcal{S}$ *on a set* S *corresponds to a unique equivalence* R *on* S *and conversely.*
- $EQ(S)$ *has a lattice structure with respect to the inclusion order. More specifically,* $\langle EQ(S), \vee, \wedge, \Delta, 1\rangle$ *is an atomic, atomistic, relatively complemented, complete, continuous, semimodular lattice with the least element* Δ *being the diagonal of* S *and the greatest element* $1 = S^2$*. Moreover, semimodularity in the above can be replaced with*

$$(a \wedge b \prec a \,\&\, a \wedge b \prec b \longrightarrow a \prec a \vee b)$$

- *The atoms of* $EQ(S)$ *have the form* $\Delta \cup \{(a, b),\ (b, a)\}$ *for distinct* $a, b \in S$*. The compact elements of* $EQ(S)$ *are finite joins of these atoms.*
- *The partition lattice* $\Pi(S)$ *is dually isomorphic to* $EQ(S)$*.*
- *Every lattice is embeddable into a lattice of the form* $EQ(S)$ *for some set* S*[145].*

Proof Most of the proof can be found in [48]. Note that continuous lattices are essentially a form of generalized algebraic lattices. □

A collection $\mathcal{A} = \{H_\alpha : \alpha \in I\}$ of subsets of S is a *normal cover* if and only if all of the following hold:

- $\mathcal{A}$ is an antichain with respect to the usual inclusion order,
- $\mathcal{A}$ is a cover for S, and
- if A is a subset of S which is not included in any H_α, then there exists a two element subset of A with the same property.

Normal covers correspond to blocks of tolerances defined on the set S in a bijective way [18, 19]. A refined version of the result is proved next. A specific version of this result for compatible tolerances on a lattice has been used by the present author in a duality for super rough sets [81] Sect. 5.2.

Theorem 46 *Every normal cover* $\mathcal{S}$ *on a set* S *is a system of blocks of a tolerance* $T \in Tol(S)$ *and conversely.*

Proof

- If $T \in Tol(S)$, then let $\mathcal{B} = \{B_j : j \in J\}$ be a system of blocks of T. As T is reflexive and B_js are maximal squares contained in T, $\mathcal{B}$ is an antichain that covers S. Suppose the third condition is false, then there exists a $H \subset S$ such that $H \nsubseteq B_j$ for all $j \in J$ and satisfies

$$(\forall a, b \in H)(\exists j \in J)\, (a, b) \in B_j.$$

This yields $H^2 \subseteq T$ and so it must be contained in a block of the form B_j—a contradiction.

- For the converse, if $\mathcal{S} = \{B_j : j \in J\}$ is a normal cover, let

$$(a, b) \in T \leftrightarrow (\exists j \in J)\, a, b \in B_j.$$

It is easy to check that T is reflexive and symmetric. Suppose that B is a block of T, but $B \notin \mathcal{S}$, then $B^2 \subset T$ and third condition contradicts the assumption. So T is exactly the tolerance corresponding to the cover $\mathcal{S}$.

□

The result can be extended to generalizations of tolerances to finite arity [77]. The generalization is very significant for rough sets and so the following concept of k-ary similarity space is introduced. Such structures can be found in application contexts, but their semantics remain an open problem.

Definition 47 A k-ary relation T on a set S is a *k-tolerance* on S if and only if all of the following hold (for all k-permutations σ):

$$(\forall (a, a, \ldots, a) \in S^k)\, T(a, a, \ldots, a) \qquad \text{(k-reflexive)}$$

$$T(a_1, a_2, \ldots, a_k) \longrightarrow T(a_{\sigma 1}, a_{\sigma 2}, \ldots, a_{\sigma k}) \qquad \text{(k-symmetric)}$$

Definition 48 A collection of subsets $\mathcal{L} = \{L_j : j \in J\}$ is a τ_k-covering if and only if all of the following hold:

- $\mathcal{L}$ is an antichain with respect to the usual inclusion order,
- $\mathcal{L}$ is a cover for S, and
- if A is a subset of S which is not included in any L_j, then there exists a k elements $\{a_i\}_1^k$ of A which are not included in any L_j.

The proof of the following theorem also relies on the same strategy used for tolerances:

Theorem 49 *Every τ_k cover $\mathcal{S}$ on a set S is a system of blocks of a k-tolerance $T \in Tol_k(S)$ and conversely. $Tol_k(S)$ being the set of all k-tolerances on S.*

6.2 Order Theoretic Representations

Given a covering approximation space derived from an information table and associated basic operators, some operators relating to information exchange and approximations can be defined. Related red results are also presented in this sub section. The basic form of these duality results is the following:

- For an operator satisfying some property there exists a cover such that it generates an equivalent approximation operator.

- A lower approximation operator is a topological interior operator if and only if the cover that generates it satisfies a property Xi.
- An upper approximation operator ui defined by a cover $\mathcal{C}$ is a topological (or Kuratowski) closure operator if and only if ui satisfies condition ϕ_i for $i = 1, 2, 3, 4$.

The following theorem was proved in [152]

Theorem 50 *When S is finite, a covering $\mathcal{C}$ is unary if and only if*

$$(\forall K_1, K_2 \in \mathcal{C})(\exists C_1, \ldots C_n \in \mathcal{C})\ K_1 \cap K_2 = \cup_1^n C_i.$$

Proof Suppose $\mathcal{C}$ is unary, then as $(\forall x \in S)\,\#(Md(x)) = 1$, let $md(x) = \{K_x\}$ and let $K_1, K_2 \in \mathcal{C}$.

- If $K_1 \subseteq K_2$, then $K_1 \cap K_2$ is obviously a union of a finite number of elements of the cover.
- Otherwise, if $x \in K_1 \cap K_2$ and $K_x \nsubseteq K_1$, then $K_1 \notin md(x)$. So $(\exists K_1^* \in \mathcal{C})\, x \in K_1^* \subset K_1$. Using the same argument on K_2 and other elements of $\mathcal{C}$, it is possible to get an infinite sequence $K_1^* \subset K_2^* \supset \ldots$ from elements of $\mathcal{C}$
- But S is finite and the contradiction means

$$K_x \subseteq K_1, K_2 \text{ and } K_1 \cap K_2 \;=\; \bigcup_{x \in K_1 \cap K_2} K_x$$

- For the converse, if there exist two elements $K_1, K_2 \in md(x)$ such that $K_1 \cap K_2$ is not a union of finite number of elements of $\mathcal{C}$, then
- it is easy to obtain a contradiction when K_1 and K_2 are comparable.

□

Proposition 51 *If $L : \wp(S) \mapsto \wp(S)$ is an abstract operator on a set S that satisfies contraction, idempotency, monotonicity and top then there exists a covering $\mathcal{C}$ of S such that the lower approximation $l1$ generated by $\mathcal{S}$ coincides with L.*

Theorem 52 *For every interior operator $L : \wp(S) \mapsto \wp(S)$ there exists a unary covering $\mathcal{C}$ on S such that the lower approximation of the first type $l1$ generated by $\mathcal{C}$ coincides with L.*

Proof Since L is an interior operator, it satisfies top, contraction, monotonicity and idempotence. By the previous lemma, there exists a cover $\mathcal{C}$ such that the lower approximation of the first type $l1$ generated by it coincides with L.

L satisfies multiplicativity, $(\forall A, B)\ L(A \cap B) = L(A) \cap L(B)$, and this together with Theorem 50 yields the result. □

In general, the first, second, third and fourth type of upper approximation operators determined by a cover $\mathcal{S}$ on a set S are not topological closure operators.

These are defined as below:

- $X^{l1} = \bigcup\{K : K \in \mathcal{S} \,\&\, K \subseteq X\}$
- $X^{u1+} = X^{l1} \cup \bigcup\{\mathrm{md}(x) : x \in X\}$,
- $X^{u1} = X^{l1} \cup \bigcup\{\mathrm{md}(x) : x \in X \setminus X^{l1}\}$ [11],
- $X^{u2+} = \bigcup\{K : K \in \mathcal{S}, K \cap X \neq \emptyset\} = \bigcup\{Fr(x) : x \in X\}$,
- $X^{u3+} = \bigcup\{\mathrm{md}(x) : x \in X\}$,
- $X^{u4+} = X^{l1} \cup \{K : K \cap (X \setminus X^{l1}) \neq \emptyset\}$,

Closely related to X^{u1} is $X^{u1+} = X^{l1} \cup \bigcup\{\mathrm{md}(x) : x \in X\}$ These have been defined many times over in the literature (see [86, 148]) and also the chapter on granular rough sets in this volume by the present author for details [98].

In [151, 153, 154], conditions for the upper approximation operators to be closure operators are proved, but the conditions do not amount to the operators being topological closure operators. In [44], the following is proved:

Theorem 53 *The following are equivalent if S is finite:*

1. *$\mathcal{C}$ is a unary cover of S.*
2. *$\mathcal{C}$ is a base for some topology τ on S*
3. $$(\forall K_1, K_2 \in \mathcal{C})(\forall x \in K_1 \cap K_2)(\exists K \in \mathcal{C})\, x \in K \subseteq K_1 \cap K_2$$
4. *$u1$ is a topological closure operator.*

Proof The equivalence of the first and third statement will be proved first. If $\mathcal{C}$ is unary, let $(\forall A, B \in \mathcal{C})(\forall x \in A \cap B)\ \mathrm{md}(x) = \{K_x\}$. x must be a representative element of K_x. So $x \in K_x \subseteq A \cap B$, with $K_x \in \mathcal{C}$.

If $\mathcal{C}$ is not unary, then $(\exists A, B \in \mathcal{C}) A, B \in \mathrm{md}(x) \,\&\, A \neq B$. But by the third statement, there must exist a $K \subset A \cap B$ satisfying $x \in K$. This contradicts $A, B \in \mathrm{md}(x)$.

From the above, it follows that $\mathcal{C}$ is a unary cover if and only if there exists a topology τ on S such that $\mathcal{C}$ is a base for the topology. □

The following example shows that it is not possible to generalize to the infinite case:

Example 54

- Let $S = [-1, 1]$
- $\mathcal{C} = \{\{x\} : x \in S \setminus \{0\}\} \cup \{(-\frac{1}{n}, \frac{1}{n}) : n \in N\} \cup \{\{-1, 0, 1\}\}$
- $\mathcal{C}$ covers S and if $x \neq 0$ then $md(x) = \{\{x\}\}$ and $md(0) = \{\{-1, 0, 1\}\}$. $\mathcal{C}$ is unary, but the theorem does not hold.

Example 55

- Let $S = R$—the set of reals and $\mathcal{C} = \{(x - \frac{1}{n}, x + \frac{1}{n}) : x \in S \,\&\, n \in N\}$.
- $\mathcal{C}$ is a base for the usual topology on S.
- $(\forall x \in S)\, md(x) = \emptyset \,\&\, \{x\}^{u1} = \emptyset$. So $u1$ is not a closure operator.

Theorem 56 *When S is a finite or an infinite set, $u2+$ is a topological closure operator if and only if $\{Fr(x) : x \in S\}$ forms a partition of S.*

Proof The converse is obvious.

- Let $u2+$ be a topological closure operator. It suffices to show that

$$(\forall a, b)(Fr(a) \neq Fr(b) \longrightarrow Fr(a) \cap Fr(b) = \emptyset).$$

  Or else there exists $z \in Fr(a) \cap Fr(b)$.
- Clearly, $Fr(z) \subseteq \bigcup\{Fr(x) : x \in Fr(b)\} = Fr(b)$ and
- $Fr(b) \subseteq \bigcup\{Fr(x) : x \in Fr(z)\} = Fr(z)$. So $Fr(b) = Fr(z)$
- Similarly, $Fr(a) = Fr(z) = Fr(b)$
- This contradicts $Fr(a) \neq Fr(b)$.

□

Theorem 57 *$u2+$ is a topological closure operator if and only if there is a closed-open topology τ (that is a union of members of a partition on S) on S such that $\{Fr(x) : x \in S\}$ is a base of τ if and only*

$$(\forall a, b \in S)\ Fr(a) \cap Fr(b) = \emptyset \text{ or } a \in Fr(b)$$

Proof The proof is by an extension of the proof of the previous theorem. □

Let $cfr(x) = \bigcup md(x)$ for any $x \in S$

Theorem 58 *For a finite or an infinite S, $u3+$ is a topological closure operator if and only if each $x \in S$ is a representative element of $cfr(x)$ for the unary cover $\{cfr(x) : x \in S\}$.*

Proof If $u3+$ is a closure operator then

- for each $a \in S$, if $a \in cfr(b)$ for some $b \in S$, then

$$cfr(a) \subseteq \bigcup\{cfr(z) : z \in cfr(b)\} = cfr(b)^{u3+} = cfr(b)$$

- So a must be a representative element of $cfr(a)$ for the cover $\{cfr(z) : z \in S\}$

If x is a representative element of $cfr(a)$ for the cover $\mathbb{C} = \{cfr(z) : z \in S\}$, then

- For each $z \in \{x\}^{u3+} = cfr(x)$, since z is a representative element of $cfr(z)$ for the cover $\mathbb{C}$,

$$\{z\}^{u3+} = cfr(z) \subseteq cfr(x) = \{x\}^{u3+}$$

- So

$$\{x\}^{u3+u3+} = \bigcup\{cfr(z) : z \in cfr(x)\} \subseteq \{x\}^{u3+}.$$

- This verifies idempotence. Rest of properties can be directly checked.

□

Theorem 59 *For a finite or an infinite S, $u3+$ is a topological closure operator if and only if $\{cfr(x) : x \in S\}$ is a base for a topology τ on S and for each $x \in S$, $\{cfr(x)\}$ is a local base at x.*

Proof The proof follows from the previous theorem and the result that for every unary cover there exists a topology τ on S for which $\{cfr(x) : x \in S\}$ is a base for (S, τ). The missing steps can be found in [44]. □

For the proof of the next three theorems, the reader is referred to [44].

Theorem 60 *For any finite or infinite S, $u4+$ is a closure operator if and only if the cover $\mathcal{C}$ satisfies For all $K_1, K_2 \in \mathcal{C}$ if $K_1 \neq K_2 \,\&\, K_1 \cap K_2 \neq \emptyset$ then $(\forall x \in K_1 \cap K_2)\, \{x\} \in \mathcal{C}$.*

Theorem 61 *For any finite or infinite S, $u4+$ is a closure operator if and only if the cover $\mathcal{C}$ is a base for a topology τ on S and*

- *(S, τ) is a union of disjoint subspaces S_1 and S_2.*
- *For any distinct $A, B \in \mathcal{C}$, either $A \cap S_2 = B \cap S_2 = \emptyset$ or $A \cap S_2 \neq B \cap S_2$ and $\{F \cap S_2 : F \in \mathcal{C}\}$ is a partition of S_2, and*
- *In the topologies τ_1, τ_2 induced on S_1 and S_2 respectively, S_1 is a discrete topological space and S_2 is a pseudo-discrete space.*

From the above results it can be deduced that

Theorem 62 *$u4+$ is a closure operator $Rightarrow$ $u1$ is a closure operator $Rightarrow$ $u3+$ is a closure operator and no other relation between similar statements hold.*

A number of if and only conditions for a covering $\mathcal{C}$ being unary are known. Some of these are summarized below:

Theorem 63 *A cover $\mathcal{C}$ of a set S is unary if and only if*

- $u3+ = u1$
- $(\forall x \in S)\, nbd(x) \in \mathcal{C}$
- $(\forall X \subseteq S)\, (X^{u4+})^{u3+} = X^{u4+}$.
- $(\forall X \subseteq S)\, (X^{u2+})^{u3+} = X^{u2+}$.

6.3 Galois Connections

A Galois connection for partial covers (that is arbitrary collections of subsets of a given set) under stringent conditions on approximations has been proved recently in [27]. The main definitions, terminology and result of [27] has been simplified and reformulated in this subsection.

A *base systems* $\mathfrak{B}$ is essentially an *an arbitrary nonempty collection of subsets* of a universe S. It is not required that

$$\bigcup \mathfrak{B} = S$$

This is used as a very restricted granulation (even relative to cover based rough sets).

A set X is said to be $\mathfrak{B}$-definable if it is a union of some elements of $\mathfrak{B}$ in at least one way. That is

$$\exists \mathfrak{H} \subseteq \mathfrak{B} \bigcup \mathfrak{H} = X$$

The set of all $\mathfrak{B}$-definable elements is denoted by $\Delta_{\mathfrak{B}}$. It is assumed that $\emptyset \in \mathfrak{B}$. A $\mathfrak{B}$-definable element will be said to be *strongly* $\mathfrak{B}$*-definable* if it is the union of exactly one set of elements of $\mathfrak{B}$.

A base system $\mathfrak{B}$ is said to be *single-layered* if and only if

$$(\forall B \in \mathfrak{B})(\forall \mathfrak{H} \subseteq \mathfrak{B} \setminus \{B\})\; B \cap \bigcup \mathfrak{H} \neq B$$

This means that

Proposition 64 *If a base system* $\mathfrak{B}$ *is single-layered, then each element of* $\mathfrak{B}$ *is a minimum description of some* $x \in S$*. Also an element* $x \in S$ *need not satisfy* $\mathrm{md}(x) \in \mathfrak{B}$.

This concept of one-layer base systems in [27] is superfluous because it is the same thing as saying that the elements of $\mathfrak{B}$ are pairwise disjoint.

The lower and upper approximation are defined by taking $\mathfrak{B}$ to be the set of granules like so

$$A^{l_b} = \bigcup \{B :\; B \in \mathfrak{B} \,\&\, B \subseteq A\}$$

$$A^{u_b} = \bigcup \{B :\; B \in \mathfrak{B} \,\&\, B \cap A \neq \emptyset\}$$

Proposition 65 *The two approximations satisfy all of*

- *Monotonicity:* $(\forall A, X \in \wp(S))(A \subseteq B \longrightarrow A^{l_b} \subseteq X^{l_b} \,\&\, A^{u_b} \subseteq X^{u_b})$
- $\emptyset^{u_b} = \emptyset$ *and* $\emptyset^{l_b} = \emptyset$
- *If* $H \in \Delta_{\mathfrak{B}}$ *then* $H^{l_b} = H$
- $(\forall X \in \wp(S))\; X^{l_b l_b} = X^{l_b} \subseteq X \,\&\, x^{l_b} \subseteq X^{u_b}$
- $\Im(u_b) \subseteq \Im(l_b) = \Delta_{\mathfrak{B}}$

The following proposition happens precisely because of the constraints imposed on definability.

Proposition 66 *The set of* $\mathfrak{B}$*-definable elements are all strongly* $\mathfrak{B}$*-definable if and only if* $\mathfrak{B}$ *is single layered.*

The main theorem proved in [27] is this:

Theorem 67 *In the context of partial covering approximation spaces, the pair of maps* (l_b, u_b) *is a Galois connection on* $\langle \wp(S), \subseteq \rangle$ *if and only if* $\mathfrak{B}$ *is single-layered and pairwise disjoint, and therefore a partition.*

Proof To ensure that $(\forall X \in \wp(S))\, X^{l_b u_b} \subseteq X$ it is necessary that $\mathfrak{B}$ be pairwise disjoint and conversely—this can be proven through a contradiction argument.

The other part $(\forall X \in \wp(S))\, X \subseteq X^{u_b l_b} \subseteq X$ holds obviously under the conditions.

A particularly strange way of partialization is also considered in [27] to improve the above result to single-layer $\mathfrak{B}$-systems. The strategy in the presence of the strong aggregation properties assumed means that *no nontrivial approximation that are below the minimal elements of* $\mathfrak{B}$ *are to be regarded as legitimate and upper approximations of sets that do not include the set are inadmissible*. Formally this translates into

Definition 68 In the context, let l_{pb}, u_{pb} be partial maps that satisfy

$$x^{l_{pb}} = \begin{cases} x^{l_b} & \text{if } x = \emptyset \text{ or } x^{l_b} \neq \emptyset \\ \text{undefined} & \text{otherwise} \end{cases} \tag{12}$$

$$x^{u_{pb}} = \begin{cases} x^{u_b} & \text{if } x \subseteq x^{u_b} \\ \text{undefined} & \text{otherwise} \end{cases} \tag{13}$$

Theorem 69 *In the context of partial covering approximation spaces, the pair of partial maps* (l_{pb}, u_{pb}) *is a partial Galois connection on* $\langle \wp(S), \subseteq \rangle$ *if and only if* $\mathfrak{B}$ *is pairwise disjoint.*

6.4 Representation of QOAS

In the chapter on algebraic approaches to granular rough sets in this volume [98], the basic algebraic semantics of quasi-ordered approximation spaces has been presented by the present author. In this subsection, the main representation results are stated with minimal remarks. Proofs of the result are in [73]. For notation and other details, the reader is referred to the same chapter.

The almost routine construction used for proving representation results is in the proof of the following theorem.

Theorem 70 *If* $L = \left\langle \underline{L}, \vee, \wedge, 0, 1 \right\rangle$ *is a bounded distributive lattice, then there exists a QOAS* $Q = \left\langle \underline{Q}, R \right\rangle$ *such that L is embeddable in its lattice of definable sets.*

Proof

- Let $\underline{Q} = Spec(L)$—the set of prime filters of L. Let $C_a = \{F : a \in F \in Spec(L)\}$
- Let $\mathcal{C} = \{C_a : a \in \underline{Q}\}$, then define R through its neighborhoods by $[x]_i = \bigcap\{C : x \in C \in \mathcal{C}\}$, then $Q = \langle \underline{Q}, R \rangle$ is a QOCAS.
- $(\forall X \in Q)\, [X]_i = \bigcap\{C_a : X \in C_a \in \mathcal{C}\} = \bigcap\{C_a : a \in X \,\&\, C_a \in \mathcal{C}\}$.
- So definable sets including C_a are open sets in the Alexandrov topology τ_R generated by R
- The map $h : L \longmapsto \delta(Q)$ defined by $h(a) = C_a$ is a lattice embedding.

□

Theorem 71 *If $L = \langle \underline{L}, \vee, \wedge, \Rightarrow, 0, 1 \rangle$ is a Heyting algebra, then there exists a QOAS $Q = \langle \underline{Q}, R \rangle$ such that L is embeddable in its lattice of definable sets.*

Proof

- Define Q as in the previous theorem. Using $\mathcal{C}$ as a sub basis, generate a new topology τ that is coarser than τ_R
- Using τ form a Heyting algebra with set union, intersection and $\Rightarrow$ as per

$$(\forall A, B \subseteq Q)\; A \Rightarrow B = int_\tau(A^c \cup B)$$

- h as defined in the proof of the previous theorem is an embedding of Heyting algebras.

□

A lattice filter F is *complete* if it is closed under arbitrary meets. It is *completely prime* if and only if $\bigvee a_i \in F$ implies there is at least one i for which $a_i \in F$.

Theorem 72 *If $L = \langle \underline{L}, \vee, \wedge, \Rightarrow, 0, 1 \rangle$ is a Heyting algebra, then there exists a QOAS $Q = \langle \underline{Q}, R \rangle$ such that L is isomorphism in its lattice of definable sets.*

Proof The main step in the proof is to take the base set of Q as the principal filters generated by join-irreducible elements. These filters would be complete, completely prime and would separate points in the lattice. □

Theorem 73 *If $L = \langle \underline{L}, \vee, \wedge, \Rightarrow, 0, 1 \rangle$ is a Heyting algebra, then there exists a QOAS Q such that L is isomorphic to a subalgebra of the Heyting algebra $\mathbb{R}_Q$ formed by Q*

Proof The proof of the result is based on that of the previous results and the properties of $\mathbb{R}_Q$. □

Theorem 74 *If $L = \langle \underline{L}, \vee, \wedge, \Rightarrow, 0, 1 \rangle$ be a completely distributive Heyting algebra in which the set of join irreducibles is join-dense, then there exists a QOAS Q such that L is isomorphic to the Heyting algebra $\mathbb{R}_Q$ formed by Q.*

Proof The proof of the result is based on that of the previous results and the properties of $\mathbb{R}_Q$. □

6.5 Tarski Algebras and Spaces

Tarski algebras are the same thing as implication algebras [123]. A few full dualities relating to classes of such algebras are known. Two related dualities are outlined in this section. One of this is a duality for finite Tarski sets [16, 17] or covering approximation spaces. But it has not been viewed in a rough perspective before. Open research problems in the context are also indicated.

The duality between Boolean algebras and Boolean spaces is an example of a topological duality—this basic result can be generalized to a duality between Tarski algebras and spaces [1]. Full dualities between the category of Boolean algebras with meet-morphisms that preserve 1 and Boolean spaces with Boolean relations are also known [52]. This is generalized to Tarski algebras in [17].

Definition 75 A *Tarski algebra* (or an *implication algebra*) is an algebra of the form $S = \left\langle \underline{S}, \cdot, 1 \right\rangle$ of type 2, 0 that satisfies (in the following, the implication $a \cdot b$ is written as ab as in [123])

$$1a = a \tag{T1}$$

$$aa = 1 \tag{T2}$$

$$a(bc) = (ab)(ac) \tag{T3}$$

$$(ab)b = (ba)a \tag{T4}$$

The variety of IAs is denoted by $\mathcal{V}_{IA}$. If X is a set, and $(\forall A, B \in \wp(S))\ A \cdot B = A^c \cup B$, then $\left\langle \underline{\wp(X)}, \cdot, X \right\rangle$ is an IA. Any subalgebra of such an algebra is said to be an *IA or Tarski algebra of sets*. A join-semilattice order $\leq$ is definable in a IA as below:

$$(\forall a, b)\ a \leq b \leftrightarrow ab = 1;\ \text{the join is } a \vee b = (ab)b$$

Filters or deductive systems of an IA S are subsets $K \subseteq S$ that satisfy

$$1 \in K \ \&\ (\forall a, b)(a, ab \in K \longrightarrow b \in K)$$

The set of all filters $\mathcal{F}(S)$ is an algebraic, distributive lattice whose compact elements are all those filters generated by finite subsets of S. A filter K is prime if and only if it satisfies $(\forall a, b)(a \vee b \in K \longrightarrow a \in K \text{ or } b \in K)$.

Theorem 76 *In a finite IA S, the following hold:*

- *A filter is prime if and only if it is a maximal filter.*
- *A filter is prime or maximal iff it is of the form* $(x\downarrow)^c$ *for a coatom* x

- *If $Spec(S)$ is the set of prime or maximal filters of S and $\sigma_S : S \longmapsto \wp(Spec(S))$ is a map into the IA of sets $\wp(Spec(S))$ and is defined by*

$$(\forall x)\, \sigma_S(x) = \{K : x \in K \in Spec(S)\},$$

then σ_S is an embedding

In the last theorem if S is a finite Boolean algebra, then it is provable that $Spec(S) \cong Spec(\wp(Spec(S))$ and in fact for any finite Boolean algebra S, $S \cong \wp(Spec(S))$. This does not hold for finite IA. But note that $Spec(S)$ is determined by the set $CoAt(S)$ of coatoms.

Definition 77 A *Tarski set* is a pair $\langle X, \mathcal{S}\rangle$ where X is a non-empty set and $\mathcal{S}$ is a nonempty subset of $\wp(X)$. It is *dense* (or a *covering approximation space* (CAS)) if and only if $\bigcup(\mathcal{S}) = X$. The *dual* of a Tarski set $\langle X, \mathcal{S}\rangle$ is the subset $\Delta(X) \subset \wp(X)$ defined as below:

$$\Delta(X) = \{U : \exists W \in \mathcal{S}\, \&\, \exists H \subseteq W\, \&\, U = W^c \cup H\}$$

Theorem 78 *Let $\langle X, \mathcal{S}\rangle$ is a Tarski set, then $\langle \Delta(X), \cdot, X\rangle$ is a Tarski subalgebra of sets.*

The proof is by direct verification.

If S is a finite Tarski algebra and $\sigma_S : S \longmapsto \wp(Spec(S))$ is the map defined earlier and $\mathcal{K}_S = \{\sigma(x)^c : x \in S\}$, then the Tarski set $\langle Spec(S), \mathcal{K}_S\rangle$ is also referred to as the *associated set* of S.

Theorem 79 *If S is a finite Tarski algebra, then $\sigma_S(S) = \Delta(Spec(S))$ and so $S \cong \Delta(Spec(S))$.*

Proof

- Since $(\forall x)\, \sigma_S(x) = \sigma(x) \cup \emptyset$, therefore $\sigma_S(x) \in \Delta(Spec(S))$.
- Let $U \in \Delta(Spec(S))$. By definition, $(\exists x \in S)(\exists H \subseteq (\sigma_S(x))^c)\, U = \sigma_S(x) \cup H$
- Let $H = \{Q_1, \ldots, Q_n\}$ For each of these maximal filters Q_i, there exists a coatom q_i that generates it.
- So $\sigma_S(q_i)^c = \{(q_i \downarrow)^c\}$. So $H = \bigcup(\sigma_S(q_i))^c$.
- This means $U = \sigma(x) \cup \bigcup(\sigma_S(q_i))^c = \sigma_S(b)$ for some b
- So $U \in \sigma_S(S)$ and $\sigma_S(S) = \Delta(Spec(S))$.

□

Theorem 80 *Let $\langle X, \mathcal{S}\rangle$ be a finite dense Tarski set or a CAS, then the map $\xi_X : X \longmapsto Spec(\Delta(X))$ defined by $\xi_X(x) = \{U : x \in U \in \Delta(X)\}$ is an injective and a surjection.*

Proof The proof is by direct verification.

- If $a, b \in X$ are distinct elements, then $b \in \{a\}^c \in Coat(\Delta(X))$. So ξ_X is injective.

- By finiteness, $(\forall Q \in Spec(\Delta(X)))(\exists U \in Coat(\Delta(X)))\; Q = (U \downarrow)^c$.
- For a specific Q and U in the last statement, $(\exists x \in X)\, U = \{x\}^c$ as $\langle X, \mathcal{S}\rangle$ is dense. Clearly then $\xi_X(x) = Q$ and $X \cong Spec(\Delta(X))$.

□

The result is an abstract representation theorem for finite Tarski algebras. The actual significance of the result has not been properly explored in the context of covering approximation spaces (even in the finite case). This is considered separately by the present author in a forthcoming paper. For one thing, every construct in a CAS has an algebraic representation.

For extending the results to the infinite case, a topological extension is necessary.

Definition 81 A *Tarski space* (T-space) is a concrete topological structure of the form $\chi = \langle X, \mathcal{K}, \tau\rangle$ that satisfies:

1. $\langle X, \tau\rangle$ is a Hausdorff, totally disconnected topological space with $\mathcal{K}$ being a basis for the compact subsets of τ.
2. $(\forall A, .B \in \mathcal{K})\, A \cap B^c \in \mathcal{K}$
3. For any two distinct $a, b \in X$, exists a $U \in \mathcal{K}$ such that $a \in U$ and $b \notin U$.
4. If F is a closed subset and $\{U_i\}_{i\in I}$ is a directed subcollection of sets in $\mathcal{K}$ and for each $i \in I$, $F \cap U_i \neq \emptyset$, then $F \cap (\bigcap U_i) \neq \emptyset$.

Given a T-space two distinct Tarski subalgebras of a set Tarski algebra are defined in [17]:

$$T_{\mathcal{K}}(X) = \{W^c \cup H : H \subseteq W \in \mathcal{K}\} \qquad \text{(T-algebra)}$$

$$\Delta_{\mathcal{K}}(X) = \{U : U^c \in \mathcal{K}\} \qquad \text{(dual T-algebra)}$$

Theorem 82

- *If* $X \in \mathcal{K}$, *then* χ *is a Boolean space and* $\Delta_{\mathcal{K}}(X)$ *is a Boolean algebra of all clopen sets of the topological space.*
- *If* X *is finite, then* $T_{\mathcal{K}}(X) = \Delta_{\mathcal{K}}(X)$

If $A, B \in \mathcal{V}_{IA}$, then a *semi-morphism* is a monotone map $f : A \longmapsto B$ that satisfies

- $f(ab) \leq f(a)f(b)$
- $f(1) = 1$

Example 83 If X and W are sets and $R \subset X \times W$, let $[x]_i = \{a : Rxa\}$. Define a map $h_r : \wp(W) \longmapsto \wp(X)$ such that for any $U \subseteq W$,

$$h_R(U) = \{x : [x]_i \subseteq U\}$$

$h_R \in SMor(\wp(W), \wp(X))$—the set of semi-morphisms : $W \longmapsto X$.

Definition 84 Let χ_X and χ_W be two T-spaces over X and W respectively, then $R \subseteq X \times W$ is a *T-relation* if and only if the following hold:

- $(\forall U \in \Delta_{\mathcal{K}_W}(W))\, h_R(U) = \{x : [x]_i \subseteq U\} \in \Delta_{\mathcal{K}_X}(X)$
- $[x]_i$ is a closed subset of W for each $x \in X$

A *T-partial* function is a partial map $f : X \longmapsto W$ such that for each $U \in \Delta_{\mathcal{K}_W}(W)$, $f^{-1}(U) \in \Delta_{\mathcal{K}_X}(X)$. The set of all T-partial functions (resp. relations) from X to W will be denoted by $TF(X, W)$ (resp. $TR(X, W)$).

Definition 85 The following categories can be defined on the basis of the above:

- $\mathfrak{TR}$ with Objects being Tarski spaces and Morphisms being sets of T-Relations.
- $\mathfrak{TF}$ with Objects being Tarski spaces and Morphisms being sets of T-partial functions.
- $\mathfrak{ST}$ with Objects being Tarski algebras and Morphisms being sets of semi-morphisms.
- $\mathfrak{HT}$ with Objects being Tarski algebras and Morphisms being sets of homomorphisms.

Theorem 86

- *$\mathfrak{HT}$ is a subcategory of $\mathfrak{ST}$,*
- *$\mathfrak{ST}$ is dually equivalent to $\mathfrak{TR}$, and*
- *$\mathfrak{HT}$ is dually equivalent to $\mathfrak{TF}$.*

Proof For the gory details, the reader is referred to [17]. Simpler proofs are of interest. □

7 Discrete Duality for Double Stone Algebras

Double Stone algebras are among the first algebras proposed as a semantics for classical rough sets. Two new discrete dualities have been proved recently for double Stone algebras in [35]. These are considered here. Dualities like these are justified by their use in applications in logic or in algebra and related logics have been considered in the same paper. The main proof in [35] has been reworked here.

Recall from [98] that a *Stone algebra* L is an algebra of type (2, 2, 1, 0, 0) of the form

$$L = \langle \underline{L}, \vee, \wedge, *, 0, 1 \rangle$$

that satisfies

- $\langle \underline{L}, \vee, \wedge, 0, 1 \rangle$ is a bounded distributive lattice.
- x^* is the pseudo-complement of x, that is $y \leq x^* \Leftrightarrow y \wedge x = 0$
- $x^* \vee x^{**} = 1$

$B(L) = \{a^* : a \in L\}$ is the *center* of L $\Delta^*(L) = \{a : a^* = 0\}$ is its set of *dense* elements.

Proposition 87 *In a Stone algebra L, the following hold:*

- *$B(L)$ is a Boolean subalgebra of L.*
- *$\Delta^*(L) \in \mathcal{F}(L)$ and $\Delta^*(L) = \{a \vee a^* : a \in L\}$*
- *A prime filter is maximal if and only if $\Delta^*(L) \subseteq F$.*
- $(\forall a, b)\ (a \wedge b)^* = a^* \vee b^*\ \&\ (a \vee b)^{**} = a^{**} \vee b^{**}$

A *double Stone algebra L* is an algebra of type (2, 2, 1, 1, 0, 0) of the form

$$L = \langle \underline{L}, \vee, \wedge, *, {}^+, 0, 1 \rangle$$

that satisfies

- $\langle \underline{L}, \vee, \wedge, *, 0, 1 \rangle$ is a Stone algebra
- x^+ is the dual pseudo-complement of x, i.e. $x^+ \leq a \Leftrightarrow a \vee x = 1$
- $x^+ \wedge x^{++} = 0$.

It is possible to replace the second and the third condition by the equations,

- $x \wedge (x \wedge b^* = x \wedge b^*, \quad x \vee (x \vee b)^+ = x \vee b^+$
- $x \wedge 0^* = x, \quad x \vee 1^+ = x$
- $0^{**} = 0\ \&\ 1^{++} = 1$.

A double Stone algebra is *regular* if and only if $x \wedge x^+ \leq b \vee b^*$ if and only if

$$(x^+ = b^+, x^* = b^* \longrightarrow x = b).$$

If Q is a subset of a partially ordered set $X = \langle \underline{X}, \leq \rangle$, then $Q \uparrow = \{x : (\exists q \in Q)\, q \leq x\}$. It is the principal o-filter generated by Q. The set of all o-filters will be denoted by $\mathcal{F}_o(X)$. It is a bounded distributive lattice under the induced set-theoretical operations. The set of maximal and minimal elements of the partially ordered set X will be denoted by $\max(X)$ and $\min(X)$ respectively. For any $a, b \in X$, $LB(a, b)$ and $UB(a, b)$ shall denote the set of common lower and common upper bounds respectively.

On a partially ordered set $X = \langle \underline{X}, \leq \rangle$, consider the following conditions:

$$(\forall a)(\exists! b \in \max(X))\, a \leq b \tag{F1}$$

$$(\forall a)(\exists! b \in \min(X))\, b \leq a \tag{F2}$$

$$(\forall a, b)(LB(a, b) \neq \emptyset \longrightarrow UB(a, b) \neq \emptyset) \tag{FW1}$$

$$(\forall a, b)(UB(a, b) \neq \emptyset \longrightarrow LB(a, b) \neq \emptyset) \tag{FW2}$$

$$(\forall a)(\exists b \in \max(X))\, a \leq b \tag{W1}$$

$$(\forall a)(\exists b \in \min(X))\, b \leq a \tag{W2}$$

Theorem 88 *In a partially ordered set X, the following hold:*

- *If F1 holds then FW1 also holds. The converse is false in general.*
- *If F2 holds then FW2 also holds. The converse is false in general.*
- *X satisfies F1 if and only if X satisfies FW1 and W1.*
- *X satisfies F2 if and only if X satisfies FW2 and W2.*

Proof The proof is easy and is left to the reader. If X is an infinite unbounded chain, then it satisfies FW1 and FW2, but none of the rest. □

Definition 89

- A *double Stone frame* (dsf) $X = \langle \underline{X}, \leq \rangle$ is a partially ordered set that satisfies F1 and F2.
- A *weak double Stone frame* (wdsf) $X = \langle \underline{X}, \leq \rangle$ is a partially ordered set that satisfies FW1 and FW2.

The main motivation for the definitions is in the following theorem [20] on the set of prime ideals $\mathfrak{I}_p(L)$ of a Stone algebra L.

Theorem 90 *A partially ordered set X is isomorphic to the partially ordered set $Spec(L)$ of a Stone algebra if and only if F2 holds and if $x \in \min(X)$ then $x \uparrow$ is a singleton or is isomorphic to $Spec(Z)$ of some distributive lattice with top.*

The second part of the condition can be seen relative to the fact that a pseudo-complemented distributive lattice is a Stone algebra if and only if the join of any two minimal prime ideals is the whole algebra.

Definition 91 The complex algebra of a double Stone frame X is the algebra

$$\mathfrak{C}(X) = \langle \underline{\mathcal{F}_o(X)}, \cap, \cup, ^*, ^+, \emptyset, \rangle$$

with extra operations being defined by

$$(\forall A \subseteq X)\; A^* = (\max(X) \cap A) \downarrow^c \tag{14}$$

$$(\forall A \subseteq X)\; A^+ = (\min(X) \cap A) \uparrow^c \tag{15}$$

$\underline{\mathcal{F}_o(X)}$ will be abbreviated by oX

Definition 92 If L is a double Stone algebra, then its *canonical frame* is the frame

$$\mathfrak{Cf}(L) = \langle \mathcal{F}_p(L), \subseteq \rangle$$

Theorem 93 *If X is a double Stone frame, then $\mathfrak{C}(X)$ is a double Stone algebra.*

Proof

- oX is a bounded distributive lattice.
- $A^* \cap A = \emptyset$ because $(\max(X) \cap A) \downarrow = A \downarrow$.

- Let $B \in oX$ and $A \cap B = \emptyset$. If $B \cap (A \downarrow) \neq \emptyset$ and $b \in B$, $a \in A$ and $b \leq a$. As B is an order filter, it follows that $a \in B \cap A$—a contradiction. So A^* is the pseudo-complement of A.
- Suppose $x \in X \setminus (A^* \cup A^{**})$, then
  -
$$x \in (A^* \cup A^{**})^c = A^{*c} \cap A^{**c} = A \downarrow \cap((A \downarrow)^c \downarrow)$$
  - So $(\exists a, b)\, x \leq a \,\&\, a \in A \,\&\, x \leq b \,\&\, b \notin A \downarrow$
  - By F1, let $x \leq h \in \max(X)$. Clearly it is necessary that $a, b \leq h$.
  - Now $a \in A \in oX$ yields $h \in A$
  - $b \leq h$ yields $b \in A \downarrow$—a contradiction
  - Therefore $A^* \cup A^{**} = X$.
- To show that A^+ is an o-filter, let $x \in (\min(X) \cap A) \uparrow^c$ and $x \leq a$
  - If $a \in (\min(X) \cap A) \uparrow$ then $(\exists b \in (\min(X) \cap A))\, b \leq a$
  - If $s \in \min(X) \,\&\, s \leq x \leq a$, hence $s = b \in A$
  - This yields the contradiction $x \in (\min(X) \cap A) \uparrow$. So A^+ is an o-filter.
- Next, it will be shown that $A = B^c \leftrightarrow A^+ \subseteq B$
  - If $A = B^c$, let $x \notin B$ and $h \in \min(X) \,\&\, h \leq x$
  - If $x \notin (\min(X) \cap A) \uparrow$, then $h \notin \min(X) \cap A$. So $h \notin A$
  - So $A \cup B = X \,\&\, h \in B$ and therefore $x \in B$—a contradiction.
  - The converse implication a direct argument suffices for $A \cup A^+ = X$
  - $x \notin A^+ \rightarrow x \in (\min(X) \cap A) \uparrow$ and $(\exists h \in \min(X) \cap A)\, h \leq x$
  - As A is an o-filter, $x \in A$

$A^+ \cap A^{++} = \emptyset$ can be verified by a contradiction argument again. In words, this reads as *let h be the minimal element of an element x in the intersection, then x cannot be in the o-filter generated by the minimal elements in A. This means h cannot be in A. On the other hand as x is in the double dual pseudo complementation of A, h cannot be a minimal element in the dual pseudo complementation of A. So h must be in the o-filter generated by the set of minimal elements in A. Minimality of h, means it must be in A. This contradiction means the original intersection must be empty.* □

Theorem 94 *If S is a double Stone algebra, then $\mathfrak{Cf}(S)$ is a double Stone frame.*

The proof of this statement is obvious.

Theorem 95

- *If S is a double Stone algebra, then S is isomorphic to a subalgebra of $\mathfrak{CCf}(L)$.*
- *If X is a double Stone frame, then it is isomorphic to a substructure of*

$$\langle \mathfrak{CfC}(X), \subseteq \rangle .$$

Proof For any prime filter F, let $F_{\max}$ be the unique maximal filter containing it and let F_m be the unique minimal filter contained in F.

- For the first part, note that the Stone embedding $w : L \longmapsto \wp(\mathcal{F}_p(L))$ defined by $w(x) = \{F : x \in F \in \mathcal{F}_p(L)\}$ is an embedding of bounded lattices. So it suffices to show that w preserves * and $^+$.
- If $F \in w(a)^*$, then $F \notin (\max(\mathcal{F}_p(L) \cap w(a))) \downarrow$ and so $a \notin F_{\max}$. So $a^* \in F_{\max}$ and $a^* \in F$. Otherwise $a^* \vee a^{**} = 1$ yields $a^{**} \in F \subseteq F_{\max}$—a contradiction. So $F \in w(a^*)$.
- For proving $w(a^*) \subseteq w(a)^*$, note that if $F \in w(a^*)$, yields $a^* \in F \subseteq F_{\max}$. So $a \notin F_{\max}$ and $F \notin (\max(\mathcal{F}_p(L) \cap w(a))) \downarrow$ or $F \in w(a)^*$.
- To show $w(a)^+ \subseteq w(a^+)$, let $F \in w(a)^+$, then $F_m \notin w(a)$ and $a \notin F_m$. Since $a \vee a^+ = 1$ and F_m is prime, $a^+ \in F_m$ and $a^+ \in F$.
- To show $w(a^+) \subseteq w(a)^+$, if $a^+ \in F$ then it suffices to show $F_m \notin w(a)$. Since F_m is minimal, $a \wedge a^+ \notin F_m$. So if $a \in F_m$, then $a^+ \notin F_m$ and $a^{++} \in F_m$ and therefore $a^{++} \in F$. But as $a^+ \in F$, $0 = a^+ \wedge a^{++} \in F$—a contradiction. Therefore $a \notin F_m$.

- For the second part of the theorem, define a map $g : X \longmapsto \mathfrak{CfC}(X)$ by

$$(\forall x)\, g(x) = \{B : x \in B \in \mathcal{F}_o(X)\}$$

- If $b \leq c$ and $A \in g(b)$, then $c \in A$ as it is an order filter. Hence $g(b) \subseteq g(c)$.
- If $b \nleq c$, then $c \uparrow \nsubseteq b \uparrow$ and there exists a prime filter F containing $c \uparrow$ but not $b \uparrow$.

□

The second part of the proof shows that the second part of theorem holds for all partially ordered sets.

A duality for weak double Stone frames can also be proved. Here only the statement of the result will be mentioned. For details the reader is referred to [35].

Definition 96 If X is a weak double Stone frame, then its complex algebra $\mathfrak{C}_w(X)$ is the algebra

$$\left\langle \underline{\mathcal{F}_o(X)}, \cap, \cup, ^*, ^+, \emptyset, \right\rangle$$

with extra operations being defined by

$$(\forall A \subseteq X)\, A^* = \{x : x \uparrow \cap A = \emptyset\} \tag{16}$$

$$(\forall A \subseteq X)\, A^+ = \{x : x \downarrow \cap (A^c) \neq \emptyset\} \tag{17}$$

It can be shown that this definition coincides with the earlier one if X is a double Stone frame. Analogue dualities hold for weak double Stone frames.

Theorem 97

- *If S is a double Stone algebra, then S is isomorphic to a subalgebra of* $\mathfrak{C}_w\mathfrak{C}\mathfrak{f}(L)$.
- *If X is a weak double Stone frame, then it is isomorphic to a substructure of* $\langle \mathfrak{C}\mathfrak{f}\mathfrak{C}_w(X), \subseteq\rangle$.

The next theorem concerns regular double Stone algebras:

Theorem 98

- *If a partially ordered set X has chains of length at most 2, then $\mathfrak{C}(X)$ is a regular double Stone algebra.*
- *If L is a regular double Stone algebra, then $\mathfrak{C}\mathfrak{f}(L)$ has chains of at most length 2*
- *The duality for double Stone algebras extends to this context as well.*

8 Preference and Discernibility in Rough Sets

In this section, the duality proved in connection to preference relations in [34] is reworked for general contexts. These have connections with rough sets but have not been indicated in the paper and so these aspects are also invented here. Importantly the approach has connections with specific versions of the antichain based approach due to the present author [89, 97].

For basics of preference relations, semi orders and interval orders, the reader is referred to [43, 115, 125]. A survey of non conventional approaches is in [76].

Definition 99 Let S be a set of concepts, attributes or alternatives. A *preference relation* π and a *indifference I* relation are binary relation on S that satisfy;

$$(\forall a, b)\,(\pi ab \longrightarrow \neg\pi ba) \tag{A1}$$

$$(\forall a, b)\,(\pi ab \longrightarrow \neg Iab) \tag{subdiscernibility}$$

$$(\forall a)\, Iaa \tag{I-reflexive}$$

$$(\forall a, b)\,(Iab \longrightarrow Iba) \tag{I-symmetry}$$

The tuple $\left\langle \underline{S}, \pi, I\right\rangle$ is said to be a *preference frame*.

In the definition, indifference is a similarity or tolerance relation, while preference is a asymmetrical, irreflexive relation. Transitivity is not required of π.

Some of the essential terminology is fixed first. Every function $f \in S^S$ on a set S induces a complex (or global) function $\bar{f} \in \wp(S)^{\wp(S)}$ that is defined as $(\forall A \in \wp(S))\bar{f}(A) = \{f(x) : x \in A\}$. For simplicity $\bar{f}(A)$ will be denoted by $f[A]$. Further, the same symbol will be used for an algebra and its underlying set.

If $L = \left\langle \underline{L}, \vee, \wedge, 0, 1\right\rangle$ is a bounded lattice, then a *modal operator* on it is a map $f \in L^L$ that satisfies

$$(\forall a, b)\; f(0) = 0\;\&\; f(a \vee b) = f(a) \vee f(b)$$

$h \in L^L$ is a *sufficiency operator* if and only if it satisfies

$$(\forall a, b)\, h(0) = 1 \,\&\, h(a \vee b) = h(a) \wedge h(b)$$

Definition 100 A *mixed algebra* (MIX) is an algebra of the form

$$B = \langle \underline{B}, \vee, \wedge, \neg, f, h, 0, 1 \rangle$$

that satisfies all of the following ($\mathcal{F}_u(B)$ is the set of all ultrafilters of B):

$$B = \langle \underline{B}, \vee, \wedge, \neg, 0, 1 \rangle \text{ is a Boolean algebra} \tag{BA}$$

$$f \text{ is a modal operator on } B \tag{modal}$$

$$h \text{ is a sufficiency operator on } B \tag{suff}$$

$$(\forall F, G \in \mathcal{F}_u(B))\,,\ F \cap h(G) \neq \emptyset \leftrightarrow f(G) \subseteq F \tag{UF}$$

A *weak mixed algebra* (wMIX) is an algebra of the form

$$B = \langle \underline{B}, \vee, \wedge, \neg, f, h, 0, 1 \rangle$$

that is a Boolean algebra with a modal operator f and a sufficiency operator h that coincide on all atoms of B. Each complete and atomic MIX is a wMIX, but the converse is false in general. Also every modal algebra cannot be extended to a MIX. To see this consider a modal algebra with the modal operator being the identity operator. It is known that the class of MIX is not first order axiomatizable [33].

Definition 101 If B is a MIX and R_B is a relation on the set of its ultrafilters $\mathcal{F}_u(B)$ defined by

$$R_B FG \leftrightarrow f[G] \subseteq F,$$

then the relational system $\mathfrak{C}(B) = \langle \mathcal{F}_u(B), R_B \rangle$ is the *canonical system* of B.

Definition 102 Let $S = \langle \underline{S}, R \rangle$ be a general approximation space, with R being an arbitrary binary relation, then on its powerset $\wp(\underline{S})$, the following operators can be defined:

$$(\forall A \in \wp(S))\ A^{\bar{u}} = \{x : [x]_i \cap A \neq \emptyset \,\&\, x \in S\} \tag{inverse-upper}$$

$$(\forall A \in \wp(S))\ A^{s} = \{x : A \subseteq [x]_i \,\&\, x \in S\} \tag{sufficiency}$$

$$(\forall A \in \wp(S))\ A^{\bar{l}} = \{x : [x]_i \subseteq A \,\&\, x \in S\} \tag{inverse-lower}$$

Theorem 103 *In the context of the above definition all of the following hold:*

- $^{\bar{u}}$ *is a complete modal operator and* ^{s} *is a complete sufficiency operator.*
- $\left\langle \underline{\wp(S)}, \cup, \cap, ^{c}, ^{\bar{u}}, ^{s}, \emptyset, S \right\rangle$ *is a MIX.*
- $^{\bar{l}}$ *is a definable operation:* $(\forall X \in \wp(S))\, X^{\bar{l}} = X^{c\bar{u}c}$.
- *R is symmetric if and only if* $(\forall X \in \wp(S))\, X \subseteq X^{ss}$
- $(\forall X \in \wp(S))\, X^{s} \subseteq X^{\bar{u}} \,\&\, X^{\bar{u}} \setminus X^{*} \subseteq X^{\bar{u}} \setminus X^{l}$
- $(\forall X \in \wp(S))\, (X^{s} \subseteq X^{\bar{l}} \longrightarrow X^{*} = X^{l})$

From this, it should be clear that the MIX algebra on a powerset is essentially the Boolean algebra with approximation operators enhanced with an additional sufficiency operator. From a rough perspective, it helps in comparing the relative size of sets and granules.

8.1 Preference Algebras

Preference algebras are basically double MIXs.

Definition 104 A preference algebra B is an algebra of the form

$$B = \left\langle \underline{B}, \vee, \wedge, \neg, f, h, f_1, h_1, 0, 1 \right\rangle$$

that satisfies $B = \left\langle \underline{B}, \vee, \wedge, \neg, f, h, 0, 1 \right\rangle$ and $B = \left\langle \underline{B}, \vee, \wedge, \neg, f_1, h_1, 0, 1 \right\rangle$ are MIXs and satisfy all of the following:

$$(\forall a \in B)\, a \wedge f(h(a)) = 0 \tag{A1}$$

$$(\forall a \in B)\, a \wedge f(h_1(a)) = 0 \tag{A2}$$

$$(\forall a \in B)\, a \leq f_1(a) \tag{A3}$$

$$(\forall a \in B)\, a \leq h_1(h_1(a)) \tag{A4}$$

The axioms $A1$–$A4$ correspond to the four axioms of preference frames in the sense of correspondences between modal frame and algebraic semantics (see [127]).

Construction
A unique preference algebra can be constructed from a given preference frame $\langle \underline{S}, \pi, I \rangle$, as below:

- Define the operators $^{\bar{u}}$ and ^{s} on $\wp(S)$ by regarding $\langle \underline{S}, \pi \rangle$ as a general approximation space as in Definition 102 and denote them by f_π and h_π respectively.
- Define the operators $^{\bar{u}}$ and ^{s} on $\wp(S)$ by regarding $\langle \underline{S}, I \rangle$ as a general approximation space as in Definition 102 and denote them by f_I and h_I respectively.
- $\mathfrak{C}(S) = \left\langle \underline{\wp(S)}, \cup, \cap, ^{c}, f_\pi, h_\pi, f_I, h_I, 0, 1 \right\rangle$ is the *complex algebra* derived from the preference frame S

Theorem 105 *In the complex algebra* $\mathfrak{C}(S)$ *derived from a preference frame* S*, both* $\left\langle \underline{\wp(S)}, \cup, \cap, ^c, f_\pi, h_\pi, 0, 1\right\rangle$ *and* $\left\langle \underline{\wp(S)}, \cup, \cap, ^c, f_I, h_I, 0, 1\right\rangle$ *are weak MIXs that satisfy* $A1$–$A4$.

Proof The proof is fairly direct. □

Definition 106 If $B = \langle \underline{B}, \vee, \wedge, \neg, f, h, f_1, h_1, 0, 1\rangle$ is a preference algebra, on the set of ultrafilters $\mathfrak{F}_u(B)$ of B, let π_B and I_B be two binary relations defined as below:

$$(\forall F, G \in \mathfrak{F}_u(B))\,(\pi_B FG \leftrightarrow f(G) \subseteq F)$$

$$(\forall F, G \in \mathfrak{F}_u(B))\,(\mathrm{i}_B FG \leftrightarrow f_1(G) \subseteq F)$$

Then the relational system $\langle \mathfrak{F}_u(B), \pi_B, I_B\rangle$ is called the *canonical frame* of the preference algebra B and is denoted by $\mathfrak{Cf}(B)$.

Using a contradiction argument, it can be proved that

Theorem 107 *The canonical frame of a preference algebra is a preference frame.*

The meaning of the following duality result will be explored after its proof:

Theorem 108 *Suppose* $S = \langle \underline{S}, \pi, I\rangle$ *is a preference frame and*

$$B = \langle \underline{B}, \vee, \wedge, \neg, f, h, f_1, h_1, 0, 1\rangle$$

is a preference algebra, then

1. *The map* $v : B \longmapsto \mathfrak{C}\mathfrak{Cf}(B)$ *defined by*

$$(\forall a \in B)\, v(a) = \{F : F \in \mathfrak{F}_u(B)\,\&\, a \in F\}$$

   is an embedding of preference algebras.
2. *The map* $w : S \longmapsto \mathfrak{Cf}\mathfrak{C}(S)$ *defined by*

$$(\forall a \in S)\, w(a) = \{X : X \in \wp(S)\,\&\, a \in X\}$$

   is an embedding of preference frames.

Proof

- Stone maps are Boolean embeddings. So v is a Boolean embedding.
- To show that v preserves $f,\ h,\ f_1$ and h_1, note that
  - $F \in h_{\pi_B}(v(x)) \leftrightarrow F \in h_B(v(x)) \leftrightarrow (\forall G \in \mathfrak{F}_u(B))(G \in v(x) \rightarrow \pi_B FG)$. The subscript on the predicate π has been used to indicate its construction.
  - So, by condition UF $f(G) \in F \leftrightarrow (\forall G \in \mathfrak{F}_u(B))(x \in G \rightarrow F \cap h(G) \neq \emptyset)$
  - Inclusion: Let $x \in G \in \mathfrak{F}_u(B)$, then as $h(x) \in F$, $F \cap h(G) \neq \emptyset$.

  - Reverse Inclusion: Suppose $h(x) \notin F$. Form the set $W_h = \{b : b \in B \,\&\, \neg(h(\neg b)) \notin F\}$. Let G^+ be the proper filter generated by $W_h \cup \{x\}$. If G^+ is not proper, then it would be yield the contradiction $h(x) \in F$. So G^+ can be extended to a prime filter G. As $x \in G$, $F \cap h(G) \neq \emptyset$. So $(\exists b \in G)\, h(b) \in F$. This yields $\neg h(b) = \neg(h(\neg(\neg b))) \notin F$, that is $\neg b \in W_h \subseteq G$ or $b \notin G$—a contradiction (by the definition of ultrafilters.)

- Preservation of modal operators is by standard methods.

  For the second part of the theorem,

- Clearly $w(x)$ is the principal ultrafilter of $\wp(S)$ generated by $\{x\}$. For any $b, c \in S$, it suffices to prove that $\pi bc \leftrightarrow \pi_{\wp(S)} w(b)w(c)$ and $Ibc \leftrightarrow I_{\wp(S)} w(b)w(c)$.
- $\pi_{\wp(S)} w(b)w(c) \leftrightarrow f_\pi(w(c)) \subset w(b)$
  - $\leftrightarrow (\forall B \subset A)\,(c \in B \to b \inf_\pi(B))$
  - $\leftrightarrow (\forall B \subset A)\,(c \in B \to [b]_i \cap B \neq \emptyset)$.
- If πbc and $c \in B$, then $[b]_i \cap B \neq \emptyset$ and by the last two way implication, $\pi_{\wp(S)} w(b)w(c)$.
- If $\pi_{\wp(S)} w(b)w(c)$ for some $b, c \in S$, then setting $B = c$ suffices to show πbc.
- This proves the second part.

□

From the theorem, it follows that

Corollary 109

- *Every preference frame is embeddable into the canonical frame of its complex algebra.*
- *Every preference algebra is embeddable into the complex algebra of its canonical frame.*

In the appendix of the paper [34], it is also shown that the above duality can be modified to first order scenario.

8.2 *Interpretation and Problems*

If *indifference* is read as similarity (as the axioms indicate), then the duality theorem apparently provides an interesting semantic bound for similarity based rough sets using pointwise approximations only. This is because in the preference algebra, it is not possible to represent the sufficiency operator in terms of other rough operators and set operations. However it is representable using additional rough approximation operators. This suggests a number of related problems.

1. Preference among single attributes may possibly generate reducts and can in any case, help with reduct computation. The fine details are an open problem.
2. What is the connection of the preference frames with dominance based rough sets?

9 Distributive Lattices with Galois Connections

If $S = \langle \underline{S}, R \rangle$ is a general approximation space, then for any subset $A \subseteq S$, let

$$A^{\blacktriangle} = \{x : x \in S \,\&\, [x]_i \subseteq A\}$$
$$A^{\vartriangle} = \{x : x \in S \,\&\, [x] \subseteq A\}$$
$$A^{\blacktriangledown} = \{x : x \in S \,\&\, [x]_i \cap A \neq \emptyset\}$$
$$A^{\triangledown} = \{x : x \in S \,\&\, [x] \cap A \neq \emptyset\}$$

Let $\mathcal{R} = \{(X^{\blacktriangle}, X^{\blacktriangledown}) : X \subseteq S\}$. Define a partial-order on it via

$$(X^{\blacktriangle}, X^{\blacktriangledown}) \leq (Z^{\blacktriangle}, Z^{\blacktriangledown}) \leftrightarrow X^{\blacktriangle} \subseteq Z^{\blacktriangle} \,\&\, X^{\blacktriangledown} \subseteq Z^{\blacktriangledown} \tag{18}$$

It is known that the pair $(^{\blacktriangle}, ^{\triangledown})$ (resp $(^{\vartriangle}, ^{\blacktriangledown})$) of pointwise rough approximation operators forms an order-preserving Galois connection for any binary relation on the powerset $\wp(S)$.

In [39], extensions of bounded distributive lattices equipped with a Galois connection are studied through concepts of GC-frames and canonical frames (of the algebras). The complex algebras of GC-frames are defined through rough approximation operators and it is proved that every bounded distributive lattice with a Galois connection (represented by rough upper and lower approximations) can be embedded into the complex algebra of its canonical frame. The result is also extended to weakly atomic Heyting-Brouwer algebras endowed with a Galois connection in the paper. The essence of the constructions are considered in this section.

9.1 Bounded Distributive Lattices with a Galois Connection

Bounded distributive lattices often arise as sets of subsets of a universe of attributes in the study of rough sets. They also relate to generalization of relation algebras, but that connection is too weak.

Definition 110 A *bounded distributive lattice with a Galois connection*, (gcd-lattice), is an algebra of the form $\mathbb{L}_a = \langle \underline{L}, \vee, \wedge, f, g, 0, 1 \rangle$ that satisfies:

- $\langle \underline{L}, \vee, \wedge, 0, 1 \rangle$
- The two maps $f, g : L \mapsto L$ form an order preserving Galois connection or adjunction: that is the following hold:

$$f \circ g \circ f = f \tag{19}$$

$$g \circ f \circ g = g \tag{20}$$

$$(\forall x, b)\, f(x \vee b) = f(x) \vee f(b) \tag{21}$$

$$(\forall x, b)\, g(x \wedge b) = g(x) \wedge g(b) \tag{22}$$

$$(\forall x)\, f(x) = \bigwedge \{b : x \leq g(b)\} \tag{23}$$

$$(\forall x)\, g(x) = \bigvee \{b : f(b) \leq q\} \tag{24}$$

Definition 111 A *Galois connection-frame* (or a *GC-frame* or an *A-frame*) $\mathbb{F} = \langle \underline{F}, <, R \rangle$ is a relational system that satisfies the following:

$$\langle \underline{F}, < \rangle \text{ is a quasi-ordered set} \tag{quasi-ordered set}$$

$$(\forall a, b, c, e)\, (a < b \,\&\, e < c \,\&\, Rac \longrightarrow Rbe) \tag{Eq:R}$$

In [39], the authors have used flawed notation: $\leq$ instead of $<$. If $>$ is defined by $(\forall a, b)(a < b \leftrightarrow b > a)$, then the condition Eq:R can be rewritten as: $> \circ R \circ > \subseteq R$. It is also possible to replace $<$ with a groupoidal operation.

Definition 112 If $\mathcal{F}_p(L)$ is the set of all prime lattice filters of a gcd-lattice $\mathbb{L}_a = (L, \vee, \wedge, f, g, 0, 1)$, then let Ξ be the relation defined by

$$\Xi FG \Leftrightarrow (\forall b)\, (b \in G \longrightarrow f(b) \in F) \Leftrightarrow (\forall b)\, (g(b) \in G \longrightarrow b \in F) \tag{$\star$}$$

then the *canonical frame* of the gcd-lattice $\mathbb{L}_a$ is the relational system $\mathfrak{Cf}(L) = \left\langle \underline{\mathcal{F}_p(L)}, \subseteq, \Xi \right\rangle$

Lemma 113 *For a gcd-lattice* $\mathbb{L}_a = \langle \underline{L}, \vee, \wedge, f, g, 0, 1 \rangle$*, its canonical frame* $\mathfrak{Cf}(L) = (\mathcal{F}_p(L), \subseteq, \Xi)$ *is a GC-frame.*

Proof Let $F \subseteq H$, ΞFG, and $J \subseteq G$. Then,

$$\begin{aligned} (\forall a)\, a \in J &\rightarrow a \in G \text{ (because } J \subseteq G) \\ &\rightarrow f(a) \in F \text{ (as } \Xi FG) \\ &\rightarrow f(a) \in H \text{ (because } F \subseteq H) \end{aligned}$$

This yields ΞHJ and also condition Eq:R holds. □

Definition 114 Let $\mathbb{F} = (X, <, \Xi)$ be a GC-frame. The algebra

$$\mathfrak{C}(\mathbb{F}) = (\tau_<, \cup, \cap, ^{\blacktriangle}, ^{\triangledown}, \emptyset, X)$$

is the *complex algebra* of $\mathbb{F}$.

Complex algebras of GC-frames are gcd-lattices

Proposition 115 *Let* $\mathbb{F} = (X, <, \Xi)$ *be a GC-frame. Then, the complex algebra* $\mathfrak{C}(\mathbb{F}) = (\tau_<, \cup, \cap, ^{\blacktriangle}, ^{\triangledown}, \emptyset, X)$ *is a gcd-lattice.*

Proof It is clear that the algebra $(\tau_<, \cup, \cap, \emptyset, X)$ is a bounded distributive lattice. It is required to show that $(\forall A \in \tau_<)\, A^{\blacktriangle},\ A^{\triangledown} \in \tau_<$.

- $(\forall x \in A^{\blacktriangle})\,(x \leq b \longrightarrow (\exists z \in A)\, \Xi xz)$
- As $\mathbb{F}$ is a GC-frame $(x \leq b \,\&\, \Xi xz \,\&\, z \leq z \longrightarrow \Xi xz)$.
- So $b \in A^{\blacktriangle}$ and $A^{\blacktriangle} \in \tau_<$.
- For the other part, let $x \in A^{\triangledown}$ and $x \leq b$.
- If Ξzb, then $z \leq z \,\&\, \Xi zb \,\&\, x \leq b$ yields Ξzx and $z \in A$.
- Hence, $b \in A^{\triangledown}$ and $A^{\triangledown} \in \tau_<$.

□

In [38], the following improved version of the prime filter theorem was proved

Theorem 116 *Let Q be a set whose complement is a join-subsemilattice of a distributive lattice. If a filter F is contained in Q, then there exists a prime filter P such that*

$$F \subseteq P \subseteq Q$$

To derive the prime filter theorem from this, it suffices to start from a $Q = L \setminus \{a\}$ satisfying $a \notin F$.

Theorem 117 *For every gcd-lattice* $\mathbb{L}_a = \langle \underline{L}, \vee, \wedge, f, g, 0, 1 \rangle$*, there exists a GC-frame* $\mathbb{F} = \langle \underline{X}, <, \Xi \rangle$ *such that* $\mathbb{L}_a$ *is isomorphic to a subalgebra of* $\mathfrak{C}(\mathbb{F})$*. If* $\mathbb{L}_a$ *is finite, then it is isomorphic to* $\mathfrak{C}(\mathbb{F})$.

Proof The long proof of [39] has been restructured here.

Define a map $h : \mathbb{L}_a \longmapsto \mathfrak{C}(\mathfrak{Cf}(L))$ as below:

$$h(x) = \{F : F \in \mathfrak{F}_p(L) \,\&\, x \in F\}$$

Since L is a distributive lattice, h is a lattice embedding (see [48], for example) that satisfies $h(0) = \emptyset$ and $h(1) = \mathfrak{F}_p(L)$.

Next it will be shown that

$$(\forall x \in L)\, h(g(x)) = h(x)^{\triangledown}$$

- Let $F \in h(g(x))$. This is equivalent to $g(x) \in F \in \mathcal{F}_p(L)$.
- Suppose $F \notin h(x)^{\triangledown}$, then $(\exists G \in \mathcal{F}_p(L))\ \Xi GF \,\&\, G \notin h(x)$.
- Now $g(x) \in F \longrightarrow x \in G$. So by the definition of Ξ $G \in h(x)$—a contradiction. Hence, $F \in h(x)^{\triangledown}$.

For the converse,

- If $F \in h(x)^{\triangledown}$, then $\Xi GF \longrightarrow G \in h(x)$ and $x \in G$.
- Suppose $F \notin h(g(x))$, that is, $g(x) \notin F$.
- The preimage $g^{-1}(F)$ of the filter F must necessarily be a filter as g is multiplicative and order-preserving.
- As $x \notin g^{-1}(F)$ there exists a prime filter G such that $g^{-1}(F) \subseteq G$ and $x \notin G$ (by the prime filter theorem).
- $g^{-1}(F) \subseteq G$ yields $(\forall z \in L)(g(z) \in F \longrightarrow z \in G)$. So ΞGF.
- This yields the contradiction $x \in G$. So, $F \in h(g(x))$.

Using a similar, but not equivalent argument it can be shown that

$$(\forall x \in L)\, h(f(x)) = h(x)^{\blacktriangle}$$

To see this,

- Suppose $F \in h(x)^{\blacktriangle}$, then $(\exists G \in h(x))\ \Xi FG$.
- Since $G \in h(x) \leftrightarrow x \in G$, therefore $f(x) \in F$ and $F \in h(f(x))$ follows.
- If $F \in h(f(x))$, that is, $f(x) \in F$, then $x\uparrow \subseteq f^{-1}(F)$. Theorem 116 ensures that there is a prime filter H such that $x\uparrow \subseteq H \subseteq f^{-1}(F)$. It follows that ΞFH, $H \in h(x)$ and $F \in h(x)^{\blacktriangle}$.

For the last part, let L be a finite gcd-lattice and let $J(L)$ be its set of all join-irreducible elements. In a finite lattice, all filters are principal order filters and a principal filter $b\uparrow$ is prime if and only if $b \in J(L)$. So $\mathcal{F}_p(L) = \{a\uparrow : a \in J(L)\}$.

- To show that the map h is onto $\mathfrak{C}(\mathfrak{Cf}(L))$, note that if $A \in \mathfrak{C}(\mathfrak{Cf}(L))$, then A is a $\subseteq$-closed subset of $\mathcal{F}_p(L)$.
- Let $x = \bigvee\{z \in J(L) : z\uparrow \in A\}$, then $h(x) = \{z\uparrow : z \leq x \,\&\, z \in J(L)\}$.
- If $c\uparrow \in h(x)$, then $c \in J(L)$ and $c \leq \bigvee\{z \in J(L) : z\uparrow \in A\}$.
- Because L is finite and c is join-irreducible, it is necessary that $c \leq y$ for some $y \in \{z \in J(L) : z\uparrow \in A\}$.
- $c \leq y \longrightarrow y\uparrow \subseteq c\uparrow$.
- So $\uparrow c \in A$ and $A \subseteq h(x)$ follows.

□

9.2 Extensions to Heyting-Brouwer Algebras

The above result can be extended to Heyting and Heyting-Brouwer algebras endowed with Galois connections.

Definition 118 A *HGC-algebra* $\mathbb{H}_a = \langle \underline{L}, \vee, \wedge, \rightarrow, f, g, 0, 1 \rangle$ is an algebra that satisfies

- $\langle \underline{L}, \vee, \wedge, f, g, 0, 1\rangle$ is a gcd lattice.
- $\langle \underline{L}, \vee, \wedge, \rightarrow, 0, 1\rangle$ is a Heyting algebra

This algebra is a model for intuitionistic logic with a Galois connection (IntGC) [38]. A formula ϕ is provable in IntGC if and only if ϕ is valid in all HGC-algebras. IntGC has the finite model property: that is a formula ϕ is provable in IntGC if and only if ϕ is valid in all finite HGC-algebras.

GC-frames introduced in Definition 111 serve also as frames for HGC-algebras. The canonical frame of an HGC-algebra is $\mathfrak{Cf}(L) = (\mathcal{F}_p(L), \subseteq, \Xi)$, where $\mathcal{F}_p(L)$ is the set of prime filters and Ξ is defined as in ($\star$). Similarly, for a GC-frame $\mathbb{F}$, its complex HGC-algebra is

$$\mathfrak{C}(\mathbb{F}) = (\tau_<, \cup, \cap, \rightarrow, {}^{\blacktriangle}, {}^{\triangledown}, \emptyset, X),$$

where $\rightarrow$ is defined as in Proposition 4. Clearly, the complex algebra $\mathfrak{C}(\mathbb{F})$ of any GC-frame $\mathcal{F}$ is an HGC-algebra, because $\tau_<$ is a Heyting algebra, and $A^{\triangledown}, A^{\blacktriangle} \in \tau_<$ for all $A \in \tau_<$.

Theorem 119 *Let $\mathbb{H}_a = \langle \underline{L}, \vee, \wedge, f, g, 0, 1\rangle$ be an HGC-algebra. Then, there exists a GC-frame $\mathbb{F} = \langle \underline{X}, \leq, \Xi\rangle$ such that $\mathbb{H}_a$ is isomorphic to a subalgebra of $\mathfrak{C}(\mathbb{F})$.*

A HGC-algebra is said to be *spatial* if its underlying lattice is spatial. In particular, any finite distributive lattice with a Galois connection determines a spatial HGC-algebra.

Theorem 120 *Let $\mathbb{H}_a = \langle \underline{L}, \vee, \wedge, f, g, 0, 1\rangle$ be a spatial HGC-algebra. Then, there exists a GC-frame $\mathbb{F} = \langle \underline{X}, \leq, \Xi\rangle$ such that $\mathbb{H}_a$ is isomorphic to $\mathfrak{C}(\mathbb{F})$.*

The proof can be found in [39].

Definition 121 A *HBGC-algebra* $L_h = \langle \underline{L}, \vee, \wedge, \rightarrow, \leftarrow, f, g, 0, 1\rangle$ is an algebra such that $\langle \underline{L}, \vee, \wedge, \rightarrow, \leftarrow, 0, 1\rangle$ is a Heyting-Brouwer algebra and (f, g) is an order-preserving Galois connection on L.

The canonical frame of an HBGC-algebra L_h is the GC-frame defined on the set of all prime filters, that is, $\mathfrak{Cf}(L) = \left\langle \underline{\mathcal{F}_p(L)}, , \subseteq, \Xi\right\rangle$. Similarly, for a frame $\mathbb{F} = \langle \underline{X}, , \leq, \Xi\rangle$, its complex HBGC-algebra is

$$\mathfrak{C}(\mathbb{F}) = \langle \tau_<, \cup, \cap, \rightarrow, \leftarrow, {}^{\blacktriangle}, {}^{\triangledown}, \emptyset, X\rangle$$

It is clear that the complex algebra $\mathfrak{C}(\mathbb{F})$ determined by any GC-frame $\mathcal{F}$ is an HBGC-algebra because the operation $\leftarrow$ for Alexandrov topologies is given in Proposition 4.

Theorem 122 *Let $L_h = \langle \underline{L}, \vee, \wedge, \rightarrow, \leftarrow, f, g, 0, 1\rangle$ be an HBGC-algebra. Then, there exists a GC-frame $\mathbb{F} = \langle X, \leq, \Xi\rangle$ such that L_h is isomorphic to a subalgebra of $\mathfrak{C}(\mathbb{F})$.*

Theorem 123 *Let* $L_h = \langle \underline{L}, \vee, \wedge, \rightarrow, \leftarrow, f, g, 0, 1\rangle$ *be a complete and weakly atomic HBGC-algebra. Then, there exists a GC-frame* $\mathbb{F} = \langle \underline{X}, \leq, \Xi\rangle$ *such that* L_h *is isomorphic to* $\mathfrak{C}(\mathbb{F})$.

Proof If $L_h = \langle \underline{L}, \vee, \wedge, \rightarrow, \leftarrow, f, g, 0, 1\rangle$ is a complete and weakly atomic HBGC-algebra, then its underlying complete lattice satisfies (JID) and (MID). Weak atomicity ensures that the lattice is isomorphic to a Alexandrov topology. It is also known that Alexandrov topologies determine complete weakly atomic HBGC-algebras. That L_h is isomorphic to $\mathfrak{C}(\mathbb{F})$ can be proved similarly as in case of Heyting GC-algebras. □

10 Other Red Results in Rough Sets

A number of representation and duality results are known in rough sets. In this section some of those that have not been covered in other sections are mentioned.

10.1 Spatial Mereology

Duality and representation theory for spatial mereology is very rich [37, 56, 141]. These have been recently used in papers concerning rough sets in [96, 141]. Proximity relations and related topologies have also been extensively used in image processing [25, 112, 113]. Connections between dependence in rough sets and subjective probability due to the present author in [93] and spatial mereology have investigated by her in a forthcoming paper. In [86], mereological aspects have been approached via parthood relations as opposed to contact relations. Related theory has been omitted for reasons of time and space.

10.2 Duality Results

Topological dualities (including discrete dualities) between the algebras and frames indicated in Table 4 are relevant for some semantics of rough sets. Details of these have been omitted.

The duality approach due to [30, 31] is also referred to as perp semantics. In the modal approach negations are also viewed as modal operators in this framework and a Kripke-style semantics has been developed for various logics. In a recent paper [74], this has been extended to rough sets. In a partially ordered set, a sub-minimal negation is a unary operation ¬ that satisfies

$$(\forall a, b)\ (a \leq b \longrightarrow \neg b \leq \neg a)$$

Table 4 Dualities-1

Name	Class of algebras	Frame/space
Kleene	$\mathcal{S}_K$	$\mathbb{F}_K$
Dual Kleene	$\mathcal{S}_{Kd}$	$\mathbb{F}_{Kd}$
Stone	$\mathcal{S}_S$	$\mathbb{F}_S$
Boolean contact	$\mathcal{S}_{Bc}$	$\mathbb{F}_{Bc}$
Boolean proximity	$\mathcal{S}_{Bp}$	$\mathbb{F}_{Bp}$

The negations used in the logics must necessarily be stronger than this sub-minimal negation (also see [140]).

10.3 Canonical Extensions

The basic idea of the approach is to extend a given semantic structure through canonical constructions and study the original ordered algebra through the properties of the extension. For this strategy to be successful it is necessary that most properties (especially equational ones) be preserved. Common completions include Dedekind-Macneille completions or natural completions and canonical extensions. Many of the results in the approach relate to pointwise approximations because of their use in modal logic. A survey can be found in [45]. Canonical extensions of convex decompositions of distributive lattice and related consequence operators have also been considered by the present author in [82]. But connections with rough approximations are not part of the paper. A major problem with the method is that many properties are not preserved in non-distributive lattices. Some of the basics of the method are recalled first.

The *canonical model approach* can be traced to early work on Boolean algebras with operators [64]. These techniques have since been adapted to different generalized modal logics admitting of the Lindenbaumm algebra construction and more recently to bounded lattices with operators [45]. In rough sets, natural extensions have been considered for pointwise approximations in [58, 139].

On a Poset, natural completions are defined in the following way.

Definition 124 Let $\langle \underline{X}, \leq \rangle$ be a Poset, then

$$(\forall A \in \wp(X))\, ub(A) = \{x : x \in X \& (\forall a \in A)\, a \leq x\} \qquad \text{(upper bound)}$$

$$(\forall A \in \wp(X))\, lb(A) = \{x : x \in X \& (\forall a \in A)\, x \leq a\} \qquad \text{(lower bound)}$$

$$dm(X) = \{A : A \in \wp(X) \,\&\, lb(ub(A)) = A\} \qquad \text{(dm-set)}$$

$\langle dm(X), \subseteq \rangle$ is the natural completion of X and is also realizable as the set of principal order ideals. The map $\varphi : X \longmapsto dm(X)$ defined by $\varphi(x) = x \downarrow$ preserves joins and meets that exist in X and $Im(\varphi)$ is both join-dense and meet-dense in $dm(X)$.

Let K be a complete lattice and L a sublattice of it, then

1. K is *join-dense* in L if every element of L is a join of elements of a subset of K.
2. K is *meet-dense* in L if every element of L is a meet of elements of a subset of K.
3. K is *dense* in L if it is both meet-dense and join-dense in L.
4. K is *compact* in L if

$$(\forall A,\ B \subseteq K)\ \bigwedge A \leq \bigvee B \longrightarrow \exists \text{finite}\, H \subseteq A,\ F \subseteq B,\ \bigwedge H \subseteq \bigvee F$$

Theorem 125 *For all lattices K there exists a unique complete lattice L (up to isomorphism) such that the following hold:*

- *K is a sublattice of L.*
- *K is both join and meet-dense in L.*

$K^* = L$ *is called the natural or Mc Neille completion of the lattice K.*

Theorem 126 *For all lattices K there exists a unique complete lattice L (up to isomorphism) such that the following hold:*

- *K is a sublattice of L.*
- *K is both join and meet-dense in L.*
- *K is compact in L.*

$K^\sigma = L$ *is known as the* canonical extension *of K.*

Definition 127 In the above the *closed elements* of K^σ are those elements that are representable as meets of elements of K. The *open* elements of K^σ are those that are representable as joins of elements of K. The corresponding sets will be denoted by $\mathfrak{C}(K^\sigma)$ and $\mathfrak{O}(K^\sigma)$.

Note that in the canonical extension, the elements of K are precisely the clopen elements.

