Geometric Methods in Physics. XXXVI Workshop 2017
Trends in Mathematics, 139–146

States in Deformation Quantisation: Hopes and Difficulties

Jaromir Tosiek

Abstract. A notion of the state in classical and in quantum physics is discussed. Several classes of continuous linear functionals over different algebras of formal series are introduced. The condition of nonnegativity of functionals over the $*$ algebra is analysed.

Mathematics Subject Classification (2010). 81S30.

Keywords. Deformation quantisation, formal series.

1. Introduction

One of the most fundamental features of physics is that it proposes transformation of the real world into numbers. From this point of view one can say that physical reality consists of two main ingredients: the quantities which are measured called *observables* and the characteristics of a system under consideration known as a *state*. These two components are then combined to give *results*.

There exist several possible realisations of this scheme. In classical statistical physics observables are identified with smooth real functions f on a phase space $\mathcal{M}$, states are represented by densities of probability ϱ and results are mean values calculated as the functional action $\langle \varrho, f \rangle$.

At the quantum level in the Hilbert space model observables are self adjoint operators $\hat{f}$ acting in a space $\mathcal{H}$, states are density operators $\hat{\varrho}$ and results are traces $\mathrm{Tr}(\hat{\varrho} \cdot \hat{f})$. The reader interested in a systematic discussion of these postulates is encouraged to see [1].

However, our expectations in physics are bigger. We not only need a suitable mapping of reality into numbers but we would also like to be able to predict new phenomena. This process of prediction is based on logic and involves mathematical structures in which the sets of observables and of states can be equipped.

We start our contribution with a sketch of connections between a class of functions representing classical observables and functionals being densities of probability. Then we introduce formal series with respect to a deformation parameter λ, substitute a nonabelian $*$ product for the 'usual' multiplication of series and finally build linear functionals representing states. We do that in order to deal with quantum problems in frames of deformation quantisation formalism [2–4].

This formalism of deformation contains some difficulties. First of all, it usually involves infinite number of terms. Thus even elementary calculations for flat systems become rather complicated. Moreover, infinite sums appearing in some expressions may not be convergent.

But on the other hand deformation quantisation works well in every reference system. It thus seems to be a remedy for difficulties present in description of quantum phenomena in gravitational fields. In addition, from the conceptual point of view, it enlightens relationship between classical (undeformed) and quantum (deformed) physics.

2. Classical statistical mechanics

As we have already mentioned, in classical physics we assume that observables are smooth real functions defined on a phase space $\mathcal{M}$ of a system being a symplectic manifold. Thus all observables are elements of a wider structure: the ring of complex-valued smooth functions $(C^{\infty}(\mathcal{M}), +, \cdot)$ which form an algebra over the field of complex numbers $\mathbb{C}$. The constant function equal to $\mathbf{1}$ at every point of the manifold $\mathcal{M}$ is the identity element of this algebra.

A definition of convergence in the set $C^{\infty}(\mathcal{M})$ has been adapted from theory of generalised functions (see [5]). We say that the sequence $\{f_n\}_{n=1}^{\infty}$ is *convergent* to a function f_0, if on every compact subset of the manifold $\mathcal{M}$, $\dim \mathcal{M} = 2r$, every sequence of partial derivatives $\left\{ \frac{\partial^{m_1+m_2+\cdots+m_{2r}}}{\partial^{m_1} q^1 \ldots \partial^{m_{2r}} q^{2r}} f_n \right\}_{n=1}^{\infty}$ is uniformly convergent to the derivative $\frac{\partial^{m_1+m_2+\cdots+m_{2r}}}{\partial^{m_1} q^1 \ldots \partial^{m_{2r}} q^{2r}} f_0$.

