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After Plancherel Formula

Yury Neretin

Abstract. We discuss two topics related to Fourier transforms on Lie groups and on homogeneous spaces: the operational calculus and the Gelfand–Gindikin problem (program) about separation of non-uniform spectra. Our purpose is to indicate some non-solved problems of noncommutative harmonic analysis that definitely are solvable.

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1. Abstract Plancherel theorem for groups

See, e.g., [2]. Let G be a type I locally compact group with a two-side invariant Haar measure dg. Denote by $\widehat{G}$ the set of all irreducible unitary representations of G (defined up to a unitary equivalence[1]). For $\rho \in \widehat{G}$ denote by H_ρ the space of the representation ρ. For $\rho \in \widehat{G}$ and $f \in L^1(G)$ we define the following operator in H_ρ:

$$\rho(f) := \int_G f(g)\, \rho(g)\, dg.$$

This determines a representation of the convolution algebra $L^1(G)$ in H_ρ,

$$\rho(f_1)\rho(f_2) = \rho(f_1 * f_2).$$

Consider a Borel measure ν on $\widehat{G}$ and the direct integral of Hilbert spaces H_ρ with respect to the measure ν. Consider the space $\mathcal{L}(\widehat{G}, \nu)$ of measurable functions Φ

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[1] For a formal definition of type I groups see, e.g., [2, Sect. 7.2]. Connected semisimple Lie groups, connected nilpotent Lie groups, classical p-adic groups have type I. This condition implies a presence of the standard Borel structure on $\widehat{G}$ and a uniqueness of a decomposition of any unitary representation of G into a direct integral of irreducible representations.

on $\widehat{G}$ sending any $\rho \in G$ to a Hilbert–Schmidt operator in H_ρ and satisfying the condition

$$\int_{\widehat{G}} \mathrm{tr}\big(\Phi(\rho)^*\Phi(\rho)\big)\, d\nu(\rho) < \infty.$$

There exists a unique measure μ on $\widehat{G}$ (the Plancherel measure*), such that for any f_1, $f_2 \in L^1 \cap L^2(G)$ we have*

$$\langle f_1, f_2\rangle_{L^2(G)} = \int_{\widehat{G}} \mathrm{tr}\big(\rho(f_2)^*\rho(f_1)\big)\, d\mu(\rho)$$

and the map $f \mapsto \rho(f)$ extends to a unitary operator from $L^2(G)$ to the space $\mathcal{L}^2(\widehat{G}, \mu)$ (F.I. Mautner, I. Segal (1950), see, e.g., [2]).

2. An example. The group $\mathrm{GL}(2, \mathbb{R})$

Let $\mathrm{GL}(2, \mathbb{R})$ be the group of invertible real matrices of order 2. Let $\mu \in \mathbb{C}$ and $\varepsilon \in \mathbb{Z}_2$. We define the function $x^{\mu /\!/ \varepsilon}$ on $\mathbb{R} \setminus 0$ by

$$x^{\mu/\!/\varepsilon} := |x|^\mu \,\mathrm{sgn}(x)^\varepsilon.$$

Denote $\Lambda := \mathbb{C} \times \mathbb{Z}_2 \times \mathbb{C} \times \mathbb{Z}_2$. For each element $(\mu_1, \varepsilon_1; \mu_2, \varepsilon_2)$ of Λ we define a representation $T_{\mu,\varepsilon}$ of $\mathrm{GL}_2(\mathbb{R})$ in the space of functions on $\mathbb{R}$ by

$$T_{\mu_1,\varepsilon_1;\mu_2,\varepsilon_2} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \varphi(t) = \varphi\left(\frac{b+td}{a+tc}\right) \cdot (a+tc)^{-1+\mu_1-\mu_2/\!/\varepsilon_1-\varepsilon_2} \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{1/2+\mu_2/\!/\varepsilon_2}.$$

This formula determines the *principal series of representations of* $\mathrm{GL}(2, \mathbb{R})$. If $\mu_1 - \mu_2 \notin \mathbb{Z}$, then representations $T_{\mu_1,\varepsilon_1;\mu_2,\varepsilon_2}$ and $T_{\mu_2,\varepsilon_2;\mu_1,\varepsilon_1}$ are irreducible and equivalent (on representations of $\mathrm{SL}(2, \mathbb{R})$, see, e.g., [4, 39]).