Definition 128 Let A, B be two lattices and let $f : A \longmapsto B$ be an order preserving map, then f^l, f^u, f^σ, f^π are the *lower Mc Neille* (or *lower natural*), *upper Mc Neille*, *lower canonical* and *upper canonical extensions* of f to the respective completions.

$$f^l = \bigvee \{f(a) : a \leq ua \in A\} \qquad \text{(l-natural)}$$

$$f^u = \bigwedge \{f(a) : u \leq aa \in A\} \qquad \text{(u-natural)}$$

$$f^\sigma(u) = \bigvee \left\{ \bigwedge \{f(a) : x \leq a \in A\} : x \leq u, x \in \mathfrak{K}(L^\sigma) \right\} \qquad \text{(lcan)}$$

$$f^\pi(u) = \bigwedge \left\{ \bigvee \{f(a) : a \leq b, a \in A\} : u \leq b \in \mathfrak{O}(L^\pi) \right\} \qquad \text{(ucan)}$$

The above canonical extensions can be generalized to arbitrary maps in the following way:

$$f^{*\sigma}(u) = \bigvee \left\{ \bigwedge \{f(a) : a \in [x, b]_A\} : x \leq u \leq b \in \mathfrak{K}(A^\sigma)\, b \in \mathfrak{O}(A^\sigma) \right\} \tag{25}$$

and

$$f^{*\pi}(u) = \bigwedge \left\{ \bigvee \{f(a) : a \in [x, b]_A\} : x \leq u \leq b \in \mathfrak{K}(A^\sigma)\, b \in \mathfrak{O}(A^\sigma) \right\} \tag{26}$$

Definition 129 By a *canonical extension* of a Boolean algebra S, is meant a complete atomic Boolean algebra S^* that satisfies,

1. $S \in Su(S^*)$
2. Each atom of S^* is a meet of elements of S.
3.

$$(\forall K \in \wp(S)) \left(\overset{S^*}{\bigvee}(K) = 1 \longrightarrow \exists F \in \wp_f(K) \overset{S^*}{\bigvee}(F) = 1 \right)$$

It is provable that

Proposition 130

1. *If $\varphi : S_1 \longmapsto S_2$ is an isomorphism of Boolean algebras, then there exists an isomorphism $\varphi^* : S_1^* \longmapsto S_2^*$ such that $\varphi^* \cap (S_1 \times S_2) \equiv \varphi$.*
2. *If S^* is a canonical extension of the Boolean algebra S, then $\forall x \in S^* \exists x_{ij} \in S\ x = \vee_i \wedge_j (x_{ij})$.*
3. *If S^* is a canonical extension of the Boolean algebra S, then $\forall x \in S^* \exists x_{ij} \in S\ x = \wedge_i \vee_j (x_{ij})$.*
4. *The canonical extension of a Boolean algebra $S = \left\langle \underline{S}, +, -, 0 \right\rangle$ is the Boolean algebra $S^* = \left\langle \underline{\wp(\dashv(S))}, \cup, \setminus, \emptyset \right\rangle$, where $\underline{\wp(\dashv(S))}$ is the powerset of all ultrafilters of S. If $\xi(x) = \{U \in \dashv(S);\ x \in U\}$ then the map $\xi : S \mapsto S^*$ is a Boolean embedding.*

A modal operator λ in any Boolean algebra with a modal operator λ, can be extended in at least two ways for forming canonical models. If S is a Boolean algebra with a operator, then the canonical extension of the forgetful Boolean algebra S^b can generated from the collection of all open sets $\mathfrak{O}(S^\sigma)$ and also from the collection of 'closed sets' $\mathfrak{C}(S^\sigma)$ respectively. Corresponding to this the natural way of extending λ is as in the following definition.

Definition 131 If S, B are Boolean algebras and $\lambda : S \mapsto B$ a monotone function, then let $\lambda^\sigma, \lambda^\pi$ be maps : $\wp(\dashv(S)) \longmapsto \wp(\dashv(B))$ defined via

$$\lambda^\sigma(A) = \bigcup_{X \supseteq F \in \mathfrak{C}(A^\sigma)} \bigcap_{F \subseteq \xi(a)} \xi(\lambda(a))$$

$$\lambda^\pi(A) = \bigcap_{X \subseteq L \in \mathfrak{O}(A^\sigma)} \bigcup_{\xi(a) \subseteq L} \xi(\lambda(a)).$$

Proposition 132 *For clopen, closed and open elements of S^σ, it is that*

- $\lambda^\sigma(\xi(a)) = \lambda^\pi(\xi(a)) = \xi(\lambda(a))$ *for all clopens* $\xi(a) \in \xi(S)$.
- $\lambda^\sigma(F) = \bigcap_{F \subseteq \xi(a)} \xi(\lambda(a))$

Proposition 133 *If S, B are Boolean algebras and $\lambda : S \mapsto B$ be a monotone function, then*

- λ^σ *and* λ^π *are monotone functions.*
- λ^σ *and* λ^π *coincide on open and closed sets.*
- *Both preserve closed and open elements respectively.*

Definition 134 If S is a Boolean algebra with a modal operator then

$$S^\sigma = \langle S^\sigma_{bool}, \lambda^\sigma \rangle \text{ and}$$
$$S^\pi = \langle S^\pi_{bool}, \lambda^\pi \rangle.$$

Proposition 135 *In the above context S is a subalgebra of both S^σ and S^π.*

Remark 136 S^σ is $\vee$ and $\wedge$-generable from $\mathfrak{cl}(S^\sigma)$ and $\mathfrak{O}(S^\sigma)$ respectively.

Proposition 137 *If S is a Boolean algebra with a modal operator λ and B a π- or σ-canonical extension of S, then S is isomorphic to a subalgebra of B.*

10.3.1 Connections with Rough Sets

In [58, 139], Dedekind-Macneille completions of the algebras derived from the pointwise approximations mentioned in Sect. 9 are considered. Assume that on a general approximation space $S = \langle \underline{S}, R \rangle$, the pointwise approximations $\blacktriangle, \blacktriangledown, \triangle, \triangledown$ are given for any subset $A \subseteq S$ (the notation does not agree with that used in [139]). Let

$$\mathfrak{R} = \{(A^\blacktriangle, A^\blacktriangledown) : A \subseteq S\} \quad \text{(RS)}$$

$$\Re = \{(A, B)\,(A, B) \in \Im(\blacktriangle) \times \Im(\blacktriangledown)\ \&\ A \subseteq B \quad \text{(Ob)}$$

Let $\mathcal{R} = \{(X^{\blacktriangle}, X^{\blacktriangledown}) : X \subseteq S\}$. Define a partial-order on it via

$$(X^{\blacktriangle}, X^{\blacktriangledown}) \leq (Z^{\blacktriangle}, Z^{\blacktriangledown}) \leftrightarrow X^{\blacktriangle} \subseteq Z^{\blacktriangle} \,\&\, X^{\blacktriangledown} \subseteq Z^{\blacktriangledown} \tag{27}$$

and similarly define a partial-order *preceq* on $\Im(\blacktriangle) \times \Im(\blacktriangledown)$. This induces a order on $\Re$. All of the partial-orders can be denoted by the same symbol $\leq$ because of inclusions of the sets.

Recall that in [46], it had been proved that

Theorem 138 *When R is an equivalence relation, then*

$$\Re \cong \left\langle 2^I \times 3^J, \leq \right\rangle,$$

with I being the set of singleton predecessor neighborhoods, J being the set of non-singleton predecessor neighborhoods, 2^I being the set of all maps from I to the two element chain and 3^J being the set of all maps from J to a three element chain.

In [58], it has been proved that

Theorem 139 *When R is a reflexive relation, then*

- *$\mathcal{R}$ is a sub partially ordered set of $\Re$*
- *$\Re$ is a sublattice of $\Im(\blacktriangle) \times \Im(\blacktriangledown)$.*
- *$\Re$ is a completion of $\mathcal{R}$*
- *$\left\langle 2^I \times 3^J, \leq \right\rangle$ is a completion of $\mathcal{R}$ with I being the set of singleton predecessor neighborhoods and J is the set of non-singleton predecessor neighborhoods.*

The smallest completion that is isomorphic to the natural completion has been constructed in [139]. However the algebraic properties are not clear because of the fact that the order structure is apparently not good enough. Let $H = \{x : \#([x]_i) = 1\}$

$$\mathcal{H}(R) = \{(A, B) : (A, B) \in \Im(\blacktriangle) \times \Im(\blacktriangledown) \,\&\, A^{\triangledown\blacktriangledown} \subseteq B \,\&\, A \cap H = B \cap H\} \tag{28}$$

Theorem 140 *In a general approximation space S, $\mathcal{H}(R)$ is isomorphic to the natural completion of $\mathcal{R}$.*

For the proof, the reader is referred to [139].

10.4 Inverse Problems

The concept of *inverse problem* was introduced by the present author in [81] and was subsequently refined in [86]. Granular operator spaces and higher order variants studied by the present author in [84, 94, 97] are important structures that can be used for its formulation. In simple terms, the problem is a generalization of the duality problem which may be obtained by replacing the semantic structures with parts thereof. In a mathematical sense, this generalization may not be proper (or conservative) in general.

The basic problem is

- Given a set of approximations, similarities and
- some relations about the objects.
- Find an information system or a set of approximation spaces that
- fits the available information according to a
- rough procedure

In this formalism, a number of information systems or approximation systems along with rough procedures may qualify. Even when a number of additional conditions like lattice orders, aggregation and commonality operations are available, the problem may not be solvable in a unique sense. In this respect, the example using the information system from Table 2 should be suggestive. The following example from [97] is more suggestive

Example 141 This example has the form of a narrative in [97] that gets progressively complex. It has been used to illustrate a number of computational contexts in the paper.

Suppose Alice wants to purchase a laptop from an on line store for electronics. Then she is likely to be confronted by a large number of models and offers from different manufacturers and sellers. Suppose also that the she is willing to spend less than €x and is pretty open to considering a number of models. This can happen, for example, when she is just looking for a laptop with enough computing power for her programming tasks.

This situation may appear to have originated from information tables with complex rules in columns for decisions and preferences. Such tables are not information systems in the proper sense. Computing power, for one thing, is a context dependent function of CPU cache memories, number of cores, CPU frequency, RAM, architecture of chipset, and other factors like type of hard disk storage.

Proposition 142 *The set of laptops $\mathbb{S}$ that are priced less than €x can be totally quasi-ordered.*

Proof Suppose $\prec$ is the relation defined according to $a \prec b$ if and only if price of laptop a is less than or equal to that of laptop b. Then it is easy to see that $\prec$ is a reflexive and transitive relation. If two different laptops a and b have the same price, then $a \prec b$ and $b \prec a$ would hold. So $\prec$ may not be antisymmetric.

□

Suppose that under an additional constraint like CPU brand preference, the set of laptops becomes totally ordered. That is under a revised definition of $\prec$ of the form: $a \prec b$ if and only if price of laptop a is less than that of laptop b and if the prices are equal then CPU brand of b must be preferred over a's.

Suppose now that Alice has more knowledge about a subset C of models in the set of laptops $\mathbb{S}$. Let these be labeled as *crisp* and let the order on C be $\prec_{|C}$. Using additional criteria, rough objects can be indicated. Though lower and upper approximations can be defined in the scenario, the granulations actually used are harder to arrive at without all the gory details.

This example once again shows that granulation and construction of approximations from granules may not be related to the construction of approximations from properties in a cumulative way.

In [97], it is also shown that the number of data sets, of the form mentioned, that fit into a rough scheme of things are relatively less than the number of those that do not fit. A number of combinatorial bounds on the form of distribution of rough objects are also proved in the paper.

Examples of approximations that are not rough in any sense are common in misjudgments and irrational reasoning guided by prejudice. So solutions to the problem can also help in judging the irrationality of reasoning and algorithms in different contexts. Development of proper algebraic methods for the problem class is an important research area.

11 Representations of General Rough and Fuzzy Sets

Some representation results between rough and fuzzy sets have been proved by the present author in [86, 91]. Her results were obtained in connection with properties of granules and possible valuations. The result in [86] includes the representation proved through membership functions for classical rough sets in [146] and so the latter is omitted. It may also be noted that a large number of algebraic structures including rough algebras, pre-rough algebras, regular double Stone algebras, semi-simple Nelson algebras, super rough algebras [81], 3-valued Lukasiewicz (Moisil) algebras and Wajsberg algebras provide semantics of classical rough sets (see [110]). Related logics [7] do not have much to say about granularity in an explicit way, but are nevertheless related to logics associated with fuzzy sets and the special cases of BL-algebras [8]. In the second chapter [12] of this volume semantics of fuzzy sets over rough sets have also been discussed in the context of BZ De Morgan algebras. The goal of this section is also to motivate investigations of granularity in these considerations.

A non-controversial definition of fuzzy sets, with the purpose of removing the problems with the 'membership function formalism', was proposed in [122]. In [86], it was shown by the present author that the definition can be used to establish a link between fuzzy sets and granulations in rough sets. The connection is essentially of a mathematical nature. In the present author's view, the results should be read as *in a certain perspective, the granularity of particular rough contexts originate from fuzzy contexts and vice versa*. The existence of any such perspective and its possible simplicity provides another classification of general rough set theories.

Definition 143 A *fuzzy subset* (or *fuzzy set*) $\mathbb{A}$ of a set S is a collection of subsets $\{A_i\}_{i\in[0,1]}$ satisfying the following conditions:

- $A_0 = S$,
- $(0 \leq a < b \leq 1 \rightarrow A_b \subseteq A_a)$,
- $A_b = \bigcap_{0\leq a<b} A_a$.

A fuzzy membership function $\mu_{\mathbb{A}} : X \mapsto [0, 1]$ is defined via $\mu_{\mathbb{A}}(x) = \mathrm{Sup}\{a : a \in [0, 1], x \in A_a\}$ for each x. The *core* of $\mathbb{A}$ is defined by $\mathrm{Core}(A) = \{x \in S : \mu_{\mathbb{A}}(x) = 1\}$. $\mathbb{A}$ is *normalized* if and only if it has non-empty core. The *support* of $\mathbb{A}$ is defined as the closure of $\{x \in S; \mu_{\mathbb{A}}(x) > 0\}$. The *height* of $\mathbb{A}$ is $H(\mathbb{A}) = \mathrm{Sup}\{\mu_{\mathbb{A}}(x); x \in S\}$. The *upper level set* is defined via $U(\mu, a) = \{x \in S : \mu_{\mathbb{A}}(x) \geq a\}$. The class of all fuzzy subsets of S will be denoted by $\mathcal{F}(S)$. The standard practice is to refer to 'fuzzy subsets of a set' as simply a 'fuzzy set'.

Proposition 144 *Every fuzzy subset $\mathbb{A}$ of a set S is a granulation for S which is a descending chain with respect to inclusion and with its first element being S.*

The cardinality of the indexing set and the second condition in the definition of fuzzy sets is not a problem for use as granulations in RST, but almost all types of upper approximations of any set will end up as S. From the results proved in the previous sections it should also be clear that many of the nice properties of granulations will not be satisfied modulo any kind of approximations. It is shown below that simple set theoretic transformations can result in better granulations. Granulations of the type described in the proposition will be called *phi-granulations*.

Construction-1

1. Let $P = \{0, p_1, \ldots p_{n-1}, 1\}$ be a finite set of rationals in the interval [0,1] in increasing order.
2. From $\mathbb{A}$ extract the collection $\mathbb{B}$ corresponding to the index P.
3. Let $B_0 \setminus B_{p_1} = C_1$, $B_{p_1} \setminus B_{p_2} = C_2$ and so on.
4. Let $\mathcal{C} = \{C_1, C_2, \ldots, C_n\}$.
5. This construction can be extended to countable and uncountably infinite P in a natural way.

Theorem 145 *The collection $\mathcal{C}$ formed in the fourth step of the above construction is a partition of S. The reverse transform is possible, provided P has been selected in an appropriate way.*

It has been shown that fuzzy sets can be corresponded to classical rough sets in at least one way and conversely by way of stipulating granules and selecting a suitable transform. But a full semantic comprehension of these transforms cannot be done without imposing a proper set of restrictions on admissibility of transformation and is context dependent. The developed axiomatic theory makes these connections clearer.

As far as granulation in the context of fuzzy sets is concerned, most approaches have been in relation to the precision based approach as in [149, 150]. These do not directly relate to the present approach.

The result also means that rough membership functions are not necessary to establish a semantics of fuzzy sets within the rough semantic domain as considered in [146]. Further as noted in [146], the semantics of fuzzy sets within rough sets is quite restricted and form a special class. The core and support of a fuzzy set is realized as lower and upper approximations. Here this need not happen, but it has been shown that any fuzzy set defined as in the above is essentially equivalent to a granulation that can be transformed into different granulations for RST. A more detailed analysis of the connections and extensions has been proved in [90] and in a forthcoming paper due to the present author. This includes an extension of L-fuzzy sets and is presented in the section following the next section.

11.1 Dualities of Rough Difference Orders

Rough difference orders were introduced in [80] by the present author as a semantic approach to handling orders on sets of attributes. The structure is closely related to BL-algebras, the algebras of Hajek-style fuzzy logic [53, 143] and hybrid fuzzy rough approaches. The approach is also relevant for dominance based rough sets. Related representation theorems are also adapted for the context in the paper.

12 L-Fuzzy Sets, Quasi Orders and Topology

In this section, connections between Alexandrov topologies, L-fuzzy sets, quasi-orders and related red results are studied. New results on direct decomposability and definable operations are also proved by the present author.

Theorem 146 *The set $QO(S)$ of quasi-order relations on a set S forms a complete lattice with respect to the induced inclusion order. The meet coincides with set intersection, while the join of quasi-orders P, Q is the least quasi-order containing the two. It is a quotient lattice of $Ref(S)$ and the continuous lattice $EQ(S)$ is a sublattice of $QO(S)$.*

Proof On $Ref(S)$, the order $<$ defined via, if $P, Q \in Ref(S)$, then

$$P < Q \text{ if and only if } P \subseteq Q \text{ in } \wp(S^2)\text{—the powerset of } S^2,$$

is a complete lattice order. Since the set $QO(S) \subseteq Ref(S)$ and if $P \in QO(S)$, there exists a subset $\tau(P)$ of $Ref(S)$ that is the largest with respect to the condition:

$$R \in \tau(P) \stackrel{\text{def}}{\longleftrightarrow} \mathbf{T}(R) = P.$$

$\mathbf{T}(R)$ being the transitive closure of R.

If $P \sim Q$ if and only if $\tau(P) = \tau(Q)$, then $\sim$ is an equivalence on $Ref(S)$ and clearly $Ref(S)| \sim = QO(S)$. $\sim$ preserves the order on $Ref(S)$, so $QO(S)$ is a quotient lattice of $Ref(S)$. □

Proposition 147 *On $QO(S)$, a complementation $'$ and involution $^{-1}$ can be respectively defined in the following way:*

$$\forall R \in QO(S)\ R' = R_1 \vee R_2$$
$$\textit{with } R_1 = \{(y, x)\,;\ Rxy \& \neg Ryx\} \cup \Delta_S$$
$$\textit{and } R_2 = \{(x, y)\,;\ [x],\ [y] \in S|\mathbb{E}(R)\,\&\,[x] \neq [y]\} \cup \Delta_S.$$

$$R^{-1} = \{(y, x);\ Rxy\}.$$

Proof $R' = R_1 \vee R_2$ means it has to be the transitive completion of $R_1 \cup R_2$ as both R_1, R_2 are reflexive relations. So it is necessarily a quasi-order.

A pair (a, b) is in the complement (or $R'ab$) if and only if ($a = b$ or Rba & Rab) or ($[a]$, $[b]$ form distinct classes of the smallest equivalence containing R) or (a, b) is in the smallest quasi-order containing the previous two relations R_1, R_2.

□

Theorem 148 *Every quasi-ordered set of the form $S = \langle \underline{S}, \leq \rangle$ induces an Alexandrov topology on S via:*

$$\tau_{\leq} = \{X\,;\ X \subseteq S\,\&\,(\forall x, y \in S)(x \in X\,\&\,x \leq y \longrightarrow y \in X)\}.$$

Conversely, if τ is an Alexandrov topology then a quasi-order $\leq_\tau$ can be defined on S via

$$y \in N_\tau(x) \longrightarrow x \leq_\tau y.$$

That is the o-filters of $\leq$ form an Alexandrov topology. Further under the above process it is that

$$\langle QO(S), \subseteq\rangle \cong \langle \mathcal{A}(S), \supseteq\rangle$$

and the corresponding operations on the two lattices are related as per

- $\leq_1 \vee \leq_2 = \leq_{\tau_1 \cap \tau_2} = \leq_{\tau_1 \vee \tau_2}$.
- $\tau_1 \vee \tau_2 = \tau_{\leq_1 \cap \leq_2}$

Let $\tau^{op} = \langle \tau, \supseteq\rangle$ be the opposite topology, then the map $\varphi_\tau : S \longmapsto \tau^{op}$ defined by $\varphi_\tau(x) = N_\tau(x)$ is a τ^{op}-fuzzy set.

If φ is an L-Fuzzy set on S, then $\varphi^* : S \longmapsto \tau_\varphi^{op}$ defined by $\varphi^*(x) = N_\varphi(x)$ is a fuzzy set such that $\leq_\varphi$ of φ is equal to $\leq_{\varphi^*}$ and $\varphi^{**} = \varphi^*$.

Proof Proof of this theorem can be found in [59]. □

It is known that algebraic operations on $\mathbb{F}(X, L)$ can be induced on the fuzzy set through point-wise operations. The most important problems in this regard relate to definability of aggregation, commonality, implicants, complementarity and negation like operations. In [59], aggregation, commonality and generalized negation are defined as below (the notation is flawed there):

- $\varphi \cup \psi = \xi$ where $<_\xi = <_\varphi \vee <_\psi$.
- $\varphi \cap \psi = \xi$ where $<_\xi = <_\varphi \wedge <_\psi$
- $\varphi^c = \xi$ where $<_\xi = <_{\varphi'}$

If $\varphi \in \mathbb{F}(X, L)$, then a quasi-order $\leq_\varphi$ can be defined on X via

$$x \leq_\varphi y \longleftrightarrow \varphi x \leq \varphi y.$$

Neighborhoods are defined by setting $N_\varphi(x) = \{y;\ \varphi x \leq \varphi y\}$.

12.1 Direct Decomposition of Quasi Orders

Definition 149 A quasi-ordered set (quasi-ordered set) S is said to be a direct sum of the subsets $\{S_\alpha\}_{\alpha \in \mathcal{A}} = \mathfrak{S}$;

$$S = \sum_{\alpha \in \mathcal{A}} S_\alpha \overset{\text{def}}{\longleftrightarrow} S = \bigcup_\alpha S_\alpha \ \&\ (\forall \alpha \neq \beta)\ S_\alpha \parallel S_\beta.$$

But $S_\alpha \parallel S_\beta \longrightarrow S_\alpha \cap S_\beta = \emptyset$. So $\mathfrak{S}$ is a partition of S with a corresponding equivalence σ.

Definition 150 A quasi-ordered set S is said to be *direct sum indecomposable* if whenever S is a direct sum of the form

$$S = \sum_{\alpha \in \mathcal{A}} S_\alpha,$$

then $\#(\mathcal{A}) = 1$. That is it admits of no non trivial direct sum decompositions.

The following problems/questions will be dealt with in this section:

- How to characterize direct sum indecomposable L-Fuzzy sets?
- Is it possible to generate all of the fuzzy sets from direct sum indecomposable ones?
- What is the connection with Alexandrov topologies?

Theorem 151 *All of the following hold:*

- *The set of double ideals $\mathcal{IF}(S)$ is the center of both $\mathcal{I}(S)$ and $\mathcal{F}(S)$ and is a complete atomic Boolean algebra.*
- *The atoms of the complete atomic Boolean algebra coincide with the principal double ideals of S and will be denoted by $At(\mathcal{IF}(S))$*
- *The principal double ideals can be generated by a recursive process.*

Some key results relating to direct sum decompositions of quasi-ordered set s are collected in the following theorem. Proofs can be found in [21].

Theorem 152

1. *The collection of all direct sum decompositions $\mathfrak{E}(S)$ of a quasi-ordered set S is a principal double ideals of the lattice $EQ(S)$. In fact it is generated by the smallest equivalence $\mathbb{E}(<)$ containing the quasi-order $<$.*
2. *If ηbc means b and c have nonempty intersection, then $\mathbb{E}(<)bc$ if and only if $(\exists x_1, x_2, \ldots x_n \in S)\, x_1 = b \,\&\, x_n = c \,\&\, \eta(\downarrow x_i)(\downarrow x_{i+1})$, $n < \infty$.*
3. *$\mathfrak{E}(S)$ is isomorphic to the lattice of complete Boolean subalgebras of the lattice $\mathcal{FI}(S)$ of double ideals of S.*
4. *Every quasi-ordered set S is representable as a direct sum of direct sum indecomposable quasi-ordered set s.*
5. *The lattice $\mathcal{I}(S)$ of o-ideals of a quasi-ordered set S is a direct product of lattices $\{L_a\}_{a\in A}$ if and only if S is a direct sum of $\{S_a\}_{a\in A}$ and $L_a = \mathcal{I}(S_a)$ for each $a \in A$.*

The following result provides some ways of identifying direct sum indecomposability.

Theorem 153 *The following are equivalent:*

1. *$\mathcal{I}(S)$ is a direct product indecomposable lattice.*
2. *S is a direct sum indecomposable quasi-ordered set.*
3. *S has no proper double o-ideals.*

Proof See [21]. □

12.2 Example: Direct Sum Indecomposable Quasi-Ordered Set

An example of a L-Fuzzy set that has L as a direct sum indecomposable quasi-ordered set is the quasi-ordered set in Fig. 1 of [59]. Another example is formulated in this subsection.

In Fig. 1, if $a \leftarrow b$ is read as a is part of b, then $\leftarrow$ is a quasi-order in the context. Let the associated quasi-ordered set be L. Then L is direct-sum indecomposable.

As L denotes aesthetic attributes, it can be used to classify art work in the associated perspective. A L-Fuzzy set would simply be a map from a set of processes (or art works) with related attributes that best describes the process (or

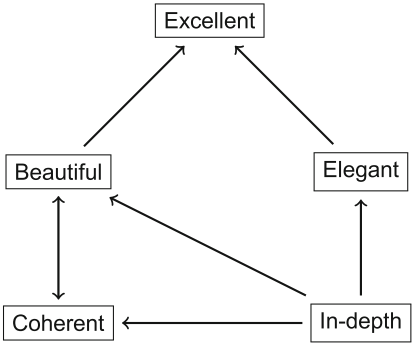

Fig. 1 Aesthetic attributes

art works). Admittedly the example is minimalist—but it is suggestive of ways of forming similar quasi-ordered set s for the analysis of art and possibly integrating them.

12.3 Connections with Alexandrov Topologies and Fuzzy Operations

The structure of direct sum indecomposable quasi-ordered set s are investigated and the properties of fuzzy operations of [59] are extended below.

Theorem 154 *If a quasi-ordered set* L *is direct sum indecomposable and* $\varphi : X \longmapsto L$ *is a L-Fuzzy set, then* φ *induces a chain connected Alexandrov topology on* X*. If* X *is also path connected and connected, then* X *is a* T_o*—Alexandrov topology.*

Proof

1. As $\varphi : X \longmapsto L$ is a L-fuzzy set,
2. A quasi-order $\ll = <_{\varphi}$ can be induced on X as per earlier definition:

$$a \ll b \longleftrightarrow \varphi(a) < \varphi(b).$$

3. Define the Alexandrov topology $\tau_{\ll}$ on X as per

$$\tau_{\ll} = \{A\,;\; A \subseteq X \,\&\, (\forall x, y \in X)(x \in A \,\&\, \varphi(x) < \varphi(y) \longrightarrow y \in A)\}.$$

   That is $\tau_{\ll}$ is the set of order filters on X.
4. If $B \in \tau_{\ll}$, then it cannot be a proper ideal.
5. If $x, y \in X$, then a finite sequence of elements $x_i : i = 1, \ldots, n$ can be determined subject to

$$x_1 = x,\; x_n = y \text{ and } (\forall i)(x_i \uparrow)\eta(x_{i+1} \uparrow)$$

6. This means the Alexandrov topology is chain connected but not path connected and connected (unless L is a partial-order). Because if it were path connected and connected, then the T_o-axiom would be deducible.

□

As the concept of direct sums (or coproducts) extends naturally to topologies, it is provable that

Theorem 155 *Every L-fuzzy set on X induces an Alexandrov topology on it that is a direct sum of chain connected Alexandrov topologies $\{\tau_\alpha\}_{\alpha\in\mathcal{A}}$. Further every L-fuzzy set is direct sum of L_α-fuzzy sets over X_α for $\alpha \in \mathcal{A}$ corresponding to the direct sum selected on L.*

Proof The proof relies on the extension of the concept of direct sum (or coproducts) to topologies.

□

The connection of the three operations of [59] on $\mathbb{F}(X, L)$ and direct sum decomposability are considered next.

Theorem 156 *All of the following hold in a $\mathbb{F}(X, L)$:*

1. *The quasi-order on L induces a quasi-order on $\mathbb{F}(X, L)$ via*

$$\varphi \Supset \psi \text{ if and only if } (\forall x \in X)\, \varphi(x) < \psi(x).$$

   Let the resulting quasi-ordered set be $\mathfrak{X}$
2. *$\mathfrak{X}$ is direct sum decomposable into $\{\mathfrak{X}_a\}_{a\in A}$.*
3. *The operations $\cup, \cap$ induce restricted operations $\cup^a, \cap^a$, on the ath direct summand. Complementation also induces a complementation on direct summands.*
4. *If $\{L_\alpha\}_{\alpha\in\mathcal{K}}$ is the direct sum decomposition of L, then the set of all admissible direct sum combinations $\{\mathfrak{Y}_b\}_{b\in B}$ of $\mathbb{F}(X, L_\alpha)$ coincides with $\{\mathfrak{X}_a\}_{a\in A}$.*
5. *For a fixed direct sum composition of $\mathfrak{X}$ into $\{\mathfrak{X}_a\}_{a\in A}$, the operations $\cap_a, \cup_a$ on each of the direct summands can respectively be extended to $\cup, \cap$ on $\mathfrak{X}$.*
6. *The complementation operation is also so extendable over summands.*

Proof Most of the theorem has already been proved.

The main thing about the operations $\cup, \cap$ are that they are defined relative the induced order on X. If $\varphi_a, \psi_a \in \mathfrak{X}_a$ for each $a \in A$, then it is necessary to be able to determine the quasi-order associated with them and ensure that the direct sum is well defined. Extension of the maps is then easy. The extension of the operations $\cup_a, \cap_a$ from the direct summand domains to the direct sum follows then.

For complementation, the proof succeeds because the complement of a quasi-order on a direct summand is a closed operation that actually coincides with the restriction of the global complementation. This is because the equivalence closure of a quasi-order restricted to a direct sum indecomposable summand is the same as the restriction of the equivalence closure of the quasi-order to the direct summand.

It is necessary to restrict the considerations to a single direct sum as two different direct sums cannot be handled simultaneously.

□

13 General Rough Sets and Dreamy Fuzzy Sets

The arguments for allowing more scope for handling ontology by avoiding numeric oversimplifications and distortions is important particularly for AI as humans express thoughts in many different ways through words. The written expression may be a refinement of their actual thinking. Evidence for this can be found in Susan Sontag's views expressed in [131] (in relation to writing literary works):

> Words have their own firmness. The word on the page may not reveal (may conceal) the flabbiness of the mind that conceived it. All thoughts are upgrades — get more clarity, definition, authority, by being in print - that is, detached from the person who thinks them. (20th Aug'1964)
>
> I think I am ready to learn how to write. Think with words, not with ideas. (5th March'1970)
>
> The function of writing is to explode one's subject — transform it into something else. (Writing is a series of transformations.)
>
> Writing means converting one's liabilities (limitations) into advantages. For example, I don't love what I'm writing. Okay, then - that's also a way to write, a way that can produce interesting results. (5th Nov'1976)

Then again in many reflective writing (as in poetry) compositionality works in ways that cannot be explained from the immediate context. It is the whole together with parts thereof that lead to a new context and meaning—in a sense, this kind of expression is dreamy. These suggest one again that it would be better to assume as little as is possible and at the same time have scope for adding additional layers of information. Associating numeric grades with linguistic hedges amounts to adding an additional layer of distortion and so should be avoided. These models are bound to have embedded concepts that can be cast in the Object-Property-Attribute-Value perspective (as encoded by information tables). Which in turn are liable to contamination when simplified with numeric valuations of vague predicates and domain of discourse are not stable [86].

Definition 157 By a *Dreamy Fuzzy Set* is meant a collection of the form $\{A_\alpha\}_{\alpha \in L}$ that satisfies all of the following:

$$L = \langle \underline{L}, \wedge, 0, 1 \rangle \text{ is a bounded directed quasi-order,} \tag{1}$$

$$(\forall \alpha, \beta \in L)(\alpha < \beta \longrightarrow A_\beta \subseteq A_\alpha), \tag{2}$$

$$A_0 = S \tag{3}$$

$$(\forall \beta \in L) A_\alpha = \bigcap_{\beta \in \downarrow \alpha} A_\beta, \tag{4}$$

$$\bigcup_{\alpha \neq 0} A_\alpha = S. \tag{5}$$

Proposition 158 *The fifth condition does not follow from the first two conditions.*

Proof This can be proved through easy counterexamples. □

Theorem 159 *All of the following hold:*

- *Every L-fuzzy set when L is a bounded directed quasi-ordered set is transformable into a dreamy fuzzy set.*
- *Every dreamy fuzzy set is not necessarily derivable from a L-fuzzy set.*

Proof Let $\varphi : X \longmapsto L$ be a L-fuzzy set with L being a bounded directed quasi-ordered set. Then dreamy fuzzy sets can be constructed in many ways from which the L-fuzzy set can be recovered through specific processes π. One way is as below:

- Induce a quasi-order $\lessdot$ on X via

$$a \lessdot b \stackrel{\text{def}}{\longleftrightarrow} \varphi(a) < \varphi(b).$$

- On $\mathfrak{I}_p(X)$ define a map

$$\hat{\varphi}(\downarrow x) = \{\varphi(x)\}.$$

  This definition is clearly always possible.
- Principal order ideals are well defined over quasi-ordered sets. So the existence of principal order ideals is guaranteed.
- The map $\hat{\varphi}$ provides the required index.
- Alternatively, on $\wp(L)$, an order $\ll$ can be defined via

$$A \ll B \stackrel{\text{def}}{\longleftrightarrow} (\forall x \in A)(\forall y \in B)\, x < y.$$

- Define suitable global extensions of φ to the lattice of all order ideals.

□

In the present author's perspective, the main problems are about the natural ways in which dreamy fuzzy sets can be viewed as generators of granulations. The earlier decomposition based algebraic perspective suggests the following:

Definition 160 A *well founded dreamy set* $\{A_i\}_{i \in L}$, will be a dreamy fuzzy set in which L is pre-well.

Definition 161

1. Let $\{A_i\}_{i\in L}$ be a well founded well dreamy fuzzy set.
2. Form a direct sum decomposition $\{L_\alpha\}_{\alpha\in\Lambda}$ of L with L_α being direct sum indecomposable for each α.
3. On each direct sum indecomposable component $L_\alpha = \{J_r^\alpha\}_{r\in K_\alpha}$, apply construction-1 on sets corresponding to each of the chains to get multiple partitions in S
4. All of the partitions in S together form a collection of sets of granules

$$\{\{C_{i\alpha}\}_{i\in J^\alpha}\}_{\alpha\in\Lambda} \text{ on } S.$$

5. The corresponding granulation $\mathcal{C}$ will be referred to as the *granulation induced by a WWF-dreamy fuzzy set* $\{A_i\}_{i\in L}$.

Definition 162 On the set S, endowed with the granulation $\mathcal{C}$, let the approximation operators l, u be defined as follows (for an arbitrary subset $B \subseteq S$):

$$B^l = \bigcup\{X : X \in \mathcal{C} \,\&\, X \subseteq B\}, \quad \text{(Lower Approximation)}$$

$$B^u = \bigcup\{X : X \in \mathcal{C} \,\&\, X \cap B \neq \emptyset\}\} \quad \text{(Upper Approximation)}$$

The defined approximations are granular and make use of the available information in the context and are therefore justified.

Theorem 163 *The granulation* $\mathcal{C}$ *satisfies:*

$$\bigcup C = S,$$

It is possible that $(\exists B, G \in \mathcal{C})\ B \cap G \neq \emptyset$,

It is possible that $(\exists B, G \in \mathcal{C})\ B \subseteq G$,

It is possible that $\mathcal{C}$ *is not a normal cover,*

It is possible that $(\emptyset \notin \mathcal{C})$.

Further $\mathcal{C}$ *satisfies the axioms of representability (RA), lower stability (LS), lower idempotence (LI) and lower full underlap (LFU) relative the approximations* l, u *in the terminology of [86, 88]. So they form an admissible set of granulation.*

Proof The counterexamples required for the proof are easy to construct.

Most of the axioms of granules including those for mereological atomicity, crispness, stability, unique underlap and idempotence do not hold in general. Representability holds because of the definition of approximations. The other properties in this context are:

$$(\forall X \in \mathcal{C})(\forall B)(X \subseteq B \longrightarrow X \subseteq B^l), \quad \text{(LS)}$$

$$(\forall X \in \mathcal{C}) X^{ll} = X^{l}, \tag{LI}$$

$$(\forall X, Y \in \mathcal{C})(\exists B) B^{l} = B. \tag{LFU}$$

A normal cover, by definition, must also be an antichain. This property fails to start with. □

The above shows that

- fuzzy sets in the non controversial view are simple transformations of granulation that contain the actual measures,
- these granulations can be realized within general rough sets,
- Even L-Fuzzy sets can be so transformed,
- fuzzy sets lose information by way of simplification (when a more involved perspective is possible),
- a unique transformation of fuzzy sets into rough sets and conversely is not a good idea—the actual situations when they become unique requires a very frequentist perspective when cardinalities can measure properties uniquely, and
- that L-Fuzzy sets defined over direct sum decompositions of directed well founded quasi-ordered set s correspond to admissible granulations.

13.1 How to Construct Examples

Examples are pretty easy to construct for the concepts defined above. This is shown through well-known results in the theory of quasi orders and related relational and algebraic systems. Dreamy fuzzy sets and L-fuzzy sets when L is a well quasi-ordered set can be visualized as special arrangements of subsets on positions corresponding to elements of the well quasi-ordered set. Well quasi-orders and closely related variants can be handled in a computationally efficient way, in principle, because they have no infinite antichains and no infinite strictly decreasing sequences.

Theorem 164 *The class of well quasi-ordered sets is closed under relational morphic images, finite unions and finite cartesian products.*

Definition 165 Let $S = \langle \underline{S}, <, \Sigma \rangle$ be a quasi-ordered algebraic system (with Σ being a set of finitary operations) with the arity $\nu(f) = r < \infty$ for each operation $f \in \Sigma$. Further assume that each of the operations are compatible with the quasi order. S is said to be *minimal* if it has no proper subalgebraic systems. $<$ is said to be a *divisibility order* on S if

$$(\forall f \in \Sigma)(\forall x_1, \ldots x_r \in S)(\forall 1 \leq j \leq r)\, x_j < f(x_1, \ldots, x_j, \ldots, x_r)$$

If Σ_r is the set of operations of arity r is quasi ordered by an order $\prec$, then $\prec$ is said to be *compatible* with $<$ if and only if

$$(\forall f, h \in \Sigma_r)(\forall x_1, \ldots, x_r \in S)\, f \prec h \rightarrow f(x_1, \ldots, x_r) < h(x_1, \ldots, x_r)$$

The following theorem (see [133]) encompasses a large number of situations that can occur in practice.

Theorem 166 *On any minimal quasi-ordered algebraic system $S = \langle \underline{S}, <, \Sigma \rangle$ with divisibility quasi-order, if each Σ_r is well quasi-ordered and compatible with $<$, then $<$ is a well quasi-order.*

The following are all special cases of the above theorem that are more common in practical situations.

Theorem 167 *All of the following hold:*

- *On any quasi-ordered algebraic system $S = \langle \underline{S}, <, \Sigma \rangle$ with divisibility quasi-order and generated by a well quasi-ordered set X and finite Σ, $<$ is a well quasi-order.*
- *On any finitely generated algebraic system $S = \langle \underline{S}, \Sigma \rangle$ with divisibility quasi-order and finite Σ, $<$ is a well quasi-order.*
- *If X is a well quasi-ordered alphabet, then the free monoid X^* on X is well quasi-ordered by for all $\{a_k\}_1^p$, $\{b_i\}_1^q$,*

$$a_1 \cdots a_p < b_1 \cdots b_q \leftrightarrow (\exists 1 \leq j_1 < \ldots < j_p \leq q)(\forall 1 \leq i \leq p)\, a_i < b_{j_i}$$

- *If X is a finite alphabet, then the free monoid X^* on X is well quasi-ordered by the subword order for all $\{a_k\}_1^p$, $\{b_i\}_1^q$,*

$$a_1 \cdots a_p < b_1 \cdots b_q \leftrightarrow (\exists 1 \leq j_1 < \ldots < j_p \leq q)(\forall 1 \leq i \leq p)\, a_i = b_{j_i}$$

Ideals and filters can be used to specify closure conditions in the context. Further sets of graphs under subgraph ordering have been studied as well quasi-orders. The same applies to directed graphs of different kinds. A survey of theoretical application areas is in [54] and more details can be found in standard texts like [133].

13.2 Extended Example: Dreamy Fuzzy Set

Dreamy fuzzy sets are plentiful, thanks to the unrealness of the so-called natural numbers, in real life. The first example is both an example of a dreamy fuzzy set and also a new approach to psychological tests of a kind. Psychological scales based on long batteries of questions have the following features:

- they tend to be very sensitive to contexts in which they are employed,
- practitioners may not agree on scale,
- get outdated over time due to changes in social conditions,
- get outdated over time due to changes in the very model that they were designed for and
- numeric summaries are intended for easy decision making and superficial diagnosis.

Human resources and psychologists professionals use various models for characterizing the personality of individuals. These use either a type or trait approach to personality. Traits are defined as relatively enduring, stable and consistent individual differences in cognition, emotions and behavior. Another view compatible with [26] is that traits are the biologically heritable, pre-cultural, and hierarchically structured aspects that are integral parts of personality like a person's beliefs, skills, and attitudes.

In the big five model due to [26], the five personality dimensions are *openness, conscientiousness, extroversion, agreeableness and neuroticism*. The last is also referred to as emotional stability is distinct from Freudian concept of the same name and is related to a person's emotional stability and amount of negative emotions harbored. Negative emotions are not the same as skeptical thought, and weak people may become moody and tense (reflecting neuroticism) due to their negative thinking. For more information on these models the reader is referred to [26, 99]. There is no general agreement as to whether these models are really usable outside of the western world.

In the associated emotional stability trait, people who score low in neuroticism are more emotionally stable and do not experience negative feelings often. But they do not need to experience positive feelings often. Those who score high in neuroticism are likely to be emotionally reactive.

In the Big Five personality traits are sub-classifiable into six subtracts. The sub-traits associated with the emotional stability trait are *anxiety, anger, depression, self-consciousness, immoderation and vulnerability*. Sub-traits can be assessed independently of the trait they belong to.

Some examples of questions used in the battery for assessment are as follows:

1. I am a 'worrier'
2. I make friends easily
3. I have a vivid imagination
4. I trust others
5. I complete tasks successfully
6. I get angry easily
7. I really enjoy large parties and gatherings
8. I think art is important
9. I use and manipulate others to get my own way
10. I don't like things to be a mess—I like to tidy up

Table 5 Value of response

Meaning	Value
Strongly disagree	−2
Disagree	−1
Neutral	0
Agree	1
Strongly agree	2

11. I often feel sad
12. I like to take charge of situations and events
13. I experience deep and varied emotions
14. I love to help others
15. I keep my promises

Let S be the set of questions, P a population of individuals and V is mapped to be the set $\mathbf{V} = \{-2, -1, 0, 1, 2\}$ as per the table (Table 5):

Response to one question is often not independent of other questions and responses. If two questions refer to a concept and its expression in a subject, then response to one question consisting of an expression can get altered by the response to the other question. Usually this does not happen in a linear way and experts can potentially differ on how the expressions should be combined. To see this consider possible ways of inferring from possible responses to *I love to help others* and *I make friends easily*. If a subject responds with $+2$ and -1 to these queries respectively, then a expert with preconceived world view of *how things should be* is bound to infer differently from another who assumes nothing in relation to the context being referred to. In fact responses to these two queries are directly related to responses of few other queries. Numeric valuations of potential integration strategies used by experts is bound to be controversial and so a noncontroversial approach is used here.

Typically the number of questions exceed 150 in the context, these are not all independent and it is possible to define dreamy fuzzy sets in more than one way—either through

1. relations defined on the set of questions through associated tags for example or
2. through relations defined on set of relevant subsets $\mathfrak{S}$ of S (relevant relates to key concepts being isolated by the queries).

In the second perspective, it is possible to define a relation $<$ corresponding to *is at least as informative as* (from a relative perspective). If $A, B \in \mathfrak{S}$, then it can happen that a $A, B \subseteq C \subseteq A \cup B$ is actually relevant and C may be at least as informative as $A \cup B$.

Proposition 168 *$<$ is a directed quasi-order on the set of subsets of questions $\mathfrak{S} \subseteq \wp(S)$.*

Proof If A is a subset of questions, then $A < A$.

If $A \subset B$, then obviously $A < B$.

If $A \subset B$ and $B \subset A$ then it does not mean that $A = B$ as the intent of the questionnaire is to isolate features through multiple perspectives—positive, negative and neutral.

Transitivity of $\subset$ relation yields transitivity of $<$—this monotonicity is implicit in the idea of *being informative* as opposed to *being clear*.

The directed aspect arises from the aggregation operation and finite boundedness of $\mathfrak{S}$:

$$(\forall A, B \in \mathfrak{S})(\exists C \in \mathfrak{S})(\forall E \in \mathfrak{S})\,(A < E \& B < E \longrightarrow A < C < E).$$

□

Proposition 169 *The above proposition holds even when the aggregation operation is set theoretic union.*

Theorem 170 *The directed quasi-ordered set of sets is transformable into a dreamy fuzzy set when the set of all relevant subsets of questions is closed under intersection.*

Proof By direct verification. □

The information provided by the valuation and query is usable to define another quasi-order on a set of subsets of pairs of the form (query, valuation). This again leads to dreamy fuzzy subsets.

14 Red Results of AntiChain Based Semantics

Antichains have been used by the present author for inventing semantics [89, 97] for almost all general rough sets. For the semantics, the minimal assumptions required are

- the requirement that the collection of objects under consideration forms a set.
- a classification of objects as crisp or non-crisp, and
- enough description of the properties satisfied and related valuation

A few distinct relations on the set of all antichains of a partially or quasi-ordered set P are of natural interest in these contexts. The antichain based approach has also been described in the chapter on granular rough sets in this volume by the present author [98] and some of the red results that have been used are included in the same chapter.

If P is a Poset, then $\max(P)$ shall denote its set of maximal elements, while the width (respectively height) of the partially ordered set will be denoted by $w(P)$ (respectively $h(P)$). The *order dimension* $dim(P)$ of P is the minimum number l for which there exists an order-preserving function from P into a direct product of l chains.

An element x in a lattice L is said to be *completely join irreducible* (respectively completely meet irreducible) if for any $X \subseteq L$, $\bigvee X = x$ (respectively $\bigwedge X = x$) implies $x \in X$.

An element x in a lattice L is said to be *completely join prime* (respectively completely meet prime) if for any $X \subseteq L$, $x \leq \bigvee X$ (respectively $\bigwedge X \leq x$) implies that $(\exists a \in X)\, x \leq a$ (respectively $(\exists a \in X)\, a \leq x$).

L is said to be *superalgebraic* if and only if every element is the join of completely join prime elements. This concept is self-dual.

The *Alexandrov completion* of a partially ordered set $P = \langle \underline{P}, \leq \rangle$ is $\mathfrak{I}_o(P) = \langle \underline{\mathfrak{I}(P)}, \subseteq \rangle$ with $\underline{\mathfrak{I}(P)}$ being the set of order ideals of P.

Theorem 171

- *Alexandrov completions are superalgebraic complete lattices.*
- *If P satisfies ascending chain condition (ACC), then $\mathfrak{I}_o(P)$ is a strongly coatomic, superalgebraic, completely distributive lattice. ACC is the condition that there are no proper infinite ascending chains.*
- *Every element of $\mathfrak{I}_o(P)$ has an irredundant join-representation by join irreducible elements of the form $x \downarrow$ for each $x \in P$.*
- *The set of completely meet-irreducible elements $M(\mathfrak{I}_o(P))$ is in injective correspondence with $\{P \setminus (\{x\} \uparrow) : x \in P\}$ as defined by $x \longmapsto P \setminus (\{x\} \uparrow)$. If P is coatomistic, then $P \setminus (\{x\} \uparrow) = \cup\{c \downarrow : c$ is a coatom $\&xnleqc\}$.*
- *Every strongly coatomic, superalgebraic, completely distributive lattice is isomorphic to the Alexandrov completion of a partially ordered set satisfying ACC. These lattices are also representable as sober T_o Alexandrov spaces. (A T_o—space is sober when point closures are the join-prime closed sets.)*

Proof The proofs can be found in [3, 48, 72, 75, 132]. □

Proposition 172 *A partially ordered set Q has no infinite anti-chains if and only if every o-ideal of Q is a finite union of o-ideals.*

Proof The proof of this result can be found in [75] for example.

□

The next theorem is an extension of the classic result due to Dilworth [48] and its converse. Many proofs of the result are known [75].

Theorem 173 *Let P be a partially ordered set with longest chains of length r, then P can be partitioned into k number of antichains implies $r \leq k$. Dually, if Z is a finite partially ordered set with k elements in its largest antichain, then a chain decomposition of Z must contain at least k chains.*

Proof The dual is proved below:

- Let $a \in \max(P)$, then $w(P \setminus \{a\}) = k \implies k \leq w(P) \leq k + 1$.
- If $\{C_i\}_{i=1}^k$ is a chain decomposition of $P \setminus \{a\}$ and X is an antichain of size k, then $\#(A \cap C_i) = 1$.
- Let $a_i = \max(C_i)$ and let it be in an antichain of size k for each i. Then $\{a_i\}_1^k$ must be an antichain. Otherwise if $a_2 > a_1$, and $\{b_1, a_2, b_3, \ldots, b_k\}$ is an antichain of size k with $b_1 \in C_1$, then $a_2 > a_1 \geq b_1$—a contradiction.

- If $\{a, a_1, \ldots, a_k\}$ is an antichain, then $w(P) = k + 1$ and $\{\{a\}, C_1, \ldots, C_k\}$ would be a partition into $k+1$ chains. Otherwise, $a_i < a$ for some i and $\{x : x \in C_1 \,\&\, x \leq a_i\} \cup \{a\} = K$ would be a chain.
- Since every antichain of size k in $P \setminus \{a\}$ contains an element of $\{z : z \in C_i \,\&\, z \leq a_i\}$ for each i, $P \setminus K$ does not contain any antichain of size k.
- By induction a partition $\{T_i\}_{i=1}^{k-1} = \Pi$ of $P \setminus K$ into antichains can be formed. $\Pi \cup K$ would then be a partition of P into k antichains.

The rest of the proof is left to the reader. □

Proposition 174 *If a finite partially ordered set P satisfies $\#(P) \geq st + 1$, then it contains a chain of length $s + 1$ or an antichain of size $t + 1$.*

Theorem 175 *If L is a finite distributive lattice then $dim(L) = w(J(L))$.*

A *normal ideal* or *cut* is an intersection of principal ideals:

$$Z_{\downarrow} = \{a : \forall b \in Za \leq b\}$$

Definition 176 On the set of all antichains $AC(P)$ of a partially ordered set P, the following relations can be defined:

$$(\forall A, B \in AC(P))\; A \trianglelefteq B \text{ if and only if } (\forall a \in A)(\exists b \in B) a \leq b$$

$$(\forall A, B \in AC(P))\; A \preceq B \text{ if and only if } A \downarrow \subseteq B \downarrow$$

$\underline{AC(P)} = \left\langle \underline{AC(p)}, \trianglelefteq \right\rangle$ is the *antichain completion* of P.

Proposition 177 *In the above definition, the two relations are equivalent join semilattice orders. The join-irreducible elements of $AC(P)$ are the singleton antichains and the unique irredundant join representation of any antichain is given by*

$$(\forall A \in AC(P))\; A = \bigvee_{x \in A} \{x\}$$

Proof The maximal elements of $A \downarrow \cup B \downarrow$ coincide with those of $A \cup B$. So the result follows. □

In any infinite ascending chain, complete join irreducibility and join completeness need not be definable.

Theorem 178 *$AC(P)$ is a lattice if and only if*

$$(\forall A, B \in AC(P))(\exists C \in AC(P))\; A \downarrow \cap B \downarrow = C \downarrow \tag{29}$$

In the situation, $A \wedge B = C$ is well defined. The lattice is necessarily distributive.

Proof If the Condition (29) is true then it can be checked that $A \wedge B = C$. The order ideals generated by $AC(P)$ form a meet semilattice with respect to $\cap$. Distributivity holds because the lattice of order ideals is a set lattice and this in turn causes $AC(P)$ to be distributive.

For the converse, if $AC(P)$ is a lattice, but $(\exists A, B \in AC(P))\, A\downarrow \cap B\downarrow = E$ is not generated by an antichain. For this to happen, it is necessary that there exists a $z \in E$ that is not bounded by any element of E. By assumption,

$$A \wedge B \subseteq (A \wedge B)\downarrow \subset E.$$

As $(A \wedge B)\downarrow$ and E are down closed sets, $(\exists x \in E)(\forall b \in A \wedge B)\, \neg(x \leq b)$ must hold. This in turn leads to $A \wedge B \lhd A \wedge B$—a contradiction.

Rest of the proof is easy. □

Corollary 179 *If $AC(P)$ is a lattice and P is a $\cdot$-semilattice, then*

$$A \wedge B = \{a \cdot b \,:\, a \in A \,\&\, b \in B\}$$

Theorem 180 *If P satisfies ACC, then the order embedding $\downarrow: AC(P) \longmapsto \mathcal{I}_o(P)$ is an isomorphism.*

Proposition 181 *In any partially ordered set P, the atoms of the Alexandrov completion $\mathcal{I}_o(P)$ are the down sets of the form $\{x\}\downarrow$, with x being minimal elements of P. The atoms of $AC(P)$ are the antichains of the form $\{x\}$ with x being a minimal element of P.*

Proof The result can be proved by a simple contradiction argument. □

The second part of the following theorem was proved in [67]

Theorem 182 *The partially ordered set $AC_m(X)$ of all maximum sized antichains of a partially ordered set X is a distributive lattice under the order induced from $AC(X)$ and for every finite distributive lattice L and every chain decomposition C of J_L (the set of join irreducible elements of L), there is a partially ordered set X_C such that $L \cong AC_m(X_C)$.*

Proof The first part of the result is fairly direct. The proof of the converse is very long and the reader is invited to try and find a simpler proof. □

Theorem 183

- *If X_1, X_2 are two partially ordered sets then $AC(X_1) \cong AC(X_2)$ if and only if $X_1 \cong X_2$.*
- *It is possible that $AC_m(X_1) \cong AC_m(X_2)$ for non isomorphic partially ordered sets X_1, X_2.*
- *For each distributive lattice L, there exist infinitely many partially ordered sets X such that $L \cong AC_m(X)$*

- *If $\alpha(L)$ is the set of partially ordered sets whose lattice of maximum sized antichains are isomorphic to L, then $\alpha(L)$ contains a set of minimal partially ordered sets relative to inclusion and size.*
- *For any partially ordered set X, the partially ordered set of join irreducible elements of the distributive lattice of antichains of X coincides with X. That is $J(AC(X)) \cong X$.*
- *For any distributive lattice L, $L \cong AC(J(L))$.*

The minimal partially ordered sets in $\alpha(L)$ are of much interest for inverse problems [94] of rough sets. A characterization of these is proved in [68].

If P is a finite partially ordered set of width $w(P) = n$, then an element $a \in P$ is said to be *essential* if there exist $\{a_i\}_{i=1}^{n-1} \in P$ such that $\{a, a_1, \ldots a_{n-1}\} \in AC_m(P)$. P is *essential* if and only if all of its elements are essential. All minimal partially ordered sets in $\alpha(L)$ must be essential, but not all essential partially ordered sets are minimal in general.

15 Conclusion and Directions

In this chapter a broad overview of some basic representation and duality results that have been used in the study of rough sets have been considered. Results that have much potential have also been adapted for rough sets. Some important directions are mentioned below:

Canonical dualities have gained some importance in recent work, but these have not been covered fully in the present chapter. Infinitary operations are commonly used in these and also in many topological algebraic. But very little work on the meaning of infinitary operations in the rough context has been done. It can be argued that they are not being grounded properly. For example, infinite attribute sets are not encountered in practice and some reformulation of the attributes involved in information tables is essential for speaking about infinite sets of attributes. This is an important foundational problem that can significantly alter the direction of the subject.

Duality and representation theory for rough sets has been shaped to a substantial extent by the concerns of modal logic and frame semantics. In the present author's opinion it is also important to change the filters and ideals used in proofs. Plenty of such concepts are available for the purpose[41, 126, 142]. This is necessitated by the needs of approximate reasoning.

The interconnections between pointwise, granular, abstract and cogranular approximations [96] are not fully understood because of the limited number of results that have been proved to date. Some of the known results have been mentioned in this chapter. It should be noted that both positive and negative results on possible interconnections are relevant in the study of rough sets, and vague reasoning in general. A subclass of the above class of problems is that of redefining co-granular and point wise approximations as granular approximations.

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Algebraic Methods for Orthopairs and Induced Rough Approximation Spaces

Gianpiero Cattaneo and Davide Ciucci

Abstract In this chapter we are interested to study the structures arising from pairs of elements from a partially ordered set (poset) which share some orthogonality between them, the so-called *orthopairs*, with respect to a unary operation of De Morgan complementation (or in the case of a lattice interpreted as De Morgan negation).

1 Part I: Orthopair Algebras from De Morgan Posets and Lattices

In this chapter we are interested to study the structures arising from pairs of elements from a partially ordered set (poset) which share some orthogonality between them, the so-called *orthopairs*, with respect to a unary operation of De Morgan complementation (or in the case of a lattice interpreted as De Morgan negation).

1.1 De Morgan Posets and Lattices

Let us start our investigation from the basic structure of *De Morgan poset* in its abstract formulation according to the following definition.

Definition 1 A *De Morgan poset* is a structure $\mathfrak{DP} = \left\langle \Sigma, \leq, ', 0, 1 \right\rangle$ where

(i) the sub-structure $\mathfrak{P} = \langle \Sigma, \leq, 0, 1\rangle$ is a poset with respect to the partial order relation $\leq$, bounded by the least element 0 and the greatest element 1, i.e., $\forall a \in \Sigma,\ 0 \leq a \leq 1$ (with $0 \neq 1$, i.e., Σ contains at least two distinct elements 0 and 1).

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(ii) the mapping $'$: $\Sigma \to \Sigma$ is a De Morgan unary operation, i.e., it satisfies the two conditions:

(dM1) $\forall a \in \Sigma, a = a''$ (involutive);
(dM2) $\forall a, b \in \Sigma, a \leq b$ implies $b' \leq a'$ (contraposition)

A De Morgan lattice is a De Morgan poset $\langle \Sigma, \leq, ', 0, 1 \rangle$ in which both the greatest lower bound (g.l.b.) and the least upper bound (l.u.b.) exist for any pair of elements $a, b \in \Sigma$; the g.l.b. is denoted by $a \wedge b$ and the l.u.b. by $a \vee b$. Hence a De Morgan lattice is denoted by $\langle \Sigma, \wedge, \vee, ', 0, 1 \rangle$.

(Let us recall that a lattice is a fortiori a poset and so all the results involving a poset can be immediately applied to a lattice).

Lemma 2 *Let Σ be a De Morgan* poset *then $0' = 1$ and $1' = 0$. Moreover the following holds:*

$$\forall a \in \Sigma, \ a' = a''' \tag{1}$$

where we have adopted the convention of writing $a'' := (a')'$ and so on by the iteration procedure.

The following proposition expresses in the poset context the equivalence among the *contraposition law* (dM2) and the *dual contraposition law* (dM2a). In the lattice case this equivalence result can be improved with the further two *De Morgan laws* (dM2b-L) and (dM2c-L). The proof of the various points of this proposition can be found in [22, Proposition 3.3].

Proposition 3 *Let Σ be a De Morgan* poset *then under condition* (dM1)*, the following properties are mutually equivalent among them:*

(dM2) $\forall a, b \in \Sigma$, $a \leq b$ *implies* $b' \leq a'$ (contraposition)*;*
(dM2a) $\forall a, b \in \Sigma$, $b' \leq a'$ *implies* $a \leq b$ (dual contraposition).

Let Σ be De Morgan lattice *then under condition* (dM1) *the following four properties are mutually equivalent among them:*

(dM2) $\forall a, b \in \Sigma$, $a \leq b$ *implies* $b' \leq a'$ (contraposition)*;*
(dM2a) $\forall a, b \in \Sigma$, $b' \leq a'$ *implies* $a \leq b$ (dual contraposition)*;*
(dM2b-L) $\forall a, b \in \Sigma$, $(a \wedge b)' = a' \vee b'$ (first De Morgan law)*;*
(dM2c-L) $\forall a, b \in \Sigma$, $(a \vee b)' = a' \wedge b'$ (second De Morgan law).

Remark 4 As usual a De Morgan lattice $\langle \Sigma, \wedge, \vee, ', 0, 1 \rangle$ is considered an algebraic model of some propositional logic where the elements $a, b, c, \ldots$ of Σ are interpreted as *propositions*, the lattice binary operations meet, $\wedge$, and join, $\vee$, as mathematical realizations of the logical connectives AND and OR, and the unary De Morgan operation $'$ as mathematical realization of the logical (De Morgan) connective NOT ([42] and see also [30]). This last consideration allows us to call the unary De Morgan operation as *De Morgan negation* in the sequel, also in the poset case, and in this interpretation the condition (dM1) realizes the *double negation law*. ■

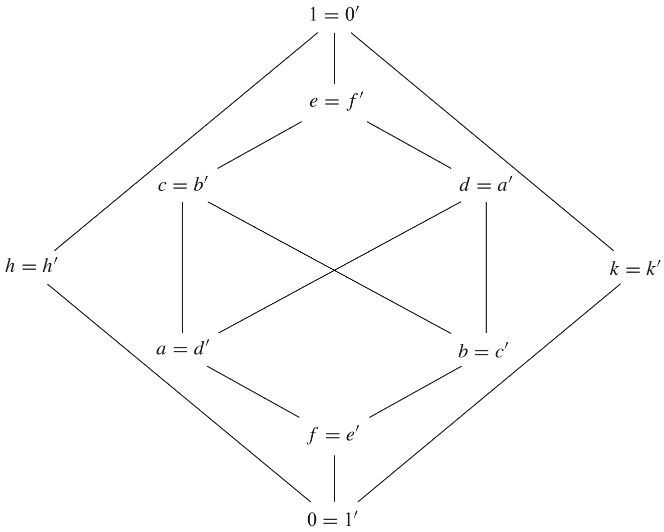


Fig. 1 The ten element De Morgan poset dMP10 with two half elements

Let Σ be a De Morgan poset then, according to Moisil [50],

(a) the set $N_0(\Sigma) := \{a \in \Sigma : a \leq a'\}$ is the 0-*kernel* of Σ, whose elements are said to be *contingent* or *of type* II. Of course, $0 \in N_0(\Sigma)$ and $a \in N_0(\Sigma)$ iff $a'' \in N_0(\Sigma)$;
(b) the set $N_1(\Sigma) := \{b \in \Sigma : b' \leq b\}$ is the 1-*kernel* of Σ, whose elements are said to be *possible* or of *type* I. Of course, $1 \in N_1(\Sigma)$ and $b \in N_1(\Sigma)$ iff $b' \in N_1(\Sigma)$;
(c) the set $N_c(\Sigma) := N_0(\Sigma) \cap N_1(\Sigma) = \{c \in \Sigma : c = c'\}$ is the *half kernel* of Σ, whose elements are said to be the *half elements*.

A De Morgan poset is said to be *genuine* iff it admits at least two half elements, i.e., iff $\exists h, k \in N_c(\Sigma), h \neq k$.

Example 5 In Fig. 1 it is drawn the Hasse diagram of the De Morgan poset dMP10 based on a ten element poset Σ_{10}.

This is a poset which is not a lattice since the collection of lower bounds (l.b.) of the pair $\{c, d\}$ consists of the subset l.b.$\{c, d\} = \{a, b, f, 0\}$ which has no greatest lower bound (g.l.b.). Similarly the pair $\{a, b\}$, whose collection of upper bounds (u.b.) is u.b.$\{a, b\} = \{c, d, e, 1\}$, has no least upper bound (l.u.b.). Furthermore it contains two half elements $h = h'$ and $k = k'$, that is $N_c(\Sigma_{10}) = \{h, k\}$, and so it is a genuine De Morgan poset.

Let us note that in particular $k \leq k'$ and $h' \leq h$, but $k \not\leq h$, and so, according to Definition 7 below, it is not a Kleene poset.

In a De Morgan poset it may happen that there is a unique half element without being a Kleene poset (look at Definition 7 below with the corresponding Lemma 9).

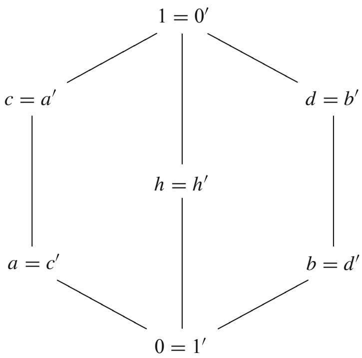


Fig. 2 The seven element De Morgan (not distributive) lattice dML7 with a unique half element h

Example 6 The Hasse diagram drawn in Fig. 2 shows the De Morgan lattice called dML7, based on a seven element lattice Σ_7, which contains a half elements: $N_c(\Sigma_7) = \{h\}$. This De Morgan lattice is neither distributive ($a \vee (h \wedge d) = a \neq 1 = (a \vee h) \wedge (a \vee d)$) nor Kleene (see the condition (K) in the following Definition 7) since $a \leq a'$ and $d' \leq d$ but $a \not\leq d$ or equivalently (see the condition (KL) in Lemma 8) $a \wedge a' = a \not\leq b' = b \vee b'$. In this De Morgan lattice it is that $a \wedge a' = a \neq 0$ and $a \vee a' = a' \neq 1$. Therefore neither the noncontradiction law nor the excluded middle law are satisfied (it is not a Boolean lattice).

Definition 7 A *Kleene poset* is a De Morgan poset which satisfies the further Kleene condition:

$$\text{(K)} \qquad \forall a, b \in \Sigma,\ a \leq a' \ \text{ and } \ b' \leq b \quad \text{imply} \quad a \leq b. \tag{2a}$$

In other words, condition (K) can also be formulated as follows:

$$\text{(KN)} \quad a \in N_0'(\Sigma) \quad \text{and} \quad b \in N_1'(\Sigma) \quad \text{imply} \quad a \leq b. \tag{2b}$$

Lemma 8 *Let Σ be a De Morgan* lattice. *Then, property (K) is equivalent to the following:*

$$\textit{(KL)} \qquad \forall a, b \in \Sigma, \quad a \wedge a' \leq b \vee b'. \tag{3}$$

Proof Let us suppose that (KL) holds. if $a \leq a'$ and $b' \leq b$, then $a = a \wedge a' \leq b \vee b' = b$. Conversely, for all a, b one has that $a \wedge a' \leq a \vee a' = (a' \wedge a)'$ and $b \wedge b' = (b \vee b')' \leq b \vee b'$; thus by (K), $a \vee a' \leq b \vee b'$. □

A Kleene poset (resp., lattice) is said to be *genuine* iff there exists an element $\frac{1}{2}$ ($\neq 0, 1$), called the *half element*, such that $\left(\frac{1}{2}\right)' = \frac{1}{2}$ and so the central kernel is not empty: $\frac{1}{2} \in N_c(\Sigma)$ (and a fortiori $\frac{1}{2} \in N_0'(\Sigma)$ and $\frac{1}{2} \in N_1'(\Sigma)$). Note that for this element $\frac{1}{2} \wedge \left(\frac{1}{2}\right)' = \frac{1}{2}$ ($\neq 0$) and $\frac{1}{2} \vee \left(\frac{1}{2}\right)' = \frac{1}{2}$ ($\neq 1$), i.e., $\frac{1}{2}$ does not satisfy the noncontradiction law and the excluded middle law.

Lemma 9 *The half element with respect to the Kleene complementation, if it exists, is unique. In other words, a Kleene poset is genuine iff* $N_c(\Sigma) = \left\{\frac{1}{2}\right\}$ *and in this case one has that:*

(i) $a \leq \frac{1}{2}$ *for all* $a \in N_0(\Sigma)$*;*
(ii) $\frac{1}{2} \leq b$ *for all* $b \in N_1(\Sigma)$.

Proof Let h be another half element, i.e., such that $h = h'$. From $\frac{1}{2} = \left(\frac{1}{2}\right)'$ and $h = h'$ it follows that in particular $\frac{1}{2} \leq \left(\frac{1}{2}\right)'$ and $h' \leq h$ and so (K) implies that $\frac{1}{2} \leq h$. Conversely, the same identities in particular also lead to $h \leq h'$ and $\left(\frac{1}{2}\right)' \leq \frac{1}{2}$ and so (K) implies that $h \leq \frac{1}{2}$. For the proof of point (i), if $a \leq a'$ and $h' = h \leq h(= \frac{1}{2})$, applying the condition (K), we get $a \leq h$. The point (ii) can be proved in a similar way. □

In a Kleene poset it may happen that there is no half element, as Example 10 shows.

Example 10 In Fig. 3 it is depicted the Hasse diagram of the Kleene poset KP6 based on a six element poset Σ_6.

This is a poset which is not a lattice. Furthermore it contains no half element since there is no element $x \in \Sigma_6$ s.t. $x = x'$, that is $N_c(\Sigma_6) = \emptyset$.

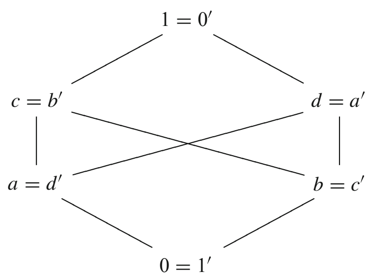
Let us note that for any pair of elements $x, y \in \Sigma_6$, conditions $x \leq x'$ and $y' \leq y$ always imply $x \leq y$ and so, according to Definition 7, it turns out to be a Kleene poset.

On the contrary the following is an example of a genuine Kleene lattice:

Example 11 With a slight modification of Fig. 3, we have drawn in Fig. 4 the Hasse diagram of the Kleene *distributive* lattice KL7 based on the seven element distributive lattice Σ_{K7} with a unique half element. Note that for any pair of elements $x, y \in \Sigma_{K7}$, conditions $x \leq x'$ and $y' \leq y$ always imply $x \leq y$ and so it is a Kleene lattice.

In Example 12 below we have a not distributive Kleene lattice with a unique half element.

Fig. 3 The six element Kleene poset KP6 with no half element



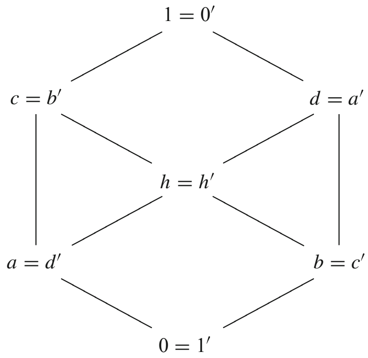


Fig. 4 The seven element distributive Kleene lattice KL7 with a unique half element

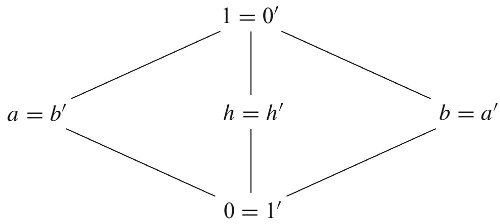


Fig. 5 The five element Hasse diagram of the not distributive Kleene lattice KL5 with a unique half element

Example 12 Let us consider the following Hasse diagram of the Kleene lattice KL5 based on the lattice Σ_5. This lattice is not distributive since $a \wedge (h \vee b) = a \neq 0 = (a \wedge h) \vee (a \wedge b)$ and has a unique half element $h = h'$ (Fig. 5).

Definition 13 A De Morgan operation $'$ on a poset Σ is said to be an *orthocomplementation* (also *standard negation*) if one, and hence both, of the following mutually equivalent properties is satisfied:

(oc-2a) For every $a \in \Sigma$ there exists in Σ the g.l.b. $a \wedge a'$ and it is $a \wedge a' = 0$ (noncontradiction);

(oc-2b) For every $a \in \Sigma$ there exists in Σ the l.u.b. $a \vee a'$ and it is $a \vee a' = 1$ (excluded middle);

Adopting the notation for partial algebras $s \overset{\omega}{=} t$ which means "if both sides s and t are defined then the two terms are equal," we can write the above definitions as follows:

$$\text{(oc-2a)} \quad \forall a \in \Sigma, \quad a \wedge a' \overset{\omega}{=} 0 \qquad \text{(noncontradiction)}$$

$$\text{(oc-2b)} \quad \forall a \in \Sigma, \quad a \vee a' \overset{\omega}{=} 1 \qquad \text{(excluded middle)}$$

In this case Σ is said to be an *orthoposet* .

Of course, if Σ is a De Morgan *lattice* the above two mutually equivalent properties reduce to the following:

$$\text{(ocL-2a,b)} \quad \forall a \in \Sigma, a \wedge a' = 0 \quad \text{(equivalently, } a \vee a' = 1\text{)}$$

and if they are satisfied we say that Σ is a (not distributive) *Boolean lattice* and in the case of distributivity of Σ we speak of *Boolean algebra* .

In the case of a De Morgan poset in general neither the *noncontradiction* nor the *excluded middle* laws for the De Morgan negation, both involving a property which must be satisfied for any element of Σ, are required to hold. But it may happen that only for some element from Σ condition (oc-2a), or equivalently (oc-2b), is verified. This allows one to make the following characterization:

- Let Σ be a De Morgan poset. An element $a \in \Sigma$ is said to be *complemented* iff its De Morgan negation $a' \in \Sigma$ is such that $a \wedge a' \overset{\omega}{=} 0$ (or, equivalently, is such that $a \vee a' \overset{\omega}{=} 1$).

The collection of all complemented elements on a De Morgan poset Σ is denoted by:

$$\Sigma_c := \{a \in \Sigma : a \wedge a' \overset{\omega}{=} 0 \text{ (equivalently } a \vee a' \overset{\omega}{=} 1)\}.$$

This set is not empty since $0, 1 \in \Sigma_c$. Moreover, if $a \in \Sigma_c$ then also $a' \in \Sigma_c$ (whose De Morgan complement is $(a')' = a$) and so the restriction of the De Morgan negation to Σ_c is in its turn a De Morgan negation $' : \Sigma_c \to \Sigma_c$. Hence, we have the structure $\langle \Sigma_c, \leq, ', 0, 1 \rangle$ (resp., $\langle \Sigma_c, \wedge, \vee, ', 0, 1 \rangle$) which is a sub De Morgan poset (resp., lattice) of the original De Morgan poset $\langle \Sigma, \leq, ', 0, 1 \rangle$ (resp., lattice $\langle \Sigma, \wedge, \vee, ', 0, 1 \rangle$).

Example 14 In the Kleene not distributive lattice KL5 of Example 12 the collection of complemented elements is $\Sigma_c(\text{KL5}) = \{0, a, a', 1\}$, which does not contain the half element h.

1.2 The Space of Orthogonal Pairs on De Morgan Structures

Following the paper [21] (and also [22]) the structure of De Morgan poset is the more natural one to introduce in an abstract context a notion of orthogonality between pairs of elements, according to the following definition.

Definition 15 Let $\langle \Sigma, \leq, ', 0, 1 \rangle$ be a De Morgan poset. Two elements $a, b \in \Sigma$ are *orthogonal*, and we write $a \perp b$, iff the following holds.

$$a \perp b \quad \text{iff} \quad a \leq b' \quad \text{(equivalently,} \quad b \leq a'). \tag{4}$$

For any pair of subsets $\mathscr{M}$ and $\mathscr{N}$ of the poset Σ we denote by $\mathscr{M} \perp \mathscr{N}$ the fact that $\forall m \in \mathscr{M}$ and $\forall n \in \mathscr{N}$ it is $m \perp n$. In the particular case of a singleton $\mathscr{M} = \{m\}$ we denote by $m \perp \mathscr{N}$ the fact that $\forall n \in \mathscr{N}$ it is $m \perp n$.

Proposition 16 *The just introduced orthogonality relation on the De Morgan poset Σ satisfies the following properties (see [21, 23]).*

$(og-0)$	$\forall a \in \Sigma,\ 0 \perp a$	*(0 full orthogonality)*
$(og-1)$	$\forall a \in \Sigma,\ \ a \perp a \ \ \textit{iff} \ \ a \in N_0(\Sigma)$	*(0-kernel irreflexive)*
$(og-2)$	$\forall a, b \in \Sigma,\ a \perp b \quad \textit{iff} \quad b \perp a$	*(symmetry)*
$(og-3)$	$\forall a, b \in \Sigma,\ a \leq b \textit{ and } b \perp c \quad \textit{imply} \quad a \perp c$	*(absorbtion)*

In the case of an orthoposet Σ condition (og-1) must be substituted by the following

$$(op-1) \quad \forall a \in \Sigma, \quad a \perp a \ \ \textit{iff} \ \ a = 0 \qquad \textit{(0-irreflexive)}$$

Proof The (og-0) is a trivial consequence of $0 \leq a'$ for any $a \in \Sigma$. Moreover, $a \perp a$ means $a \leq a'$, i.e., $a \in N_0(\Sigma)$, i.e., the (og-1). The (og-2) is trivial consequence of the fact that $a \leq b'$ iff $b = b'' \leq a'$. On the other hand, $a \leq b$ and $b \perp c$ means $a \leq b$ and $b \leq c'$ and so, for the transitivity of the order relation $a \leq c'$, i.e., $a \perp c$, that is (og-3). Finally, in the case of an orthoposet the two conditions $a \leq a'$ and $a \wedge a' = 0$ imply $a \wedge a \leq a \wedge a' = 0$, i.e., $a = a \wedge a = 0$, i.e., (op-1). On the other hand, if $a = 0$ then $a = 0 \leq 1 = a'$. □

1.2.1 Minimal (or Pre) BZ Posets of Orthopairs Induced from De Morgan Posets

Let Σ be a De Morgan poset. The collection of all orthopairs generated by Σ will be denoted by

$$\mathbb{A}(\Sigma) := \{(a_1, a_0) \in \Sigma \times \Sigma : a_1 \perp a_0\}. \tag{5}$$

(Sometimes, $\mathbb{A}(\Sigma)$ will also be written as $(\Sigma \times \Sigma)_\perp = (\Sigma^2)_\perp$.)

The following is trivial.

Proposition 17 *Let $\langle \Sigma, \leq, ', 0, 1 \rangle$ be a De Morgan poset. Then, the collection $\mathbb{A}(\Sigma)$ of all its orthopairs turns out to be a poset with respect to the following partial order relation:*

$$(a_1, a_0) \sqsubseteq (b_1, b_0) \quad \textit{iff} \quad a_1 \leq b_1 \ \textit{ and } \ b_0 \leq a_0 \tag{6}$$

The poset $\langle \mathbb{A}(\Sigma), \sqsubseteq \rangle$ is bounded by the least element $\mathbf{0} := (0, 1)$ and the greatest element $\mathbf{1} := (1, 0)$. For any (a_1, a_0) let us set $-(a_1, a_0) := (a_0, a_1)$ as the result of a unary operation whose properties will be discussed in the following.

The following subsets of $\mathbb{A}(\Sigma)$ will be very useful in the sequel

(0A) Since $\forall a \in \Sigma$, $0 \leq a'$, i.e., $0 \perp a$, we can define

$$\mathbb{A}_{oa}(\Sigma) := \{(0, a) \in \Sigma \times \Sigma : a \in \Sigma\} \subseteq \mathbb{A}(\Sigma). \tag{7}$$

(A0) Since $\forall a \in \Sigma$, $a \leq 0' = 1$, i.e., $a \perp 0$, we can define

$$\mathbb{A}_{ao}(\Sigma) := \{(a, 0) \in \Sigma \times \Sigma : a \in \Sigma\} \subseteq \mathbb{A}(\Sigma). \tag{8}$$

Let us note that $(a, 0) \in \mathbb{A}_{ao}(\Sigma)$ iff $-(a, 0) = (0, a) \in \mathbb{A}_{oa}(\Sigma)$, so the list of all the elements belonging to $\mathbb{A}_{oa}(\Sigma)$ allows one to immediately obtaining the corresponding list of elements from $\mathbb{A}_{ao}(\Sigma)$. We denote this fact by $\mathbb{A}_{ao}(\Sigma) = -\mathbb{A}_{oa}(\Sigma)$. Of course, $(0, 1) \in \mathbb{A}_{oa}(\Sigma)$ and $(1, 0) \in \mathbb{A}_{ao}(\Sigma)$ and so both of them are not empty. Moreover, $(0, 0) \in \mathbb{A}_{ao}(\Sigma) \cap \mathbb{A}_{oa}(\Sigma)$.

(AB) Making use of these we can define also

$$\mathbb{A}_{ab}(\Sigma) : = \mathbb{A}(\Sigma) \setminus (\mathbb{A}_{ao}(\Sigma) \cup \mathbb{A}_{oa}(\Sigma)) \tag{9a}$$

$$= \{(a_1, a_0) \in \Sigma \times \Sigma : (0 \neq) a_1 \perp a_0 (\neq 0)\} \tag{9b}$$

Let us note that $(a_1, a_0) \in \mathbb{A}_{ab}(\Sigma)$, with $a_1 \leq a_0'$, implies $(a_0, a_1) = -(a_1, a_0) \in \mathbb{A}_{ab}(\Sigma)$, since from $a_1 \leq a_0'$ it can be deduced that $a_0 = (a_0')' \leq a_1'$.