States are represented by the functionals called densities of probabilities ϱ. They are elements of the space of linear continuous functionals $\mathcal{E}'(\mathcal{M})$ over the set of functions $C^{\infty}(\mathcal{M})$. Every density of probability ϱ is a real functional

$$\forall \ C^{\infty}(\mathcal{M}) \ni f = \overline{f} \Rightarrow \langle \varrho, f \rangle \in \mathbb{R}. \tag{1}$$

Moreover, the density ϱ has to be nonnegative

$$\forall \ C^{\infty}(\mathcal{M}) \ni f \ \ \langle \varrho, f \cdot \overline{f} \rangle \geq 0 \tag{2}$$

and normalised

$$\langle \varrho, \mathbf{1} \rangle = 1. \tag{3}$$

A sequence of densities $\{\varrho_n\}_{n=1}^{\infty}$ tends to a functional ϱ_0 if

$$\forall \ C^{\infty}(\mathcal{M}) \ni f \ \ \lim_{n \to \infty} \langle \varrho_n, f \rangle = \langle \varrho_0, f \rangle. \tag{4}$$

The postulate saying that every density of probability belongs to the space $\mathcal{E}'(\mathcal{M})$ implies that ϱ is of compact support. Many widely used distributions of probability **do not belong** to $\mathcal{E}'(\mathcal{M})$, e.g., the Gaussian distribution. We accept this limitation because the richness of mathematical properties of functionals from $\mathcal{E}'(\mathcal{M})$ provides a perfect opportunity to apply them in modeling of reality.

3. Physical background of formal series calculus

The fundamental difference between classical and quantum physics arises from the fact that observables and states in quantum mechanics depend on a special parameter – *the Planck constant* $\hbar$. Its crucial role is illustrated by the Heisenberg uncertainty principle for the position x and the canonically conjugated momentum p, which for series of independent measurements in classical physics is of trivial form

$$\Delta x \Delta p \geq 0$$

while in quantum mechanics one obtains

$$\Delta x \Delta p \geq \frac{\hbar}{2}.$$

By Δ we denote the mean square deviation.

For technical reasons in quantum calculations it is convenient to represent observables by their expansions in power series with respect to $\hbar$

$$f \sim \sum_{l=-z}^{\infty} \hbar^l f_l.$$

Notice that at this stage we accept only a finite set of negative powers of $\hbar$.

This series representation usually simplifies considerations but it is the source of two serious problems. The first one is that there is no one to one mapping between smooth functions and their respective power series. The second difficulty is the loss of convergence. Therefore to deal with power series we need to develop a special method known as the *formal series calculus*.

Since foundations of the formal series calculus are purely mathematical, instead of the Planck constant $\hbar$ we will use a parameter λ. We assume that this parameter is real and positive.

At the beginning we extend the field of complex numbers $\mathbb{C}$, namely we introduce a field of formal series of complex numbers

$$\mathbb{C}[\lambda^{-1}, \lambda]] \ni c[[\lambda]] = \sum_{l=-z}^{\infty} \lambda^l c_l, \quad \forall\, l \quad c_l \in \mathbb{C}, \quad z \in \mathcal{N}. \tag{5}$$

A sequence $\{(\sum_{l=-z}^{\infty} \lambda^l c_l)_n\}_{n=1}^{\infty}$ of elements from the field $\mathbb{C}[\lambda^{-1}, \lambda]]$ is convergent to an element $\sum_{l=-z}^{\infty} \lambda^l c_{l0}$, if for every index l the sequence $\{(c_l)_n\}_{n=1}^{\infty}$ of complex numbers approaches c_{l0}.

The set of formal series of smooth functions $C^\infty[\lambda^{-1},\lambda]](\mathcal{M})$ being a stage for constituting the formal series calculus, consists of elements which are of the form

$$\varphi[[\lambda]] = \sum_{l=-z}^{\infty} \lambda^l \varphi_l, \quad \forall\, l \quad \varphi_l \in C^\infty(\mathcal{M}), \quad z \in \mathcal{N}. \tag{6}$$

In $C^\infty[\lambda^{-1},\lambda]](\mathcal{M})$ we define multiplication by scalars from the field $\mathbb{C}[\lambda^{-1},\lambda]]$, complex conjugation and multiplication of series. All of these operations are natural extensions of their $C^\infty(\mathcal{M})$ counterparts. Hence we quote only a formula expressing the product of series.

Multiplication of formal series being a straightforward generalisation of multiplication of functions can be written as

$$\sum_{l=-z}^{\infty} \lambda^l \varphi_l \bullet \sum_{k=-s}^{\infty} \lambda^k \psi_k = \frac{1}{\lambda^{z+s}} \sum_{l=0}^{\infty} \lambda^l \sum_{k=0}^{l} \varphi_{k-z} \psi_{l-k-s}. \tag{7}$$

The set of formal series with the $\bullet$ product constitutes a commutative ring $(C^\infty[\lambda^{-1},\lambda]](\mathcal{M}),\bullet)$.

