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Star Exponentials in Star Product Algebra

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Dedicated to the memory of Syed Twareque Ali

Abstract. A star product is an associative product for certain function space on a manifold, which is given by deforming a usual multiplication of functions. The star product we consider is given on $\mathbb{C}^n$ in non-formal sense. In the star product algebra we consider exponential elements, which are called star exponentials. Using star exponentials we construct star functions, which are regarded as sections of star algebra bundle over a space of complex matrices. In this note we give a brief review on star products.

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1. Star products

The origin of star products can be traced back to Weyl [8], Wigner [9], Moyal [3], related to quantum mechanics. In 1970's, Bayen–Flato–Fronsdal–Licherowicz–Sternheimer [1] gave a concept of deformation quantization or star product, where formal star products are discussed. Formal means that the deformation is constructed in formal power series with respect to the deformation parameter. Many results are published with various applications by means of formal deformation quantization, which is a very general concept and its existence on any Poisson manifold is proved by M. Kontsevich ([2]).

A star product we consider in this note is a star product for certain functions on $\mathbb{R}^n$ or $\mathbb{C}^n$. The star product on $\mathbb{R}^n$ or $\mathbb{C}^n$ can be considered also in non-formal sense, for example we can consider non formal star products for polynomials. We introduce a family of star products which contains noncommutative star products, and also commutative star products. This note is on this product and its extension.

1.1. Definition of star products

First we introduce a biderivation acting on functions as follows.

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Biderivation. Let Λ be an arbitrary $n \times n$ complex matrix. We then consider a biderivation

$$\overleftarrow{\partial_w}\Lambda\overrightarrow{\partial_w} = (\overleftarrow{\partial_{w_1}}, \ldots, \overleftarrow{\partial_{w_n}})\Lambda(\overrightarrow{\partial_{w_1}}, \ldots, \overrightarrow{\partial_{w_n}}) = \sum_{k,l=1}^{n} \Lambda_{kl}\overleftarrow{\partial_{w_k}}\overrightarrow{\partial_{w_l}}$$

where $(w_1, \ldots, w_n)$ are the coordinates of $\mathbb{C}^n$. Here the over left (resp. right) arrow means that the derivative $\overleftarrow{\partial}$ (resp. $\overrightarrow{\partial}$) acts to the left (resp. right) function, namely,

$$f\overleftarrow{\partial_w}\Lambda\overrightarrow{\partial_w}g = f\left(\sum_{k,l=1}^{n} \Lambda_{kl}\overleftarrow{\partial_{w_k}}\overrightarrow{\partial_{w_l}}\right) g = \sum_{k,l=1}^{n} \Lambda_{kl}\, \partial_{w_k}\, f\, \partial_{w_l}\, g.$$

Since Λ is a constant matrix, we can easily calculate the power of the biderivation, for example

$$f(\overleftarrow{\partial_w}\Lambda\overrightarrow{\partial_w})^2 g = \sum_{k_1,k_2,l_1,l_2=1}^{n} \Lambda_{k_1l_1}\, \Lambda_{k_2l_2}\, \partial_{w_{k_1}}\, \partial_{w_{k_2}}\, f\, \partial_{w_{l_1}}\, \partial_{w_{l_1}}\, g.$$

Star product. Now for functions f, g we define a star product $f *_{_\Lambda} g$ by means of the power series of the above biderivation such that

Definition 1.

$$\begin{aligned} f *_{_\Lambda} g &= f \exp \tfrac{i\hbar}{2}\left(\overleftarrow{\partial_w}\Lambda\overrightarrow{\partial_w}\right)\, g = f\sum_{k=0}^{\infty} \tfrac{1}{k!}\left(\tfrac{i\hbar}{2}\right)^k \left(\overleftarrow{\partial_w}\Lambda\overrightarrow{\partial_w}\right)^k g \\ &= fg + \tfrac{i\hbar}{2} f\left(\overleftarrow{\partial_w}\Lambda\overrightarrow{\partial_w}\right) g + \cdots + \tfrac{1}{k!}\left(\tfrac{i\hbar}{2}\right)^k f\left(\overleftarrow{\partial_w}\Lambda\overrightarrow{\partial_w}\right)^k g + \cdots \end{aligned}$$

where $\hbar$ is a positive parameter.

Then we see easily

Theorem 2. *For an arbitrary Λ, the star product $*_{_\Lambda}$ is well defined on polynomials, and is associative.*

Remark 3.