Of course, trivially,

$$\mathbb{A}(\Sigma) = \mathbb{A}_{ao}(\Sigma) \cup \mathbb{A}_{oa}(\Sigma) \cup \mathbb{A}_{ab}(\Sigma).$$

From (0A) and (A0) we obtain the following partial order chain in $\mathbb{A}(\Sigma)$:

$$\forall a \in \Sigma, \quad (0, 1) \sqsubseteq (0, a) \sqsubseteq (0, 0) \sqsubseteq (a, 0) \sqsubseteq (1, 0) \tag{10}$$

Furthermore, from (AB) we have the following two partial order chains in $\mathbb{A}(\Sigma)$:

$$\forall (a_1, a_0) \in \mathbb{A}_{ab}(\Sigma), \quad (0, 1) \sqsubseteq \{(a_1, a_0),\ (a_0, a_1)\} \sqsubseteq (1, 0) \tag{11}$$

These two situations can be compacted in the *partial* Hasse diagram of Fig. 6 relative to a generic possible element $a \in \Sigma$ (of course with $a \neq 0, 1$) and a possible single orthopair $(a_1, a_0) \in \mathbb{A}_{ab}(\Sigma)$, which must be present in each *complete* Hasse diagram of $\mathbb{A}(\Sigma)$ involving all possible (and different among them) orthopairs $(a_1, a_0) \in \mathbb{A}_{ab}(\Sigma)$.

In this section we study the algebraic poset structure shared by the collection of all orthopairs from a De Morgan poset.

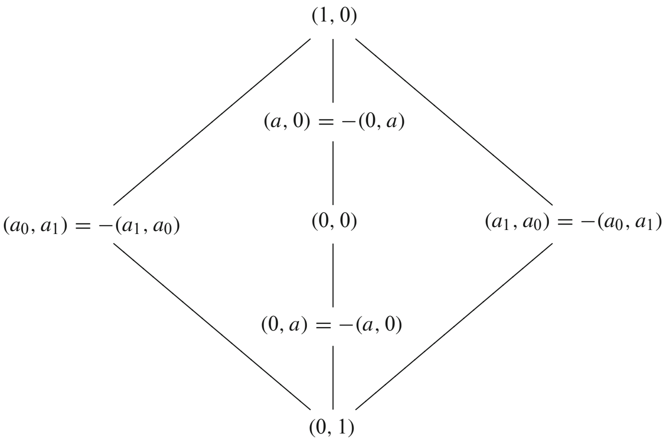


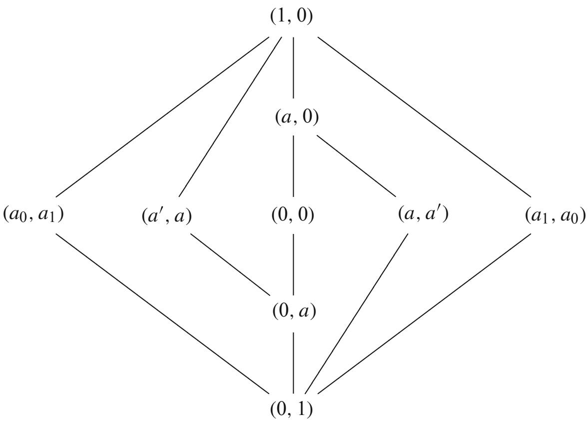
Fig. 6 Partial Hasse diagram that must be present in every complete Hasse diagram of $\mathbb{A}(\Sigma)$ if there exists $a \neq 0$ and at least an orthopair $(a_1, a_0) \in \mathbb{A}_{ab}(\Sigma)$

Lemma 18 *Let* $\langle \Sigma, \leq, ', 0, 1 \rangle$ *be a De Morgan poset. For any element* $a \in \Sigma$*, the pairs* $(a, a') \in \mathbb{A}(\Sigma)$ *and* $(a', a) \in \mathbb{A}(\Sigma)$*.*

Proof For any $a \in \Sigma$ we have that from condition (dM1) $a = (a')'$, which in particular satisfies the inequality $a \leq (a')'$, i.e., $a \perp a'$. On the other hand, from this last, by condition (og-2) we obtain that also $a' \perp a$. Thus, $\forall a \in \Sigma$, $(a, a'), (a', a) \in \mathbb{A}(\Sigma)$. □

Hence the partial Hasse diagram of Fig. 6 must be completed by Fig. 7 always under the existence of an element $a \neq 0$ and at least one orthopair $(a_1, a_0) \in \mathbb{A}_{ab}(\Sigma)$.

Fig. 7 Further partial Hasse diagram that must be present in every complete Hasse diagram of $\mathbb{A}(\Sigma)$



In Proposition 17 we have seen that the collection of all orthopairs from a De Morgan poset Σ has a structure of bounded poset $\langle \mathbb{A}(\Sigma), \sqsubseteq, \mathbf{0} = (0, 1), \mathbf{1} = (1, 0) \rangle$ with respect to the partial order relation:

$$(a_1, a_0) \sqsubseteq (b_1, b_0) \quad \text{iff} \quad a_1 \leq b_1 \ \text{ and } \ b_0 \leq a_0.$$

Then, the following can be proved.

Theorem 19 *Let* $\left\langle \Sigma, \leq, ', 0, 1 \right\rangle$ *be a De Morgan poset.*

This poset structure can be equipped with the following two mappings $\mathbb{A}(\Sigma) \to \mathbb{A}(\Sigma)$ *associating with any element* $(a_0, a_1) \in \mathbb{A}(\Sigma)$ *the following element in* $\mathbb{A}(\Sigma)$*, respectively:*

$$-(a_1, a_0) := (a_0, a_1) \tag{12a}$$

$$\sim (a_1, a_0) := (a_0, (a_0)') \tag{12b}$$

The mapping $- : \mathbb{A}(\Sigma) \to \mathbb{A}(\Sigma)$ *is such that it admits the half element* $\mathbf{1/2} = (0, 0) = -\mathbf{1/2}$*, moreover all the properties of a* De Morgan negation *on posets are true since for any orthopairs* $\alpha = (a_1, a_0)$ *and* $\beta = (b_1, b_0)$ *one has the following:*

(dM1) $\alpha = -(-\alpha)$ (double negation law)*;*
(dM2) $\alpha \sqsubseteq \beta$ *implies* $-\beta \sqsubseteq -\alpha$ (contraposition)*;*

The mapping $\sim: \mathbb{A}(\Sigma) \to \mathbb{A}(\Sigma)$ *satisfies the properties of a* minimal Brouwer negation *on posets true for any orthopairs* $\alpha = (a_1, a_0)$ *and* $\beta = (b_1, b_0)$*:*

(B1) $\alpha \sqsubseteq \sim (\sim \alpha)$ (weak double negation law)
(B2) $\alpha \sqsubseteq \beta$ *implies* $\sim \beta \sqsubseteq \sim \alpha$ (B-contraposition)

Further, the (strong) interconnection rule *is satisfied:*

(IR) *Let* $\alpha = (a_1, a_0) \in \mathbb{A}(\Sigma)$*, then* $- \sim \alpha = \sim\sim \alpha$*.*

Note that the orthopair $(a_1, a_0) \in \mathbb{A}(\Sigma)$ *is a half element* ($(a_1, a_0) = -(a_1, a_0)$) *iff* $a_1 = a_0$*, that is the half elements are of the form* (a, a) *with* $a \leq a'$*. Denoting by* $\mathbb{A}_h(\Sigma)$ *the collection of all* half elements *from* $\mathbb{A}(\Sigma)$ *we have that*

$$\mathbb{A}_h(\Sigma) := \{(a, a) \in \Sigma \times \Sigma : a \in N_0(\Sigma)\} \subseteq \mathbb{A}(\Sigma).$$

In particular we have already seen that $\mathbf{1/2} = (0, 0)$ *is one of the possible half elements:* $(0, 0) \in \mathbb{A}_h(\Sigma)$.

Proof As to the De Morgan properties of $-$, we have that $\mathbf{1/2} = (0, 0) = -(0, 0) = -\mathbf{1/2}$. Furthermore, trivially $-(-(a_1, a_0)) = (a_1, a_0)$, i.e., (dM1).

Condition $(a_1, a_0) \sqsubseteq (b_1, b_0)$, i.e., $a_1 \leq b_1$ and $b_0 \leq a_0$, leads to $(b_0, b_1) \sqsubseteq (a_0, a_1)$, i.e., $-(b_1, b_0) \sqsubseteq -(a_1, a_0)$, i.e., (dM2).

As to the Brouwer properties of $\sim$, first of all we have that $\sim (\sim (a_1, a_0)) = \sim (a_0, (a_0)') = \left((a_0)', ((a_0)')'\right) = ((a_0)', a_0)$ and from $a_1 \leq (a_0)'$ and $a_0 \leq a_0$ it follows that $(a_1, a_0) \sqsubseteq \sim (\sim (a_1, a_0))$, which is the (B1). Moreover, if $(a_1, a_0) \sqsubseteq (b_1, b_0)$ then in particular $b_0 \leq a_0$, which for the contraposition law (dM2) implies $(a_0)' \leq (b_0)'$. So conditions $b_0 \leq a_0$ and $(a_0)' \leq (b_0)'$ mean that $\sim (b_1, b_0) = (b_0, (b_0)') \sqsubseteq (a_0, (a_0)') = \sim (a_1, a_0)$, i.e., the (B2).

For the interconnection rule (IR), let us observe that $-(\sim (a_1, a_0)) = -\big(a_0, (a_0)'\big) = \big((a_0)', a_0\big)$ and $\sim (\sim (a_1, a_0)) = \sim \big(a_0, (a_0)'\big) = \big((a_0)', a_0\big)$. □

In this way, and summarizing the results of Theorem 19, we have the following.

(mBZ) Starting from a De Morgan poset $\langle \Sigma, \leq, ', 0, 1\rangle$ one obtains a structure $\langle \mathbb{A}(\Sigma), \sqsubseteq, -, \sim, \mathbf{0}, \mathbf{1}\rangle$ of *minimal Brouwer Zadeh (mBZ) poset* , since

- (Z) the operation $-$ satisfies both the conditions (dM1) and (dM2) for a De Morgan (Zadeh) negation;
- (mB) the operation $\sim$ satisfies only the two conditions (B1) and (B2) of an intuitionistic (Brouwer) negation, but not the condition (B3) required by intuitionistic logic. Let us recall that a negation satisfying the only two conditions (B1) and (B2) is called "minimal" by Dunn in [38] whereas in [23] it is called "pre Brouwer".

Furthermore the mBZ poset $\mathbb{A}(\Sigma)$ is genuine since the orthopair $\mathbf{1/2} := (0, 0) \in \mathbb{A}(\Sigma)$ is one of the possible half elements of this poset, $-\mathbf{1/2} = \mathbf{1/2}$, with the further identity $\sim \mathbf{1/2} = (0, 1) = \mathbf{0}$.
Finally, the two negations are interconnected by condition (IR), necessary in order to have a BZ structure.

Let us note that relative to the De Morgan (Zadeh) negation $-$ one has that $\mathbf{1/2} \sqcap -\mathbf{1/2} = \mathbf{1/2} \neq \mathbf{0}$ and $\mathbf{1/2} \sqcup -\mathbf{1/2} = \mathbf{1/2} \neq \mathbf{1}$, and so both the algebraic versions of noncontradiction and excluded middle principles in order to have a *Boolean negation* on orthopairs do not hold. On the other hand, $\mathbf{1/2} \sqcap \sim \mathbf{1/2} = \mathbf{0}$. But we have seen that for any $a \in N_0(\Sigma)$, i.e., such that $a \leq a'$, the orthopair (a, a) is a half element ($-(a, a) = (a, a)$) for which $\sim (a, a) = (a, a')$ ($a \leq (a')' = a$) and so for $a \neq 0$ we have $(a, a) \sqcap \sim (a, a) = (a, a \vee a') \neq (0, 1) = \mathbf{0}$.

Summarizing,

(N1) the operation $-$ *is not* an algebraic realization of a *standard Boolean negation*. Moreover,

(N2) in general the operation $\sim$ *is not* an algebraic realization of the *intuitionistic Brouwer negation* (it is sufficient that $N_0(\Sigma)$ contains an element $a \neq 0$).

As to the point (N2), the noncontradiction principle $\forall \alpha, \alpha \sqcap \sim \alpha = \mathbf{0}$, usually denoted as (B3), combined with the previous ones (B1) and (B2) completely characterizes the intuitionistic (Brouwer) negation as can be verified in the book of Heyting [44] where, after the claim that "the main differences between classical and intuitionistic logics are in the properties of the negation," it is listed the accepted principles of this negation whose formulation in terms of the algebraic model, as shown in [19], are just the conditions (B1), (B2), and (B3). In the same book the principles that this negation rejects are also listed, in particular the dual of the (B1), which would lead to the strong version of the double negation law, and also the excluded middle principle $\forall\alpha,\ \alpha \sqcup \sim \alpha = \mathbf{1}$ is rejected.

We have seen that the mBZ poset $\mathbb{A}(\Sigma)$ induced from a De Morgan poset Σ always admits at least a half element, the orthopair $\mathbf{1/2} = (0, 0)$. But in general this is not the unique half element since any orthopair (a, a), for a running in $N_0(\Sigma)$, is a half element of $\mathbb{A}(\Sigma)$.

Example 20 In the Hasse diagram of the Kleene poset KP6 treated in Example 10 the corresponding structure of orthopairs $\mathbb{A}$(KP6) is decomposed into the following three parts:

$$\mathbb{A}_{oa}(KP6) = \big\{(0, 0), (0, a), (0, b), (0, c = b'), (0, d = a'), (0, 1)\big\} = -\mathbb{A}_{ao}(KP6)$$

$$\mathbb{A}_{ab}(KP6) = \{(a, a), (b, b), (a, a'), (a', a), (b, b'), (b', b), (a, b), (b, a)\}$$

The Hasse diagram of the induced minimal BZ poset structure $\mathbb{A}$(KP6) of all orthopairs is drawn in Fig. 8, from which one realizes that in addition to the standard

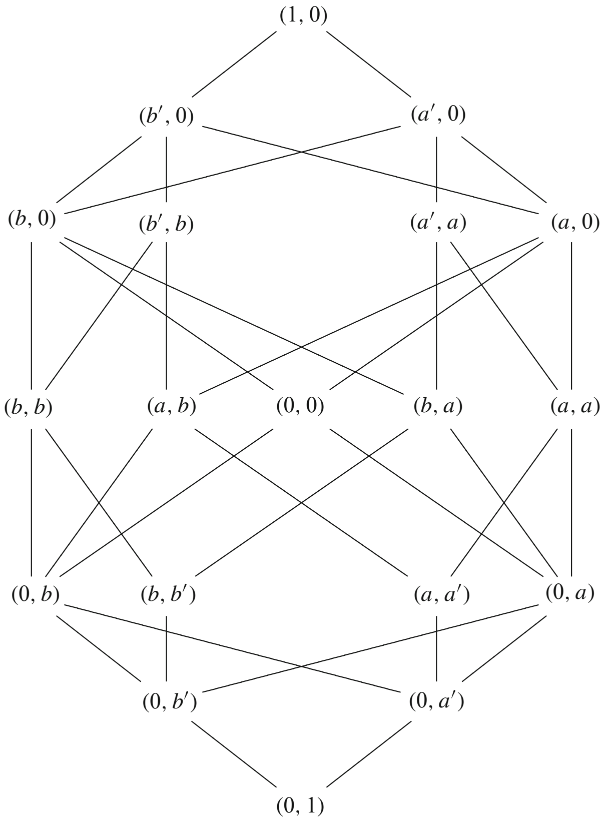


Fig. 8 Hasse diagram of the mBZ structure $\mathbb{A}$(KP6) with three half elements

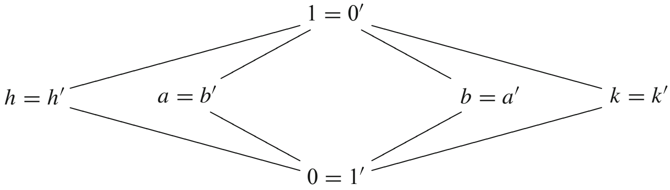


Fig. 9 The six element De Morgan lattice dML6 with two half elements h and k

half element $(0, 0)$ it contains two other half elements (a, a) and (b, b), although KP6 has no half element.

Theorem 19 assures that the mapping $\sim$ defined by Eq. (12b) is a minimal Brouwer negation which in general does not satisfy the condition (B3). That the negation $\sim$ of $\mathbb{A}$(KP6) is a minimal Brouwer negation which is not full Brouwer can be directly proved for instance considering that $(0, b) \sqcap \sim (0, b) = (0, b) \sqcap (b, b') = (0, b') \neq (0, 1)$, i.e., in the present case the condition (B3) is not satisfied.

This poset is not a lattice since, for instance, the upper bounds of the two orthopairs $\{(a, 0), (b, 0)\}$ are $\left\{(a', 0), (b', 0), (1, 0)\right\}$ which do not admit the l.u.b. Analogously we have that the lower bounds of the two orthopairs $\{(0, a), (0, b)\}$ are $\left\{(0, a'), (0, b'), (0, 1)\right\}$ which do not admit the g.l.b.

Example 21 Let us consider the De Morgan (not distributive) lattice dML6 of the Fig. 9 with two half elements h, k. The corresponding structure of orthopairs $\mathbb{A}$(dML6) is decomposed into the following three parts:

$$\mathbb{A}_{oa}(\text{dML6}) = \left\{(0, 0), (0, a), (0, b = a'), (0, h), (0, k), (0, 1)\right\} = -\mathbb{A}_{ao}(\text{dML6})$$

$$\mathbb{A}_{ab}(\text{dML6}) = \left\{(h, h), (k, k), (a, a'), (a' = b, a = b')\right\}$$

The Hasse diagram for the minimal BZ lattice $\mathbb{A}$(dML6) is drawn in Fig. 10.

Note that $\mathbb{A}$(dML6) is a lattice which is not distributive since for instance $(h, 0) \sqcap [(a', 0) \sqcup (a, 0)] = (h, 0) \neq (0, 0) = [(h, 0) \sqcap (a', 0)] \sqcup [(h, 0) \sqcap (a, 0)]$. Moreover it does not satisfy the condition (KL) since for instance $(h, h) \sqcap -(h, h) = (h, h)$ is incomparable with $(k, k) \sqcup -(k, k) = (k, k)$. Also the condition (B3) is not satisfied since for instance $(h, h) \sqcap \sim (h, h) = (h, h) \neq (0, 1)$.

Definition 22 Owing to condition (B1) it is possible to define an orthopair $\alpha = (a_1, a_0)$ as *exact* (or *B-crisp*) with respect to the quasi Brouwer negation if and only if (iff) $(a_1, a_0) = \sim\sim (a_1, a_0)$ and since $\sim\sim (a_1, a_0) = ((a_0)', a_0)$ this happens iff $a_0 = (a_1)'$, that is iff they are of the form $\alpha = (a_1, (a_1)')$ for $a_1 \in \Sigma$, i.e., setting $a := a_1$ iff $\alpha = (a, a')$ for $a \in \Sigma$. The collection of all orthopairs which are Brouwer exact (or B-crisp) is denoted as $\mathbb{A}_e(\Sigma)$ and so:

$$\mathbb{A}_e(\Sigma) := \left\{(a_1, a_0) \in \mathbb{A}(\Sigma) : (a_1, a_0) = \sim\sim (a_1, a_0) = ((a_0)', a_0)\right\} \quad (13a)$$

$$= \left\{(a, a') \in \mathbb{A}(\Sigma) : a \in \Sigma\right\}. \quad (13b)$$

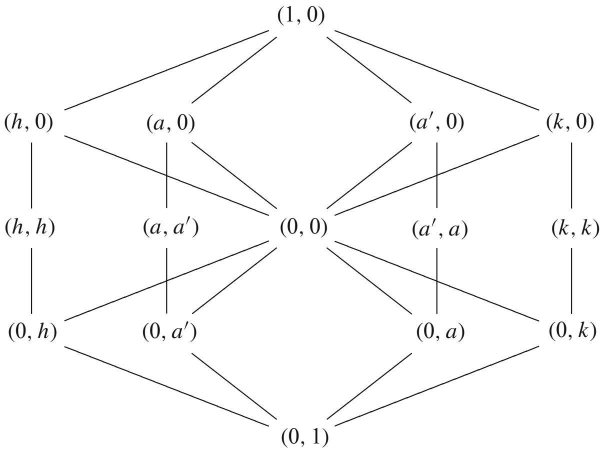


Fig. 10 Hasse diagram of $\mathbb{A}$(dML6) with the three half elements (h, h), $(0, 0)$, and (k, k)

Once observed that for every $a \in \Sigma$ the inequality $a \leq (a')'$ holds, i.e., $\forall a \in \Sigma$, $(a, a') \in \mathbb{A}(\Sigma)$, the above collection of Brouwer exact orthopairs $\mathbb{A}_e(\Sigma)$ must not be confused with the collection:

$$\mathbb{A}(\Sigma_c) := \left\{(a_1, (a_1)') \in \mathbb{A}(\Sigma) : \exists a_1 \wedge (a_1)' = 0 \text{ and } \exists a_1 \vee (a_1)' = 1\right\} \tag{14a}$$

$$= \left\{(c, c') \in \mathbb{A}(\Sigma) : c \in \Sigma_c\right\}. \tag{14b}$$

Comparing (13b) with (14b) we have the inclusions:

$$\mathbb{A}(\Sigma_c) \subseteq \mathbb{A}_e(\Sigma) \subseteq \mathbb{A}(\Sigma).$$

Example 23 In Example 11 of the seven element Kleene distributive lattice KL7 based on the lattice Σ_{K7} we have that

$$\mathbb{A}_e(\Sigma_{K7}) = \left\{(0, 1), (a, a'), (a', a), (b, b'), (b', b), (h, h), (1, 0)\right\}$$

which is the distributive lattice (Fig. 11).

On the contrary, $\mathbb{A}((\Sigma_{K7})_c) = \{(0, 1), (1, 0)\}$ is a two elements Boolean algebra.

Example 24 Let us consider the six element Boolean algebra BA6 drawn in Fig. 12.

In this case we have that

$$\mathbb{A}_e(\text{BA6}) = \mathbb{A}((\text{BA6})_c) = \left\{(0, 1), (a, a'), (a', a), (b, b'), (b', b), (1, 0)\right\}$$

The latter two are represented by the six element Boolean algebra whose Hasse diagram is given in Fig. 13.

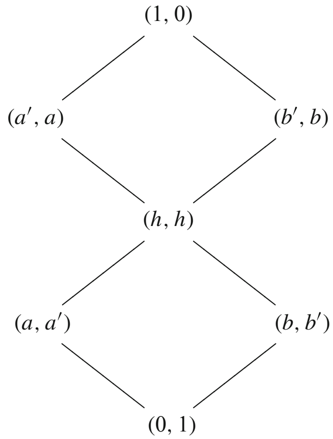


Fig. 11 Hasse diagram of the distributive lattice $\mathbb{A}_e(\Sigma_{K7})$

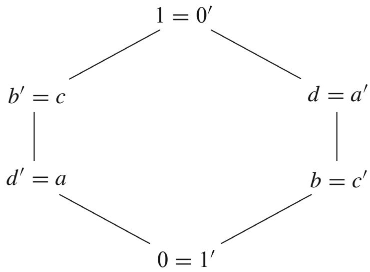


Fig. 12 Hasse diagram of the six element Boolean algebra BA6

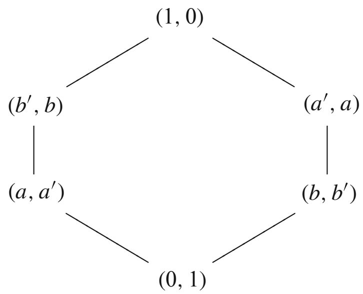


Fig. 13 Hasse diagram of the two Boolean algebras $\mathbb{A}_e(\text{BA6}) = \mathbb{A}((\text{BA6})_c)$

Note that $x \longleftrightarrow (x, x')$ defines an isomorphism between the Boolean algebras BA6 and $\mathbb{A}$(BA6).

Rough Approximation Spaces Induced from Minimal BZ Posets of Orthopairs

In the minimal BZ poset $\mathbb{A}(\Sigma)$ of all orthopairs induced from the De Morgan poset Σ, the *interior* and the *closure* of any orthopair $(a_1, a_0) \in \mathbb{A}(\Sigma)$ are defined in the following way:

$$I\,(a_1, a_0) :=\sim -(a_1, a_0) = \big(a_1, (a_1)'\big) \tag{15a}$$

$$C\,(a_1, a_0) := - \sim (a_1, a_0) = \big((a_0)', a_0\big) \tag{15b}$$

Trivially the following chain of inclusions with respect to the partial order $\sqsubseteq$ of orthopairs holds, corresponding to the satisfaction of the following roughness coherence condition:

$$\text{(RC1)} \qquad \forall (a_1, a_0) \in \mathbb{A}(\Sigma),\ \ I(a_1, a_0) \sqsubseteq (a_1, a_0) \sqsubseteq C(a_1, a_0) \tag{16}$$

That is the interior (resp., closure) orthopair $I(a_1, a_0)$ (resp., $C(a_1, a_0)$) is an *inner (resp., outer) approximation* of the given orthopair (a_1, a_0).

Proposition 25 *Let $\mathbb{A}(\Sigma)$ be the minimal BZ poset of all orthopairs from Σ. Then, the corresponding set of open elements $\mathscr{O}(\mathbb{A}(\Sigma)) := \{(a_1, a_0) \in \mathbb{A}(\Sigma) : I(a_1, a_0) = (a_1, a_0)\}$ and closed elements $\mathscr{C}(\mathbb{A}(\Sigma)) := \{(a_1, a_0) \in \mathbb{A}(\Sigma) : C(a_1, a_0) = (a_1, a_0)\}$ coincide and are equal to the collection of all exact (or B-crisp) orthopairs introduced by Eq.* (22) *of Definition 22:*

$$\mathscr{O}(\mathbb{A}(\Sigma)) = \mathscr{C}(\mathbb{A}(\Sigma)) = \mathbb{A}_e(\Sigma) = \big\{(a, a') : a \in \Sigma\big\}$$

On $\mathbb{A}_e(\Sigma)$ the two negations coalesce in a unique negation $\forall (a, a') \in \mathbb{A}_e(\Sigma)$, $-(a, a') \;=\sim\; (a, a') = (a', a)$, denoted by $-$, and the structure $\langle \mathbb{A}_e(\Sigma), \sqsubseteq, -, (0, 1), (1, 0)\rangle$ is a De Morgan bounded poset. Furthermore, the mapping

$$\varphi : \langle \mathbb{A}_e(\Sigma), \sqsubseteq, -, (0, 1), (1, 0)\rangle \to \big\langle \Sigma, \leq, {}', 0, 1\big\rangle, \quad (a, a') \to \varphi(a, a') := a$$

is a bijection preserving the De Morgan structures in the sense that

(Is1) $(a, a') \sqsubseteq (b, b')$ *iff* $\varphi(a, a') \leq \varphi(b, b')$ *iff* $a \leq b$

(Is2) $\varphi(-(a, a')) = a'$

(Is3) $\varphi(0, 1) = 0$ *and* $\varphi(1, 0) = 1$.

This De Morgan poset isomorphism between $\mathbb{A}_e(\Sigma)$ and Σ, also denoted by $(a, a') \longleftrightarrow a$, will be extended to the Cartesian products $\mathbb{A}_e(\Sigma) \times \mathbb{A}_e(\Sigma)$ and $\Sigma \times \Sigma$ by the one-to-one correspondence denoted simply as $((a, a'), (b, b')) \longleftrightarrow (a, b)$.

The *exterior* of an orthopair $(a_1, a_0) \in \mathbb{A}(\Sigma)$ is defined as

$$E(a_1, a_0) := -C(a_1, a_0) = (a_0, (a_0)') \tag{17}$$

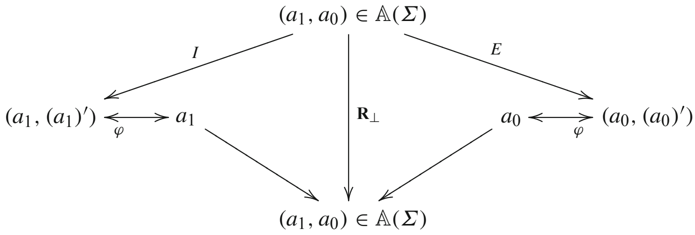


Fig. 14 Ortho-rough approximation of the orthopair $(a_1, a_0) \in \mathbb{A}(\Sigma)$

Trivially, $(a_1, a_0) \sqsubseteq -E(a_1, a_0) = ((a_0)', a_0)$, that is

$$\forall (a_1, a_0) \in \mathbb{A}(\Sigma), \quad (a_1, a_0) \perp E(a_1, a_0) \tag{18}$$

The ortho-rough approximation of any orthopair $(a_1, a_0) \in \mathbb{A}(\Sigma)$ is

$$\mathbf{R}_\perp(a_1, a_0) := (I(a_1, a_0), E(a_1, a_0)) = ((a_1, (a_1)'), (a_0, (a_0)') \tag{19}$$

whose diagrammatic representation is given by Fig. 14.

Hence, the ortho-rough approximation of (a_1, a_0) is identifiable with the pair itself, $\mathbf{R}_\perp(a_1, a_0) = ((a_1, (a_1)'), (a_0, (a_0)') \longleftrightarrow (a_1, a_0)$. Therefore, this approximation does not bring any knowledge increase.

1.2.2 Minimal (or Pre) BZ^{dM} Lattices of Orthopairs Induced from De Morgan Lattices

Let us now investigate what happens when the starting structure is a De Morgan lattice.

Theorem 26 *Let* $\langle \Sigma, \wedge, \vee, ', 0, 1 \rangle$ *be a De Morgan lattice. Then, the collection* $\mathbb{A}(\Sigma)$ *of all its orthopairs is a lattice with respect to the following meet and join operations:*

$$(a_1, a_0) \sqcap (b_1, b_0) := (a_1 \wedge b_1, a_0 \vee b_0) \tag{20a}$$

$$(a_1, a_0) \sqcup (b_1, b_0) := (a_1 \vee b_1, a_0 \wedge b_0) \tag{20b}$$

The partial order induced from these operations according to

$$(a_1, a_0) \sqsubseteq (b_1, b_0) \quad \textit{iff} \quad (a_1, a_0) = (a_1, a_0) \sqcap (b_1, b_0) \tag{21a}$$

$$\textit{iff} \quad (b_1, b_0) = (a_1, a_0) \sqcup (b_1, b_0) \tag{21b}$$

is just the one of Eq. (6)*: iff* $a_1 \leq b_1$ *and* $b_0 \leq a_0$.

The unary operations of De Morgan and minimal Brouwer negations are always the ones defined by Eq. (12)*:*

$$-(a_1, a_0) := (a_0, a_1)$$
$$\sim (a_1, a_0) := (a_0, (a_0)')$$

Therefore, we have a lattice structure $\langle \mathbb{A}(\Sigma), \sqcap, \sqcup, -, \sim, \boldsymbol{0}, \boldsymbol{1}\rangle$ *based on the collection* $\mathbb{A}(\Sigma)$ *of all orthopairs from the De Morgan lattice* Σ *whose operation* $-$ *satisfies the conditions (dM1) and (dM2) of a De Morgan negation and whose operation* $\sim$ *satisfies the conditions (B1) and (B2) of a minimal Brouwer negation . The two negations are linked by the interconnection rule:*

$$\text{(IR)} \quad - \sim (a_1, a_0) = \sim\sim (a_1, a_0)$$

Moreover, the following laws for the Brouwer negation turn out to be mutually equivalent among them:

$$\text{(B2)} \quad (a_1, a_0) \sqsubseteq (b_1, b_0) \ \textit{implies} \ \sim (b_1, b_0) \sqsubseteq \sim (a_1, a_0),$$
$$\text{(B2a)} \quad \sim ((a_1, a_0) \sqcup (b_1, b_0)) = \sim (a_1, a_0) \sqcap \sim (b_1, b_0)\,,$$
$$\text{(B2b)} \quad \sim ((a_1, a_0) \sqcap (b_1, b_0)) = \sim (a_1, a_0) \sqcup \sim (b_1, b_0)\,.$$

In other words $\mathbb{A}(\Sigma)$ *is a* minimal BZ lattice *whose Brouwer negation satisfies both the De Morgan laws, i.e., a minimal BZ*^{dM} *lattice.*

Recall that the half elements are of the form (a, a) *with* $a \leq a'$*. Precisely,*

$$\textit{Let } a \leq a' \ \ (a \in N_0(\Sigma)) \ \ \textit{then} \ \ -(a, a) = (a, a) \ \ \textit{and} \ \ \sim (a, a) = (a, a').$$

In particular $(0, 0)$ *is a half element such that* $-(0, 0) = (0, 0)$ *and* $\sim (0, 0) = (0, 1)$*. Furthermore, if h is a half element,* $h = h'$*, then* $-(h, h) = \sim (h, h) = (h, h)$*.*

Proof of (20a) Let us show that if $(a_1, a_0), (b_1, b_0) \in \mathbb{A}(\Sigma)$ then $(a_1 \wedge b_1)$ is orthogonal to $(a_0 \vee b_0)$, i.e., $((a_1 \wedge b_1), (a_0 \vee b_0)) \in \mathbb{A}(\Sigma)$. From the hypothesis we get $a_1 \leq a_0'$ and $b_1 \leq b_0'$ and so $a_1 \wedge b_1 \leq a_0' \wedge b_0' = (a_0 \vee b_0)'$. Thus $(a_1 \wedge b_1, a_0 \vee b_0) \in \mathbb{A}(\Sigma)$.

Let us show that $(a_1 \wedge b_1, a_0 \vee b_0)$ is a lower bound of the pair $\{(a_1, a_0), (b_1, b_0)\}$. From $a_1 \leq a_1 \wedge b_1$ and $a_0 \leq a_0 \vee b_0$ it follows that $(a_1 \wedge b_1, a_0 \vee b_0) \sqsubseteq (a_1, a_0)$. In a similar way one obtains that $(a_1 \wedge b_1, a_0 \vee b_0) \sqsubseteq (b_1, b_0)$.

Let $(x_1, x_0) \in \mathbb{A}(\Sigma)$ be a lower bound of the pair $(a_1, a_0)\,, (b_1, b_0) \in \mathbb{A}(\Sigma)$, i.e., $x_1 \leq \{a_1, b_1\}$ and $\{a_0, b_0\} \leq x_0$. From these it follows that $x_1 \leq a_1 \wedge b_1$ and $a_0 \vee b_0 \leq x_0$ and so $(x_1, x_0) \sqsubseteq (a_1 \wedge b_1, a_0 \vee b_0)$, i.e., $(a_1 \wedge b_1, a_0 \vee b_0) = (a_1, a_0) \sqcap (b_1, b_0)$. The $\sqcup$ case is similar. □

Proof of (21a) $(a_1, a_0) \sqcap (b_1, b_0) = (a_1 \wedge b_1, a_0 \vee b_0) = (a_1, a_0)$ iff $a_1 = a_1 \wedge b_1$ and $a_0 = a_0 \vee b_0$ iff $a_1 \leq b_1$ and $b_0 \leq a_0$ iff $(a_1, a_0) \sqsubseteq (b_1, b_0)$. □

Proof of (B2) Trivially, let $(a_1, a_0), (b_1, b_0) \in \mathbb{A}(\Sigma)$, i.e., $a_1 \leq a_0'$ and from $b_1 \leq b_0'$ it follows $b_0 \leq b_1'$. Since, $(a_1, a_0) \sqsubseteq (b_1, b_0)$ means in particular that $b_0 \leq a_0$, with this last equivalent to $a_0' \leq b_0'$, we get that $(b_0, b_0') \sqsubseteq (a_0, a_0')$ and so from $\sim (b_1, b_0) = (b_0, b_0')$ and $\sim (a_1, a_0) = (a_0, a_0')$ we conclude that $\sim (b_1, b_0) \sqsubseteq \sim (a_1, a_0)$. □

Proof of (B2b)

$$\sim ((a_1, a_0) \sqcap (b_1, b_0)) = \sim (a_1 \wedge b_1, a_0 \vee b_0) =$$
$$= \big(a_0 \vee b_0, (a_0 \vee b_0)'\big) = \big(a_0 \vee b_0, (a_0)' \wedge (b_0)'\big) =$$
$$= \big(a_0, (a_0)'\big) \sqcup \big(b_0, (b_0)'\big) = \sim (a_1, a_0) \sqcup (b_1, b_0)\,.$$

The proof of (B2a) is similar and the equivalence of (B2), (B2a), and (B2b) is trivial. □

Some *negative* results of the structure of minimal BZ^{dM} lattices of orthopairs generated by De Morgan lattices are collected in the following theorem.

Theorem 27 *Let $\mathbb{A}(\Sigma)$ be the minimal BZ^{dM} lattice of orthopairs generated by the De Morgan lattice Σ. Then,*

(NG1) Condition (K) characterizing the Kleene complementation in general does not hold. Indeed let $\alpha = (a_1, a_0)$, $\beta = (b_1, b_0) \in \mathbb{A}(\Sigma)$, then from $\alpha \sqcap -\alpha = (a_1 \wedge a_0, a_1 \vee a_0)$ and $\beta \sqcup -\beta = (b_1 \vee b_0, b_1 \wedge b_0)$ one has that the condition inside $\mathbb{A}(\Sigma)$, "(K) $\alpha \sqcap -\alpha \sqsubseteq \beta \sqcup -\beta$," assumes the form:

$$\text{(K)} \qquad (a_1 \wedge a_0, a_1 \vee a_0) \sqsubseteq (b_1 \vee b_0, b_1 \wedge b_0)$$

This is satisfied under the two conditions "$a_1 \wedge a_0 \leq b_1 \vee b_0$ and $b_1 \wedge b_0 \leq a_1 \vee a_0$," which in general do not hold since from the orthogonality condition $a_0 \leq a_1'$ we can only state that $0 \leq a_1 \wedge a_0 \leq a_1 \wedge a_1' \neq 0$.

(NG2) Noncontradiction principle (oc-2a) and excluded middle principle (oc-2b) characterizing the orthocomplementation of an orthoposet in general are not satisfied. Indeed, for any element $\alpha = (a_1, a_0) \in \mathbb{A}(\Sigma)$ one has that

$$\text{(woc-2a)} \quad (a_1, a_0) \sqcap - (a_1, a_0) = (a_1 \wedge a_0, a_1 \vee a_0) \neq (0, 1)$$
$$\text{(woc-2b)} \quad (a_1, a_0) \sqcup - (a_1, a_0) = (a_1 \vee a_0, a_1 \wedge a_0) \neq (1, 0)$$

(NG3) Also the condition (B3) of the Brouwer complementation in general does not hold but the following weaker form is satisfied:

$$\text{(wB3)} \quad (a_1, a_0) \sqcap \sim (a_1, a_0) = \big(a_1 \wedge a_0, a_0 \vee (a_0)'\big)\,.$$

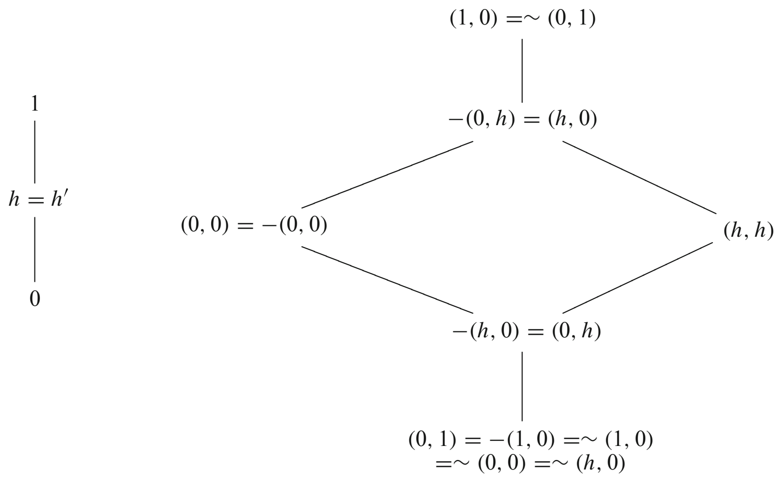


Fig. 15 The three valued totally ordered Kleene distributive lattice MVL3 (at the left side) with corresponding minimal BZ^{dM} distributive lattice $\mathbb{A}$(MVL3) (at the right side), where $(h,h) = -(h,h) = \sim (h,h) = \sim (0,h)$

(NG4) *Also in the case of a Kleene lattice Σ (and so with a unique half element) the orthopair **1/2** =* $(0,0)$ *is a half element which in general, as in the case of a Kleene poset, is not unique (see the Example 28 of the Kleene lattice MVL3).*

Example 28 Let us consider the totally ordered *three valued* Kleene lattice MVL3 of the Fig. 15, which can be considered as a part of the *three valued logic of Łukasiewicz*: "A possible step beyond the simple case of two-valued logic is the introduction of a third, "intermediate" or "neutral" or "indeterminate" truth value h. This step was first taken by J. Łukasiewicz [...] with a paper of 1920" [62, p. 22].

The three valued totally ordered Kleene lattice MVL3 is such that we can select the following three subsets of $\mathbb{A}$(MVL3):

$$\mathbb{A}_{oa}(\text{MVL3}) = \{(0,0), (0,h), (0,1)\} = -\mathbb{A}_{ao}(\text{MVL3})$$
$$\mathbb{A}_{ab}(\text{MVL3}) = \{(h,h)\}$$

Once constructed the lattice $\mathbb{A}$(MVL3) depicted at the right side of Fig. 15 we can say that

- The totally ordered three valued Kleene distributive lattice MVL3 contains the unique half element h (intermediate, neutral, indeterminate truth value).
- The minimal BZ^{dM} distributive lattice $\mathbb{A}$(MVL3) contains two half elements $(0,0)$ and (h,h) (and so it is not minimal Brouwer Kleene, i.e., $\neg(\text{mBK}^{dM})$).
- The structure is minimal since condition (B3) for the Brouwer negation does not hold. For instance $(0,h) \sqcap \sim (0,h) = (0,h) \neq (0,1)$.

The next proposition improves on Theorem 26.

Proposition 29 *Let Σ be a* Boolean lattice *(that is the De Morgan negation is such that condition $\forall a \in \Sigma$, $a \wedge a' = 0$ and its dual $a \vee a' = 1$ both hold, defining in this way a* Boolean *orthocomplementation on a not necessary distributive lattice). Then, under these conditions:*

(BL-K) *The (KL) condition holds for any element $(a_1, a_0) \in \mathbb{A}(\Sigma)$ since in the case of a Boolean lattice it assumes the form $(0, a_1 \vee a_0) \sqsubseteq (b_1 \vee b_0, 0)$ with respect to which the two relations $0 \leq b_1 \vee b_0$ and $0 \leq a_1 \vee a_0$ hold.*

(BL-H) *It contains a unique half element $(0, 0)$. Indeed, conditions (a, a) and $a \leq a'$ imply $a = a \wedge a' = 0$.*

(BL-B3) *The (B3) condition holds for any element $(a_1, a_0) \in \mathbb{A}(\Sigma)$ since in the case of a Boolean lattice it assumes the form $(a_1, a_0) \sqcap \sim (a_1, a_0) = (0, 1)$. But from the orthogonality condition $a_0 \leq a_1'$ we have that $0 \leq a_1 \wedge a_0 \leq a_1 \wedge a_1' \leq 0$, and from the Boolean lattice condition we have also that $a_0 \vee a_0' = 1$.*

As a consequence of these results, if Σ is a Boolean lattice different from the trivial *one, $\Sigma \neq \{0, 1\}$, then $\mathbb{A}(\Sigma)$ is a Brouwer Kleene (BK^{dM}) lattice since the negation $-$ is a Kleene complementation satisfying (dM1), (dM2), and (KL), and the $\sim$ is a real Brouwer negation satisfying (B1), (B2)–(B2a,b), and (B3), the two are linked by the interconnection rule (IR).*

Furthermore,

(BA) *If Σ is a Boolean algebra (that is a distributive Boolean lattice) then the BK^{dM} lattice $\mathbb{A}(\Sigma)$ is* distributive *(its Kleene negation $-$ is not Boolean in general). Summarizing, $\mathbb{A}(\Sigma)$ is a BK^{dM} algebra (distributive lattice).*

Proof The proofs of points (BL-K), (BL-oc2), (BL-B3), and (BL-H) are quite simple. Let us prove the only condition (BA). Indeed, $(a, b) \sqcap ((c, d) \sqcup (e, f)) = (a, b) \sqcap (c \vee e, d \wedge f) = (a \wedge (c \vee e), b \vee (d \wedge f))$. On the other hand, $((a, b) \sqcap (c, d)) \sqcup ((a, b) \sqcap (e, f)) = (a \wedge c, b \vee d) \sqcup (a \wedge e, b \vee f) = ((a \wedge c) \vee (a \wedge e), (b \vee d) \wedge (b \vee f)) =$ (distributivity) $= (a \wedge (c \vee e), b \vee (d \wedge f))$. From these two results we get $(a, b) \sqcap ((c, d) \sqcup (e, f)) = ((a, b) \sqcap (c, d)) \sqcup ((a, b) \sqcap (e, f))$, whatever be (a, b), (c, d), and (e, f). □

Example 30 Let us consider the four element Boolean algebra B4 of Fig. 16.

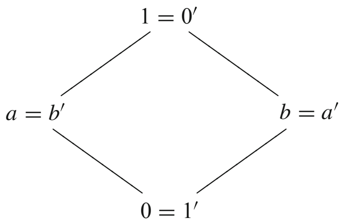


Fig. 16 The four element Boolean algebra B4

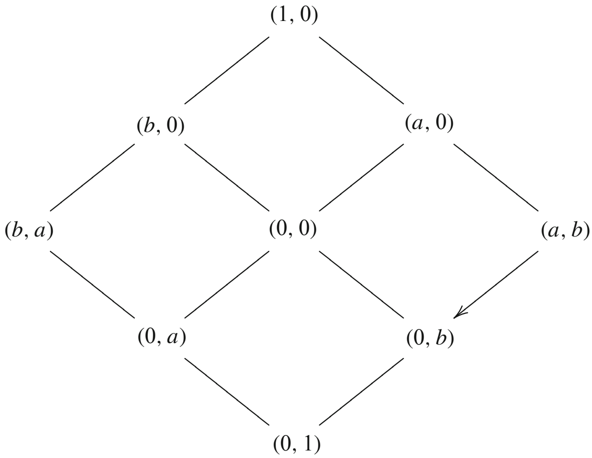


Fig. 17 Hasse diagram of the BK^{dM} distributive lattice $\mathbb{A}(B4)$ with unique half element (0, 0)

The corresponding BK^{dM} *distributive* lattice structure of orthopairs $\mathbb{A}(B4)$ can be decomposed into the three subsets

$$\mathbb{A}_{a0}(B4) = \{(0, 0), (0, a), (0, b), (0, 1)\} = -\mathbb{A}(B4)$$
$$\mathbb{A}_{ab}(B4) = \left\{(a, a') = (a, b), (b, b') = (b, a)\right\}$$

The corresponding Hasse diagram is drawn in Fig. 17 where the tabular description of the two negations of $\mathbb{A}(B4)$ is the following.

$\mathbb{A}(B4)$	$(0,0)$	$(0,a)$	$(0,b)$	$(a,0)$	$(b,0)$	$(0,1)$	$(1,0)$	(a,a')	(b,b')
$-(x,y)$	$(0,0)$	$(a,0)$	$(b,0)$	$(0,a)$	$(0,b)$	$(1,0)$	$(0,1)$	(a',a)	(b',b)
$\sim(x,y)$	$(0,1)$	(a,a')	(b,b')	$(0,1)$	$(0,1)$	$(1,0)$	$(0,1)$	(a',a)	(b',b)

1.3 Concrete Models of Orthopair Structures

In this section we consider concrete models of orthopair structures from the strongest to the weakest one, in order to give concrete examples in which this structure plays an important role. All these models, unlike the examples considered earlier, admit of infinite universes.

1.3.1 Orthopairs from the Distributive Lattice of Subsets of a Universe

Let us consider a concrete (nonempty) universe of points X. It is well know that:

(P1) The corresponding *power set*, $\mathscr{P}(X)$, consisting of the collection of all its subsets $A \subseteq X$, is an *atomic distributive (complete) lattice* with respect to the set theoretical intersection ($\cap$) and union ($\cup$), bounded by the empty set $\emptyset$ as the least element and the whole universe X as the greatest element of the lattice. The lattice atoms are the singletons $\{x\}$ formed by elements from X.

(P2) The lattice $\mathscr{P}(X)$ can be equipped with the orthocomplementation mapping (Boolean negation) associating with any subset $A \in \mathscr{P}(X)$ the set theoretical complement $A^c := X \setminus A \in \mathscr{P}(X)$.

(P3) In conclusion, the structure $\mathfrak{P}(X) = \langle \mathscr{P}(X), \cap, \cup, ^c, \emptyset, X\rangle$ is an orthocomplemented distributive (complete) lattice, i.e., a Boolean algebra.

The orthogonality relation on $\mathscr{P}(X)$, according to (4) of Definition 15 is then

$$\forall A, B \in \mathscr{P}(X), \quad A \perp B \quad \text{iff} \quad A \subseteq B^c \quad \text{iff} \quad A \cap B = \emptyset. \tag{22}$$

So the collection of all orthopairs of subsets from X is

$$\begin{aligned}\mathbb{P}(X) : &= \left\{(A_1, A_0) \in \mathscr{P}(X) \times \mathscr{P}(X) : A_1 \subseteq A_0^c\right\} = \\ &= \{(A_1, A_0) \in \mathscr{P}(X) \times \mathscr{P}(X) : A_1 \cap A_0 = \emptyset\}\end{aligned}$$

Sometimes, in order to stress that we have to do with orthopairs, we also denote this collection by $(\mathscr{P}(X) \times \mathscr{P}(X))_\perp$.

The collection $\mathbb{P}(X)$, according to the point (BA) of Proposition 29, has a structure of BK^{dM} algebra containing the half element $(\emptyset, \emptyset)$

$$\mathfrak{OP} := (\mathbb{P}(X), \sqcap, \sqcup, -, \sim, (\emptyset, X), (X, \emptyset))\ .$$

Since this case will play an important role in the following, we will explicitly list all the operations that distinguish its algebraic behavior.

$$(A_1, A_0) \sqcap (B_1, B_0) = (A_1 \cap B_1, A_0 \cup B_0) \tag{23a}$$

$$(A_1, A_0) \sqcup (B_1, B_0) = (A_1 \cup B_1, A_0 \cap B_0) \tag{23b}$$

$$-(A_1, A_0) = (A_0, A_1) \tag{23c}$$

$$\sim (A_1, A_0) = (A_0, A_0^c) \tag{23d}$$

with induced operations

$$I(A_1, A_0) = (A_1, A_1^c) \tag{23e}$$

$$C(A_1, A_0) = (A_0^c, A_0) \tag{23f}$$

$$E(A_1, A_0) = (A_0, A_0^c) \tag{23g}$$

Of course, the following chain of order inclusions holds.

$$I(A_1, A_0) = (A_1, A_1^c) \sqsubseteq (A_1, A_0) \sqsubseteq (A_0^c, A_0) = C(A_1, A_0)$$

Note that the Kleene negation $-$ is not Boolean. For instance for any arbitrary orthopair (A_1, A_0), with $A_1 \cap A_0 = \emptyset$, we have that $(A_1, A_0) \sqcap -(A_1, A_0) = (\emptyset, A_1 \cup A_0)$ which in general is different from the least lattice element $(\emptyset, X)$ unless $A_1 \cup A_0 = X$, i.e., $A_0 = A_1^c$.

An orthopair of subsets $(A_1, A_0) \in \mathbb{P}(X)$ is exact (or Brouwer crisp) iff $A_1 = A_0$ and so their collection is $\mathbb{P}_e(X) = \{(A, A^c) : A \in \mathscr{P}(X)\}$. Hence, $\mathbb{P}_c(X)$ and $\mathscr{P}(X)$ are isomorphic by the one-to-one correspondence

$$(A, A^c) \in \mathbb{P}_c(X) \longleftrightarrow \mathscr{P}(X) \ni A \tag{24}$$

We remark that the many other operations can be defined on orthopairs of (Boolean) sets. A more general survey can be found in [32], whereas for a link among these orthopairs and other knowledge representation systems, the reader can refer to [34].

1.3.2 Orthopairs from the Distributive Lattice of Fuzzy Sets on a Universe

According to the seminal paper of Zadeh [70], let us consider a universe X and the collection of fuzzy sets on it, represented by their functions: $f : X \mapsto [0, 1]$, whose collection will be denoted by $\mathscr{F}(X) := [0, 1]^X$. Hence,

(F1) The family $\mathscr{F}(X)$ of all fuzzy sets on the universe X is a *distributive (complete) lattice* with respect to the following operations on pairs of fuzzy sets $f, g \in \mathscr{F}(X)$ defined $\forall x \in X$ by the laws:

$$(f \wedge g)(x) = \min\{f(x), g(x)\} \qquad \text{(meet)}$$

$$(f \vee g)(x) = \max\{f(x), g(x)\} \qquad \text{(join)}$$

This lattice is bounded by the *least fuzzy set* $\mathbf{0}(x) = 0$ and the *greatest fuzzy set* $\mathbf{1}(x) = 1$.

(F2) The lattice $\mathscr{F}(X)$ can be equipped by the following operation:

$$\forall x \in X, \quad f'(x) = 1 - f(x) \qquad \text{(Kleene negation)}$$

which turns out to be a *Kleene negation*, i.e., the conditions (dM1), (dM2), and (KL) are satisfied.

(F3) In conclusion, the structure $\mathfrak{F}(X) := \langle \mathscr{F}(X), \wedge, \vee, ', \mathbf{0}, \mathbf{1}, \mathbf{1/2} \rangle$ is a Kleene-complemented distributive (complete and so bounded) lattice, i.e., *Kleene algebra*, with half element **1/2**.

Remark 31

- About the three properties which characterize the negation $f \in \mathscr{F}(X) \to f' = \mathbf{1} - f \in \mathscr{F}(X)$ the not trivial one is the Kleene condition (KL) which we prove now. This result is a consequence of the fact that if r is a real number with $0 \leq r \leq 1$, then $\min\{r, 1-r\} \leq 1/2$. Indeed, if $r \leq 1/2$ there is nothing to prove. But if $1/2 \leq r$ then $1 - r \leq 1/2$ and so $\min\{r, 1-r\} \leq 1/2$; of course, if $r = f(x)$ then from $f(x) + f'(x) = 1$ it follows that $f'(x) = 1 - r$ and so $\forall x \in X, \min\left\{f(x), f'(x)\right\} \leq 1/2$.

  Analogously, it is trivial to prove that $\max\{r, 1-r\} \geq 1/2$ from which in the case of the generic fuzzy set $g \in \mathscr{F}(X)$ we get that $\forall x \in X, \max\left\{g(x), g'(x)\right\} \geq 1/2$. Therefore, $\forall x \in X, (f \wedge f')(x) \leq 1/2 \leq (g \vee g')(x)$.
- Note that the Kleene negation $f \in \mathscr{F}(X) \to f' = \mathbf{1} - f \in \mathscr{F}(X)$ cannot be a Boolean complementation since in general for any fuzzy set $f \in \mathscr{F}(X)$ we have that $\forall x \in X, (f \wedge f')(x) = \min\{f(x), 1 - f(x)\}$, which is equal to $\mathbf{0}(x)$ iff $\forall x \in X, f(x) \in \{0, 1\}$, i.e., iff $f \in \{0, 1\}^X$ is a two-valued set (so the noncontradiction law does not hold in $\mathscr{F}(X)$ since it must be verified for every fuzzy set f).

  Similarly, $f \vee f' = \mathbf{1}$ iff $f \in \{0, 1\}^X$, i.e., iff f is a two-valued fuzzy set (so also the excluded middle law does not hold in $\mathscr{F}(X)$).
- Recall that for any subset $A \in \mathscr{P}(X)$ of the universe X the corresponding characteristic function $\chi_A(x) = 1$ if $x \in A$ and $= 0$ otherwise, is a two-valued fuzzy set which in the fuzzy set tradition is called *crisp set*.

Note that at this moment the notion of crisp set has nothing to do with the notion of *exact element* according to the Definition 22 which requires the definition of a quasi Brouwer negation $\sim f$.

Conversely, given a crisp set $f : X \to \{0, 1\}$ its *certainty-yes* domain is the subset $A_1(f) := \{x \in X : f(x) = 1\}$ of the universe X, and so $f = \chi_{A_1(f)}$. If we denote by $\mathscr{F}_c(X) := \{0, 1\}^X$ the collection of all crisp sets we have that $\mathscr{F}_c(X) = \{\chi_A : A \in \mathscr{P}(X)\}$ obtaining in this way a one-to-one correspondence

$$\chi_A \in \mathscr{F}_c(X) \longleftrightarrow \mathscr{P}(X) \ni A$$

which allows one to identify crisp sets on X with subsets of X. ■

The partial order relation induced from the lattice operations ($f \leq g$ iff $f = f \wedge g$, or equivalently iff $g = f \vee g$) is the usual point-wise order:

$$f \leq g \quad \text{iff} \quad \forall x \in X, f(x) \leq g(x). \tag{25}$$

The orthogonality relation between fuzzy sets $f, g \in \mathscr{F}(X)$ is

$$f \perp g \quad \text{iff} \quad f \leq g' \quad \text{iff} \quad f + g \leq \mathbf{1} \quad \text{iff} \quad f + g \in \mathscr{F}(X) \tag{26}$$

Hence, we can consider the collection of all orthopairs of fuzzy sets:

$$\begin{aligned} \mathbb{F}(X) &= \{(f_1, f_0) \in \mathscr{F}(X) \times \mathscr{F}(X) : f_0 \leq f_1'\} = \\ &= \{(f_1, f_0) \in \mathscr{F}(X) \times \mathscr{F}(X) : f_1 + f_0 \in \mathscr{F}(X)\} \end{aligned}$$

Also in this case, in order to stress the orthogonality condition, we sometimes use to denote this collection by $(\mathscr{F}(X) \times \mathscr{F}(X))_\perp$.

According to Theorem 26 $\mathbb{F}(X)$ has a minimal BZ^{dM} lattice structure (with two half elements $(\mathbf{0}, \mathbf{0})$ and $(\mathbf{1/2}, \mathbf{1/2})$):

$$\mathfrak{OF}(X) := \langle \mathbb{F}(X), \sqcap, \sqcup, -, \sim, (\mathbf{0}, \mathbf{1}), (\mathbf{1}, \mathbf{0}) \rangle .$$

While the formal versions of the lattice operations $\sqcap$ and $\sqcup$ for orthopairs of fuzzy sets according to Eq. (20) are straightforward (with respect to which $\mathfrak{OF}(X)$ turns out to be a *distributive (complete) lattice*), let us dwell a little in the formulation of the two negations $-$ and $\sim$ in the present case of orthopairs of fuzzy sets.

About the Two Negations on Orthopairs of Fuzzy Sets: The De Morgan and the Minimal Brouwer

Let us recall the two definitions of negation specialized to the case of orthopairs of fuzzy sets:

(dM)	$-(f_1, f_0) := (f_1, f_0)$	(De Morgan negation on orthopairs)
(mB)	$\sim (f_1, f_0) := (f_0, f_0')$	(minimal Brouwer negation on orthopairs)

Note that

¬(KL) The negation (dM) is not Kleene. First of all, for any real number $k \in [0, 1]$ let us define the constant fuzzy set $\mathbf{k} \in \mathscr{F}(X)$ defined by the law $\forall x \in X$, $\mathbf{k}(x) = k$. Then, let us consider the two orthopairs of fuzzy sets $(\mathbf{0.4}, \mathbf{0.5}) \in \mathbb{F}(X)$ and $(\mathbf{0}, \mathbf{0.2}) \in \mathbb{F}(X)$. Trivially, $(\mathbf{0.4}, \mathbf{0.5}) \sqcap -(\mathbf{0.4}, \mathbf{0.5}) = (\mathbf{0.4}, \mathbf{0.5})$ and $(\mathbf{0}, \mathbf{0.2}) \sqcup -(\mathbf{0}, \mathbf{0.2}) = (\mathbf{0.2}, \mathbf{0})$.
But $(\mathbf{0.4}, \mathbf{0.5}) \sqcap -(\mathbf{0.4}, \mathbf{0.5}) \not\sqsubseteq (\mathbf{0}, \mathbf{0.2}) \sqcup -(\mathbf{0}, \mathbf{0.2})$, since $0.4 \nleq 0.2$.

¬(B3) The best which we can obtain is that
$\forall x \in X$, $(0, 1) \leq [(f_1, f_0) \sqcap \sim (f_1, f_0)](x) = [(f_1, f_0) \sqcap (f_0, f_0')](x) = (f_1 \wedge f_0, f_0 \vee f_0')(x) \leq (1/2, 1/2)$. The last inequality is a consequence of the inequalities true for any $x \in X$ (see the remark 2) $(f_1 \wedge f_0)(x) \leq (f_1 \wedge f_1')(x) \leq 1/2$ and $(f_0 \vee f_0')(x) = \max\{f_0(x), 1 - f_0(x)\} \geq 1/2$.

Therefore, according to Theorem 26 we can conclude with the following result.

Proposition 32 *The structure $\mathfrak{OF}(X)$ based on the collection $\mathbb{F}(X)$ of all orthopairs of fuzzy sets is a minimal Brouwer Zadeh (mBZ^{dM}) distributive* *(complete) lattice, whose minimal Brouwer negation besides the weak double negation law (B1) satisfies both the De Morgan laws (B2-a,b).*

Furthermore, an orthopair of fuzzy sets $(f_1, f_0) \in \mathbb{F}(X)$ *is an exact element in the sense that according to Definition 22 it satisfies the condition* $(f_1, f_0) = \sim\sim (f_1, f_0) = (f_0', f_0)$ *iff* $\forall x \in X,\ f_1(x) + f_0(x) = 1$.

Orthopairs of fuzzy sets (f_1, f_0) are the basic elements of the Atanassov approach introduced in his paper [3] (and see also [4]). This approach has been the cause of a terminological debate started for the first time in [23, p. 183], at that time without any resonance, and subsequently re-proposed in [15, 16]. An explicit discussion about this terminological controversy has been published in [37], with the consequent answer in [39], and by the same Atanassov in [5]. This debate ended definitively with the article [18].

To summarize the question, we must bear in mind that Atanassov introduces the terminology of intuitionistic fuzzy sets (IFS) taking into account the lattice sub-structure $\langle \mathbb{F}(X), \sqcap, \sqcup, -, (\mathbf{0}, \mathbf{1}), (\mathbf{1}, \mathbf{0}) \rangle$ of $\mathfrak{O}\mathfrak{F}(X)$ in which the only negation $-(f_1, f_0) = (f_0, f_1)$ is considered, completely neglecting the other minimal intuitionistic (Brouwer) negation $\sim (f_1, f_0) = (f_0, f_0')$.

In [3] Atanassov claims that since for the operation $-$ "the logical law of excluded middle is not valid, similarly to the case of intuitionistic mathematics, one can assert that $-$ is an intuitionistic negation". Our answer was that the negation $-$ is a De Morgan negation which satisfies the strong double negation law $-(-(f_1, f_0)) = (f_1, f_0)$, rejected by intuitionistic mathematics (recall the before quoted Heyting book [44]), and does not satisfy the algebraic version of noncontradiction law (in general $(f_1, f_0) \sqcap -(f_1, f_0) \neq (\mathbf{0}, \mathbf{1})$) which on the contrary is assumed to hold in intuitionistic logic.

About a Real Brouwer Negation

In [18] it is shown that another negation $\approx$ in $\mathbb{F}(X)$ can be defined as follows:

$$\approx (f_1, f_0) = (\chi_{A_1(f_0)}, \chi_{A_1(f_0)^c})$$

Then it is proved that

- the negation $\approx$ is a full Brouwer negation in the sense that it satisfies not only the two expected conditions (B1) and (B2) (also in this case equivalent to the two B De Morgan laws (B2-a) and (B2-b)), but also the noncontradiction law (B3). Let us note that as stated by Heyting in [44, p. 100] condition (B2-b) is the algebraic version of a principle which cannot be asserted by the intuitionistic logic. In some sense the operation $\approx$ relative to orthopairs of fuzzy sets is a stronger version of the standard intuitionistic negation.

1.3.3 Unsharp (or Fuzzy) Quantum Mechanics in Hilbert Spaces

In this section we treat the argument of *unsharp quantum mechanics* based on the set of effect operators realizing the *formal analogy*, but underlying also the profound differences, with respect to the discussion of fuzzy sets on a universe X treated in Sect. 1.3.2.

First of all let us recall that "a (complex) Hilbert space is a vector space over the complex numbers in which there is given a complex valued function of two variables $\langle\phi|\psi\rangle$ such that: (1) For fixed ψ, $\langle\phi|\psi\rangle$ is a linear function on ϕ, (2) $\langle\phi|\psi\rangle = \overline{\langle\psi|\phi\rangle}$, (3) $\langle\phi|\phi\rangle > 0$ unless $\phi = 0$." Setting $\|\phi\| = \sqrt{\langle\phi|\phi\rangle}$, under $d(\phi, \psi) = \|\phi - \psi\|$, (4) $\mathscr{H}$ is a complete metric space (from [48, section 6]; a more complete treatment can be found in [43]).

An effect operator is a linear operator F satisfying the condition of being positive and absorbing: $\forall\psi \in \mathscr{H}$, $0 \leq \langle\psi|F\psi\rangle \leq \|\psi\|^2$. Let us denote by $\mathscr{F}(\mathscr{H})$ the collection of all effect operators on $\mathscr{H}$, whose elements are interpreted as describing physical apparatuses which measure a yes-no effect on any individual sample of the physical system udder observation [45, 46]. As particular interesting cases of effect operators we consider the zero operator $O : \psi \in \mathscr{H} \to O(\psi) = \mathbf{0} \in \mathscr{H}$, the identity operator $I : \psi \in \mathscr{H} \to I(\psi) = \psi \in \mathscr{H}$, and the *semi-transparent effect operator* $(1/2)I : \psi \in \mathscr{H} \to (1/2)I(\psi) := (1/2)\psi$.

The first analogy is that it is possible to introduce a partial order relation on $\mathscr{F}(\mathscr{H})$:

$$\text{Let } F, G \in \Phi(\mathscr{H}), \quad \text{then} \quad F \leq G \;\text{ iff }\; \forall\psi \in \mathscr{H},\; \langle\psi|F\psi\rangle \leq \langle\psi|G\psi\rangle \tag{27}$$

Let us denote by $\mathscr{H}_0$ the collection of all non zero vectors of the Hilbert space $\mathscr{H}$, whose elements are interpreted as *preparation procedures* of identical physical systems under well defined and repeatable conditions [45, 46]. Since any effect operator is positive and absorbing, it is possible to introduce the *probability of occurrence* of the effect $F \in \mathscr{F}(\mathscr{H})$ in the preparation $\psi \in \mathscr{H}_0$ as the quantity

$$p(\psi, F) := \frac{\langle\psi|F\psi\rangle}{\|\psi\|^2} \in [0, 1] \tag{28}$$

With this definition of occurrence probability the above partial order relation can be formulated in the following equivalent way:

$$\text{Let } F, G \in \mathscr{F}(\mathscr{H}), \quad \text{then} \quad F \leq G \;\text{ iff }\; \forall\psi \in \mathscr{H}_0,\; p(\psi, F) \leq p(\psi, G) \tag{29}$$

If for a fixed effect $F \in \mathscr{F}(\mathscr{H})$ one introduces the mapping

$$f_F : \mathscr{H}_0 \to [0, 1],\; \psi \to f_F(\psi) := p(\psi, F) = \frac{\langle\psi|F\psi\rangle}{\|\psi\|^2}$$

then the mapping $f_F \in [0,1]^{\mathcal{H}_0}$ can be considered as a quantum fuzzy set. Hence $\mathcal{F}(\mathcal{H}_0) := \{f_F : F \in \mathcal{F}(\mathcal{H})\}$ is the collection of all fuzzy representations of quantum effect operators.

We can now explore the analogy of the *representation* of effect operators $F \in \mathcal{F}(\mathcal{H})$ by fuzzy sets $f_F \in \mathcal{F}(\mathcal{H}_0) = [0,1]^{\mathcal{H}_0}$ on the universe $\mathcal{H}_0$.

(Qm-F1) The above partial ordering on effect operators F and G (29) can be translated into the following partial ordering of the corresponding quantum fuzzy sets f_F and f_G (and compare with the fuzzy case of Eq. (25)):

$$f_F \leq f_G \text{ iff } \forall \psi \in \mathcal{H}_0,\ f_F(\psi) \leq f_G(\psi) \tag{30}$$

But it is well known that different from the standard fuzzy set theory on the universe X, the structure $\langle \mathcal{F}(\mathcal{H}), \leq, O, I\rangle$ is not a lattice but it is a poset bounded by the least element O and the greatest element I: $\forall F \in \mathcal{F}(\mathcal{H}), O \leq F \leq I$.

(Qm-F2) For any effect $F \in \mathcal{F}(\mathcal{H})$, the linear operator $F' := I - F$ is an effect too. This means that the mapping $' : \mathcal{F}(\mathcal{H}) \to \mathcal{F}(\mathcal{H})$ associating with any effect $F \in \mathcal{F}(\mathcal{H})$ the effect $F' \in \mathcal{F}(\mathcal{H})$ is a Kleene negation on a poset (not a lattice) since not only the two conditions (dM1) $F = F''$ and (dM2b) $F \leq G$ implies $G' \leq F'$ hold, but also the Kleene condition on posets is verified: (K) $F \leq F'$ and $G' \leq G$ implies $F \leq (1/2)I \leq G$.

In particular we have that $((1/2)I)' = (1/2)I$ and so there exist the meet $(1/2)I \wedge ((1/2)I)' = (1/2)I \neq O$ (the noncontradiction law does not hold) and the join $(1/2)I \vee ((1/2)I)' = (1/2)I \neq I$ (the excluded middle law does not hold). This means that the operator $'$ is a *genuine* Kleene negation on a poset, which cannot be an orthocomplementation.

(Qm-F3) The orthogonality relation on effect operators is the usual

$$\begin{aligned} \forall F, G \in \mathcal{F}(\mathcal{H}),\ F \perp G \quad &\text{iff} \quad F \leq G' \\ &\text{iff} \quad F + G \leq I \\ &\text{iff} \quad F + G \in \mathcal{F}(\mathcal{H}) \end{aligned}$$

(compare with (26)).

(Qm-F4) No possible analogy with respect to the lattice operations on fuzzy sets on the universe X can be done since $\mathcal{F}(\mathcal{H})$ is a poset which is not a lattice.

Furthermore, there is a deep different interpretation of the mathematical objects treated in the two theories: in the Zadeh interpretation for any point x of the universe X the real quantity $f(x) \in [0,1]$ represents the *degree of membership* of the point to the fuzzy set $f \in \mathcal{F}(X)$ [70], whereas in the axiomatic version of unsharp quantum mechanics for any preparation ψ of the universe $\mathscr{H}_0$ the real quantity $f_F(\psi)$ represents the probability of occurrence of the effect F when the physical system is prepared according to ψ [10].

Anyway, we can introduce the collection $\mathbb{A}(\mathscr{H})$ of all orthopairs $(F_1, F_0)\mathscr{F}(\mathscr{H}) \times \mathscr{F}(\mathscr{H})$, with $F_1 \perp F_0$ $(F_1 + F_0 \in \mathscr{F}(\mathscr{H}))$, of effect operators whose collection will be denoted by $\mathbb{F}(\mathscr{H}) := \{(F_1, F_0) \in \mathscr{F}(\mathscr{H}) \times \mathscr{F}(\mathscr{H}) : F_1 + F_0 \in \mathscr{F}(\mathscr{H})\}$.

$\mathbb{F}(\mathscr{H})$ has a structure of minimal (pre) BZ poset (with half element (O, O)) according to the results of Theorem 19:

$$\langle \mathbb{F}(\mathscr{H}), \sqsubseteq, -, \sim, (O, I), (I, O) \rangle$$

2 Part II: Orthopair Algebras from Minimal Brouwer Zadeh (BZ) Posets and Lattices

2.1 Brouwer Zadeh (BZ) Structures

In the previous section we have seen as the collection $\mathbb{A}(\Sigma)$ of all orthopairs induced from a de Morgan poset (resp., lattice) naturally presents a structure of minimal (pre) BZ poset (resp., minimal (pre) BZ^{dM} lattice). This result induces to consider this kind of structure from a pure abstract point of view, in itself interesting, regardless of where and how it can be obtained from other structures, thus neglecting their eventual induced generation. This will be the argument of the present section starting from the following abstract definition based on a De Morgan poset.

2.1.1 Minimal (Pre) BZ Posets with Analysis of the Induced Structures

Now we are going to discuss an abstract system based on a De Morgan poset according to the following definition.

Definition 33 A system $\mathfrak{BZ} := \langle \Sigma, \leq, ', {}^{\sim}, 0, 1 \rangle$ is a *minimal (pre) Brouwer Zadeh (BZ) poset* (resp., *lattice*) iff the following hold:

(1) The sub-structure $\mathfrak{DM} := \langle \Sigma, \leq, ', 0, 1 \rangle$ is a De Morgan poset (resp., lattice).
(2) This De Morgan poset (resp., lattice) is equipped with a unary operation ${}^{\sim} : \Sigma \mapsto \Sigma$ which is a *minimal (pre) Brouwer* complementation. In other words for arbitrary $a, b \in \Sigma$:

(B1) $a \leq a^{\sim\sim}$ *(weak double negation law)*
(B2) $a \leq b$ implies $b^{\sim} \leq a^{\sim}$ *(B contraposition law)*

(3) The two complementations are linked by the interconnection rule which must hold for arbitrary $a \in \Sigma$:

(IR) $a^{\sim\sim} = a^{\sim\prime}$.

A *Brouwer Kleene (BK) lattice* is a minimal BZ lattice in which the further following conditions hold for arbitrary $a, b \in \Sigma$.

(KL) $a \wedge a' \leq b \vee b'$;
(B3) $a \wedge a^{\sim} = 0$.

In this case $a \to a'$ is a *Kleene* negation (i.e., (dM1), (dM2), and (KL) hold) and $a \to a^{\sim}$ is a true *Brouwer* (also *intuitionistic*) negation (i.e., (B1), (B2), and (B3) hold).

A *Brouwer Boolean (BB) lattice* is a minimal BZ lattice in which the further following conditions hold for arbitrary $a, b \in \Sigma$.

(ocL-2a,b) $a \wedge a' = 0$ (equivalently $a \vee a' = 1$);
(B3) $a \wedge a^{\sim} = 0$.

In this case $a \to a'$ is a *Boolean* negation (i.e., (dM1), (dM2), and (oc-2a,b)) and $a \to a^{\sim}$ is a true *Brouwer* (also *intuitionistic*) negation (i.e., (B1), (B2), and (B3)).

Example 34 The three valued Łukasiewicz logic treated in Example 28 can be equipped with a Brouwer negation obtaining in this case the Brouwer Kleene (BK) distributive lattice drawn in Fig. 18.

In particular, it is $h \leq 1 = h^{\sim\sim}$ according to the Brouwer condition (B1). Condition (B2) is also trivially verified: $0 \leq h$ implies $h^{\sim} = 0 \leq 1 = 0^{\sim}$ and $h \leq 1$ implies $1^{\sim} \leq h^{\sim}$. Furthermore, $h \wedge h^{\sim} = h \wedge 0 = 0$, i.e., condition (B3) is true. Finally, the interconnection condition between the two negations (IR) is verified since $h^{\sim\prime} = 1 = h^{\sim\sim}$.

Let us now prove some properties of BZ poset structures. Some of them can be found in [12], but here they are repeated in such a way as to make the reading of this chapter self-sufficient, even for readers who are not interested in all the arguments treated in [12].

Lemma 35 *In any minimal (pre) BZ poset the following holds.*

$$0' = 0^{\sim} = 1 \quad \textit{and} \quad 1' = 1^{\sim} = 0 \tag{31}$$

Proof Since for every $x \in \Sigma$, $0 \leq x'$, by De Morgan contraposition (dM2) and double negation (dM1) laws, $x = x'' \leq 0'$ holds for every x. Taking in particular $x = 1$ we have that $1 = 0'$. On the other hand, for every $x \in \Sigma$, $0 \leq x^{\sim}$ from which,

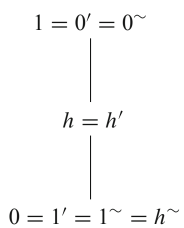


Fig. 18 The three valued BK distributive lattice

by Brouwer contraposition (B2) and weak double negation (B1) laws, it follows that $x \leq x^{\sim\sim} \leq 0^{\sim}$ holds for every x. Taking $x = 1$ we obtain that $1 = 0^{\sim}$.

Since, for every $x \in \Sigma$, $x' \leq 1$, by De Morgan contraposition, double negation and (IR), $1^{\sim} \leq 1' \leq x$ is true for every x. Choosing $x = 0$ we obtain $1^{\sim} = 1' = 0$. □

Let us prove some results about Brouwer complementation which will be useful in the sequel.

Lemma 36 *Let Σ be a minimal (pre) BZ lattice. Then, under condition (B1), the following are equivalent for every pair of elements $a, b \in \Sigma$.*

$$\text{(B2)} \quad a \leq b \quad \textit{implies} \quad b^{\sim} \leq a^{\sim} \qquad \textit{(B-contraposition)}$$

$$\text{(B2a)} \quad (a \vee b)^{\sim} = a^{\sim} \wedge b^{\sim} \qquad \textit{(B-De Morgan law)}$$

Proof First of all, let us assume that the contraposition (B2) is true. From $a, b \leq a \vee b$, by contraposition, $(a \vee b)^{\sim} \leq a^{\sim}, b^{\sim}$, i.e., $(a \vee b)^{\sim}$ is a lower bound of the pair $a^{\sim}, b^{\sim}$. Now, let c be any lower bound of this pair, $c \leq a^{\sim}, b^{\sim}$, then by contraposition and (B1) $a, b \leq a^{\sim\sim}, b^{\sim\sim} \leq c^{\sim}$ from which it follows that $a \vee b \leq c^{\sim}$ and by contraposition and (B1) $c \leq c^{\sim\sim} \leq (a \vee b)^{\sim}$, i.e., this last is the greatest lower bound of the pair $a^{\sim}, b^{\sim}$.

On the contrary, let $(a \vee b)^{\sim} = a^{\sim} \wedge b^{\sim}$ be true, then if $a \leq b$ we have that $b^{\sim} = (a \vee b)^{\sim} = a^{\sim} \wedge b^{\sim} \leq a^{\sim}$. □

In general, the "dual" contraposition law for a minimal (pre) BZ poset, "$b^{\sim} \leq a^{\sim}$ implies $a \leq b$", and the "dual" De Morgan law for a minimal (pre) BZ lattice, "$(a \wedge b)^{\sim} = a^{\sim} \vee b^{\sim}$," do not hold for the Brouwer negation. As expressed by property (B3), the noncontradiction law is satisfied by all elements of a BK lattice, whereas, in general, the excluded middle law, "$\forall a \in \Sigma, a \vee a^{\sim} = 1$," is not required to hold.

Lemma 37 *In any minimal (pre) BZ poset the following holds.*

$$\forall a \in \Sigma, \quad a^{\sim\sim\sim} = a^{\sim} \tag{32}$$

In any minimal (pre) BZ lattice the following holds.

$$a^{\sim} \vee b^{\sim} \leq (a \wedge b)^{\sim} \tag{33}$$

Proof Let Σ be a minimal BZ poset. Since (B1) is true for any element of Σ, if we apply it to the element $a^{\sim}$ we obtain $a^{\sim} \leq a^{\sim\sim\sim}$; on the other hand, applying the contraposition law of the Brouwer complementation to (B1) we obtain $a^{\sim\sim\sim} \leq a^{\sim}$.

Let Σ be a minimal BZ lattice. Applying B-contraposition to $a \wedge b \leq a, b$ we get $a^{\sim}, b^{\sim} \leq (a \wedge b)^{\sim}$ from which it follows that $a^{\sim} \vee b^{\sim} \leq (a \wedge b)^{\sim}$. □

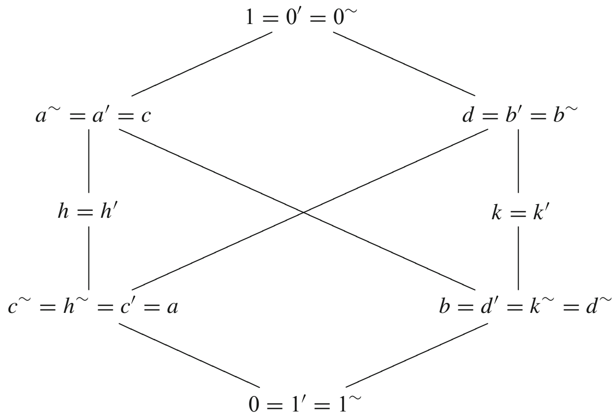


Fig. 19 Genuine minimal BZ poset of eight element mBZP8

Let us recall that an element $h \in \Sigma$ of a poset with De Morgan complementation has been called a *half element* iff it is such that $h = h'$; this half element cannot be equal to 0 or 1, $h \neq 0, 1$, since $0 \neq 0' = 1$ and $1 \neq 1' = 0$. We have seen that for this element $\exists h \wedge h = h \neq 0$ and $\exists h \vee h = h \neq 1$. A minimal (pre) BZ poset is said to be *genuine* iff there exist at least two half elements for the De Morgan complementation.

Example 38 In Fig. 19, it is drawn a genuine minimal BZ poset. This poset has two half elements $h = h'$ and $k = k'$ and so it cannot be Kleene. Let us note that $\forall x \neq h, k,\ x = x^{\sim\sim}$ whereas it is $h \leq c = h^{\sim\sim}$ and $k \leq d = k^{\sim\sim}$. The minimal Brouwer negation does not satisfy the condition (B3) since for instance $a \wedge a^{\sim} = a \neq 0$.