If $\mu_1 = i\tau_1$, $\mu_2 = i\tau_2 \in i\mathbb{R}$, then a representation $T_{\mu_1,\varepsilon_1;\mu_2,\varepsilon_2}$ is unitary in $L^2(\mathbb{R})$ (they are called representations the *unitary principal series*).

Next, we define *representations of the discrete series*. Let $n = 1, 2, 3, \dots$ Consider the Hilbert space H_n of holomorphic functions φ on $\mathbb{C} \setminus \mathbb{R}$ satisfying

$$\int_{\mathbb{C}\setminus\mathbb{R}} |\varphi(z)|^2 |\operatorname{Im} z|^{n-1}\, d\operatorname{Re} z\, d\operatorname{Im} z < \infty.$$

In fact, φ is a pair of holomorphic functions determined on half-planes $\operatorname{Im} z > 0$ and $\operatorname{Im} z < 0$. For $\tau \in \mathbb{R}$, $\delta \in \mathbb{Z}_2$ we define the unitary representation $D_{n,\tau,\delta}$ of $\mathrm{GL}_2(\mathbb{R})$ in H_n by

$$D_{n,\tau,\delta} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \varphi(z) = \varphi\Big(\frac{b+zd}{a+zc}\Big)(a+zc)^{-1-n} \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{1/2+n/2+i\tau/\!/\delta}.$$

There exists also the complementary series of unitary representations, which does not participate in the Plancherel formula.

Remark. The expression for $D_{n,\tau,\delta}$ is contained in the family $T_{\mu_1,\varepsilon_1;\mu_2,\varepsilon_2}$, but we change the space of the representations.

The Plancherel measure for $\mathrm{SL}(2,\mathbb{R})$ was explicitly evaluated in 1952 by Harish-Chandra, it is supported by the principal and discrete series. On the principal series the density is given by the formula (see, e.g., [39])

$$d\mathcal{P} = \frac{1}{16\pi^3}(\tau_1 - \tau_2)\tanh \pi(\tau_1 - \tau_2)/2\, d\tau_1\, d\tau_2, \qquad \text{if } \varepsilon_1 - \varepsilon_2 = 0;$$
$$d\mathcal{P} = \frac{1}{16\pi^3}(\tau_1 - \tau_2)\coth \pi(\tau_1 - \tau_2)/2\, d\tau_1\, d\tau_2 \qquad \text{if } \varepsilon_1 - \varepsilon_2 = 1.$$

On nth piece of the discrete series the measure is given by

$$d\mathcal{P} = \frac{n}{8\pi^3} d\tau.$$

3. Homogeneous spaces, etc.

The Plancherel formula for complex classical groups was obtained by I.M. Gelfand and M.A. Naimark [5] in 1948–50, for real semisimple groups by Harish-Chandra in 1965 (see, e.g., [11, 13]), there is also a formula for nilpotent groups (A.A. Kirillov [12], L. Pukanszky [37]).

During 1950–early 2000s there was obtained a big zoo of explicit spectral decompositions of L^2 on homogeneous spaces, of tensor products of unitary representations, of restrictions of unitary representations to subgroups. We present some references, which can be useful for our purposes [1, 5, 9, 11, 16, 23, 27, 38, 41]. Unfortunately, texts about groups of rank > 1 are written for experts and are heavy for exterior readers. See also the paper [29] on some spectral problems (deformations of L^2 on pseudo-Riemannian symmetric spaces), which apparently are solvable but are not solved.

However, a development of the last decades seems strange. The Plancherel formula for Riemannain symmetric spaces [7] (see, e.g., [10]) and Bruhat–Tits buildings [14] had a general mathematical influence (for instance to theory of special functions and to theory of integrable systems). Usually, Plancherel formulas are heavy results (with impressive explicit formulas) without further continuation even inside representation theory and noncommutative harmonic analysis.

4. Operational calculus for $\mathrm{GL}(2,\mathbb{R})$, see [33], 2017

Denote by Gr_4^2 the Grassmannian of all two-dimensional linear subspaces in $\mathbb{R}^4$. The natural action of the group $\mathrm{GL}(4,\mathbb{R})$ in $\mathbb{R}^4$ induces the action on Gr_4^2, therefore we have a unitary representation of the group $\mathrm{GL}(4,\mathbb{R})$ in L^2 on Gr_4^2 (this is an irreducible representation of a degenerate principal series) and the corresponding action of the Lie algebra $\mathfrak{gl}(4)$.