(i) The star product $*_{_\Lambda}$ is a generalization of the well-known products in physics. For example suppose $n = 2m$ and if we put $\Lambda = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ (blockwise), then we have the Moyal product, and similarly we have the normal product for $\Lambda = \left(\begin{smallmatrix} 0 & 0 \\ 2 & 0 \end{smallmatrix}\right)$, and the anti-normal product for $\Lambda = \left(\begin{smallmatrix} 0 & -2 \\ 0 & 0 \end{smallmatrix}\right)$, respectively.

(ii) If Λ is a symmetric matrix, the star product $*_{_\Lambda}$ is commutative. Furthermore, if Λ is a zero matrix, then the star product is nothing but a usual commutative product.

1.2. Equivalence, Star product algebra bundle and flat connection

Equivalence. Let Λ be an arbitrary $n \times n$ complex matrix. Then $(\mathbb{C}[w], *_{_\Lambda})$ is an associative algebra where $\mathbb{C}[w]$ is the set of complex polynomials of the coordinate system $w = (w_1, w_2, \dots, w_n)$. The algebraic structure of $(\mathbb{C}[w], *_{\Lambda})$ depends only on the skewsymmetric part of Λ. Namely, let Λ_1, Λ_2 be $n \times n$ complex matrices with common skew-symmetric part. Then we have the decomposition

$$\Lambda_1 = \Lambda_- + K_1, \quad \Lambda_2 = \Lambda_- + K_2,$$

where Λ_- is a skew-symmetric matrix and K_1, K_2 are symmetric matrices. Then we have

Theorem 4. *The algebras* $(\mathbb{C}[u,v], *_{\Lambda_1})$ *and* $(\mathbb{C}[u,v], *_{\Lambda_2})$ *are isomorphic with an isomorphism* $I_{K_1}^{K_2} : (\mathbb{C}[u,v], *_{\Lambda_1}) \to (\mathbb{C}[u,v], *_{\Lambda_2})$ *given by the power series of the differential operator* $\partial_w(K_2 - K_1)\partial_w$ *such that*

$$I_{K_1}^{K_2}\,(f) = \exp\left(\tfrac{i\hbar}{4}\partial_w(K_2 - K_1)\partial_w\right)\,(f) = \sum_{n=0}^{\infty} \tfrac{1}{n!}\left(\tfrac{i\hbar}{4}\right)^n (\partial_w(K_2 - K_1)\partial_w)^n f$$

where $\partial_w(K_2 - K_1)\partial_w = \sum_{kl}(K_2 - K_1)_{kl}\partial_{w_k}\partial_{w_l}$.

For star products $*_{\Lambda_k}$, $k = 1, 2, 3$ with common skew-symmetric part of Λ_k, a direct calculation gives

Theorem 5. *The isomorphisms satisfy the following chain rule:*

(i) $I_{K_3}^{K_1} I_{K_2}^{K_3} I_{K_1}^{K_2} = Id$, (ii) $\left(I_{K_1}^{K_2}\right)^{-1} = I_{K_2}^{K_1}$

Star product algebra bundle and flat connection. Let us fix a skew-symmetric matrix Λ_- and consider a family of matrices $\{\Lambda = \Lambda_- + K\}$ with common skew-symmetric part Λ_- where K denotes its symmetric part. Then, by the above theorems we have a family of star products $\{*_{\Lambda}\}$ parameterized by $\{K\}$ whose elements are mutually isomorphic, and since $*_{\Lambda}$ depends only on the symmetric part K we write as $*_{\Lambda} = *_{K}$.

Here we regard this family of star products in the following way: we have an associative algebra $(\mathcal{P}, *)$ determined by Λ_- such that an each algebra $(\mathbb{C}[w], *_{K})$ of the family is regarded as a local expression of $(\mathcal{P}, *)$ at K. Each element $p \in \mathcal{P}$ has a polynomial expression at every K, which is denoted by $: p :_{K}$. Due to the previous theorem of the chain rules of $I_{K_1}^{K_2}$, we have a geometric picture: we have an algebra bundle over the space of symmetric matrices $\pi : \cup_K(\mathbb{C}[w], *_{K}) \to \mathcal{S} = \{K\}$ such that the fiber at K is the algebra $\pi^{-1}(K) = (\mathbb{C}[w], *_{K})$. The bundle has a flat connection ∇ and the element $p \in (\mathcal{P}, *)$ is regarded as a parallel section of the bundle and $: p :_{K}$ is the value at K.