In the framework of minimal (pre) BZ posets, one can naturally introduce the *anti-Brouwer complement* $^\flat : \Sigma \mapsto \Sigma$ defined for every $a \in \Sigma$ as: $a^\flat := a'^{\sim\prime}$. It can be easily shown that the operation $^\flat$ satisfies the following properties:

(AB1) $a^{\flat\flat} \leq a$ (dual weak double negation law)
(AB2) $a \leq b$ implies $b^\flat \leq a^\flat$ (contraposition law)

In a BZ poset, the complementation $^\flat$ satisfies also the further property:

(AB3) $a \vee a^\flat = 1$ (excluded middle law)

2.1.2 Rough Approximation Spaces Induced from Minimal BZ Posets

We have seen that in a minimal (pre) BZ poset three complementations, playing the role of De Morgan, Brouwer, and anti Brouwer negations, are involved. These three negations turn out to be connected according to the following result:

Lemma 39 *Let Σ be a minimal (pre) BZ poset. Then, the following order chain holds:*

$$\forall a \in \Sigma\ , \quad a^{\sim} \le a' \le a^{\flat}\ .$$

Proof Indeed, from (B1) $a \le a^{\sim\sim}$ by (dM2) we get $a^{\sim\sim\prime} \le a'$, then by (IR) it follows $a^{\sim\sim\sim} \le a'$, i.e., $a^{\sim} \le a'$. We have just proved that $\forall a \in \Sigma$, $a^{\sim} \le a'$; if we apply this last to the element a' we get $a'^{\sim} \le a$ from which it follows $a' \le a'^{\sim\prime} = a^{\flat}$. □

Let us now introduce three unary operations on Σ defined for any $a \in \Sigma$ by the laws (recall the interconnection rule (IR)):

$$\mathbf{i}(a) := a^{\flat\flat} = a'^{\sim} \qquad \text{(interior=necessity)} \tag{34a}$$

$$\mathbf{c}(a) := a^{\sim\sim} = a^{\sim\prime} \qquad \text{(closure = possibility)} \tag{34b}$$

$$\mathbf{e}(a) := \mathbf{c}(a)' = a^{\sim} \qquad \text{(exterior=impossibility)} \tag{34c}$$

From another point of view, since in the minimal BZ poset approach the negation $^{\sim}$ plays a primitive role, these relationships can also be formulated in the following way.

$$\mathbf{e}(a) := a^{\sim} \qquad \text{(exterior=impossibility)} \tag{35a}$$

$$\mathbf{i}(a) := \mathbf{e}(a') \qquad \text{(interior=necessity)} \tag{35b}$$

$$\mathbf{c}(a) := \mathbf{e}(a)' \qquad \text{(closure = possibility)} \tag{35c}$$

Lemma 40 *The following are true.*

$$\forall a \in \Sigma, \quad \mathbf{i}(a) \le a \le \mathbf{c}(a) \tag{36}$$

$$\forall a \in \Sigma, \quad \mathbf{i}(a) \perp \mathbf{e}(a) \quad \textit{and} \quad \mathbf{c}(a) \perp \mathbf{e}(a) \tag{37}$$

Proof Applying $\forall a \in \Sigma$, $a^{\sim} \le a'$ to the element a' we get $\mathbf{i}(a) = a'^{\sim} \le a'' = a$. On the other hand, from $a^{\sim} \le a'$, by the (dM2) contraposition, we obtain $a = a'' \le a^{\sim\prime} = \text{(IR)} = a^{\sim\sim} = \mathbf{c}(a)$.

Applying $\forall a \in \Sigma$, $a^{\sim} \le a'$ to the element $a^{\sim}$ we get $\mathbf{c}(a) = a^{\sim\sim} \le a^{\sim\prime} = \mathbf{e}(a)'$, i.e., $\mathbf{c}(a) \perp \mathbf{e}(a)$. On the other hand, $\mathbf{i}(a) \le \mathbf{c}(a)$ and $\mathbf{c}(a) \le \mathbf{e}(a)'$ imply $\mathbf{i}(a) \le \mathbf{e}(a)'$, i.e., $\mathbf{i}(a) \perp \mathbf{e}(a)$. □

Since in general $\mathbf{i}(a) \leq a$, an element $e \in \Sigma$ is said to be *open* iff $\mathbf{i}(e) = e$. Analogously, since in general $a \leq \mathbf{c}(a)$, an element $f \in \Sigma$ is said to be *closed* iff $f = \mathbf{c}(f)$. We will denote by $\mathscr{O}(\Sigma)$ the collection of all *open* elements of the space, while $\mathscr{C}(\Sigma)$ will represent the collection of all *closed* elements.

Formally:

$$\mathscr{O}(\Sigma) = \{e \in \Sigma : e = \mathbf{i}(e)\} \quad \text{and} \quad \mathscr{C}(\Sigma) = \{f \in \Sigma : f = \mathbf{c}(f)\}$$

Proposition 41 *Let Σ be a minimal BZ poset. Then, the collection of all open elements coincides with the collection of all closed elements, $\mathscr{O}(\Sigma) = \mathscr{C}(\Sigma)$, and in this case this common set of* clopen *elements will be denoted by*

$$\mathscr{E}(\Sigma) := \mathscr{O}(\Sigma) = \mathscr{C}(\Sigma)$$

Any element of $\mathscr{E}(\Sigma)$ is said to be exact*, or* crisp.

Of course for any $a \in \Sigma$, all the elements $\mathbf{i}(a)$, $\mathbf{c}(a)$, and $\mathbf{e}(a)$ are exact. Formally.

$$\mathbf{i}(a), \mathbf{c}(a), \mathbf{e}(a) \in \mathscr{E}(\Sigma) \tag{38}$$

Finally, also $0, 1 \in \mathscr{E}(\Sigma)$.

Proof Let $e \in \mathscr{O}(\Sigma)$, i.e., $e = \mathbf{i}(e) = e'^{\sim}$, then $(e)^{\sim\prime} = (e'^{\sim})^{\sim\prime} = ((e')^{\sim\sim})' =$ (IR) $= e'^{\sim\prime\prime} = e'^{\sim} = e$, i.e., $\mathbf{c}(e) = e^{\sim\prime} = e$, in other words $e \in \mathscr{C}(\Sigma)$.

Conversely, let $e \in \mathscr{C}(\Sigma)$, i.e., $e = \mathbf{c}(e) = e^{\sim\sim}$, then $(e)'^{\sim} = (e^{\sim\sim})'^{\sim} =$ (IR) $= (e^{\sim\prime})'^{\sim} = e^{\sim\sim} = e$, i.e., $\mathbf{i}(e) = e'^{\sim} = e$, in other words $e \in \mathscr{O}(\Sigma)$.

Furthermore, $(\mathbf{i}(a))^{\sim\sim} = (a'^{\sim})^{\sim\sim} = (a')^{\sim\sim\sim} = a'^{\sim} = \mathbf{i}(a)$, i.e., $\mathbf{i}(a) \in \mathscr{C}(\Sigma) = \mathscr{E}(\Sigma)$. Similarly, $(\mathbf{c}(a))^{\sim\sim} = (a^{\sim\sim})^{\sim\sim} = a^{\sim\sim} = \mathbf{c}(a)$, i.e., $\mathbf{c}(a) \in \mathscr{C}(\Sigma) = \mathscr{E}(\Sigma)$. Finally, $(\mathbf{e}(a))^{\sim\sim} = (a^{\sim})^{\sim\sim} = a^{\sim} = \mathbf{e}(a)$, i.e., $\mathbf{e}(a) \in \mathscr{C}(\Sigma) = \mathscr{E}(\Sigma)$. □

Lemma 42 *In any minimal BZ poset the following relations hold:*

$$\mathbf{i}(a') = \mathbf{e}(a) \quad \textit{and} \quad \mathbf{e}(a') = \mathbf{i}(a) \tag{39a}$$

$$a \leq b \quad \textit{implies} \quad \mathbf{e}(b) \leq \mathbf{e}(a) \tag{39b}$$

Proof Trivially, $\mathbf{i}(a') = (a')'^{\sim} = a^{\sim} = \mathbf{e}(a)$. Analogously, for the second identity. The (39b) is nothing else than the condition (B2) under the definition $\mathbf{e}(x) = x^{\sim}$. □

The collection of all exact elements has an interesting lattice structure according to the following results.

Proposition 43 *Let $(\Sigma, \wedge, \vee, ', ^{\sim}, 0, 1)$ be a minimal (pre) BZ* lattice. *Then, the set of all exact elements is a De Morgan lattice $\langle \mathscr{E}(\Sigma), \wedge_o, \vee_o, ', 0, 1\rangle$ with respect to the following.*

1. *$\mathscr{E}(\Sigma)$ is closed under the join and this join $\vee_o$ coincides with the join $\vee$ of the lattice Σ. In other words:*

$$\forall e, f \in \mathscr{E}(\Sigma) : \quad e \vee_o f = e \vee f \in \mathscr{E}(\Sigma)$$

2. *$\mathscr{E}(\Sigma)$ is closed under the meet and this meet $\wedge_o$ coincides with the meet $\vee$ of the lattice Σ. In other words:*

$$\forall e, f \in \mathscr{E}(\Sigma) : \quad e \wedge_o f = e \wedge f \in \mathscr{E}(\Sigma)$$

3. *The two negations coincide on elements from $\mathscr{E}(\Sigma)$,*

$$\forall e \in \mathscr{E}(\Sigma) : \quad e' = e^{\sim} \in \mathscr{E}(\Sigma)$$

and the mapping $' : \mathscr{E}(\Sigma) \mapsto \mathscr{E}(\Sigma),\ \ e \to e'$ turns out to be a De Morgan complementation in the sense that the following are satisfied:

(SC1) $\quad e'' = e,$
(SC2) $\quad (e \vee f)' = e' \wedge f',$

Furthermore, if Σ is a BZ lattice (so besides conditions (B1) and (B2) the Brouwer negation satisfies also the condition (B3)), then

(SC3) $\quad \forall e \in \mathscr{E}(\Sigma),\ e \wedge e' = 0$ *(equivalently, $e \vee e' = 1$).*

i.e., $\mathscr{E}(\Sigma)$ is a standard complemented lattice (Boolean lattice).

Proof Let $e, f \in \mathscr{E}(\Sigma)$, then from the lattice property of Σ the join $e \vee f$ exists in $\langle \Sigma, \leq \rangle$. To prove that the join $e \vee_o f$ exists in the poset $\langle \mathscr{E}(\Sigma), \leq_o \rangle$ it is sufficient to show that $e \vee f$ is exact. Now, the involved elements are such that $\{e, f\} \leq e \vee f$. Thus, by the isotonicity condition (L2) of the interior operation, $\{e = \mathbf{i}(e),\ f = \mathbf{i}(f)\} \leq \mathbf{i}(e \vee f)$, i.e., $\mathbf{i}(e \vee f)$ is an upper bound of the pair $\{e, f\}$. Hence, $e \vee f \leq \mathbf{i}(e \vee f)$ so that, since $\mathbf{i}(e \vee f) \leq e \vee f$, we obtain $e \vee f = \mathbf{i}(e \vee f)$.

Let $e, f \in \mathscr{E}(\Sigma)$ be two exact elements whose meet $e \wedge f$ exists in $\langle \Sigma, \leq \rangle$. We have to show that $\mathbf{i}(e \wedge f)$ is their meet in $\mathscr{E}(\Sigma)$. We have $\mathbf{i}(e \wedge f) \leq e \wedge f \leq \{e, f\}$, i.e., it is a lower bound of the pair $\{e, f\}$. Let $x \in \mathscr{E}(\Sigma)$ be any lower bound $x \leq \{e, f\}$, then $x \leq e \wedge f$, from this it follows that $x = \mathbf{i}(x) \leq \mathbf{i}(e \wedge f)$, i.e., $\mathbf{i}(e \wedge f) = e \wedge f$.

The element e is exact iff $e = e^{\sim\sim}$, from which it follows that $e' = (e^{\sim})^{\sim\prime} =$ (IR) $= e^{\sim\sim\sim} = e^{\sim}$, i.e., $\forall e \in \mathscr{E}(\Sigma),\ e' = e^{\sim}$. Moreover, we have that $e^{\sim} = (e^{\sim})^{\sim\sim}$, i.e., $e^{\sim} \in \mathscr{C}(\Sigma) = \mathscr{E}(\Sigma)$, and so also $e' = e^{\sim}$ is exact.

From this result, we have that (SC1) and (SC2) are conditions (dM1) and (dM2) which hold in Σ. Whereas (SC3) is condition (B3), $\forall a \in \Sigma,\ a \wedge a^{\sim} = 0$, applied to any exact element $e \in \mathscr{E}(\Sigma)$ under the condition $e^{\sim} = e'$. □

The above considerations lead to the definition of an abstract approximation space generated by a minimal (pre) BZ poset.

Definition 44 Let $\left\langle \Sigma, \leq, ', {}^{\sim}, 0, 1 \right\rangle$ be a minimal BZ poset. The induced *rough approximation space* is the structure

$$\mathfrak{RAS}(\Sigma) = \langle \Sigma, \mathscr{E}(\Sigma), \mathbf{i}, \mathbf{c} \rangle$$

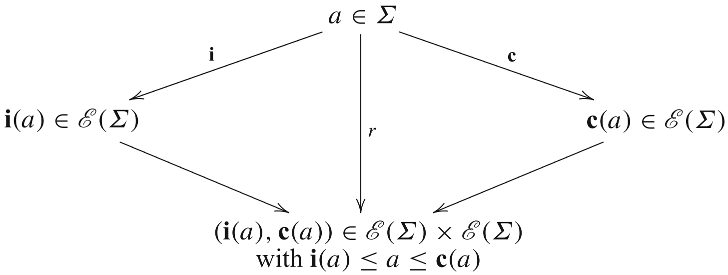
where

- Σ is the set of *approximable* elements;
- $\mathscr{E}(\Sigma) \subseteq \Sigma$ is the set of *exact* (*crisp*) elements;
- $\mathbf{i} : \Sigma \to \mathscr{E}(\Sigma)$ is the *lower approximation map*, associating with any approximable element a, its (exact) *interior* $\mathbf{i}(a) = a^{\flat\flat}$;
- $\mathbf{c} : \Sigma \to \mathscr{E}(\Sigma)$ is the *upper approximation map*, associating with any approximable element a, its (exact) *closure* $\mathbf{c}(a) = a^{\sim\sim}$;

For any element $a \in \Sigma$, its *rough approximation* is defined as the pair:

$$r(a) := (\mathbf{i}(a), \mathbf{c}(a)) \in \mathscr{E}(\Sigma) \times \mathscr{E}(\Sigma) \qquad [\text{with} \quad \mathbf{i}(a) \leq a \leq \mathbf{c}(a)]$$

drawn in the following diagram:



This approximation is the best approximation by interior–closure pairs which is possible to introduce on a minimal BZ structure. To be precise, for any element $a \in \Sigma$ the following requirements in order to have a “good” lower approximation hold:

(L1) $\mathbf{i}(a)$ is an exact element, i.e., $\mathbf{i}(a) \in \mathscr{E}(\Sigma)$;
(L2) $\mathbf{i}(a)$ is a *lower* approximation of a, i.e., $\mathbf{i}(a) \leq a$;
(L3) $\mathbf{i}(a)$ is the best lower approximation of a by exact elements, i.e., let $e \in \mathscr{E}(\Sigma)$ be such that $e \leq a$, then $e \leq \mathbf{i}(a)$.

From properties (L1)–(L3), it follows that the interior of an element a can be expressed in the following compact form:

$$\mathbf{i}(a) = \max\{x \in \mathscr{E}(\Sigma) : x \leq a\}$$

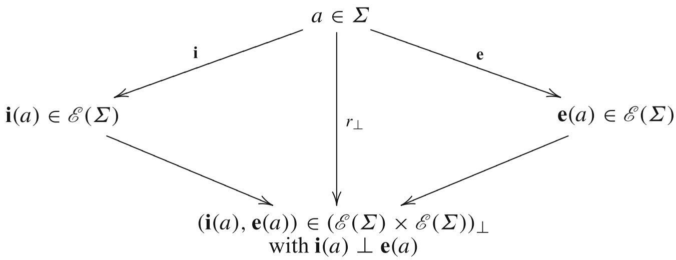


Fig. 20 Ortho-rough approximation of the element $a \in \Sigma$

Analogously, for any approximable element $a \in \Sigma$ the following minimal requirements in order to have a "good" upper approximation hold:

(U1) $\mathbf{c}(a)$ is an exact element, i.e., $\mathbf{c}(a) \in \mathscr{E}(\Sigma)$;
(U2) $\mathbf{c}(a)$ is an *upper* approximation of a, i.e., $a \leq \mathbf{c}(a)$;
(U3) $\mathbf{c}(a)$ is the best upper approximation of a by exact elements, i.e., let $f \in \mathscr{E}(\Sigma)$ be such that $a \leq f$, then $\mathbf{c}(a) \leq f$.

By properties (U1)–(U3), it follows that the upper approximation of an element a can be expressed in the following compact form:

$$\mathbf{c}(a) = \min\{y \in \mathscr{E}(\Sigma) : a \leq y\}$$

An equivalent way to define a rough approximation is to consider the interior–exterior pair instead of the interior–closure pair:

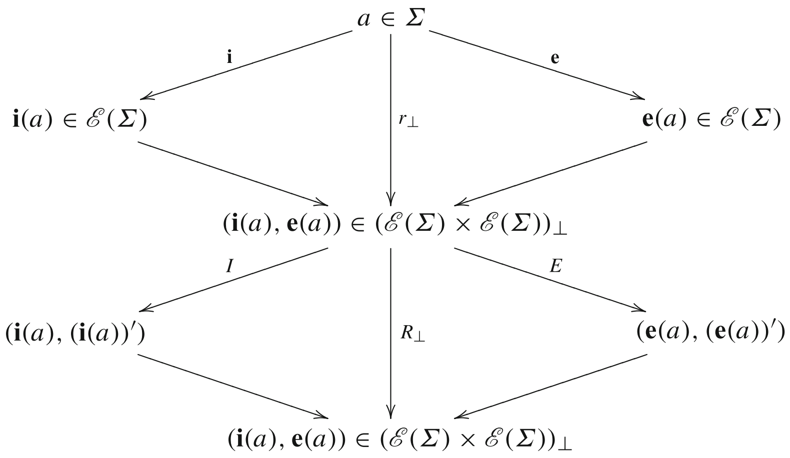
$$r_\perp(a) := (\mathbf{i}(a), \mathbf{e}(a)) = (\mathbf{i}(a), \mathbf{c}(a)') \qquad [\text{with } \mathbf{i}(a) \perp \mathbf{e}(a)]$$

Since a minimal BZ (mBZ) poset Σ is a De Morgan poset with respect to the negation $'$, also in the present case we can consider the collection $\mathbb{A}(\Sigma) = (\Sigma \times \Sigma)_\perp$ of all (De Morgan) orthopair introduced in Sect. 1.2.1 by Eq. (5). However, in the case of a mBZ poset one can select, according to Proposition 41, the collection $\mathscr{E}(\Sigma)$ of all its exact elements and then construct the collection $(\mathscr{E}(\Sigma) \times \mathscr{E}(\Sigma))_\perp := \{(h_1, h_0) \in \mathscr{E}(\Sigma) \times \mathscr{E}(\Sigma) : h_1 \perp h_0\}$ of all orthopairs formed by exact elements. Of course, $(\mathscr{E}(\Sigma) \times \mathscr{E}(\Sigma))_\perp \subseteq (\Sigma \times \Sigma)_\perp$. Moreover, according to Eqs. (37) and (38), any $(\mathbf{i}(a), \mathbf{e}(a)) \in (\mathscr{E}(\Sigma) \times \mathscr{E}(\Sigma))_\perp$ is an orthopair of exact elements.

All this discussion can be summarized by the following diagram.

Let us remark that given the interior–exterior approximation $r_\perp(a)$, the interior–closure approximation $r(a)$ can be obtained in an obvious way, and vice versa, through the one-to-one correspondence $\mathbf{e}(a) \leftrightarrow \mathbf{c}(a) (= \mathbf{e}(a)')$ established by the De Morgan mapping which is a bijection on Σ.

Furthermore, from the fact that $(\mathscr{E}(\Sigma) \times \mathscr{E}(\Sigma))_\perp \subseteq (\Sigma \times \Sigma)_\perp = \mathbb{A}(\Sigma)$ and taking into account the diagram of Fig. 14, we can complete the diagram of Fig. 20 in the following way.



2.1.3 Minimal (Pre) BZ Lattices with Analysis of the Induced Structures

In this subsection we prove some interesting properties in the case of minimal BZ *lattices* recalling that, since a lattice is in particular a poset, all the properties about posets proved in the previous sections of the Part II are immediately true for the case of lattices.

Proposition 45 *Let* $\langle \Sigma, \wedge, \vee, ', {}^\sim, 0, 1 \rangle$ *be a minimal BZ* lattice. *Then, the mapping*

$$\mathbf{i} : \Sigma \to \Sigma, \quad \mathbf{i}(a) := a^{\flat\flat} = a'^{\sim} \tag{40a}$$

is a Halmos *interior operator. That is, the following are satisfied:*

$(I0)$	$1 = \mathbf{i}(1)$	*(normalized = N modal principle)*
$(I1)$	$\mathbf{i}(a) \leq a$	*(decreasing = T modal principle)*
$(I2K)$	$\mathbf{i}(a \wedge b) = \mathbf{i}(a) \wedge \mathbf{i}(b)$	*(multiplicative= M and C modal principles)*
$(sI3)$	$\mathbf{i}(a) = (\mathbf{i}(\mathbf{i}(a)'))'$	*(interconnection = 5 modal principle)*

Dually, the mapping

$$\mathbf{c} : \Sigma \to \Sigma, \quad \mathbf{c}(a) := a^{\sim\sim} \tag{40b}$$

is a Halmos *closure operator. That is, the following are satisfied:*

$(C0)$	$0 = \mathbf{c}(0)$	*(normalized = P modal principle)*
$(C1)$	$a \leq \mathbf{c}(a)$	*(increasing = T modal principle)*

$(C2K)$ $\mathbf{c}(a \vee b) = \mathbf{c}(a) \vee \mathbf{c}(b)$ *(additive = M and C modal principles)*

$(sC3)$ $\mathbf{c}(a) = (\mathbf{c}(\mathbf{c}(a))')'$ *(interconnection = 5 modal principle)*

The two operators are linked by the relationships [30, p. 7]:

$$DF\Box \quad \mathbf{i}(a) = (\mathbf{c}(a'))' \quad \textit{and} \quad DF\Diamond \quad \mathbf{c}(a) = (\mathbf{i}(a'))'$$

Let us note that the two conditions (sI3) and (sC3) can be equivalently formulated in the following forms usually considered in the algebraic approach to modal logic.

$$\textit{(5 modal principle)} \qquad \mathbf{i}(a) = \mathbf{c}(\mathbf{i}(a)) \quad \textit{and} \quad \mathbf{c}(a) = \mathbf{i}(\mathbf{c}(a))$$

Proof Let us prove only the properties (C0)–(C3K) relative to $\mathbf{c}$; the case of $\mathbf{i}$ is dual.

(C0) is a trivial consequence of (31).
(C1) From (B1) we get $a \le a^{\sim\sim} = \mathbf{c}(a)$.
(C2K) From $\mathbf{c}(a \vee b) = (a \vee b)^{\sim\prime} =$ (B2a) $= (a^{\sim} \wedge b^{\sim})' =$ (dM2b-L) $= a^{\sim\prime} \vee b^{\sim\prime} = \mathbf{c}(a) \vee \mathbf{c}(b)$.
(sC3) From $\mathbf{c}(a) = a^{\sim\prime}$ we get $(\mathbf{c}(\mathbf{c}(a)'))' = ((\mathbf{c}(a)')^{\sim\prime})' = (\mathbf{c}(a)')^{\sim} = (a^{\sim\prime})'^{\sim} = a^{\sim\sim} =$ (IR) $= a^{\sim\prime} = \mathbf{c}(a)$. □

As stressed in the brackets, both the interior and closure operators satisfy the algebraic versions of axioms and rules of some celebrated modal principles (see [30]), once interpreted the interior as a *necessity* operator and the closure as a *possibility* operator and the partial order relation $a \le b$ as the algebraic counterpart of the statement "$A \to B$ is true" with respect to some implication connective $\to$ involving sentences A and B of the modal language. As a consequence of the results of Proposition 45 we can state the following.

(AM) *Let* $\langle \Sigma, \wedge, \vee, ', ^{\sim}, 0, 1 \rangle$ *be a minimal BZ lattice. Then, the structure* $\langle \Sigma, \wedge, \vee, ', \mathbf{i}, 0, 1 \rangle$ *is an algebraic model (AM) of a necessity modal logical system,*

(a) in general based on a De Morgan lattice instead of on a Boolean algebra,
(b) in which modal principles for necessity N, T, M – C, 5, and DF◇ are satisfied.

Let us note that the following further principles of modalities are satisfied (as can be directly proved).

$a \le \mathbf{i}(\mathbf{c}(a))$ B principle

$\mathbf{i}(a) = \mathbf{i}(\mathbf{i}(a))$ and $\mathbf{c}(a) = \mathbf{c}(\mathbf{c}(a))$ 4 principle

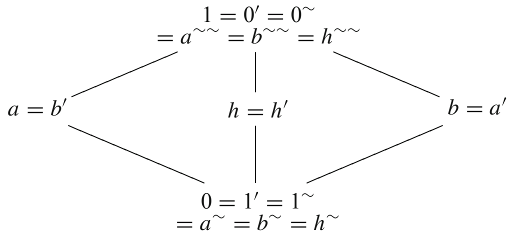


Fig. 21 Brouwer Kleene lattice of five element BKL5 with one half element

Example 46 The Hasse diagram drawn in Fig. 21 is a Brouwer Kleene (BK) not distributive lattice of five elements named BKL5 with a unique half element $h = h' = h^{\sim}$. The satisfaction of the Kleene condition (KL) follows from the inequalities:

$$\forall x \neq h, \quad x \wedge x' = 0 \leq h = h \vee h'$$
$$\forall y \neq h, \quad h \wedge h' = h \leq 1 = y \vee y'$$
$$\forall x, y \neq h, \quad x \wedge x' = 0 \leq 1 = y \vee y'$$

This BK lattice is not minimal since for any x condition (B3) $x \wedge x^{\sim} = 0$ is satisfied.

A Result on the BZ Distributive Lattice Case

In the abstract BZ *distributive* lattice context we can "translate" a very interesting result proved by Bonikowski in the concrete case of Pawlak rough set theory based on the power set of an approximation space (Theorem 2.6 of [8]).

Proposition 47 *Let Σ be a BZ* distributive *lattice. If either a or b is exact, i.e., it belongs to $\mathscr{E}(\Sigma)$, then*

$$\mathbf{i}(a \vee b) = \mathbf{i}(a) \vee \mathbf{i}(b) \tag{41a}$$

$$\mathbf{c}(a \wedge b) = \mathbf{c}(a) \wedge \mathbf{c}(b) \tag{41b}$$

Proof Without loss in generality, let us assume that $e \in \mathscr{E}(\Sigma)$ and $b \in \Sigma$.

First of all, from Eq. (33) we have $e^{\sim} \vee b^{\sim} \leq (e \wedge b)^{\sim}$ from which, by (dM2), we get $(e \wedge b)^{\sim\prime} \leq (e^{\sim} \vee b^{\sim})' = \text{(dM2c-L)} = e^{\sim\prime} \wedge b^{\sim\prime}$, that is we have obtained

$$\mathbf{c}(e \wedge b) \leq \mathbf{c}(e) \wedge \mathbf{c}(b) \tag{*}$$

Let us set $c := \mathbf{c}(e \wedge b)$. By $e \wedge b \leq e$ and isotonicity of $\mathbf{c}$ we have $c = \mathbf{c}(e \wedge b) \leq \mathbf{c}(e) = e$, i.e., $c \leq e$. Hence, taking into account the (sC3) of Proposition 43,

$$c = e \wedge c = (e \wedge e') \vee (e \wedge c) = e \wedge (e' \vee c) \tag{**}$$

From point (3) of Proposition 43 we have that e' is exact, moreover from the idempotency of the modal principle 4 we have that $c = \mathbf{c}(e \wedge b)$ is exact; hence, by point (1) of Proposition 43, also $e' \vee c$ is exact. So, recalling that for exact elements of a BZ lattice the (SC3) holds: $e \vee e' = 1$,

$$b \leq \mathbf{c}(b) = \mathbf{c}(b \wedge (e \vee e')) = \text{distributivity} = \mathbf{c}[(b \wedge e) \vee (b \wedge e')] = \text{(C2K)}$$
$$= \mathbf{c}(b \wedge e) \vee \mathbf{c}(b \wedge e') = c \vee \mathbf{c}(b \wedge e') = \text{isotonicity} =$$
$$\leq c \vee \mathbf{c}(e') = c \vee e'$$

Hence, $\mathbf{c}(b) \leq c \vee e'$, from which it follows that $e \wedge \mathbf{c}(b) \leq e \wedge (c \vee e') = (**) = c = \mathbf{c}(e \wedge b)$. Since e is exact, this last can be written as

$$\mathbf{c}(e) \wedge \mathbf{c}(b) \leq \mathbf{c}(e \wedge b) \qquad (***)$$

From (*) e (***) we obtain the result (41b).

From (41b), applied to the pair e' (exact) and b', it follows that $(e' \wedge b')^{\sim\prime} = e'^{\sim\prime} \wedge b'^{\sim\prime}$, that is $(e' \wedge b')^{\sim} = (e'^{\sim\prime} \wedge b'^{\sim\prime})' = e'^{\sim} \vee b'^{\sim}$. Finally, $\mathbf{i}(e \vee b) = (e \vee b)'^{\sim} = (e' \wedge b')^{\sim} = e'^{\sim} \vee b'^{\sim} = \mathbf{i}(e) \vee \mathbf{i}(b)$, that is (41a). □

Example 48 Let us consider the minimal BKL6 distributive lattice of Fig. 22. The Brouwer negation is minimal (pre) since $b \wedge b^{\sim} = b \neq 0$ so the Brouwer condition (B3) is not satisfied. The two negations satisfy the interconnection rule (IR) since $\forall x, x^{\sim\prime} = x^{\sim\sim}$.

Let us recall that the Kleene negation cannot be an orthocomplementation if it does not satisfy the noncontradiction law ($b \wedge b' = b \neq 0$) and the excluded middle law ($d \vee d' = d \neq 1$). This BZ lattice has no De Morgan half element ($\nexists x$ s.t. $x = x'$), while it has c such that $c^{\sim} = 0$. Moreover, the contraposition law dual with respect to (B2a) is not verified in this example: indeed, it is $c^{\sim} \leq a^{\sim}$, but $a \nleq c$.

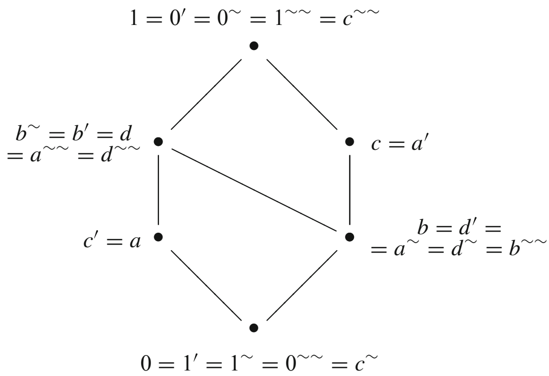


Fig. 22 The six element minimal BKL6 lattice

2.2 Minimal BZ Posets of Orthopairs Induced from Minimal BZ Posets

In Sect. 1.2, we have seen that,

(mBZP) from a De Morgan poset structure $\mathfrak{DM}(\Sigma) = \langle \Sigma, \leq, ', 0, 1 \rangle$ based on a De Morgan negation $' : \Sigma \to \Sigma$, the collection $\mathbb{A}(\Sigma)$ of all orthopairs $(a_1, a_0) \in \Sigma \times \Sigma$ such that $a_1 \perp a_0$ (i.e., $a_1 \leq a_0'$)) has a natural structure of minimal BZ poset;

(mBZL) moreover, in the particular case of a De Morgan lattice this structure of orthopairs $\mathbb{A}(\Sigma)$ turns out to be a *minimal* BZ^{dM} lattice, i.e., a minimal BZ lattice whose Brouwer negation satisfies both the De Morgan laws (B2a) and (B2b) (contrary to the standard intuitionistic negation in which only one of the De Morgan laws, precisely (B2a), is accepted whereas the other (B2b) is rejected). This structure is minimal since condition (B3) in general does not hold.
Furthermore, if we start from a Boolean lattice then the structure of orthopairs is a BK^{dM} lattice, i.e., the negation $\sim$ is *full* Brouwer one, since besides the conditions (B1), (B2a)–(B2b), it satisfies also (B3); whereas the other negation $-$ is Kleene, i.e., it satisfies the conditions (dM1), (dM2), and (KL).

In Sect. 2.1.2, we have seen that given an element a of a minimal BZ poset (resp., lattice) Σ, the ortho-rough approximation of a is given by $r_\perp(a) = (\mathbf{i}(a), \mathbf{e}(a))$, with $\mathbf{i}(a) \perp \mathbf{e}(a)$. Now, we show that it is possible to give to the collection of all such ortho-rough approximations, $r_\perp(a)$ for a running in Σ, a structure which is embedded into the minimal BZ poset $\mathbb{A}(\Sigma)$ (resp., minimal BZ^{dM} lattice).

Let $\langle \Sigma, \leq, ', ^\sim, 0, 1 \rangle$ (resp., $\langle \Sigma, \wedge, \vee, ', ^\sim, 0, 1 \rangle$) be a minimal BZ poset (resp., lattice). For the sake of simplicity, in the sequel we set

$$\forall a \in \Sigma \quad a_i := \mathbf{i}(a) \quad \text{and} \quad a_e := \mathbf{e}(a) = \mathbf{c}(a)'$$

We recall that both a_i and a_e are exact elements, $a_i, a_e \in \mathscr{E}(\Sigma)$, such that $a_i \perp a_e$.

Let us now introduce the collection of all *ortho-rough approximations* from Σ as the set

$$\mathbb{R}(\Sigma) := \{(\mathbf{i}(a), \mathbf{e}(a)) = (a_i, a_e) : a \in \Sigma\}$$

which is the imagine under the rough approximation map $r_\perp$ of the minimal BZ poset Σ: $\mathbb{R}(\Sigma) = \text{Range}(r_\perp)$. Since a minimal BZ poset Σ is in particular a De Morgan poset, also in this case it is possible to introduce the collection $\mathbb{A}(\Sigma)$ which contains $\mathbb{R}(\Sigma)$ owing to the fact that its generic element $(\mathbf{i}(a), \mathbf{e}(a))$ is such that $\mathbf{i}(a) \perp \mathbf{e}(a)$. Therefore, $\mathbb{R}(\Sigma) \subseteq \mathbb{A}(\Sigma)$. But at the beginning of this section we have stressed that $\mathbb{A}(\Sigma)$ is characterized by a particular BZ structure according to the points (mBZP) and (mBZL). Precisely, it is a *minimal BZ poset* in the poset case of Σ and a *minimal BZ^{dM} lattice* in the lattice case of Σ.

Now, it is interesting to investigate whether BZ-like structures can also be generated in the case of $\mathbb{R}(\Sigma)$. Recalling the two negations defined in $\mathbb{A}(\Sigma)$, the De Morgan and the minimal Brouwer, according to

$$\forall (a_1, a_0) \in \mathbb{A}(\Sigma), \quad -(a_1, a_0) = (a_0, a_1) \quad \text{and} \quad \sim (a_1, a_0) = (a_0, (a_0)')$$

the following theorem holds.

Theorem 49 *Let* $\langle \Sigma, \leq, ', ^{\sim}, 0, 1 \rangle$ *be a minimal BZ poset. Then, the structure* $\langle \mathbb{R}(\Sigma), \sqsubseteq, -, \sim, (0,1), (1,0) \rangle$ *is a minimal BZ poset (which in general is not a lattice) bounded by the least element* $(0, 1)$ *and the greatest element* $(1, 0)$ *with respect to the partial order:*

$$(a_i, a_e) \sqsubseteq (b_i, b_e) \quad \textit{iff} \quad a_i \leq b_i \textit{ and } b_e \leq a_e \tag{42}$$

and the De Morgan and minimal Brouwer negations, given respectively by:

$$-(a_i, a_e) := \big(\mathbf{i}(a'), \mathbf{e}(a')\big) = (a_e, a_i) \tag{43a}$$

$$\sim (a_i, a_e) := (\mathbf{i}(a^{\sim}), \mathbf{e}(a^{\sim})) = (a_e, (a_e)') \tag{43b}$$

Moreover, the following hold.

1. *The poset* $\mathbb{R}(\Sigma)$ *satisfies the interconnection rule:*

$$\text{(IR)} \quad - \sim (a_i, a_e) = \sim\sim (a_i, a_e).$$

2. *If* Σ *has a half element* $h = h' = h^{\sim}$ *(it is genuine) then also* $\mathbb{R}(\Sigma)$ *has the half element (it is genuine too):*

$$(\mathbf{i}(h), \mathbf{e}(h)) = (h, h)$$

with $(\mathbf{i}(h), \mathbf{e}(h)) = -(\mathbf{i}(h), \mathbf{e}(h)) = \sim (\mathbf{i}(h), \mathbf{e}(h))$.

Proof Trivially, $(0, 1) \sqsubseteq (a_i, a_e) \sqsubseteq (1, 0)$, whatever be (a_i, a_e).

Let Σ be a minimal BZ poset. As to the unary mapping $-$, relations $\mathbf{i}(a') = a_e$ and $\mathbf{e}(a') = a_i$ are nothing else than the (39a) written in another form. Moreover, we have the following results:

(dM1) $-(-(a_i, a_e)) = -(a_e, a_i) = (a_i, a_e)$.

(dM2) Let us suppose that $(a_i, a_e) \sqsubseteq (b_i, b_e)$, i.e., $a_i \leq b_i$ and $b_e \leq a_e$. It easily follows that $(b_e, b_i) \sqsubseteq (a_e, a_i)$, i.e., $-(b_i, b_e) \sqsubseteq -(a_i, a_e)$.

Hence, $-$ is a De Morgan negation.

As to the unary mapping $\sim$ we have the following: $\mathbf{i}(a^\sim) = (a^\sim)^{\sim\prime} = \text{(IR)} = a^{\sim\sim\sim} = a^\sim$, i.e., $\mathbf{i}(a^\sim) = a^\sim$. On the other hand, $\mathbf{e}(a^\sim) = (a^\sim)^\sim = \text{(IR)} = a^{\sim\prime}$, i.e., $\mathbf{e}(a^\sim) = a^{\sim\prime}$. Therefore from these results we get $\sim (a_i, a_e) = (a_e, (a_e)') = (a^\sim, (a^\sim)' = (\mathbf{i}(a^\sim), \mathbf{e}(a^\sim))$, i.e., $\sim (a_i, a_e) = (\mathbf{i}(a^\sim)), \mathbf{e}(a^\sim))$.

Furthermore, the following hold.

(B1) We have $\sim (a_i, a_e) = (a_e, (a_e)') = \left(a^\sim, a^{\sim\prime}\right)$, and so $\sim (\sim (a_i, a_e)) =\sim (a_e, (a_e)') = ((a_e)', ((a_e)')') = ((a_e)', a_e)$ i.e., $\sim (\sim (a_i, a_e)) = ((a_e)', a_e)$. This being stated, from $a_e \leq ((a_e)')'$ it follows that $(a_i, a_e) \sqsubseteq \sim\sim (a_i, a_e)$.

(B2) Let $(a_i, a_e) \sqsubseteq (b_i, b_e)$, i.e., $a_1 \leq b_i$ and $b_e \leq a_e$, from which in particular we have the following $b_e \leq a_e$ and $(a_e)' \leq (b_e)'$. Hence, $\sim (b_i, b_e) = (b_e, (b_e)') \sqsubseteq (a_e, (a_e)') =\sim (a_i, a_e)$.

Hence, $\sim$ is a minimal Brouwer negation.

Let us prove the interconnection rule.

(IR) We have $- \sim (a_i, a_e) = -(a_e, (a_e)') = ((a_e)', a_e)$. On the other hand, $\sim\sim (a_i, a_e) =\sim (a_e, (a_e)') = ((a_e)', (a_e)')') = ((a_e)', a_e)$. From these two results we can conclude that $- \sim (a_i, a_e) =\sim\sim (a_i, a_e)$, i.e., the rule (IR).

Now, if the minimal BZ poset Σ has the half element $h = h' = h^\sim$, then $\mathbf{i}(h) = h'^\sim = h$. On the other hand $\mathbf{e}(h) = h^\sim = h$. Hence, $(\mathbf{i}(h), \mathbf{e}(h)) = (h, h)$ with $-(h, h) = (h, h) = (h, h') =\sim (h, h)$. □

Since in general $\mathbb{R}(\Sigma)$ is not a lattice it will be interesting to have some result about the possible existence of upper bounds and lower bounds with respect to (42).

Proposition 50 *In the poset* $\mathbb{R}(\Sigma)$ *the element* $(\mathbf{i}(a_i \wedge b_i), \mathbf{e}(a_i \wedge b_i))$ *(which is equal to* $\left(\mathbf{i}(a_i \wedge b_i), \mathbf{i}(a_i' \vee b_i')\right)$*) is a lower bound of the pair* (a_i, a_e) *,* $(b_i, b_e) \in \mathbb{R}(\Sigma)$*. That is,*

$$(\mathbf{i}(a_i \wedge b_i), \mathbf{e}(a_i \wedge b_i)) \sqsubseteq \{(a_i, a_e)\,,\ (b_i, b_e)\}\,.$$

In general the greatest lower bound of the pair (a_i, a_e) *,* (b_i, b_e) *does not exist.*

Proof On the poset based on the partial order relation (42) let us consider a generic pair of elements $(a_i, a_e)\,, (b_i, b_e) \in \mathbb{R}(\Sigma)$. Let us prove that $(\mathbf{i}(a_i \wedge b_i), \mathbf{e}(a_i \wedge b_i)) =$ (39a), (dM2b-L) $= \left(\mathbf{i}(a_i \wedge b_i), \mathbf{i}(a_i' \vee b_i')\right)$ is a lower bound of this pair.

We have that $a_i \wedge b_i = \mathbf{i}(a) \wedge \mathbf{i}(b) = \text{(I2K)} = \mathbf{i}(a \wedge b) = \text{(modal 4)} = \mathbf{i}(\mathbf{i}(a \wedge b)) = \text{(I2K)} = \mathbf{i}(\mathbf{i}(a) \wedge \mathbf{i}(b)) = \mathbf{i}(a_i \wedge b_i)$, i.e., $a_i \wedge b_i = \mathbf{i}(a_i \wedge b_i)$.

So $\mathbf{i}(a_i \wedge b_i) \leq \{a_i, b_i\}$. Moreover, from $\left\{a_i', b_i'\right\} \leq a_i' \vee b_i'$, by (I1) and (39a), one gets that $\left\{\mathbf{e}(a_i) = \mathbf{i}(a_i'),\ \mathbf{e}(b_i) = \mathbf{i}(b_i')\right\} \leq \mathbf{i}(a_i' \vee b_i')$. But, since by (I1) it is $a_i \leq a$, using (39b) we arrive to the relation $\{\mathbf{e}(a),\ \mathbf{e}(b)\} \leq \mathbf{i}(a_i' \vee b_i')$. In conclusion, $(\mathbf{i}(a_i \wedge b_i), \mathbf{e}(a_i \wedge b_i)) \sqsubseteq \{(a_i, a_e)\,,\ (b_i, b_e)\}$. □

2.3 BZ^{dM} Lattice Structures of Orthopairs Induced from BZ Lattice Structures

We will investigate now a class of BZ structures in which the existence of the lattice meet and join is assured for any pair of elements from $\mathbb{R}(\Sigma)$.

Let us recall that in Lemma 36 we have proved that in any minimal BZ lattice Σ the required contraposition law (B2) is equivalent to the first B-De Morgan law:

$$\text{(B2a)} \quad \forall a, b \in \Sigma, (a \vee b)^{\sim} = a^{\sim} \wedge b^{\sim}.$$

In general the dual B-De Morgan law does not hold (as Example 48 shows, where, for instance, $a^{\sim} \vee b^{\sim} = b < 1 = (a \wedge b)^{\sim}$; note that also $\mathbf{i}(a) \vee \mathbf{i}(b) = b < d = \mathbf{i}(a \vee b)$).

In this regard, we can prove the following result recalling that the condition (DD) plays an important role in the paper [28].

Proposition 51 *In any minimal BZ lattice the following are equivalent.*

(B2b) $\forall a, b \in \Sigma$, $(a \wedge b)^{\sim} = a^{\sim} \vee b^{\sim}$;
(DD) $\forall a, b \in \Sigma$, $\mathbf{i}(a \vee b) = \mathbf{i}(a) \vee \mathbf{i}(b)$;
(DD_c) $\forall a, b \in \Sigma$, $\mathbf{c}(a \wedge b) = \mathbf{c}(a) \wedge \mathbf{c}(b)$.

In literature the condition (DD) is also called the distributivity principle for modality.

Proof Let (B2b) be true. $\mathbf{i}(a \vee b) = (a \vee b)'^{\sim} = \text{(dM2)} = (a' \wedge b')^{\sim} = \text{(B2b)} = a'^{\sim} \vee b'^{\sim} = \mathbf{i}(a) \vee \mathbf{i}(b)$.

Conversely, let (DD) be true. $(a \wedge b)^{\sim} = \text{(dM1), (dM2)} = (a' \vee b')'^{\sim} = \mathbf{i}(a' \vee b') = \text{(DD)} = \mathbf{i}(a') \vee \mathbf{i}(b') = (a')'^{\sim} \vee (b')'^{\sim} = a^{\sim} \vee b^{\sim}$.

The equivalence between (DD) and (DD_c) is a trivial consequence of the identities $\mathbf{i}(a) = (\mathbf{c}(a'))'$ and $\mathbf{c}(a) = (\mathbf{i}(a'))'$. □

This result allows one to introduce the following definition.

Definition 52 A *minimal B-De Morgan BZ (minimal BZ^{DM}) lattice* is a minimal BZ lattice satisfying also the dual De Morgan property for the Brouwer negation:

$$\text{(B2b)} \quad \forall a, b \in \Sigma, (a \wedge b)^{\sim} = a^{\sim} \vee b^{\sim}$$

Now, the following results hold.

Theorem 53 *Let* $\langle \Sigma, \wedge, \vee, ', {}^{\sim}, 0, 1 \rangle$ *be a* minimal B–De Morgan *Brouwer Zadeh (minimal BZ^{DM}) lattice with corresponding* minimal BZ^{DM} lattice *structure* $\langle \mathbb{A}(\Sigma), \sqcap, \sqcup, -, \sim, (0,1), (1,0) \rangle$ *of all its orthopairs induced from the De Morgan lattice sub-structure* $\langle \Sigma, \wedge, \vee, ', 0, 1 \rangle$ *(recall Theorem 26 of Sect. 1.2.2).*

(1) The collection $\mathbb{R}(\Sigma)$ of all rough representations $r_\perp(a) = (\mathbf{i}(a), \mathbf{e}(a)) = (a_i, a_e)$, for a running in Σ, is closed with respect to the BZ operations of $\mathbb{A}(\Sigma)$:

$$(a_i, a_e) \sqcap (b_i, b_e) := (a_i \wedge b_i, a_e \vee b_e) = ((a \wedge b)_i, (a \wedge b)_e) \in \mathbb{R}(\Sigma)$$

$$(a_i, a_e) \sqcup (b_i, b_e) := (a_i \vee b_i, a_e \wedge b_e) = ((a \vee b)_i, (a \vee b)_e) \in \mathbb{R}(\Sigma)$$

$$-(a_i, a_e) := (a_e, a_i) = \big((a')_i, (a')_e\big) \in \mathbb{R}(\Sigma)$$

$$\sim (a_i, a_e) := (a_e, (a_e)') = ((a^\sim)_i, (a^\sim)_e) = (a_i, a_e) \sim \in \mathbb{R}(\Sigma)$$

(Note that the lattice meet and join in $\mathbb{R}(\Sigma)$ coincide with the lattice meet and join in $\mathbb{A}(\Sigma)$).

In particular the negation $\sim$ is a minimal Brouwer negation satisfying the conditions (B1) and (B2). Therefore the structure $\langle \mathbb{R}(\Sigma), \sqcap, \sqcup, -, \sim, (0,1), (1,0)\rangle$ is a minimal BZ^{DM} lattice *whose induced partial order is the usual one:*

$$(a_i, a_e) \sqsubseteq (b_i, b_e) \quad \textit{iff} \quad a_i \le b_i \textit{ and } b_e \le a_e$$

(2) If Σ is a BZ lattice (and so the negation $^\sim$ on Σ satisfies besides conditions (B1) and (B2) also condition (B3)) then we have also:

$$\text{(B3)} \quad (a_i, a_e) \sqcap \sim (a_i, a_e) = ((a \wedge a^\sim)_i, (a \wedge a^\sim)_e) = (0, 1)\,.$$

In this case the structure $\langle \mathbb{R}(\Sigma), \sqcap, \sqcup, -, \sim, (0,1), (1,0)\rangle$ is a BZ^{DM} lattice.

(3) Moreover, if Σ is Boolean with respect to the De Morgan negation (and so the negation $'$ on Σ besides conditions (dM1) and (dM2) satisfies also the two conditions (oc-2ab)), then also $\mathbb{R}(\Sigma)$ is Boolean with respect to $-$, i.e.,

$$\text{(oc-2a)} \quad (a_i, a_e) \sqcap - (a_i, a_e) = (0, 1)$$

$$\text{(oc-2b)} \quad (a_i, a_e) \sqcup - (a_i, a_e) = (1, 0)$$

In this case the structure $\langle \mathbb{R}(\Sigma), \sqcap, \sqcup, -, \sim, (0,1), (1,0)\rangle$ is a BB^{DM} lattice.

Proof Let Σ be a minimal BZ^{DM} lattice whose operation $^\sim$ is a minimal De Morgan Brouwer negation, i.e., it satisfies all the conditions (B1), (B2) equivalent to (B2a), and (B2b).

(1) With respect to the lattice meet we have $a_i \wedge b_i = \mathbf{i}(a) \wedge \mathbf{i}(b) = \text{(I2K)} = \mathbf{i}(a \wedge b) = (a \wedge b)_i$. On the other hand, $a_e \vee b_e = a^\sim \vee b^\sim = \text{(B2b)} = (a \wedge b)^\sim = (a \wedge b)_e$. The lattice join leads to $a_i \vee b_i = \mathbf{i}(a) \vee \mathbf{i}(b) = \text{(DD)} = \mathbf{i}(a \vee b) = (a \vee b)_i$. On the other hand, $a_e \wedge b_e = a^\sim \wedge b^\sim = \text{(B2b)} = (a \vee b)^\sim = (a \vee b)_e$.

As to the De Morgan negation $a_e = a^\sim = \text{(dM1)} = (a')'^\sim = (a')_i$ and $a_i = a'^\sim = (a')^\sim = (a')_e$.

As to the minimal Brouwer negation, in Theorem 49 it has been shown that $\sim (a_i, a_e) = \big(a_e, (a_e)'\big)$, with the proof of the interconnection rule (IR) at the point (1).

(2) Let us now suppose that the negation $^\sim$ on Σ satisfies besides the minimal conditions (B1) and (b2) also the condition (B3): $\forall a \in \Sigma$, $a \wedge a^\sim = 0$. Let us now consider $(a_i, a_e) \sqcap \sim (a_i, a_e) = \big(a_i \wedge a_e, a_e \vee (a_e)'\big)$. Then, $a_i \wedge a_e = a'^\sim \wedge a^\sim =$ (32) $= a'^\sim \wedge (a^{\sim\sim})^\sim =$ (IR) $= a'^\sim \wedge a^{\sim\prime\sim} = \mathbf{i}(a) \wedge \mathbf{i}(a^\sim) =$ (I2K) $= \mathbf{i}(a \wedge a^\sim) =$ (B3) $= \mathbf{i}(0) = 0$. Similarly, $a_e \vee (a_e)' = a^\sim \vee a^{\sim\prime} =$ (IR) $= a^\sim \vee a^{\sim\sim} =$ (B2b) $= (a \wedge a^\sim)^\sim =$ (B3) $= 0^\sim = 1$.

(3) Finally, let "(oc-2a) $\forall a \in \Sigma$, $a \wedge a' = 0$" and "(oc-2b) $\forall a \in \Sigma$, $a \vee a' = 1$" be true. Then $(a_i, a_e) \sqcap - (a_i, a_e) = (a_i \wedge a_e, a_i \vee a_e)$. But, $a_i \wedge a_e = \mathbf{i}(a) \wedge a^\sim = \mathbf{i}(a) \wedge \mathbf{i}(a') =$ (I2K) $= \mathbf{i}(a \wedge a') =$ (oc-2a) $= \mathbf{i}(0) = 0$. Dually $a_i \vee a_e = \mathbf{i}(a) \vee a^\sim = \mathbf{i}(a) \vee \mathbf{i}(a') =$ (DD) $= \mathbf{i}(a \vee a') =$ (oc-2b) $= \mathbf{i}(1) = 1$.

The excluded middle law can be proved similarly. □

2.3.1 Zadeh Fuzzy Sets and Induced BZ Structures

In Sect. 1.3.2, we have introduced the notion of fuzzy set on a universe X as a mapping $f : X \to [0, 1]$, whose collection has been denoted as $\mathscr{F}(X) := [0, 1]^X$. Let us recall that in any $\mathscr{F}(X)$ it is possible to single out the subset $\mathscr{F}_c(X) := \{0, 1\}^X$ of all two-valued fuzzy sets consisting of the characteristic functions χ_A of all the subsets A of X: $\mathscr{F}_c(X) = \{\chi_A : A \in \mathscr{P}(X)\}$, whose elements are said to be *exact* sets.

Let us now follow the points of Definition 33 in order to give to $\mathscr{F}(X)$ an interesting algebraic structure.

(BKF1) We have proved that the collection of all fuzzy sets on the universe X has a structure of Kleene algebra with half element $\mathfrak{F} = \big\langle \mathscr{F}(X), \wedge, \vee, {}', \mathbf{0}, \mathbf{1}, \mathbf{1/2} \big\rangle$, i.e., it is a distributive (complete) lattice bounded by the fuzzy sets $\mathbf{0}(x) = 0$ and $\mathbf{1}(x) = 1$, with half element $\mathbf{1/2}(x) = 1/2$, equipped with a Kleene negation, i.e., a unary operation $f \in \mathscr{F}(X) \to f' = (\mathbf{1} - f) \in \mathscr{F}(X)$ satisfying the De Morgan conditions (dM1) and (dM2), plus the Kleene condition (KL).

(BKF2) For any fixed fuzzy set $f \in \mathscr{F}(X)$ let us introduce the following subsets of the universe X:

$$A_0(f) := \{x \in X : f(x) = 0\} \qquad \text{(the certainty-no domain)}$$

$$A_1(f) := \{x \in X : f(x) = 1\} \qquad \text{(the certainty-yes domain)}$$

$$A_p(f) := \{x \in X : f(x) \neq 0\} = (A_0(f))^c \qquad \text{(the possibility domain)}$$

Then we can associate with any fuzzy set $f \in \mathscr{F}(X)$ the exact set $f^\sim := \chi_{A_0(f)} \in \mathscr{F}_c(X)$. It is now easy to prove that the mapping $^\sim : \mathscr{F}(X) \to$

$\mathscr{F}(X)$, $f \to f^{\sim} = \chi_{A_0(f)}$, is a Brouwer negation since it satisfies the conditions (B1), (B2) (equivalent to (B2a)), and (B3).

(BKF3) In this way we have obtained the structure $\langle \mathscr{F}(X), \wedge, \vee, {}', {}^{\sim}, \mathbf{0}, \mathbf{1}, \mathbf{1/2} \rangle$ of Brouwer Kleene distributive lattice since the interconnection rule (IR) $\forall f \in \mathscr{F}(X)$, $f^{\sim\prime} = f^{\sim\sim}$ holds.

In Sect. 1.3.2 we have shown that $f \wedge f' = \mathbf{0}$ iff $f \vee f' = \mathbf{1}$ iff $f \in \{0,1\}^X$, excluding in this way that the Kleene negation $'$ can be Boolean. Similarly, we have that $f \vee f^{\sim} = \mathbf{1}$, i.e., $\forall x \in X$, $\max \{ f(x), \chi_{A_0(f)}(x) \} = 1$, iff $f \in \{0,1\}^X$.

On the basis of this BK distributive lattice, according to Eq. (34), the interior, closure, and exterior of any fuzzy set $f \in \mathscr{F}(X)$ assume the forms:

$$\mathbf{i}(f) := \chi_{A_1(f)} \qquad \text{(fuzzy interior)}$$

$$\mathbf{c}(f) := \chi_{A_p(f)} \qquad \text{(fuzzy closure)}$$

$$\mathbf{e}(f) := \chi_{A_0(f)} \qquad \text{(fuzzy exterior)}$$

From these definitions we have that

$$f = \mathbf{i}(f) \quad \text{iff} \quad f = \mathbf{c}(f) \quad \text{iff} \quad \exists A \in \mathscr{P}(X) \text{ s.t. } f = \chi_A$$

In other words, the collection $\mathscr{E}(X)$ of all *exact elements* from the BK lattice $\mathscr{F}(X)$ coincides with the collection of all two valued functions (*crisp sets*) on X. Formally,

$$\mathscr{E}(X) = \{0,1\}^X.$$

Then, according to the results of Proposition 43, we have that the collection of all crisp sets is a Boolean algebra $\langle \mathscr{E}(X), \wedge, \vee, {}', \chi_\emptyset, \chi_X \rangle$ where, in particular, we have that for any pair of crisp sets $\chi_A, \chi_B \in \mathscr{E}(X)$ and any point of the universe $x \in X$ it is

$$(\chi_A \wedge \chi_B)(x) = \min\{\chi_A(x), \chi_B(x)\} = \chi_{A\cap B}(x)$$

$$(\chi_A \vee \chi_B)(x) = \max\{\chi_A(x), \chi_B(x)\} = \chi_{A\cup B}(x)$$

$$(\chi_A{}')(x) = (\mathbf{1} - \chi_A)(x) = \chi_{A^c}(x)$$

$$\chi_\emptyset(x) = \mathbf{0}(x) = 0$$

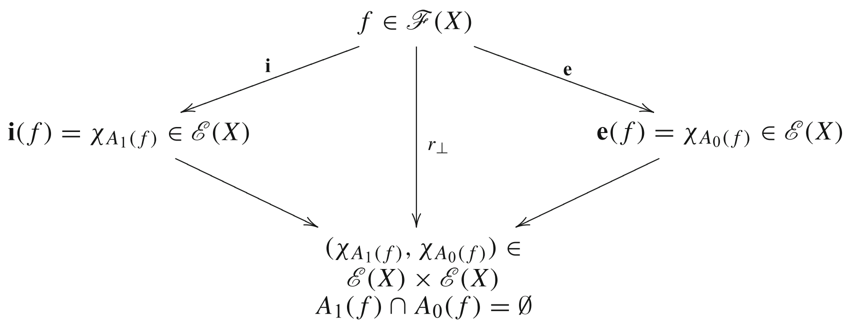
$$\chi_X(x) = \mathbf{1}(x) = 1$$

Now, the mapping $\Phi : \mathscr{E}(X) \to \mathscr{P}(X)$, $\chi_A \to \Phi(\chi_A) := A$ is an isomorphism between the two Boolean algebras $\langle \mathscr{E}(X), \wedge, \vee, {}', \chi_\emptyset, \chi_X \rangle$ and $\langle \mathscr{P}(X), \cap, \cup, {}^c, \emptyset, X \rangle$ since it is a one-to-one and onto mapping preserving the

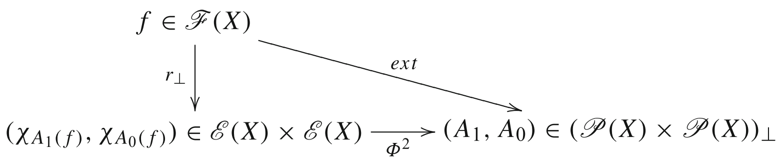
Boolean operations:

$$\Phi(\chi_A \wedge \chi_B) = \Phi(\chi_{A\cap B}) = A \cap B$$
$$\Phi(\chi_A \vee \chi_B) = \Phi(\chi_{A\cup B}) = A \cup B$$
$$\Phi(\chi_A{}') = \Phi(\chi_{A^c}) = A^c$$
$$\Phi(\chi_\emptyset) = \emptyset$$
$$\Phi(\chi_X) = X$$

This result allows one to identify the two Boolean algebra structures, written as $\mathscr{E}(X) \longleftrightarrow \mathscr{P}(X)$. In the present case of fuzzy sets, the orthopair description of the rough approximation of a fuzzy set $f \in \mathscr{F}(X)$ by the interior–exterior crisp pair is drawn by the diagram:



Extending the above isomorphism $\Phi : \mathscr{E}(X) \to \mathscr{P}(X)$ to the Cartesian product, $\Phi^2 : \mathscr{E}(X) \times \mathscr{E}(X) \to (\mathscr{P}(X) \times \mathscr{P}(X))$, $(\chi_A, \chi_B) \to \Phi^2(\chi_A, \chi_B) := (A, B)$, the ortho-rough approximation of the fuzzy set $f \in \mathscr{F}(X)$ can be represented by the following diagram:



where we have introduced the *extensional mapping* $ext = (\Phi^2 \circ r_\perp) : \mathscr{F}(X) \to (\mathscr{P}(X) \times \mathscr{P}(X))_\perp$ associating with any fuzzy set $f \in \mathscr{F}(X)$ its *extension* $ext(f) = (\Phi^2 \circ r_\perp)(f) = (A_1(f), A_0(f))$.

We have adopted here the notation

$$(\mathscr{P}(X) \times \mathscr{P}(X))_\perp := \{(A_1, A_0) \in \mathscr{P}(X) \times \mathscr{P}(X) : A_1 \cap A_0 = \emptyset\}$$

for the collection of all orthogonal pairs of subsets of the universe X which in Sect. 1.3.1 we have denoted by $\mathbb{P}(X)$.

Let us recall that $\mathbb{P}(X)$ has a natural structure of BK^{DM} distributive lattice with respect to the operations defined by Eq. (23) where the Kleene negation $-(A_1, A_0) = (A_0, A_1)$ cannot be Boolean since we can only state that for every orthopair it is $(A_1, A_0) \sqcap -(A_1, A_0) = (\emptyset, A_1 \cup A_0)$ which in general is different from the least element $(\emptyset, X)$ unless it is $A_0 = (A_1)^c$, belonging to the set of all exact orthopairs (A, A^c).

The mapping $ext : \mathscr{F}(X) \to (\mathscr{P}(X) \times \mathscr{P}(X))_\perp$ is an *epimorphism*, i.e., a morphism from the BK distributive lattice $\langle \mathscr{F}(X), \wedge, \vee, ', {}^\sim, \mathbf{0}, \mathbf{1} \rangle$ *onto* the BK^{DM} distributive lattice $\langle \mathbb{P}(X), \sqcap, \sqcup, -, \sim, (\emptyset, X), (X, \emptyset) \rangle$ (for this structure recall the Sect. 1.3.1 with related operations (23)). Indeed, we have the following results.

Proposition 54 *Let $f, g \in \mathscr{F}(X)$. Then,*

$$\text{(EP1)} \qquad ext(f \wedge g) = ext(f) \sqcap ext(g)$$

$$\text{(EP2)} \qquad ext(f \vee g) = ext(f) \sqcup ext(g)$$

$$\text{(EP3)} \qquad ext(f') = -ext(f)$$

$$\text{(EP4)} \qquad ext(f^\sim) = \sim ext(f)$$

$$\text{(EP5)} \qquad ext(\boldsymbol{0}) = (\emptyset, X) \quad \text{and} \quad ext(\boldsymbol{1}) = (X, \emptyset).$$

Furthermore, the mapping $ext : \mathscr{F}(X) \to \mathbb{P}(X)$ is surjective.

Proof (EP1) First of all, we have that

$$A_1(f \wedge g) = \{x \in X : (f \wedge g)(x) = 1\} = \{x \in X : f(x) = 1 \text{ and } g(x) = 1\} = \\ \{x \in X : f(x) = 1\} \cap \{x \in X : g(x) = 1\} = A_1(f) \cap A_1(g)$$

$$A_0(f \wedge g) = \{x \in X : (f \wedge g) = 0\} = \{x \in X : f(x) = 0 \text{ or } g(x) = 0\} = \\ \{x \in X : f(x) = 0\} \cup \{x \in X : g(x) = 0\} = A_0(f) \cup A_0(g)$$

From these two results we get that

$$ext(f \wedge g) = (A_1(f \wedge g), A_0(f \wedge g)) = (A_1(f) \cap A_1(g), A_0(f) \cup A_0(g)) = \\ (A_1(f), A_0(f)) \sqcap (A_1(g), A_0(g)) = ext(f) \sqcap ext(g)$$

The proof of(EP 2) is similar. Let us now prove (EP3). We have that $A_1(f') = \left\{x \in X : f'(x) = 1\right\} = \{x \in X : f(x) = 0\} = A_0(f)$ and similarly $A_0(f') = A_1(f)$. Hence, $ext(f') = (A_1(f'), A_0(f')) = (A_0(f), A_1(f)) = -(A_1(f), A_0(f)) = -ext(f)$. For the proof of (EP4) let us note that $f^\sim = \chi_{A_0(f)}$ from which we have that $A_1(f^\sim) = \left\{x \in X : \chi_{A_0(f)}(x) = 1\right\} = A_0(f)$ and

$A_0(f^\sim) = \{x \in X : \chi_{A_0(f)}(x) = 0\} = (A_0(f))^c$. So, $ext(f^\sim) = (A_1(f^\sim), A_0(f^\sim)) = (A_0(f)$, and $(A_0(f))^c) =\sim (A_1(f), A_0(f)) =\sim ext(f)$.

Furthermore, $ext(\mathbf{0}) = (A_1(\mathbf{0}), A_0(\mathbf{0})) = (\emptyset, X)$ and $ext(\mathbf{1}) = (A_1(\mathbf{1}), A_0(\mathbf{1})) = (X, \emptyset)$.

Let $(A_1, A_0) \in \mathbb{P}(X)$. Let us now consider the fuzzy set $f := \frac{1}{2}\left(\chi_{A_1} + \chi_{A_0^c}\right)$, then $f(x) = 1$ iff $\chi_{A_1}(x) = 1$ and $\chi_{A_0^c}(x) = 1$ iff $\chi_{A_1}(x) = 1$ and $\chi_{A_0}(x) = 0$, i.e., $f(x) = 1$ iff $x \in A_1$; similarly, $f(x) = 0$ iff $x \in A_0$. Hence, $ext(f) = (A_1, A_0)$. □

2.3.2 Pawlak Rough Sets and Induced BZ Structures

The standard Pawlak's notion of rough approximation space (see, for instance, [57, 58, 60]) is essentially based on an equivalence space $(X, \mathscr{R})$, where X is an universe equipped with an equivalence (reflexive, symmetric and transitive) relation $\mathscr{R}$, sometimes denoted also by $\equiv$. In this context, the binary relation $\equiv$ is said to be an *indiscernibility* relation and two elements x, y of the universe X which are in the relation $x \equiv y$ are called *indiscernible*.

As a consequence, the equivalence relation of indiscernibility $\equiv$ will determine a partition π of the universe X into a set of equivalence classes G, each of which is called *elementary set* or also *granule of knowledge*.

(In) Two elements x and y will be *indiscernible* ($x \equiv y$) iff they belong to the same equivalence class G:

$$\text{Let } x, y \in X, \quad \text{then } x \equiv y \quad \text{iff} \quad \exists G \in \pi \ \text{ s.t. } \ x \in G \text{ and } y \in G.$$

Given an element $x \in X$ we can define the *equivalence class* generated by x as the subset of the universe X defined as

$$G(x) = \{y \in X : x \equiv y\}$$

Obviously, $G(x)$ is not empty because x belongs to it and it constitute a *granule* of knowledge about x. Let us stress that two knowledge granules are either disjoint or equal between them.

Following Definition 33, in the case of the partition π of X generated by an indiscernibility (equivalence) relation $\equiv$ we have the following results

Proposition 55 *Let X be a (nonempty) universe of points equipped with a partition π.*

(1) The power set $\mathscr{P}(X)$ of the universe X equipped with the has a structure $\langle \mathscr{P}(X), \cap, \cup, {}^c, \emptyset, X\rangle$ of Boolean algebra *with respect to the set theoretical operations of intersection $\cap$, union $\cup$ and complementation $A^c := X \setminus A$, bounded by the least element $\emptyset$ and the greatest element X.*

(2) The Boolean algebra $\mathscr{P}(X)$ can be equipped with an operation $^\sim$ assigning to any subset of the universe X the subset

$$\forall H \in \mathscr{P}(X), \quad H^\sim := \cup\left\{G \in \pi : G \subseteq H^c\right\}. \tag{44}$$

This operation is a Brouwer negation since the following hold for arbitrary $H, K \in \mathscr{P}(X)$:

$$\begin{aligned} &(B1) \quad H \subseteq H^{\sim\sim} \\ &(B2) \quad H \subseteq K \quad \text{implies} \quad K^\sim \subseteq H^\sim \\ &(B3) \quad H \cap H^\sim = \emptyset \end{aligned}$$

(3) The interconnection rule between the two negations holds:

$$(IR) \quad \forall H \in \mathscr{P}(X), \; H^{\sim\sim} = H^{\sim c}.$$

In other words, $\langle \mathscr{P}(X), \cap, \cup, {}^c, {}^\sim, \emptyset, X\rangle$ is a Brouwer Boolean (BB) algebra (distributive lattice).

Proof Since the proofs of points (B1)–(B3) are quite trivial, let us prove the interconnection rule (IR).

Since Eq. (44) defining $H^\sim$ is true for every subset of X we can apply it to the same subset $H^\sim$ obtaining

$$(H^\sim)^\sim = \cup\left\{\widehat{G} \in \pi : \widehat{G} \subseteq (H^\sim)^c\right\} \tag{*}$$

On the other hand we have that $y \in (H^\sim)^c$ iff "$\forall\, G \subseteq H^c, \;\; y \notin G$" iff "$\neg\exists\, G \subseteq H^c$ s.t. $y \in G$" iff "$\exists\, \widehat{G} \subseteq (H^\sim)^c$ s.t. $y \in \widehat{G}$." From this result we get

$$(H^\sim)^c = \cup\left\{\widehat{G} \in \pi : \widehat{G} \subseteq (H^\sim)^c\right\} \tag{**}$$

Comparing (*) and (**) we obtain (IR). □

In order to stress that this BB algebraic structure rises from the partition space (X, π), at least in defining the Brouwer negation $^\sim$ by Eq. (44), in the sequel we denote this power set as $\mathscr{P}(X, \pi)$.

As usual in BZ structures, taking into account the condition (B1) of Brouwer negation, one can introduce the *exact elements*, or *definable sets* in the Pawlak terminology, in order to distinguish them from all the other subsets called *undefinable* [59], as those subsets $E \in \mathscr{P}(X, \pi)$ such that $E = E^{\sim\sim}(= E^{\sim c})$. The collection of all definable sets from the partition space (X, π) will be denoted by $\mathscr{E}(X, \pi)$

Proposition 56 *Let $E \in \mathscr{E}(X, \pi)$, then there exists a subfamily of knowledge granules $\left\{G_j \in \pi : j \in J\right\} \subseteq \pi$ s.t. $E = \cup\left\{G_j \in \pi : j \in J\right\}$. Furthermore the following hold.*

(CS1) $E \in \mathscr{E}(X,\pi)$ *iff* $E^c \in \mathscr{E}(X,\pi)$,
(CS2) $\emptyset \in \mathscr{E}(X,\pi)$ *and* $X \in \mathscr{E}(X,\pi)$,
(CS3) $\forall \left\{E_j : j \in J\right\} \subseteq \mathscr{E}(X,\pi),\ \ \cap\left\{E_j : j \in J\right\} \in \mathscr{E}(X,\pi)$,
(CS4) $\forall \left\{E_j : j \in J\right\} \subseteq \mathscr{E}(X,\pi),\ \ \cup\left\{E_j : j \in J\right\} \in \mathscr{E}(X,\pi)$.

In other words, $\mathscr{E}(X,\pi)$ is an Alexandroff topology [1, 2] in which, owing to (CS1), the collections of open sets and closed sets coincide with $\mathscr{E}(X,\pi)$.

Proof From $\widehat{E} = \widehat{E}^{\sim c}$ we get $\widehat{E}^c = \widehat{E}^{\sim} = \cup\{G \in \pi : G \subseteq \widehat{E}^c\}$. Setting $E = \widehat{E}^c$ the previous identity can be written as $E = \cup\{G \in \pi : G \subseteq E\}$. Conditions (CS1)–(Cs4) are easy to prove. □

We are now ready to introduce on $\mathscr{P}(X,\pi)$ the usual operations of interior, closure, and exterior, according to the general theory developed in Sect. 2.1.2, according to Eq. (35).

$$\mathbf{e}(H) = H^{\sim} \ \ = \cup\{G \in \pi : G \subseteq H^c\} \tag{45a}$$

$$\mathbf{i}(H) = H^{c\sim} = \cup\{G \in \pi : G \subseteq H\} \tag{45b}$$

$$\mathbf{c}(H) = H^{\sim c} = \cup\{G \in \pi : H \cap G \neq \emptyset\} \tag{45c}$$

So, given a subset of objects from the universe $H \in \mathscr{P}(X\pi)$, its rough approximation is the interior–closure pair $r(H) = (\mathbf{i}(H), \mathbf{c}(H))$, with $\mathbf{i}(H) \subseteq H \subseteq \mathbf{c}(H)$. Equivalently the rough approximation of the subset H can be expressed as the interior–exterior pair $r_\perp(H) = (\mathbf{i}(H), \mathbf{e}(H)) = (\mathbf{i}(H), \mathbf{u}(H)^c)$, with $\mathbf{i}(H) \cap \mathbf{e}(H) = \emptyset$.

Now we can apply the general results proved before.

(OP1) The collection $\mathbb{A}(X,\pi)$ of all orthopairs (H_1, H_0), with H_1, H_0 subsets of X such that $H_1 \subseteq H_0^c$ (i.e., $H_1 \cap H_0 = \emptyset$), according to Theorem 26 and Proposition 29, has a structure of BB^{DM} algebra (distributive lattice) with respect to the lattice operations (20) and the two negations (12).

(OP2) The collection $\mathscr{P}(X,\pi)$ of all subsets of the partition space (X,π) in general does not satisfy the dual B De Morgan law (B2b), and so neither (DD) and (DD_c).

Example 57 In the universe $X = \{a, b, c, d, e, f, g, h\}$ let us consider the partition $\pi = \{G_1 = \{a, b\}, G_2 = \{c\}, G_3 = \{d, e, f\}, G_4 = \{g, h\}\}$.

Let us consider the two subsets $H = \{a, b, d\}$ and $K = \{b, f, g\}$, then $H^{\sim} = \{c, g, h\}$, $K^{\sim} = \{c\}$ and so $H^{\sim} \cup K^{\sim} = \{c, g, h\}$. On the other hand $(H \cap K)^{\sim} = \{b\}^{\sim} = \{c, d, e, f, g, h\}$. Hence $H^{\sim} \cup K^{\sim} \subsetneqq (H \cap K)^{\sim}$.

(OP3) Since $\mathscr{P}(X,\pi)$ is not (B2b)-De Morgan, we cannot apply the results of Theorem 53 to the collection $\mathbb{R}(X,\pi)$ of all ortho-rough approximations $(\mathbf{i}(H), \mathbf{e}(H))$, for H running on $\mathscr{P}(X,\pi)$.
In particular, we cannot state that the structure $\langle \mathbb{R}(\mathscr{P}(X,\pi)), \sqsubseteq, -, \sim, (\emptyset, X), (X, \emptyset)\rangle$, is a lattice, but only that it is a BK poset equipped with a Kleene negation $-(\mathbf{i}(H), \mathbf{e}(H)) =$

$(\mathbf{e}(H), \mathbf{i}(H))$ and a full Brouwer negation $\sim (\mathbf{i}(H), \mathbf{e}(H)) = (\mathbf{e}(H), \mathbf{e}(H)^c)$ interconnected by the rule (IR).
That negation $-$ is Kleene follows from the fact that $(\mathbf{i}(H), \mathbf{e}(H)) \sqcap -(\mathbf{i}(H), \mathbf{e}(H)) = (\mathbf{i}(H) \cap \mathbf{e}(H)), \mathbf{i}(H) \cup \mathbf{e}(H)) = (\emptyset, \mathbf{i}(H) \cup \mathbf{e}(H))$ and $(\mathbf{i}(K), \mathbf{e}(K)) \sqcup -(\mathbf{i}(K), \mathbf{e}(K)) = (\mathbf{i}(K) \cup \mathbf{e}(K)), \mathbf{i}(K) \cap \mathbf{e}(K)) = (\mathbf{i}(K) \cap \mathbf{e}(K), \emptyset)$; hence,

$$(\mathbf{i}(H), \mathbf{e}(H)) \sqcap -(\mathbf{i}(H), \mathbf{e}(H)) = (\emptyset, \mathbf{i}(H) \cup \mathbf{e}(H)) \sqsubseteq \\ (\mathbf{i}(K) \cap \mathbf{e}(K), \emptyset) = (\mathbf{i}(K), \mathbf{e}(K)) \sqcup -(\mathbf{i}(K), \mathbf{e}(K)).$$

For the Brouwer condition (B3) we have that
$(\mathbf{i}(H) \cup \mathbf{e}(H)) \sqcap \sim (\mathbf{i}(H), \mathbf{e}(H)) = (\mathbf{i}(H) \cup \mathbf{e}(H)) \sqcap (\mathbf{e}(H), \mathbf{e}(H)^c) = (\emptyset, X)$.

Let us now discuss two standard partitions for *any* universe X.

Example 58 Let X be a not empty universe. The *trivial partition* of X is the collection of its subsets $\pi_t = \{\emptyset, X\}$. Of course the corresponding collection of definable sets is $\mathscr{E}(X, \pi_t) = \{\emptyset, X\} = \pi_t$. The corresponding Brouwer negation, according to Eq. (44) is then

$$\forall H \in \mathscr{P}(X, \pi_t), \quad H^{\sim_t} = \begin{cases} \emptyset & \text{if } H \neq \emptyset \\ X & \text{if } H = \emptyset \end{cases} \quad \text{with } H^{\sim_t \sim_t} = \begin{cases} X & \text{if } H \neq \emptyset \\ \emptyset & \text{if } H = \emptyset \end{cases}$$

Therefore, for any subset $H \in \mathscr{P}(X, \pi_t) \setminus \{\emptyset, X\}$ we have that

$$\mathbf{e}(H) = \emptyset \qquad \mathbf{i}(H) = \emptyset \qquad \mathbf{c}(H) = X$$

from which it follows that $r(H) = (\emptyset, X)$ and $r_\perp(H) = (\emptyset, \emptyset)$.
Moreover, for the two extremal cases $\emptyset$ and X we have that

$$\mathbf{e}(\emptyset) = X \qquad \mathbf{i}(\emptyset) = \emptyset \qquad \mathbf{c}(\emptyset) = \emptyset$$

from which it follows that $r(\emptyset) = (\emptyset, \emptyset)$ and $r_\perp(\emptyset) = (\emptyset, X)$.
On the other hand,

$$\mathbf{e}(X) = \emptyset \qquad \mathbf{i}(X) = X \qquad \mathbf{c}(X) = X$$

from which it follows that $r(X) = (X, X)$ and $r_\perp(X) = (X, \emptyset)$.
Note the following chain of inclusions:

$$\forall H \in \mathscr{P}(X, \pi_t), \quad (\emptyset, X) \sqsubseteq r_\perp(H) = (\emptyset, \emptyset) \sqsubseteq (X, \emptyset).$$

Example 59 Let X be a not empty universe. The *discrete partition* π_d of X consists of all singletons $\{x\} \in \mathscr{P}(X, \pi_d)$, for x running in X, plus the empty set $\emptyset$. In this case we have that the collection of all definable sets coincides with the power set: $\mathscr{E}(X, \pi_d) = \mathscr{P}(X, \pi_d)$.

Hence, for any subset $H \in \mathscr{P}(X, \pi_d)$ we have that $H^{\sim_d} = H^c$, the set theoretical complement of the subset itself. So, we have that in this case for any $H \in \mathscr{P}(X, \pi_d)$

$$\mathbf{i}(H) = H \qquad \mathbf{c}(H) = H \qquad \mathbf{e}(H) = H^c$$

From these results it follows that $r(H) = (H, H)$ and $r_\perp(H) = (H, H^c)$.

The Classical Pawlak Approach

In particular, in the classical Pawlak's approach to rough sets the equivalence relation is obtained from an *information table* $\mathscr{K}(X) = \langle X, Att(X), Val, F\rangle$, where X is a nonempty universe of *objects*, $Att(X)$ is a nonempty set of *attributes* related to the objects of X, and $F : X \times Att(X) \to Val$ the *information mapping* associating with any pair consisting of an object $x \in X$ and an attribute $a \in Att$ the *value* $F(x, a) \in Val$ which the attribute a assigns to the object x.

Precisely, once fixed a subset of attributes $D \subseteq Att(X)$, the equivalence indiscernibility relation of any two objects is defined as:

$$x \equiv y \quad \text{iff} \quad \forall a \in D,\ F(x, a) = F(y, a) \tag{46}$$

Following the partition space results, we have that the structure $\langle \mathscr{P}(X), \cap, \cup, ^c, ^\sim, \emptyset, X\rangle$ is a BB^{DM} lattice where the preclusive complementation $^\sim$ is the mapping associating with any subset H of the universe its preclusive complement

$$H^\sim = \{x \in X : \forall y \in H,\ \exists a \in D \text{ s.t. } F(x, a) \neq F(y, a)\}$$

Indeed, we have that "$x \in H^\sim$ iff $\forall y \in H$, $y \notin G(x)$." But "$y \in G(x)$ iff $\forall a \in D$, $F(x, a) = F(y, a)$" and so "$y \notin G(x)$ iff $\exists a \in D$ s.t. $F(x, a) \neq F(y, a)$." In conclusion, "$x \in H^\sim$ iff $\forall y \in H, \exists a \in D$ s.t. $F(x, a) \neq F(y, a)$."

Notwithstanding the drawbacks of points (OP2) and (OP3), a very interesting and important result of Bonikowski in [8] assures that in the case of a partition space (X, π) based on a universe X of *finite cardinality* and a partition π induced from an equivalence relation of indiscernibility $\equiv$, the structure $\mathbb{R}(X, \pi) := \{(\mathbf{i}(H), \mathbf{e}(H)) : H \in \mathscr{P}(X, \pi)\}$ is a distributive lattice with respect to the partial order relation $\sqsubseteq$.