For $g \in \mathrm{GL}(2,\mathbb{R})$ its graph is a linear subspace in $\mathbb{R}^2 \oplus \mathbb{R}^2 = \mathbb{R}^4$. In this way we get an embedding

$$\mathrm{GL}(2,\mathbb{R}) \to \mathrm{Gr}_4^2.$$

The image of the embedding is an open dense subset in Gr_4^2. Thus we have an identification of Hilbert spaces

$$L^2\big(\mathrm{GL}(2,\mathbb{R})\big) \simeq L^2\big(\mathrm{Gr}_4^2\big)$$

(since natural measures on $\mathrm{GL}(2,\mathbb{R})$ and Gr_4^2 are different, we must multiply functions by an appropriate density to obtain a unitary operator). Therefore we get a canonical action of the group $\mathrm{GL}(4,\mathbb{R})$ in $L^2\big(\mathrm{GL}(2,\mathbb{R})\big)$. It is easy to see that the block diagonal subgroup $\mathrm{GL}(2,\mathbb{R}) \times \mathrm{GL}(2,\mathbb{R}) \subset \mathrm{GL}(4,\mathbb{R})$ acts by left and right shifts on $\mathrm{GL}(2,\mathbb{R})$.

We wish to evaluate the action of the Lie algebra $\mathfrak{gl}(4)$ in the Fourier-image.

Consider the space $C_0^\infty\big(\mathrm{GL}(2,\mathbb{R})\big)$ of smooth compactly supported functions on $\mathrm{GL}(2,\mathbb{R})$. For any $F \in C_0^\infty\big(\mathrm{GL}(2,\mathbb{R})\big)$ consider the operator-valued function $T_{\mu_1,\varepsilon_1;\mu_2,\varepsilon_2}(F)$ depending on $(\mu_1,\varepsilon_1;\mu_2,\varepsilon_2) \in \Lambda$. We write these operators in the form

$$T_{\mu_1,\varepsilon_1;\mu_2;\varepsilon_2}(F)\varphi(t) = \int_{-\infty}^{\infty} K(t,s|\mu_1,\varepsilon_1;\mu_2,\varepsilon_2)\,\varphi(s)\,ds.$$

The kernel K is smooth in t, s and holomorphic in μ_1, μ_2.

On the other hand we have the Hilbert space $\mathcal{L}^2\big(\widehat{\mathrm{GL}(2,\mathbb{R})}, d\mathcal{P}\big)$. The norm in this Hilbert space is given by

$$\begin{aligned}\|K\|^2 = \int \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \big|K(t,s|\mu_1,\varepsilon_1;\mu_2,\varepsilon_2)\big|^2 dt\,ds\,d\mathcal{P}(\mu) + \\ + \Big\{\text{summands corresponding to the discrete series}\Big\}.\end{aligned} \tag{1}$$

We must write the action of the Lie algebra $\mathfrak{gl}(4)$. Denote by e_{kl} the standard generators of $\mathfrak{gl}(4)$ acting in smooth compactly supported functions on $\mathrm{GL}(2,\mathbb{R})$ and by E_{kl} the same generators acting in the space of functions of variables t, s, μ_1, ε_1, μ_2, ε_2. The action of the subalgebra $\mathfrak{gl}(2)\oplus\mathfrak{gl}(2)$ is clear from the definition of the Fourier transform, this Lie algebra acts by first-order differential operators. For instance

$$e_{12} = -b\frac{\partial}{\partial a} - d\frac{\partial}{\partial b}, \qquad E_{12} = \frac{\partial}{\partial t};$$
$$e_{43} = b\frac{\partial}{\partial a} + d\frac{\partial}{\partial c}, \qquad E_{43} = -s^2\frac{\partial}{\partial s} + (-1-\mu_1+\mu_2)s.$$

Define shift operators V_1^+, V_1^-, V_2^+, V_2^- by

$$V_1^{\pm}K(t,s|\mu_1,\varepsilon_1;\mu_2,\varepsilon_2) = K(t,s|\mu_1\pm 1,\varepsilon_1+1;\mu_2,\varepsilon_2); \tag{2}$$
$$V_2^{\pm}K(t,s|\mu_1,\varepsilon_1;\mu_2,\varepsilon_2) = K(t,s|\mu_1,\varepsilon_1;\mu_2\pm 1,\varepsilon_2+1). \tag{3}$$

