Thrice Critical Case in Singularly Perturbed Control Problems

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Abstract The aim of the paper is to describe the special critical case in the theory of singularly perturbed optimal control problems and to give an example which is typical for slow/fast systems. The theory has traditionally dealt only with perturbation problems near normally hyperbolic manifold of singularities and this manifold is supposed to be isolated. We reduce the original singularly perturbed problem to a regularized one such that the existence of slow integral manifolds can be established by means of the standard theory.

1 Introduction

Consider singularly perturbed differential systems of the type

$$\frac{dx}{dt} = f(x, y, t, \varepsilon), \quad \varepsilon \frac{dy}{dt} = g(x, y, t, \varepsilon), \tag{1}$$

where x and y are vectors, and ε is a small positive parameter.

Such systems play an important role as mathematical models of numerous nonlinear phenomena in different fields; see, e.g., [3, 5].

A usual approach in the qualitative study of (1) is to consider first the so called degenerate system $dx/dt = f(x, y, t, 0), 0 = g(x, y, t, 0)$ and then to draw conclusions for the qualitative behavior of the full system (1) for sufficiently small ε. In order to recall a basic result of the geometric theory of singularly perturbed systems, we introduce the following notation and assumptions for sufficiently small positive ε_0, $0 \le \varepsilon \le \varepsilon_0$:

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(A_1) the functions f and g are sufficiently smooth and uniformly bounded together with all their derivatives;

(A_2) there are some region $\mathbf{G} \in R^m$ and a function $h(x, t, \varepsilon)$ of the same smoothness as g such that $g(x, h(x, t), t, 0) \equiv 0$, for all $(x, t) \in \mathbf{G} \times R$;

(A_3) the spectrum of the Jacobian matrix $B(x, t) = g_y(x, h(x, t), t, 0)$ is uniformly separated from the imaginary axis for all $(x, t) \in \mathbf{G} \times R$, i.e., the eigenvalues $\lambda_i(x, t)$, $i = 1, \ldots, n$, of the matrix $B(x, t)$ satisfy the inequality $|Re\lambda_i(x, t)| \geq \gamma$, for some positive number γ.

Then the following result is valid (see, e.g., [6, 8]):

Proposition 1 *Under the assumptions* $(A_1) - (A_3)$ *there is a sufficiently small positive* ε_1, $\varepsilon_1 \leq \varepsilon_0$, *such that for* $\varepsilon \in \overline{I_1}$ *system* (1) *has a smooth integral manifold* $\boldsymbol{M}_\varepsilon$ *(slow integral manifold) with the representation*

$$\boldsymbol{M}_\varepsilon := \{(x, y, t) \in R^{n+m+1} : y = \psi(x, t, \varepsilon),\ (x, t) \in \boldsymbol{G} \times R\},$$

and with the asymptotic expansion $\psi(x, t, \varepsilon) = h(x, t) + \varepsilon\psi_1(x, t) + \cdots$.

The motion on this manifold is described by the *slow* differential equation $\dot{x} = f(x, \psi(x, t, \varepsilon), t, \varepsilon)$.

Remark 2 The global boundedness assumption in (A_1) with respect to (x, y) can be relaxed by modifying f and g outside some bounded region of $R^n \times R^m$.

Remark 3 In applications, it is usually assumed that the spectrum of the Jacobian matrix $g_y(x, h(x, t), t, 0)$ is located in the left half plane. Under this additional hypothesis, the manifold $\mathbf{M}_\varepsilon$ is exponentially attracting for $\varepsilon \in I_1$.

The case when assumption (A_3) is violated is called critical. We distinguish three subcases:

(i) The Jacobian matrix $g_y(x, y, t, 0)$ is singular on some subspace of $R^m \times R^n \times R$. In that case, system (1) is referred to as a singular singularly perturbed system; see [1]. This subcase has been treated in [1–3, 5].

(ii) The Jacobian matrix $g_y(x, y, t, 0)$ has eigenvalues on the imaginary axis with nonvanishing imaginary parts. A similar case has been investigated in [3, 5, 7].