Before obtaining this result we give here a new proof of a preliminary result which is totally inspired by the Bonikowski paper (Lemma 4.11 of [8]), but which is more compact and self-consistent.

Proposition 60 *Let (X, π) be a partition space whose universe is finite ($|X| < \infty$). Then, for every pair $H, K \in \mathscr{P}(X, \pi)$ there exists a subset $Z \in \mathscr{P}(X, \pi)$ such that:*

$$\mathbf{i}(Z) = \mathbf{i}(H) \cap \mathbf{i}(K) \tag{47a}$$

$$\mathbf{c}(Z) = \mathbf{c}(H) \cap \mathbf{c}(K) \tag{47b}$$

$$\mathbf{e}(Z) = \mathbf{e}(H) \cup \mathbf{e}(K) \tag{47c}$$

Moreover, as to the anti-Brouwer negation, we have that

$$Z^\flat = H^\flat \cup H^\flat \tag{47d}$$

Proof For an arbitrary pair of subsets H, K of X, let us consider the subset $A := \mathbf{c}(H) \cap \mathbf{c}(K)$. We discuss two cases.

(a) Let $A = \mathbf{c}(H) \cap \mathbf{c}(K) = \emptyset$. In this case a fortiori $\mathbf{i}(H) \cap \mathbf{i}(K) = \emptyset$, and from the general property (I2K) $\mathbf{i}(H \cap K) = \mathbf{i}(H) \cap \mathbf{i}(K) = \emptyset$. On the other hand, from (*) of the proof of Proposition 47 it follows that $\emptyset \subseteq \mathbf{c}(H \cap K) \subseteq \mathbf{c}(H) \cap \mathbf{c}(K) = \emptyset$.

(b) Let $A = \mathbf{c}(H) \cap \mathbf{c}(K) \neq \emptyset$. Then A is an exact (clopen) nonempty set which can be expressed as the set theoretic union of mutually disjoint nonempty elementary sets (knowledge granules): $A = \mathbf{c}(H) \cap \mathbf{c}(K) = G_1(A) \cup \ldots \cup G_k(A)$, where for every i it is $G_i(A) \neq \emptyset$ and for $i \neq j$ it is $G_i(A) \cap G_j(A) = \emptyset$. Let us choose in any elementary set $G_i(A)$ a single element $x_i \in G_i(A)$ and let us collect them in the set $Y = \{x_1, \ldots, x_k\} \subseteq \mathbf{c}(H) \cap \mathbf{c}(K)$. Of course, according to (45c), $\mathbf{c}(\{x_i\}) = G_i(A)$ and from this result one gets that Y is a *minimal lower sample* according to the terminology of Bonikowski:

$$\mathbf{c}(Y) = \mathbf{c}(\{x_1\} \cup \ldots \cup \{x_k\}) = G_1(A) \cup \ldots \cup G_k(A) = \mathbf{i}(A)$$

Since A is exact, i.e., $A = \mathbf{i}(A)$, we can also write this result as follows:

$$\mathbf{c}(Y) = A = \mathbf{c}(H) \cap \mathbf{c}(K) \tag{48}$$

As to $\mathbf{i}(Y)$, if its interior is empty, $\mathbf{i}(Y) = \emptyset$, then trivially $\mathbf{i}(Y) \subseteq \mathbf{i}(H) \cap \mathbf{i}(K)$. Let us suppose that $\mathbf{i}(Y) \neq \emptyset$, then there exists $z \in \mathbf{i}(Y) \subseteq Y$ and so there exists j such that $z = x_j \in Y$. In particular, $z \in G_j(A) = [x_j]$ and for every $i \neq j$ it is $z \notin G_i(A)$, or in other words $[z] \cap G_i(A) = \emptyset$. Moreover, $z \in \mathbf{i}(Y)$ implies $[z] \subseteq \mathbf{i}(Y) \subseteq Y$, i.e., $[z] \subseteq \{x_1, \ldots, x_k\}$, and so necessarily $G_j(A) = [x_j] = [z] = \{x_j\}$. Hence, $x_j = z \in \mathbf{i}(Y) \subseteq Y \subseteq \mathbf{c}(H) \cap \mathbf{c}(K)$. In particular, $x_j \subseteq \mathbf{c}(H)$ and this means that $\emptyset \neq [x_j] \cap H = \{x_j\} \cap H$, i.e., $\{x_j\} = [x_j] \subseteq H$ which is a condition assuring that $x_j \in \mathbf{i}(H)$. Therefore, we have obtained that for every $x_j \in \mathbf{i}(Y)$ necessarily $x_j \in \mathbf{i}(H)$. In a similar way one obtains that for every $x_j \in \mathbf{i}(Y)$ necessarily $x_j \in \mathbf{i}(K)$. Thus, also in this case $\mathbf{i}(Y) \subseteq \mathbf{i}(H) \cap \mathbf{i}(K)$.

As a conclusion, whatever be the intersection $A = \mathbf{c}(H) \cap \mathbf{c}(K)$ one always has that

$$\mathbf{i}(Y) = \mathbf{i}(\{x_1, \dots, x_k\}) \subseteq \mathbf{i}(H) \cap \mathbf{i}(K) \tag{49}$$

Now, let us consider the subset $Z := [\mathbf{i}(H) \cap \mathbf{i}(K)] \cup Y$. Then, making use of Proposition 47 applied to the pair formed by the exact subset $\mathbf{i}(H) \cap \mathbf{i}(K)$ and the subset Y we get

$$\begin{aligned}\mathbf{i}(Z) &= \mathbf{i}\,\{\,[\mathbf{i}(H) \cap \mathbf{i}(K)] \cup Y\} = \text{(41a)} = \mathbf{i}(\mathbf{i}(H) \cap \mathbf{i}(K)) \cup \mathbf{i}(Y) = \\ &= (\mathbf{i}(H) \cap \mathbf{i}(K)) \cup \mathbf{i}(Y) = (2) = \mathbf{i}(H) \cap \mathbf{i}(K)\end{aligned} \tag{50}$$

Similarly, applying Proposition 47 to the pair of exact sets $\mathbf{i}(H)$ and $\mathbf{i}(K)$

$$\begin{aligned}\mathbf{c}(Z) &= \mathbf{c}\,\{\,[\mathbf{i}(H) \cap \mathbf{i}(K)] \cup Y\} = \mathbf{c}(\mathbf{i}(H) \cap \mathbf{i}(K)) \cup \mathbf{c}(Y) = \text{(41b)} \\ &= [\mathbf{c}(\mathbf{i}(H)) \cap \mathbf{c}(\mathbf{i}(K))] \cup \mathbf{c}(Y) = (1) \\ &= [\mathbf{i}(H) \cap \mathbf{i}(K)] \cup [\mathbf{c}(H) \cap \mathbf{c}(K)] = \mathbf{c}(H) \cap \mathbf{c}(K)\end{aligned} \tag{51}$$

From $\mathbf{c}(Z) = \mathbf{c}(H) \cap \mathbf{c}(K)$ it follows that $X \setminus \mathbf{c}(Z) = X \setminus \mathbf{c}(H) \cap \mathbf{c}(K) = [X \setminus \mathbf{c}(H)] \cup [X \setminus \mathbf{c}(K)]$, that is $\mathbf{e}(Z) = \mathbf{e}(H) \cup \mathbf{e}(K)$. Finally, $H^\flat \cup K^\flat = H^{c\sim c} \cup H^{c\sim c} = (H^{c\sim} \cap H^{c\sim})^c = (\mathbf{i}(H) \cap \mathbf{i}(K))^c =$ (47a) $= \mathbf{i}(Z)^c = (Z^{c\sim})^c = Z^\flat$. □

We discuss now the main difficulty in asserting that $\mathbb{R}(X, \pi)$ is a lattice. If one considers the rough sets generated by a subset H, $r_\perp(H) = (\mathbf{i}(H), \mathbf{e}(H))$ and by a subset K, $r_\perp(K) = (\mathbf{i}(K), \mathbf{e}(K))$, then mimicking the lattice operations (20) involving orthopairs of the abstract approach, one is tempted to define $(\mathbf{i}(H) \cap \mathbf{i}(K), \mathbf{e}(H) \cup \mathbf{e}(K))$ and $(\mathbf{i}(H) \cup \mathbf{i}(K), \mathbf{e}(H) \cap \mathbf{e}(K))$ as possible lattice operations of the concrete Pawlak case. The delicate point consists in the fact that in order to assure that these two pairs are elements of $\mathbb{R}(X, \pi)$ it is necessary to prove that there exists two subsets Z and W of X such that $r_\perp(Z) = (\mathbf{i}(H) \cap \mathbf{i}(K), \mathbf{e}(H) \cup \mathbf{e}(K))$ and $r_\perp(W) = (\mathbf{i}(H) \cup \mathbf{i}(K), \mathbf{e}(H) \cap \mathbf{e}(K))$.

The following result is exactly a reformulation of the Bonikowski proof of [8, p. 417] assuring that in the case of a finite universe X this condition is satisfied.

Proposition 61 *Let (X, π) be a finite partition space ($|X| < \infty$). The poset $\mathbb{R}(X, \pi)$ of all rough representations of subsets from X is a distributive lattice.*

Precisely, for any pair of subsets $H, K \in \mathscr{P}(X, \pi)$ the meet and join operations are given respectively by the following (where Z is the subset introduced in the proof of Proposition 60 relatively to the pair H and K, and W^c is the subset corresponding to the application of Proposition 60 to the pair H^c and K^c):

$$r_\perp(H) \sqcap r_\perp(K) = r_\perp(Z) = (\mathbf{i}(H) \cap \mathbf{i}(K), \mathbf{e}(H) \cup \mathbf{e}(K)) \tag{52a}$$

$$r_\perp(H) \sqcup r_\perp(K) = r_\perp(W) = (\mathbf{i}(H) \cup \mathbf{i}(K), \mathbf{e}(H) \cap \mathbf{e}(K)) \tag{52b}$$

Proof Let us consider the element of $\mathbb{R}(X,\pi)$ corresponding to the rough representation of the subset Z of Proposition 60: $r_\perp(Z) = (\mathbf{i}(Z), \mathbf{e}(Z)) =$ (47a) and (47c) $= (\mathbf{i}(H) \cap \mathbf{i}(K), \mathbf{e}(H) \cup \mathbf{e}(K))$. We have that $r_\perp(Z) \sqsubseteq r_\perp(H) = (\mathbf{i}(H), \mathbf{e}(H))$ since from (47a) one has that $\mathbf{i}(Z) \subseteq \mathbf{i}(H)$ and $\mathbf{e}(H) \subseteq \mathbf{e}(Z)$. Similarly, from (47c) it follows that $r_\perp(Z) \sqsubseteq r_\perp(K) = (\mathbf{i}(K), \mathbf{e}(K))$. That is $r_\perp(Z)$ is a lower bond in the poset $\mathbb{R}(X,\pi)$ of the pair $r_\perp(H), r_\perp(K)$.

Let now $r_\perp(W) = (\mathbf{i}(W), \mathbf{e}(W))$ be a generic lower bond of the same pair in the poset $\mathbb{R}(X,\pi)$, then

$$\mathbf{i}(W) \subseteq \mathbf{i}(H),\ \mathbf{i}(K) \quad \text{and} \quad \mathbf{e}(H), \mathbf{e}(K) \subseteq \mathbf{e}(W)$$

From these, using (3) and (4) of the proof of Proposition 60, it follows that

$$\mathbf{i}(W) \subseteq \mathbf{i}(H) \cap \mathbf{i}(K) = \mathbf{i}(Z)$$

$$\mathbf{e}(Z) = X \setminus \mathbf{c}(Z) = X \setminus [\mathbf{c}(H) \cap \mathbf{c}(K)] = \mathbf{e}(H) \cup \mathbf{e}(K) \subseteq \mathbf{e}(W)$$

Hence, we have proved that $r_\perp(W) \sqsubseteq r_\perp(Z)$, and so $r_\perp(Z)$ is the greatest lower bond of the pair $r_\perp(H),\ r_\perp(K)$.

This being stated, let us consider $\mathbf{i}(H) \cup \mathbf{i}(K) = [\mathbf{i}(H)^c \cap \mathbf{i}(K)^c]^c = [(H^c)^{\sim c} \cap (K^c)^{\sim c}\,]^c = [\mathbf{c}(H^c) \cap \mathbf{c}(K^c)\,]^c$. If we denote by W^c the subset of X corresponding to the application of Proposition 60 (in particular Eq. (47b)) relatively to the pair H^c and K^c, we have that $\mathbf{i}(H) \cup \mathbf{i}(K) = [\mathbf{c}(W^c)]^c = [(W^c)^{\sim c}]^c = W^{c\sim} = \mathbf{i}(W)$. On the other hand, $\mathbf{c}(H) \cap \mathbf{c}(K) = H^\sim \cap K^\sim = (H^{\sim c} \cup K^{\sim c})^c = [(H^c)^{c\sim c} \cup (K^c)^{c\sim c}]^c = [(H^c)^\flat \cup (H^c)^\flat]^c$. With respect to the same pair H^c and K^c if we apply (47d) of Proposition 60 we obtain $\mathbf{c}(H) \cap \mathbf{c}(K) = [(W^c)^\flat]^c = W^\sim$ $= \mathbf{e}(W)$. □

Summarizing,

1. In the case of a partition space (X,π) consisting of a universe X of *finite* cardinality equipped with a partition π induced from an indiscernibility (equivalence) relation $\equiv$, the collection $\mathbb{R}(X,\pi)$ of all rough representations of subsets $H \subseteq X$ gives rise to a BK algebra (distributive lattice)

$$(\mathbb{R}(X,\pi), \sqcap, \sqcup, -, \sim, (\emptyset, X), (X, \emptyset)).$$

2. This algebra is a sub BK algebra of the BB^{DM} algebra $\mathbb{A}(X)$ of all mutually disjoint pairs (A_1, A_0) of subsets A_1 and A_0 (with $A_1 \cap A_0 = \emptyset$) of the finite universe X. In particular all the operations $\sqcap, \sqcup, -, \sim$ are preserved and so also $\mathbb{R}(X,\pi)$ is a BK^{DM} algebra, whose operations are the following:

$$(\mathbf{i}(H), \mathbf{e}(H)) \sqcap (\mathbf{i}(K), \mathbf{e}(K)) = (\mathbf{i}(H) \cap \mathbf{i}(K), \mathbf{e}(H) \cup \mathbf{e}(K))$$

$$(\mathbf{i}(H), \mathbf{e}(H)) \sqcup (\mathbf{i}(K), \mathbf{e}(K)) = (\mathbf{i}(H) \cup \mathbf{i}(K), \mathbf{e}(H) \cap \mathbf{e}(K))$$

$$-(\mathbf{i}(H), \mathbf{e}(H)) = (\mathbf{e}(H), \mathbf{i}(H))$$

$$\sim (\mathbf{i}(H), \mathbf{e}(H)) = (\mathbf{e}(H), \mathbf{c}(H))$$

2.4 Heyting Wajsberg (HW) Algebras

Before entering into the details involving Heyting Wajsberg (HW) algebras, we need to introduce some notions about implications and their algebraic axiomatization. This structures will indeed be used to characterize the collection of different kind of orthopairs.

This problem was addressed by the authors in the paper [14] and subsequently expanded and completed in [26, 27] (after some preliminary works in weaker structures [20, 24, 25]), where it is introduced a structure based on two operations $\rightarrow_L$ and $\rightarrow_G$ called Wajsberg and Heyting (HW) implications, respectively. These names are justified on the basis of the papers [63, 66, 67] for Wajsberg algebra and [52] for Heyting algebra (recalling that what is here called Heyting algebra in literature sometimes appears with the name of Brouwerian lattice [7, p. 45] or Brouwer algebra [51]).

The discussion just made leads to investigate abstract algebraic structures based on two primitive implication connectives by the following structure [26].

Definition 62 A system $\mathfrak{HW}(A) = \langle A, \rightarrow_L, \rightarrow_G, 0\rangle$ is a *Heyting Wajsberg* (HW) *algebra* if A is a nonempty set, $0 \in A$, and $\rightarrow_L, \rightarrow_G$ are binary operations, such that, once defined the further operations

(Op1) $a \vee b := (a \rightarrow_L b) \rightarrow_L b$
(Op2) $a \wedge b := \neg((\neg a \rightarrow_L \neg b) \rightarrow_L \neg b)$
(Op3) $\neg a := a \rightarrow_L 0$
(Op4) $\sim a := a \rightarrow_G 0$
(Op5) $1 := \neg 0$

the following axioms are satisfied:

(HW1) $a \rightarrow_G a = 1$
(HW2) $a \rightarrow_G (b \wedge c) = (a \rightarrow_G c) \wedge (a \rightarrow_G b)$
(HW3) $a \wedge (a \rightarrow_G b) = a \wedge b$
(HW4) $(a \vee b) \rightarrow_G c = (a \rightarrow_G c) \wedge (b \rightarrow_G c)$
(HW5) $1 \rightarrow_L a = a$
(HW6) $a \rightarrow_L (b \rightarrow_L c) = \neg(a \rightarrow_L c) \rightarrow_L \neg b$
(HW7) $\neg \sim a \rightarrow_L \sim\sim a = 1$
(HW8) $(a \rightarrow_G b) \rightarrow_L (a \rightarrow_L b) = 1$

Let us recall that from any HW algebra it is possible to induces the algebraic structures discussed in the following Sects. 2.4.1 and 2.4.2 (see also [26]).

2.4.1 Wajsberg Algebras Induced from HW Algebras

(W) The structure $\mathfrak{W} = \langle A, \rightarrow_L, 1\rangle$ obtained from a HW algebra neglecting the implication connective $\rightarrow_G$ is a Wajsberg algebra in the sense that the following axioms are satisfied

(W1) $1 \to_L a = a$
(W2) $(a \to_L b) \to ((b \to_L c) \to_L (a \to_L c)) = 1$
(W3) $(a \to_L b) \to_L b = (b \to_L a) \to_L a$
(W4) $(a' \to_L b') \to_L (b \to_L a) = 1$

The notion of Wajsberg algebra (W algebra for short) was introduced by Wajsberg in order to give an algebraic axiomatization to many valued logic [66, 67], taking inspiration from the Łukasiewicz approach to many-valued logic [9] (see also [62]).

Furthermore, the important result about Wajsberg algebras with respect to the Kleene complementation is the following one.

Proposition 63 *Let* $\mathfrak{W} = \langle A, \to_L, 1\rangle$ *be a Wajsberg algebra. Let us define* $\wedge, \vee$ *as in equations (OP1), (OP2), the unary operation* $\neg a$ *as in equation (OP3), and according to (OP5) let us set* $0 = \neg 1$. *Then,*

- *the algebraic structure* $\mathfrak{K} = \langle A, \wedge, \vee, \neg, 0, 1\rangle$ *is a Kleene complemented lattice bounded by the least element 0 and the greatest element 1 whose induced partial order relation is given by*

$$(\mathrm{E}^*)_L \qquad a \leq b \quad \textit{iff} \quad a \to_L b = 1.$$

which according to Hardegree [41, 42] expresses one of the minimal implicative conditions*, called* law of entailment*, relating the implication connective with the partial order relation describing the binary* implication relation.

Another algebraic approach to Łukasiewicz logic is the one proposed by Chang in [29] with the name of MV algebra. This algebra represents a weakening of Boolean algebras, where the notion of *disjunction* (resp., *conjunction*) is split into two different operations. The first kind of operation behaves like a Łukasiewicz disjunction $\odot$ (resp., conjunction $\oplus$) which in general is not idempotent; the second kind of operation is a lattice meet $\wedge$ (resp., lattice join $\vee$).

As noticed by Chang "it is clear that the axiom system [formalized in [29]] is not the most economical one; they are given in the above form for their intuitive contents". A more economical and independent axiomatization of MV algebras was given in [20] according to the following.

Theorem 64 *Let* $\left\langle A, \oplus, {}', 0\right\rangle$ *be an algebra satisfying the following axioms*

(MV1) $(a \oplus b) \oplus c = b \oplus (c \oplus a)$
(MV2) $a \oplus 0 = a$
(MV3) $a \oplus 0' = 0'$
(MV4) $(0')' = 0$
(MV5) $(a' \oplus b)' \oplus b = (a \oplus b')' \oplus a$

Then, once defined $a \odot b := (a' \oplus b')'$ *and* $1 = 0'$, *the structure* $\left\langle A, \oplus, \odot, ', 0, 1\right\rangle$ *is a MV algebra according to Definition originally proposed by Chang. The vice versa is also true, i.e., any Chang formalization of MV algebra satisfies all properties (MV1)–(MV5).*

In any MV algebra it is possible to induce a *Kleene lattice* structure $\langle A, \wedge, \vee, ', 0, 1\rangle$ where the $\wedge$ and $\vee$ operations are defined as $a \vee b = (a \odot b') \oplus b$ and $a \wedge b = (a \oplus b') \odot b$, and $' : A \to A$ is the unary operation of the original MV structure. The order relation induced on A from this Kleene lattice is:

$$a \leq b \quad \text{iff} \quad a \wedge b = a \quad \text{iff} \quad a' \oplus b = 1$$

Thus in this framework, the lattice operations turn out to be defined in terms of the Kleene complement and of the Łukasiewicz operations. Whenever the two disjunctions–meet (resp., conjunctions–join) collapse into one and the same operation, $\odot = \wedge$ ($\oplus = \vee$), one obtains a Boolean algebra.

The proof of the following result can be found in [31], or also in [64, pp. 41, 44, 45]

Theorem 65

1. *Let $\mathscr{A} = \langle A, \oplus, ', 0\rangle$ be a MV algebra. Once defined $1 = 0'$ and the operator $\to_L$: $A \mapsto A$ as $a \to_L b = a' \oplus b$ and setting $\neg a = a'$, then the structure $\mathscr{A}_W = \langle A, \to_L, \neg, 1\rangle$ is a Wajsberg algebra.*
2. *Let $\mathscr{A} = \langle A, \to_L, \neg, 1\rangle$ be a Wajsberg algebra. Once defined $0 = \neg 1$ and the operator $\oplus : A \mapsto A$ as $a \oplus b = \neg a \to_L b$ and setting $a' = \neg a$, then the structure $\mathscr{A}_C = \langle A, \oplus, ', 0\rangle$ is a MV algebra.*
3. *Let $\mathscr{A} = \langle A, \oplus, ', 0\rangle$ be a MV algebra. Then $\mathscr{A} = \mathscr{A}_{WC}$.*
4. *Let $\mathscr{A} = \langle A, \to_L, \neg, 1\rangle$ be a Wajsberg algebra. Then $\mathscr{A} = \mathscr{A}_{CW}$.*

In other words, there is a one-to-one correspondence between MV algebras and Wajsberg algebras [64, p. 45, Theorem 9].

Furthermore, the Kleene lattice structure induced from $\mathscr{A}_W$ coincides with the Kleene lattice structure induced from $\mathscr{A}_C$: that is $(a \oplus b') \odot b = \neg((\neg a \to_L \neg b) \to_L \neg b)$ for the meet and an analogous result for the join.

In all the above discussion, either relative to Wajsberg algebras or to Chang MV algebras, there is no mention to the possible distributivity of the induced Kleene lattice structure. This result is proved in the context of MV algebras, whose proof can be found in [64] as consequence of proposition 54 taken into account proposition 21.

Proposition 66 *Chang MV algebras, and so Wajsberg algebras, are distributive lattices.*

2.4.2 Heyting Algebras Induced from HW Algebras

(H) The structure $\mathfrak{H} = \langle A, \wedge, \vee, \to_G, 0\rangle$ obtained from a HW algebra neglecting the implication connective $\to_L$ is a Gödel algebra that is an Heyting algebra plus the Dummett condition [27]. We recall that a Heyting algebra satisfies the following axioms

(H1) $a \to_G a = b \to_G b$
(H2) $(a \to_G b) \wedge b = b$
(H3) $a \to_G (b \wedge c) = (a \to_G c) \wedge (a \to_G b)$
(H4) $a \wedge (a \to_G b) = a \wedge b$
(H5) $(a \vee b) \to_G c = (a \to_G c) \wedge (b \to_G c)$
(H6) $0 \wedge a = 0$

and the Dummett (or pre-linearity) condition reads as:

$$\text{(D)} \qquad (a \to_G b) \vee (b \to_G a) = 1$$

We also remark that a Gödel algebra can be seen also as a residuated lattice [69], i.e. a structure $(A, \wedge, \vee, \star, \to, 0, 1)$ such that the pair $(\star, \to)$ satisfies the adjoint condition

$$\text{(Adj)} \qquad c \leq a \to b \text{ iff } a \star c \leq b,$$

where the two operators $\wedge$ and $\star$ coincide.

Heyting algebras have also been called by Birkhoff in [7, p. 45] *Brouwerian lattices*, i.e., lattices in which the *relative pseudo-complement* exists for any pair of its elements. Rasiowa and Sikorski call these algebraic structures as *relatively pseudo-complemented lattices*, whereas "every relatively pseudo-complemented lattice with zero element is called a *pseudo-Boolean algebra*" [61, p. 59], where it is remarked that the name "pseudo Boolean algebra" is due to the fact that this notion generalizes the one of Boolean algebra, since any Boolean algebra is also a pseudo Boolean algebra. It is interesting to note that this notion, in its variety of different adopted terminology, is defined in a non-equational way.

Rasiowa and Sikorski in [61, p. 59] asserted that "Obviously every *relatively pseudo-complemented lattice* can be conceived as an algebra [...] Similarly every *pseudo-Boolean algebra* can be conceived as an algebra." The equational formalization of these algebras, here only enunciated, is then dealt with in [61, Chapter IV], with all the necessary proofs (point **1.1** p. 123 for relatively pseudo-complemented lattices and point **1.2** p. 124 for pseudo Boolean algebras).

Furthermore in the footnote of [61, p. 124] it is recognised that "a simpler [equational] set of axioms for pseudo-Boolean algebras" has been introduced by Monteiro in [51]. Precisely, it is discussed an *equational* set of axioms based on lattices without 0 element with the name of *generalized Brouwer algebra* and a generalized Brouwer algebra equipped with the least element 0 is called *Brouwer algebra*. In this paper it is proved the equivalence between generalized Brouwer algebras and relatively pseudo-complemented lattices, together with the independence of the introduced axioms, and of Brouwer algebras with pseudo-Boolean lattices. In a successive paper of Monteiro [52] the same structure is called *Heyting algebra*.

Let us remark that in a footnote of the page 59 of [61] in which pseudo-Boolean lattices are introduced, Rasiowa and Sikorski underline that the structures dual to the latter are called *Brouwer algebras* by McKinsey and Tarski in [49]).

The main result about Heyting algebras (pseudo-Boolean algebras) with respect to the Brouwer complementation is the following one.

Proposition 67 *Let* $\mathfrak{H} = \langle \mathscr{A}, \wedge, \vee, \to_G, 0\rangle$ *be a Heyting algebra (pseudo-Boolean algebra). Let the negation of any* $a \in \mathscr{A}$ *be defined as* $\sim a := a \to_G 0$ *with the further definition* $1 :=\sim 0 = 0 \to_G 0$. *Then,*

- *the algebraic structure* $\mathfrak{B} = \langle \mathscr{A}, \wedge, \vee, \sim, 0, 1\rangle$ *is a Brouwer complemented lattice with* 0 *as the least and* 1 *as the greatest elements of the lattice whose induced partial order relation is given by*

$$(\mathrm{E}^*)_G \qquad a \le b \quad \textit{iff} \quad a \to_G b = 1.$$

Recalling that the non-equational way to introduce Heyting algebras (pseudo-Boolean lattices) is given by the existence for any pair of elements a, b of an element $a \to_G b$, called the *pseudo-complement* of a *relative* to b, such that,

$$(\mathrm{I}) \qquad a \wedge x \le b \quad \text{if and only if} \quad x \le a \to_G b$$

then, the (I) applied to the case $x = 1$ leads immediately to the minimal implicative condition for Heyting algebras $(\mathrm{E}^*)_G$.

In Proposition 67 there is no mention to the distributivity of the involved lattice. This as a consequence of the following result, whose proof can be found for instance in [61, p. 59].

Proposition 68 *Every pseudo-Boolean algebra (Heyting algebra) is necessarily distributive.*

As a final result (without entering in formal details for which we refer to [61]), let us recall that the idea of treating the set of all formulas of a formalized language as an abstract algebra with operations corresponding to logical connectives was first used by A. Lindenbaum and A. Tarski. Then it is possible to prove the following [61, p. 382].

Theorem 69 *Let* $\mathscr{T}$ *be a formalized intuitionistic theory, then the Lindenbaum–Tarski algebra* $\mathscr{U}(\mathscr{T})$ *associated to* $\mathscr{T}$ *is a pseudo-Boolean algebra (Heyting algebra).*

Thus, "the metatheory of the intuitionistic logic coincides with the theory of pseudo-Boolean [Heyting] algebras in the same sense as the metatheory of classical logic coincides with the theory of Boolean algebras" [61, p. 380], or simply "Heyting algebras play for the intuitionistic propositional calculus the same role played by the Boolean algebras for the classical propositional calculus." [52].

2.4.3 An Important Result About HW Algebras

Given a Heyting Wajsberg algebra it turns out that the following results can be proved [26].

Proposition 70 *Let $\mathfrak{HW}(A)$ be a Heyting Wajsberg algebra. Then*

(HW1) *the operations $\wedge$ and $\vee$ are just the meet and join operators of a* distributive *lattice structure whose partial order is as usual defined as $a \leq b$ iff $a \wedge b = a$ (equivalently, iff $a \vee b = b$).*

(HW2) *It is easy to prove that under the above axioms the following holds:*

$$a \leq b \quad \textit{iff} \quad a \to_L b = 1 \quad | iff \quad a \to_G b = 1 \tag{53}$$

(HW3) *The structure $\mathfrak{BZ}(A) := \langle A, \wedge, \vee, \neg, \sim, 0\rangle$ is a* Brouwer Kleene algebra *(*distributive lattice*) whose Brouwer negation satisfies the B-De Morgan condition (B2b) (i.e., it is a BK^{DM} algebra). The induced interior and closure operations are $\mathbf{i}(a) = \sim\neg(a)$ and $\mathbf{c}(a) = \neg\sim(a)$, respectively. Hence the exterior of a is then $\mathbf{e}(a) = \sim a$.*

Some formal remarks. What in HW algebras is denoted by $\neg a$ and $\sim a$ for a generic element $a \in A$ coincides with what in the BZ context is denoted by a' and $a^{\sim}$. In this section we maintain the orthodox notation of the HW algebras, considering it is easy to translate this last to the BZ notation.

Let us recall the results of Proposition 51 which assure that the interior (resp., closure) operation satisfies the condition (DD) (resp., (DD_c)).

As a consequence of the point (HW3) we can apply to HW algebras all the results obtained in the Part II. Precisely,

1. Since $\mathfrak{BZ}(A)$ is in particular a Boolean algebra with respect to the negation $\neg$, point (BA) of Proposition 29 assures that the collection $\mathbb{A}(\mathfrak{BZ}(A))$ of all orthopairs $(a_1, a_0) \in \mathfrak{BZ}(A) \times \mathfrak{BZ}(A)$ such that $a_1 \leq \neg a_0$ is a Brouwer Kleene (BK) algebra (distributive lattice) whose operation $\sim (a_1, a_0) = (a_0, \neg a_0)$ is a Brouwer negation as consequence of point (BL-B3) of Proposition 29 and whose operation $-(a_1, a_0) = (a_0, a_1)$ is a Kleene negation as proved in point (BA) of the same proposition, which in general is not Boolean. Furthermore, the Brouwer negation $\sim$ satisfies the B-De Morgan condition (B2b) of Proposition 51 and so

$$\mathbb{A}(\mathfrak{BZ}(A)) \text{ is a BK}^{DM} \text{ algebra.}$$

2. Point (3) of Theorem 53 holds, which assures that the collection $\mathbb{R}(\mathfrak{BZ}(A))$ of all ortho-rough approximations $(\mathbf{i}(a), \mathbf{e}(a)) = (\sim\neg(a), \sim(a))$, for $a \in \mathfrak{BZ}(A)$, is a Brouwer Boolean algebra (distributive lattice) whose Brouwer negation $\sim$ satisfies the B-De Morgan condition (B2b) of Proposition 51 and so

$$\mathbb{R}(\mathfrak{BZ}(A)) \text{ is a BB}^{DM} \text{ algebra.}$$

The development of the abstract theory of Heyting Wajsberg (HW) algebra as an algebraic method developed in the lattice context would be too wasteful of space to be developed in this chapter (obviously interested persons can access the papers [26, 27]). We only mention without proofs some interesting results.

First of all, let us remark that HW algebras are equivalent to other well-known algebras such as

- De Morgan BZMV^{DM} algebras, which are a pasting of BZ^{DM} lattices and MV algebras [25];
- De Morgan BZ Wajsberg algebras, a pasting of BZ^{DM} lattices and Wajsberg algebras [13];
- Stonean MV algebras [6];
- MV$_\Delta$ algebras [40].

Finally, in [27] a logical calculus has been introduced such that the corresponding Lindembaum–Tarski algebra is a HW algebra and its semantical completeness has been proved.

2.5 *Concrete Models of HW Algebras*

HW algebras are the best suitable structures able to fully characterize some interesting concrete models. For instance, as we will see in the following subsections, the real unit interval or the collection of all fuzzy sets on a universe X can be endowed with a HW algebraic structure.

2.5.1 HW Algebra Based on Real Unit Interval

In this subsection we discuss, as a useful model, the particular case of the concrete HW algebra based on the unit real interval $[0, 1]$. This approach is in itself interesting and not particularly reductive because all the results on the HW algebra $[0, 1]$ that will be exposed here are valid and immediately extendable to the abstract algebraic case. Indeed, owing to [27, Theorem 2.8] and some equivalence theorems [27, Section 2], we can state the very important result about the *strong form of completeness theorem*:

(SCT) Let ϕ and ψ be well defined terms, in the traditional way, on the language of a HW algebra (that is well formed formulas formed using the primitive and derived unary and binary connectives).

If the identity $\phi = \psi$, as expression of a HW principle, holds in the [0,1]–model.

Then the same HW principle holds in *all* HW algebras.

From the terminological point of view, we must stress that in the concrete case of the real unit interval [0,1] the structure of Wajsberg algebra is obtained through an implication connective introduced by Łukasiewicz while the structure of Heyting algebra is obtained through an implication connective introduced by Gödel (see the important book of Rescher on multi-valued logic [62]).

In conclusion as we say in this long quotation from the introduction of [26]:

> Taking inspiration by [. . .] considerations about [0, 1], we are going to study algebraic structures provided by [. . .] more than one implication and we will focus our attention on new added operators definable by composition of the previous ones. All these structures are algebraic counterparts of corresponding logical systems. [. . .]
>
> At the top level of our construction we introduce Heyting Wajsberg (HW) algebras. This new structure is characterized by the presence of both and only the Gödel and Łukasiewicz implications as primitive operators. It was introduced with the aim of giving a rich and complete algebraic approach to rough sets [14, 17], and it revealed a great connection with other existing algebras of many valued logics.

So let us investigate the real unit interval as a standard environment of an algebraic model for many valued logic. Technically speaking, truth values of a logical system are defined just as syntactic labels, with no numerical meaning. In a subsequent step, it is possible to give an interpretation of the logical system in terms of an algebraic structure; only during such a process, the truth values are associated with elements of the structure, that can be mathematical objects more abstract than real numbers. Therefore, always quoting from [26]:

> The numbers of [0, 1] are interpreted, after Łukasiewicz [47], as the possible truth values which the logical sentences can be assigned to. As usually done in literature, the values 1 and 0 denote respectively truth and falseness, whereas all the other values are used to indicate different degrees of indefiniteness.

Now the following result holds.

(U-HW) Let us consider the structure $\mathfrak{H}\mathfrak{W}([0,1]) = \langle [0,1], \to_L, \to_G, 0\rangle$ consisting of the totally ordered unit interval [0, 1] equipped with the two primitive *implication* connectives defined by the following equations:

$$a \to_L b := \min\{1, 1 - a + b\}$$

$$a \to_G b := \begin{cases} 1 & \text{if } a \le b \\ b & \text{if } a > b \end{cases}$$

Then introduced according to the definitions (Op1)–(Op5) the derived connectives

$$a \vee b := \neg((\neg a \to_L \neg b) \to_L \neg b) = \max\{a, b\} \tag{54a}$$

$$a \wedge b := (a \to_L b) \to_L b = \min\{a, b\} \tag{54b}$$

$$\neg a := a \to_L 0 = 1 - a \tag{54c}$$

$$\sim a := a \to_G 0 = \begin{cases} 1 & \text{if } a = 0 \\ 0 & \text{otherwise} \end{cases} \tag{54d}$$

$$1 := 0 \to_L 0 \tag{54e}$$

it turns out that $\mathfrak{HW}([0,1])$ is a HW algebra since all the eight axioms (HW1)–(HW8) of Definition 62 characterizing this algebraic structure are satisfied (see also the section 4 of [14]).

The connective $\to_L$ is the implication introduced by Łukasiewicz in his infinite valued logic L_∞ in 1930, while the connective $\to_G$ is the implication introduced by Gödel in his infinite valued logic G_∞ (see [62]).

Let us stress that the standard total order of [0, 1] can be recovered by these implications since it is

$$\forall a, b \in [0,1], \quad a \le b \quad \text{iff} \quad a \to_L b = 1 \quad \text{iff} \quad a \to_G b = 1$$

This means that both these implication connectives satisfy the law of entailment (E*), one of the minimal implication conditions (see [41, 42]).

According to the general theory of HW algebras the MV-*disjunction* and MV-*conjunction* connectives are the following.

$$a \oplus b = \neg a \to_L b = \min\{1, a + b\} \tag{55a}$$

$$a \odot b = \neg(a \to_L \neg b) = \neg(\neg a \oplus \neg b) = \max\{0, a + b - 1\} \tag{55b}$$

Let us now consider the two unary operations defined inside the HW algebra structure [0, 1] by the two implication connectives $\to_L$ and $\to_G$ according to the following two definitions:

$$\mathbf{c}(a) = (a \to_G 0) \to_L 0 = \begin{cases} 0 & \text{if } a = 0 \\ 1 & \text{otherwise} \end{cases} \tag{56a}$$

$$\mathbf{i}(a) = (a \to_L 0) \to_G 0 = \begin{cases} 1 & \text{if } a = 1 \\ 0 & \text{otherwise} \end{cases} \tag{56b}$$

Then the following is easy to prove.

(U-KC) The structure $\mathfrak{KC}([0,1]) := \langle [0,1], \wedge, \vee, \neg, \mathbf{c}, 0, 1 \rangle$ based on the unit interval [0, 1] is a Kuratowski closure (Kleene) lattice whose Kuratowski closure operation $\mathbf{c} : [0,1] \to [0,1]$ is defined by Eq. (56a). The corresponding set of closed elements is $\mathscr{C}([0,1]) = \{0, 1\}$, the two values Boolean algebra, since $\mathbf{c}(a) = a$ in [0, 1] iff either $a = 0$ or $a = 1$.

(U-KI) The Kuratowski interior operation induced from the closure $\mathbf{c}$ is just the operation given by (56b), that is $\mathbf{i} = \neg\mathbf{c}\neg(a)$. The structure of Kuratowski interior (Kleene) lattice is defined consequently and the corresponding set of open elements is $\mathscr{O}([0,1]) = \{0,1\}$, i.e., always the two values Boolean algebra, since also in this case $\mathbf{i}(a) = a$ in $[0,1]$ iff either $a = 0$ or $a = 1$.

(U-RAS) The rough approximation space based on $[0,1]$ is then obtained by the rough approximation map $r : [0,1] \to \{0,1\} \times \{0,1\}$ assigning to any real number $a \in [0,1]$ the Boolean pair $r(a) = (\mathbf{i}(a), \mathbf{c}(a)) \in \{0,1\} \times \{0,1\}$.

Making use of the closure operation it is possible to introduce another binary connective considered by Monteiro in [52] and defined as follows:

$$a \to_F b = \mathbf{c}(\neg a) \vee b = \begin{cases} b & \text{if } a = 1 \\ 1 & \text{otherwise} \end{cases}$$

with respect to which we have a further negation connective:

$$\flat a = a \to_F 0 = \begin{cases} 0 & \text{if } a = 1 \\ 1 & \text{otherwise} \end{cases}$$

Let us stress that the two binary operations $\odot$ and $\wedge$ are paradigmatic examples of *continuous* t-norms, i.e., continuous mappings $\mathbf{t} : [0,1] \times [0,1] \mapsto [0,1]$ fulfilling the following properties for all $a, b, c \in [0,1]$:

(T1) $a\mathbf{t}b = b\mathbf{t}a$ (commutativity)
(T2) $(a\mathbf{t}b)\mathbf{t}c = x\mathbf{t}(b\mathbf{t}c)$ (associativity)
(T3) $a \le b$ implies $a\mathbf{t}c \le b\mathbf{t}c$ (monotonicity)
(T4) $a\mathbf{t}1 = 1\mathbf{t}a = a$

The t-norm $\wedge$ is called the Gödel t-norm, while the t-norm $\odot$ is the Łukasiewicz t-norm.

Let $\mathbf{t}$ be a continuous t-norm. The *implication* (*residuum, quasi-inverse*) operation induced by $\mathbf{t}$ is the map $\to_{\mathbf{t}}$: $[0,1] \times [0,1] \mapsto [0,1]$ defined for arbitrary $a, b \in [0,1]$ as follows:

$$a \to_{\mathbf{t}} b = \sup\{c \in [0,1] : a\mathbf{t}c \le b\}$$

The Gödel and Łukasiewicz t-norms induce the above two implication connectives: $a \to_\wedge b = a \to_G b$ and $a \to_\odot b = a \to_L b$ respectively.

Remark 71 As proved in [26], there does not exist a t-norm whose residuum in the Monteiro implication $\to_F$ defined above.

Let $\mathbf{t}$ be a t-norm with associated implication operation $\to_{\mathbf{t}}$. The *negation* induced from $\mathbf{t}$ is the unary operation $\neg_{\mathbf{t}} : [0,1] \mapsto [0,1]$ defined as:

$$\neg_{\mathbf{t}} a := a \to_{\mathbf{t}} 0 = \sup\{c \in [0,1] : a\mathbf{t}c = 0\}$$

It is worth noting that the negation induced by the Gödel t-norm is $a \to_G 0 = \sim a$, while the negation induced by the Łukasiewicz t-norm is $a \to_L 0 = \neg a$.

Finally, a $\mathbf{t}$-conorm is a mapping $\mathbf{s} : [0,1] \times [0,1] \mapsto [0,1]$ fulfilling properties (T1), (T2), (T3) and the boundary condition:

(S4) $a\mathbf{s}0 = a$ for all $a \in [0,1]$.

Given a t-norm $\mathbf{t}$, the *dual t-conorm* is defined through the formula

$$\mathbf{s_t}(a,b) := 1 - (\mathbf{t}((1-a),(1-b))$$

The dual t-conorms of Łukasiewicz and Gödel t-norms, are respectively, the mappings $\oplus$ and $\vee$.

2.5.2 HW Algebra Based on Fuzzy Sets

In this section we show that, similarly to the previous case of real unit interval, the collection of all fuzzy sets $\mathscr{F}(X) = [0,1]^X$ on the universe X, can be equipped with a structure of a HW algebra once defined the suitable two implication operators.

Proposition 72 *The structure $\langle [0,1]^X, \to_L, \to_G, \mathbf{0} \rangle$ is a HW algebra once defined the implication operators as follows:*

$$(f_1 \to_L f_2)(x) := \min\{1, 1 - f_1(x) + f_2(x)\}$$

$$(f_1 \to_G f_2)(x) := \begin{cases} 1 & f_1(x) \leq f_2(x) \\ f_2(x) & \textit{otherwise} \end{cases}$$

The lattice meet and join operations defined on $\mathscr{F}(X)$ by the fuzzy set HW structure according to the above (Op1) and (Op2) are the following:

$$(f_1 \vee f_2)(x) = \max\{f_1(x), f_2(x)\}$$

$$(f_1 \wedge f_2)(x) = \min\{f_1(x), f_2(x)\}$$

whose corresponding partial order relation is the usual pointwise ordering on fuzzy sets:

$$\forall f_1, f_2 \in \mathscr{F}(X), \quad f_1 \leq f_2 \text{ iff } \forall x \in X, f_1(x) \leq f_2(x)$$

The Kleene and Brouwer negations induced on the HW structure of $\mathscr{F}(X)$ according to the above definition (Op3) and (OP4) are given respectively by:

$$\neg f(x) := 1 - f(x)$$
$$\sim f(x) := \begin{cases} 1, & \text{if } f(x) = 0 \\ 0, & \text{otherwise} \end{cases}$$

In Sect. 2.3.1 these two negations have been denoted respectively as $f' = \neg f$ and $f^{\sim} = \sim f$.

The MV operations of conjunction and disjunction are respectively:

$$(f_1 \oplus f_2)(x) = \min\{1, f_1(x) + f_2(x)\}$$
$$(f_1 \odot f_2)(x) = \max\{0, f_1(x) + f_2(x) - 1\}$$

Recalling that in point (BKF2) of Sect. 2.3.1 we have introduced the certainty-yes domain (resp., possibility domain) of a fuzzy set f as the subset of the universe $A_1(f) := \{x \in X : f(x) = 1\}$ (resp., $A_p(f) := \{x \in X : f(x) \neq 0\}$), we have the following.

The interior of a fuzzy set f is the characteristic function of the certainty-yes domain of f:

$$\mathbf{i}(f) = \chi_{A_1(f)} = \begin{cases} 1 & \text{if } f(x) = 1 \\ 0 & \text{otherwise} \end{cases}$$

The closure of a fuzzy set f is the characteristic function of the possibility domain of f:

$$\mathbf{c}(f) = \chi_{A_p(f)} = \begin{cases} 1 & \text{if } f(x) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Let us recall that in Sect. 2.3.1 we have introduced the isomorphism $\Phi^2 : (\mathscr{E}(X) \times \mathscr{E}(X))_\perp \to (\mathscr{P}(X) \times \mathscr{P}(X))_\perp$, $(\chi_A, \chi_B) \to \Phi^2(\chi_A, \chi_B) := (A, B)$, which allows the identification between the collection $(\mathscr{E}(X) \times \mathscr{E}(X))_\perp$ of all orthopairs of crisp sets $(\chi_{A_1(f)}, \chi_{A_0(f)})$ (with $\chi_{A_1(f)} \perp \chi_{A_0(f)}$) and the collection $(\mathscr{P}(X) \times \mathscr{P}(X))_\perp$ of all the orthopair of subsets of the universe $(A_1(f), A_0(f))$ (with $(A_1(f) \cap A_0(f)) = \emptyset$). Formally,

$$(\chi_{A_1(f)}, \chi_{A_0(f)}) \in (\mathscr{E}(X) \times \mathscr{E}(X))_\perp \longleftrightarrow (\mathscr{P}(X) \times \mathscr{P}(X))_\perp \ni (A_1(f), A_0(f))$$

Furthermore, in Sect. 1.3.1 we have seen that the collection $\mathbb{P}(X) = (\mathscr{P}(X) \times \mathscr{P}(X))_\perp$ of all the orthopairs from the power set $\mathscr{P}(X)$ of the universe X (which is a Boolean algebra) has a structure of BK^{DM} algebra (distributive lattice). From another point of view, the Boolean algebra $\mathscr{P}(X)$ of subsets generates the BK^{DM} algebra $\mathbb{P}(X)$ of orthopairs of subsets.

But in point (HW3) of Proposition 70 we have shown that in general any abstract structure of HW algebra $\mathfrak{HW}(A)$ generates a structure of BK^{DM} algebra $\mathfrak{BZ}(A)$. Thus it is of some interest to investigate whether in the concrete case of $\mathbb{P}(X)$ the BK^{DM} structure can also be obtained from a corresponding concrete HW algebra defined on it. We shall give a positive answer to this problem in the forthcoming Corollary 74 in the general context of the family $\mathbb{A}(\Sigma)$ of all orthopairs from an abstract Boolean algebra Σ.

This topic will be dealt with in the following section where all the operations involved have been inspired by the works of Pagliani [54, 55], also treated by us in [17] and for the only Gödel implication case in [18].

2.6 HW Algebra of All Orthopairs Induced by Boolean Algebras

Now, we give to the collection of all orthopairs from a Boolean algebra the structure of a HW algebra.

Theorem 73 *Let* $\langle \Sigma, \wedge, \vee, \neg, 0, 1\rangle$ *be a Boolean algebra with associated collection of all orthopairs* $\mathbb{A}(\Sigma) = \{(a_1, a_0) \in \Sigma \times \Sigma : a_1 \perp a_0\}$*, equipped with the negation* $-(a_1, a_0) = (a_0, a_1)$.

Then, the structure $\langle \mathbb{A}(\Sigma), \to_L, \to_G, \mathbf{0}\rangle$*, where* $\to_L, \to_G$ *and* $\mathbf{0}$ *are defined as follows:*

$$(a_1, a_0) \to_L (b_1, b_0) := ((\neg a_1 \wedge \neg b_0) \vee a_0 \vee b_1, a_1 \wedge b_0) \tag{57a}$$

$$(a_1, a_0) \to_G (b_1, b_0) := ((\neg a_1 \wedge \neg b_0) \vee a_0 \vee b_1, \neg a_0 \wedge b_0) \tag{57b}$$

$$\mathbf{0} := (0, 1) \tag{57c}$$

is the HW *algebra of orthopairs.*

According to points (OP1) and (OP2) of Definition 62 the lattice operations are defined as

$$\begin{aligned}(a_1, a_0) \sqcup (b_1, b_0) &:= ((a_1, a_0) \to_L (b_1, b_0)) \to_L (b_1, b_0)\\ &= (a_1 \vee b_1, a_0 \wedge b_0)\\ (a_1, a_0) \sqcap (b_1, b_0) &:= -((-(a_1, a_0) \to_L -(b_1, b_0)) \to_L -(b_1, b_0))\\ &= (a_1 \wedge b_1, a_0 \vee b_0)\end{aligned}$$

The partial order induced from these lattice operations is:

$$(a_1, a_0) \sqsubseteq (b_1, b_0) \quad \textit{iff} \quad a_1 \le b_1 \textit{ and } b_0 \le a_0$$

According to points (OP3) and (OP4) of Definition 62 the two negations are defined as

$$-(a_1, a_0) := (a_1, a_0) \to_L \mathbf{0} = (a_0, a_1)$$

$$\sim (a_1, a_0) := (a_1, a_0) \to_G \mathbf{0} = (a_0, \neg a_0)$$

Therefore, in agreement with point (HW3) of Proposition 70, the negation $-$ is Kleene and $\sim$ is a Brouwer negation satisfying the B-De Morgan condition (B2b).

So, the anti–Brouwer negation is:

$$\flat(a_1, a_0) := - \sim -(a_1, a_0) = (\neg a_1, a_1)$$

and according to point (Op5) the greatest lattice element is $\mathbf{1} := (1, 0)$.

The MV conjunction and disjunction operations are then the following:

$$(a_1, a_0) \oplus (b_1, b_0) = ((\neg a_0 \wedge \neg b_0) \vee (a_1 \vee b_1), a_0 \wedge b_0)$$

$$(a_1, a_0) \odot (b_1, b_0) = (a_1 \wedge b_1, (\neg a_1 \wedge \neg b_1) \vee (a_0 \vee b_0))$$

Proof Trivially, $\to_L$ and $\to_G$ are closed operators on $\mathbb{A}(\Sigma)$, i.e., for any pair of elements $h, k \in \mathbb{A}(\Sigma)$, $(h \to_L k) \in \mathbb{A}(\Sigma)$ and $(h \to_G k) \in \mathbb{A}(\Sigma)$. That axioms (HW1)–(HW8) are satisfied is proved in Appendix 2.9. □

Let us synthesize the following particular aspect of the previous Theorem 73 stated as suitable corollary formed by two steps.

Corollary 74 *Starting from the Boolean algebra* $\langle \Sigma, \wedge, \vee, \neg, 0, 1 \rangle$

(Step 1) *it is possible to construct the induced HW algebra based on the collection* $\mathbb{A}(\Sigma) = \{(a_1, a_0) \in \Sigma \times \Sigma : a_1 \perp a_0\}$ *of all its orthopairs equipped with the two implications defined by Eq.* (57)*:*

$$\langle \mathbb{A}(\Sigma), \to_L, \to_G, \mathbf{0} = (0, 1) \rangle$$

(Step 2) *then, it is possible, according to point (HW3) of Proposition 70, to induce the BK*^{DM} *algebra discussed in Theorem 26*

$$\langle \mathbb{A}(\Sigma), \sqcap, \sqcup, -, \sim, (0, 1), (1, 0) \rangle$$

Indeed, the lattice operations $\sqcap, \sqcup$ *and the negations* $-, \sim$ *are the same:*

$$(a_1, a_0) \sqcup (b_1, b_0) = (a_1 \vee b_1, a_0 \wedge b_0)$$

$$(a_1, a_0) \sqcap (b_1, b_0) = (a_1 \wedge b_1, a_0 \vee b_0)$$

$$-(a_1, a_0) := (a_0, a_1)$$

$$\sim (a_1, a_0) := (a_0, \neg a_0)$$

Consequently, the interior and closure operators are also the same:

$$\mathbf{i}(a_1, a_0) = (a_1, \neg a_1)$$
$$\mathbf{c}(a_1, a_0) = (\neg a_0, a_0)$$

with the obvious relationships:

$$\mathbf{i}(a_1, a_0) = (a_1, \neg a_1) \sqsubseteq (a_1, a_0) \sqsubseteq (\neg a_0, a_0) = \mathbf{c}(a_1, a_0).$$

Furthermore, the Kleene algebra sub-structure $\langle \mathbb{A}(\Sigma), \sqcap, \sqcup, -, \mathbf{0}, \mathbf{1} \rangle$ *cannot be a Boolean algebra owing to the presence of the half element* $\frac{\mathbf{1}}{\mathbf{2}} := (0, 0)$ *such that* $\frac{\mathbf{1}}{\mathbf{2}} \sqcap \frac{\mathbf{1}}{\mathbf{2}} = \frac{\mathbf{1}}{\mathbf{2}} \sqcup \frac{\mathbf{1}}{\mathbf{2}} = \frac{\mathbf{1}}{\mathbf{2}}$.

From the modal point of view the structure $\langle \mathbb{A}(\Sigma), \sqcap, \sqcup, -, \mathbf{i}, (0, 1), (1, 0) \rangle$ satisfies the (AM) statement discussed in Sect. 2.1.3 and the further (DD) principle, since this is true for any HW algebra. Summarizing,

1. it based on a Kleene algebra (distributive lattice), which cannot be a Boolean algebra;
2. it is S5-like since the modal principles N, T, M-C, and 5 are satisfied by the necessity operator $\mathbf{i}$;
3. it satisfies the "spurious" axiom (DD).

Example 75 (The HW Algebra of Orthopairs from the Power Set $\mathscr{P}(X)$*)* For any fixed universe of points X, its power set has a structure $\langle \mathscr{P}(X), \cap, \cup, {}^c, \emptyset, X \rangle$ of Boolean algebra and so one can apply the two steps of Corollary 74 of Theorem 73 in order to obtain the HW algebra based on the collection $\mathbb{P}(X)$ of all orthopairs (A_1, A_0) of subsets of X (with $A_1 \cap A_0 = \emptyset$) and containing the half element $(\emptyset, \emptyset)$:

$$\langle \mathbb{P}(X), \rightarrow_L, \rightarrow_G, (\emptyset, X) \rangle$$

This according to the operations defined by Eq. (57):

$$(A_1, A_0) \rightarrow_L (B_1, B_0) := ((A_1^c \cup B_0^c) \cup A_0 \cup B_1,\ A_1 \cap B_0) \tag{58a}$$
$$(A_1, B_0) \rightarrow_G (B_1, B_0) := ((A_1^c \cup B_0^c) \cup A_0 \cup B_1,\ A_0^c \cap B_0) \tag{58b}$$

The induced (according to the step 2) BK^{DM} algebra

$$\langle \mathbb{P}(X), \sqcap, \sqcup, -, \sim, (\emptyset, X), (X, \emptyset) \rangle .$$

is characterized by the following operations:

$$(A_1, A_0) \sqcap (B_1, B_0) = (A_1 \cap B_1, A_0 \cup B_0) \quad (59a)$$

$$(A_1, A_0) \sqcup (B_1, B_0) = (A_1 \cup B_1, A_0 \cap B_0) \quad (59b)$$

$$-(A_1, A_0) = (A_0, A_1) \quad (59c)$$

$$\sim (A_1, A_0) = (A_0, A_0^c) \quad (59d)$$

This in agreement with what we have discussed in Sect. 1.3.1.

We remark that orthopairs have been considered also in other algebraic settings by different authors:

- Walker studied orthopairs on a Boolean algebra Σ [68] and proved that the structure $\langle \mathbb{A}(\Sigma), \sqcap, \sqcup, \sim, (0, 1)\rangle$ is a Stone algebra, where $\sqcap, \sqcup$ and $\sim$, are the operations defined as above.
- The structure of nested pairs of a Boolean algebra, i.e., pairs (a, b) such that $a \leq b$, was studied also by Monteiro [52, p. 199]. Translating his results to orthopairs, he showed that $\langle \mathbb{A}(\Sigma), \sqcap, \sqcup, -, \mathbf{c}, (1, 0)\rangle$ is a three-valued Łukasiewicz algebra.
- If we consider as starting point an Heyting algebra $\langle \Sigma, \wedge, \vee, \rightarrow, 0, 1\rangle$ instead of a Boolean one, Vakarelov [65] showed that $\langle \mathbb{A}(\Sigma), \sqcap, \sqcup, \rightsquigarrow, \neg, \mathbf{c}, (1, 0)\rangle$ where $(a, b) \rightsquigarrow (c, d) := (a \rightarrow c, a \wedge d)$ is a Nelson algebra.

Further historical remarks can be found in [56, Frame 10.11].

2.7 HW Algebra $\mathbb{R}(\Sigma)$ of All Rough Representatives Induced from BB^{DM} Algebras

In the previous Sect. 2.6 we have seen that for a given Boolean algebra $\langle \Sigma, \wedge, \vee, \neg, 0, 1\rangle$, the collection of all its orthopairs $\mathbb{A} = \{(a_1, a_0) \in \Sigma^2 : a_1 \perp a_0\}$ has a structure of HW algebra with respect to the two implication connectives of Eq. (57) introduced by Theorem 73.

In this section we want to explore another possibility of constructing a HW algebra on the basis of a BB^{DM} structure. As this structure was quickly mentioned earlier, we want now give it a complete definition.

Definition 76 A BB^{DM} algebra is a structure $\langle \Sigma, \wedge, \vee, \neg, \sim, 0, 1\rangle$ such that

(BB1) the sub-structure $\langle \Sigma, \wedge, \vee, 0, 1\rangle$ is a distributive bounded lattice;

(BB2) the sub-structure $\langle \Sigma, \wedge, \vee, \neg, 0, 1\rangle$ is a Boolean algebra, i.e., a De Morgan algebra satisfying the two conditions

(oc-1,2) $\forall a \in \Sigma, a \wedge \neg a = 0$ and $a \vee \neg a = 1$;

(BB3) the mapping $\sim: \Sigma \to \Sigma$ is a Brouwer negation satisfying besides conditions (B1) and (B3) also the further two B-De Morgan conditions

(B2a,b) $\forall a, b \in \Sigma, \sim(a \vee b) = \sim a \wedge \sim b$ and $\sim(a \wedge b) = \sim a \vee \sim b$.

Since a BB^{DM} algebra is in particular a Boolean algebra (see point (BB2)) it is always possible to construct on its basis the HW algebra $\mathbb{A}(\Sigma)$ of its orthopairs. But in this particular case it is also possible to construct the collection $\mathbb{R}(\Sigma) = \{r_\perp(a) = (\mathbf{i}(a), \mathbf{e}(a)) : a \in \Sigma\}$ of all ortho-rough approximations $r_\perp(a)$ for a running in Σ. In the next theorem we show that it is possible to easily define a HW algebraic structure on $\mathbb{R}(\Sigma)$. Recall that setting $a_i = \mathbf{i}(a) = \sim\neg(a)$ and $a_e = \mathbf{e}(a) = \sim a$ we usually write $r_\perp(a) = (a_i, a_e)$.

Theorem 77 *Let Σ be a BB^{DM} algebra and let $(a_i, a_e), (b_i, b_e) \in \mathbb{R}(\Sigma)$. Since from point (BB2) we have a sub-structure of Boolean algebra, the following two implication operations defined similarly as in Eq. (57) can be introduced*

$$(a_i, a_e) \to_L (b_i, b_e) := ((\neg a_i \wedge \neg b_e) \vee a_e \vee b_i, a_i \wedge b_e) \tag{60a}$$

$$(a_i, a_e) \to_G (b_i, b_e) := ((\neg a_i \wedge \neg b_e) \vee a_e \vee b_i, \neg a_e \wedge b_e) \tag{60b}$$

They are closed on $\mathbb{R}(\Sigma)$ and the following hold:

$$(a_i, a_e) \to_L (b_i, b_e) = r_\perp(\neg(a_i \vee b_e) \vee \neg a \vee b),$$

$$(a_i, a_e) \to_G (b_i, b_e) = r_\perp(\neg(a_i \vee b_e) \vee \sim a \vee b).$$

The structure $\langle \mathbb{R}(\Sigma), \to_L, \to_G, (0, 1)\rangle$ is a HW algebra whose lattice operations defined according points (OP1) and (OP2) of Definition 62 are:

$$(a_i, a_e) \sqcap (b_i, b_e) = (a_i \wedge b_i, a_e \vee b_e) = r_\perp(a \wedge b)$$

$$(a_i, a_e) \sqcup (b_i, b_e) = (a_i \vee b_i, a_e \wedge b_e) = r_\perp(a \vee b)$$

The Kleene, Brouwer and anti-Brouwer negations of any element $(a_i, a_e) \in \mathbb{R}(\Sigma)$ are respectively:

$$-(a_i, a_e) := (a_e, a_i) = r_\perp(\neg a)$$

$$\sim(a_i, a_e) := (a_e, \neg a_e) = r_\perp(\sim a)$$

$$\flat(a_i, a_e) := (\neg a_i, a_i) = r_\perp(\flat a)$$

The MV operations are defined as:

$$(a_i, a_e) \oplus (b_i, b_e) := ((\neg a_e \wedge \neg a_e) \vee (a_i \vee b_i), a_e \wedge b_e)$$

$$(a_i, a_e) \odot (b_i, b_e) := (a_i \wedge b_i, (\neg a_i \wedge \neg b_i) \vee (a_e \vee b_e))$$

Finally, the interior and closure operations are defined as

$$\mathbf{i}(a_i, a_e) := (a_i, \neg a_i) = r_\perp(\mathbf{i}(a))$$

$$\mathbf{c}(a_i, a_e) := (\neg a_e, a_e) = r_\perp(\mathbf{c}(a))$$

Proof If (a_i, a_e) , $(b_i, b_e) \in \mathbb{R}(\Sigma)$ then,

$$\begin{aligned}
& r_\perp(\neg(a_i \vee b_e) \vee \neg a \vee b) = \\
& = (\sim \neg[(\neg \sim \neg(a) \wedge \neg \sim b) \vee \neg a \vee b], \sim[(\neg \sim \neg(a) \wedge \neg \sim b) \vee \neg a \vee b]) = \\
& = (\sim [(\sim \neg a \vee \sim b) \wedge a \wedge \neg b], \sim[(\neg \sim \neg(a) \wedge \neg \sim b) \vee \neg a \vee b]) = \\
& = ((\sim\sim \neg a \wedge \sim\sim b) \vee \sim a \vee \mathbf{i}(b), (\sim \neg \sim \neg a \vee \sim \neg \sim b) \wedge \sim \neg a \wedge \sim b) = \\
& = ((\neg \sim \neg a \wedge \neg \sim b) \vee \sim a \vee \mathbf{i}(b), (\sim\sim\sim \neg a \vee \sim\sim\sim b) \wedge \sim \neg a \wedge \sim b) = \\
& = ((\neg \mathbf{i}(a) \wedge \neg b_e) \vee a_e \vee \mathbf{i}(b), (\neg a \vee \sim b) \wedge \sim \neg a \wedge \sim b) = \\
& = ((\neg a_i \wedge \neg b_e) \vee a_e \vee b_i, a_i \wedge b_e) = (a_i, a_e) \rightarrow_L (b_i, b_e).
\end{aligned}$$

Dually, for the Gödel implication.

Let us consider $r_\perp(a \wedge b) = (\mathbf{i}(a \wedge b), \sim (a \wedge b)) =$ (B2b) $= (\sim \neg(a \wedge b), \sim a \vee \sim b) =$ (DM2b) $= (\sim (\neg a \vee \neg b), a_e \vee b_e) =$ (B2a) $= (\sim \neg a \wedge \sim \neg b, a_e \vee b_e) = (a_i \wedge b_i, a_e \vee b_e)$. Dually, for the $\sqcup$ case.

Let us consider $r_\perp(\mathbf{i}(a)) = (\mathbf{i}(\mathbf{i}(a)), \mathbf{e}(\mathbf{i}(a))) = (\mathbf{i}(a), \sim\sim \neg(a))$. But from the interconnection rule (IR) we have that $\forall \alpha \in \Sigma$, $\sim\sim \alpha = \neg \sim \alpha$, which applied to $\alpha = \neg a$ leads to $\sim\sim \neg a = \neg(\sim \neg a) = \neg(\mathbf{i}(a))$. Thus we obtain that $r_\perp(\mathbf{i}(a)) = (\mathbf{i}(a), \neg(\mathbf{i}(a))$. Dually, for the other identity, or directly $r_\perp(\mathbf{c}(a)) = (\mathbf{i}(\mathbf{c}a), \sim \mathbf{c}a) = (\sim \neg\neg \sim a, \sim \neg \sim a) =$ (IR) $= (\neg \sim a, \sim a) = (\neg a_e, a_e)$.

All the other identities are easy to prove. □

2.7.1 Implication Operations in Concrete Pawlak Rough Set Theory

In Sect. 2.3.2 we have treated the Pawlak approach to *rough sets* on a given (not empty and finite) universe of points X based to an *equivalence space* $(X, \equiv)$, where $\equiv$ is an *equivalence* (reflexive, symmetric, and transitive) binary relation on X. This relation is interpreted as an *indiscernibility* relation since for any pair of points $a, b \in X$ the formula $a \equiv b$ means that there is a *system of information* generating $(X, \equiv)$ with respect to which "a cannot be distinguished from b" (for a general discourse about this rough sets argument see [11]).

Of course, as usual, from any equivalence relation it can be induced a partition $\pi := \{G_j : j \in J\}$ of the universe X by the equivalence classes G_j generated from $\equiv$, in such a way that (X, π) is a *partition space*. This partition space is interpreted as

- an *approximation space* with respect to which any subset A of X can be approximated by the bottom (lower approximation $\mathbf{l}(A) = \cup\{G \in \pi : G \subseteq A\}$) and the top (upper approximation $\mathbf{u}(A) := \cup\{G \in \pi : A \cap G \neq \emptyset\}$).

From this point of view any such equivalence class $G \in \pi$ is also considered as a *granule of knowledge* supported/furnished a priori by the information system. In order to underline its role of lower (resp., upper) approximation we will indicate with $\mathbf{l}(A)$ (resp., $\mathbf{u}(A)$) what in the general treatment we have denoted with $\mathbf{i}$ as interior (resp., $\mathbf{c}(A)$ as closure). The denotation of $\mathbf{e}(A) := \cup\{G \in \pi : G \subseteq A^c\}$ as exterior remains unchanged, but it is also denoted as $A^\sim = \mathbf{e}(A)$.

Let us recall that an *exact* or *definable* set $E \in \mathscr{P}(X)$ is a union of a collection of granules (equivalence classes) from π. We denote now the collection of all such exact sets by $\mathscr{E}(X)$ (instead of $\mathscr{E}(X, \pi)$ if this does not involve any confusion). Let us recall that $\mathscr{E}(X)$ has a structure of Alexandroff topology (it is closed with respect to the intersection (resp., union) of any arbitrary family open (resp., closed) exact sets).

In the case of the ortho-rough representations $r_\perp(H) := (\mathbf{l}(H), \mathbf{e}(H))$, with $\mathbf{l}(H) \cap \mathbf{e}(H) = \emptyset$, of any subset H of X, whose collection has been denoted as $\mathbb{R}(X, \pi) := (\mathscr{E}(X) \times \mathscr{E}(X))_\perp$, one can apply the operations introduced in Theorem 77 by Eq. (60):

$$(\mathbf{l}(H), \mathbf{e}(H)) \rightarrow_L (\mathbf{l}(K), \mathbf{e}(K)) \tag{61a}$$
$$:= ([\mathbf{l}^c(H) \cap \mathbf{e}^c(K)] \cup [\mathbf{e}(H) \cup \mathbf{l}(K)],\ \mathbf{l}(H) \cap \mathbf{e}(K))$$
$$(\mathbf{l}(H), \mathbf{e}(H)) \rightarrow_G \langle \mathbf{l}(K), \mathbf{e}(K)\rangle \tag{61b}$$
$$:= ([\mathbf{l}^c(H) \cap \mathbf{e}^c(K)] \cup [\mathbf{e}(H) \cup \mathbf{l}(K)],\ \mathbf{e}^c(H) \cap \mathbf{e}(K))$$

It is in this context that Pagliani introduced the operator $\rightarrow_L$ in [53, Definition 1.2] and $\rightarrow_G$ in [54, Proposition 3.8], and all the results concerning the approach of Pagliani to the algebraic structure of rough sets have been collected in [55].

The above operators (61) are closed on $\mathbb{R}(X, \pi)$ since the following proposition can be proved.

Proposition 78 *Let* $\langle \mathbf{l}(H), \mathbf{e}(H)\rangle$, $(\mathbf{l}(K), \mathbf{e}(K)\rangle$ *be two elements from* $\mathbb{R}(X, \pi)$*, and let* $\rightarrow_L$ *and* $\rightarrow_G$ *be defined as in* (61)*. Then*

$$(\mathbf{l}(H), \mathbf{e}(H)) \rightarrow_L (\mathbf{l}(K), \mathbf{e}(K)) = [r_\perp(\mathbf{l}(H)^c) \sqcap r_\perp(\mathbf{l}(K))] \sqcup r_\perp(K) \sqcup r_\perp(K^c)$$
$$(\mathbf{l}(H), \mathbf{e}(H)) \rightarrow_G (\mathbf{l}(K), \mathbf{e}(K)) = [r_\perp(\mathbf{l}(H)^c) \sqcap r_\perp(\mathbf{l}(K))] \sqcup r_\perp(K) \sqcup r_\perp(K^\sim)$$

Thus, according to Proposition 60, there exists two subsets Z_L and Z_G of the universe X such that

$$(\mathbf{l}(H), \mathbf{e}(H)) \rightarrow_L (\mathbf{l}(K), \mathbf{e}(K)) = r_\perp(Z_L)$$
$$(\mathbf{l}(H), \mathbf{e}(H)) \rightarrow_G \langle\mathbf{l}(K), \mathbf{e}(K)) = r_\perp(Z_G)$$

In this way $(\mathbf{l}(H), \mathbf{e}(H)) \rightarrow_L (\mathbf{l}(H), \mathbf{e}(H))$ *and* $(\mathbf{l}(H), \mathbf{e}(H)) \rightarrow_G \langle\mathbf{l}(H), \mathbf{e}(H))$, *as rough representations of two subsets Z_L and Z_G of X, are both elements of* $\mathbb{R}(X, \pi)$.

We note that other implication operators on rough sets and orthopairs have been studied in [33], where also a discussion on their interpretability is put forward.

2.8 A Negative Theorem and a Consequent Open Problem

If one looks at the concrete models of HW algebras described in the previous sections, they realize that the case of the collection $\mathbb{F}(X)$ of orthopairs of fuzzy sets does not appear as a possible candidate for such a structure. This happens not for some forgetfulness, but rather for an impossibility due to two facts:

Fact 1. The theorem on orthopairs 73 requires that the initial basic structure must be a Boolean algebra (see the proofs in Appendix 2.9 strongly dependent from the Boolean properties). But $\mathbb{F}(X)$ has a negation $-(f_1, f_0) = (f_0, f_1)$ which is not Boolean (it is Kleene).

Fact 2. An interesting negative theorem on which we discuss now.

In order to clarify this second fact let us make a summary of the involved theoretical situation.

In Theorem 73 one starts from a Boolean algebra Σ and then, through the induced relation of orthogonality, builds the collection $\mathbb{A}(\Sigma)$ of all the orthopairs of elements from Σ. Then, a structure of HW is assigned to $\mathbb{A}(\Sigma)$ through the definition of the two implication operations $\rightarrow_L$ and $\rightarrow_G$ according to the definitions (57) of Theorem 73.

Let us stress that the so obtained HW algebra $\mathbb{A}(\Sigma)$ is the merge of a Wajsberg algebra linked to the Łukasiewicz implication $\rightarrow_L$ and a Heyting algebra linked to the Gödel implication $\rightarrow_G$. In its turn, this HW algebra $\mathbb{A}(\Sigma)$ induces a BK^{DM} algebra as a merge of a Kleene algebra based on the negation $-(a_1, a_0) = (a_1, a_0) \rightarrow_L (0, 1)$ and a Brouwer algebra based on the negation $\sim (a_1, a_0) = (a_1, a_0) \rightarrow_G (0, 1)$.

The important point for our discussion is that the obtained Wajsberg algebra induces a Kleene negation $-(a_1, a_0)$ (see Proposition 63).

Once made these general premises, let us recall that in [18] we have introduced on $\mathbb{F}(X)$ the following Gödel implication (proposed in [35, p. 64] relatively to the unit interval [0, 1] and extended by us to $\mathbb{F}(X)$ on [16]):

$$((f_A, g_A) \to_G (f_B, g_B))(x) := \begin{cases} (1, 0) & \text{if } f_A(x) \leq f_B(x) \\ & \text{and } g_A(x) \geq g_B(x) \\ (1 - g_B(x), g_B(x)) & \text{if } f_A(x) \leq f_B(x) \\ & \text{and } g_A(x) < g_B(x) \\ (f_B(x), 0) & \text{if } f_A(x) > f_B(x) \\ & \text{and } g_A(x) \geq g_B(x) \\ (f_B(x), g_B(x)) & \text{if } f_A(x) > f_B(x) \\ & \text{and } g_A(x) < g_B(x) \end{cases}$$

The structure $\langle \mathbb{F}(X), \sqcap, \sqcup, \to_G, (\mathbf{0}, \mathbf{1}) \rangle$ is a Heyting algebra. The Brouwer negation induced from the Gödel implication connective $\to_G$ in the usual manner $\sim \langle f_A, g_A \rangle = \langle f_A, g_A \rangle \to_G \langle \mathbf{0}, \mathbf{1} \rangle$ is the following one, defined whatever be $x \in X$ by the law:

$$\sim (f_A, g_A)(x) = \begin{cases} (1, 0) & \text{if} \quad g_A(x) = 1 \\ (0, 1) & \text{if} \quad g_A(x) \neq 1 \end{cases}$$

With this result it would seem that the path to attribute to $\mathbb{F}(X)$ some "kind" of HW algebra has been traveled in half. It would be enough to introduce on it an appropriate Łukasiewicz-like implication. But unfortunately due to the following interesting negative theorem proved in [36] this is not possible:

- Any de Morgan negation on $\mathbb{F}(X)$, whatever be its concrete definition, *cannot* satisfy the Kleene condition (K).

Thus, since the negation induced from a Wajsberg algebra is necessarily Kleene, it is impossible to give to the set of fuzzy orthopairs $\mathbb{F}(X)$ a structure of Wajsberg algebra.

This result poses as conclusion of this Chapter the following interesting

Open Problem: whether it is possible to weaken one of the (W1)–(W4) conditions that define a Wajsberg algebra in such a way as to obtain a weak Łukasiewicz implication, say $\to_{wL}$, such that:

1. $a \to_L b$ implies $a \to_{wL} b$.
2. The structure $\langle A, \to_{wL}, 1 \rangle$ satisfies the conditions (W1)–(W3) and a weak form (wW4) of (W4), i.e. it is a weak Wajsberg algebra, in such a way that $\neg_w a := a \to_{wL} 0$ turns out to be a de Morgan negation without condition (K). This is due to the fact that if one looks, for example in [64], the proof of how it is possible to obtain from a Wajsberg algebra the structure of de Morgan lattice, only the axioms (W1)–(W3) are needed plus some weak form of (W4), while it is the condition (W4) that strongly enters in the proof of the Kleene condition (K).

3. It is possible to give an axiomatization of the structure $\langle A, \to_{wL}, \to_G, 0\rangle$ in such a way that the structure $\langle A, \to_{wL}, 1\rangle$ turns out to be a weak Wajsberg algebra, while the structure $\langle A, \wedge, \vee, \to_G, 0\rangle$ of Heyting algebra must be unchanged.

As real conclusion of this chapter, let us note that also if in the case of orthopairs of fuzzy sets $(f_1, f_0) \in \mathbb{F}(X) = (\mathscr{F}(X) \times \mathscr{F}(X))_\perp$ only the Gödel implication can be introduced, in Sect. 1.3.2 $\mathbb{F}(X)$ has been equipped with a structure of minimal BZ^{DM} algebra with respect to the two negations

$$-(f_1, f_0) = (f_0, f_1) \quad \text{and} \quad \sim (f_1, f_0) = (f_0, \neg f_0)$$

The induced interior and closure operations are then the following

$$\mathbf{i}(f_1, f_0) = (f_1, \neg f_1) \quad \text{and} \quad \mathbf{c}(f_1, f_0) = (\neg f_0, f_0)$$

From this it follows that an orthopair $(f_1, f_0) \in \mathbb{F}(X)$ is exact, i.e., it satisfies the conditions $\mathbf{i}(f_1, f_0) = \mathbf{c}(f_1, f_0) = (f_1, f_0)$, iff it satisfies the condition $\forall x \in X,\ f_1(x) + f_0(x) = 1$, contrary to the general condition of being an orthopair characterized by $\forall x \in X,\ f_1(x) + f_0(x) \leq 1$.

A particular sub-family of orthopairs of fuzzy sets from $\mathscr{F}(X)$ is the collection of all orthopairs consisting of crisp sets from $\mathscr{F}_c(X)$, i.e., pairs $(\chi_{A_1}, \chi_{A_0}) \in (\mathscr{F}_c(X) \times \mathscr{F}_c(X))_\perp$ such that $A_1 \cap A_0 = \emptyset$. Let us denote this family as $\mathbb{C}(X) \subseteq \mathbb{F}(X)$. This family can be put in a one-to-one correspondence with the HW algebra $\mathbb{P}(X) = (\mathscr{P}(X) \times \mathscr{P}(X))_\perp$ of orthopairs of subsets of X discussed in Example 75. Formally, $(\chi_{A_1}, \chi_{A_0}) \longleftrightarrow (A_1, A_0)$.

Therefore also $\mathbb{C}(X)$ is a HW algebra with respect to the two operations of implications that it can inherit from $\mathbb{P}(X)$.

2.9 *Proof of Theorem 73*

Let $\mathbb{A}(\Sigma) = \{(a, b) \in \Sigma^2 : a \perp b, a \vee a' = 1, b \vee b' = 1\}$ be the collection of all exact orthogonal pairs of elements from a Boolean algebra (*distributive* (condition (D)) *ortholattice* (conditions (dM1), (dM2), and (oc-2a) and (oc-2b)) Σ. Let us define the following operators on $\mathbb{A}(\Sigma)$

$$(a_1, a_0) \to_L (b_1, b_0) := ((a_1' \wedge b_0') \vee a_0 \vee b_1, a_1 \wedge b_0) \tag{62a}$$

$$(a_1, a_0) \to_G (b_1, b_0) := ((a_1' \wedge b_0') \vee a_0 \vee b_1, a_0' \wedge b_0) \tag{62b}$$

$$\mathbf{0} := (0, 1) \tag{62c}$$

Further, if according to Definition 62, we derive the negation and lattice operators we obtain the following result.

Lemma 79 *Let* $\to_L, \to_G$ *and* $\mathbf{0}$ *be defined as in Eq.* (62). *Then the following hold:*

1. $-(a_1, a_0) := (a_1, a_0) \to_L \mathbf{0} = (a_0, a_1)$
2. $\sim (a_1, a_0) = (a_1, a_0) \to_G \mathbf{0} = (a_0, a_0')$
3. $(a_1, a_0) \sqcup (b_1, b_0) = ((a_1, a_0) \to_L (b_1, b_0)) \to_L (b_1, b_0) = (a_1 \vee b_1, a_0 \wedge b_0)$
4. $(a_1, a_0) \sqcap (b_1, b_0) = -(-(a_1, a_0) \to_L -(b_1, b_0)) \to_L -(b_1, b_0)) =$ $= (a_1 \wedge b_1, a_0 \vee b_0)$
5. $\mathbf{1} = \mathbf{0}^- = (1, 0)$

Proof Let us denote by $(\perp)$ the condition of orthopair $a_0 \leq a_1'$. Moreover, from this we get $a_1' \leq a_1' \vee a_0 \leq (\perp) \leq a_1' \vee a_1' = a_1'$ i.e., $a_1' \vee a_0 = a_1'$.

(1) So we have the following.

$$
\begin{aligned}
(a_1, a_0) \to_L (0, 1) = & \\
& = ((a_1' \wedge 0) \vee a_0 \vee 0, a_1 \wedge 1) = \text{ (D) } = \\
& = ((a_1' \vee a_0) \wedge (0 \vee a_0), a_1) = \text{ } (\perp) \text{ } = \\
& = (a_1' \wedge a_0, a_1) = \text{ } (\perp) = \\
& = (a_0, a_1)
\end{aligned}
$$

Note that in this proof the conditions (oc-2a,b) are not used.