This is a simple translation of the equivalence among the star product algebras. However, this picture plays an important role when we consider star exponentials and star functions below.

2. Star exponential

Now we consider general star product $*_{\Lambda}$, and consider exponential elements of polynomials in star product algebras.

Idea of definition. For a polynomial H of the star product algebra $(\mathbb{C}[w], *_{_\Lambda})$, we want to define a star exponential

$$e_{*_\Lambda}^{t\frac{H}{\mathrm{i}\hbar}} = \sum_n \frac{t^n}{n!}\left(\tfrac{H}{\mathrm{i}\hbar}\right)^n_{*_\Lambda}$$

where $\left(\frac{H}{\mathrm{i}\hbar}\right)^n_{*_\Lambda}$ is an nth power of $\frac{H}{\mathrm{i}\hbar}$ with respect to the star product $*_{_\Lambda}$. However, the expansion $\sum_n \frac{t^n}{n!}\left(\frac{H}{\mathrm{i}\hbar}\right)^n_{*_\Lambda}$ is not convergent in general, and then we consider a star exponential by means of a differential equation.

Definition 6. The star exponential $e_{*_\Lambda}^{t\frac{H}{\mathrm{i}\hbar}}$ is given as a solution of the differential equation

$$\tfrac{d}{dt}F_t = \tfrac{H}{\mathrm{i}\hbar} *_{_\Lambda} F_t, \quad F_0 = 1.$$

2.1. Star exponential of linear and quadratic polynomials

We are interested in the star exponentials of linear, and quadratic polynomials. For these, we can solve the differential equation explicitly.

Linear case. We denote a linear polynomial by $\sum_{j=1}^n a_j w_j = \langle \boldsymbol{a}, \boldsymbol{w}\rangle$, $a_j \in \mathbb{C}$. This case naive expansion $\sum_n \frac{t^n}{n!}\left(\frac{\langle \boldsymbol{a},\boldsymbol{w}\rangle}{\mathrm{i}\hbar}\right)^n_{*_\Lambda}$ is convergent. Actually we see directly that the nth power with respect to $*_{_\Lambda}$ is

$$\langle \boldsymbol{a}, \boldsymbol{w}\rangle^n_{*_\Lambda} = \sum_{k=0}^{[n/2]} \tfrac{1}{k!}\left(\tfrac{i\hbar}{4}\boldsymbol{a}\Lambda\boldsymbol{a}\right)^k \frac{n!}{(n-2k)!}\langle \boldsymbol{a}, \boldsymbol{w}\rangle^{n-2k}$$

where $\boldsymbol{a}\Lambda\boldsymbol{a} = \sum_{ij}\Lambda_{ij}a_i a_j$ and the expansion is convergent. Then we have

Proposition 7. *For* $\sum_j a_j w_j = \langle \boldsymbol{a}, \boldsymbol{w}\rangle$

$$e_{*_\Lambda}^{t\langle \boldsymbol{a},\boldsymbol{w}\rangle/(\mathrm{i}\hbar)} = e^{t^2\boldsymbol{a}\Lambda\boldsymbol{a}/(4\mathrm{i}\hbar)}e^{t\langle \boldsymbol{a},\boldsymbol{w}\rangle/(\mathrm{i}\hbar)} = e^{t^2\boldsymbol{a}K\boldsymbol{a}/(4\mathrm{i}\hbar)}e^{t\langle \boldsymbol{a},\boldsymbol{w}\rangle/(\mathrm{i}\hbar)}$$

where K is the symmetric part of Λ.

Thus the star exponentials are analytic and satisfy the exponential law with respect to the parameter t. By direct calculation we see

Proposition 8. *The star product of the star exponentials is convergent and it holds*

$$e_{*_\Lambda}^{\langle \boldsymbol{a},\boldsymbol{w}\rangle/(\mathrm{i}\hbar)} *_{_\Lambda} e_{*_\Lambda}^{\langle \boldsymbol{b},\boldsymbol{w}\rangle/(\mathrm{i}\hbar)} = e^{\boldsymbol{a}(\Lambda_-)\boldsymbol{b}/(2\mathrm{i}\hbar)}e_{*_\Lambda}^{\langle \boldsymbol{a}+\boldsymbol{b},\boldsymbol{w}\rangle/(\mathrm{i}\hbar)}.$$

Thus star exponentials of linear polynomials form a group.