(iii) The Jacobian matrix $g_y(x, y, t, 0)$ is singular on the set $\mathbf{M}_0 := \{(x, y, t) \in R^m \times R^n \times R : y = h(x, t), (x, t) \in \mathbf{G} \times R\}$. In that case, $y = h(x, t)$ is generically an isolated root of $g = 0$ but not a simple one.

Other critical cases were considered, for example, in [3–5].

The critical case (i) was considered as applied to the high-gain control problem, the case (ii) was considered as applied to the manipulator control, and the case (iii) was considered as applied to the partially cheap control problem; see, for example, [3, 5]. It is not inconceivable that combinations of other pairs of critical cases and even triple critical cases are of interest as well and possibly they will be considered later.

2 Critical Case

Consider the control system $\varepsilon \dot{x} = A(t, \varepsilon)x + \varepsilon B(t, \varepsilon)u$, $x \in R^{n+m}$, $x(0) = x_0$, with the cost functional

$$J = \frac{1}{2}x^T(1)Fx(1) + \frac{1}{2}\int_0^1 (x^T(t)Q(t)x(t) + \varepsilon u^T(t)R(t)u(t))dt,$$

where A, F_1, and Q are $(n \times n)$-matrices, and B is a $(n \times m)$-matrix, and R is a $(m \times m)$-matrix. Suppose that all these matrices have the following asymptotic presentations with respect to ε:

$$A(t, \varepsilon) = \sum_{j \geq 0} \varepsilon^j Aj(t), \quad B(t, \varepsilon) = \sum_{j \geq 0} \varepsilon^j B_j(t), \quad Q(t, \varepsilon) = \sum_{j \geq 0} \varepsilon^j Q_j(t),$$
$$R(t, \varepsilon) = \sum_{j \geq 0} \varepsilon^j R_j(t), \quad F(\varepsilon) = \sum_{j \geq 0} \varepsilon^j F_j,$$

with matrix coefficients smooth on t, for $t \in [0, 1]$.

The solution to this problem is the optimal linear feedback control law

$$u = -\varepsilon^{-1}R^{-1}B^T P(t, \varepsilon)x,$$

where P satisfies the differential matrix Riccati equation

$$\varepsilon \dot{P} = -PA - A^T P + PSP - \varepsilon Q, \quad P(1, \varepsilon) = F. \tag{2}$$

Setting $\varepsilon = 0$, we obtain from (2) the matrix algebraic equation $-MA_0 - A_0^T M + MS_0 M - Q_0 = 0$, where $S_0 = B_0 R_0^{-1} B_0^T$. For systems with low energy dissipation the matrices S_0 and Q_0 are equal to zero and the main role plays the linear operator $\mathbf{L}X = XA_0 + A_0^T X$. For this class of systems the eigenvalues of A_0 are pure imaginary and the spectrum of the linear operator $\mathbf{L}$ has a nontrivial kernel, since sums $(\lambda_i(t) + \lambda_j(t))$, $i, j = 1, \ldots, n$, form its spectrum. This means that the Eq. (2) is singular singularly perturbed. Thus, the dimension of the slow integral manifold of (2) is greater than zero and the problem under consideration is critical in this sense. Moreover, under taking into account that zero eigenvalues are multiple and all other, nonzero eigenvalues of $\mathbf{L}$, are pure imaginary, it is possible to say that this problem is thrice critical.

3 Example

Let

$$A = \begin{pmatrix} -\varepsilon & 1 \\ -1 & -\varepsilon \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad R = (1), \quad Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$S = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix},$$

and consider the corresponding differential system

$$\begin{aligned} \varepsilon \dot{p}_1 &= 2p_2 + 2\varepsilon p_1 + p_2^2 - \varepsilon, \\ \varepsilon \dot{p}_2 &= 2\varepsilon p_2 - p_1 + p_3 + p_2 p_3, \\ \varepsilon \dot{p}_3 &= -2p_2 + 2\varepsilon p_3 + p_3^2. \end{aligned}$$

First, we need to separate it into a slow and a fast subsystem. At first glance, all three equations are singularly perturbed. However, setting $\varepsilon = 0$, we obtain $p_1 = p_2 = p_3 = 0$, and we should consider the matrix of leading terms on the right hand side of the system, which has the form

$$\begin{pmatrix} 0 & 2 & 0 \\ -1 & 0 & 1 \\ 0 & -2 & 0 \end{pmatrix}.$$

Obviously, this matrix has a zero eigenvalue and two pure imaginary eigenvalues, i.e., the problem under consideration is twice critical. Moreover, the trivial solution is multiple. This means that we have thrice critical case.