(2)

$$
\begin{aligned}
(a_1, a_0) \to_G (0, 1) = & \\
& = ((a_1' \wedge 0) \vee a_0 \vee 0, a_0' \wedge 1) = \text{ (D) } = \\
& = ((a_1' \vee a_0) \wedge (0 \vee a_0), a_0') = \text{ } (\perp) \text{ } = \\
& = (a_1' \wedge a_0, a_0') = \text{ } (\perp) = \\
& = (a_0, a_0')
\end{aligned}
$$

Also in this case there is no use of conditions (oc-2a,b).

(3) Let us note that at the step (*), in the right part we use the following results: $a_1' \wedge (b_0 \wedge b_0') = \text{ (co-2a) } = a_1' \wedge 0 = 0$; on the other hand, from the condition of orthopair $b_1 \leq b_0'$ we get $0 \leq b_0 \wedge b_1 \leq b_0 \wedge b_0' = \text{ (oc-2) } = 0$, i.e., $b_0 \wedge b_1 = 0$.

Hence, we have

- $((a_1, a_0) \to_L (b_1, b_0)) \to_L (b_1, b_0) =$
- $((a_1' \wedge b_0') \vee a_0 \vee b_1, a_1 \wedge b_0) \to_L (b_1, b_0) =$
- $((((a_1' \wedge b_0') \vee a_0 \vee b_1)' \wedge b_0') \vee (a_1 \wedge b_0), ((a_1' \wedge b_0') \vee a_0 \vee b_1) \wedge b_0) =$ (dM), (D) $=$
- $(((a_1 \vee b_0) \wedge a_0' \wedge b_1' \wedge b_0') \vee b_1 \vee (a_1 \wedge b_0), (a_1' \wedge b_0' \wedge b_0) \vee (a_0 \wedge b_0) \vee (b_0 \wedge b_1)) =$
- (Distributivity with respect to b_0' at the left and (*) at the right)
- $((((a_1 \wedge b_0') \vee (b_0 \wedge b_0')) \wedge a_0' \wedge b_1') \vee b_1 \vee (a_1 \wedge b_0), a_0 \wedge b_0) =$

- (Distributivity with respect to b_1 and (oc-2a))
- $(([(a_1 \wedge b_0') \vee b_1] \wedge (b_1 \vee a_0') \wedge (b_1' \vee b_1)) \vee (a_1 \wedge b_0), a_0 \wedge b_0) =$ (D), (oc-2b) $=$
- $([(a_1 \vee b_1) \wedge (b_0' \vee b_1) \wedge (b_1 \vee a_0')] \vee (a_1 \wedge b_0), a_0 \wedge b_0) =$
- From the orthogonality $b_1 \leq b_0'$ we get $b_0' \vee b_1 = b_0'$ and from (D)
- $([b_0' \wedge (b_1 \vee (a_1 \wedge a_0'))] \vee (a_1 \wedge b_0), a_0 \wedge b_0) =$
- (From (D) and orthogonality $a_1 \wedge a_0' = a_1,\ b_1 \wedge b_0 = b_1$) $=$
- $[(b_1 \vee (b_0' \wedge a_1))] \vee (a_1 \wedge b_0), a_0 \wedge b_0) =$
- $(b_1 \vee (a_1 \wedge (b_0 \vee b_0')), a_0 \wedge b_0) =$ (oc-2b) $= (a_1 \vee b_1, a_0 \wedge b_0)$

(4) Can be deduced by (3) and definition of $\sqcup, \sqcap, -$.
(5) By definition of $\mathbf{0}$ and $-$. □

Now we have the instruments to show that $\mathbb{A}(\Sigma)$ has a HW algebraic structure.

Theorem 80 *The structure $\langle \mathbb{A}(\Sigma), \to_L, \to_G, \mathbf{0}\rangle$, where $\mathbb{A}(\Sigma)$ is the collection of all orthogonal pairs on a Boolean algebra, $\to_L, \to_G$ and $\mathbf{0}$ are defined as in (62) is a HW algebra.*

Proof We show that all axioms $(HW1)$–$(HW8)$ are satisfied.

(HW1)

$$\begin{aligned}(a_1, a_0) \to_G (a_1, a_0) &= ((a_1' \wedge a_0') \vee a_0 \vee a_1, a_0' \wedge a_0) = \\ &= ([(a_1' \vee a_0) \wedge (a_0' \vee a_0)] \vee a_1, 0) = \\ &= (1, 0) = \mathbf{1}.\end{aligned}$$

(HW2)

- $(a_1, a_0) \to_G ((b_1, b_0) \sqcap (c_1, c_0)) =$
- $((a_1' \wedge (b_0 \vee c_0)') \vee a_0 \vee (b_1 \wedge c_1), a_0' \wedge (b_0 \vee c_0)) =$
- $((a_1' \wedge (b_0 \vee c_0)') \vee (a_1' \wedge a_0) \vee (a_0 \wedge b_1 \wedge c_1) \vee (b_0' \wedge b_1 \wedge c_0' \wedge c_1), a_0' \wedge (b_0 \vee c_0)) =$
- $((a_1' \wedge (a_0 \vee (b_0 \vee c_0)')) \vee ((b_1 \wedge c_1) \wedge (a_0 \vee (b_0' \vee c_0))), a_0' \wedge (b_0 \vee c_0)) =$
- $((a_1' \vee (b_1 \wedge c_1)) \wedge (a_0 \vee (b_0 \vee c_0)'), a_0' \wedge (b_0 \vee c_0)) =$
- $((a_1' \vee b_1) \wedge (a_1' \vee c_1) \wedge (a_0 \vee b_0') \wedge (a_0 \vee c_0'), a_0' \wedge (b_0 \vee c_0)) =$
- $((a_1' \vee b_1) \wedge (a_0 \vee b_0' \vee b_1) \wedge (a_1' \vee c_1) \wedge (a_0 \vee c_0' \vee c_1), a_0' \wedge (b_0 \vee c_0)) =$
- $(((a_1' \wedge (b_0' \vee a_0)) \vee b_1) \wedge (c_1 \vee (a_1' \wedge (c_0' \vee a_0))), a_0' \wedge (b_0 \vee c_0)) =$
- $(((a_1' \wedge b_0') \vee a_0 \vee b_1) \wedge ((a_1' \wedge c_0') \vee a_0 \vee c_1), (a_0' \wedge b_0) \vee (a_0' \wedge c_0)) =$
- $((a_1, a_0) \to_G (b_1, b_0)) \sqcap ((a_1, a_0) \to_G (c_1, c_0))$.

(HW3)

$$\begin{aligned}(a_1, a_0) \sqcap ((a_1, a_0) \to_G (b_1, b_0)) &= \\ (a_1 \wedge ((a_1' \wedge b_0') \vee a_0 \vee b_1), (a_0' \wedge b_0) \vee a_0) &= \\ ((a_1 \wedge a_1' \wedge b_0') \vee (a_1 \wedge a_0) \vee (a_1 \wedge b_1), (a_0 \vee a_0') \wedge (a_0 \vee b_0)) &= \\ (a_1 \wedge b_1, a_0 \vee b_0) &= \\ (a_1, a_0) \sqcap (b_1, b_0).&\end{aligned}$$

(HW4) Similar to $(HW3)$.

(HW5)

$$(1, 0) \to_L (a_1, a_0) = ((1' \wedge a_0') \vee 0 \vee a_1, 1 \wedge a_0) = (a_1, a_0).$$

(HW6)

$$\begin{aligned}
(a_1, a_0) \to_L ((b_1, b_0) \to_L (c_1, c_0)) &= \\
((a_1' \wedge (b_1 \wedge c_0)') \vee a_0 \vee ((b_1' \wedge c_0') \vee b_0 \vee c_1), a_1 \wedge b_1 \wedge c_0) &= \\
((a_1' \wedge (b_1' \vee c_0')) \vee a_0 \vee ((b_1' \wedge c_0') \vee b_0 \vee c_1), a_1 \wedge b_1 \wedge c_0) &= \\
((a_1' \wedge b_1') \vee (a_1' \wedge c_0') \vee a_0 \vee (b_1' \wedge c_0') \vee b_0 \vee c_1, a_1 \wedge b_1 \wedge c_0) &= \\
((b_1' \wedge (a_1' \vee c_0')) \vee (a_1' \wedge c_0') \vee a_0 \vee b_0 \vee c_1, a_1 \wedge b_1 \wedge c_0) &= \\
= -((a_1, a_0) \to_L (c_1, c_0)) \to_L -(b_1, b_0).
\end{aligned}$$

(HW7)

$$\begin{aligned}
- \sim (a_1, a_0) \to_L \sim\sim (a_1, a_0) &= (a_0', a_0) \to_L (a_0', a_0) = \\
&= (((a_0)'' \wedge a_0') \vee a_0 \vee a_0', a_0' \wedge a_0) = \\
&= (1, 0).
\end{aligned}$$

(HW8)

$$\begin{aligned}
((a_1, a_0) \to_G (b_1, b_0)) \to_L ((a_1, a_0) \to_L (b_1, b_0)) &= \\
((a_1' \wedge b_0') \vee a_0 \vee b_1, a_0' \wedge b_0) \to_L ((a_1' \wedge b_0') \vee a_0 \vee b_1, a_1 \wedge b_0) &= \\
((((a_1' \wedge b_0') \vee a_0 \vee b_1)' \wedge (a_1 \wedge b_0)') \vee (a_0' \wedge b_0) \vee ((a_1' \wedge b_0') \vee a_0 \vee b_1), & \\
((a_1' \wedge b_0') \vee a_0 \vee b_1) \wedge b_0 \wedge a_1) &= \\
(a_1' \vee b_0' \vee (a_0' \wedge b_0) \vee a_0 \vee b_1 \vee (a_1' \wedge b_0'), ((a_0 \vee b_1) \wedge b_0) \wedge a_1) &= \\
(a_1' \vee ((b_0' \vee a_0') \wedge (b_0' \vee b_0)) \vee a_0 \vee b_1 \vee (a_1' \wedge b_0'), & \\
((a_0 \wedge b_0) \vee (b_1 \wedge b_0)) \wedge a_1) &= (1, 0).
\end{aligned}$$

□

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Rough Objects in Monoidal Closed Categories

Patrik Eklund and María-Ángeles Galán-García

Abstract This chapter will build upon previous achievements on monadic rough objects over the category `Set`, and show how rough object approximation and algebraic manipulation in general can be enriched by extending constructions to work similarly over monoidal closed categories embracing both algebraic as well as order structures. The chapter will also show how the rough information model in this monoidal closed category extension connects with other information models being relational in their basic original structures. Additionally, the chapter will discuss the potential of real world applications.

1 Introduction

In [11–13], we initiated work on category theoretic extension of rough sets, constructed by set functors, i.e., functors over the category `Set` of sets and functions. Based on the observation that a relation $R \subseteq X \times X$ is equivalently represented as a morphism $\rho_X : X \longrightarrow \mathsf{P}X$, the Kleisli morphism of the corresponding Kleisli category for the powerset monad, approximations for rough sets can be equivalently defined using the natural transformations, respectively, the unit $\eta : \mathsf{id} \longrightarrow \mathsf{P}$ and multiplication $\mu : \mathsf{PP} \longrightarrow \mathsf{P}$ of the corresponding powerset monad $(\mathtt{P}, \eta, \mu)$.

A key property needed in the case of considering monads over `Set` is that the monad is a partially ordered monad [24], as developed in [11]. The powerset monad is partially ordered, and partially ordered monads composed with monad make the composition partially ordered. This opens up the possibility to define a wide range of monads to be used as a monadic generalization of rough sets [14, 17]. A typical

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composition is the powerset monad composed, and, more generally, many-valued powerset monads, composed with the term monad, as originally proposed in [15].

The syntax of relations as extendable to monadic representation appears also in other similar computational and logical models like in formal concept analysis and description logic [9, 20].

Examples of applications in data analysis were outlined in [16, 18], and monadic constructions appear in several information structure oriented applications, e.g., in manufacturing and production [22] and health [6, 21].

Development on rough monads, and, more generally, on rough objects in a category theory settings, have so far focused on functors and monads exclusively over the category `Set`. In order to generalize from just using partially ordered monads to enable order structures, we need to consider functors and monads over categories embracing order structures. The term functor construction has been provided over Goguen's category, and, more generally, over monoidal closed categories [19].

These algebraically enriched categories provide suitable structure for management of algebraic developments related to objects like those represented by generalized rough sets. In fact, as shown in [23], they enable to provide the algebraic foundations of many-valued structures, combining algebraic operations in presence of order structures. This in particular involves the use of quantales and related categorical objects, starting with most general categories like the category `Sup` of complete lattices and join preserving maps.

2 Information Granularity and Classification

The basic relational model starts with a binary relation

$$R \subseteq X \times X$$

which equivalently can be represented as a mapping

$$\rho : X \times X \to 2$$

where 2 denotes the two-pointed set $\{0, 1\}$ (or $\{false, true\}$), i.e., representing binary (two-valued) truth. The relation has initially no (algebraic) properties. Neither are elements $x \in X$ assumed to embrace any features.

In this context, information granularity has therefore several different aspects, and possibilities to increase that granularity.

Firstly, two valuedness can (and in many applications should) be extended to many valuedness. If we extend 2 to $\mathfrak{Q}$, typically with $\mathfrak{Q} = (Q, *)$ as a non-commutative quantale, we have a relation that can take multiple truth values:

$$\rho : X \times X \to Q.$$

In this case, non-commutativity of the quantale means that aggregations will need to consider the order among elements in Q.

Secondly, we may indeed add algebraic properties (to ρ). Here we observe that relations $R \subseteq X \times X$ correspond precisely to functions (in form of substitutions) $\sigma_R : X \longrightarrow \mathsf{P}X$, where P is the powerset functor over the category of sets and functions. This is then the basis for viewing *generalized relations* as morphisms (substitutions) in the Kleisli category over *generalized powerset monads*. In the many-valued case we have many-valued relations in form of $\sigma_R : X \longrightarrow \mathsf{Q}X$. The powerset functors P (in the case of 2) and Q (in the case of $\mathfrak{Q}$) are both extendable to monads, and they generate respective Eilenberg-Moore categories, where the algebraic structure of the monad become explicit. Here we should distinguish clearly between algebraic structure and algebraic properties. Algebraic structures comes from the monad, whereas algebraic properties are imposed over that structure. An algebraic operator provides structure and we may have properties for that operator. For instance, commutativity is a property of a binary operator.

Thirdly, we may unravel the features of $x \in X$, so that x is not just a point or a name of an element, but an expression. Now we should not confuse with algebraic expressions enabled by $\mathfrak{Q}$, but rather start with an underlying signature Σ, so that $x \in X$ are just variables, and the term set $\mathsf{T}_\Sigma X$ is now the base set for which relations

$$\rho : \mathsf{T}_\Sigma X \times \mathsf{T}_\Sigma X \to 2$$

are explored. This brings us to the many-valued

$$\sigma : X \longrightarrow \mathsf{Q}\mathsf{T}_\Sigma X$$

where the term composition makes $\mathsf{Q} \circ \mathsf{T}_\Sigma$ extendable to a monad. The algebraic structure of that monad is much more elaborate.

A generalized relation for a monad $\mathbf{F} = (\mathsf{F}, \eta, \mu)$ is thus equivalent to a *variable substitution* $\sigma : X \longrightarrow \mathsf{F}X$. Doing so immediately brings us to *monads over categories*. A functor $\mathsf{F} : \mathtt{Set} \longrightarrow \mathtt{Set}$ over the category of sets is rather rudimentary, and functors $\mathsf{F} : \mathtt{C} \longrightarrow \mathtt{C}$ more generally over monoidal closed categories can embrace much more structure. This shows how 'structure and properties' can be invoked in several ways. From application point of view it is often critical where and how such structure and properties are added.

As an example, if we only say 'crankshaft' as a name for a component in an automotive system-of-systems, 'crankshaft' is just an x in some X, but if we include the attributes $attr_1, \ldots, attr_n$ attached to a crankshaft, it becomes a logical term. Using logical notation, $crankshaft$ is a logical constant (of zero arity), whereas $crankshaft(attr_1, \ldots, attr_n)$ is a term, with

$$crankshaft : \mathtt{s}_1 \times \cdots \times \mathtt{s}_n \longrightarrow \mathtt{s}$$

being an operator (of arity n) and $\mathtt{s}_i$, $i = 1, \ldots, n$, and $\mathtt{s}$ are types (sorts).

Where terms in $\mathsf{T}_\Sigma X$ are one-sorted [10], this example calls for the need to describe the many-sorted term functor. Indeed, so far we have only provided the term functor $\mathsf{T}_\Sigma : \mathtt{Set} \longrightarrow \mathtt{Set}$, which for a "set of variables" X (an object in $\mathtt{Set}$) provides $\mathsf{T}_\Sigma X$, the "set of all terms over Σ". Following the notation in [19], in the many-sorted (and crisp) case we need the "sorted category of sets" for the many-sorted term functor. Let the sort set S, as an object of $\mathtt{Set}$, be an index set (in ZFC). For a category $\mathtt{C}$, we write $\mathtt{C}_S$ for the product category $\prod_S \mathtt{C}$. The objects of $\mathtt{C}_S$ are tuples $(X_{\mathtt{s}})_{\mathtt{s}\in S}$ such that $X_{\mathtt{s}} \in \mathsf{Ob}(\mathtt{C})$ for all $\mathtt{s} \in S$. We could use X_S as a shorthand notation for these tuples. Based on our signature Σ, we will have $X_{\mathtt{s}}$ and $X_{\mathtt{bool}}$ as separate variable sets, so that $\mathsf{T}_\Sigma X_S$ is the tuple $(\mathsf{T}_{\Sigma,\mathtt{s}} X_{\mathtt{s}})_{\mathtt{s}\in S}$. See [19] for the formal categorical term construction. It corresponds, in $\mathtt{Set}$, to the traditional view of the "set of terms". In the case of λ-terms, the term construction avoids the need to use renaming.

3 Partially Ordered Monads

Partially ordered monads were introduced in [24] based on the notion of *basic triples*, i.e., triples $\boldsymbol{\Phi} = (\phi, \leq, \eta)$, where ϕ is a covariant functor from $\mathtt{Set}$, the category of sets and functions, to $\mathtt{acSLAT}$, the category of almost complete semilattices and non-empty suprema preserving maps, and $\eta : \mathsf{id} \longrightarrow \phi$ is a natural transformation. The functor ϕ takes objects (sets) X in $\mathtt{Set}$ and produces objects $(\phi X, \leq)$ in $\mathtt{acSLAT}$, so that if X is empty, then also ϕX is empty. Further, $\eta(x) \wedge \eta(y)$ exists only in the case $x = y$.

Instead of the more general category $\mathtt{acSLAT}$, we may obviously use the specific $\mathtt{Sup}$ category, and benefit from properties as described in [23].

Example 1 Let L be a completely distributive lattice. The covariant powerset functor L is obtained by $\mathsf{L}X = L^X$, i.e. the set of mappings (or L-fuzzy sets) $A : X \to L$, and following [25], for a morphism $f : X \to Y$ in $\mathtt{Set}$, by defining $\mathsf{L}f(A)(y) = \bigvee_{f(x)=y} A(x)$. Further, define $\eta_X : X \to \mathsf{L}X$ so that $\eta_X(x)(x')$ is 1, if $x = x'$, and 0 otherwise. Then $(\mathsf{L}, \leq, \eta)$ is a basic triple, and can be extended to a partially ordered monad $(\mathsf{L}, \leq, \eta, \mu)$ by $\mu_X(\mathscr{M})(x) = \bigvee_{A\in\mathsf{L}X} A(x) \wedge \mathscr{M}(A)$ [11]. A special case indeed is $L = 2$, where $\mathsf{L} = \mathsf{P}$, the ordinary powerset functor, and the structure of L can also be that of a quantale.

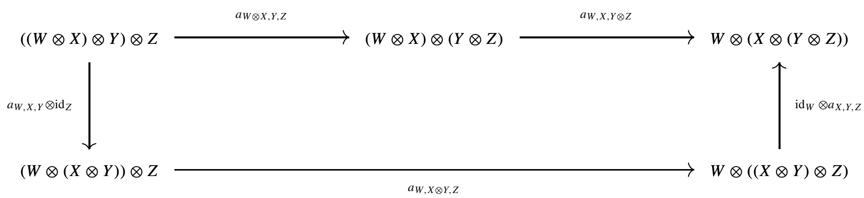
Example 2 Let $\Sigma = (S, \Omega)$ be a signature of sorts in S and operators in Ω, and let $\mathsf{T}_\Sigma : \mathtt{Set} \longrightarrow \mathtt{Set}$ be the term functor over Σ [19]. Then the composed functor $\mathsf{L} \circ \mathsf{T}_\Sigma$ can be extended to a monad, and further to a partially ordered monad [11].

4 Monoidal Biclosed Categories

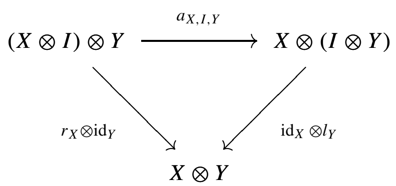
The history of monoidal closed categories goes back to the study of such natural equivalences [8], and further back to theories of linear operators [1] and homology theory. Monoidal closed categories became formally defined in [28]. Monoidal categories were called *categories with multiplication* in [2, 3] and [29]. The notion and name of *monoidal closed category* attains its final formulation in [30].

In monoidal closed categories, the notion of a product is weakened. Products of objects are related to exponential objects, and the Hom-set $\mathrm{Hom}(X, Y)$ in a category, with X and Y as objects, itself and object of the category, is such an exponential object. The natural equivalence $\mathrm{Hom}(A \times B, C) \cong \mathrm{Hom}(A, \mathrm{Hom}(B, C))$ is fundamental. In Set, $\mathrm{Hom}(B, C)$ is the exponential object, and the natural equivalence means that Set is a cartesian closed category. The cartesian product is indeed strong, and therefore products can be weakened using bifunctors $\otimes : \mathtt{C} \times \mathtt{C} \longrightarrow \mathtt{C}$. We write $A \otimes B$ instead of $\otimes(A, B)$, for objects A and B in $\mathrm{Ob}(\mathtt{C})$.

In defining monoidal closed categories, we follow the notational style in [27]. Let $\mathtt{C}$ be a category, $\otimes : \mathtt{C} \times \mathtt{C} \longrightarrow \mathtt{C}$ a bifunctor, and I a *unit object* in $\mathtt{C}$. If there are natural isomorphisms $a_{X,Y,Z} : (X \otimes Y) \otimes Z \longrightarrow X \otimes (Y \otimes Z)$, $l_X : I \otimes X \longrightarrow X$ and $r_X : X \otimes I \longrightarrow X$ making the diagrams



and



commute, we say that $\mathtt{C}$, equipped with the bifunctor, a unit object, and these natural isomorphisms, is a monoidal category. A monoidal category becomes a *monoidal (left) closed category*, if the functor $\_ \otimes B : \mathtt{C} \longrightarrow \mathtt{C}$ has a right adjoint, denoted $[B, \_]$, for all objects B. It is right closed, if $A \otimes \_ : \mathtt{C} \longrightarrow \mathtt{C}$, for all objects A, has a right adjoint. A monoidal closed category is *biclosed*, if it is both left and right closed, and a *symmetric monoidal category*, whenever the tensor product is commutative.

Example 3 If a quantale $\mathfrak{Q}$ is commutative and unital, then the Goguen category $\texttt{Set}(\mathfrak{Q})$ is a symmetric monoidal closed category and it is therefore also biclosed. See [19] for detail on how the term functor is constructed over $\texttt{Set}(\mathfrak{Q})$. Note also how $L_{id} \circ \mathsf{T}_\Sigma$ over $\texttt{Set}$ is very different from the term functor T_Σ over $\texttt{Set}(\mathfrak{Q})$. This difference is intuitively the difference between "computing with fuzzy" and "fuzzy computing".

5 Rough Monads over Monoidal Closed Categories

Let R be a relation on X, i.e. $R \subseteq X \times X$, and $\rho_X : X \longrightarrow \mathsf{P}X$ the corresponding mapping, where $\rho_X(x) = \{y \in X | xRy\}$. The inverse relation R^{-1} is $\rho_X^{-1}(x) = \{y \in X | xR^{-1}y\}$.

In [12] we showed, for the powerset monad $\mathbf{P} = (\mathsf{P}, \eta, \mu)$ over $\texttt{Set}$, how the lower approximation of a set $A \subseteq X$ is obtained by

$$A^{\downarrow} = \{x \in X | \rho(x) \subseteq A\} = \bigvee_{\rho_X(x) \wedge A > 0} \eta_X(x) = \mu_X \circ \mathsf{P}\rho_X^{-1}(A)$$

and the upper approximation correspondingly by

$$A^{\uparrow} = \{x \in X | \rho(x) \cap A \neq \emptyset\} = \bigvee_{\rho_X(x) \leq A} \eta_X(x).$$

This indeed works because the monad $\mathbf{P} = (\mathsf{P}, \eta, \mu)$ is a partially ordered monad $\mathbf{P} = (\mathsf{P}, \leq, \eta, \mu)$ with $\leq$ being $\subseteq$.

We can now note the distinction between "sets as rough" over $\texttt{Set}$ and "rough sets" over $\texttt{Set}(\mathfrak{Q})$. The former is basically based on a composed monad $\boldsymbol{\Phi} \circ \mathbf{T}_\Sigma$ over $\texttt{Set}$, whereas the latter is based on a monad over $\texttt{Set}(\mathfrak{Q})$. Almost all of fuzzy rough set theory is not actually about 'rough sets' but rather 'sets as rough'. This restricted understanding is due to not using categorical constructions, and indeed doing so means being simply over the category of sets.

We now formally describe the extension from *over the category of sets* to generally *over a monoidal closed category*. The first step is extending ϕ : $\texttt{Set} \longrightarrow \texttt{acSLAT}$ to $\phi : \texttt{C} \longrightarrow \texttt{SUP}$, where $\texttt{C}$ is a monoidal biclosed category, and $\texttt{SUP}$ of complete lattices and join-preserving maps. The category $\texttt{SUP}$ is considered as the underlying category for the *algebraic foundations of many-valuedness* in [23].

For a generalized relation $\rho : \mathsf{id} \longrightarrow \phi$, and a well-defined corresponding inverse transformation $\rho^{-1} : \mathsf{id} \longrightarrow \phi$, rough monads over $\texttt{C}$ then enable to define approximations. Since we now do not have 'elements' like x in X, we need to corresponding 'singleton objects' defined in $\texttt{C}$. In the special case of $\texttt{Set}$, a $x \in X$ can be identified with its one-pointed set $\{x\}$ as the singleton object. Given the powerset monad (P, η, μ), this singleton object is $\eta(x)$.

The situation over $\mathtt{Set}$ indeed provides the *generalized lower approximation* of an object A in $\mathtt{Set}$, as a monoidal biclosed category, according to

$$A^{\downarrow} = \mu_X \circ \phi\rho_X^{-1}(A),$$

and the upper approximation correspondingly according to

$$A^{\uparrow} = \bigvee_{\rho_X(x) \leq A} \eta_X(x).$$

This situation can be generalized to become over any monoidal biclosed category $\mathtt{C}$. In [23], Example 2.3.4 shows the technique for handling singleton objects, and this technique can be generalized to enable approximations over any monoidal biclosed category.

Example 4 Since $\mathtt{Set}(\mathfrak{Q})$ is a monoidal biclosed category [26], we may consider $\phi : \mathtt{Set}(\mathfrak{Q}) \longrightarrow \mathtt{SUP}$, providing a suitable "rough sets" over $\mathtt{Set}(\mathfrak{Q})$ as a useful model in applications.

6 Applications

In this section we show how the notion of approximations can be understood in the context of relations between drugs, and in particular for drug interactions, where the interaction between respective chemical substances (coded on ATC level V) is many-valued and classified given a quantale.

We will use the drug interaction example outlined in [21] and see how rough monad approximations can be used to model interactions for conglomerates of drugs.

Interventions aim to create transitions between 'states of condition', where the objective of an intervention is based on the fundamental principle of at least not to make things worse ('primum non nocere'). The three-valuedness used in [21] includes

- condition under control or problem removed ($\top$),
- condition that requires intervention (a),
- condition not improvable by intervention ($\bot$).

Interventions having desirable effect will change a condition state from a to $\top$, where interventions having no effect leaves the condition states unchanged. The state set $C_3 = \{\top, a, \bot\}$ is used as a partial order. The unitalization $\widehat{C_3}$ was in [21] shown to correspond to levels of evidence, and interventions were identified as (condition state) transitions in an action $C_3 \otimes \mathfrak{Q} \xrightarrow{\boxdot} C_3$.

In the following we briefly discuss the structure of classifications of drugs. The Anatomic Therapeutic Chemical ATC/DDD (Anatomical Therapeutic Chem-

Table 1 Classification of nitrazepam

N	Nervous system	1st level, main anatomical group
N05	Psycholeptics	2nd level, therapeutic subgroup
N05C	Hypnotics and sedatives	3rd level, pharmacological subgroup
N05CD	Benzodiazepine derivatives	4th level, chemical subgroup
N05CD02	Nitrazepam	5th level, chemical substance

ical/Defined Daily Dose) classification system is one of the Related Classifications in the WHO (World Health Organization) Family of International Classifications[1] (FIC). The drugs are classified using ATC codes appearing in five levels.

For drug utilization statistics, a unit of measurement called defined daily dose (DDD) has also been developed to complement ATC. A DDD is the average dose per day for a drug that is used for its main indication when treating adults. This is not to be confused with the guideline or recommendation of dosage. Indeed, the DDD could be in the middle between two commonly prescribed dosages and as such never be an actual prescribed dosage.

Table 1 presents an example using nitrazepam (code N05CD02) as a drug, typically used for short term sleeping problems (insomnia).

ATC encodes drugs and drug interventions, where interventions in general stem from diseases and also targets functioning. From WHO classification point of view, diseases are encoded in ICD (International Classification Diseases), and functioning is encoded in ICF (International Classification of Functioning, Disability and Health). As an example, the ICD code for insomnia is G47.0, where insomnia is a sleep disorder (ICD code G47). Nitrazepam (ATC code N05CD02) is therapeutically *indicated* for the short-term treatment of insomnia (ICD code G47.0). An example *contraindication* is acute pulmonary insufficiency (ICD code J95.2). Common side-effects of the use of nitrazepam are dizziness and unsteadiness. Note how 'dizziness' is basically undefined and uncoded when appearing in the context of listed side-effects for nitrazepam. However, 'dizziness' is formally encoded as a functioning aspect under ICF. The ICF code for Dizziness is b2401, and characterized as *Sensation of motion involving either oneself or one's environment; sensation of rotating, swaying or tilting*. Dizziness falls under ICF code b240 'Sensations associated with hearing and vestibular function', in turn part of 'Sensations of dizziness, falling, tinnitus and vertigo, in turn part of 'Hearing and vestibular functions' (ICF codes b230-b249).

[1]WHO website for classifications http://www.who.int/classifications/en/.

This discussion, drawn from [21], shows the need to relate diseases, functioning and (drug) treatments, and how one domain may act on another.

From rough sets point of view, drug-drug interactions are bivalent or multivalent relations. Pharmacological societies do not share a common view on this valuedness issue. However, there are shared models, where SFINX[2] [4] is one of them. The information in SFINX is divided into five different parts describing each pair of drugs involving an interaction: medical consequence, recommendation, mechanism, background and references.

In SFINX, classifications A, B, C and D are defined, respectively, as 'Minor interaction of no clinical relevance.', 'Clinical outcome of the interaction is uncertain and/or may vary.', 'Clinically relevant interaction that can be handled e.g. by dose adjustments.' and 'Clinically relevant interaction. The combination is best avoided.', respectively. The classifications A and B are obviously related since they mean no or uncertain clinical relevance, whereas C and D represent clinically relevant interactions. In other words, A and B are closer to allow prescription, whereas C and D basically means not to prescribe. Further, A is stronger in favour of prescription (despite interaction) than B since A is no evidence and B is uncertain clinical outcome. Similarly, D is stronger against prescription that C since D is generally best avoided, whereas C opens up a possibility to manage a clinically relevant interaction with dose adjustment.

If L is the lattice with A–D, and ATC is some structure representing the ATC classification, then SFINX is basically a mapping $\sigma_{SFINX} : ATC \times ATC \to L$. Whereas A–D classifies clinical relevance, the SFINX interaction model additionally include levels of documentation (0–4), which represent strength of 'evidence'. Since L represents level of evidence, the question is now which quantale homomorphism best model the situation. In this case we would naturally choose $L = \mathrm{Im}(h_3)$ with natural order. For the definition of h_3, see [21]. If two conditions x and y are treated with respective drugs d_x and d_y, as interventions notated, respectively, as $(x, 1)$ and $(y, 1)$, then we expect

$$\sigma_{SFINX}(d_x, d_y) \geq h_3((x, 1) \mathbin{\widehat{*_\ell}} (y, 1)) = h_3((x, 1)) \circ h_3((y, 1))$$

i.e., the drug interaction must be proportionally less as compared to the aggregated evidence of respective drugs interventions. Or the other way around, a strong interaction between d_x and d_y will jeopardize the advantage of the aggregation of the simultaneous intervention for conditions x and y. The equality establishes a connection between order in interventions and sequentializing evidence-based treatment guidelines.

Similarly we can now imagine many-valued relation in connection with $ICD \times ATC$ for drug interventions related to disease, or with $ATC \times ICF$ for functioning

[2] SFINX as a database and corresponding support system is in use in almost all pharmacies in Finland.

related side-effects of drugs, or ICD, ICF and ATC appearing in other products and combinations.

Upper and lower approximations can potentially be added as granularity features in the ATC classification, where the upper approximation is moving upwards in ATC levels, and lower approximation is correspondingly moving downwards. Concepts like 'cholinergic drugs' embrace a set of drugs with similar pharmacological properties mostly as inhibitors in the nervous system. Rough operators and related algebra is well suited to instruments such an analysis of drug features. In this paper we provide only some suggestions and indications for further studies.

Drug interactions is part of the issue of 'polypharmacy', where withdrawal of drugs is becoming subjected to guideline development, e.g. for FRIDs (fall risk inducing drugs) [7, 32, 33]. Falls prevention is one of the activities within EIP AHA (European Innovation Partnership for Active and Healthy Ageing) [6], where good practices, e.g. as related with drugs management, are implemented with respective reference site [31].

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Appendix: Category Theory Notations and Constructions

Basic Concepts and Notations

In a category $\mathtt{C}$ with objects A and B, morphisms f from A to B are typically denoted by $f : A \longrightarrow B$ or $A \xrightarrow{f} B$. The (A-)identity morphism is denoted $A \xrightarrow{\mathrm{id}_A} A$ and morphism composition uses $\circ$. The set of $\mathtt{C}$-morphisms from A to B is written as $\mathrm{Hom}_{\mathtt{C}}(A, B)$ or $\mathrm{Hom}(A, B)$.

The category of sets, $\mathtt{Set}$, is the most typical example of a category, and consists of sets as objects and functions (in ZFC) as morphisms together with the ordinary composition and identity. Other categories may be defined, for example, using $\mathtt{Set}$ as a basis: a structure, defined by the given metalanguage, is added on $\mathtt{Set}$-objects, and then morphisms are defined as $\mathtt{Set}$-morphisms preserving these structures. A typical example is to add uncertainty, modelled by a quantale $\mathfrak{Q}$, on $\mathtt{Set}$-objects: The objects of the Goguen category $\mathtt{Set}(\mathfrak{Q})$ are pairs (X, α), where X is an object of $\mathtt{Set}$ and $\alpha \colon X \longrightarrow Q$ is a function (in ZFC). The morphisms $(X, \alpha) \xrightarrow{f} (Y, \beta)$ are $\mathtt{Set}$-morphisms $X \xrightarrow{f} Y$ satisfying $\alpha \leq \beta \circ f$. The composition of morphisms is defined as composition of $\mathtt{Set}$-morphisms. Originally, Goguen considered a completely distributive lattice as the underlying lattice in [25] and further properties for Goguen categories can be found in [34].

A (covariant) *functor* $\mathsf{F} : \mathtt{C} \longrightarrow \mathtt{D}$ between categories is a mapping that assigns each $\mathtt{C}$-object A to a $\mathtt{D}$-object $\mathsf{F}(A)$ and each $\mathtt{C}$-morphism $A \xrightarrow{f} B$ to a $\mathtt{D}$-morphism $\mathsf{F}(A) \xrightarrow{\mathsf{F}(f)} \mathsf{F}(B)$, such that $\mathsf{F}(f \circ g) = \mathsf{F}(f) \circ \mathsf{F}(g)$ and $\mathsf{F}(\mathrm{id}_A) = \mathrm{id}_{\mathsf{F}(A)}$. Composition of functors is denoted $\mathsf{G} \circ \mathsf{F} : \mathtt{C} \longrightarrow \mathtt{E}$ and the identity functor is written $\mathrm{id}_{\mathtt{C}} : \mathtt{C} \longrightarrow \mathtt{C}$. The (covariant) powerset functor $\mathsf{P} : \mathtt{Set} \longrightarrow \mathtt{Set}$ is the typical example of a functor, and is defined by $\mathsf{P}A$ being the powerset of A, i.e., the set of subsets of A, and $\mathsf{P}f(X)$, for $X \subseteq A$, being the image of X under f, i.e., $\mathsf{P}f(X) = \{f(x) \mid x \in X\}$. A contravariant functor $\mathsf{F} : \mathtt{C} \longrightarrow \mathtt{D}$ maps to each $\mathtt{C}$-morphism $A \xrightarrow{f} B$ a $\mathtt{D}$-morphism $\mathsf{F}(B) \xrightarrow{\mathsf{F}(f)} \mathsf{F}(A)$, and for the contravariant powerset functor $\overline{\mathsf{P}} : \mathtt{Set} \longrightarrow \mathtt{Set}$ we have $\overline{\mathsf{P}}A = \mathsf{P}A$ and $\overline{\mathsf{P}}f(Y) = \{x \in X \mid \exists y \in Y : f(x) = y\}$.

A *natural transformation* $\tau : \mathsf{F} \longrightarrow \mathsf{G}$ between functors assigns to each $\mathtt{C}$-object A a $\mathtt{D}$-morphism $\tau_A : \mathsf{F}A \longrightarrow \mathsf{G}A$ such that $\mathsf{G}f \circ \tau_A = \tau_B \circ \mathsf{F}f$, for any $f : A \longrightarrow B$. The identity natural transformation $\mathsf{F} \xrightarrow{\mathrm{id}_{\mathsf{F}}} \mathsf{F}$ is defined by $(\mathrm{id}_{\mathsf{F}})_A = \mathrm{id}_{\mathsf{F}A}$. If all τ_A are isomorphisms, τ is called a *natural isomorphism*, or *natural equivalence*. For functors F and natural transformations τ we often write $\mathsf{F}\tau$ and $\tau\mathsf{F}$ to mean $(\mathsf{F}\tau)_A = \mathsf{F}\tau_A$ and $(\tau\mathsf{F})_A = \tau_{\mathsf{F}A}$, respectively. It is easy to see that $\eta : \mathrm{id}_{\mathtt{Set}} \longrightarrow P$ given by $\eta_X(x) = \{x\}$, and $\mu : P \circ P \longrightarrow P$ given by $\mu_X(\mathscr{B}) = \bigcup \mathscr{B} (= \bigcup_{B \in \mathscr{B}} B)$ are natural transformations. The (vertical) composition $\sigma \circ \tau : \mathsf{F} \longrightarrow \mathsf{H}$ of natural transformations is defined by $(\sigma \circ \tau)_A = \sigma_A \circ \tau_A$, for all $\mathtt{D}$-objects A.

Whereas morphisms are typically seen as 'mappings' between objects in a category, functors are 'mappings' between categories, i.e., morphisms in (quasi-) categories of categories, and natural transformations are 'mappings' between functors, i.e., morphisms in functor categories. These notions clearly lead to views on hierarchies of sets, classes and conglomerates, where foundational issues enter the scene, and our approach roughly follows Grothendieck's [5] view of set-theoretic foundations for category theory.

A *monad* (or triple, or algebraic theory) over a category $\mathtt{C}$ is written as $\mathbf{F} = (\mathsf{F}, \eta, \mu)$, where $\mathsf{F} : \mathtt{C} \longrightarrow \mathtt{C}$ is a (covariant) functor, and $\eta : \mathrm{id} \longrightarrow \mathsf{F}$ and $\mu : \mathsf{F} \circ \mathsf{F} \longrightarrow \mathsf{F}$ are natural transformations for which $\mu \circ \mathsf{F}\mu = \mu \circ \mu\mathsf{F}$ and $\mu \circ \mathsf{F}\eta = \mu \circ \eta\mathsf{F} = \mathrm{id}_{\mathsf{F}}$ hold. A Kleisli category $\mathtt{C}_{\mathbf{F}}$ for a monad $\mathbf{F}$ over a category $\mathtt{C}$ is defined as follows: Objects in $\mathtt{C}_{\mathbf{F}}$ are the same as in $\mathtt{C}$, and the morphisms are defined as $\mathrm{Hom}_{\mathtt{C}_{\mathbf{F}}}(X, Y) = \mathrm{Hom}_{\mathtt{C}}(X, \mathsf{F}Y)$, that is morphisms $f : X \rightharpoondown Y$ in $\mathtt{C}_{\mathbf{F}}$ are simply morphisms $f : X \longrightarrow \mathsf{F}Y$ in $\mathtt{C}$, with $\eta_X : X \longrightarrow \mathsf{F}X$ being the identity morphism on X. Composition of morphisms is defined as

$$(X \rightharpoondown^{f} Y) \diamond (Y \rightharpoondown^{g} Z) = X \xrightarrow{\mu_Z \circ \mathsf{F}g \circ f} \mathsf{F}Z.$$

The category $\mathtt{Rel}$ with sets as objects and binary relations as morphisms, is isomorphic with the Kleisli category of the powerset monad over $\mathtt{Set}$. This invites to viewing Kleisli morphisms as a general notion for relations in the sense of intuitively being "substitutions".

Powerset monads and their many-valued extensions are in close connection to fuzzification and are good candidates to represent situations with incomplete or imprecise information. The many-valued covariant powerset functor L for a completely distributive lattice $\mathfrak{L} = (L, \vee, \wedge)$ is obtained by $\mathsf{L}X = L^X$, i.e. the set of functions (or $\mathfrak{L}$-sets) $\alpha : X \longrightarrow L$, and following [25], for a morphism $f : X \longrightarrow Y$ in $\mathtt{Set}$, by defining $\mathsf{L}f(\alpha)(y) = \bigvee_{f(x)=y} \alpha(x)$. Further, if we define $\eta_X : X \longrightarrow \mathsf{L}X$ by

$$\eta_X(x)(x') = \begin{cases} \top & \text{if } x = x' \\ \bot & \text{otherwise} \end{cases}$$

and $\mu : \mathsf{L} \circ \mathsf{L} \longrightarrow \mathsf{L}$ by

$$\mu_X(\mathscr{M})(x) = \bigvee_{\alpha \in \mathsf{L}X} A(x) \wedge \mathscr{M}(\alpha)$$

then $\mathbf{L} = (\mathsf{L}, \eta, \mu)$ is a monad.

Sorted Categories

In the one-sorted (and crisp) case for signatures we typically work in $\mathtt{Set}$, but in the many-sorted (and crisp) case we need the "sorted category of sets" for the many-sorted term functor. We start this section by a more general view by considering "a sorted category of objects".

Let S be an index set (in ZFC), the indices are called *sorts* (or types), and we do not assume any order on S. For a category $\mathtt{C}$, we write $\mathtt{C}_S$ for the product category $\prod_S \mathtt{C}$. The objects of $\mathtt{C}_S$ are tuples $(X_{\mathtt{s}})_{\mathtt{s}\in S}$ such that $X_{\mathtt{s}} \in \mathrm{Ob}(\mathtt{C})$ for all $\mathtt{s} \in S$. We will also use X_S as a shorthand notation for these tuples. The morphisms between objects $(X_{\mathtt{s}})_{\mathtt{s}\in S}$ and $(Y_{\mathtt{s}})_{\mathtt{s}\in S}$ are tuples $(f_{\mathtt{s}})_{\mathtt{s}\in S}$ such that $f_{\mathtt{s}} \in \mathrm{Hom}_{\mathtt{C}}(X_{\mathtt{s}}, Y_{\mathtt{s}})$ for all $\mathtt{s} \in S$, and similarly we will use f_S as a shorthand notation. The composition of morphisms is defined sortwise (componentwise), i.e., $(g_{\mathtt{s}})_{\mathtt{s}\in S} \circ (f_{\mathtt{s}})_{\mathtt{s}\in S} = (g_{\mathtt{s}} \circ f_{\mathtt{s}})_{\mathtt{s}\in S}$.

Functors $\mathsf{F}_{\mathtt{s}} : \mathtt{C} \longrightarrow \mathtt{D}$ are lifted to functors $\mathsf{F} = (\mathsf{F}_{\mathtt{s}})_{\mathtt{s}\in S}$ from $\mathtt{C}_S$ to $\mathtt{D}_S$. so that e.g. the regular powerset functor $\mathsf{P}_S = (\mathsf{P})_{\mathtt{s}\in S}$ and the regular many-valued powerset functor $\mathsf{L}_S = (\mathsf{L})_{\mathtt{s}\in S}$, both are lifted to functors on $\mathtt{Set}_S$.

Products and coproducts, $\prod$ and $\coprod$, are handled sortwise. We also have a "subobject relation", thus, $(X_{\mathtt{s}})_{\mathtt{s}\in S} \subseteq (Y_{\mathtt{s}})_{\mathtt{s}\in S}$ if and only if $X_{\mathtt{s}} \subseteq Y_{\mathtt{s}}$ for all $\mathtt{s} \in S$. It is clear that all limits and colimits exist in $\mathtt{Set}_S$, because operations on $\mathtt{Set}_S$-

objects are defined sortwise for sets. Further, the product $\prod_{i\in I} \mathsf{F}_i$ and coproduct $\coprod_{i\in I} \mathsf{F}_i$ of covariant functors F_i over $\mathtt{Set}_S$ are defined as

$$(\prod_{i\in I} \mathsf{F}_i)(X_{\mathtt{s}})_{\mathtt{s}\in S} = \prod_{i\in I} \mathsf{F}_i(X_{\mathtt{s}})_{\mathtt{s}\in S}$$

and

$$(\coprod_{i\in I} \mathsf{F}_i)(X_{\mathtt{s}})_{\mathtt{s}\in S} = \coprod_{i\in I} \mathsf{F}_i(X_{\mathtt{s}})_{\mathtt{s}\in S}$$

with morphisms being handled accordingly.

The category $\mathtt{Set}(\mathfrak{Q})_S$ is called the *many-sorted Goguen category*. Objects in this category are tuples of pairs $((X_{\mathtt{s}}, \alpha_{\mathtt{s}}))_{\mathtt{s}\in S}$ as objects, where for each $\mathtt{s} \in S$, $\alpha_{\mathtt{s}}\colon X_{\mathtt{s}} \longrightarrow Q$ is a function (in ZFC). So, fixing $\mathtt{s} \in S$ we consider pairs $(X_{\mathtt{s}}, \alpha_{\mathtt{s}})$ as objects in $\mathtt{Set}(\mathfrak{Q})$. Now, the $\mathtt{Set}(\mathfrak{Q})$-morphisms $(X_{\mathtt{s}}, \alpha_{\mathtt{s}}) \xrightarrow{f_{\mathtt{s}}} (Y_{\mathtt{s}}, \beta_{\mathtt{s}})$ form morphisms $((X_{\mathtt{s}}, \alpha_{\mathtt{s}}))_{\mathtt{s}\in S} \xrightarrow{(f_{\mathtt{s}})_{\mathtt{s}\in S}} ((Y_{\mathtt{s}}, \beta_{\mathtt{s}}))_{\mathtt{s}\in S}$.

Term Constructions

Here we recall the term functor construction and for clarity we present it in the one sorted situation, using the construction presented in [18]. The many sorted extension is found in [19].

Let $\Omega = \bigcup_{n=0}^{\infty} \Omega_n$ be an operator domain, where Ω_n contains n-ary operators. The term functor $\mathsf{T}_\Omega\colon \mathtt{Set} \to \mathtt{Set}$ is given as $\mathsf{T}_\Omega(X) = \bigcup_{k=0}^{\infty} T_\Omega^k(X)$, where

$$\begin{aligned} \mathsf{T}_\Omega^0(X) &= X, \\ \mathsf{T}_\Omega^{k+1}(X) &= \{(n, \omega, (m_i)_{i\leq n}) \mid \omega \in \Omega_n, n \in N, m_i \in \mathsf{T}_\Omega^k(X)\}. \end{aligned}$$

In this context it is more convenient to write terms as $(n, \omega, (x_i)_{i\leq n})$ instead of the more common $\omega(x_1, \ldots, x_n)$. It is clear that $(\mathsf{T}_\Omega X, (\sigma_\omega)_{\omega\in\Omega})$ is an Ω-algebra, if $\sigma_\omega((m_i)_{i\leq n}) = (n, \omega, (m_i)_{i\leq n})$ for $\omega \in \Omega_n$ and $m_i \in \mathsf{T}_\Omega X$. Morphisms $X \xrightarrow{f} Y$ in $\mathtt{Set}$ are extended in the usual way to the corresponding Ω-homomorphisms $(\mathsf{T}_\Omega X, (\sigma_\omega)_{\omega\in\Omega}) \xrightarrow{\mathsf{T}_\Omega f} (\mathsf{T}_\Omega Y, (\tau_\omega)_{\omega\in\Omega})$, where $\mathsf{T}_\Omega f$ is given as the Ω-extension of $X \xrightarrow{f} Y \hookrightarrow T_\Omega Y$ associated to $(\mathsf{T}_\Omega Y, (\tau_{n\omega})_{(n,\omega)\in\Omega})$. To obtain the term monad, define $\eta_X^{\mathsf{T}_\Omega}(x) = x$, and let $\mu_X^{\mathsf{T}_\Omega} = id^{\star}_{T_\Omega X}$ be the Ω-extension of $id_{T_\Omega X}$ with respect to $(\mathsf{T}_\Omega X, (\sigma_{n\omega})_{(n,\omega)\in\Omega})$. This gives us the (one-sorted) term monad $\mathbf{T}_\Omega = (\mathsf{T}_\Omega, \eta^{T_\Omega}, \mu^{T_\Omega})$.

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Rough Algebraic Structures Corresponding to Ring Theory

Bijan Davvaz

Abstract The concept of rough set was originally proposed by Pawlak in 1982. Since then the subject has been investigated in many papers. Some authors studied algebraic properties of rough sets. The lattice theoretical approach has been suggested by Iwinski. Pomykala and Pomykala showed that the set of rough sets forms a Stone algebra. Comer presented an interesting discussion of rough sets and various algebras related to the study of algebraic logic, such as Stone algebras and relation algebras. It is a natural question to ask what does happen if we substitute an algebraic structure instead of the universe set. Biswas and Nanda introduced the notion of rough subgroups. Kuroki introduced the notion of a rough ideal in a semigroup. Kuroki and Wang gave some properties of the lower and upper approximations with respect to the normal subgroups. Also, Kuroki and Mordeson studied the structure of rough sets and rough groups. Jun applied the rough set theory to BCK-algebras. The present author applied the concept of approximation spaces in ring theory, module theory and algebraic hyperstructures. A key notion in Pawlak rough set model is an equivalence relation. The equivalence classes are the building blocks for the construction of the lower and upper approximations. An equivalence relation is sometimes difficult to be obtained in real-world problems due to the vagueness and incompleteness of human knowledge. From this point of view, the author introduced the concept of lower inverse and upper inverse of a set under a set-valued map, which is a generalization of the lower and upper approximations. Using this the concept of a set-valued homomorphism for groups, rings, modules and lattices was introduced. The concept of uniform set-valued homomorphism was introduced and it was shown by the present author that every set-valued homomorphism is uniform. The overall aim of this chapter is to present an introduction to some of these results, methods and ideas about rough algebraic structures. Most of the focus will be on rough rings and their generalizations.

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1 Introduction

The concept of rough set was originally proposed by Pawlak [24, 25] as a formal tool for modelling and processing in complete information in information systems. Since then the subject has been investigated in many papers. The theory of rough set is an extension of set theory, in which a subset of a universe is described by a pair of ordinary sets called the lower and upper approximations. A key notion in Pawlak rough set model is an equivalence relation, i.e., a reflexive, symmetric and transitive relation. The equivalence classes are the building blocks for the construction of the lower and upper approximations. The lower approximation of a given set is the union of all the equivalence classes which are subsets of the set, and the upper approximation is the union of all the equivalence classes which have a non-empty intersection with the set. The lattice theoretical approach has been suggested by Iwinski [12]. He defined the family $\mathscr{R}$ of rough subsets of a universe U as an inclusion relation restricted to certain Boolean complete subalgebra $\mathscr{B}$ of $P(U)$ and proved that it is a complete, atomic, distributive lattice.

This chapter is organized as follows: After an introduction, Sect. 2 begins with defining upper and lower approximations in terms of an equivalence ρ and Proposition 1 lists the essential and well-known properties of rough approximations. After that, rough equality relation is defined and rough sets are defined to be equivalence classes of rough equality. Then we define a pair (A, B) of subsets of U as a rough set if and only if $(A, B) = app(X)$ for some X in $\mathscr{P}(U)$. Then, we recall the result by J. Pomykala and J.A Pomykala stating the rough sets form a complete atomic distributive Stone lattice. The section ends by recalling the result by Comer which says that rough set lattice forms a regular double Stone algebra. Section 3 presents results on the *relationship between rough sets and ring theory*. In applied mathematics we encounter many examples of mathematical objects that can be added to each other and multiplied to each other. First of all, the real numbers themselves are such objects. Other examples are real valued functions, the complex numbers and infinite series. We introduce the notion of rough subring (respectively, ideal) with respect to an ideal of a ring which is an extended notion of a subring (respectively, ideal) in a ring, and we give some properties of the lower and the upper approximations in a ring. Section 4 begins by recalling Zadeh's basic definitions of fuzzy sets and their *set-theoretical* operations. We also recall Liu's (1982) definitions of fuzzy subrings and fuzzy ideals of a ring. The theory of rough set and the theory of fuzzy set are seen as complementary generalizations of classical set theory. By using the concept of fuzzy sets, we introduce and discuss the concept of fuzzy rough subrings and ideals of a ring. Section 5 concerns a relationship between rough sets, fuzzy sets and ring theory. We consider a ring as a universal set and we assume that the knowledge about objects is restricted by a fuzzy ideal. In fact, we apply the notion of fuzzy ideal of a ring for definitions of the lower and upper approximations in a ring. Some characterizations of the above approximations are made and some examples are presented. In Sect. 6, we consider the relation α and its transitive closure α^*. The relation α is the smallest equivalence relation on a ring R so that R/α^* is a commutative ring. Based on the relation α, we define a neighborhood system for each element of R, and we present a general framework of the study of approximations in rings. The connections between rings and operators are examined. In the last section, the concepts of set-valued homomorphism and strong set-valued homomorphism of a ring are presented, and related properties are investigated.

2 Rough Sets and Stone Algebra

Let U be a universe of objects and ρ be an equivalence relation on U. Given an arbitrary set $A \subseteq U$, a concept in U, it may be impossible to describe A precisely using the equivalence classes of ρ. That is, the available information is not sufficient to give a precise representation of A. In this case, one may characterize A by a pair of lower and upper approximations

$$\underline{app}(A) := \bigcup_{[a]_\rho \subseteq A} [a]_\rho \quad \text{and} \quad \overline{app}(A) := \bigcup_{[a]_\rho \cap A \neq \emptyset} [a]_\rho,$$

where $[a]_\rho = \{b \mid a \ \rho \ b\}$ is the equivalence class containing a. The lower approximation $\underline{app}(A)$ is the union of all the elementary sets which are subsets of A. The upper approximation $\overline{app}(A)$ is the union of all the elementary sets which have a non-empty intersection with A. An element in the lower approximation necessarily belongs to A, while an element in the upper approximation possibly belong to A. We can express lower and upper approximations as follows:

$$\underline{app}(A) = \{a \in U \mid [a]_\rho \subseteq A\} \quad \text{and} \quad \overline{app}(A) = \{a \in U \mid [a]_\rho \cap A \neq \emptyset\}.$$

A subset X of U is called definable if $\underline{app}(X) = \overline{app}(X)$. If $X \subseteq U$ is given by a predicate P and $x \in U$, then

(1) $x \in \underline{app}(X)$ means that x certainly has property P
(2) $x \in \overline{app}(X)$ means that x possibly has property P
(3) $x \in U \backslash \overline{app}(X)$ means that x definitely does not have property P.

Proposition 1 *We have*

(1) $\underline{app}(A) \subseteq A \subseteq \overline{app}(A)$
(2) $\underline{app}(\emptyset) = \emptyset = \overline{app}(\emptyset)$
(3) $\underline{app}(U) = U = \overline{app}(U)$;
(4) *If* $A \subseteq B$, *then* $\underline{app}(A) \subseteq \underline{app}(B)$ *and* $\overline{app}(A) \subseteq \overline{app}(B)$
(5) $\underline{app}(\underline{app}(A)) = \underline{app}(A)$
(6) $\overline{app}(\overline{app}(A)) = \overline{app}(A)$
(7) $\overline{app}(\underline{app}(A)) = \underline{app}(A)$
(8) $\underline{app}(\overline{app}(A)) = \overline{app}(A)$

(9) $\underline{app}(A) = (\overline{app}(A^C))^C$
(10) $\overline{app}(A) = (\underline{app}(A^C))^C$;
(11) $\underline{app}(A \cap B) = \underline{app}(A) \cap \underline{app}(B)$
(12) $\overline{app}(A \cap B) \subseteq \overline{app}(A) \cap \overline{app}(B)$
(13) $\underline{app}(A \cup B) \supseteq \underline{app}(A) \cup \underline{app}(B)$
(14) $\overline{app}(A \cup B) = \overline{app}(A) \cup \overline{app}(B)$.

A pair (U, ρ) where $U \neq \emptyset$ and ρ is an equivalence relation on U, is called an approximation space. For an approximation space (U, ρ), by a rough approximation in (U, ρ) we mean a mapping $app : \mathscr{P}(U) \to \mathscr{P}(U) \times \mathscr{P}(U)$ defined by for every $X \in \mathscr{P}(U)$,

$$app(X) = (\underline{app}(X), \overline{app}(X)).$$

The rough equality between sets is defined in the following way: for any $A, B \subseteq U$

$$A \approx B \;\Leftrightarrow\; \underline{app}(A) = \underline{app}(B) \text{ and } \overline{app}(A) = \overline{app}(B).$$

Obviously, $\approx$ is an equivalence relation on $P(U)$. Any equivalence class of the relation $\approx$ is called a rough set. We denote by

$$\mathscr{R}^0 = \{[X]_{\approx} \mid X \subseteq U\}$$

the family of all rough sets

Therefore, for a given approximation space (U, ρ), a pair $(A, B) \in \mathscr{P}(U) \times \mathscr{P}(U)$ is a rough set in (U, ρ) if and only if $(A, B) = app(X)$ for some $X \in P(U)$.

Many properties of rough equality between sets as well as suggestions for their applications are to be found in [24]. An algebraic characterization of the relation $\approx$ is given in [22, 23].

Let $app(A) = (\underline{app}(A), \overline{app}(A))$ and $app(B) = (\underline{app}(B), \overline{app}(B))$ be any two rough sets in the approximation space (U, ρ). Then, we set the union, intersection, inclusion relation, complement, and set difference between rough sets as follows:

(1) $app(A) \sqcup app(B) := (\underline{app}(A) \cup \underline{app}(B), \overline{app}(A) \cup \overline{app}(B))$
(2) $app(A) \sqcap app(B) := (\underline{app}(A) \cap \underline{app}(B), \overline{app}(A) \cap \overline{app}(B))$
(3) $app(A) \sqsubseteq app(B) :\Leftrightarrow app(A) \sqcap app(B) = app(A)$.

When $app(A) \sqsubseteq app(B)$, we say that $app(A)$ is a rough subset of $app(B)$. Thus, in the case of rough sets $app(A)$ and $app(B)$,

$$app(A) \sqsubseteq app(B) \text{ if and only if } \underline{app}(A) \subseteq \underline{app}(B) \text{ and } \overline{app}(A) \subseteq \overline{app}(B).$$

This property of rough inclusion has all the properties of set inclusion. The rough complement of $app(A)$ denoted by $app^C(A)$ is defined by

$$app^C(A) := (U\backslash\overline{app}(A),\ U\backslash\underline{app}(A)).$$

Also, we can define $app(A)\backslash app(B)$ as follows:

$$\begin{aligned} app(A)\backslash app(B) &:= app(A) \cap app^C(B) \\ &= (\underline{app}(A)\backslash\overline{app}(B),\ \overline{app}(A)\backslash\underline{app}(B)). \end{aligned}$$

Pomykala and Pomykala [26] showed that the set of rough sets forms a Stone algebra.

Lemma 2 ([26, Lemma 4]) *Let (U, ρ) be an approximation space and $\mathscr{R}^0$ the family of rough sets. Then, the algebra $(\mathscr{R}^0, \sqcup, \sqcap)$ is a complete distributive lattice.*

The lattice $(\mathscr{R}^0, \sqcup, \sqcap)$ is bounded, where $0 = [\emptyset]_\approx$ is the least element and $1 = [U]_\approx$ is the greatest element.

Let us recall that in a bounded lattice $(L, \vee, \wedge, 0, 1)$, x is a complement of y if and only if $x \wedge y = 0$ and $x \vee y = 1$.

It is easy to notice that $\mathscr{R}^0$ is not a complemented lattice [12].

Let L be a lattice with 0; an element x^* is a pseudo-complement of $x \in L$ if and only if $x \wedge x^* = 0$ and $x \wedge z = 0$ implies that $z \leq x^*$. A pseudo-complemented lattice is one in which every element has a pseudo-complement.

We may define a pseudo-complement operation on $\mathscr{R}^*$, as follows: for any $\mathscr{X} \in \mathscr{R}^*$,

$$\mathscr{X}^* = [U - \overline{\mathscr{X}}]_\approx,$$

$\mathscr{X}^*$ is a pseudo-complement of $\mathscr{X}$.

Lemma 3 ([26, Lemma 7]) $(\mathscr{R}^0, \sqcup, \sqcap, *)$ *is pseudo-complemented.*

A distributive lattice with pseudo-complementation is called a Stone algebra if and only if it satisfies the Stone identity $a^* \vee a^{**} = 1$.

Lemma 4 ([26, Lemma 8]) *In the algebra $(\mathscr{R}^0, \sqcup, \sqcap, *, 0, 1)$ the Stone identity is valid.*

Now, we can summarize the above results as follows.

Theorem 5 ([26, Theorem 1]) *Suppose that $\mathscr{R}^0$ is the family of rough sets. Then, the algebra $(\mathscr{R}^0, \sqcup, \sqcap, *, 0, 1)$ is a complete, atomic Stone algebra.*

Comer [2] remarked upon some relationships between the ideas of an approximation space and rough sets and algebras related to the study of algebraic logic, namely, cylindric algebras, relation algebras, and Stone algebras. He considered three separate cases. The first deals with the family of approximation spaces induced by the indiscernibility relation for different sets of attributes of an information system. The family of closure operators defining these approximation spaces is abstractly characterized as a certain type of Boolean algebra with operators. An alternate formulation in terms of a general class of diagonal-free cylindric algebras is given. The second observation concerns the lattice theoretic approach to the study of rough sets suggested by Iwinski [12] and the result by Pomykala and Pomykala [26] that the collection of rough sets of an approximation space forms a Stone algebra. It is shown that every regular double Stone algebra is embeddable into the algebra of all rough subsets of an approximation space.

A double Stone algebra is an algebra $(L, +, \cdot, *, +, 0, 1)$ such that $(L, +, \cdot, 0, 1)$ is a bounded distributive lattice, $*$ is a pseudo-complement, Stone's law holds (i.e., $a^* + a^{**} = 1$), $+$ is a dual pseudo-complement (i.e., $x \geq a^+ \Leftrightarrow x + a = 1$), and the dual Stone law (i.e., $a^+ \cdot a^{++} = 0$) holds. A double Stone algebra is regular if $a^+ = b^+$ and $a^* = b^*$ imply $a = b$. See Gratzer [11] for basic facts about (double) Stone algebras.

Theorem 6 ([2, Theorem 2.1]) *$\mathscr{R}^0$ is a regular Stone algebra for every approximation space.*

3 Roughness in Rings

Biswas and Nanda [1] studied rough sets in algebraic structures. They gave the notion of rough subgroups. Because their notation depends on the upper approximation and does not depend on the lower approximation, Kuroki and Wang [17] discussed the lower and upper approximation of a group. Kuroki in [15], introduced the notion of a rough ideal in a semigroup. Also, Kuroki and Mordeson [16] studied the structure of rough sets and rough groups. Jun [13] applied the rough set theory to BCK-algebras.

Davvaz [3] concerned a relationship between rough sets and ring theory, also see [14]. He introduce the notion of rough subring (resp. ideal) with respect to an ideal of a ring which is an extended notion of a subring (resp. ideal) in a ring, and he gave some properties of the lower and the upper approximations in a ring.

A non-empty set R is said to be a ring if in R there are defined two binary operations, denoted by $+$ and $\cdot$ respectively, such that for all a, b, c in R: (1) $a+b = b + a$, (2) $(a + b) + c = a + (b + c)$, (3) there is an element 0 in R such that $a + 0 = a$, (4) there exists an element $-a$ in R such that $a + (-a) = 0$, (5) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$, (6) $\cdot$ is distributive with respect to $+$, i.e., $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(x + y) \cdot z = x \cdot z + y \cdot z$. Axioms (1) through (4) merely state that R is an abelian group under the operation $+$. The additive identity of a ring is called the zero element. If in addition: $a \cdot b = b \cdot a$, for all a, b in R, then R is said to be a commutative ring. If R contains an element 1 such that $1 \cdot a = a \cdot 1 = a$ for all a in R, then R is said to be a ring with unit element. Let R be a ring and S be a non-empty subset of R, which is closed under the addition and multiplication in R. If S is itself a ring under these operations then S is called a subring of R; more formally, S is a subring of R if the following conditions hold: $a, b \in S$ implies that

$a - b \in S$ and $a \cdot b \in S$. A non-empty subset I of a ring R is said to be an ideal of R if (1) I is a subgroup of R under addition, (2) for every $a \in I$ and $r \in R$, both ar and ra are in I. Clearly, each ideal is a subring.

Given a ring R and an ideal I, the underlying equivalence relation $\equiv_I$ induced by I on R is given by $x \equiv_I y :\Longleftrightarrow x - y \in I$. In such a case we have that the equivalence class of x with respect of $\equiv_I$ is exactly the coset $x + I$. Therefore, properly speaking, the corresponding approximation space is $(R, \equiv_I)$. Now, this notation can be naturally substitute with (R, I).

Throughout this paper R is a ring. Let I is an ideal of R and X is a non-empty subset of R. Then, the sets

$$\underline{app}_I(X) := \{x \in R |\ x + I \subseteq X\}, \quad \overline{app}_I(X) := \{x \in R |\ (x + I) \cap X \neq \emptyset\},$$

are called, respectively, lower and upper approximations of the set X with respect to the ideal I. The pair $(R, I$ will be referred to as a *rough ring*.

Proposition 7 *In a rough ring (R, I), for any two subsets $A, B \subseteq R$, we have:*

(1) $\underline{app}_I(A) \subseteq A \subseteq \overline{app}_I(A)$
(2) $\underline{app}_I(\emptyset) = \emptyset = \overline{app}_I(\emptyset)$
(3) $\underline{app}_I(R) = R = \overline{app}_I(R)$
(4) *If $A \subseteq B$, then $\underline{app}_I(A) \subseteq \underline{app}_I(B)$ and $\overline{app}_I(A) \subseteq \overline{app}_I(B)$*
(5) $\underline{app}_I(\underline{app}_I(A)) = \underline{app}_I(A)$
(6) $\overline{app}_I(\overline{app}_I(A)) = \overline{app}_I(A)$
(7) $\overline{app}_I(\underline{app}_I(A)) = \underline{app}_I(A)$
(8) $\underline{app}_I(\overline{app}_I(A)) = \overline{app}_I(A)$
(9) $\underline{app}_I(A) = (\overline{app}_I(A^C))^C$
(10) $\overline{app}_I(A) = (\underline{app}_I(A^C))^C$
(11) $\underline{app}_I(A \cap B) = \underline{app}_I(A) \cap \underline{app}_I(B)$
(12) $\overline{app}_I(A \cap B) \subseteq \overline{app}_I(A) \cap \overline{app}_I(B)$
(13) $\underline{app}_I(A \cup B) \supseteq \underline{app}_I(A) \cup \underline{app}_I(B)$;
(14) $\overline{app}_I(A \cup B) = \overline{app}_I(A) \cup \overline{app}_I(B)$
(15) $\underline{app}_I(x + I) = \overline{app}_I(x + I)$ *for all $x \in R$.*

For every approximation space (R, I),

(1) for every $A \subseteq R$, $\underline{app}_I(A)$ and $\overline{app}_I(A)$ are definable sets
(2) for every $x \in R$, $x + I$ is definable set.

If A and B are non-empty subsets of R, let AB denote the set of all finite sums $\{a_1b_1 + a_2b_2 + \ldots + a_nb_n |\ n \in \mathbb{N},\ a_i \in A,\ b_i \in B\}$. Moreover,

$$A + B := \{a + b \mid a \in A, b \in B\}.$$

Proposition 8 *Let I be an ideal of R, and A, B non-empty subsets of R. Then,*

$$\overline{app}_I(A) + \overline{app}_I(B) = \overline{app}_I(A + B).$$

Proposition 9 *Let I be an ideal of R, and A, B non-empty subsets of R. Then,*

$$\underline{app}_I(A) + \underline{app}_I(B) \subseteq \underline{app}_I(A + B).$$

The following example shows that

$$\underline{app}_I(A + B) \subseteq \underline{app}_I(A) + \underline{app}_I(B)$$

does not hold in general.

Example 10 Let $R = \mathbb{Z}_{12}$, $I = \{0, 6\}$, $A = \{0, 1, 2, 5, 6, 8\}$ and $B = \{0, 3, 4, 6, 9\}$. Then

$$\begin{aligned}
&\underline{app}_I(A) = \{0, 2, 6, 8\},\\
&\underline{app}_I(B) = \{0, 3, 6, 9\},\\
&\underline{app}_I(A) + \underline{app}_I(B) = \{0, 2, 3, 5, 6, 8, 9, 11\},\\
&\underline{app}_I(A + B) = \{0, 2, 3, 4, 5, 6, 8, 9, 10, 11\}.
\end{aligned}$$

Let I be an ideal of R and A any non-empty subset in R. It is easy to see that $\overline{app}_I(A)$ coincides with the subset $I + A$.

Proposition 11 ([32, Theorem 3.2]) *Let I be an ideal of R and A, B non-empty subsets of R. Then,*

$$\overline{app}_I(A) \cdot \overline{app}_I(B) \subseteq \overline{app}_I(A \cdot B).$$

Proposition 12 ([32, Theorem 3.3]) *Let I be an ideal of R and A, B non-empty subsets of R. If $I^2 = I$, then*

(1) $\overline{app}_I(A) \cdot \overline{app}_I(B) = \overline{app}_I(A \cdot B)$;
(2) $\underline{app}_I(A) \cdot \underline{app}_I(B) \subseteq \underline{app}_I(A \cdot B)$.

Proposition 13 ([32, Proposition 3.4]) *Let R be an idempotent ring, I, J two ideals of R and K a subring of R. Then,*

$$\underline{app}_I(K) \cdot \underline{app}_J(K) = \underline{app}_{(I+J)}(K).$$

Lemma 14 *Let I, J be two ideals of R such that $I \subseteq J$ and let A be a non-empty subset of R, then*

(1) $\underline{app}_J(A) \subseteq \underline{app}_I(A)$
(2) $\overline{app}_I(A) \subseteq \overline{app}_J(A)$.

Lemma 15 *Let I, J be two ideals of R and A a non-empty subset of R. Then,*

(1) $\underline{app}_I(A) \cap \underline{app}_J(A) \subseteq \underline{app}_{(I\cap J)}(A)$
(2) $\overline{app}_{(I\cap J)}(A) \subseteq \overline{app}_I(A) \cap \overline{app}_J(A)$.

Proposition 16 *If I, J are two ideals of R, then $\overline{app}_I(J)$ is an ideal of R.*

Proof Suppose that $a, b \in \overline{app}_I(J)$ and $r \in R$, then $(a+I) \cap J \neq \emptyset$ and $(b+I) \cap J \neq \emptyset$. So there exist $x \in (a+I) \cap J$ and $y \in (b+I) \cap J$. Since J is an ideal of R, we have $x - y \in J$ and $x - y \in (a+I) - (b+I) = a - b + I$. Hence, $(a - b + I) \cap J \neq \emptyset$, which implies $a - b \in \overline{app}_I(J)$. Also, we have $rx \in J$ and $rx \in r(a+I) = ra + I$. So $(ra + I) \cap J \neq \emptyset$, which implies $ra \in \overline{app}_I(J)$. Therefore, $\overline{app}_I(J)$ is an ideal of R. □

Similarly, if I is an ideal and J is a subring of R, then $\overline{app}_I(J)$ is a subring of R.

Proposition 17 *If I, J be two ideals of R, then $\underline{app}_I(J)$ is an ideal of R.*

Proof Suppose $a, b \in \underline{app}_I(J)$ and $r \in R$. Then, $a + I \subseteq J$ and $b + I \subseteq J$. It is easy to see that $(a - b + I) \subseteq J$ and $(ra + I) \subseteq J$. Hence, $a - b \in \underline{app}_I(J)$ and $ra \in \underline{app}_I(J)$. □

Similarly, if I is an ideal and J is a subring of R, then $\underline{app}_I(J)$ is a subring of R.

Let I be an ideal of R and $app_I(A) = (\underline{app}_I(A), \overline{app}_I(A))$ a rough set in the approximation space (R, I). If $\underline{app}_I(A)$ and $\overline{app}_I(A)$ are ideals (resp. subrings) of R, then we call $app_I(A)$ a rough ideal (resp. subring). Note that a rough subring also is called a rough ring.

Corollary 18

(1) *If I, J are two ideals of R, then $app_I(J)$ and $app_J(I)$ are rough ideals.*
(2) *If I is an ideal and J is a subring of R, then $app_I(J)$ is a rough ring.*

Proposition 19 *Let I, J be two ideals of R and K a subring of R. Then,*

$$\overline{app}_I(K) \cdot \overline{app}_J(K) \subseteq \overline{app}_{(I+J)}(K).$$

Proof Suppose x be any element of $\overline{app}_I(K) \cdot \overline{app}_J(K)$. Then, $x = \sum_{i=1}^{n} a_i b_i$ for some $a_i \in \overline{app}_I(K)$ and $b_i \in \overline{app}_J(K)$. Hence, $(a_i + I) \cap K \neq \emptyset$ and $(b_i + J) \cap K \neq \emptyset$, and so there exist elements $x_i, y_i \in R$ such that $x_i \in (a_i + I) \cap K$ and $y_i \in (b_i + J) \cap K$. Since K is a subring of R, we have $\sum_{i=1}^{n} x_i y_i \in K$. On the other hand we have

$$\sum_{i=1}^{n} x_i y_i \in \sum_{i=1}^{n} (a_i + I)(b_i + J) = \sum_{i=1}^{n} a_i b_i + I + J.$$

Therefore, we have

$$(\sum_{i=1}^{n} a_i b_i + I + J) \cap K \neq \emptyset,$$

which implies that $\sum_{i=1}^{n} a_i b_i \in \overline{app}_{(I+J)}(K)$. □

If we strengthen the condition, the inclusion symbol "⊆" of Proposition 19 may be replaced by an equal sign.

Proposition 20 ([32, Theorem 3.6]) *Let R be a ring with identity 1, I, J two ideals of R and K a subring of R such that $1 \in K$. Then,*

$$\overline{app}_I(K) \cdot \overline{app}_J(K) = \overline{app}_{(I+J)}(K).$$

Proof Since $1 \in K$, we have $IK = I$, $KJ = J$ and $K^2 = K$. Thus, we obtain

$$\begin{aligned}\overline{app}_I(K) \cdot \overline{app}_J(K) &= (I + K) \cdot (J + K) = IJ + IK + KJ + K^2 \\ &= I + J + K = \overline{app}_{(I+J)}(K).\end{aligned}$$ □

Proposition 21 ([32, Theorem 3.4]) *Let R be an idempotent ring, I, J two ideals of R and K a subring of R. Then,*

$$\underline{app}_I(K) \cdot \underline{app}_J(K) = \underline{app}_{(I+J)}(K).$$

Proposition 22 *Let I, J be two ideals of R and K a subring of R. Then,*

$$\underline{app}_I(K) + \underline{app}_J(K) = \underline{app}_{(I+J)}(K).$$

Proof Since $I \subseteq I + J$ and $J \subseteq I + J$, it follows that $\underline{app}_{(I+J)}(K) \subseteq \underline{app}_I(K)$ and $\underline{app}_{(I+J)}(K) \subseteq \underline{app}_J(K)$ and so $\underline{app}_{I+J}(K) \subseteq \underline{app}_I(K) + \underline{app}_J(K)$.

Now, let $x \in \underline{app}_I(K) + \underline{app}_J(K)$, then $x = a + b$ for some $a \in \underline{app}_I(K)$ and $b \in \underline{app}_J(K)$. Hence, $a + I \subseteq K$ and $b + J \subseteq K$. So

$$x + I + J = a + b + I + J = a + I + b + J \subseteq K + K = K$$

which yields $x \in \underline{app}_{(I+J)}(K)$. □

Proposition 23 *Let I, J be two ideals of R and K a subring of R. Then,*

$$\overline{app}_I(K) + \overline{app}_J(K) \subseteq \overline{app}_{(I+J)}(K).$$

Now, let R and R' be two rings and $\varphi : R \to R'$ a homomorphism from R to R'. It is well known, $ker\varphi$ is an ideal of R.

Theorem 24 *Let R and R' be two rings and f a homomorphism from R to R'. If A is a non-empty subset of R, then*

$$f(\overline{app}_{ker\varphi}(A)) = f(A).$$

Proof Since $A \subseteq \overline{app}_{ker\varphi}(A)$, it follows that $f(A) \subseteq f(\overline{app}_{ker\varphi}(A))$.

Conversely, let $y \in f(\overline{app}_{ker\varphi}(A))$. Then, there exists an element $x \in \overline{app}_{ker\varphi}(A)$ such that $f(x) = y$, so we have $(x + ker\varphi) \cap A \neq \emptyset$. Thus, there exists an element $a \in (x + ker\varphi) \cap A$. Then, $a = x + b$ for some $b \in ker\varphi$, that is, $x = a - b$. Then, we have

$$y = f(x) = f(a - b) = f(a) - f(b) = f(a) \in f(A),$$

and so $f(\overline{app}_{ker\varphi}(A)) \subseteq f(A)$. □

The lower and upper approximations can be presented in an equivalent form as follows:

Let I be an ideal of R, and A a non-empty subset of R. Then,

$$\underline{\underline{app}}_I(A) = \{a+I \in R/I \mid a+I \subseteq A\}, \quad \overline{\overline{app}}_I(A) = \{a+I \in R/I \mid (a+I)\cap A \neq \emptyset\}.$$

Proposition 25 *Let I, J be two ideals of R, then $\overline{\overline{app}}_I(J)$ is an ideal of R/I.*

Proof Suppose that $a+I, b+I \in \overline{\overline{app}}_I(J)$ and $r+I \in R/I$. Then, $(a+I)\cap J \neq \emptyset$ and $(b + I) \cap J \neq \emptyset$, so there exist $x \in (a + I) \cap J$ and $y \in (b + I) \cap J$. Since J is an ideal of R, we have $x - y \in J$ and $rx \in J$. Also, we have

$$x - y \in (a + I) - (b + I) = a - b + I,$$
$$rx \in r(a + I) = ra + I.$$

Therefore, $(a-b+I)\cap J \neq \emptyset$ and $(ra+I)\cap J \neq \emptyset$, which imply $(a+I)-(b+I) \in \overline{\overline{app}}_I(J)$ and $(r + I)(a + I) \in \overline{\overline{app}}_I(J)$. Therefore, $\overline{\overline{app}}_I(J)$ is an ideal of R/I. □

Proposition 26 *If I, J be two ideals of R, then $\underline{\underline{app}}_I(J)$ is an ideal of R/I.*

Proof It is straightforward. □

Similarly, if I is an ideal and J is a subring of R, then $\underline{\underline{app}}_I(J)$ and $\overline{\overline{app}}_I(J)$ are subrings of R/I.

4 Fuzzy Sets and Fuzzy Rough Sets

Zadeh in [30] introduced the notion of a fuzzy subset A of a non-empty set U as a membership function $\mu_A : U \to [0, 1]$ which associates with each point $x \in U$ its "degree of membership" $\mu_A(x) \in [0, 1]$.

Let A and B are fuzzy subsets in U. Then,

(1) $A = B \;:\Leftrightarrow\; \mu_A(x) = \mu_B(x)$, for all $x \in U$
(2) $A \subseteq B \;:\Leftrightarrow\; \mu_A(x) \leq \mu_B(x)$, for all $x \in U$
(3) $C = A \cup B \;:\Leftrightarrow\; \mu_C(x) = \max\{\mu_A(x), \mu_B(x)\}$ for all $x \in U$

(4) $D = A \cap B \;:\Leftrightarrow\; \mu_D(x) = \min\{\mu_A(x), \mu_B(x)\}$, for all $x \in U$
(5) The complement of A denoted by A^C, is defined by $\mu_{A^C}(x) := 1 - \mu_A(x)$ for all $x \in U$.

Rosenfeld [27] introduced fuzzy sets in the realm of group theory and formulated the concept of a fuzzy subgroup of a group. Since then many researchers are engaged in extending the concepts of abstract algebra to the broader framework of the fuzzy setting. In 1982, Liu [19] defined and studied fuzzy subrings and fuzzy ideals of a ring.

A fuzzy subset A of a ring R is called a fuzzy subring of R if, for all $x, y \in R$

(1) $\mu_A(x - y) \geq \min\{\mu_A(x), \mu_A(y)\}$;
(2) $\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y)\}$.