To be definite, we present formulas for two nontrivial generators e_{kl} and their Fourier images E_{kl}:

$$\begin{aligned}
e_{14} &= \frac{\partial}{\partial b} + \frac{c}{ad-bc},\\
E_{14} &= \frac{-1/2+\mu_1}{\mu_1-\mu_2}\,\frac{\partial}{\partial s}\,V_1^- + \frac{-1/2+\mu_2}{\mu_1-\mu_2}\,\frac{\partial}{\partial t}\,V_2^-,\\
e_{32} &= -\left(ac\frac{\partial}{\partial a} + ad\frac{\partial}{\partial b} + c^2\frac{\partial}{\partial c} + cd\frac{\partial}{\partial d}\right) - c,\\
E_{32} &= \frac{1/2+\mu_1}{\mu_1-\mu_2}\,\frac{\partial}{\partial t}\,V_1^+ + \frac{1/2+\mu_2}{\mu_1-\mu_2}\,\frac{\partial}{\partial s}\,V_2^+.
\end{aligned}$$

There is also a correspondence for operators of multiplication by functions. For instance, the operator of multiplication by c in $C_0^\infty\big(\mathrm{GL}(2,\mathbb{R})\big)$ corresponds to

$$\frac{1}{\mu_1-\mu_2}\left(\frac{\partial}{\partial t}V_1^+ + \frac{\partial}{\partial s}V_2^+\right)$$

in the Fourier-image. There are similar formulas for multiplications by a, b, d. The operator of multiplication by $(ad-bc)^{-1}$ corresponds to $V_1^-V_2^-$ (the last statement is trivial). The operator $\frac{\partial}{\partial b}$ corresponds to

$$\frac{\mu_1}{\mu_1-\mu_2}\,\frac{\partial}{\partial s}\,V_1^- + \frac{\mu_2}{\mu_1-\mu_2}\,\frac{\partial}{\partial t}\,V_2^-,$$

There are similar formulas for other partial derivatives.

We emphasize that **our formulas contain shifts in imaginary directions** (the shifts in (2)–(3) are transversal to the contour of integration in (1)).

5. Difference operators in imaginary direction and classical integral transforms

The operators iE_{kl} are symmetric in the sense of the spectral theory. The question about domains of self-adjointness is open.

There exist elements of spectral theory of self-adjoint difference operators in $L^2(\mathbb{R})$ of the type

$$Lf(s) = a(s)f(s+i) + b(s)f(s) + c(s)f(s-i), \qquad i^2=-1, \tag{4}$$

see [8, 30]. Recall that several systems of classical hypergeometric orthogonal polynomials (Meixner–Polaszek, continuous Hahn, continuous dual Hahn, Wilson) are eigenfunctions of operators of this type. In the polynomial cases the problems are algebraic. The simplest nontrivial analytic example is the operator

$$Mf(s) = \frac{1}{is}\big(f(s+i) - f(s-i)\big)$$

in $L^2\big(\mathbb{R}_+, |\Gamma(is)|^{-2}ds\big)$. We define M on the space of functions f holomorphic in a strip $|\operatorname{Im} s| < 1+\delta$ and satisfying the condition

$$|f(s)| \leqslant \exp\{-\pi|\operatorname{Re} s|\}|\operatorname{Re} s|^{-3/2-\varepsilon}$$

in this strip. The spectral decomposition of M is given by the inverse Kontorovich–Lebedev integral transform. Recall that the direct Kontorovich–Lebedev transform

$$\mathcal{K}f(s) = \int_0^\infty K_{is}(x)f(x)\frac{dx}{x},$$

where K_{is} is the Macdonald–Bessel function, gives the spectral decomposition of a second-order differential operator, namely

$$D := \left(x\frac{d}{dx}\right)^2 - x^2, \qquad x > 0.$$

The transform $\mathcal{K}$ is a unitary operator $L^2(\mathbb{R}_+, dx/x) \to L^2\big(\mathbb{R}_+, |\Gamma(is)|^{-2}ds\big)$. It sends D to the multiplication by s^2, and $\mathcal{K}^{-1}$ sends M to the multiplication by $2/x$. So we get so-called *bispectral problem.*

Now there is a zoo of explicit spectral decompositions of operators (4). The similar bispectrality appears for some other integral transforms: the index hypergeometric transform (another names of this transform are: the Olevsky transform, the Jacobi transform, the generalized Mehler–Fock transform) [25], the Wimp transform with Whittaker kernel [30], for a continuous analog of expansion in Wilson polynomials proposed by W. Groenevelt [8], etc.