Moreover, we say that the sequence $\{(\sum_{l=-z}^{\infty} \lambda^l \varphi_l)_n\}_{n=1}^{\infty}$ tends to a series $\sum_{l=-z}^{\infty} \lambda^l \varphi_{l0}$, if for every l the sequence $\{(\varphi_l)_n\}_{n=1}^{\infty}$ is convergent to the function φ_{l0} in the sense of convergence in the space of functions $C^\infty(\mathcal{M})$.

A partial derivative of a series $\sum_{l=-z}^{\infty} \lambda^l \varphi_l$ is calculated as

$$\frac{\partial^{m_1+m_2+\cdots+m_{2r}}}{\partial^{m_1} q^1 \cdots \partial^{m_{2r}} q^{2r}} \sum_{l=-z}^{\infty} \lambda^l \varphi_l := \sum_{l=-z}^{\infty} \lambda^l \frac{\partial^{m_1+m_2+\cdots+m_{2r}}}{\partial^{m_1} q^1 \cdots \partial^{m_{2r}} q^{2r}} \varphi_l$$

and its integral equals

$$\int_{\mathcal{M}} \left(\sum_{l=-z}^{\infty} \lambda^l \varphi_l \right) \omega^r := \sum_{l=-z}^{\infty} \lambda^l \int_{\mathcal{M}} \varphi_l \, \omega^r$$

providing all functions φ_l are summable.

4. States over the commutative ring $(C^\infty[\lambda^{-1},\lambda]](\mathcal{M}),\bullet)$

Let us start from a generalisation of action of any element $T \in \mathcal{E}'(\mathcal{M})$ on a formal series $\sum_{k=-z}^{\infty} \lambda^k \varphi_k$ from $C^\infty[\lambda^{-1},\lambda]](\mathcal{M})$. This generalisation is of the form

$$\left\langle T, \sum_{k=-z}^{\infty} \lambda^k \varphi_k \right\rangle := \sum_{k=-z}^{\infty} \lambda^k \left\langle T, \varphi_k \right\rangle \in \mathbb{C}[\lambda^{-1},\lambda]]. \tag{8}$$

To be able to talk about the states the three properties have to be satisfied. Reality of functional T means that implication holds

$$\sum_{k=-z}^{\infty} \lambda^k \overline{\varphi_k} = \sum_{k=-z}^{\infty} \lambda^k \varphi_k \Longrightarrow \left\langle T, \sum_{k=-z}^{\infty} \lambda^k \varphi_k \right\rangle \in \mathbb{R}[\lambda^{-1},\lambda]].$$

Normalisation is natural. It requires only extension of multiplication of functionals by numbers to multiplication by series from $\mathbb{C}[\lambda^{-1}, \lambda]]$.

The notion of nonnegativity is in conflict with the idea of formal series because on one hand we deal with specific real numbers, on the other hand we avoid the question about summability. We propose the following (compromising) definition of nonnegativity.

A generalised function $T \in \mathcal{E}'(\mathcal{M})$ is *nonnegative* if for every admissible value of the parameter λ and every finite series $\sum_{l=-z}^{s} \lambda^l \varphi_l$

$$\left\langle T, \sum_{k_1=-z}^{s} \lambda^{k_1} \overline{\varphi}_{k_1} \bullet \sum_{k_2=-z}^{s} \lambda^{k_2} \varphi_{k_2} \right\rangle \geq 0. \tag{9}$$

This formulation is stronger than the one proposed by Waldmann [6].

It seems to be natural that linear functionals over the ring $(C^\infty[\lambda^{-1},\lambda]](\mathcal{M}),\bullet)$ also may depend on λ. Let us first consider formal series of generalised functions of the form $\sum_{l=-s}^{\infty} \lambda^l T_l$. Their functional action is of the form

$$\left\langle \sum_{l=-s}^{\infty} \lambda^l T_l, \sum_{k=-z}^{\infty} \lambda^k \varphi_k \right\rangle := \frac{1}{\lambda^{s+z}} \sum_{u=0}^{\infty} \lambda^u \sum_{l=0}^{u} \left\langle T_{l-s}, \varphi_{u-l-z} \right\rangle. \tag{10}$$

It is required that all supports are contained in a common compact set. Notions of reality, nonnegativity and normalisation condition can be easily adapted to them.