If the condition (2) is replaced by

$$\mu_A(xy) \geq \max\{\mu_A(x), \mu_A(y)\},$$

then A is called a fuzzy ideal of R.

The reader will find in [8, 20] some basic definitions and results about the fuzzy algebra.

Let (U, θ) is an approximation space and $app(X)$ a rough set in (U, θ). A fuzzy rough set $app(A) = (\underline{app}(A), \overline{app}(A))$ in $app(X)$ is characterized by a pair of maps

$$\mu_{\underline{app}(A)} : \underline{app}(X) \to [0, 1] \text{ and } \mu_{\overline{app}(A)} : \overline{app}(X) \to [0, 1].$$

with the property that

$$\mu_{\underline{app}(A)}(x) \leq \mu_{\overline{app}(A)}(x) \text{ for all } x \in \underline{app}(X).$$

Dubois and Prade [9] introduced the problem of fuzzification of a rough set. Also, Nanda and Majumdar in [21] investigated and discussed the concept of fuzzy rough sets.

For two fuzzy rough sets

$$app(A) = (\underline{app}(A), \overline{app}(A)) \text{ and } app(B) = (\underline{app}(B), \overline{app}(B))$$

in $app(X)$ we define

(1) $app(A) = app(B)$ iff

$$\mu_{\underline{app}(A)}(x) = \mu_{\underline{app}(B)}(x) \text{ for all } x \in \underline{app}(X),$$

$$\mu_{\overline{app}(A)}(x) = \mu_{\overline{app}(B)}(x) \text{ for all } x \in \overline{app}(X);$$

(2) $app(A) \subseteq app(B)$ iff

$$\mu_{\underline{app}(A)}(x) \subseteq \mu_{\underline{app}(B)}(x) \text{ for all } x \in \underline{app}(X),$$

$$\mu_{\overline{app}(A)}(x) \subseteq \mu_{\overline{app}(B)}(x) \text{ for all } x \in \overline{app}(X);$$

(3) $app(C) = app(A) \cup app(B)$ iff

$$\mu_{\underline{app}(C)}(x) = \max\{\mu_{\underline{app}(A)}(x),\ \mu_{\underline{app}(B)}(x)\} \text{ for all } x \in \underline{app}(X),$$

$$\mu_{\overline{app}(C)}(x) = \max\{\mu_{\overline{app}(A)}(x),\ \mu_{\overline{app}(B)}(x)\} \text{ for all } x \in \overline{app}(X);$$

(4) $app(D) = app(A) \cap app(B)$ iff

$$\mu_{\underline{app}(D)}(x) = \min\{\mu_{\underline{app}(A)}(x),\ \mu_{\underline{app}(B)}(x)\} \text{ for all } x \in \underline{app}(X),$$

$$\mu_{\overline{app}(D)}(x) = \min\{\mu_{\overline{app}(A)}(x),\ \mu_{\overline{app}(B)}(x)\} \text{ for all } x \in \overline{app}(X);$$

(5) We define the complement $app^C(A)$ of $app(A)$ by the ordered pair $(\underline{app}^C(A), \overline{app}^C(A))$ of membership functions where

$$\mu_{\underline{app}^C(A)}(x) = 1 - \mu_{\underline{app}(A)}(x) \text{ for all } x \in \underline{app}(X),$$

$$\mu_{\overline{app}^C(A)}(x) = 1 - \mu_{\overline{app}(A)}(x) \text{ for all } x \in \overline{app}(X).$$

Let I is an ideal of R and $app_I(X) = (\underline{app}_I(X),\ \overline{app}_I(X))$ a rough ring. The difference $\widehat{app}_I(X) = \overline{app}_I(X) \backslash \underline{app}_I(X)$ is called the boundary region of X. Let $app_I(A) = (\underline{app}_I(A),\ \overline{app}_I(A))$ is a fuzzy rough set of $app_I(X)$. We define $\overline{\mu}_{\underline{app}_I(A)} : \overline{app}_I(X) \to [0, 1]$ as follows:

$$\overline{\mu}_{\underline{app}_I(A)}(x) = \begin{cases} \mu_{\underline{app}_I(A)}(x) & \text{if } x \in \underline{app}_I(X) \\ 0 & \text{if } \widehat{app}_I(X). \end{cases}$$

Definition 27 Let $app_I(X)$ is a rough ring. An interval-valued fuzzy subset $\mathscr{A}$ is given by

$$\mathscr{A} = \{(x, [\overline{\mu}_{\underline{app}_I(A)}(x),\ \mu_{\overline{app}_I(A)}(x)] \mid x \in \overline{app}_I(X)\}$$

where $(\underline{app}_I(A),\ \overline{app}_I(A))$ is a fuzzy rough set of $app_I(X)$.

Suppose that $\widetilde{\mu_{\mathscr{A}}}(x) = [\overline{\mu}_{\underline{app}_I(A)}(x),\ \mu_{\overline{app}_I(A)}(x)]$ for all $x \in \overline{app}_I(X)$, and $D([0, 1])$ denotes the family of all closed subintervals of $[0, 1]$. If $\overline{\mu}_{\underline{app}_I(A)}(x) = \mu_{\overline{app}_I(A)}(x) = c$ where $0 \le c \le 1$, then we have $\widetilde{\mu_{\mathscr{A}}}(x) = [c, c]$ which we also assume, for the sake of convenience, to belong to $D([0,1])$. Thus, $\widetilde{\mu_{\mathscr{A}}}(x) \in D([0,1])$ for all $x \in \overline{app}_I(X)$.

Definition 28 Let $D_1 = [a_1, b_1],\ D_2 = [a_2, b_2]$ be elements of $D([0,1])$ then we define

$$rmax(D_1, D_2) = [a_1 \vee a_2,\ b_1 \vee b_2],$$

$$rmin(D_1, D_2) = [a_1 \wedge a_2,\ b_1 \wedge b_2].$$

We call $D_2 \leq D_1$ if and only if $a_2 \leq a_1$ and $b_2 \leq b_1$.

Definition 29 Let $app_I(X)$ be a rough ring. A fuzzy rough set $app_I(A) = (\underline{app}_I(A), \overline{app}_I(A))$ in $\overline{app}_I(X)$ is called a fuzzy rough subring if for each $x, y \in \overline{app}_I(X)$, the following hold:

$$\widetilde{\mu_{\mathscr{A}}}(x+y) \geq rmin\{\widetilde{\mu_{\mathscr{A}}}(x), \widetilde{\mu_{\mathscr{A}}}(y)\} \tag{1}$$

$$\widetilde{\mu_{\mathscr{A}}}(xy) \geq rmin\{\widetilde{\mu_{\mathscr{A}}}(x), \widetilde{\mu_{\mathscr{A}}}(y)\} \tag{2}$$

If the condition (2) is replaced by

$$\widetilde{\mu_{\mathscr{A}}}(xy) \geq rmax\{\widetilde{\mu_{\mathscr{A}}}(x), \widetilde{\mu_{\mathscr{A}}}(y)\}$$

then $\mathscr{A}$ is called a fuzzy rough ideal.

Lemma 30 *Let $app_I(X)$ is a rough ring. If $app_I(A) = (\underline{app}_I(A),\ \overline{app}_I(A))$ and $app_I(B) = (\underline{app}_I(B),\ \overline{app}_I(B))$ are two fuzzy rough subrings (resp. ideals) of $app_I(X)$ then $A \cap B$ is a fuzzy rough subring (resp. ideal) of $app_I(X)$.*

Definition 31 Let $app(X)$ be a rough ring and $app_I(A) = (\underline{app}_I(A),\ \overline{app}_I(A))$ a fuzzy rough set of $app(X)$. Then, we define

$$\underline{\mathscr{A}}_t = \{x \in \underline{app}_I(X) \mid \mu_{\underline{app}_I(A)}(x) \geq t\},$$

$$\overline{\mathscr{A}}_t = \{x \in \overline{app}_I(X) \mid \mu_{\overline{app}_I(A)}(x) \geq t\}.$$

$(\underline{\mathscr{A}}_t, \overline{\mathscr{A}}_t)$ is called a level rough set.

Theorem 32 *Let $app_I(X)$ is a rough ring and $app_I(A) = (\underline{app}_I(A),\ \overline{app}_I(A))$ a fuzzy rough set of $app_I(X)$. Then, $app_I(A)$ is a fuzzy rough subring of $app_I(X)$ if and only if for every $0 \leq t \leq 1$, $(\underline{\mathscr{A}}_t, \overline{\mathscr{A}}_t)$ is a rough subring of $app_I(X)$.*

5 Roughness in Rings Based on Fuzzy Ideals

Davvaz [4] considered a ring as a universal set and assumed that the knowledge about objects is restricted by a fuzzy ideal. In fact, he applied the notion of fuzzy ideal of a ring for definitions of the lower and upper approximations in a ring. Here we review some definitions and results.

Let μ and λ be two fuzzy subsets of a ring R. Then, the sum $\mu + \lambda$ is defined by

$$(\mu + \lambda)(x) := \sup_{x=a+b} \{ \min\{\mu(a), \lambda(b)\} \} \text{ for all } x \in R.$$

This definition is obtained from Zadeh's extension principle [31].

A fuzzy subset μ of a ring R is called a fuzzy ideal of R if it has the following properties:

(1) $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in R$
(2) $\mu(xy) \geq \max\{\mu(x), \mu(y)\}$ for all $x, y \in R$.

For a fuzzy ideal μ of a ring R, we have the following:

(1) $\mu(x) \leq \mu(0)$ and $\mu(x) = \mu(-x)$ for all $x \in R$, where 0 denotes the additive identity of R
(2) $\mu(x - y) = \mu(0)$ implies $\mu(x) = \mu(y)$, where $x, y \in R$.

The following statement is well known and easily seen.

Let μ and λ be fuzzy ideals of a ring R. Then, $\mu \cap \lambda$ is also a fuzzy ideal of R.

When μ is any fuzzy subset of R. The $\mu_t = \{x \in R \mid \mu(x) \geq t\}$, where $t \in [0, 1]$ is called a t-level subset of μ. The concept of t-level subset is very important in the relationship between fuzzy sets and crisp sets. It is well known that each fuzzy set can be uniquely represented by the family of all its t-level subsets. Also, a fuzzy subset μ of a ring R is a fuzzy ideal of R, if and only if the t-level subsets μ_t, $t \in Im\mu$ are ordinary ideals of R.

Definition 33 Let μ be a fuzzy ideal of R. For each $t \in [0, 1]$, the set $U(\mu, t) := \{(a, b) \in R \times R \mid \mu(a - b) \geq t\}$ is called a t-level relation of μ.

An equivalence relation θ on a ring R is a congruence relation if $(a, b) \in \theta$ implies $(a + x, b + x) \in \theta$ and $(x + a, x + b) \in \theta$ for all $x \in R$.

Lemma 34 *Let μ be a fuzzy ideal of a ring R, and let $t \in [0, 1]$. Then, $U(\mu, t)$ is a congruence relation on R.*

Proof For any element a of R, $\mu(a - a) = \mu(0) \geq t$ and so $(a, a) \in U(\mu, t)$. If $(a, b) \in U(\mu, t)$, then $\mu(a - b) \geq t$. Since μ is a fuzzy ideal of R, $\mu(b - a) = \mu(-(b - a)) = \mu(a - b) \geq t$ which yields $(b, a) \in U(\mu, t)$. If $(a, b) \in U(\mu, t)$ and $(b, c) \in U(\mu, t)$, then since μ is a fuzzy ideal of R,

$$\mu(a - c) = \mu((a - b) + (b - c)) \geq \min\{\mu(a - b), \mu(b - c)\} \geq \min\{t, t\} = t,$$

and so $(a, c) \in U(\mu, t)$. Therefore, $U(\mu, t)$ is an equivalence relation on R. Now, let $(a, b) \in U(\mu, t)$ and x be any element of R. Then, since $\mu(a - b) \geq t$,

$$\mu((a + x) - (b + x)) = \mu((a + x) + (-x - b)) = \mu(a + (x - x) - b)$$
$$= \mu(a + 0 - b) = \mu(a - b) \geq t,$$

and so $(a + x, b + x) \in U(\mu, t)$. Since $(R, +)$ is an abelian group, it follows that $(x + a, x + b) \in U(\mu, t)$. Therefore, $U(\mu, t)$ is a congruence relation on R. □

In this case, we say that a is congruent to b mod μ, written $a \equiv_t b(mod\mu)$ if $\mu(a - b) \geq t$.

Lemma 35 *Let μ and λ be fuzzy ideals of a ring R, and $t \in [0, 1]$. Then,*

$$U(\mu \cap \lambda, t) = U(\mu, t) \cap U(\lambda, t).$$

We denote by $[x]_{(\mu,t)}$ the equivalence class of $U(\mu, t)$ containing x of R.

Lemma 36 *Let μ be a fuzzy ideal of a ring R. If $a, b \in R$ and $t \in [0, 1]$, then*

(1) $[a]_{(\mu,t)} + [b]_{(\mu,t)} = [a + b]_{(\mu,t)}$;
(2) $[-a]_{(\mu,t)} = -([a]_{(\mu,t)})$.

Proof

(1) Suppose that $x \in [a]_{(\mu,t)} + [b]_{(\mu,t)}$. Then, there exist $y \in [a]_{(\mu,t)}$ and $z \in [b]_{(\mu,t)}$ such that $x = y + z$. Since $(a, y) \in U(\mu, t)$ and $(b, z) \in U(\mu, t)$, it follows that $(a + b, y + z) \in U(\mu, t)$ or $(a + b, x) \in U(\mu, t)$, and so $x \in [a + b]_{(\mu,t)}$.
Conversely, let $x \in [a + b]_{(\mu,t)}$ then $(x, a + b) \in U(\mu, t)$. Hence, $(x - b, a) \in U(\mu, t)$ and so $x - b \in [a]_{(\mu,t)}$ or $x \in [a]_{(\mu,t)} + b$, which implies that $x \in [a]_{(\mu,t)} + [b]_{(\mu,t)}$.
(2) We have

$$x \in [-a]_{(\mu,t)} \Leftrightarrow (x, -a) \in U(\mu, t) \Leftrightarrow (0, -a - x) \in U(\mu, t)$$
$$\Leftrightarrow (a, -x) \in U(\mu, t) \Leftrightarrow -x \in [a]_{(\mu,t)} \Leftrightarrow x \in -([a]_{(\mu,t)}).$$

□

Proposition 37

(1) *Let μ be a fuzzy ideal of a ring R and $t \in [0, 1]$. For any $a \in R$, we have $a + [0]_{(\mu,t)} = [a]_{(\mu,t)}$.*
(2) *Let μ and λ be two fuzzy ideals of a ring R such that $\lambda \subseteq \mu$ and $t \in [0, 1]$. Then, $[x]_{(\lambda,t)} \subseteq [x]_{(\mu,t)}$ for all $x \in R$.*

Proof

(1) Assume that $a \in R$, then we have

$$x \in a + [0]_{(\mu,t)} \Leftrightarrow x - a \in [0]_{(\mu,t)} \Leftrightarrow (x - a, 0) \in U(\mu, t) \\ \Leftrightarrow (x, a) \in U(\mu, t) \Leftrightarrow x \in [a]_{(\mu,t)}.$$

(2) We have

$$y \in [x]_{(\lambda,t)} \Rightarrow (x, y) \in U(\lambda, t) \Rightarrow \lambda(x - y) \geq t \\ \Rightarrow \mu(x - y) \geq t \Rightarrow (x, y) \in U(\mu, t) \\ \Rightarrow y \in [x]_{(\mu,t)}.$$

□

Let μ and λ be two fuzzy ideals of a ring R. The composition of congruence relations $U(\mu, t)$ and $U(\lambda, t)$ is defined as follows:
$U(\mu, t) \circ U(\lambda, t) =$
$\{(a, b) \in R \times R \mid \exists y \in R \text{ such that } (a, c) \in U(\mu, t),\ (c, b) \in U(\lambda, t)\}$.
It is no difficult to see that $U(\mu, t) \circ U(\lambda, t)$ is also a congruence relation. We denote this congruence relation by $U(\mu \circ \lambda, t)$.

Proposition 38

(1) *Let μ and λ be two fuzzy ideals of a ring R and $t \in [0, 1]$. Then,* $U(\mu \circ \lambda, t) \subseteq U(\mu + \lambda, t)$.
(2) *Let μ and λ be fuzzy ideals of a ring R with finite images, and $t \in [0, 1]$. Then,* $U(\mu \circ \lambda, t) = U(\mu + \lambda, t)$.

Proof

(1) Suppose that (a, b) be an arbitrary element of $U(\mu \circ \lambda, t)$. Then, there exists an element $c \in R$ such that $(a, c) \in U(\mu, t)$ and $(c, b) \in U(\lambda, t)$. Therefore, we have $\mu(a - c) \geq t$ and $\lambda(c - b) \geq t$. Then,

$$(\mu + \lambda)(a - b) = \sup_{u+v=a-b} \{ \min\{\mu(u), \lambda(v)\}\} \geq \min\{\mu(a - c), \lambda(c - b)\} \\ \geq \min\{t, t\} = t,$$

and so $(a, b) \in U(\mu + \lambda, t)$.
(2) By item (1), we have $U(\mu \circ \lambda, t) \subseteq U(\mu + \lambda, t)$, Therefore, we show that $U(\mu + \lambda, t) \subseteq U(\mu \circ \lambda, t)$. Assume that $(x, y) \in U(\mu + \lambda, t)$, then $(\mu + \lambda)(x - y) \geq t$. We have

$$\sup_{x-y=a+b} \{ \min\{\mu(a), \lambda(b)\} \} \geq t.$$

Since $Im\mu$ and $Im\lambda$ are finite, then

$$\min\{\mu(a_0), \lambda(b_0)\} \geq t \text{ for some } a_0, b_0 \in R$$

such that $x - y = a_0 + b_0$. Thus, $\mu(a_0) \geq t$ and $\lambda(b_0) \geq t$. Now, we have $\mu(a_0 - 0) \geq t$ and $\lambda(x - y - a_0) \geq t$, which imply $(a_0, 0) \in U(\mu, t)$ and $(x - y, a_0) \in U(\lambda, t)$. Therefore, $(x - y, 0) \in U(\mu \circ \lambda, t)$. Since $U(\mu \circ \lambda, t)$ is a congruence relation, we get $(x, y) \in U(\mu \circ \lambda, t)$. □

Let μ be a fuzzy ideal of a ring R and $t \in [0, 1]$, we know $U(\mu, t)$ is an equivalence relation (congruence relation) on R. Therefore, when $U = R$ and θ is the above equivalence relation, then we use (R, μ, t) instead of approximation space (U, θ).

Let μ be a fuzzy ideal of a ring R and $U(\mu, t)$ be a t-level congruence relation of μ on R. Let X be a non-empty subset of R. Then, the sets

$$\underline{U}(\mu, t, X) := \{x \in R \mid [x]_{(\mu,t)} \subseteq X\},$$

$$\overline{U}(\mu, t, X) := \{x \in R \mid [x]_{(\mu,t)} \cap X \neq \emptyset\},$$

are called, respectively, the lower and upper approximations of the set X with respect to $U(\mu, t)$.

The following proposition serve as the starting point for our analysis in the present paper.

Proposition 39 *For every approximation space (R, μ, t) and every subsets A, B of R, we have:*

(1) $\underline{U}(\mu, t, A) \subseteq A \subseteq \overline{U}(\mu, t, A)$
(2) $\underline{U}(\mu, t, \emptyset) = \emptyset = \overline{U}(\mu, t, \emptyset)$
(3) $\underline{U}(\mu, t, R) = R = \overline{U}(\mu, t, R)$
(4) *If $A \subseteq B$, then $\underline{U}(\mu, t, A) \subseteq \underline{U}(\mu, t, B)$ and $\overline{U}(\mu, t, A) \subseteq \overline{U}(\mu, t, B)$*
(5) $\underline{U}(\mu, t, \underline{U}(\mu, t, A)) = \underline{U}(\mu, t, A)$
(6) $\overline{U}(\mu, t, \overline{U}(\mu, t, A)) = \overline{U}(\mu, t, A)$;
(7) $\overline{U}(\mu, t, \underline{U}(\mu, t, A)) = \underline{U}(\mu, t, A)$
(8) $\underline{U}(\mu, t, \overline{U}(\mu, t, A)) = \overline{U}(\mu, t, A)$
(9) $\underline{U}(\mu, t, A) = (\overline{U}(\mu, t, A^c))^c$
(10) $\overline{U}(\mu, t, A) = (\underline{U}(\mu, t, A^c))^c$
(11) $\underline{U}(\mu, t, A \cap B) = \underline{U}(\mu, t, A)) \cap \underline{U}(\mu, t, B)$
(12) $\overline{U}(\mu, t, A \cap B) \subseteq \overline{U}(\mu, t, A)) \cap \overline{U}(\mu, t, B)$
(13) $\underline{U}(\mu, t, A \cup B) \supseteq \underline{U}(\mu, t, A)) \cup \underline{U}(\mu, t, B)$
(14) $\overline{U}(\mu, t, A \cup B) = \overline{U}(\mu, t, A)) \cup \overline{U}(\mu, t, B)$
(15) $\underline{U}(\mu, t, [x]_{(\mu,t)}) = \overline{U}(\mu, t, [x]_{(\mu,t)})$ *for all $x \in R$.*

The following example shows that the converse of (12) and (13) in Proposition 39 is not true.

Example 40 Let $R = \{0, a, b, c\}$. Define addition and multiplication by Cayley tables:

$+$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	a	b	c
c	0	0	0	0

Then, R is a ring. Clearly, $a = -a,\ b = -b$ and $c = -c$. We define

$$\mu(0) = t_0, \quad \mu(c) = t_1, \quad \mu(a) = \mu(b) = t_2,$$

where $t_i \in [0, 1]$, $i = 0, 1, 2$ and $t_2 < t_1 < t_0$. It is no difficult to see that μ is a fuzzy ideal of R. We have

$$U(\mu, t_0) = \{(0, 0),\ (a, a),\ (b, b),\ (c, c)\};$$
$$U(\mu, t_1) = \{(0, 0),\ (a, a),\ (b, b),\ (c, c),\ (a, b),\ (b, a),\ (0, c),\ (c, 0)\};$$
$$U(\mu, t_2) = R \times R.$$

Now, let $A = \{0, a\}$ and $B = \{0, b, c\}$. Then,

$$\overline{U}(\mu, t_1, A) = R; \quad \overline{U}(\mu, t_1, B) = R; \quad \overline{U}(\mu, t_1, A \cap B) = \{0, c\};$$

and

$$\underline{U}(\mu, t_1, A) = \emptyset; \quad \underline{U}(\mu, t_1, B) = \{0, c\}; \quad \underline{U}(\mu, t_1, A \cup B) = \underline{U}(\mu, t_1, R) = R.$$

Therefore,

$$\overline{U}(\mu, t_1, A) \cap \overline{U}(\mu, t_1, B) \not\subseteq \overline{U}(\mu, t_1, A \cap B),$$

$$\underline{U}(\mu, t_1, A \cup B) \not\subseteq \underline{U}(\mu, t_1, A) \cup \underline{U}(\mu, t_1, B).$$

From Proposition 39 and Example 40, we can draw the following conclusions.

(1) The certain information of $A \cup B$ may be more than the union of the certain information of A and B.
(2) The uncertain information of $A \cap B$ may be less than the intersection of the uncertain information of A and B.

Proposition 41 *Let μ and λ be fuzzy ideals of a ring R, and $t \in [0, 1]$. If X is a non-empty subset of R, then*

$$\overline{U}(\mu \cap \lambda, t, X) \subseteq \overline{U}(\mu, t, X) \cap \overline{U}(\lambda, t, X).$$

Proof We have

$$\begin{aligned}
x \in \overline{U}(\mu \cap \lambda, t, X) &\Rightarrow [x]_{(\mu\cap\lambda,t)} \cap X \neq \emptyset \\
&\Rightarrow \exists a \in [x]_{(\mu\cap\lambda,t)} \cap X \\
&\Rightarrow (a, x) \in U(\mu \cap \lambda, t) \text{ and } a \in X \\
&\Rightarrow (\mu \cap \lambda)(a - x) \geq t \text{ and } a \in X \\
&\Rightarrow \min\{\mu(a - x), \lambda(a - x)\} \geq t \text{ and } a \in X \\
&\Rightarrow \mu(a - x) \geq t \text{ and } \lambda(a - x) \geq t \text{ and } a \in X \\
&\Rightarrow (a, x) \in U(\mu, t) \text{ and } (a, x) \in U(\lambda, t) \text{ and } a \in X \\
&\Rightarrow (a, x) \in U(\mu, t),\ a \in X \text{ and } (a, x) \in U(\lambda, t),\ a \in X \\
&\Rightarrow a \in [x]_{(\mu,t)} \cap X \text{ and } a \in [x]_{(\lambda,t)} \cap X \\
&\Rightarrow [x]_{(\mu,t)} \cap X \neq \emptyset \text{ and } [x]_{(\lambda,t)} \cap X \neq \emptyset \\
&\Rightarrow x \in \overline{U}(\mu, t, X) \text{ and } x \in \overline{U}(\lambda, t, X).
\end{aligned}$$

Therefore, $\overline{U}(\mu\cap\lambda, t, X) \subseteq \overline{U}(\mu, t, X)\cap\overline{U}(\lambda, t, X)$. This completes the proof. □

The following example shows that the converse of Proposition 41 is not true.

Example 42 Let $R = \mathbb{Z}_6$ (the ring of integers modulo 6). Define fuzzy subsets $\mu : \mathbb{Z}_6 \to [0, 1]$ and $\lambda : \mathbb{Z}_6 \to [0, 1]$ by

$$\begin{aligned}
&\mu(0) = t_0,\ \ \mu(1) = \mu(2) = \mu(4) = \mu(5) = t_3,\ \ \mu(3) = t_2; \\
&\lambda(0) = t_1,\ \ \lambda(1) = \lambda(3) = \lambda(5) = t_4,\ \ \lambda(2) = \lambda(4) = t_2,
\end{aligned}$$

where $t_i \in [0, 1]$, $0 \leq i \leq 4$ and $t_4 < t_3 < t_2 < t_1 < t_0$. It follows that μ and λ are fuzzy ideals of $\mathbb{Z}_6$. We have

$$\begin{aligned}
(\mu \cap \lambda)(0) &= t_1, \\
(\mu \cap \lambda)(2) = (\mu \cap \lambda)(4) &= t_3, \\
(\mu \cap \lambda)(1) = (\mu \cap \lambda)(3) = (\mu \cap \lambda)(5) &= t_4.
\end{aligned}$$

Also, we have

$$\begin{aligned}
U(\mu, t_0) &= \{(0, 0),\ (1, 1),\ (2, 2),\ (3, 3),\ (4, 4),\ (5, 5)\}; \\
U(\mu, t_2) &= \{(0, 0),\ (1, 1),\ (2, 2),\ (3, 3),\ (4, 4),\ (5, 5), \\
&\qquad (5, 2),\ (2, 5),\ (4, 1),\ (1, 4),\ (0, 3),\ (3, 0)\}; \\
U(\mu, t_3) &= \mathbb{Z}_6 \times \mathbb{Z}_6, \\
U(\lambda, t_1) &= \{(0, 0),\ (1, 1),\ (2, 2),\ (3, 3),\ (4, 4),\ (5, 5)\}; \\
U(\lambda, t_2) &= \{(0, 0),\ (1, 1),\ (2, 2),\ (3, 3),\ (4, 4),\ (5, 5); \\
&\qquad (5, 3),\ (3, 5),\ (4, 2),\ (2, 4),\ (1, 3), (3, 1), \\
&\qquad (0, 2),\ (2, 0),\ (5, 1),\ (1, 5),\ (0, 4),\ (4, 0)\}; \\
U(\lambda, t_4) &= \mathbb{Z}_6 \times \mathbb{Z}_6;
\end{aligned}$$

and

$$\begin{aligned}
U(\mu\cap\lambda, t_1) &= \{(0,0),\ (1,1),\ (2,2),\ (3,3),\ (4,4),\ (5,5)\};\\
U(\mu\cap\lambda, t_3) &= \{(0,0),\ (1,1),\ (2,2),\ (3,3),\ (4,4),\ (5,5),\\
&\qquad (5,3),\ (3,5),\ (4,2),\ (2,4),\ (1,3),\ (3,1),\\
&\qquad (0,2),\ (2,0),\ (5,1),\ (1,5),\ (0,4),\ (4,0)\};\\
U(\mu\cap\lambda, t_4) &= \mathbb{Z}_6\times\mathbb{Z}_6.
\end{aligned}$$

Now, let $X = \{1, 2, 3\}$, then

$$\overline{U}(\mu, t_2, X) = \mathbb{Z}_6,\quad \overline{U}(\lambda, t_2, X) = \mathbb{Z}_6,\quad \overline{U}(\mu\cap\lambda, t_2, X) = \{1, 2, 3\},$$

and so $\overline{U}(\mu\cap\lambda, t, X) \neq \overline{U}(\mu, t, X)\cap\overline{U}(\lambda, t, X)$.

Proposition 43 *Let μ and λ be fuzzy ideals of a ring R, and $t \in [0, 1]$. If X is a non-empty subset of R, then*

$$\underline{U}(\mu, t, X)\cap\underline{U}(\lambda, t, X) \subseteq \underline{U}(\mu\cap\lambda, t, X).$$

Proof We have

$$\begin{aligned}
x\in\underline{U}(\mu, t, X)\cap\underline{U}(\lambda, t, X) &\Rightarrow x\in\underline{U}(\mu, t, X)\quad\text{and}\quad x\in\underline{U}(\lambda, t, X)\\
&\Rightarrow [x]_{(\mu,t)}\subseteq X\quad\text{and}\quad [x]_{(\lambda,t)}\subseteq X\\
&\Rightarrow [x]_{(\mu\cap\lambda,t)}\subseteq X\\
&\Rightarrow x\in\underline{U}(\mu\cap\lambda, t, X).
\end{aligned}$$

Therefore, $\underline{U}(\mu, t, X)\cap\underline{U}(\lambda, t, X) \subseteq \underline{U}(\mu\cap\lambda, t, X)$. □

The inclusion symbol $\subseteq$ in Proposition 43 may not be replaced by an equals sign, as the next example shows.

From Propositions 41, 43, we can draw the following conclusions.

(1) The certain information of X with respect to the intersection of two fuzzy ideals μ and λ may be more than the intersection of the certain information of X with respect to the fuzzy ideals μ and λ.
(2) The uncertain information of X with respect to the intersection of two fuzzy ideals μ and λ may be less than the intersection of the uncertain information of X with respect to fuzzy ideals μ and λ.

A non-empty subset A of a ring R is called an upper rough ideal of R if $\overline{U}(\mu, t, A)$ is an ideal of R.

Proposition 44 *Let μ be a fuzzy ideal of a ring R and $t \in [0, 1]$. If A is an ideal of R, then A is an upper rough ideal of R.*

Proof Suppose that $a, b \in \overline{U}(\mu, t, A)$ and $r \in R$, then $[a]_{(\mu,t)}\cap A \neq \emptyset$ and $[b]_{(\mu,t)}\cap A \neq \emptyset$. So there exist $x \in [a]_{(\mu,t)}\cap A$ and $y \in [b]_{(\mu,t)}\cap A$. Since A

is an ideal of R, it follows that $x - y \in A$ and $rx \in A$. Now, we have

$$x - y \in [a]_{(\mu,t)} - [b]_{(\mu,t)} = [a - b]_{(\mu,t)}.$$

Hence, $[a - b]_{(\mu,t)} \cap A \neq \emptyset$, which implies $a - b \in \overline{U}(\mu, t, A)$.

Since $(x, a) \in U(\mu, t)$, it follows that $\mu(x - a) \geq t$. Now, we have

$$\mu(rx - ra) = \mu(r(x - a)) \geq \max\{\mu(r), \mu(x - a)\} \geq \mu(x - a) \geq t.$$

Hence, $(rx, ra) \in U(\mu, t)$ or $rx \in [ra]_{(\mu,t)}$, thus $rx \in [ra]_{(\mu,t)} \cap A$ which implies $[ra]_{(\mu,t)} \cap A \neq \emptyset$. Therefore, $ra \in \overline{U}(\mu, t, A)$. In a similar way, we get $ar \in \overline{U}(\mu, t, A)$. Therefore, $\overline{U}(\mu, t, A)$ is an ideal of R. □

The above proposition shows that the notion of an upper rough ideal is an extended notion of a usual ideal of a ring. It is no difficult to see that the converse of Proposition 44 does not hold in general.

Lemma 45 *Let μ be a fuzzy ideal of a ring R and $t \in [0, 1]$. If $\underline{U}(\mu, t, A)$ is a non-empty set, then $[0]_{(\mu,t)} \subseteq A$.*

Proof Suppose that $\underline{U}(\mu, t, A) \neq \emptyset$, then there exists $x \in \underline{U}(\mu, t, A)$ or $[x]_{(\mu,t)} \subseteq A$. So $-([x]_{(\mu,t)}) \subseteq -A = \{-a \mid a \in A\} = A$. Now, we have

$$\begin{aligned}
[0]_{(\mu,t)} &= [x + (-x)]_{(\mu,t)} \\
&= [x]_{(\mu,t)} + [-x]_{(\mu,t)} \\
&= [x]_{(\mu,t)} + (-[x]_{(\mu,t)}) \\
&\subseteq A + A = A.
\end{aligned}$$

□

Proposition 46 *Let μ be a fuzzy ideal of a ring R and $t \in [0, 1]$. Let A be an ideal of R. If $\underline{U}(\mu, t, A)$ is a non-empty set, then it is equal to A.*

Proof By Proposition 39(1), we have $\underline{U}(\mu, t, A) \subseteq A$. We show that $A \subseteq \underline{U}(\mu, t, A)$. Assume that a is an arbitrary element of A. By Lemma 45, we have $[0]_{(\mu,t)} \subseteq A$. Since A is an ideal of R, it follows that

$$a + [0]_{(\mu,t)} \subseteq a + A \subseteq A.$$

Now, we obtain $[a]_{(\mu,t)} \subseteq A$, which implies $a \in \underline{U}(\mu, t, A)$. □

Let μ be a fuzzy ideal of a ring R and $(\underline{U}(\mu, t, A), \overline{U}(\mu, t, A))$ a rough set in the approximation space (R, μ, t). If $\underline{U}(\mu, t, A)$ and $\overline{U}(\mu, t, A)$ are ideals of R, then we call $(\underline{U}(\mu, t, A), \overline{U}(\mu, t, A))$ a rough ideal. Therefore, we have

Corollary 47 *Let μ be a fuzzy ideal of a ring R and $t \in [0, 1]$. If A is an ideal of R, then $(\underline{U}(\mu, t, A), \overline{U}(\mu, t, A))$ is a rough ideal of R.*

Proposition 48 *Let μ and λ be two fuzzy ideals of a ring R such that $\lambda \subseteq \mu$. If A is a non-empty subset of R and $t \in [0, 1]$, then*

(1) $\overline{U}(\lambda, t, A) \subseteq \overline{U}(\mu, t, A)$
(2) $\underline{U}(\mu, t, A) \subseteq \underline{U}(\lambda, t, A)$.

Proof

(1) Suppose that x be an arbitrary element of $\overline{U}(\lambda, t, A)$, then $[x]_{(\lambda,t)} \cap A \neq \emptyset$. Since $[x]_{(\lambda,t)} \subseteq [x]_{(\mu,t)}$, it follows that $[x]_{(\mu,t)} \cap A \neq \emptyset$ which implies $x \in \overline{U}(\mu, t, A)$.
(2) Assume that $x \in \underline{U}(\mu, t, A)$, then $[x]_{(\mu,t)} \subseteq A$. Now, we obtain $[x]_{(\lambda,t)} \subseteq A$ which implies $x \in \underline{U}(\lambda, t, A)$. □

Lemma 49 *Let μ and λ be fuzzy ideals of a ring R, and $t \in [0, 1]$. Let X be a non-empty subset of R. If $U(\lambda, t) \subseteq U(\mu, t)$, then*

(1) $\overline{U}(\lambda, t, X) \subseteq \overline{U}(\mu, t, X)$
(2) $\underline{U}(\mu, t, X) \subseteq \underline{U}(\lambda, t, X)$.

Proof

(1) Suppose that x is an arbitrary element of $\overline{U}(\lambda, t, X)$, then there exists $a \in [x]_{(\lambda,t)} \cap X$. Then, $a \in X$ and $(a, x) \in U(\lambda, t) \subseteq U(\mu, t)$. Therefore, $a \in [x]_{(\mu,t)} \cap X$, and so $x \in \overline{U}(\mu, t, X)$.
(2) Suppose that x is an arbitrary element of $\underline{U}(\mu, t, X)$, then $[x]_{(\mu,t)} \subseteq X$. Since $[c]_{(\lambda,t)} \subseteq [x]_{(\mu,t)}$, it follows that $[x]_{(\lambda,t)} \subseteq X$ which implies that $x \in \underline{U}(\lambda, t, X)$. □

Now, we consider the relationships between the lower and upper approximations with respect to the sum of fuzzy ideals and the composition of congruence relations.

Proposition 50 *Let μ and λ be fuzzy ideals of a ring R, and $t \in [0, 1]$. If X is a non-empty subset of R, then*

(1) $\overline{U}(\mu \circ \lambda, t, X) \subseteq \overline{U}(\mu + \lambda, t, X)$
(2) $\underline{U}(\mu + \lambda, t, X) \subseteq \underline{U}(\mu \circ \lambda, t, X)$.

Proposition 51 *Let μ and λ be fuzzy ideals of a ring R with finite images, and $t \in [0, 1]$. If X is a non-empty subset of R, then*

(1) $\overline{U}(\mu \circ \lambda, t, X) = \overline{U}(\mu + \lambda, t, X)$
(2) $\underline{U}(\mu \circ \lambda, t, X) = \underline{U}(\mu + \lambda, t, X)$.

If A and B are non-empty subsets of R, Let $A \cdot B$ denote the set of all finite sums $\{a_1b_1 + a_2b_2 + \ldots + a_nb_n \;\; n \in \mathbb{N}, \; a_i \in A, \; b_i \in B\}$.

Proposition 52 *Let μ and λ be fuzzy ideals of a ring R, and $t \in [0, 1]$. If A is an ideal (or sub-ring) of R, then*

$$\overline{U}(\mu, t, A) \cdot \overline{U}(\lambda, t, A) \subseteq \overline{U}(\mu \circ \lambda, t, A).$$

Proof Suppose that z be any element of $\overline{U}(\mu, t, A) \cdot \overline{U}(\lambda, t, A)$. Then,

$$z = \sum_{i=1}^{n} a_i b_i \text{ for some } a_i \in \overline{U}(\mu, t, A) \text{ and } b_i \in \overline{U}(\lambda, t, A).$$

Thus, $[a_i]_{(\mu,t)} \cap A \neq \emptyset$ and $[b_i]_{(\lambda,t)} \cap A \neq \emptyset$ for $i = 1, \ldots, n$. So, there exist elements x_i and y_i in R such that

$$x_i \in [a_i]_{(\mu,t)} \cap A \quad \text{and} \quad y_i \in [b_i]_{(\lambda,t)} \cap A \quad \text{for } i = 1, \ldots, n.$$

Since A is an ideal (or sub-ring) of R, it follows that $\sum_{i=1}^{n} x_i y_i \in A$. Since $(x_i, a_i) \in U(\mu, t)$ and $(y_i, b_i) \in U(\lambda, t)$, we have $\mu(x_i - a_i) \geq t$ and $\lambda(y_i - b_i) \geq t$. Then, $\mu(x_i b_i - a_i b_i) = \mu((x_i - a_i)b_i) \geq \max\{\mu(x_i - a_i), \mu(b_i)\} \geq \mu(x_i - a_i) \geq t$, and $\lambda(x_i y_i - x_i b_i) = \lambda(x_i(y_i - b_i)) \geq \max\{\lambda(x_i), \lambda(y_i - b_i)\} \geq \lambda(y_i - b_i) \geq t$. Hence, $(x_i b_i, a_i b_i) \in U(\mu, t)$ and $(x_i y_i, x_i b_i) \in U(\lambda, t)$, and so

$$(x_i y_i, a_i b_i) \in U(\mu \circ \lambda, t) \text{ for all } i = 1, \ldots n.$$

Since $U(\mu \circ \lambda, t)$ is a congruence relation, it follows that

$$(\sum_{i=1}^{n} x_i y_i, \sum_{i=1}^{n} a_i b_i) \in U(\mu \circ \lambda, t),$$

and so $\sum_{i=1}^{n} x_i y_i \in [\sum_{i=1}^{n} a_i b_i]_{(\mu \circ \lambda, t)}$. Therefore, $[\sum_{i=1}^{n} a_i b_i]_{(\mu \circ \lambda, t)} \cap A \neq \emptyset$ which implies that $x = \sum_{i=1}^{n} a_i b_i \in \overline{U}(\mu \circ \lambda, t, A)$. □

Corollary 53 *Let μ and λ be fuzzy ideals of a ring R, and $t \in [0, 1]$. If A is an ideal (or sub-ring) of R, then*

$$\overline{U}(\mu, t, A) \cdot \overline{U}(\lambda, t, A) \subseteq \overline{U}(\mu + \lambda, t, A).$$

The inclusion symbol $\subseteq$ in Proposition 52 may not be replaced by an equal sign, as the next example shows.

Example 54 Let $R = \mathbb{Z}_{15}$ and

$$\mu(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and } \lambda(x) = \begin{cases} 1 & \text{if } x = 0, 5, 10 \\ 0 & \text{otherwise} \end{cases}$$

and let $A = \{0, 3, 6, 9, 12\}$. Then, routine calculations give that

$\overline{U}(\mu, \frac{1}{2}, A) = \{0, 3, 6, 9, 12\} = A; \qquad \overline{U}(\lambda, \frac{1}{2}, A) = \mathbb{Z}_{15};$
$\overline{U}(\mu, \frac{1}{2}, A) \cdot \overline{U}(\lambda, \frac{1}{2}, A) = A \cdot \mathbb{Z}_{15} = A.$

On the other hand, it is easy to see that $U(\mu \circ \lambda, \frac{1}{2}) = U(\lambda, \frac{1}{2})$, and so $\overline{U}(\mu \circ \lambda, \frac{1}{2}) = \mathbb{Z}_{15}$. This shows that $\overline{U}(\mu, t, A) \cdot \overline{U}(\lambda, t, A) = \overline{U}(\mu \circ \lambda, t, A)$ is not true in general.

From Proposition 52 and Example 54, we see that the uncertain information of A with respect to the sum of two fuzzy ideals μ and λ may be more than the product of the uncertain information of A with respect to the fuzzy ideals μ and λ.

Proposition 55 *Let μ and λ be fuzzy ideals of a ring R, and $t \in [0, 1]$. If A is a subgroup of $(R, +)$, then*

$$\overline{U}(\mu, t, A) + \overline{U}(\lambda, t, A) = \overline{U}(\mu \circ \lambda, t, A).$$

Proof Suppose that z be any element of $\overline{U}(\mu, t, A) + \overline{U}(\lambda, t, A)$. Then, $z = a + b$ for some $a \in \overline{U}(\mu, t, A)$ and $b \in \overline{U}(\lambda, t, A)$. Thus,

$$[a]_{(\mu,t)} \cap A \neq \emptyset \quad \text{and} \quad [b]_{(\lambda,t)} \cap A \neq \emptyset.$$

So there exist elements x and y in R such that $x \in [a]_{(\mu,t)} \cap A$ and $y \in [b]_{(\lambda,t)} \cap A$. Since A is a subgroup of $(R, +)$, it follows that $x + y \in A$. Since $(x, a) \in U(\mu, t)$ and $(y, b) \in U(\lambda, t)$, we obtain $(x - a, 0) \in U(\mu, t)$ and $(0, b - y) \in U(\lambda, t)$, and so

$$(x - a, b - y) \in U(\mu \circ \lambda, t) \quad \text{or} \quad (x + y, a + b) \in U(\mu \circ \lambda, t).$$

Hence, $x + y \in [a + b]_{(\mu\circ\lambda,t)}$. Now, we have $x + y \in [z]_{(\mu\circ\lambda,t)} \cap A$, which implies $z \in \overline{U}(\mu \circ \lambda, t, A)$. Therefore, $\overline{U}(\mu, t, A) + \overline{U}(\lambda, t, A) \subseteq \overline{U}(\mu \circ \lambda, t, A)$.

Conversely, assume that x be any element of $\overline{U}(\mu \circ \lambda, t, A)$, then $[x]_{(\mu\circ\lambda,t)} \cap A \neq \emptyset$. So there exists $a \in R$ such that $a \in [x]_{(\mu\circ\lambda,t)} \cap A$. Thus, $(a, x) \in U(\mu \circ \lambda, t)$. Since $U(\mu \circ \lambda, t)$ is a congruence relation, there exists $y \in R$ such that $(a, y) \in U(\mu, t)$ and $(y, x) \in U(\lambda, t)$. Since $a \in A$ and $a \in [y]_{(\mu,t)}$, we have $[y]_{(\mu,t)} \cap A \neq \emptyset$, and so $y \in \overline{U}(\mu, t, A)$. From $(y, x) \in U(\lambda, t)$, we get $(x - y, 0) \in U(\lambda, t)$ or $0 \in [x - y]_{(\lambda,t)}$. Since A is a subgroup of $(R, +)$, it follows that $0 \in A$. Thus, $0 \in [x - y]_{(\lambda,t)} \cap A$, which implies $x - y \in \overline{U}(\lambda, t, A)$. So

$$x = y + (x - y) \in \overline{U}(\mu, t, A) + \overline{U}(\lambda, t, A).$$

Therefore, we get $\overline{U}(\mu \circ \lambda, t, A) \subseteq \overline{U}(\lambda, t, A) + \overline{U}(\lambda, t, A)$. □

6 Approximations in a Ring by Using a Neighborhood System

Lin [18] proposed a more general framework for the study of approximation operators by using the so-called neighborhood systems from a topological space. In a neighborhood system, each element of a universe is associated with a family of subsets of the universe. This family is called a neighborhood system of the element, and each member in the family is called a neighborhood of the element. Any subset of the universe can be approximated based on neighborhood systems of all elements in the universe, also see [28]. In [10], Freni considered the relation γ and its transitive closure γ^* on a semigroup.

Let S be a semigroup. Then, we set $\gamma_1 = \{(x, x) \mid x \in S\}$ and for every integer $n > 1$, γ_n is the relation defined as follows:

$$x\gamma_n y \;\Leftrightarrow\; \exists(z_1, \ldots, z_n) \in S^n,\ \exists\sigma \in \mathbb{S}_n \;:\; x = \prod_{i=1}^{n} z_i,\ \ y = \prod_{i=1}^{n} z_{\sigma(i)},$$

where $\mathbb{S}_n$ is the symmetric group on n letters. The relation γ is the smallest equivalence relation on S so that S/γ^* is a commutative semigroup. Based on the relation γ, Davvaz in [6] defined a neighborhood system for each element of S, and presented a general framework of the study of approximations in semigroups. Davvaz [7] considered the relation α and its transitive closure α^*. The relation α is the smallest equivalence relation on a ring R so that R/α^* is a commutative ring. Based on the relation α, he defined a neighborhood system for each element of R.

Definition 56 Let R be a ring. A congruence relation ρ on R is an equivalence relation that satisfies

$$r_1 + s_1\ \rho\ r_2 + s_2 \ \text{ and } r_1 s_1\ \rho\ r_2 s_2,$$

whenever $r_1 \rho r_2$ and $s_1 \rho s_2$.

For a congruence on a ring, the equivalence class containing 0 is always a two-sided ideal, and the two operations on the set of equivalence classes define the corresponding quotient ring.

Lemma 57 *Let R be a ring and ρ be an equivalence relation on R. Then, ρ is a congruence relation on R if and only if for every $x, y, a \in R$,*

$$x\rho y \;\Rightarrow\; \begin{cases} x + a\ \rho\ y + a, & a + x\ \rho\ a + y, \\ x \cdot a\ \rho\ y \cdot a, & a \cdot x\ \rho\ a \cdot y. \end{cases}$$

Proof It is straightforward. □

Definition 58 Let R be a (non-commutative) ring. We define the relation α as follows:

$x \; \alpha \; y \iff \exists n \in \mathbb{N}, \; \exists (k_1, \ldots, k_n) \in \mathbb{N}^n$ and $[\exists (x_{i1}, \ldots, x_{ik_i}) \in R^{k_i}, \exists \sigma_i \in \mathbb{S}_{k_i}, \; (i = 1, \ldots, n)]$ such that

$$x = \sum_{i=1}^{n} (\prod_{j=1}^{k_i} x_{ij}) \quad \text{and} \quad y = \sum_{i=1}^{n} (\prod_{j=1}^{k_i} x_{i\sigma_i(j)}).$$

The relation α is reflexive and symmetric. Let α^* be the transitive closure of α.

Theorem 59 *α^* is a congruence relation on R.*

Proof If $x\alpha y$, then $\exists n \in \mathbb{N}, \; \exists (k_1, \ldots, k_n) \in \mathbb{N}^n$, and $[\exists (x_{i1}, \ldots, x_{ik_i}) \in R^{k_i}$, $\exists \sigma_i \in \mathbb{S}_{k_i}, \; (i = 1, \ldots, n)]$ such that

$$x = \sum_{i=1}^{n} (\prod_{j=1}^{k_i} x_{ij}) \quad \text{and} \quad y = \sum_{i=1}^{n} (\prod_{j=1}^{k_i} x_{i\sigma_i(j)}).$$

and so

$$x + a = \sum_{i=1}^{n} (\prod_{j=1}^{k_i} x_{ij}) + a \quad \text{and} \quad y + a = \sum_{i=1}^{n} (\prod_{j=1}^{k_i} x_{i\sigma_i(j)}) + a.$$

Now, let $k_{n+1} = 1, x_{n+1\;1} = a, \sigma_{n+1} = id$. Thus,

$$x + a = \sum_{i=1}^{n+1} (\prod_{j=1}^{k_i} x_{ij}) \quad \text{and} \quad y + a = \sum_{i=1}^{n+1} (\prod_{j=1}^{k_i} x_{i\sigma_i(j)}).$$

Therefore, $x + a \; \alpha \; y + a$. In the same way, we can show that $a + x \; \alpha \; a + y$. Now, it is easy to see that

$$x + a \; \alpha^* \; y + a \quad \text{and} \quad a + x \; \alpha^* \; a + y.$$

Now, note that

$$xa = \Big(\sum_{i=1}^{n} (\prod_{j=1}^{k_i} x_{ij})\Big)a \quad \text{and} \quad ya = \Big(\sum_{i=1}^{n} (\prod_{j=1}^{k_i} x_{i\sigma_i(j)})\Big)a,$$

which yields that

$$xa = \sum_{i=1}^{n} \Big((\prod_{j=1}^{k_i} x_{ij})a\Big) \quad \text{and} \quad ya = \sum_{i=1}^{n} \Big((\prod_{j=1}^{k_i} x_{i\sigma_i(j)})a\Big).$$

We put $k_i' = k_i + 1$, $x_{ik_i'} = a$ and define $\tau_i(r) = \sigma_i(r)$ for all $r = 1, \ldots, k_i$ and $\tau_i(k_i + 1) = k_i + 1$. In this case, $\tau_i \in \mathbb{S}_{k_i'}$ $(i = 1, \ldots, n)$. Thus,

$$xa = \sum_{i=1}^{n}(\prod_{j=1}^{k_i'} x_{ij}) \quad \text{and} \quad ya = \sum_{i=1}^{n}(\prod_{j=1}^{k_i'} x_{i\tau_i(j)}).$$

Therefore, $xa \ \alpha \ ya$ and so $xa \ \alpha^* \ ya$. Similarly, we obtain $ax \ \alpha^* \ ay$. This completes the proof. □

We define $\oplus$ and $\odot$ on R/α^* in the usual manner:

$$\alpha^*(a) \oplus \alpha^*(b) = \alpha^*(a + b),$$
$$\alpha^*(a) \odot \alpha^*(b) = \alpha^*(ab).$$

Corollary 60 *The quotient R/α^* is a commutative ring.*

Proof Since α^* is a congruence relation, it follows that R/α^* is a ring. Suppose that σ is the permutation of $\mathbb{S}_2$ such that $\sigma(1) = 2$. Clearly, we have $x_1x_2 \ \alpha \ x_{\sigma(1)}x_{\sigma(2)}$. Then, $x_1x_2 \ \alpha^* \ x_{\sigma(1)}x_{\sigma(2)}$. Therefore, R/α^* is a commutative ring. □

Theorem 61 *The relation α^* is the smallest equivalence relation such that the quotient R/α^* is a commutative ring.*

Proof Let θ be an equivalence relation such that R/θ is a commutative ring and let $\varphi : R \to R/\theta$ be the canonical projection. If $x\alpha y$, then there exist $n \in \mathbb{N}$, $(k_1, \ldots, k_n) \in \mathbb{N}^n$ and there exist $(x_{i1}, \ldots, x_{ik_i}) \in R^{k_i}$ and $\sigma_i \in \mathbb{S}_{k_i}$ $(i = 1, \ldots, n)$ such that

$$x = \sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij}) \quad \text{and} \quad y = \sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{i\sigma_i(j)}).$$

Hence,

$$\varphi(x) = \sum_{i=1}^{n}(\prod_{j=1}^{k_i} \varphi(x_{ij})) \quad \text{and} \quad \varphi(y) = \sum_{i=1}^{n}(\prod_{j=1}^{k_i} \varphi(x_{i\sigma_i(j)})).$$

By the commutativity of R/θ, it follows that $\varphi(x) = \varphi(y)$. Thus, $x\alpha y$ implies that $x\theta y$. Finally, let $x\alpha^* y$. Then, there exist $z_1, \ldots, z_m \in R$ such that $x = z_1\alpha z_2$, $z_2\alpha z_3, \ldots, z_{n-1}\alpha z_n = y$, and so $x = z_1\theta z_2, z_2\theta z_3, \ldots, z_{n-1}\theta z_n = y$

Since θ is transitively closed, it follows that $x\theta y$. Hence,

$$x \in \alpha^*(y) \ \Rightarrow \ x \in \theta(y).$$

Therefore, $\alpha^* \subseteq \theta$. □

Definition 62 For the relation α on R and a positive integer k, we now define a notion of binary relation α^k called the k-step-relation of α as follows:

(1) $\alpha^1 = \alpha$
(2) $\alpha^k = \{(x, y) \in R \times R \mid$ there exist $y_1, y_2, \ldots, y_i \in R$, $1 \le i \le k-1$, such that $x\alpha y_1, y_1\alpha y_2, \ldots, y_i\alpha y\} \cup \alpha^1$, $k \ge 2$.

It is easy to see that

$$\alpha^{k+1} = \alpha^k \cup \{ (x, y) \in R \times R \mid \text{there exist } y_1, \ldots, y_k \in R, \text{ such that } x\alpha y_1,\ y_1\alpha y_2, \ldots, y_k\alpha y\}.$$

Obviously, $\alpha^k \subseteq \alpha^{k+1}$, and there exists $n \in \mathbb{N}$ such that $\alpha^k = \alpha^n$ for all $k \ge n$. (In fact $\alpha^n = \alpha^*$ is nothing else but the transitive closure of α). Of course α^* is transitive. The relation α^k can be conveniently expressed as a mapping from R to $\wp(R)$, $N_k(x) = \{y \in R \mid x\alpha^k y\}$ by collecting all α^k-related elements for each element $x \in R$. The set $N_k(x)$ may be viewed as a α^k-neighborhood of x defined by the binary relation α^k.

Based on the relation α^k on R, we can obtain a neighborhood system for each element x: $\{N_k(x) \mid k \ge 1\}$. This neighborhood system is monotonically increasing with respect to k. We can also observe that

$$N_k(x) = \{y \in R \mid \text{there exist } y_1, y_2, \ldots, y_i \in R \text{ such that } x\alpha y_1,\ y_1\alpha y_2, \ldots, y_i\alpha y,\ 1 \le i \le k-1, \text{ or } x\alpha^k y\}.$$

If A and B are non-empty subsets of a ring R, then $A + B = \{a + b \mid a \in A, b \in B\}$ and AB denote the set of all finite sums $\{a_1b_1 + a_2b_2 + \ldots + a_nb_n \mid n \in \mathbb{N},\ a_i \in A,\ b_i \in B\}$.

Theorem 63 *For each $a, b \in R$ and natural numbers k, l we have*

$$N_k(a) + N_l(b) \subseteq N_{k+l-1}(a + b).$$

Proof Suppose that $x \in N_k(a)+N_l(b)$. Then, there exist $a' \in N_k(a)$ and $b' \in N_l(b)$ such that $x = a' + b'$. Since $a' \in N_k(a)$, it follows that $a'\alpha^k a$ and so there exist $\{x_1, \ldots, x_{k+1}\} \subseteq R$ with $x_1 = a'$, $x_{k+1} = a$ such that $x_1\ \alpha\ x_2$, $x_2\ \alpha\ x_3$, …, $x_k\ \alpha\ x_{k+1}$. Hence, for $t = 1, \ldots, k$,

$x_t\ \alpha\ x_{t+1} \Leftrightarrow \exists n_t \in \mathbb{N}$, $\exists (h_{t1}, \ldots, h_{tn_t}) \in \mathbb{N}^{n_t}$ and $[\exists (u_{ti1}, \ldots, u_{tih_{t_i}}) \in R^{h_{ti}}$, $\exists \sigma_i \in \mathbb{S}_{h_{ti}}$, $(i = 1, \ldots, n_t)]$ such that

$$x_t = \sum_{i=1}^{n_t} (\prod_{j=1}^{h_{ti}} u_{tij}) \quad \text{and} \quad x_{t+1} = \sum_{i=1}^{n_t} (\prod_{j=1}^{h_{ti}} u_{ti\sigma_i(j)}).$$

Also, since $b' \in N_l(b)$, it follows that $b' \, \alpha^l \, b$ and so there exist $\{y_1, \ldots, y_{l+1}\} \subseteq R$ with $y_1 = b'$, $y_{l+1} = b$ such that $y_1 \, \alpha \, y_2$, $y_2 \, \alpha \, y_3$, ..., $y_l \, \alpha \, y_{l+1}$. Hence, for $s = 1, \ldots, l$,

$y_s \, \alpha \, y_{s+1} \;\Leftrightarrow\; \exists m_s \in \mathbb{N},\ \exists (h'_{s1}, \ldots, h'_{sm_s}) \in \mathbb{N}^{m_s}$ and $[\exists (v_{si1}, \ldots, v_{sih'_{s_i}}) \in R^{h'_{si}},\ \exists \sigma_i \in \mathbb{S}_{h'_{si}},\ (i = 1, \ldots, m_s)]$ such that

$$y_s = \sum_{i=1}^{m_s} (\prod_{j=1}^{h'_{si}} v_{sij}) \quad \text{and} \quad y_{s+1} = \sum_{i=1}^{m_s} (\prod_{j=1}^{h'_{si}} v_{si\sigma_i(j)}).$$

Therefore, we obtain

$$x_t + y_1 = \sum_{i=1}^{n_t} (\prod_{j=1}^{h_{ti}} u_{tij}) + \sum_{i=1}^{m_1} (\prod_{j=1}^{h'_{1i}} v_{1ij}),$$
$$x_{t+1} + y_1 = \sum_{i=1}^{n_t} (\prod_{j=1}^{h_{ti}} u_{ti\sigma_i(j)}) + \sum_{i=1}^{m_1} (\prod_{j=1}^{h'_{1i}} v_{1ij}),$$

and

$$x_{k+1} + y_s = \sum_{i=1}^{n_k} (\prod_{j=1}^{h_{ki}} u_{ki\sigma_i(j)}) + \sum_{i=1}^{m_s} (\prod_{j=1}^{h'_{si}} v_{sij}),$$
$$x_{k+1} + y_{s+1} = \sum_{i=1}^{n_k} (\prod_{j=1}^{h_{ki}} u_{ki\sigma_i(j)}) + \sum_{i=1}^{m_s} (\prod_{j=1}^{h'_{si}} v_{si\sigma_i(j)}).$$

If we pick up elements $z_1, \ldots, z_{k+l}$ such that

$$z_i = x_i + y_1, \qquad i = 1, \ldots, k,$$
$$z_{k+j} = x_{k+1} + y_{j+1}, \; j = 1, \ldots, l.$$

Then, $z_1 \, \alpha^{k+l-1} \, z_{m+1}$. So $x = a' + b' = x_1 + y_1 \, \alpha^{k+l-1} \, x_{k+1} + y_{l+1} = a + b$. Therefore, $x \in N_{k+l-1}(a + b)$. □

For a neighborhood operator N_k on R, we can extend N_k from $\wp(R)$ to $\wp(R)$ by: $N_k(X) = \bigcup_{x \in X} N_k(x)$ for all $X \subseteq R$. So, we can directly deduce that

Proposition 64 *We have*

(1) $A \subseteq B \;\Rightarrow\; N_k(A) \subseteq N_k(B)$
(2) *for all* $k, l \geq 1$, *we have* $N_l(N_k(x)) \subseteq N_{l+k}(x)$.

If θ^* is a congruence relation on R such that R/θ^* is a commutative ring, then $\alpha^* \subseteq \theta^*$.

Let R be a ring and A be a non-empty subset of R. We define the lower and upper approximations of A with respect to α^* as follows:

$$\underline{\alpha^*(A)} := \{x \in R \mid \alpha^*(x) \subseteq A\} \text{ and } \overline{\alpha^*(A)} := \{x \in R \mid \alpha^*(x) \cap A \neq \emptyset\}.$$

Similarly, we can define the lower and upper approximations of A with respect to η^*. In this case, we have

$$\underline{\theta^*(A)} \subseteq \underline{\alpha^*(A)} \subseteq A \subseteq \overline{\alpha^*(A)} \subseteq \overline{\theta^*(A)}.$$

Definition 65 For the relation α, by substituting equivalence class $\alpha^*(x)$ with α^k-neighborhood $N_k(x)$ in the previous definition, we can define a pair of lower and upper approximation operators with respect to N_k as follows:

$$\underline{apr}_k(A) := \{x \in R \mid N_k(x) \subseteq A\} \text{ and } \overline{apr}_k(A) := \{x \in R \mid N_k(x) \cap A \neq \emptyset\}.$$

The set $\underline{apr}_k(A)$ consists of those elements whose α^k-neighborhoods are contained in A, and $\overline{apr}_k(A)$ consists of those elements whose α^k-neighborhoods have a non-empty intersection with A.

Proposition 66 *If A is a non-empty subset of R, then we have*

(1) $\underline{apr}_{k+1}(A) \subseteq \underline{apr}_k(A)$
(2) $\overline{apr}_k(A) \subseteq \overline{apr}_{k+1}(A)$.

Therefore:

Corollary 67 *We have*

$$\bigcup\{x \mid x \in \underline{\alpha^*(A)}\} = \bigcap_k \underline{apr}_k(A) \quad \text{and} \quad \bigcup\{x \mid x \in \overline{\alpha^*(A)}\} = \bigcup_k \overline{apr}_k(A).$$

Proposition 68 *If A and B are non-empty subsets of R, then the pair of approximation operators satisfies the following properties:*

(1) $\underline{apr}_k(A) \subseteq A \subseteq \overline{apr}_k(A)$
(2) $\underline{apr}_k(A) = (\overline{apr}_k(A^c))^c$
(3) $\overline{apr}_k(A) = (\underline{apr}_k(A^c))^c$
(4) $\underline{apr}_k(A \cap B) = \underline{apr}_k(A) \cap \underline{apr}_k(B)$
(5) $\overline{apr}_k(A \cup B) = \overline{apr}_k(A) \cup \overline{apr}_k(B)$
(6) $\underline{apr}_k(A \cup B) \supseteq \underline{apr}_k(A) \cup \underline{apr}_k(B)$;
(7) $\overline{apr}_k(A \cap B) \subseteq \overline{apr}_k(A) \cap \overline{apr}_k(B)$
(8) $A \subseteq B \Rightarrow \underline{apr}_k(A) \subseteq \underline{apr}_k(B)$
(9) $A \subseteq B \Rightarrow \overline{apr}_k(A) \subseteq \overline{apr}_k(B)$.

Proposition 69 *Let A be a non-empty subset of R. For all $k \geq l \geq 1$, we have*

(1) $A \subseteq \underline{apr}_l(\overline{apr}_k(A))$
(2) $\overline{apr}_l(\underline{apr}_k(A)) \subseteq A$.

Proof

(1) Suppose that $a \in A$. If $N_l(a) = \emptyset$. Then, it is clear that $N_l(a) \subseteq \overline{apr}_k(A)$, which implies that $a \in \underline{apr}_l(\overline{apr}_k(A))$, and so $A \subseteq \underline{apr}_l(\overline{apr}_k(A))$. If $N_l(a) \neq \emptyset$, then for each $b \in N_l(a)$, we have $a \in N_l(b)$. Hence, $N_l(b) \cap A \neq \emptyset$. Now, we have $b \in \overline{apr}_l(A)$, and then we obtain $b \in \overline{apr}_k(A)$. Therefore, $N_l(a) \subseteq \overline{apr}_k(A)$, which implies that $a \in \underline{apr}_l(\overline{apr}_k(A))$, and so $A \subseteq \underline{apr}_l(\overline{apr}_k(A))$.
(2) Suppose that $a \in \overline{apr}_l(\underline{apr}_k(A))$. Then, we have $N_l(a) \cap \underline{apr}_k(A) \neq \emptyset$, and so there exists $b \in N_l(a) \cap \underline{apr}_k(A)$. Therefore, $a \in N_l(b)$ and $N_k(b) \subseteq A$. Hence, $a \in N_l(b) \subseteq N_k(b) \subseteq A$, and so we conclude that $\overline{apr}_l(\underline{apr}_k(A)) \subseteq A$. □

Proposition 70 *For all $k, l \geq 1$ and $A \subseteq R$, we have*

(1) $\underline{apr}_{l+k}(A) \subseteq \underline{apr}_l(\underline{apr}_k(A))$
(2) $\overline{apr}_{l+k}(A) \supseteq \overline{apr}_l(\overline{apr}_k(A))$.

Proof

(1) Suppose that $a \in \underline{apr}_{l+k}(A)$. Then, $N_{l+k}(a) \subseteq A$. We have $N_k(N_l(a)) \subseteq N_{k+l}(a) \subseteq A$, which implies that $N_l(a) \subseteq \underline{apr}_k(A)$. Therefore, $a \in \underline{apr}_l(\underline{apr}_k(A))$.
(2) Suppose that $a \in \overline{apr}_l(\overline{apr}_k(A))$. Then, $N_l(a) \cap \overline{apr}_k(A) \neq \emptyset$, and so there exists $b \in N_l(a) \cap \overline{apr}_k(A)$. Since $b \in \overline{apr}_k(A)$, it follows that $N_k(b) \cap A \neq \emptyset$. Now, we have

$$\emptyset \neq N_k(b) \cap A \subseteq N_k(N_l(a)) \cap A \subseteq N_{l+k}(a) \cap A,$$

and so $N_{l+k}(a) \cap A \neq \emptyset$, which implies that $a \in \overline{apr}_{l+k}(A)$. □

Proposition 71 *If A, B are non-empty subsets of R, then*

$$\overline{apr}_k(A) + \overline{apr}_l(B) \subseteq \overline{apr}_{k+l-1}(A + B).$$

Proof Suppose that z be any element of $\overline{apr}_k(A) + \overline{apr}_l(B)$. Then, there exist $x \in \overline{apr}_k(A)$ and $y \in \overline{apr}_l(B)$ such that $z = x+y$. Since $x \in \overline{apr}_k(A)$ and $y \in \overline{apr}_l(B)$, it follows that there exist $a, b \in R$ such that $a \in N_k(x) \cap A$ and $b \in N_l(y) \cap B$. So, $a \in N_k(x)$ and $b \in N_l(y)$. Now, we have $N_k(x)+N_l(y) \subseteq N_{k+l-1}(z)$. Since $a+b \in A + B$, it follows that $a + b \in N_{k+l-1}(z) \cap A + B$, and so $z \in \overline{apr}_{k+l-1}(A + B)$. This completes the proof. □

7 Generalized Lower and Upper Approximations with Respect to Ideals

Yamak et al. [29], introduced a general framework for the study of generalized rough sets in which both constructive and axiomatic approaches are used, also see [5]. They introduced the concept of a set-valued homomorphism for rings, which is a generalization of ordinary homomorphism. Then, by using the definitions of lower inverse and upper inverse, they presented the definition of uniform set-valued homomorphism. In the next paragraph we review some of their results.

Let X and U be two finite universes. We can define a set-valued function $T : X \to \mathscr{P}(U)$ where $\mathscr{P}(U)$ denotes the set of all subsets of U. The triple (X, U, T) is referred to as a generalized approximation space. It can be defined a relation from X to U by setting $\rho_T = \{(x, y) \mid x \in T(y)\}$. Obviously, suppose that ρ is an arbitrary relation from X to U. It can be defined a set-valued function $T_\rho : X \to \mathscr{P}(U)$ by $T_\rho(x) = \{y \in U \mid (x, y) \in \rho,\ x \in X\}$.

Two trivial generalized approximation space are the null generalized approximation space and the total generalized approximation space which are respectively defined as follows:

(1) The null generalized approximation space (T, X, U): $T(x) = \emptyset$ for all $x \in X$.
(2) The total generalized approximation space (T, X, U): $T(x) = U$ for all $x \in X$.

Suppose A be a fuzzy set of a universe U, we take the parameter set $X = [0, 1]$, and define the mapping $T : X \to \mathscr{P}(U)$ as follows:

$$T(\alpha) := \{x \in X \mid \alpha \leq A(x)\}$$

Definition 72 Let (X, U, T) be a generalized approximation space. For any set $B \subseteq U$, a pair of lower and upper approximations, $T^-(B)$ and $T^+(B)$, are defined by

$$T^-(B) := \{x \in X \mid T(x) \subseteq B\} \text{and } T^+(B) := \{x \in X \mid T(x) \cap B \neq \emptyset\}.$$

The pair $(T^-(B), T^+(B))$ is referred to as a generalized rough set.

Proposition 73 *Let (X, U, T) be a generalized approximation space. If A and B are subsets of U, then the following hold:*

(1) $T^-(A) =\sim T^+(\sim A)$, $T^+(A) =\sim T^-(\sim A)$
(2) $T^-(Y) = X$, $T^+(\emptyset) = \emptyset$
(3) *If* $A \subseteq B$ *implies* $T^-(A) \subseteq T^-(B)$;
(4) *If* $A \subseteq B$ *implies* $T^+(A) \subseteq T^+(B)$;
(5) $T^+(A \cup B) = T^+(A) \cup T^+(B)$
(6) $T^+(A \cap B) \subseteq T^+(A) \cap T^+(B)$
(7) $T^-(A) \cup T^-(B) \subseteq T^-(A \cup B)$
(8) $T^-(A \cap B) = T^-(A) \cap T^-(B)$

where $\sim A$ is the complement of A. With respect to certain special types, say, serial, reflexive, symmetric, and transitive binary relation on the universe U, the approximation operators have the following additional properties.

(9) *If ρ_T is a serial relation, then $T^+(U) = U$, $T^-(\emptyset) = \emptyset$, $T^-(A) \subseteq T^+(A)$*
(10) *If ρ_T is a reflexive relation, then $T^-(A) \subseteq A \subseteq T^+(A)$*
(11) *If ρ_T is a symmetric relation, then $T^+(T^-A) \subseteq A \subseteq T^-(T^+A)$*
(12) *If ρ_T is a transitive relation, then $T^-(A) \subseteq T^-(T^-(A))$, $T^+(A) \supseteq T^+(T^+(A))$.*

Proposition 74 *Let (X, U, T) be a generalized approximation space. Let $\{A_i\}_{i\in J}$ be an arbitrary family in U. Then,*

(1) $T^-(\bigcap A_i) = \bigcap T^-(A_i)$
(2) $T^+(\bigcup A_i) \subseteq \bigcup T^+(A_i)$.

Proposition 75 *Let X and Y be two non-empty sets and let (X, Y, T) be a generalized approximation space. Then,*

(1) $\{T^-(A) \mid A \in \mathscr{P}(Y)\}$ *is a complete lattice relative to the relation $\subseteq$;*
(2) $\{T^+(A) \mid A \in \mathscr{P}(Y)\}$ *is a complete lattice relative to the relation $\subseteq$;*
(3) $\{(T^-(A), T^+(A)) \mid A \in \mathscr{P}(Y)\}$ *is a complete lattice relative to the relation $\subseteq$.*

Every Pawlak rough set may be consider as a generalized rough set.

Suppose that R, S are rings. First, we define a set-valued homomorphism from R to $\mathscr{P}(S)$ and then we show that every set-valued homomorphism is uniform.

Definition 76 Let $T : R \to \mathscr{P}(S)$ be a set-valued function. The mapping T is said to be a set-valued homomorphism if

(1) $T(a) + T(b) \subseteq T(a + b)$;
(2) $-T(a) \subseteq T(-a)$
(3) $T(a) \cdot T(b) \subseteq T(a \cdot b)$

for all $a, b \in R$. A set-valued function T is called a strong set-valued homomorphism if

(1) $T(a + b) = T(a) + T(b)$
(2) $T(-a) = -T(a)$
(3) $T(a \cdot b) = T(a) \cdot T(b)$

for all $a, b \in R$.

It is easy to verify that $T(0)$ $(\neq \emptyset)$ is a subring of S for every set-valued homomorphism T.

Example 77

(1) Let F be a field. Consider the ring $(F\backslash\{0\}, \cdot, \odot)$ where $a \odot b = 1$. Then, the generalized approximation space $(T, F\backslash\{0\}, F\backslash\{0\})$ defined by $T(x) = \{x, -x\}$ is a strong set-valued homomorphism.

(2) Let I be an ideal of a ring S, and let $T : R \to \mathscr{P}(R)$ be a set-valued function defined as $T(r) = r + I$. Then, T is a set-valued homomorphism and $T^+(B) = \overline{app_I}(B),\ T^-(B) = \underline{app_I}(B)$.
This example shows that the lower and upper approximations of the set B with respect to the ideal I is a generalized rough set.
(3) Let R and S be rings. If $T : R \to \mathscr{P}(S)$ is a total set-valued function, then T is a set-valued homomorphism. If S is a ring with unity 1_S, then T is a strong set-valued homomorphism.
(4) Let R, S be rings. Then, the set-valued function $T : R \to \mathscr{P}(S)$ defined as $T(r) = \{0\}$ is a strong set-valued homomorphism.
(5) Let $f : R \to S$ be a ring homomorphism. Then, the set-valued function $T : R \to \mathscr{P}(S)$ defined as $T(r) = \{f(r)\}$ is a strong set-valued homomorphism.
(6) Let R be a ring, I an ideal of R; for $a, b \in R$ we define $a \equiv b(mod I)$ if and only if $a - b \in I$. Then, the relation $\equiv$ is a congruence.
(7) Let R be a ring, I and J ideals of R; for $x, y \in R$ we define $x \sim y$ if and only if $x = a + y + b$ for some $a \in I$ and $b \in J$. Then, the relation $\sim$ is a congruence.

Let θ be a congruence on a ring R. Define $T_\theta : R \to \mathscr{P}(R)$ by $T_\theta(x) = [x]_\theta$. Then, T_θ is a set-valued homomorphism. Note that T_θ is not a strong set-valued homomorphism in general.

Proposition 78 *Let R and S be two rings, B be a subset of S. Then,*

(1) *Let $T : R \to \mathscr{P}(S)$ be a set-valued homomorphism. Let B be a subring of S, and $T^+(B)$ a non-empty subset of R. Then, $T^+(B)$ is a subring of R;*
(2) *Let $T : R \to \mathscr{P}(S)$ be a strong set-valued homomorphism. Let B be a subring of S, and $T^-(B)$ a non-empty subset of R. Then, $T^-(B)$ is a subring of R;*
(3) *Let $T : R \to \mathscr{P}^*(S)$ be a set-valued homomorphism where $\mathscr{P}^*(S)$ denotes the set of all non-empty subsets of S. Let B be an ideal of S, and $T^+(B)$ a non-empty subset of R. Then, $T^+(B)$ is an ideal of R;*
(4) *Let $T : R \to \mathscr{P}^*(S)$ be a strong set-valued homomorphism where $\mathscr{P}^*(S)$ denotes the set of all non-empty subsets of S. Let B be an ideal of S, and $T^-(B)$ a non-empty subset of R. Then, $T^-(B)$ is an ideal of R,*

Proposition 79 *Let $T : R \to \mathscr{P}(S)$ be a set-valued homomorphism. If A, B are non-empty subsets of S, then*

(1) $T^+(A) * T^+(B) \subseteq T^+(A * B)$
(2) $T^+(A) + T^+(B) \subseteq T^+(A + B)$
(3) $T^+(A) \cdot T^+(B) \subseteq T^+(A \cdot B)$.

Proposition 80 *Let $T : R \to \mathscr{P}(S)$ be a strong set-valued homomorphism. If A, B are non-empty subsets of S, then*

(1) $T^-(A) * T^-(B) \subseteq T^-(A * B)$;
(2) $T^-(A) + T^-(B) \subseteq T^-(A + B)$
(3) $T^-(A) \cdot T^-(B) \subseteq T^-(A \cdot B)$.

Proposition 81 *Let $T : R \to \mathscr{P}(S)$ be a (strong) set-valued homomorphism. Let $f : R' \to R$ be a ring homomorphism. Then, Tof is a (strong) set-valued homomorphism from R' to S and $(Tof)^-(B) = f^{-1}(T^-(B))$, $(Tof)^+(B) = f^{-1}(T^+(B))$ for all $B \in \mathscr{P}(S)$.*

Proposition 82 *Let $T : R \to \mathscr{P}(S)$ be a (strong) set-valued homomorphism. Let $f : S \to S'$ be a ring homomorphism. Then, T_f is a (strong) set-valued homomorphism from R to S' defined by $T_f(r) = f(T(r))$ and $(T_f)^-(B) = T^-(f^{-1}(B))$, $(T_f)^+(B) = T^+(f^{-1}(B))$ for all $B \in \mathscr{P}(S')$.*

Proposition 83 *Let $T : R \to \mathscr{P}(S)$ be a (strong) set-valued homomorphism. Let I be an ideal of S. Define $T_I : R \to \mathscr{P}(S/I)$ by $T_I(r) = \{a + I \mid a \in T(r)\}$. Then, T_I is a (strong) set-valued homomorphism.*

Definition 84 Let (T, R, S) be a generalized approximation space. The mapping T is said to be lower semiuniform if, for each subring B in S, the set $T^-(B)$ is a subring of R or empty set. The mapping T is said to be upper semiuniform if, for each subring B in S, the set $T^+(B)$ is a subring of R or empty set. A set-valued mapping T is said to be uniform if it is upper and lower semiuniform.

Every strong set-valued homomorphism is uniform.

Definition 85 Let R and S be two rings, B be a subset of S. Let $T : R \to \mathscr{P}(S)$ be a set-valued function. If $T^-(B)$ and $T^+(B)$ are subrings (resp. ideals) of R, then we call $(T^-(B), T^+(B))$ a generalized rough subring (resp. ideal).

Corollary 86 *Let R, S be two rings, B be a subset of S. Let $T : R \to \mathscr{P}^*(S)$ be a strong set-valued homomorphism. Let B be a subring (resp. ideal) of S. Then, $(T^-(B), T^+(B))$ a generalized rough subring (resp. ideal).*

Suppose that R, S are rings and $T : R \to \mathscr{P}(S)$ is a set-valued function. Let I be an ideal of S and X be a non-empty subset of S. Then, the sets

$$A_I^-(X) = \{a \in R \mid T(x) + I \subseteq X\} \text{ and } A_I^+(X) = \{a \in R \mid (T(x) + I) \cap X \neq \emptyset\},$$

are called, respectively, generalized lower and generalized upper approximations of the set X with respect to the ideal I.

Lemma 87 *Let I, J be two ideals of S such that $I \subseteq J$ and let A be a non-empty subset of S. Then,*

(1) $A_J^-(A) \subseteq A_I^-(A)$;
(2) $A_I^+(A) \subseteq A_J^+(A)$.

Corollary 88 *Let I, J be two ideals of S and B be a non-empty subset of S. Then,*

(1) $A_J^-(B) \cap A_I^-(B) \subseteq A_{(I\cap J)}^-(B)$
(2) $A_{(I\cap J)}^+(B) \subseteq A_J^+(B) \cap A_I^+(B)$.

Proposition 89 *Let R, S be two rings, I be an ideal of S and B be a non-empty subset of S. Then,*

(1) *Let $T : R \to \mathscr{P}(S)$ be a set-valued homomorphism. Let B be a subring of S, and $A_I^+(B)$ a non-empty subset of R. Then, $A_I^+(B)$ is a subring of R;*
(2) *Let $T : R \to \mathscr{P}(S)$ be a strong set-valued homomorphism. Let B be a subring of S, and $A_I^-(B)$ a non-empty subset of R. Then, $A_I^-(B)$ is a subring of R;*
(3) *Let $T : R \to \mathscr{P}^*(S)$ be a set-valued homomorphism where $\mathscr{P}^*(S)$ denotes the set of all non-empty subsets of S. Let B be a ideal of S, and $A_I^+(B)$ a non-empty subset of R. Then, $A_I^+(B)$ is a ideal of R;*
(4) *Let $T : R \to \mathscr{P}^*(S)$ be a strong set-valued homomorphism where $\mathscr{P}^*(S)$ denotes the set of all non-empty subsets of S. Let B be a ideal of S, and $A_I^-(B)$ a non-empty subset of R. Then, $A_I^-(B)$ is a ideal of R,*

Proposition 90 *Let I be an ideal of S, and B, C non-empty subsets of S. Then,*

(1) *Let $T : R \to \mathscr{P}(S)$ be a set-valued homomorphism. Then,*

$$A_I^+(B) * A_I^+(C) \subseteq A_I^+(B * C).$$

(2) *Let $T : R \to \mathscr{P}(S)$ be a strong set-valued homomorphism. Then,*

$$A_I^-(B) * A_I^-(C) \subseteq A_I^-(B * C).$$

(3) *Let $T : R \to \mathscr{P}(S)$ be a set-valued homomorphism. Then,*

$$A_I^+(B) + A_I^+(C) \subseteq A_I^+(B + C).$$

(4) *Let $T : R \to \mathscr{P}(S)$ be a strong set-valued homomorphism. Then,*

$$A_I^-(B) + A_I^-(C) \subseteq A_I^-(B + C).$$

Proposition 91 *Let I, J be two ideals of S, and B be a subring of S. Then,*

(1) *Let $T : R \to \mathscr{P}(S)$ be a set-valued homomorphism. Then,*

$$A_I^+(B) * A_J^+(B) \subseteq A_{I+J}^+(B).$$

(2) *Let $T : R \to \mathscr{P}(S)$ be a strong set-valued homomorphism. Then,*

$$A_I^-(B) * A^-(B) = A_{I+J}^-(B).$$

(3) *Let $T : R \to \mathscr{P}(S)$ be a set-valued homomorphism. Then,*

$$A_I^+(B) + A_J^+(B) \subseteq A_{I+J}^+(B).$$

(4) *Let $T : R \to \mathscr{P}(S)$ be a strong set-valued homomorphism. Then,*

$$A_I^-(B) + A^-(B) = A_{I+J}^-(B).$$

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S-Approximation Spaces

Ali Shakiba

Abstract In this paper, the concept of S-approximation spaces is surveyed at first and then, the combination of different S-approximation spaces with different decider mappings S is considered, i.e. combining S-approximation spaces $G_i = (U_i, W_i, T_i, S_i)$ for $i = 1, \ldots, k$. Moreover, the problem of preserving the corresponding properties of the lower and upper approximation operators as well as the three regions of the 3WD in the combination of different S-approximation spaces is considered in the paper.