Let $\varepsilon = \mu^2$. Introducing the new variables $p_1 = \mu^2 q_1 + \mu$, $p_2 = \mu^2 q_2 + \mu^2/2$, and $p_3 = \mu^2 q_3 + \mu$, and then $s = q_1 + q_3$, we obtain the differential system

$$\begin{aligned} \mu \dot{s} &= 2q_3 + \mu q_2 + 2\mu s + \mu q_2^2 + \mu q_3^2 + 4 + \mu/4, \\ \mu^2 \dot{q}_2 &= -s + 2\mu^2 q_2 + 2q_3 + \mu q_2 + \mu^2 q_2 q_3 + \mu/2 + \mu^2, \\ \mu^2 \dot{q}_3 &= -2q_2 + 2\mu q_3 + 2\mu^2 q_3 + \mu^2 q_3^2 + 2\mu, \end{aligned}$$

with the slow variable s and two fast variables $q_2, \ q_3$.

The last system possesses one-dimensional slow invariant manifold which is weakly attractive with respect to the argument $1 - t$ because the main matrix of the fast subsystem has eigenvalues $3\mu/2 \pm i\sqrt{2 - \mu^2/4}$.

Thus, the dimension of the system of Riccati differential equations can be reduced from three to one. Let us construct the slow integral manifold using the fact that it can be asymptotically expanded in powers of the small parameter. Setting $q_2 = \varphi(s, \mu) = \mu\varphi_1(s) + \mu^2 \ldots$, and $q_3 = \psi(s, \mu) = \psi_0(s) + \mu\psi_1(s) + \mu^2 \ldots$, we obtain $\psi_0(s) = s/2$, $\varphi_1(s) = s/2$, and $\psi_1(s) = -1/4$. Thus, we obtain the slow invariant manifold $q_2 = \mu s/2 + O(\mu^2)$, $q_3 = s/2 - \mu/4 + O(\mu^2)$, with the equation on the integral manifold $\mu \dot{s} = s + 2\mu s + \mu s^2/4 + O(\mu^2)$.

Numerical experiments demonstrate the closeness of solutions of the original system and the system on the slow invariant manifolds.

4 Conclusion

The slow integral manifolds for the matrix Riccati equation of linear-quadratic control problem are constructed and it is shown that the method of integral manifolds allows us to reduce the dimension of control problems.

References

1. Z. Gu, N.N. Nefedov, R.E. O'Malley, On singular singularly perturbed initial values problems. SIAM J. Appl. Math. **49**, 1–25 (1989)
2. L.I. Kononenko, V.A. Sobolev, Asymptotic expansion of slow integral manifolds. Sib. Math. J. **35**, 1119–1132 (1994)
3. M.P. Mortell, R.E. O'Malley, A. Pokrovskii, V.A. Sobolev, *Singular Perturbation and Hysteresis* (SIAM, Philadelphia, 2005)
4. E. Shchepakina, Black swans and canards in self–ignition problem. Nonlinear Anal.: Real World Appl. **4**, 45–50 (2003)
5. E. Shchepakina, V.A. Sobolev, M.P. Mortell, *Singular Perturbations: Introduction to System Order Reduction Methods with Applications*. Lecture Notes in Mathematics (Springer, Berlin, 2014)
6. V.A. Sobolev, Integral manifolds and decomposition of singularly perturbed systems. Syst. Control Lett. **5**, 169–179 (1984)
7. V.A. Sobolev, V.V. Strygin, Permissibility of changing over to precession equations of gyroscopic systems. Mech. Solids **5**, 7–13 (1978)
8. V.V. Strygin, V.A. Sobolev, Effect of geometric and kinetic parameters and energy dissipation on orientation stability of satellites with double spin. Cosm. Res. **14**, 331–335 (1976)