For the linear case, the intertwiners are convergent. Namely, if we write the decomposition as $\Lambda = \Lambda_- + K_1$ we have

Proposition 9. *For any symmetric matrices K_1, K_2, the intertwiner*

$$I_{K_1}^{K_2} = \sum_{n=0}^{\infty}\tfrac{1}{n!}\left(\tfrac{\mathrm{i}\hbar}{4}\right)^n\left(\partial_w(K_2-K_1)\partial_w\right)^n$$

is convergent for a star exponential of linear polynomial and satisfies

$$I_{K_1}^{K_2}(e_{*_\Lambda}^{\langle \boldsymbol{a},\boldsymbol{w}\rangle/(\mathrm{i}\,\hbar)}) = e_{*_{\Lambda'}}^{\langle \boldsymbol{a},\boldsymbol{w}\rangle/(\mathrm{i}\,\hbar)}, \quad (\Lambda' = \Lambda_- + K_2).$$

Remark 10. By the above propositions, similarly as polynomial case, for a fixed Λ_- the family of groups $\{e_{*_K}^{\langle \boldsymbol{a},\boldsymbol{w}\rangle/(\mathrm{i}\,\hbar)}; a \in \mathbb{C}^n\}_{K\in\mathcal{S}}$ determines a group $\mathcal{G}$. Also we have a group bundle $\pi : \cup_K\{e_{*_K}^{\langle \boldsymbol{a},\boldsymbol{w}\rangle/(\mathrm{i}\,\hbar)}; a \in \mathbb{C}^n\} \to \mathcal{S}$ such that the each fiber is the group $\pi^{-1}(K) = \{e_{*_K}^{\langle \boldsymbol{a},\boldsymbol{w}\rangle/(\mathrm{i}\,\hbar)}; a \in \mathbb{C}^n\}$. And an element of $\mathcal{G}$ is regarded as a parallel section denoted by $e_*^{\langle \boldsymbol{a},\boldsymbol{w}\rangle/(\mathrm{i}\,\hbar)}$ of this bundle and a value at K is given by $: e_*^{\langle \boldsymbol{a},\boldsymbol{w}\rangle/(\mathrm{i}\,\hbar)} :_K = e_{*_K}^{\langle \boldsymbol{a},\boldsymbol{w}\rangle/(\mathrm{i}\,\hbar)} = e^{\boldsymbol{a}K\boldsymbol{a}/(4\mathrm{i}\hbar)+\langle \boldsymbol{a},\boldsymbol{w}\rangle/(\mathrm{i}\hbar)}$

Quadratic case. For simplicity of formula, we consider the case where Λ is a $2m \times 2m$ complex matrices with the skew symmetric part $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

Proposition 11. *For a quadratic polynomial $Q = \langle \boldsymbol{w}A, \boldsymbol{w}\rangle$ where A is a $2m \times 2m$ complex symmetric matrix, we have*

$$e_{*_\Lambda}^{t(Q/\mathrm{i}\hbar)} = \frac{2^m}{\sqrt{\det(I-\kappa+e^{-2t\alpha}(I+\kappa))}} e^{\frac{1}{\mathrm{i}\hbar}\langle \boldsymbol{w}\frac{1}{I-\kappa+e^{-2t\alpha}(I+\kappa)}(I-e^{-2t\alpha})J,\boldsymbol{w}\rangle}$$

where $\kappa = KJ$, $\alpha = AJ$ and K is the symmetric part of Λ.

Remark 12. The star exponentials of quadratic polynomials have branching, essential singularities, and also satisfy exponential law with respect to the parameter t whenever they are defined. From these singularities we are trying to derive relations for commutative or noncommutative algebras.

Proposition 13. *We have an explicit formula of the product of star exponentials of quadratic polynomials which contains a square root.*

$$e_{*_\Lambda}^{\langle \boldsymbol{w}A_1,\boldsymbol{w}\rangle/(\mathrm{i}\hbar)} *_\Lambda e_{*_\Lambda}^{\langle \boldsymbol{w}A_2,\boldsymbol{w}\rangle/(\mathrm{i}\hbar)} = \frac{1}{\sqrt{\det(1-\alpha(A_1,A_2))}} e^{\frac{1}{\mathrm{i}\hbar}\langle \boldsymbol{w}\frac{1}{1-\alpha(A_1,A_2))}A_3(A_1,A_2),\boldsymbol{w}\rangle}$$

where $\alpha(A_1, A_2)$, $A_3(A_1, A_2)$ are certain matrix-valued functions of A_1, A_2 which are explicitly written by means of Cayley transforms of A_1, A_2.