Since we need the formal series calculus to deal with quantum problems, let us consider another set of formal series of functionals.

For systems represented on the phase space $\mathbb{R}^{2r}$ we know that quantum states are represented by the Wigner functions which may contain arbitrary negative powers of λ. Thus it seems to be natural that formal series of generalised functions

$$\sum_{k=1}^{\infty} \lambda^{-k} T_{-k} + \sum_{k=0}^{\infty} \lambda^k T_k$$

should be considered. Unfortunately, such extension is not possible because the functional action

$$\left\langle \sum_{k=1}^{\infty} \lambda^{-k} T_{-k} + \sum_{k=0}^{\infty} \lambda^k T_k, \sum_{l=-z}^{\infty} \lambda^l \varphi_l \right\rangle$$

is not well defined. This observation is probably the weakest point of proposed calculus.

5. States over the algebra $(C^\infty[\lambda^{-1}, \lambda]](\mathcal{M}), *)$

One of the consequences of the Heisenberg uncertainty relation is the fact that the product of quantum observables is noncommutative. Therefore to deal with quantum problems we need another method of multiplication of formal series. This is the so-called $*$ product. Its general form is

$$\varphi * \psi := \sum_{k=0}^{\infty} \lambda^k B_k(\varphi, \psi), \quad \forall\, k \ \ B_k(\varphi,\psi) \in C^{\infty}(\mathcal{M}). \tag{11}$$

We omit here the list of axioms imposed on $\mathbb{C}[\lambda^{-1},\lambda]]$ – bilinear operators $B_k(\cdot,\cdot)$. This information can be found, e.g., in [4, 7, 8]. An extension of the $*$ product on formal series of functions is straightforward. The space of formal series equipped with the $*$ multiplication constitutes an algebra denoted as $(C^{\infty}[\lambda^{-1},\lambda]](\mathcal{M}),*)$.

The trace in algebra $(C^{\infty}[\lambda^{-1},\lambda]](\mathcal{M}),*)$ is of the form

$$\mathrm{Tr}\left(\sum_{k=-z}^{\infty}\lambda^k\varphi_k\right) := \frac{1}{\lambda^r}\int_{\mathcal{M}}\left(\sum_{k=-z}^{\infty}\lambda^k\varphi_k\right)\bullet t[[\lambda]]\,\omega^r,$$

where the series $t[[\lambda]] = \sum_{k=0}^{\infty}\lambda^k t_k$ is called *trace density*.

Since our goal is to introduce quantum states, i.e., some linear continuous functionals over the algebra $(C^{\infty}[\lambda^{-1},\lambda]](\mathcal{M}),*)$, following Schwartz [5] we consider first functionals which are of the integral form.

$$\mathbb{C}[\lambda^{-1},\lambda]] \ni \langle\psi,\varphi\rangle_* := \frac{1}{\lambda^r}\int_{\mathcal{M}}(\psi*\varphi)\bullet t[[\lambda]]\,\omega^r.$$

Notice that in general $\langle\psi,\varphi\rangle_* \neq \langle\psi,\varphi\rangle$.

However one can see that this new functional calculus is equivalent to the standard theory of generalised functions with an identification

$$\psi \sim T_{\psi}[[\lambda]] = \frac{1}{\lambda^r}t[[\lambda]]\bullet\sum_{l=0}^{\infty}\lambda^l T_{\psi\, l} \in \mathcal{E}'[\lambda^{-1},\lambda]](\mathcal{M}), \text{ i.e.,}$$

$$\forall\,\varphi\in C^{\infty}(\mathcal{M}) \ \ \langle\psi,\varphi\rangle_* = \langle T_{\psi}[[\lambda]],\varphi\rangle.$$

Let us see what might be the meaning of states in terms of the $*$ formal series calculus.

Reality of a series $\sum_{l=-s}^{\infty}\lambda^l T_l$ means that if

$$\sum_{k=-z}^{\infty}\lambda^k\overline{\varphi_k} = \sum_{k=-z}^{\infty}\lambda^k\varphi_k$$

then there is

$$\left\langle\sum_{l=-s}^{\infty}\lambda^l T_l, \sum_{k=-z}^{\infty}\lambda^k\varphi_k\right\rangle_* = \overline{\left\langle\sum_{l=-s}^{\infty}\lambda^l T_l, \sum_{k=-z}^{\infty}\lambda^k\varphi_k\right\rangle_*}, \tag{12}$$

To discuss nonnegativity we need the notion of nonnegativity of a formal series of real numbers.