This subject is now a list of examples (which certainly can be extended), but there are no a priori theorems.

6. A general problem about overalgebras

Let G be a Lie group, $\mathfrak{g}$ the Lie algebra. Let $H \subset G$ be a subgroup. Let σ be an irreducible unitary representation of G. Assume that we know an explicit spectral decomposition of restriction of ρ to a subgroup H. To write the action of the overalgebra $\mathfrak{g}$ in the spectral decomposition.

Remarks.

1) Above we have $G = \mathrm{GL}(4,\mathbb{R})$, its representation σ in L^2 on the Grassmannian Gr_4^2, and $H = \mathrm{GL}(2,\mathbb{R}) \times \mathrm{GL}(2,\mathbb{R})$. The restriction problem is equivalent to the decomposition of regular representation of $\mathrm{GL}(2,\mathbb{R}) \times \mathrm{GL}(2,\mathbb{R})$ in $L^2\big(\mathrm{GL}(2,\mathbb{R})\big)$. The Fourier transform is the spectral decomposition of the regular representation.

2) It is important that similar overgroups exist for all 10 series of classical real Lie groups[2]. Moreover, a decomposition of L^2 on any classical symmetric space[3] G/M can be regarded as a certain restriction problem, see [24].
3) Next, consider a tensor product $\rho_1 \otimes \rho_2$ of two unitary representations of a group G. Then we have the action of $G \times G$ in the tensor product, so the problem of decomposition of tensor products can be regarded as a problem of a restriction from the group $G \times G$ to the diagonal subgroup G.

The question under the discussion was formulated in [30]. Several problems of this kind were solved [18–20, 30, 31, 33]. In all the cases we get differential-difference operators including shifts in imaginary direction. Expressions also include differential operators of high order, even for $\mathrm{SL}(2,\mathbb{R})$-problems we usually get operators of order 2.

Conjecture. *All problems of this kind are solvable* (*if we are able to write a spectral decomposition*).

7. The Gelfand–Gindikin problem, [3], 1977

The set $\widehat{H}$ of unitary representations of a semisimple group H naturally splits into different types (series).

Let H be a semisimple group, M a subgroup. Consider the space $L^2(H/M)$. Usually its H-spectrum contains different series. To write explicitly decomposition of L^2 into pieces with uniform spectrum.

A variant of the problem: let G be a Lie group, $H \subset G$ a semisimple subgroup, ρ is a unitary representation of G. Answer to the same question.

8. Example: separation of series for the one-sheet hyperboloid

Consider the space $\mathbb{R}^3$ equipped with an indefinite inner product

$$\langle u, v\rangle = -u_1v_1 + u_2v_2 + u_3v_3.$$

Consider the pseudo-orthogonal group preserving the form $\langle\cdot,\cdot\rangle$, denote by $\mathrm{SO}_0(2,1)$ its connected component. Recall that $\mathrm{SO}_0(2,1)$ is isomorphic to the quotient $\mathrm{PSL}(2,\mathbb{R})$ of $\mathrm{SL}(2,\mathbb{R})$ by the center $\{\pm 1\}$.

Consider a one-sheet hyperboloid H defined by $x_1^2 - x_2^2 - x_3^2 = 1$. It is an $\mathrm{SO}_0(2,1)$-homogeneous space admitting a unique (up to a scalar factor) invariant measure. Decomposition of $L^2(H)$ into irreducible representations of $\mathrm{SO}_0(2,1)$ is well known. The spectrum is a sum of all representations of the discrete series

[2]More precisely, an overgroup $\widetilde{G}$ exists for $G = \mathrm{GL}(n,\mathbb{R})$, $\mathrm{GL}(n,\mathbb{C})$, $\mathrm{GL}(n,\mathbb{H})$, $\mathrm{O}(p,q)$, $\mathrm{U}(p,q)$, $\mathrm{Sp}(p,q)$, $\mathrm{Sp}(2n,\mathbb{R})$, $\mathrm{Sp}(2n,\mathbb{C})$, $\mathrm{O}(n,\mathbb{C})$, $\mathrm{SO}^*(2n)$ (and not for $\mathrm{SL}(n,\cdot)$, $\mathrm{SU}(p,q)$). For instance, for $g \in \mathrm{Sp}(2n,\mathbb{R})$ its graph is a Lagrangian subspace in $\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}$, this determines a map from $\mathrm{Sp}(2n,\mathbb{R})$ to the Lagrangian Grassmannian with an open dense image. We set $\widetilde{G} := \mathrm{Sp}(4n,\mathbb{R})$.