Hence the product is defined when $\det(1 - \alpha(A_1, A_2)) \neq 0$ *and associativity holds when $\{A\}$ are sufficiently small. Thus star exponentials of quadratic polynomials form a group-like object, or local group.*

For a quadratic case, since the intertwiner is a parallel transport of a section, we can obtain the intertwiner by solving a certain differential equation. If we write the decomposition as $\Lambda = \Lambda_- + K_1$ we have

Proposition 14. *For any symmetric matrix K_2, the intertwiner $I_{K_1}^{K_2}$ for a star exponential of quadratic polynomial is given as*

$$I_{K_1}^{K_2}(e_{*_\Lambda}^{\langle \boldsymbol{w}A,\boldsymbol{w}\rangle/(\mathrm{i}\,\hbar)}) = \frac{1}{\sqrt{\det(1-\beta(A)(K_2-K_1))}} e_{*_{\Lambda'}}^{\frac{1}{\mathrm{i}\hbar}\langle \boldsymbol{w}\frac{1}{1-\beta(A)(K_2-K_1)}\beta(A),\boldsymbol{w}\rangle}$$

where $\beta(A)$ is a certain matrix-valued function of A and $\Lambda' = \Lambda_- + K_2$.

Remark 15. By the above propositions, similarly as linear case, for a fixed Λ_- the family of group-like objects $\{e_{*_K}^{\langle \boldsymbol{w}A,\boldsymbol{w}\rangle/(\mathrm{i}\,\hbar)}; A \text{ symmetric}\}_{K\in\mathcal{S}}$ determines a group-like object $\mathcal{Q}$.

Also we have a group-like object bundle $\pi : \cup_K \{e_{*_K}^{\langle \boldsymbol{w}A,\boldsymbol{w}\rangle/(\mathrm{i}\hbar)}; A \text{symmetric}\} \to \mathcal{S}$ such that the each fiber is $\pi^{-1}(K) = \{e_{*_K}^{\langle \boldsymbol{w}A,\boldsymbol{w}\rangle/(\mathrm{i}\,\hbar)}; A \text{ symmetric}\}$. And an element of $\mathcal{Q}$ is regarded as a parallel section denoted by $e_*^{\langle \boldsymbol{w}A,\boldsymbol{w}\rangle/(\mathrm{i}\,\hbar)}$ of this bundle, and a value at K is given by : $e_*^{\langle \boldsymbol{w}A,\boldsymbol{w}\rangle/(i\,\hbar)} :_K = e_{*K}^{\langle \boldsymbol{w}A,\boldsymbol{w}\rangle/(\mathrm{i}\,\hbar)}$

2.2. Star functions

By the same way as in the ordinary exponential functions, we can obtain several noncommutative or commutative functions using star exponentials, which we call *star functions.* As is stated in the previous sections, these star functions are given as parallel sections $\mathcal{G}$ or $\mathcal{Q}$ of the group bundle or the group-like object bundle over $\mathcal{S}$, respectively. In this subsection we show some concrete examples of star functions. For more details see Omori–Maeda–Miyazaki–Yoshioka [4, 5].

2.2.1. Linear case. Here we show examples of the simplest case using star product of one variable. We consider functions $f(w), g(w)$ of one variable $w \in \mathbb{C}$ and consider a commutative star product $*_\tau$ with complex parameter τ such that

$$f(w) *_\tau g(w) = f(w) e^{\frac{\tau}{2} \overleftarrow{\partial}_w \overrightarrow{\partial}_w} g(w).$$

Applying the previous general formulas to the product $*_\tau$ gives

Proposition 16. *For a linear polynomial aw, $a \in \mathbb{C}$, the star exponential and the intertwiner satisfy*

$$\exp_{*_\tau} aw = \exp(aw + (\tau/4)a^2), \quad I_\tau^{\tau'}(\exp_{*_\tau} aw) = \exp_{*_{\tau'}} aw,$$

respectively.

Hence we have the space of parallel sections $\mathcal{G} = \{e_*^{aw}\}$ of the bundles of group over the parameter space $\mathbb{C} = \{\tau\}$.