A formal series $\sum_{l=-z}^{\infty}\lambda^l c_l, \ \forall\, l\ c_l\in\mathbb{R}$ of real numbers is *nonnegative* if

$$\forall\,\lambda>0\ \exists\, k\in\mathcal{N}\ \forall\, m>k\ \ \sum_{l=-z}^{m}\lambda^l c_l \geq 0.$$

It is disappointing that we again have to refer to values of sums but at this moment we have no idea how to introduce the notion of nonnegativity for formal series without a reference to numbers.

Applying this suggestion we say that the series $\sum_{l=-s}^{\infty} \lambda^l T_l$ is nonnegative if for every formal series of functions $\sum_{m=-z}^{\infty} \lambda^m \varphi_m$ the inequality

$$\left\langle \sum_{l=-s}^{\infty} \lambda^l T_l, \sum_{m_1=-z}^{\infty} \lambda^{m_1} \overline{\varphi}_{m_1} * \sum_{m_2=-z}^{\infty} \lambda^{m_2} \varphi_{m_2} \right\rangle_* \geq 0$$

holds.

Finally the normalisation condition states that

$$\langle \sum_{l=-s}^{\infty} \lambda^l T_l, \mathbf{1} \rangle_* = 1.$$

What is amazing when we test this list of properties for the most popular example of the $*$ product, i.e., the Moyal product at $\mathbb{R}^2$ [9, 10]

$$\varphi *_M \psi := \sum_{n_1,n_2=0}^{\infty} \frac{1}{n_1! n_2!} \left(-\frac{i\lambda}{2}\right)^{n_1} \left(\frac{i\lambda}{2}\right)^{n_2} \frac{\partial^{n_1+n_2}\varphi}{\partial p^{n_1} \partial q^{n_2}} \frac{\partial^{n_1+n_2}\psi}{\partial q^{n_1} \partial p^{n_2}}$$

we arrive at shocking conclusion that generalised functions with compact supports cannot be positive! This observation probably remains true for any local $*$ product. Therefore we deduce that states over formal series cannot be built in a way analogous to classical statistical physics.

6. Conclusions

As we can see, it is extremely difficult to introduce a coherent formal series calculus admitting quantum states. Two crucial facts – impossibility of building formal series of functionals with arbitrary large negative powers of λ and necessity of dealing with functionals with noncompact supports question whether formal series calculus can be successfully incorporated in quantum physics.

Thus the best solution would be to apply convergent expressions. Unfortunately, realisation of such a postulate requires a strict quantisation method which has not been formulated yet.

On the other hand the formal series are frequently useful. Thus at this moment we suggest a compromise – let us use them but simultaneously let us watch if calculations make sense.

References

[1] I. Bengtsson and K. Zyczkowski, *Geometry of Quantum States*, Cambridge University Press, Cambridge 2006.

[2] C.K. Zachos, D.B. Fairlie, T.L. Curtright (Eds.), *Quantum Mechanics in Phase Space*, World Scientific Series in 20th Century Physics, vol. 34, World Scientific, Singapore 2005.

[3] Y.S. Kim, M.E. Noz, *Phase Space Picture of Quantum Mechanics*, World Scientific, Singapore 1991.

[4] S. Waldmann, *Poisson-Geometrie und Deformationsquantisierung*, Springer-Verlag, Berlin 2007 (in German).

[5] L. Schwartz, *Méthodes mathématiques pour les sciences physiques*, Hermann, Paris 1965 (in French).

[6] S. Waldmann, *Rev. Math. Phys.* **17**, 15 (2005).

[7] B. Fedosov, *J. Diff. Geom.* **40**, 213 (1994).

[8] B. Fedosov, *Deformation Quantization and Index Theory*, Akademie Verlag, Berlin 1996.

[9] H.J. Groenewold, *Physica* **12**, 405 (1946).

[10] J.F. Plebański, M. Przanowski and J. Tosiek, *Acta Phys. Pol. B* **27**, 1961 (1996).

Jaromir Tosiek
Institute of Physics
Łódź University of Technology
Wólczanska 219
PL-90-924 Łódź, Poland
e-mail: tosiek@p.lodz.pl