[3]The groups G, M must be from the list of the previous footnote, M must be a symmetric subgroup in G.

of $\mathrm{PSL}(2,\mathbb{R})$ and the integral over the whole principal series with multiplicity 2. The separation of series was proposed by V.F. Molchanov [15] in 1980 (we use a modification from [22]).

Denote by $\overline{\mathbb{C}} = \mathbb{C} \cup \infty$ the Riemann sphere, by $\overline{\mathbb{R}} = \mathbb{R} \cup \infty$ denote the real projective line, $\overline{\mathbb{R}} \subset \overline{\mathbb{C}}$. Consider the diagonal action of $\mathrm{SL}(2,\mathbb{R})$ on $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$,

$$(x_1, x_2) \mapsto \left(\frac{b + dx_1}{a + cx_1}, \frac{b + dx_2}{a + cx_2} \right).$$

Consider the subset H' in $\overline{\mathbb{R}} \times \overline{\mathbb{R}}$ consisting of points x_1, x_2 such that $x_1 \neq x_2$. It is easy to verify that H' is an orbit of $\mathrm{SL}(2,\mathbb{R})$, it is equivalent to the hyperboloid H as a homogeneous space[4]. It is easy to verify that the invariant measure on H' is given by the formula

$$d\nu(x_1, x_2) = |x_1 - x_2|^{-2}\, dx_1\, dx_2.$$

We identify the space $L^2(H', d\nu)$ with the standard $L^2(\mathbb{R} \times \mathbb{R})$ by the *unitary* operator

$$Jf(x_1, x_2) = f(x_1, x_2)(x_1 - x_2)^{-1}.$$

Now our representation in $L^2(H)$ transforms to the following unitary representation in the standard $L^2(\mathbb{R}^2)$:

$$Q \begin{pmatrix} a & b \\ c & d \end{pmatrix} f(x_1, x_2) = f \left(\frac{b + dx_1}{a + cx_1}, \frac{b + dx_2}{a + cx_2} \right) (a + cx_1)^{-1} (a + cx_2)^{-1}. \tag{5}$$

Next, consider a unitary representation of $\mathrm{SL}(2,\mathbb{R})$ in $L^2(\mathbb{R})$ given by

$$T \begin{pmatrix} a & b \\ c & d \end{pmatrix} f(x) = f \left(\frac{b + xd}{a + xc} \right) (a + xc)^{-1}.$$

Obviously, we have $Q = T \otimes T$. The representation T is contained in the unitary principal series and it is a unique reducible element of this series (see, e.g., [4]).

Denote by $\Pi_\pm$ the upper and lower half-planes in $\overline{\mathbb{C}}$. The Hardy space $H^2(\Pi_+)$ consists of functions F_+ holomorphic in Π_+ that can be represented in the form

$$F_+(x) = \int_0^\infty \varphi(t) e^{itx}\, dt, \qquad \text{where } \varphi(t) \in L^2(\mathbb{R}_+).$$

Obviously, F is well defined also on $\mathbb{R}$ and is contained in L^2. The space $H^2(\Pi_-)$ consists of functions F_- holomorphic in Π_- of the form

$$F_-(x) = \int_{-\infty}^0 \varphi(t) e^{itx}\, dt, \qquad \text{where } \varphi(-t) \in L^2(\mathbb{R}_+).$$

Evidently,

$$L^2(\mathbb{R}) = H^2(\Pi_+) \oplus H^2(\Pi_+).$$

It can be shown that the subspaces $H^2(\Pi_\pm) \subset L^2(\mathbb{R})$ are invariant with respect to operators $T(\cdot)$, and therefore T splits into two summands $T_+ \oplus T_-$ (one of them

[4] Two families of lines on the hyperboloid correspond to two families of lines $x_1 = \text{const}$ and $x_2 = \text{const}$ on $\overline{\mathbb{R}} \times \overline{\mathbb{R}}$.

has a highest weight, another a lowest weight). Hence $Q = (T_+ \oplus T_-) \otimes (T_+ \oplus T_-)$ splits into 4 summands. It can be shown that this is the desired decomposition:

- the space $H^2(\Pi_+) \otimes H^2(\Pi_+)$ consists of functions in $L^2(\mathbb{R}^2)$ continued holomorphically to the domain $\Pi_+ \times \Pi_+$; the representation $T_+ \otimes T_+$ in $H^2(\Pi_\pm) \subset L^2(\mathbb{R})$ is a direct sum of all highest weight representations of representation of $\mathrm{PSL}(2,\mathbb{R})$;
- $T_- \otimes T_-$ is a direct sum of all lowest weight representations;
- in $T_+ \oplus T_-$ we have the direct integral of all representations of the principal series (and the same integral in $T_- \otimes T_+$).