Star Hermite function. Recall a naive expansion of star exponential for the linear case is convergent, namely

$$: \exp_*(\sqrt{2}tw) :_\tau = \sum_{n=0}^{\infty} : (\sqrt{2}w)_*^n :_\tau \frac{t^n}{n!}.$$

Note, that the explicit formula of star exponential evaluated at $\tau = -1$ gives the generating function of the Hermite polynomials $H_n(w)$, namely

$$: \exp_*(\sqrt{2}tw) :_{\tau=-1} = \exp\left(\sqrt{2}tw - \tfrac{1}{2}t^2\right) = \sum_{n=0}^{\infty} H_n(w) \tfrac{t^n}{n!}.$$

Then comparing the both expansions and we obtain

$$H_n(w) =: (\sqrt{2}w)_*^n :_{\tau=-1} .$$

We define star Hermite function (one-parameter deformation of $H_n(w)$) by using parallel sections

$$H_n(w,\tau) =: (\sqrt{2}\,w)^n_* :_\tau, \quad (n=0,1,2,\dots).$$

Then the evaluation of the parallel section $e_*^{\sqrt{2}tw}$ at τ gives a generating function of star Hermite functions, namely

$$:\exp_*(\sqrt{2}\,tw):_\tau = \sum_{n=0}^{\infty} H_n(w,\tau)\,\tfrac{t^n}{n!}.$$

Trivial identity $\frac{d}{dt}\exp_*(\sqrt{2}\,tw) = \sqrt{2}w{*}\exp_*(\sqrt{2}\,tw)$ evaluated at τ yields the identity

$$\tfrac{\tau}{\sqrt{2}}H_n'(w,\tau) + \sqrt{2}wH_n(w,\tau) = H_{n+1}(w,\tau), \quad (n=0,1,2,\dots)$$

for every $\tau \in \mathbb{C}$, and the exponential law

$$\exp_*(\sqrt{2}sw) * \exp_*(\sqrt{2}tw) = \exp_*(\sqrt{2}(s+t)w)$$

yields the identity

$$\sum_{k+l=n} \tfrac{n!}{k!l!} H_k(w,\tau) *_\tau H_l(w,\tau) = H_n(w,\tau).$$

Star theta function. We can express the Jacobi's theta functions by using parallel sections of star exponentials $\in \mathcal{G}$. The formula

$$:\exp_* n\ \mathrm{i}\ w :_\tau = \exp(n\ \mathrm{i}\ w - (\tau/4)n^2)$$

shows that for $\mathrm{Re}\,\tau > 0$, the star exponential $:\exp_* n\mathrm{i}\ w:_\tau$ is rapidly decreasing with respect to integer n. Then we can consider summations for τ such that $\mathrm{Re}\,\tau > 0$

$$:\sum_{n=-\infty}^{\infty} \exp_* 2n\mathrm{i}\ w :_\tau = \sum_{n=-\infty}^{\infty} \exp\left(2n\mathrm{i}\ w - \tau n^2\right) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2ni\ w}, \quad (q = e^{-\tau})$$

which is convergent and gives Jacobi's theta function $\theta_3(w,\tau)$. Then the infinite sums of parallel sections of $\mathcal{G}$ such as

$$\theta_{1_*}(w) = \tfrac{1}{\mathrm{i}} \sum_{n=-\infty}^{\infty} (-1)^n \exp_*(2n+1)\mathrm{i}\ w, \quad \theta_{2*}(w) = \sum_{n=-\infty}^{\infty} \exp_*(2n+1)\mathrm{i}\ w,$$

$$\theta_{3*}(w) = \sum_{n=-\infty}^{\infty} \exp_* 2ni\ w, \qquad \theta_{4*}(w) = \sum_{n=-\infty}^{\infty} (-1)^n \exp_* 2ni\ w$$

are called *star theta functions*. Actually the evaluation of $:\theta_{k*}(w):_\tau$ at τ with $\mathrm{Re}\,\tau > 0$ gives the Jacobi's theta function $\theta_k(w,\tau)$, $k=1,2,3,4$ respectively. The exponential law of star exponential yields trivial identities

$$\exp_* 2i\ w * \theta_{k*}(w) = \theta_{k*}(w) \qquad (k=2,3),$$

$$\exp_* 2i\ w * \theta_{k*}(w) = -\theta_{k*}(w) \quad (k=1,4).$$

Then using the evaluation formula : $\exp_* 2i\ w\ :_\tau = e^{-\tau} e^{2i\ w}$ and the product formula directly we see the above trivial identities are equivalent to the quasi periodicity

$$e^{2i\ w-\tau}\theta_k(w+i\ \tau)=\theta_k(w) \qquad (k=2,3),$$
$$e^{2i\ w-\tau}\theta_k(w+i\ \tau)=-\theta_k(w) \quad (k=1,4).$$