1 Introduction

Uncertainty is present in many practical decision making applications due to the incompleteness of knowledge. There are several different approaches to handle uncertainty in these applications such as the theory of rough sets and its extensions [1, 9–12, 25, 28], fuzzy set theory [29–31], Dempster-Shafer theory of evidence or belief functions [2, 16] and S-approximation spaces [3, 17–20]. The concept of S-approximation spaces was introduced in [3] as a generalization of Dempster-Shafer theory of evidence and rough set theory. Then, it was studied from a three-way decision [27] viewpoint in [17]. It is then extended to neighborhood systems [26] in [18]. It is also extended to fuzzy sets [20] as well as intuitionistic fuzzy set theory [19].

S-approximation spaces are shown to be a generalization of the belief structures in [21], i.e. any belief structure can be represented by an S-approximation space, however, the converse does not hold. For any belief structure, there exists an S-approximation space where the quality of the lower and the upper approximations induce the corresponding belief and plausibility functions. On the other hand, every irreducible partial monotone S-approximation space induces a belief structure. An

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irreducible partial monotone S-approximation space is one which does not contain trivial elements, an element $x \in U$ which either $\mathscr{I}\mathscr{P}_G(x) = \{\emptyset\}$ or $\mathscr{I}\mathscr{P}_G(x) = \emptyset$ is trivial in a partial monotone S-approximation space.

S-approximation spaces can be viewed as a generalization of T-rough sets [1] as well as the generalization of neighborhood systems [6] to two universal sets.

Finally, S-approximation spaces are equivalent to concrete neighborhood spaces defined in [8]. Let $N : U \to \mathscr{P}(\mathscr{P}(W))$ be a neighborhood map defined as

$$N(x) = \{X \subseteq W \mid S(T(x), X) = 1\}, \tag{1}$$

for $x \in U$. Then, the core and the vicinity maps induced by $\mathscr{N}(U)$ are exactly the same as the lower and the upper approximations of X in $G = (U, W, T, S)$, respectively. On the other hand, given a concrete neighborhood space $(U, \mathscr{N}(U))$, then the core and the vicinity maps induced by $\mathscr{N}(U)$ are equivalent to the lower and the upper approximations in $G = (U, W, T, S)$ where $W = \mathscr{P}(U')$, $T(x) = n(x)$, and

$$S(A, B) = \begin{cases} 1 \text{ if } C \in A \text{ for some } C \in B, \\ 0 \text{ otherwise}, \end{cases} \tag{2}$$

for $A, B \subseteq W$. However, S-approximation spaces provide another useful representation for concrete neighborhood spaces. An S-approximation space considers the key elements used to construct a neighborhood system, that is the knowledge and the decider mappings, and studies their impact on the properties of the concrete neighborhood space. The decider mapping in an S-approximation space can be almost anything, a decision tree, a classifier such as an artificial neural network or a similarity measure. On the other hand, the knowledge mapping can be used to represent almost all types of data representation and was constructed based on a key-value representation, i.e. W can be considered as pairs of attributes and values. Such a representation makes S-approximations applicable to consider almost all data representations.

In this paper, the concept of S-approximation spaces is surveyed at first and then, the combination of different S-approximation spaces with different decider mappings S is considered, i.e. combining S-approximation spaces $G_i = (U_i, W_i, T_i, S_i)$ for $i = 1, \ldots, k$. Note that the problem of combining $G_i = (U, W, T_i, S)$ for $i = 1, \ldots, k$ is considered in [18]. We will also consider the problem of preserving the corresponding properties of the lower and upper approximation operators as well as the 3WD regions in the combination of different S-approximation spaces. The intuition behind studying this combination is considering the problem of combining different knowledge mappings and deciders, e.g. different experts.

The remainder of this paper is organized as follows: In Sect. 2, we survey the concept of S-approximation spaces and its corresponding properties. In Sect. 3.1, we first survey recent results on the combination of S-approximations in terms of their knowledge mappings. Then, the problem of combining S-approximation spaces in terms of their deciders is considered in Sect. 3.2. Section 3.3 studies the problem of combining S-approximation spaces in terms of both knowledge and decider mappings. The general problem of combining S-approximation spaces in considered in Sect. 3.4. Finally, we conclude the paper and give some future research directions in Sect. 4.

2 S-Approximation Spaces

An S-approximation spaces is simply a quadruple $G = (U, W, T, S)$ where U and W are two finite non-empty sets and T and S are two mappings[3, 17]. This structure is extend to represent fuzzy sets in [20] and intuitionistic fuzzy sets in [19]. It is shown in [3, 17, 19, 20] that S-approximation spaces capture almost all the extensions of rough set theory, up to the authors' knowledge. The mapping T in all variants of S-approximation spaces can be interpreted as a granular mapping of knowledge and the mapping S can be interpreted as a decider of similarity or satisfiability or any other kind of decision based on the application. Using these two mappings, any set $X \subseteq W$ can be approximated by two subsets of U which are called the upper and the lower approximation sets of X with respect to G.

An ordinary S-approximation space, which is simply called S-approximation space, is defined formally in Definition 1.

Definition 1 (S-Approximation Spaces[3, 17]) An *S-approximation space* is a quadruple $G = (U, W, T, S)$ where U and W are finite non-empty sets, $T : U \to \mathscr{P}(W)$ and $S : \mathscr{P}(W) \times \mathscr{P}(W) \to \{0, 1\}$ are two mappings.

The lower and the upper approximations of a set $X \subseteq W$ are defined as

$$\underline{G}(X) = \{x \in U \mid S(T(x), X) = 1\}, \quad (3)$$

and

$$\overline{G}(X) = \left\{x \in U \mid S(T(x), X^c) = 0\right\}, \quad (4)$$

respectively, where X^c is the complement of X with respect to W.

The intuition behind defining Eqs. (3) and (4) is as follows:

- an element $x \in U$ which the decider can decide it as 1 with respect to the knowledge T to be in the set X, belongs to the lower approximation of X,
- and an element $x \in U$ which the decider S cannot decide it as 1 with respect to the knowledge T to be in the set X^c, belongs to the upper approximation of X.

The properties of the lower and upper approximation operators mostly depend on the properties of the mapping S. In [3, 17], it is shown that most of the properties of these operators are the same as the corresponding properties in rough set theory if the decider is of certain class. A mapping S satisfies the S-min property defined in [3] whenever for all $X, Y, Z \subseteq W$, we have $S(X, Y \cap Z) = \min\{S(X, Y), S(X, Z)\}$.

Whenever $S(Z, X) = 1$ and $X \subseteq Y$ imply $S(Z, Y) = 1$ for all $X, Y, Z \subseteq W$, the mapping S is called partial monotone[17]. It is shown in [17] that partial monotonicity is much broader than the S-min property in the sense that every S-min structure is partial monotone, however, the other side does not necessarily hold. Sometimes, the properties of the lower and upper approximation operators depends on the mapping T in addition to the mapping S. An S-approximation space $G = (U, W, T, S)$ is called complement compatible if we have $S(T(x), X) \times S(T(x), X^c) = 0$ for all $x \in U$ and $X \subseteq W$.

Remark 2 Note that the S-min and partial monotonicity properties corresponds to the properties "($N3$) if $N, N' \in \mathscr{N}_x$, then $N \cap N' \in \mathscr{N}_x$" and "($N2$) if $N \in \mathscr{N}_x$ and $N \subseteq N'$, then $N' \in \mathscr{N}_x$" in neighborhood systems defined in [8], respectively.

Proposition 3 ([17]) *Let $G = (U, W, T, S)$ be a partial monotone S-approximation space. For all $X, Y \subseteq W$, the followings hold:*

($PS1$) $X \subseteq Y$ *implies* $\overline{G}(X) \subseteq \overline{G}(Y)$,
($PS2$) $X \subseteq Y$ *implies* $\underline{G}(X) \subseteq \underline{G}(Y)$,
($PS3$) $\overline{G}(X \cup Y) \supseteq \overline{G}(X) \cup \overline{G}(Y)$,
($PS4$) $\overline{G}(X \cap Y) \subseteq \overline{G}(X) \cap \overline{G}(Y)$,
($PS5$) $\underline{G}(X \cup Y) \supseteq \underline{G}(X) \cup \underline{G}(Y)$,
($PS6$) $\underline{G}(X \cap Y) \subseteq \underline{G}(X) \cap \underline{G}(Y)$,
($PS7$) $\overline{G}(X) = (\underline{G}(X^c))^c$,
($PS8$) $\underline{G}(X) = (\overline{G}(X^c))^c$.

Remark 4 As it is shown in [3], if S satisfies the S-min property, then ($PS3$) and ($PS6$) turn into equations.

Proposition 5 ([17]) *Let $G = (U, W, T, S)$ be a complement compatible S-approximation space. Then, for any $X \subseteq W$, $\underline{G}(X) \subseteq \overline{G}(X)$.*

Since the mapping S can be arbitrarily chosen in S-approximation spaces, the lower and upper approximations of a set $X \subseteq W$ do not directly lead us toward any decision. So, these approximations are used to construct three-way decisions, or 3WDs for short, as in Definition 6.

Definition 6 ([17]) Let $G = (U, W, T, S)$ be an S-approximation space and $X \subseteq W$. Then, the 3WD regions of X with respect to G are defined as

$$\mathrm{POS}_G(X) = \left\{x \in U | S(T(x), X) = 1 \wedge S(T(x), X^c) = 0\right\}, \tag{5}$$

$$\mathrm{NEG}_G(X) = \left\{x \in U | S(T(x), X) = 0 \wedge S(T(x), X^c) = 1\right\}, \tag{6}$$

$$\mathrm{BR}_G(X) = \left\{x \in U | S(T(x), X) = S(T(x), X^c)\right\}, \tag{7}$$

where $\mathrm{POS}_G(X)$, $\mathrm{NEG}_G(X)$ and $\mathrm{BR}_G(X)$ denote the positive, negative and boundary regions.

The Definition 6 is an extension to the one in [27] (Equation 1 in there). It is notable that the properties of these three regions also depends on the properties of the decider map S.

Proposition 7 ([17]) *Let $G = (U, W, T, S)$ be a partial monotone S-approximation space. Then, for any $X, Y \subseteq W$, the followings hold:*

1. $X \subseteq Y$ *implies* $POS_G(X) \subseteq POS_G(Y)$,
2. $X \subseteq Y$ *implies* $NEG_G(Y) \subseteq NEG_G(X)$,
3. $POS_G(X \cup Y) \supseteq POS_G(X) \cup POS_G(Y)$,
4. $NEG_G(X \cup Y) \subseteq NEG_G(X) \cup NEG_G(Y)$,
5. $POS_G(X \cap Y) \subseteq POS_G(X) \cap POS_G(Y)$,
6. $NEG_G(X \cap Y) \supseteq NEG_G(X) \cap NEG_G(Y)$,
7. $POS_G(X) \cap NEG_G(Y) \subseteq POS_G(X) \cap NEG_G(X \cap Y)$.

3 The Combination of S-Approximation Spaces

The concept of S-approximation spaces shows a great flexibility in representing and handling distributed knowledge bases. In [18], the S-approximation spaces are combined in the sense of their knowledge mappings based on ideas in neighborhood systems and multi-granular rough set theory. These results are summarized in Sect. 3.1. The S-approximation spaces can be combined in terms of their decider mappings, too. Some preliminary results are given in [3], which we extend in Sect. 3.2.

3.1 Combining Knowledge Mappings in S-Approximation Spaces

Data might be aggregated from different providers or sources. Hence, we need a model to represent and handle this situation. To address the issue, there are several models such as multi-granulation rough sets [13], neighborhood-based multi-granulation rough sets [7], multi-granulation rough sets over two universal sets [22], multigranulation decision-theoretic rough sets in [15], pessimistic multigranulation rough sets in [14], multigranulation fuzzy rough sets in [23, 24], intuitionistic fuzzy multigranulation rough sets in [4], composite rough sets [32], and neighborhood system S-approximation spaces [18].

Given a collection of S-approximation spaces $G_i = (U, W, T_i, S)$ for $i = 1, \ldots, \ell$ and a set $X \subseteq W$, how can one compute the lower or upper approximation of the set X with respect to these S-approximation spaces? In [18], two different approaches are studied to answer this problem, e.g. optimistic and pessimistic interpretations. For simplicity and clearance in expression, we can consider $\ell = 2$ without loss of generality. Each of these two approaches are formally defined in Definitions 8 and 9.

Definition 8 (Optimistic Neighborhood System S-Approximation Spaces[18]) Let $G_1 = (U, W, T_1, S)$ and $G_2 = (U, W, T_2, S)$ be two S-approximation spaces and $X \subseteq W$. Then, $G_{1\oplus 2} = (U, W, T_1 \oplus T_2, S)$ is called an optimistic neighborhood system S-approximation space of G_1 and G_2, or ONS S-approximation space of G_1 and G_2 for short. The lower and upper approximations of the set X with respect to $G_{1\oplus 2}$ are defined as

$$\underline{G_{1\oplus 2}}(X) = \{x \in U \mid S(T_1(x), X) = 1 \vee S(T_2(x), X) = 1\}, \tag{8}$$

and

$$\overline{G_{1\oplus 2}}(X) = \left\{x \in U \mid S(T_1(x), X^c) = 0 \wedge S(T_2(x), X^c) = 0\right\}, \tag{9}$$

respectively.

Definition 9 (Pessimistic Neighborhood System S-Approximation Spaces[18]) Let $G_1 = (U, W, T_1, S)$ and $G_2 = (U, W, T_2, S)$ be two S-approximation spaces and $X \subseteq W$. Then, $G_{1\otimes 2} = (U, W, T_1 \otimes T_2, S)$ is called an pessimistic neighborhood system S-approximation space of G_1 and G_2, or PNS S-approximation space of G_1 and G_2 for short. The lower and upper approximations of the set X with respect to $G_{1\otimes 2}$ are defined as

$$\underline{G_{1\otimes 2}}(X) = \{x \in U \mid S(T_1(x), X) = 1 \wedge S(T_2(x), X) = 1\}, \tag{10}$$

and

$$\overline{G_{1\otimes 2}}(X) = \left\{x \in U \mid S(T_1(x), X^c) = 0 \vee S(T_2(x), X^c) = 0\right\}, \tag{11}$$

respectively.

The properties of the lower and upper approximations of a set $X \subseteq W$ with respect to either ONS or PNS of S-approximation spaces depend on the properties of the mapping S.

Proposition 10 (Properties of ONS of Partial Monotone S-Approximations[18]) *Suppose that $G_{1\oplus 2} = (U, W, T_1 \oplus T_2, S)$ is an ONS S-approximation space of $G_1 = (U, W, T_1, S)$ and $G_2 = (U, W, T_2, S)$ such that G_1 and G_2 are partial monotone and $X, Y \subseteq W$. Then,*

(OL2) $\underline{G_{1\oplus 2}}(X) \cap \underline{G_{1\oplus 2}}(Y) \supseteq \underline{G_{1\oplus 2}}(X \cap Y)$,
(OU2) $\overline{G_{1\oplus 2}}(X) \cap \overline{G_{1\oplus 2}}(Y) \supseteq \overline{G_{1\oplus 2}}(X \cap Y)$,
(OL3) $\underline{G_{1\oplus 2}}(X) \cup \underline{G_{1\oplus 2}}(Y) \subseteq \underline{G_{1\oplus 2}}(X \cup Y)$,
(OU3) $\overline{G_{1\oplus 2}}(X) \cup \overline{G_{1\oplus 2}}(Y) \subseteq \overline{G_{1\oplus 2}}(X \cup Y)$,
(OL4) $X \subseteq Y$ *implies* $\underline{G_{1\oplus 2}}(X) \subseteq \underline{G_{1\oplus 2}}(Y)$,
(OU4) $X \subseteq Y$ *implies* $\overline{G_{1\oplus 2}}(X) \subseteq \overline{G_{1\oplus 2}}(Y)$.

Proposition 11 (Properties of PNS of Partial Monotone S-Approximations[18]) *Suppose that $G_{1\otimes 2} = (U, W, T_1 \otimes T_2, S)$ is a PNS S-approximation space of $G_1 = (U, W, T_1, S)$ and $G_2 = (U, W, T_2, S)$ such that G_1 and G_2 are partial monotone and $X, Y \subseteq W$. Then,*

(PL2) $\underline{\underline{G_{1\otimes 2}}}(X) \cap \underline{\underline{G_{1\otimes 2}}}(Y) \supseteq \underline{\underline{G_{1\otimes 2}}}(X \cap Y)$,
(PU2) $\overline{\overline{G_{1\otimes 2}}}(X) \cap \overline{\overline{G_{1\otimes 2}}}(Y) \supseteq \overline{\overline{G_{1\otimes 2}}}(X \cap Y)$.
(PL3) $\underline{\underline{G_{1\otimes 2}}}(X) \cup \underline{\underline{G_{1\otimes 2}}}(Y) \subseteq \underline{\underline{G_{1\otimes 2}}}(X \cup Y)$,
(PU3) $\overline{\overline{G_{1\otimes 2}}}(X) \cup \overline{\overline{G_{1\otimes 2}}}(Y) \subseteq \overline{\overline{G_{1\otimes 2}}}(X \cup Y)$,
(PL4) $X \subseteq Y$ *implies* $\underline{\underline{G_{1\otimes 2}}}(X) \subseteq \underline{\underline{G_{1\otimes 2}}}(Y)$,
(PU4) $X \subseteq Y$ *implies* $\overline{\overline{G_{1\otimes 2}}}(X) \subseteq \overline{\overline{G_{1\otimes 2}}}(Y)$.

Definition 12 ([18]) Let $G_1 = (U, W, T_1, S)$ and $G_2 = (U, W, T_2, S)$ be S-approximation spaces and $X \subseteq W$. Then, the 3WD regions of X with respect to $G_{1\oplus 2}$ are defined as

$$\mathrm{POS}_{G_{1\oplus 2}}(X) = \underline{G_{1\oplus 2}}(X) \cap \overline{G_{1\oplus 2}}(X), \tag{12}$$

$$\mathrm{NEG}_{G_{1\oplus 2}}(X) = U \setminus \left(\underline{G_{1\oplus 2}}(X) \cup \overline{G_{1\oplus 2}}(X)\right), \tag{13}$$

$$\mathrm{BR}_{G_{1\oplus 2}}(X) = \underline{G_{1\oplus 2}}(X) \mathbin{\Delta} \overline{G_{1\oplus 2}}(X), \tag{14}$$

where $\mathrm{POS}(\cdot)$, $\mathrm{NEG}(\cdot)$ and $\mathrm{BR}(\cdot)$ denote the positive, negative and boundary regions. Similarly, the 3WD regions of X with respect to $G_{1\otimes 2}$ are defined as

$$\mathrm{POS}_{G_{1\otimes 2}}(X) = \underline{G_{1\otimes 2}}(X) \cap \overline{G_{1\otimes 2}}(X), \tag{15}$$

$$\mathrm{NEG}_{G_{1\otimes 2}}(X) = U \setminus \left(\underline{G_{1\otimes 2}}(X) \cup \overline{G_{1\otimes 2}}(X)\right), \tag{16}$$

$$\mathrm{BR}_{G_{1\otimes 2}}(X) = \underline{G_{1\otimes 2}}(X) \mathbin{\Delta} \overline{G_{1\otimes 2}}(X). \tag{17}$$

Corresponding to Eqs. (12)–(15), one may ask about the relation between the ONS 3WD regions of a set $X \subseteq W$ and the 3WD regions of the set X with respect to each G_1 and G_2 separately. A similar question can be asked about the PNS.

Proposition 13 ([18]) *Suppose that $G = (U, W, T, S)$ is either an ONS or PNS S-approximation space for $G_1 = (U, W, T_1, S)$ and $G_2 = (U, W, T_2, S)$. Then for every set $X \subseteq W$, we have*

1. $POS_{G_1}(X) \cap POS_{G_2}(X) \subseteq POS_{G_{1\oplus 2}}(X)$,
2. $POS_{G_{1\oplus 2}}(X) \subseteq POS_{G_1}(X) \cup POS_{G_2}(X)$,
3. $NEG_{G_1}(X) \cap NEG_{G_2}(X) \subseteq NEG_{G_{1\oplus 2}}(X)$,
4. $NEG_{G_{1\oplus 2}}(X) \subseteq NEG_{G_1}(X) \cup NEG_{G_2}(X)$,
5. $BR_{G_1}(X) \cap BR_{G_2}(X) \subseteq BR_{G_{1\oplus 2}}(X)$.

As it can be observed, since the behaviors of the regions are identical for ONS and PNS of S-approximation spaces, then this bound cannot be improved anymore

[18]. However, if the S-approximations participating in the ONS and PNS satisfy the non-contradictory knowledge mappings introduced in [18], then we can state stronger results.

Definition 14 (Non-contradictory Knowledge Mappings [18]) Given a decider mapping $S : \mathscr{P}(W) \times \mathscr{P}(W) \to \{0, 1\}$ and two knowledge mappings $T_1, T_2 : U \to \mathscr{P}(W)$, T_1 and T_2 are called *contradictory knowledge mappings* with respect to S, if at least one of the following holds:

1. $\exists X \subseteq W, \exists x \in U,\ S(T_1(x), X) = S(T_2(x), X^c) = 1$,
2. $G_1 = (U, W, T_1, S)$ is not complement compatible,
3. $G_2 = (U, W, T_2, S)$ is not complement compatible.

Otherwise, T_1 and T_2 are called *non-contradictory knowledge mappings* with respect to S.

Example 15 Let $U = \{u_1, \dots, u_5\}$, $W = \{w_1, \dots, w_{10}\}$ and S is defined as

$$S(X, Y) = \begin{cases} 1 \text{ if } X \subseteq Y, \\ 0\ otherwise. \end{cases} \tag{18}$$

Knowledge mappings T_1 and T_2 are defined as in Tables 1 and 2. Given set $X = \{w_1, w_3, w_5, w_6, w_9\}$, then

$$\begin{aligned} \text{POS}_{G_1}(X) &= \{u_1\}, & \text{POS}_{G_1}(X) &= \{u_2\}, \\ \text{NEG}_{G_1}(X) &= \{u_3\}, & \text{NEG}_{G_1}(X) &= \{u_1\}, \\ \text{BR}_{G_1}(X) &= \{u_2, u_4, u_5\}, & \text{BR}_{G_1}(X) &= \{u_3, u_4, u_5\}, \end{aligned}$$

Table 1 Knowledge T_1

$x \in U$	$T_1(x)$
u_1	$\{w_1, w_3, w_5, w_6, w_9\}$
u_2	$\{w_2, w_3, w_4, w_6, w_7, w_8, w_{10}\}$
u_3	$\{w_8, w_{10}\}$
u_4	$\{w_1, w_3, w_5, w_6, w_8, w_{10}\}$
u_5	$\{w_2, w_4, w_5, w_6\}$

Table 2 Knowledge T_2

$x \in U$	$T_2(x)$
u_1	$\{w_2, w_4, w_7, w_8, w_{10}\}$
u_2	$\{w_1, w_5, w_9\}$
u_3	$\{w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_9\}$
u_4	$\{w_2, w_4, w_7, w_9\}$
u_5	$\{w_1, w_3, w_7, w_8, w_9, w_{10}\}$

where $G_1 = (U, W, T_1, S)$ and $G_2 = (U, W, T_2, S)$. It is easy to see that T_1 and T_2 are contradictory knowledge mappings since for x_1,

$$S(T_1(x_1), X) = S(T_2(x_1), X^c) = 1.$$

Proposition 16 ([18]) *Suppose that $G_{1\oplus 2} = (U, W, T_1 \oplus T_2, S)$ is an ONS S-approximation space of $G_1 = (U, W, T_1, S)$ and $G_2 = (U, W, T_2, S)$ such that T_1 and T_2 are non-contradictory knowledge mappings. Then,*

1. $POS_{G_{1\oplus 2}}(X) = POS_{G_1}(X) \cup POS_{G_2}(X)$,
2. $NEG_{G_{1\oplus 2}}(X) = NEG_{G_1}(X) \cup NEG_{G_2}(X)$,
3. $BR_{G_{1\oplus 2}}(X) = BR_{G_1}(X) \cap BR_{G_2}(X)$,

for every $X \subseteq W$.

Proposition 17 ([18]) *Suppose that $G_{1\otimes 2} = (U, W, T_1 \otimes T_2, S)$ is a PNS S-approximation space of $G_1 = (U, W, T_1, S)$ and $G_2 = (U, W, T_2, S)$ such that T_1 and T_2 are non-contradictory knowledge mappings. Then,*

1. $POS_{G_{1\otimes 2}}(X) = POS_{G_1}(X) \cap POS_{G_2}(X)$,
2. $NEG_{G_{1\otimes 2}}(X) = NEG_{G_1}(X) \cap NEG_{G_2}(X)$,
3. $BR_{G_{1\otimes 2}}(X) = BR_{G_1}(X) \cup BR_{G_2}(X)$,

for every $X \subseteq W$.

Note the difference between Propositions 16 and 17. The boundary region in ONS is minimized if the knowledge mappings are non-contradictory, however, it is maximized in the PNS. To maximize the boundary region of the PNS, a weaker notion of non-contradictory is needed.

Proposition 18 ([18]) *Let $G_{1\otimes 2} = (U, W, T_1 \otimes T_2, S)$ be a PNS S-approximation space of $G_1 = (U, W, T_1, S)$ and $G_2 = (U, W, T_2, S)$, $X \subseteq W$ and $x \in U$ such that*

$$x \notin BR_{G_1}(X) \cup BR_{G_2}(X) \implies S(T_1(x), X) \times S(T_2(x), X^c) = 0 \\ \wedge\, S(T_2(x), X) \times S(T_1(x), X^c) = 0, \tag{19}$$

and

$$x \in BR_{G_i}(X) \implies S(T_i(x), X) = S(T_i(x), X^c) = 1, \tag{20}$$

for $i = 1, 2$. Then

1. $POS_{G_{1\otimes 2}}(X) = POS_{G_1}(X) \cup POS_{G_2}(X)$,
2. $NEG_{G_{1\otimes 2}}(X) = NEG_{G_1}(X) \cup NEG_{G_2}(X)$,
3. $BR_{G_{1\otimes 2}}(X) = BR_{G_1}(X) \cap BR_{G_2}(X)$.

Similarly, the properties of these three regions depends on the properties of the mapping S.

Proposition 19 ([18]) *Suppose that $G_{1\oplus 2} = (U, W, T_1 \oplus T_2, S)$ is an ONS S-approximation space of $G_1 = (U, W, T_1, S)$ and $G_2 = (U, W, T_2, S)$ such that G_1 and G_2 are partial monotone and $X, Y \subseteq W$. Then,*

1. *$X \subseteq Y$ implies $POS_{G_{1\oplus 2}}(X) \subseteq POS_{G_{1\oplus 2}}(Y)$,*
2. *$X \subseteq Y$ implies $NEG_{G_{1\oplus 2}}(Y) \subseteq NEG_{G_{1\oplus 2}}(X)$,*
3. *$POS_{G_{1\oplus 2}}(X) \cup POS_{G_{1\oplus 2}}(Y) \subseteq POS_{G_{1\oplus 2}}(X \cup Y)$,*
4. *$POS_{G_{1\oplus 2}}(X) \cap POS_{G_{1\oplus 2}}(Y) \supseteq POS_{G_{1\oplus 2}}(X \cap Y)$,*
5. *$NEG_{G_{1\oplus 2}}(X) \cup NEG_{G_{1\oplus 2}}(Y) \supseteq NEG_{G_{1\oplus 2}}(X \cup Y)$,*
6. *$NEG_{G_{1\oplus 2}}(X) \cap NEG_{G_{1\oplus 2}}(Y) \subseteq NEG_{G_{1\oplus 2}}(X \cap Y)$.*

Proposition 20 ([18]) *Suppose that $G_{1\otimes 2} = (U, W, T_1 \otimes T_2, S)$ is a PNS S-approximation space of $G_1 = (U, W, T_1, S)$ and $G_2 = (U, W, T_2, S)$ such that G_1 and G_2 are partial monotone and $X, Y \subseteq W$. Then,*

1. *$X \subseteq Y$ implies $POS_{G_{1\otimes 2}}(X) \subseteq POS_{G_{1\otimes 2}}(Y)$,*
2. *$X \subseteq Y$ implies $NEG_{G_{1\otimes 2}}(Y) \subseteq NEG_{G_{1\otimes 2}}(X)$,*
3. *$POS_{G_{1\otimes 2}}(X) \cup POS_{G_{1\otimes 2}}(Y) \subseteq POS_{G_{1\otimes 2}}(X \cup Y)$,*
4. *$POS_{G_{1\otimes 2}}(X) \cap POS_{G_{1\otimes 2}}(Y) \supseteq POS_{G_{1\otimes 2}}(X \cap Y)$,*
5. *$NEG_{G_{1\otimes 2}}(X) \cup NEG_{G_{1\otimes 2}}(Y) \supseteq NEG_{G_{1\otimes 2}}(X \cup Y)$,*
6. *$NEG_{G_{1\otimes 2}}(X) \cap NEG_{G_{1\otimes 2}}(Y) \subseteq NEG_{G_{1\otimes 2}}(X \cap Y)$.*

Knowing these two approaches to combine knowledge mappings in S-approximation spaces, one may ask about the relation among these two approaches.

Theorem 21 ([18]) *Suppose that $G_{1\oplus 2} = (U, W, T_1 \oplus T_2, S)$ and $G_{1\otimes 2} = (U, W, T_1 \otimes T_2, S)$ are ONS and PNS S-approximation spaces of $G_1 = (U, W, T_1, S)$ and $G_2 = (U, W, T_2, S)$. Then for any $X \subseteq W$, we have*

1. *$\underline{G_{1\otimes 2}}(X) \subseteq \underline{G_{1\oplus 2}}(X)$,*
2. *$\overline{G_{1\oplus 2}}(X) \subseteq \overline{G_{1\otimes 2}}(X)$.*

An interesting result would be the following. For any number of S-approximation spaces, there exists a single S-approximation space which acts identically the same with the ONS (PNS) combination of the S-approximation spaces. These results are called completion results.

Theorem 22 (Completion Result for ONS S-Approximation Spaces [18]) *Assume that $G_i = (U, W, T_i, S)$ are ℓ S-approximation spaces. Then, a single S-approximation space such as $G = (U, \mathbf{W}, T, \mathbf{S})$ can be constructed such that the lower and upper approximations of any set $X \subseteq W$ with respect to G and $G_{\oplus_{i=1}^{\ell}}$ are identical.*

Proof The proof is a constructive proof. We will define knowledge mapping T as

$$T : U \to \mathscr{P}(\mathbf{W}),$$

$$T(x) \mapsto \{T_1(x) \times T_2(x) \times \ldots \times T_\ell(x)\}.$$

By this definition, it is clear that

$$\mathbf{W} = \underbrace{\mathscr{P}(W) \times \ldots \times \mathscr{P}(W)}_{\ell \text{ times}}.$$

For the decider $\mathbf{S}$, we have

$$\mathbf{S} : \mathscr{P}(\mathbf{W}) \times \mathscr{P}(\mathbf{W}) \to \{0, 1\},$$

$$\mathbf{S}(X, Y) = \begin{cases} 1 \text{ if } \sum_{i=1}^{\ell} S\left(Proj_i\left(X\right), Proj_1(Y)\right) \neq 0 \text{ and } |X| = |Y| = 1 \\ 0 \; otherwise \end{cases},$$

where $Proj_i(X) = A_i$ for $X = \{(A_1, \ldots, A_\ell)\}$. Up to now, we have completely defined the S-approximation G. For $X \subseteq W$, we construct a counterpart $\mathbf{X} \subseteq \mathbf{W}$ as

$$\mathbf{X} = \left\{ (X, \underbrace{\emptyset, \ldots, \emptyset}_{\ell-1 \text{ times}}) \right\}.$$

Let show this way of obtaining $\mathbf{X}$ from X by a mapping $f : \mathscr{P}(W) \to \mathscr{P}(\mathbf{W})$, i.e. $f(X) = \mathbf{X}$. Next, we will show that for every $X \subseteq W$, we have

1. $\underline{G}(f(X)) = \underline{G_{\oplus_{i=1}^{\ell} i}}(X)$,
2. $\overline{G}(f(X)) = \overline{G_{\oplus_{i=1}^{\ell} i}}(X)$.

For the first case, let $x \in \underline{G_{\oplus_{i=1}^{\ell} i}}(X)$. Then, by the definition there exists an index $1 \leq i \leq \ell$ such that $S(T_i(x), X) = 1$. This way, by the definition of $\mathbf{S}$ we have $\mathbf{S}(T(x), f(X)) = 1$ and hence, $x \in \underline{G}(f(X))$. On the other hand, assume that $x \in \underline{G}(f(X))$, hence $\mathbf{S}(T(x), f(X)) = 1$. By the definition of $\mathbf{S}$, it is necessary that there exists an index $1 \leq i \leq \ell$ such that $S(T_i(x), X) = 1$, hence $x \in \underline{G_{\oplus_{i=1}^{\ell} i}}(X)$. So, we have shown the first part. The second part can be also obtained in a similar way. These two parts conclude the proof.

Theorem 23 (Completion Result for PNS S-Approximation Spaces [18]) *Assume that $G_i = (U, W, T_i, S)$ are ℓ S-approximation spaces. Then, a single S-approximation space of the form $G = (U, \mathbf{W}, T, \mathbf{S})$ can be constructed such that the lower and upper approximations of any set $X \subseteq W$ with respect to G and $G_{\otimes_{i=1}^{\ell}}$ are identical.*

Proof The proof and construction is similar to Theorem 22, except for $\mathbf{S}$ which is constructed as

$$\mathbf{S}(X, Y) = \begin{cases} 1 \text{ if } \prod_{i=1}^{\ell} S\left(Proj_i\left(X\right), Proj_1\left(Y\right)\right) \neq 0 \text{ and } |X| = |Y| = 1, \\ 0 \; otherwise. \end{cases}$$

A naive, however hard to answer, question would be on the necessity or redundancy of a knowledge map in either ONS or PNS of S-approximation spaces. This is called knowledge significance in [18].

Definition 24 **($\oplus\mathscr{T}$-Significant Knowledge Set[18])** Suppose that $G_i = (U, W, T_i, S)$ be ℓ S-approximation spaces ($i = 1, \ldots, \ell$) and $\mathscr{T} = \{T_1, \ldots, T_\ell\}$. A knowledge T_j is called $\oplus\mathscr{T}$*-significant* whenever there exists at least some $X \subseteq W$ such that

$$\underline{G_{\oplus\mathscr{T}\setminus\{T_j\}}}(X) \neq \underline{G_{\oplus\mathscr{T}}}(X), \tag{21}$$

or

$$\overline{G_{\oplus\mathscr{T}\setminus\{T_j\}}}(X) \neq \overline{G_{\oplus\mathscr{T}}}(X). \tag{22}$$

Otherwise, the knowledge T_j is called $\oplus\mathscr{T}$*-redundant*. If all the knowledge mappings in $\mathscr{A} \subseteq \mathscr{T}$ are $\oplus\mathscr{A}$-significant and for all $X \subseteq W$,

$$\underline{G_{\oplus\mathscr{A}}}(X) = \underline{G_{\oplus\mathscr{T}}}(X), \tag{23}$$

and

$$\overline{G_{\oplus\mathscr{A}}}(X) = \overline{G_{\oplus\mathscr{T}}}(X). \tag{24}$$

then $\mathscr{A}$ is called an $\oplus\mathscr{T}$*-significant knowledge set*. The set of all $\oplus\mathscr{T}$-significant knowledge sets is denoted by $SG_{\oplus}(\mathscr{T})$.

Definition 25 **($\otimes\mathscr{T}$-Significant Knowledge Set[18])** Suppose that $G_i = (U, W, T_i, S)$ be ℓ S-approximation spaces ($i = 1, \ldots, \ell$) and $\mathscr{T} = \{T_1, \ldots, T_\ell\}$. A knowledge T_j is called $\otimes\mathscr{T}$*-significant* whenever there exists at least some $X \subseteq W$ such that

$$\underline{G_{\otimes\mathscr{T}\setminus\{T_j\}}}(X) \neq \underline{G_{\otimes\mathscr{T}}}(X), \tag{25}$$

or

$$\overline{G_{\otimes\mathscr{T}\setminus\{T_j\}}}(X) \neq \overline{G_{\otimes\mathscr{T}}}(X). \tag{26}$$

Otherwise, the knowledge T_j is called $\otimes\mathscr{T}$*-redundant*. If all of knowledge mappings in $\mathscr{A} \subseteq \mathscr{T}$ are $\otimes\mathscr{A}$-significant and for all $X \subseteq W$,

$$\underline{G_{\otimes\mathscr{A}}}(X) = \underline{G_{\otimes\mathscr{T}}}(X), \tag{27}$$

and

$$\overline{G_{\otimes\mathscr{A}}}(X) = \overline{G_{\otimes\mathscr{T}}}(X). \tag{28}$$

then $\mathscr{A}$ is called a $\otimes\mathscr{T}$*-significant knowledge set*. The set of all $\otimes\mathscr{T}$-significant knowledge sets is denoted by $SG_{\otimes}(\mathscr{T})$.

Finding a minimum cardinality $\oplus\mathscr{T}$-significant ($\otimes\mathscr{T}$-significant) knowledge mappings of $\mathscr{T}$ is an **NP**-hard problem.

Theorem 26 ([18]) *For a given set of knowledge mappings $\mathscr{T} = \{T_1, \ldots, T_\ell\}$ where U and W are finite non-empty sets, $T_i : U \to \mathscr{P}(W)$, finding a minimum $\oplus\mathscr{T}$-significant knowledge set is* ***NP****-hard.*

Theorem 27 ([18]) *For a given set of knowledge mappings $\mathscr{T} = \{T_1, \ldots, T_\ell\}$ where U and W are finite non-empty sets, $T_i : U \to \mathscr{P}(W)$, finding a minimum $\otimes\mathscr{T}$-significant knowledge set is* ***NP****-hard.*

The proofs of Theorems 26 and 27 are quite technical and an interested reader is advised to consult [18].

Although finding either a minimal $\oplus\mathscr{T}$-significant or $\otimes\mathscr{T}$-significant knowledge set is shown to be **NP**-hard, there exist exponential-time exact algorithms to find such knowledge sets. Algorithm 1 decides whether or not a knowledge mapping $T_j \in \mathscr{T}$ is $\oplus\mathscr{T}$-significant. Algorithm 1 can be modified to answer the same question for $\otimes\mathscr{T}$-significance. These modifications include substituting 0s instead of 1s in lines 7, 10, 17 and 31 of Algorithm 1 and also changing the line 1 to ISPESSIMISTICREDUNDANT$(U, W, T_j, \mathscr{T}, S)$.

To illustrate the usefulness of the combination of knowledge mappings, consider the following illustrative example taken from [18].

Example 28 ([18]) Suppose that there are historical records of five doctors curing different patients with some medicines and we want to construct a medical expert system. Let $U = \{u_1, \ldots, u_5\}$ be a set of five medicines and $W = \{w_1, \ldots, w_{10}\}$ be a set of ten symptoms. For each doctor i, we have extracted the set of symptoms which caused him to prescribe a medicine u and denoted it by $T_i(u)$. In other words, $T_i : U \to \mathscr{P}(W)$. These mappings are shown in Table 3. Now, the expert system is fed with some clinical observations of two patients, e.g. $X = \{w_3, w_6, w_8, w_{10}\}$ and $Y = \{w_1, w_3, w_9, w_8\}$. The goal of the expert system is to recommend a set of medicines for each of these patients. Since the medicines in U have strong side effects, the expert system is supposed to use a pessimistic approach to prescribe a medicine. Moreover, the expert system uses Jaccards similarity measure [5], i.e.

$$J(A, B) = \frac{|A \cap B|}{|A \cup B|}, \tag{29}$$

Algorithm 1 $\oplus\mathscr{T}$-redundancy decision problem

```
procedure IsOptimisticRedundant(U, W, T_j, 𝒯, S)
    IsRedundant ← True
    for all X ⊆ W do
        lower ← ∅
        upper ← ∅
        for all x ∈ U do
            if S(T_j(x), X) = 1 then
                lower ← lower ∪ {x}
            end if
            if S(T_j(x), X^c) = 1 then
                upper ← upper ∪ {x}
            end if
        end for
        for all x ∈ lower do
            flag ← True
            for all T_i ∈ 𝒯 \ {T_j} do
                if S(T_i(x), X) = 1 then
                    flag ← False
                    break
                end if
            end for
            if flag then
                IsRedundant ← False
                upper ← ∅
                break
            end if
        end for
        for all x ∈ upper do
            flag ← True
            for all T_i ∈ 𝒯 \ {T_j} do
                if S(T_i(x), X^c) = 1 then
                    flag ← False
                    break
                end if
            end for
            if flag then
                IsRedundant ← False
                upper ← ∅
                break
            end if
        end for
        if not IsRedundant then
            break
        end if
    end for
    return IsRedundant
end procedure
```

Table 3 Medicines and symptoms [18]

W	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	w_{10}
T_1										
u_1	1	1	1	1	1	1	1	1	1	0
u_2	1	0	1	1	1	1	1	1	1	0
u_3	0	0	1	0	1	0	1	0	1	0
u_4	1	1	1	1	1	1	1	1	1	1
u_5	1	1	1	1	1	1	1	1	0	1
T_2										
u_1	0	1	0	0	1	1	0	0	0	0
u_2	1	0	0	0	0	0	1	1	1	1
u_3	0	1	1	0	1	0	0	1	0	1
u_4	0	0	1	0	0	0	0	0	0	0
u_5	1	1	1	1	1	0	1	1	1	1
T_3										
u_1	1	1	0	0	1	1	1	1	0	1
u_2	0	0	0	0	0	0	0	1	0	0
u_3	0	0	0	0	0	0	0	0	0	0
u_4	1	1	0	1	0	1	1	1	1	1
u_5	0	0	0	0	0	0	0	0	1	0
T_4										
u_1	1	1	1	0	1	1	1	0	1	1
u_2	1	0	1	0	0	0	1	1	0	1
u_3	1	1	0	0	0	1	0	0	0	1
u_4	0	1	1	1	1	0	1	1	1	1
u_5	0	0	0	0	0	0	0	0	0	0
T_5										
u_1	1	1	1	0	0	1	1	0	1	1
u_2	1	1	1	0	0	1	0	1	1	1
u_3	1	1	1	1	1	1	1	0	1	1
u_4	0	0	0	0	0	0	0	0	0	0
u_5	0	0	0	0	0	0	0	0	0	0

for $\emptyset \neq A, B \subseteq W$. Hence, $S : \mathscr{P}(W) \times \mathscr{P}(W) \rightarrow \{0, 1\}$ is defined as

$$S(A, B) = \begin{cases} 1 \text{ if } J(A, B) \geq 0.1 \text{ and } A \cup B \neq \emptyset \\ 0 \; otherwise \end{cases},$$

for $A, B \subseteq W$.

In this case, the system would suggest the patient with symptoms X the following set of medications

$$\begin{aligned}\mathrm{POS}_{\otimes\mathscr{T}}(X) &= \{u_2\},\\ \mathrm{NEG}_{\otimes\mathscr{T}}(X) &= \emptyset,\\ \mathrm{BR}_{\otimes\mathscr{T}}(X) &= \{u_1, u_3, u_4, u_5\},\end{aligned}$$

that is, it suggests taking medicine u_2, but it cannot decide about medicines u_1, u_3 through u_5. Similarly, for patient with symptoms Y,

$$\begin{aligned}\mathrm{POS}_{\otimes\mathscr{T}}(Y) &= \{u_2\},\\ \mathrm{NEG}_{\otimes\mathscr{T}}(Y) &= \{u_1\},\\ \mathrm{BR}_{\otimes\mathscr{T}}(Y) &= \{u_3, u_4, u_5\},\end{aligned}$$

that is, it suggests medicine u_2, but prohibits medicine u_1. For medicines u_3 through u_5, it cannot make a decision.

By applying modified version of Algorithm 1 for pessimistic combination of knowledge mappings, it can be seen that knowledge mappings T_1 and T_4 are $\otimes$-redundant in $\mathscr{T} = \{T_1, \ldots, T_5\}$, i.e. $\underline{\otimes\mathscr{T}}(X) = \underline{\otimes\mathscr{T}'}(X)$ and $\overline{\otimes\mathscr{T}}(X) = \overline{\otimes\mathscr{T}'}(X)$ for $X \subseteq W$ and either $\mathscr{T}' = \{T_1, T_2, T_3, T_5\}$ or $\mathscr{T}' = \{T_4, T_2, T_3, T_5\}$. Considering each of these $\mathscr{T}'$ knowledge mappings sets, we can run the modified algorithm again and again to obtain all minimal $\otimes\mathscr{T}$-significant knowledge mappings sets. Finally, we can choose a minimum cardinality $\otimes\mathscr{T}$-significant knowledge mappings set.

To illustrate the optimistic combination of knowledge mappings, consider designing a medical expert system for suggesting further tests based on historical records of five doctors examining different patients which is taken from [18]. Let $U = \{u_1, \ldots, u_5\}$ be a set of five tests and $W = \{w_1, \ldots, w_{10}\}$ be a set of ten symptoms. For each doctor i, we have extracted the set of symptoms which caused him to recommend a test u and denoted it by $T_i(u)$ and are shown in Table 4. Now, the expert system is fed with some clinical observations of two patients the same as the previous example. The goal of the expert system is to recommend a set of tests for each of these patients. Since the patients want to be assured of their possible disease and the tests in U do not have any side effects, the expert system is supposed to use an optimistic approach to suggest a test. Moreover, the expert system uses Jaccards similarity measure, Eq. (29), but with a different setting as follows

$$S'(A, B) = \begin{cases} 1 \text{ if } J(A, B) \geq 0.6 \text{ and } A \cup B \neq \emptyset \\ 0 \; otherwise \end{cases},$$

for $A, B \subseteq W$.

Table 4 Tests and symptoms [18]

W	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	w_{10}
T_1										
u_1	0	1	1	1	1	1	1	1	1	1
u_2	0	0	0	1	0	0	1	0	0	1
u_3	0	1	1	1	1	1	1	1	0	1
u_4	0	0	0	0	0	0	0	1	0	0
u_5	0	0	0	1	0	0	0	1	0	0
T_2										
u_1	1	0	0	1	1	0	1	0	1	1
u_2	0	0	0	0	0	1	1	0	0	0
u_3	0	0	0	0	1	0	0	1	0	0
u_4	0	0	1	0	0	0	1	1	1	0
u_5	1	1	1	1	1	1	1	1	1	1
T_3										
u_1	1	0	0	1	1	0	1	0	1	1
u_2	0	1	0	1	0	0	0	0	0	0
u_3	0	1	1	0	1	0	1	0	1	0
u_4	1	0	1	1	1	1	1	1	1	1
u_5	1	0	1	1	1	1	1	1	1	1
T_4										
u_1	0	0	0	0	0	0	0	0	0	0
u_2	0	1	0	1	0	0	0	0	0	0
u_3	1	1	1	1	1	1	1	1	1	1
u_4	0	0	0	0	0	0	0	0	0	0
u_5	1	1	0	1	1	1	1	1	1	1
T_5										
u_1	1	0	1	1	1	1	1	1	0	1
u_2	0	0	0	1	0	0	0	0	0	1
u_3	1	1	1	1	1	1	1	1	1	1
u_4	1	1	1	0	1	0	1	0	1	1
u_5	1	0	0	0	0	0	0	1	0	0

In this case, for the patient with symptoms X, the system has the following suggestions

$$\mathrm{POS}_{\oplus\mathscr{T}}(X) = \emptyset,$$

$$\mathrm{NEG}_{\oplus\mathscr{T}}(X) = \{u_1, u_3, u_4, u_5\},$$

$$\mathrm{BR}_{\oplus\mathscr{T}}(X) = \{u_2\},$$

that is, it cannot make any recommendations for this patient, but suggests that tests u_1, u_3 through u_5 are not useful. Similarly, for patient with symptoms Y,

$$\begin{aligned} \mathrm{POS}_{\oplus\mathscr{T}}(Y) &= \{u_4\}, \\ \mathrm{NEG}_{\oplus\mathscr{T}}(Y) &= \{u_1, u_3, u_5\}, \\ \mathrm{BR}_{\oplus\mathscr{T}}(Y) &= \{u_2\}, \end{aligned}$$

that is, it recommends taking the test u_2, but tests u_1, u_3 and u_5 are not useful. For test u_2, it cannot make a decision.

3.2 Combining Decision Mappings in S-Approximation Spaces

The results surveyed in Sect. 3.1 are applicable whenever we want to combine identical S-approximation spaces, except for the knowledge mapping. Sometimes, the S-approximations just differ in the decider mappings, i.e. $G = (U, W, T, S_i)$ for $i = 1, \ldots, \ell$. In this case, the approach of Sect. 3.1 is not applicable. Some basic results on this situation are stated in [3].

Definition 29 (Complement S-Approximation Space[3]) Suppose $G = (U, W, T, S)$ is an S-approximation space. The complement of G is defined as

$$G^c = (U, W, T, S^c) \text{ where } S^c(\cdot, \cdot) = 1 - S(\cdot, \cdot).$$

Definition 30 (Conjunction of S-Approximations[3]) Let $G_1 = (U, W, T, S_1)$ and $G_2 = (U, W, T, S_2)$ be two S-approximation spaces. The conjunction of G_1 and G_2, denoted by $G_1 \wedge G_2$ is defined as $G_1 \wedge G_2 = (U, W, T, S_\wedge)$ where $S_\wedge(\cdot, \cdot) = S_1(\cdot, \cdot) \times S_2(\cdot, \cdot)$.

Definition 31 (Disjunction of S-Approximations[3]) Let $G_1 = (U, W, T, S_1)$ and $G_2 = (U, W, T, S_2)$ be two S-approximation spaces. The disjunction of G_1 and G_2, denoted by $G_1 \vee G_2$ is defined as $G_1 \vee G_2 = (U, W, T, S_\vee)$ where $S_\vee(\cdot, \cdot) = S_1(\cdot, \cdot) + S_2(\cdot, \cdot) \mod 2$.

Note that the operations defined in Definitions 29–31 are functionally complete set of Boolean operators. In other words, one can construct arbitrary S-approximation spaces from simpler ones by combining decider mappings.

Proposition 32 ([3]) *Suppose that $G = (U, W, T, S)$ is an S-approximation space and $G^c = (U, W, T, S^c)$ is its complement. Then, for any $X \subseteq W$ we have $\underline{G^c}(X) = \overline{G}(X^c)$ and $\overline{G^c}(X) = \underline{G}(X^c)$.*

Proposition 33 ([3]) *Let $G_1 = (U, W, T, S_1)$ and $G_2 = (U, W, T, S_2)$ be two S-approximation spaces. Then, for any $X \subseteq W$, we have*

$$\underline{G_1 \wedge G_2}(X) = \underline{G_1}(X) \cap \underline{G_2}(X), \tag{30}$$

$$\overline{G_1 \wedge G_2}(X) = \overline{G_1}(X) \cup \overline{G_2}(X), \tag{31}$$

$$\underline{G_1 \vee G_2}(X) = \underline{G_1}(X) \cup \underline{G_2}(X), \tag{32}$$

and

$$\overline{G_1 \vee G_2}(X) = \overline{G_1}(X) \cap \overline{G_2}(X), \tag{33}$$

where $G_1 \wedge G_2$ and $G_1 \vee G_2$ denote the conjunction and disjunction of G_1 and G_2, respectively.

Another basic question can be asked on the relation of the 3WD regions of S-approximation spaces with their complement, their conjunction and their disjunction.

Proposition 34 *Suppose that $G = (U, W, T, S)$ is an S-approximation space and $G^c = (U, W, T, S^c)$ is its complement. Then, for any $X \subseteq W$, we have*

- $POS_{G^c}(X) = NEG_G(X)$,
- $NEG_{G^c}(X) = POS_G(X)$,
- $BR_{G^c}(X) = BR_G(X)$.

Proof The proof is obvious by considering Definition 29.

Proposition 35 *Let $G_1 = (U, W, T, S_1)$ and $G_2 = (U, W, T, S_2)$ be two S-approximation spaces. Then, for any $X \subseteq W$, we have*

1.

$$POS_{G_1 \wedge G_2}(X) \subseteq \left(BR_{G_1}(X) \cap POS_{G_2}(X)\right) \cup \left(POS_{G_1}(X) \cap BR_{G_2}(X)\right) \cup \left(POS_{G_1}(X) \cap POS_{G_2}(X)\right),$$

2.

$$POS_{G_1 \wedge G_2}(X) \supseteq \left(POS_{G_1}(X) \cap POS_{G_2}(X)\right),$$

3.

$$NEG_{G_1 \wedge G_2}(X) \subseteq \left(BR_{G_1}(X) \cap NEG_{G_2}(X)\right) \cup \left(NEG_{G_1}(X) \cap BR_{G_2}(X)\right) \cup \left(NEG_{G_1}(X) \cap NEG_{G_2}(X)\right),$$

4.

$$NEG_{G_1 \wedge G_2}(X) \supseteq \left(NEG_{G_1}(X) \cap NEG_{G_2}(X)\right),$$

5.

$$BR_{G_1 \wedge G_2}(X) \supseteq \left(NEG_{G_1}(X) \cap POS_{G_2}(X)\right) \cup \left(POS_{G_1}(X) \cap NEG_{G_2}(X)\right) \cup \left(BR_{G_1}(X) \cap BR_{G_2}(X)\right),$$

6.

$$POS_{G_1 \vee G_2}(X) \subseteq \left(BR_{G_1}(X) \cap POS_{G_2}(X)\right) \cup \left(POS_{G_1}(X) \cap BR_{G_2}(X)\right) \cup \left(POS_{G_1}(X) \cap POS_{G_2}(X)\right),$$

7.

$$POS_{G_1 \vee G_2}(X) \supseteq \left(POS_{G_1}(X) \cap POS_{G_2}(X)\right),$$

8.

$$NEG_{G_1 \vee G_2}(X) \subseteq \left(BR_{G_1}(X) \cap NEG_{G_2}(X)\right) \cup \left(NEG_{G_1}(X) \cap BR_{G_2}(X)\right) \cup \left(NEG_{G_1}(X) \cap NEG_{G_2}(X)\right),$$

9.

$$NEG_{G_1 \vee G_2}(X) \supseteq \left(NEG_{G_1}(X) \cap NEG_{G_2}(X)\right),$$

10.

$$BR_{G_1 \vee G_2}(X) \supseteq \left(NEG_{G_1}(X) \cap POS_{G_2}(X)\right) \cup \left(POS_{G_1}(X) \cap NEG_{G_2}(X)\right) \cup \left(BR_{G_1}(X) \cap BR_{G_2}(X)\right),$$

where $G_1 \wedge G_2$ and $G_1 \vee G_2$ denote the conjunction and disjunction of G_1 and G_2, respectively.

Proof The proof is immediate from Table 5.

Note that the bounds of Proposition 35 are strict. It is interesting that these bounds can be made exact if both G_1 and G_2 are complement compatible.

Proposition 36 *Let $G_1 = (U, W, T, S_1)$ and $G_2 = (U, W, T, S_2)$ be two complement compatible S-approximation spaces. Then, for any $X \subseteq W$, we have*

- $POS_{G_1 \wedge G_2}(X) = POS_{G_1}(X) \cap POS_{G_2}(X)$,
- $NEG_{G_1 \wedge G_2}(X) = NEG_{G_1}(X) \cap NEG_{G_2}(X)$,

Table 5 Different situations of the 3WD regions in disjunction and conjunction of S-approximation spaces

$S_1(T(x), X)$	$S_1(T(x), X^c)$	3WD	$S_2(T(x), X)$	$S_2(T(x), X^c)$	3WD	$S_{1\wedge 2}(T(x), X)$	$S_{1\wedge 2}(T(x), X^c)$	3WD	$S_{1\vee 2}(T(x), X)$	$S_{1\vee 2}(T(x), X^c)$	3WD
1	1	$\text{BR}_{G_1}(X)$	1	1	$\text{BR}_{G_2}(X)$	1	1	$\text{BR}_{G_1\wedge G_2}(X)$	1	1	$\text{BR}_{G_1\vee G_2}(X)$
1	1	$\text{BR}_{G_1}(X)$	1	0	$\text{POS}_{G_2}(X)$	1	0	$\text{POS}_{G_1\wedge G_2}(X)$	1	1	$\text{BR}_{G_1\vee G_2}(X)$
1	1	$\text{BR}_{G_1}(X)$	0	1	$\text{NEG}_{G_2}(X)$	0	1	$\text{NEG}_{G_1\wedge G_2}(X)$	1	1	$\text{BR}_{G_1\vee G_2}(X)$
1	1	$\text{BR}_{G_1}(X)$	0	0	$\text{BR}_{G_2}(X)$	0	0	$\text{BR}_{G_1\wedge G_2}(X)$	1	1	$\text{BR}_{G_1\vee G_2}(X)$
1	0	$\text{POS}_{G_1}(X)$	1	1	$\text{BR}_{G_2}(X)$	1	0	$\text{POS}_{G_1\wedge G_2}(X)$	1	1	$\text{BR}_{G_1\vee G_2}(X)$
1	0	$\text{POS}_{G_1}(X)$	1	0	$\text{POS}_{G_2}(X)$	1	0	$\text{POS}_{G_1\wedge G_2}(X)$	1	0	$\text{POS}_{G_1\vee G_2}(X)$
1	0	$\text{POS}_{G_1}(X)$	0	1	$\text{NEG}_{G_2}(X)$	0	0	$\text{BR}_{G_1\wedge G_2}(X)$	1	1	$\text{BR}_{G_1\vee G_2}(X)$
1	0	$\text{POS}_{G_1}(X)$	0	0	$\text{BR}_{G_2}(X)$	0	0	$\text{BR}_{G_1\wedge G_2}(X)$	1	0	$\text{POS}_{G_1\vee G_2}(X)$
0	1	$\text{NEG}_{G_1}(X)$	1	1	$\text{BR}_{G_2}(X)$	0	1	$\text{NEG}_{G_1\wedge G_2}(X)$	1	1	$\text{BR}_{G_1\vee G_2}(X)$
0	1	$\text{NEG}_{G_1}(X)$	1	0	$\text{POS}_{G_2}(X)$	0	0	$\text{BR}_{G_1\wedge G_2}(X)$	1	1	$\text{BR}_{G_1\vee G_2}(X)$
0	1	$\text{NEG}_{G_1}(X)$	0	1	$\text{NEG}_{G_2}(X)$	0	1	$\text{NEG}_{G_1\wedge G_2}(X)$	0	1	$\text{NEG}_{G_1\vee G_2}(X)$
0	1	$\text{NEG}_{G_1}(X)$	0	0	$\text{BR}_{G_2}(X)$	0	0	$\text{BR}_{G_1\wedge G_2}(X)$	0	1	$\text{NEG}_{G_1\vee G_2}(X)$
0	0	$\text{BR}_{G_1}(X)$	1	1	$\text{BR}_{G_2}(X)$	0	0	$\text{BR}_{G_1\wedge G_2}(X)$	1	1	$\text{BR}_{G_1\vee G_2}(X)$
0	0	$\text{BR}_{G_1}(X)$	1	0	$\text{POS}_{G_2}(X)$	0	0	$\text{BR}_{G_1\wedge G_2}(X)$	1	0	$\text{POS}_{G_1\vee G_2}(X)$
0	0	$\text{BR}_{G_1}(X)$	0	1	$\text{NEG}_{G_2}(X)$	0	0	$\text{BR}_{G_1\wedge G_2}(X)$	0	1	$\text{NEG}_{G_1\vee G_2}(X)$
0	0	$\text{BR}_{G_1}(X)$	0	0	$\text{BR}_{G_2}(X)$	0	0	$\text{BR}_{G_1\wedge G_2}(X)$	0	0	$\text{BR}_{G_1\vee G_2}(X)$

- $POS_{G_1 \vee G_2}(X) = \big(POS_{G_1}(X) \cap POS_{G_2}(X)\big) \cup \big(POS_{G_1}(X) \cap BR_{G_2}(X)\big) \cup \big(BR_{G_1}(X) \cap POS_{G_2}(X)\big)$,
- $NEG_{G_1 \vee G_2}(X) = \big(NEG_{G_1}(X) \cap NEG_{G_2}(X)\big) \cup \big(NEG_{G_1}(X) \cap BR_{G_2}(X)\big) \cup \big(BR_{G_1}(X) \cap NEG_{G_2}(X)\big)$,
- $BR_{G_1 \vee G_2}(X) = \big(POS_{G_1}(X) \cap NEG_{G_2}(X)\big) \cup \big(NEG_{G_1}(X) \cap POS_{G_2}(X)\big) \cup \big(BR_{G_1}(X) \cap BR_{G_2}(X)\big)$,

where $G_1 \wedge G_2$ and $G_1 \vee G_2$ denote the conjunction and disjunction of G_1 and G_2, respectively.

Proof The proof is easy to verify by removing the rows which contain either $S_1(T(x), X) = S_1(T(x), X^c) = 1$ or $S_2(T(x), X) = S_2(T(x), X^c) = 1$.

Remark 37 Note that the boundary region in the conjunction of two complement compatible S-approximation spaces is maximized whereas it is minimized in their disjunction.

Proposition 38 *Let $G_1 = (U, W, T, S_1)$ and $G_2 = (U, W, T, S_2)$ be two partial monotone S-approximation spaces. Then, $G_1 \wedge G_2 = (U, W, T, S_\wedge)$ is a partial monotone S-approximation space.*

Proof Let $X, Y, A \subseteq W$, $X \subseteq Y$ and $S_\wedge(A, X) = 1$. Then, by Definition 30, we have $S_1(A, X) = S_2(A, X) = 1$. Given that $X \subseteq Y$ and G_1 and G_2 are partial monotone, we have $S_1(A, Y) = S_2(A, Y) = 1$. This implies $S_\wedge(A, Y) = 1$ which concludes the proof.

Proposition 39 *Let $G_1 = (U, W, T, S_1)$ and $G_2 = (U, W, T, S_2)$ be two partial monotone S-approximation spaces. Then, $G_1 \vee G_2 = (U, W, T, S_\vee)$ is a partial monotone S-approximation space.*

Proof Let $X, Y, A \subseteq W$, $X \subseteq Y$ and $S_\vee(A, X) = 1$. Then, by the Definition 31, we have either $S_1(A, X) = 1$ or $S_2(A, X) = 1$. Given that $X \subseteq Y$ and G_1 and G_2 are partial monotone, we have either $S_1(A, Y) = 1$ or $S_2(A, Y) = 1$. This implies $S_\vee(A, Y) = 1$ which concludes the proof.

Considering Propositions 38 and 39, one may expect that G_c of a partial monotone S-approximation space G be partial monotone, too. However, as it is shown in Example 40, this is not the case.

Example 40 Let $G = (U, W, T, S)$ be an S-approximation space, $G^c = (U, W, T, S^c)$ be its complement and $S(A, B) \equiv A \subseteq B$. It is obvious that S is partial monotone. However, S^c is not partial monotone. Let $A = \{x_1, x_2, x_3\}$, $X = \{x_1, x_2, x_4\}$ and $Y = \{x_1, x_2, x_3, x_4\}$. Therefore, $X \subseteq Y$ and $S^c(A, X) = 1$ because $A \nsubseteq X$ and $S^c(A, Y) = 0$, i.e. $A \subseteq Y$.

The following example illustrates one possible application of combining decider mappings in ensemble decision making.

Table 6 Knowledge mapping used by two doctors in Example 41

T	w_1	w_2	w_3	w_4	w_5	w_6
u_1	0	1	1	1	1	1
u_2	0	0	0	1	0	0
u_3	1	0	1	0	0	1
u_4	0	1	1	0	0	0
u_5	1	1	0	0	0	0

Example 41 Consider two doctors each specialized in diagnosing certain diseases. These two doctors use the same knowledge mapping T over a set of diseases $U = \{u_1, \ldots, u_5\}$ with symptoms $W = \{w_1, \ldots, w_6\}$ as in Table 6, however, they differ in their decision mappings S_1 and S_2 defined as

$$S_1(A, B) = \begin{cases} 1 \text{ if } \{w_3, w_6\} \subseteq A \cap B, \\ 0 \; otherwise, \end{cases} \tag{34}$$

and

$$S_2(A, B) = \begin{cases} 1 \text{ if } \frac{|A \cap B|}{|W|} \geq 0.3, \\ 0 \; otherwise, \end{cases} \tag{35}$$

where $A, B \subseteq W$. Note that the doctors can be represented by S-approximation spaces $G_1 = (U, W, T, S_1)$ and $G_2 = (U, W, T, S_2)$. Consider two patients with symptoms $X = \{w_1, w_3, w_6\}$ and $Y = \{w_1, w_2, w_6\}$. If we use the conjunction of these two doctors, i.e. $G_{1\wedge 2}$, then we have

$$\begin{aligned} &\text{POS}_{G_{1\wedge 2}}(X) = \{u_1, u_3\}, \; \text{NEG}_{G_{1\wedge 2}}(X) = \emptyset, \; \text{BR}_{G_{1\wedge 2}}(X) = \{u_2, u_4, u_5\}, \\ &\text{POS}_{G_{1\wedge 2}}(Y) = \emptyset, \qquad \text{NEG}_{G_{1\wedge 2}}(Y) = \emptyset, \; \text{BR}_{G_{1\wedge 2}}(Y) = U. \end{aligned}$$

Using the disjunction of these two doctors, we have

$$\begin{aligned} &\text{POS}_{G_{1\vee 2}}(X) = \{u_3\}, \qquad \text{NEG}_{G_{1\vee 2}}(X) = \emptyset, \; \text{BR}_{G_{1\vee 2}}(X) = \{u_1, u_2, u_4, u_5\}, \\ &\text{POS}_{G_{1\vee 2}}(Y) = \{u_3, u_5\}, \; \text{NEG}_{G_{1\vee 2}}(Y) = \emptyset, \; \text{BR}_{G_{1\vee 2}}(Y) = \{u_1, u_2, u_4\}. \end{aligned}$$

3.3 Combining Knowledge and Decision Mappings in S-Approximation Spaces

Now, it is time to consider the combination of S-approximation spaces in terms of both the decider mappings and the knowledge mappings. Suppose that $G_i = (U, W, T_i, S_i)$ be S-approximation spaces for $i = 1, \ldots, \ell$. We investigate the following four cases:

- $G_{\oplus\mathscr{T},\vee\mathscr{S}} = \big(U, W, T_{\oplus\mathscr{T}}, S_{\vee\mathscr{S}}\big)$ which is the ONS combination of the knowledge mappings and the disjunction of the decider mappings.
- $G_{\otimes\mathscr{T},\vee\mathscr{S}} = \big(U, W, T_{\otimes\mathscr{T}}, S_{\vee\mathscr{S}}\big)$ which is the PNS combination of the knowledge mappings and the disjunction of the decider mappings.
- $G_{\oplus\mathscr{T},\wedge\mathscr{S}} = \big(U, W, T_{\oplus\mathscr{T}}, S_{\wedge\mathscr{S}}\big)$ which is the ONS combination of the knowledge mappings and the conjunction of the decider mappings.
- $G_{\otimes\mathscr{T},\wedge\mathscr{S}} = \big(U, W, T_{\otimes\mathscr{T}}, S_{\wedge\mathscr{S}}\big)$ which is the PNS combination of the knowledge mappings and the conjunction of the decider mappings.

Proposition 42 *Suppose that $G_i = (U, W, T_i, S_i)$ are S-approximation spaces for $i = 1, \ldots, \ell$ and $X \subseteq W$. Then,*

1. $\underline{G_{\oplus\mathscr{T},\wedge\mathscr{S}}}(X) = \Big(\underline{G_1}(X) \cap \underline{G_2^1}(X)\Big) \cup \Big(\underline{G_2}(X) \cap \underline{G_1^2}(X)\Big),$
2. $\overline{G_{\oplus\mathscr{T},\wedge\mathscr{S}}}(X) = \Big(\overline{G_1}(X) \cup \overline{G_2^1}(X)\Big) \cap \Big(\overline{G_1^2}(X) \cup \overline{G_2}(X)\Big),$
3. $\underline{G_{\oplus\mathscr{T},\vee\mathscr{S}}}(X) = \underline{G_1}(X) \cup \underline{G_1^2}(X) \cup \underline{G_2^1}(X) \cup \underline{G_2}(X),$
4. $\overline{G_{\oplus\mathscr{T},\vee\mathscr{S}}}(X) = \overline{G_1}(X) \cap \overline{G_1^2}(X) \cap \overline{G_2^1}(X) \cap \overline{G_2}(X),$
5. $\underline{G_{\otimes\mathscr{T},\wedge\mathscr{S}}}(X) = \underline{G_1}(X) \cap \underline{G_1^2}(X) \cap \underline{G_2^1}(X) \cap \underline{G_2}(X),$
6. $\overline{G_{\otimes\mathscr{T},\wedge\mathscr{S}}}(X) = \overline{G_1}(X) \cup \overline{G_1^2}(X) \cup \overline{G_2^1}(X) \cup \overline{G_2}(X),$
7. $\underline{G_{\otimes\mathscr{T},\vee\mathscr{S}}}(X) = \Big(\underline{G_1}(X) \cap \underline{G_1^2}(X)\Big) \cup \Big(\underline{G_2}(X) \cap \underline{G_2^1}(X)\Big),$
8. $\overline{G_{\otimes\mathscr{T},\vee\mathscr{S}}}(X) = \Big(\overline{G_1}(X) \cup \overline{G_1^2}(X)\Big) \cap \Big(\overline{G_2^1}(X) \cup \overline{G_2}(X)\Big),$

where G_i and G_i^j denote the S-approximation spaces (U, W, T_i, S_i) and (U, W, T_j, S_i), respectively.

Proof Without loss of generality, we can assume $\mathscr{T} = \{T_1, T_2\}$ and $\mathscr{S} = \{S_1, S_2\}$. These proofs can be generalized by induction.

1. By the definitions, we have

$$\begin{aligned}
\underline{G_{\oplus\mathscr{T},\wedge\mathscr{S}}}(X) &= \{x \in U \mid S_\wedge\left(\oplus\mathscr{T}(x), X\right) = 1\} \\
&= \{x \in U \mid S_\wedge\left(T_1(x), X\right) = 1 \vee S_\wedge\left(T_2(x), X\right) = 1\} \\
&= \{x \in U \mid S_1\left(T_1(x), X\right) = S_2\left(T_1(x), X\right) = 1\} \cup \\
&\quad\ \{x \in U \mid S_1\left(T_2(x), X\right) = S_2\left(T_2(x), X\right) = 1\} \\
&= \Big(\underline{G_1}(X) \cap \underline{G_2^1}(X)\Big) \cup \Big(\underline{G_2}(X) \cap \underline{G_1^2}(X)\Big).
\end{aligned}$$

2. By the definitions, we have

$$\begin{aligned}
\overline{G_{\oplus\mathscr{T},\wedge\mathscr{S}}}(X) &= \big\{x \in U \mid S_\wedge\left(\oplus\mathscr{T}(x), X^c\right) = 0\big\} \\
&= \big\{x \in U \mid S_\wedge\left(T_1(x), X^c\right) = 0 \wedge S_\wedge\left(T_2(x), X^c\right) = 0\big\}
\end{aligned}$$

$$= \left\{x \in U \mid \left[S_1\left(T_1(x), X^c\right) = 0 \vee \left(S_2\left(T_1(x), X^c\right) = 0\right)\right] \right.$$
$$\left. \wedge \left[S_1\left(T_2(x), X^c\right) = 0 \vee \left(S_2\left(T_2(x), X^c\right) = 0\right)\right]\right\}$$
$$= \left(\overline{G_1}(X) \cup \overline{G_2^1}(X)\right) \cap \left(\overline{G_1^2}(X) \cup \overline{G_2}(X)\right).$$

3. By the definitions, we have

$$\begin{aligned}
\underline{G_{\oplus\mathscr{T},\vee\mathscr{S}}}(X) &= \{x \in U \mid S_\vee\left(\oplus\mathscr{T}(x), X\right) = 1\} \\
&= \{x \in U \mid S_1\left(\oplus\mathscr{T}(x), X\right) = 1 \vee S_2\left(\oplus\mathscr{T}(x), X\right) = 1\} \\
&= \{x \in U \mid (\ S_1\left(T_1(x), X\right) = 1 \vee S_1\left(T_2(x), X\right) = 1) \\
&\quad \vee (S_2\left(T_1(x), X\right) = 1 \vee S_2\left(T_2(x), X\right) = 1)\} \\
&= \underline{G_1}(X) \cup \underline{G_1^2}(X) \cup \underline{G_2^1}(X) \cup \underline{G_2}(X).
\end{aligned}$$

4. By the definitions, we have

$$\begin{aligned}
\overline{G_{\oplus\mathscr{T},\vee\mathscr{S}}}(X) &= \left\{x \in U \mid S_\vee\left(\oplus\mathscr{T}(x), X^c\right) = 0\right\} \\
&= \left\{x \in U \mid S_1\left(\oplus\mathscr{T}(x), X^c\right) = 0 \wedge S_2\left(\oplus\mathscr{T}(x), X^c\right) = 0\right\} \\
&= \left\{x \in U \mid \left(\ S_1\left(T_1(x), X^c\right) = 0 \wedge S_1\left(T_2(x), X^c\right) = 0\right)\right. \\
&\quad \left.\wedge \left(S_2\left(T_1(x), X^c\right) = 0 \wedge S_2\left(T_2(x), X^c\right) = 0\right)\right\} \\
&= \overline{G_1}(X) \cap \overline{G_1^2}(X) \cap \overline{G_2^1}(X) \cap \overline{G_2}(X).
\end{aligned}$$

5. By the definitions, we have

$$\begin{aligned}
\underline{G_{\otimes\mathscr{T},\wedge\mathscr{S}}}(X) &= \{x \in U \mid S_\wedge\left(\otimes\mathscr{T}(x), X\right) = 1\} \\
&= \{x \in U \mid S_1\left(\otimes\mathscr{T}(x), X\right) = 1 \wedge S_2\left(\otimes\mathscr{T}(x), X\right) = 1\} \\
&= \{x \in U \mid S_1\left(T_1(x), X\right) = 1 \wedge S_1\left(T_2(x), X\right) = 1 \\
&\quad \wedge S_2\left(T_1(x), X\right) = 1 \wedge S_2\left(T_2(x), X\right) = 1\} \\
&= \underline{G_1}(X) \cap \underline{G_1^2}(X) \cap \underline{G_2^1}(X) \cap \underline{G_2}(X).
\end{aligned}$$

6. By the definitions, we have

$$\begin{aligned}
\overline{G_{\otimes\mathscr{T},\wedge\mathscr{S}}}(X) &= \left\{x \in U \mid S_\wedge\left(\otimes\mathscr{T}(x), X^c\right) = 0\right\} \\
&= \left\{x \in U \mid S_1\left(\otimes\mathscr{T}(x), X^0\right) = 0 \vee S_2\left(\otimes\mathscr{T}(x), X^c\right) = 0\right\} \\
&= \left\{x \in U \mid S_1\left(T_1(x), X^c\right) = 0 \vee S_1\left(T_2(x), X^c\right) = 0\right.
\end{aligned}$$

$$\vee S_2\left(T_1(x), X^c\right) = 0 \vee S_2\left(T_2(x), X^c\right) = 0\big\}$$
$$= \overline{G_1}(X) \cup \overline{G_1^2}(X) \cup \overline{G_2^1}(X) \cup \overline{G_2}(X).$$

7. By the definitions, we have

$$\begin{aligned}\underline{G_{\otimes\mathscr{T},\vee\mathscr{S}}}(X) &= \{x \in U \mid S_\vee\left(\otimes\mathscr{T}(x), X\right) = 1\} \\ &= \{x \in U \mid \left(S_1\left(T_1(x), X\right) = 1 \wedge S_1(T_2(x), X) = 1\right) \\ &\quad \vee \left(S_2\left(T_1(x), X\right) = 1 \wedge S_2(T_2(x), X) = 1\right)\} \\ &= \left(\underline{G_1}(X) \cap \underline{G_1^2}(X)\right) \cup \left(\underline{G_2}(X) \cap \underline{G_2^1}(X)\right).\end{aligned}$$

8. By the definitions, we have

$$\begin{aligned}\overline{G_{\otimes\mathscr{T},\vee\mathscr{S}}}(X) &= \left\{x \in U \mid S_\vee\left(\otimes\mathscr{T}(x), X^c\right) = 0\right\} \\ &= \left\{x \in U \mid \left(S_1\left(T_1(x), X^c\right) = 0 \vee S_1(T_2(x), X^c) = 0\right)\right. \\ &\quad \left.\wedge \left(S_2\left(T_1(x), X^c\right) = 0 \vee S_2(T_2(x), X^c) = 0\right)\right\} \\ &= \left(\overline{G_1}(X) \cup \overline{G_1^2}(X)\right) \cap \left(\overline{G_2^1}(X) \cup \overline{G_2}(X)\right).\end{aligned}$$

3.4 General Combination of S-Approximation Spaces

Next, we consider the problem of combining S-approximations $G_i = (U_i, W_i, T_i, S_i)$ for $i = 1, \ldots, \ell$. We propose two approaches: (1) the union denoted by $G_\cup$ and (2) the intersection denoted by $G_\cap$.

Definition 43 (Union of S-Approximation Spaces) Let $G_i = (U_i, W_i, T_i, S_i)$ be ℓ S-approximation spaces where $T_i : U_i \to \mathscr{P}(W_i)$ and $S_i : \mathscr{P}(W_i) \times \mathscr{P}(W_i) \to \{0, 1\}$. Then, the union of these S-approximation spaces is the quadruple $G_\cup = (U, W, T, S)$ where $U = \cup_{i=1,\ldots,\ell} U_i$, $W = \cup_{i=1,\ldots,\ell} W_i$. The knowledge mapping $T : U \to \mathscr{P}(W)$ is defined as

$$T(x) \mapsto \bigcup_{i=1}^{\ell} T_i(x), \tag{36}$$

for $x \in U$ where $T_i(x) = \emptyset$ if $x \notin U_i$. Finally. the decider $S : \mathscr{P}(W) \times \mathscr{P}(W) \to \{0, 1\}$ is defined as

$$S(X, Y) = \bigvee_{i=1}^{\ell} S_i(X \cap W_i, Y \cap W_i). \tag{37}$$

Definition 44 (Intersection of S-Approximation Spaces) Let $G_i = (U_i, W_i, T_i, S_i)$ be ℓ S-approximation spaces where $T_i : U_i \to \mathscr{P}(W_i)$ and $S_i : \mathscr{P}(W_i) \times \mathscr{P}(W_i) \to \{0, 1\}$. Then, the intersection of these S-approximation spaces is the quadruple $G_\cap = (U, W, T, S)$ where $U = \cap_{i=1,\dots,\ell} U_i$, $W = \cap_{i=1,\dots,\ell} W_i$. The knowledge mapping $T : U \to \mathscr{P}(W)$ is defined as

$$T(x) \mapsto \bigcap_{i=1}^{\ell} T_i(x), \tag{38}$$

for $x \in U$. Finally. the decider $S : \mathscr{P}(W) \times \mathscr{P}(W) \to \{0, 1\}$ is defined as

$$S(X, Y) = \bigwedge_{i=1}^{\ell} S_i(X, Y). \tag{39}$$

Corollary 45 *Let $G_\cup = (U, W, T, S)$ be the union of partial monotone S-approximation spaces $G_i = (U_i, W_i, T_i, S_i)$ for $i = 1, \dots, \ell$. Then, $G_\cap$ is also partial monotone.*

Proof The proof is easily concluded from Proposition 38.

Corollary 46 *Let $G_\cup = (U, W, T, S)$ be the union of partial monotone S-approximation spaces $G_i = (U_i, W_i, T_i, S_i)$ for $i = 1, \dots, \ell$. Then, $G_\cup$ is also partial monotone.*

Proof The proof is easily concluded from Proposition 39.

4 Conclusion

In this paper, we have surveyed recent results on S-approximation spaces as a generalization of rough set theory and three-way decisions. We have also studied the combination of S-approximation spaces as a concise tool to manage distributed uncertain data in several ways: (1) Combining them in terms of their knowledge mapping, (2) combining them in terms of their decision mappings and (3) combining them in general. The results obtained can be used to formalize the management of distributed uncertainty as well as the invention of novel distributed uncertain data processing algorithms. As for future research directions, one may consider the following questions: (1) Different methods of combining either the knowledge mappings or the decider mappings or both, (2) Different methods of combining S-approximation spaces in general, (3) Extending the combination of S-approximation spaces to manage fuzzy or intuitionistic datasets like [19, 20] (4) Investigating other conditions under which the lower, upper and 3WD regions preserve their properties.

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Index

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