Remark. S.G. Gindikin [6] used a similar argument (restriction from a reducible representation of an overgroup) for multi-dimensional hyperboloids.

9. Splitting off the complementary series, see [35]

Consider the pseudo-orthogonal group $\mathrm{O}(1,q)$ consisting of operators preserving the following indefinite inner product in $\mathbb{R}^{1+q}$,

$$\langle x, y\rangle = -x_0y_0 + x_1y_1 + \cdots + x_qy_q.$$

We write elements of this group as block $(1+q)\times(1+q)$ matrices $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Denote by $\mathrm{SO}_0(1,q)$ its connected component, it consists of matrices satisfying two additional conditions $\det g = +1$, $a > 0$. Denote by S^{q-1} the unit sphere in $\mathbb{R}^n$. The group $\mathrm{O}(1,q)$ acts on S^{q-1} by conformal transformations $x \mapsto (a+xc)^{-1}(b+xd)$ (they preserve the sphere), the coefficient of a dilation equals to $(a+xc)^{-1}$.

For $\lambda \in \mathbb{C}$ we define a representation $T_\lambda = T_\lambda^q$ of $\mathrm{SO}_0(1,q)$ in a space of functions on S^{q-1} by

$$T_\lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix} f(x) = (a+xc)^{-(q-1)/2+\lambda} f\big((a+xc)^{-1}(b+xd)\big).$$

If $\lambda = i\sigma \in i\mathbb{R}$, then our representation is unitary in $L^2(S^{q-1})$, in this case $T_{i\sigma}$ is called a representation of the *unitary spherical principal series*, representations $T_{i\sigma}$ and $T_{-i\sigma}$ are equivalent (on these representations see, e.g., [40]). If $0 < s < (q-1)/2$, then T_s is unitary in the Hilbert space H_s with the inner product

$$\langle f_1, f_2\rangle_s = \int_{S^{q-1}} \int_{S^{q-1}} \frac{f_1(x_1)\,\overline{f_2(x_2)}\,dx_1\,dx_2}{\|x_1 - x_2|^{(q-1)/2-s}}.$$

More precisely, $\langle, \cdot, \cdot\rangle$ determines a positive definite Hermitian form on the space $C^\infty(S^{q-1})$ (this is not obvious), we get a pre-Hilbert space and consider its completion H_s. Such representations form the *spherical complementary series.* The spaces H_s are Sobolev spaces[5].

[5]In the standard notation, H_s is the Sobolev space $H^{-s,2}(S^{q-1})$. Notice that Sobolev spaces $H^{\sigma,2}(\cdot)$ are Hilbert spaces but inner product are defined not canonically. In our case the inner products are uniquely determined from the $\mathrm{SO}_0(1,q)$-invariance. For semisimple groups of rank > 1 complementary series are realized in functional Hilbert spaces that are not Sobolev spaces.

Consider a restrictions of $T_{i\sigma}$ to the subgroup $\mathrm{SO}_0(1, q-1)$. The group $\mathrm{SO}_0(1, q-1)$ has the following orbits on S^{q-1}: the equator $Eq = S^{q-2}$ defined by the equation $x_q = 0$, the upper hemisphere H_+ and the lower hemisphere H_-. The equator has zero measure and can be forgotten. Therefore $L^2(S^{q-1}) = L^2(H_+) \oplus L^2(H_-)$. On the other hand, hemispheres as homogeneous spaces are equivalent to $\mathrm{SO}_0(1, q-1)/\mathrm{SO}(q-1)$, i.e., to the $(q-1)$-dimensional Lobachevsky space. The decomposition of L^2 is a classical problem, in each summand $L^2(H_\pm)$ we get a multiplicity-free direct integral over the whole spherical principal series.