$*$-delta functions. Since the star exponential : $\exp_*(itw)\ :_\tau = \exp(itw-\frac{\tau}{4}t^2)$ is rapidly decreasing with respect to t when $\mathrm{Re}\,\tau>0$. Then the integral of star exponential evaluated at τ

$$:\int_{-\infty}^{\infty}\exp_*(it(w-a)_*)\,dt\ :_\tau = \int_{-\infty}^{\infty}\exp(it(w-a)-\tfrac{\tau}{4}t^2)dt$$

converges for any $a\in\mathbb{C}$. We put a star δ-function

$$\delta_*(w-a)=\int_{-\infty}^{\infty}\exp_*(it(w-a)_*)dt,$$

which has a meaning at τ with $\mathrm{Re}\,\tau>0$. It is easy to see for any parallel section of polynomials $p_*(w)\in\mathcal{P}$,

$$p_*(w)*\delta_*(w-a)=p(a)\delta_*(w-a),\ w*\delta_*(w)=0.$$

Using the Fourier transform we have

$$\theta_{1*}(w)=\tfrac{1}{2}\sum_{n=-\infty}^{\infty}(-1)^n\delta_*(w+\tfrac{\pi}{2}+n\pi),\quad \theta_{2*}(w)=\tfrac{1}{2}\sum_{n=-\infty}^{\infty}(-1)^n\delta_*(w+n\pi)$$
$$\theta_{3*}(w)=\tfrac{1}{2}\sum_{n=-\infty}^{\infty}\delta_*(w+n\pi),\qquad \theta_{4*}(w)=\tfrac{1}{2}\sum_{n=-\infty}^{\infty}\delta_*(w+\tfrac{\pi}{2}+n\pi).$$

Now, we consider the τ satisfying the condition $\mathrm{Re}\,\tau>0$. Then we calcultate the integral and obtain $\delta_*(w-a)=\frac{2\sqrt{\pi}}{\sqrt{\tau}}\exp\left(-\frac{1}{\tau}(w-a)^2\right)$ and we have

$$\begin{aligned}\theta_3(w,\tau)&=\tfrac{1}{2}\sum_{n=-\infty}^{\infty}\delta_*(w+n\pi)=\tfrac{\sqrt{\pi}}{\sqrt{\tau}}\exp\left(-\tfrac{1}{\tau}\right)\sum_{n=-\infty}^{\infty}\exp\left(-2n\tfrac{1}{\tau}w-\tfrac{1}{\tau}n^2\tau^2\right)\\&=\tfrac{\sqrt{\pi}}{\sqrt{\tau}}\exp\left(-\tfrac{1}{\tau}\right)\theta_{3*}(\tfrac{2\pi w}{i\tau},\tfrac{\pi^2}{\tau}).\end{aligned}$$

We also have similar identities for other $*$-theta functions by the similar way.

2.3. Star exponentials of quadratic polynomials

Different from linear case, star exponentials of quadratic polynomials have singularities which are moving, branching, and essential singularities.

Proposition 17. *For a quadratic polynomial $aw^2_{*_\tau}=aw^2+\frac{a\tau}{2}$, $a\in\mathbb{C}$, the star exponential and the intertwiner satisfy*

$$\exp_{*_\tau} aw^2_{*_\tau}=\frac{1}{\sqrt{1-a\tau}}\exp\left(\frac{1}{1-a\tau}aw^2\right), I_\tau^{\tau'}(\exp_{*_\tau} aw^2_{*_\tau})=\exp_{*_{\tau'}} aw^2_{*_{\tau'}},$$

respectively, when the star exponential and the intertwiner contain terms of square root then this equality includes a $\pm$ *umbiguity.*

We thus have the space of parallel sections $\mathcal{Q} = \{e_*^{aw_*^2}\}$ of the bundles of group-like objects over the parameter space $\mathbb{C} = \{\tau\}$, respectively. Hence the star exponentials of quadratic polynomials, that is, parallel sections of $\mathcal{Q}$ behave strangely, but are interesting. Here I will show several concrete examples for the simple case, for more examples and details, see the references already cited above.