The restriction of a representation T_s of the complementary series is more interesting, it contains several summands of the complementary series and is equivalent to

$$\bigoplus\nolimits_{k:\, s-k>1/2} T^{q-1}_{s-k} \bigoplus L^2(H_+) \bigoplus L^2(H_-). \tag{6}$$

This spectrum was obtained by Ch. Boyer (1973), our purpose is to visualize summands of the complementary series.

According to the *trace theorems* Sobolev spaces of negative order can contain distributions supported by submanifolds. Denote by δ_{Eq} the delta-function of the equator, $\delta_{Eq} := \delta(x_q)$. Let φ be a smooth function on Eq.

$$\|\varphi\delta_{Eq}\|^2_s = \langle \varphi\delta_{Eq}, \varphi\delta_{Eq}\rangle_s = \int_{S^{q-2}} \int_{S^{q-2}} \frac{\varphi(y_1)\,\overline{\varphi(y_2)}\, dy_1\, dy_2}{\|y_1 - y_2|^{-(q-1)/2+s}}.$$

If $s > 1/2$ the integral converges and $\varphi\delta_{Eq} \in H_s$. The representation of $\mathrm{SO}_0(1, q)$ in the space of such functions is T^{q-1}_s.

Denote by $\frac{\partial}{\partial n}\delta_{Eq} := \delta'(x_q)$ the derivative of δ_{Eq} in the normal direction. Similar arguments show that for $s > 3/2$ and smooth ψ we have $\psi\frac{\partial}{\partial n}\delta_{Eq} \in H_s$. The space of functions of the form

$$\varphi\delta_{Eq} + \psi\frac{\partial}{\partial n}\delta_{Eq}$$

again is invariant. It contains the subspace T^{q-1}_s and we get the representation T^{q-1}_{s+1} in the quotient. Since our representation is unitary, T^{q-1}_{s+1} must be direct summand, etc...

Next, we consider the operator $J : H_s \mapsto L^2(S^{q-1})$ given by

$$Jf(x) = |x_q|^{(q-1)/2-s} f(x).$$

It intertwines restrictions of T_s and T_0, the kernel of J consists of distributions supported by Eq and the image is dense[6]. This gives us (6).

[6]More precisely, we consider this operator as an operator on smooth functions compactly supported outside Eq, take the closure Γ of its graph in $H_s \oplus L^2$, and examine projection operators $\Gamma \to H_s$, $\Gamma \to L^2$.

10. The modern status of the problem

We mention the following works:

a) G.I. Olshanski [36] (1990) proposed a way to split off highest weight and lowest weight representations.

b) The author in [21] (1986) proposed a way to split off complementary series (see proofs and further examples in [35], the paper [28] contains an example with separation of direct integrals of different complementary series).

c) S.G. Gindikin [6] (1993) and V.F. Molchanov [17] (1998) obtained a separation of spectra for multi-dimensional hyperboloids.

These old works had continuations, in particular there were many further works with splitting off highest weight representations (for more references, see [32]).

The recent paper [32] (2017) contains formulas for projection operators separating spectrum for L^2 on pseudo-unitary groups $\mathrm{U}(p,q)$. In this case we can consider separation into series (if we fix the number r of continuous parameters of a representation, $r \leqslant \min(p,q)$), subsubseries (if we fix all discrete parameters of a representation) and intermediate subseries. All these question are solvable. The solution was obtained by a summation of all characters corresponding to a given type of spectrum, certainly this way must be available for all semisimple Lie groups.

In [34] the problem was solved for L^2 on pseudo-Riemannian symmetric spaces $\mathrm{GL}(n,\mathbb{C})/\mathrm{GL}(n,\mathbb{R})$. The calculation is based on an explicit summation of spherical distributions. Apparently, this can be extended to all symmetric spaces of the form $G_\mathbb{C}/G_\mathbb{R}$, where $G_\mathbb{C}$ is a complex semisimple Lie group and $G_\mathbb{R}$ is a real form of $G_\mathbb{C}$ (on Plancherel formulas for such spaces, see [1, 9, 38]).

For arbitrary semisimple symmetric spaces the problem does not seem to be well formulated, see a discussion of multi-dimensional hyperboloids in [17].

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Yury Neretin
Math. Dept., University of Vienna
and
Institute for Theoretical and Experimental Physics (Moscow)
MechMath Dept., Moscow State University
Institute for Information Transmission Problems
URL: http://mat.univie.ac.at/∼neretin/
e-mail: yurii.neretin@univie.ac.at