2.3.1. "Double covering" group. Let us consider a parallel section $e_*^{tw_*^2} \in \mathcal{Q}$. This section has a singular point depending on the parameter τ, actually we see by the evaluation formula at τ that the star exponential

$$:\exp_* tw_*^2 :_\tau = \frac{1}{\sqrt{1-t\tau}} \exp\left(\frac{1}{1-t\tau}\ tw^2\right)$$

has a singularity at $t = 1/\tau$. Thus for small t, the section $e_*^{tw_*^2}$ satisfies the exponential law for every τ, i.e., $\{e_*^{tw_*^2}, t \in \mathbb{C}\}$ forms a local group. On the other hand, for each t, taking an appropriate path in $\tau \in \mathbb{C}$, the parallel transform $I_\tau^{\tau'}$ along the path gives $: e_*^{tw_*^2} :_\tau \mapsto : -e_*^{tw_*^2} :_\tau$. Hence the group-like object $e_*^{tw_*^2} \in \mathcal{Q}$ looks like a double covering group of $\mathbb{C}$.

This also appears when we consider multi-variable case $w = (w_1, \ldots, w_n)$. For example, if we assume that the number of variables is $n = 2$, and the skew-symmetric part is fixed such that $\Lambda_- = J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, then for a complex matrix $\Lambda = J + K$, (K symmetric), the associative algebra of polynomial parallel sections $\mathcal{P}$ includes the Lie algebra of $SL(2,\mathbb{C})$, which are given by quadratic polynomials. Exponentiating these quadratic elements one obtains a set of parallel sections $\widetilde{SL(2,\mathbb{C})} \subset \mathcal{Q}$ of the bundle of group-like objects over the space of all symmetric matrices $\{K\}$. The object $\widetilde{SL(2,\mathbb{C})}$ also behaves like a "double covering" group of $SL(2,\mathbb{C})$, which is called a blurred Lie group $\widetilde{SL(2,\mathbb{C})}$. (For more details, see [5]).

2.3.2. Vacuum. Consider a Weyl algebra W of two canonical generators u, v, namely $[v, u] = i\hbar$. An element $\varpi \in W$ satisfying the relation $\varpi\varpi = \varpi$ and $v\varpi = \varpi u = 0$ is called a vacuum. Vacuum plays an important role in quantum mechanics.

We can construct vacuums in the set of parallel sections $\mathcal{Q}$. For example we consider $n = 2$ and fix the skew-symmetric part of Λ to be J and we set $\Lambda = J+K$, (K symmetric). We write the generators of $\mathcal{P}$ as $w_1 = u, w_2 = v$. Then we see $[v, u]_* = v * u - u * v = i\hbar$. Then in the group-like parallel sections $\mathcal{Q}$ of star exponentials, we can construct an element $\varpi_{00} \in \mathcal{Q}$ having a property such that $\varpi_{00} * \varpi_{00} = \varpi_{00}$ and $v * \varpi_{00} = \varpi_{00} * u = 0$. We construct ϖ_{00} in the following way. We take a parallel section of star exponential such that $e_*^{2t\frac{u*v}{i\hbar}} \in \mathcal{Q}$. Then we have $\varpi_{00} = \lim_{t\to-\infty} e_*^{2t\frac{u*v}{i\hbar}}$. For example, for $K = \left(\begin{smallmatrix} 0 & \kappa \\ \kappa & \tau \end{smallmatrix}\right)$, we see

$$:\varpi_{00}:_K = \lim_{t\to-\infty} :e_*^{2t\frac{u*v}{i\hbar}}:_K = \frac{2}{1+\kappa}\exp\left(-\frac{1}{i\hbar(1+\kappa)}(2uv - \tfrac{\tau}{1+\kappa}u^2)\right).$$

Further using this vacuum we can construct generators of Clifford algebra in $\mathcal{Q}$, so we can construct Clifford algebra using parallel sections $\mathcal{Q}$ and $\mathcal{P}$. (See for details, H. Omori, Y. Maeda [6], T. Tomihisa, A. Yoshioka [7].)

Instead of taking a limit, we also obtain a vacuum by a contour integral of a parallel section of $\mathcal{Q}$ around singularities. (For details, see [5].)

2.3.3. Contour integral around singularites. An element of $\mathcal{Q}$, parallel section of star exponential of quadratic polynomials, has branching, essential singularities. Then it is natural to consider the derivation of meaningful relations from these singularities as residues of elements of $\mathcal{Q}$.

As an example, we can construct the Virasoro algebra by using residues. For details, see H. Omori, Y. Maeda, N. Miyazaki, A. Yoshioka [4].

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