Chapter 4
Fractional Differential Equations

4.1 The Existence and Uniqueness Theorem for Initial Value Problems

Definition 1 Let be the fractional differential equation (FDE)

$$(D^{\alpha}_{a_+} y)(t) = f[t, y(t)], \quad \alpha > 0, \quad t > a,$$

with the conditions:

$$(D^{\alpha-k}_{a+} y)(a+) = b_k, \quad k = 1, \ldots, n,$$

called also Riemann–Liouville FDE.

Definition 2 Let the FDE

$$(D^{\alpha}_{a+} y)(t) = f[t, y(t)], \quad \alpha > 0, \quad t > a,$$

with the initial conditions:

$$(D^k y)(0) = b_k, \quad k = 0, 1, \ldots, n-1,$$

called also Caputo FDE.

Lemma 1 *Let $y(t)$ be a function with continuous derivative in the interval $I_h(0) = [0, h]$ with values in $[y_0 - \eta, y_0 + \eta]$, then $y(t)$ satisfies the Caputo type*

$$D^{\alpha} y(t) = f(t, y(t)), \quad 0 < \alpha \leq 1, \qquad t > 0,$$

$$y(0) = y_0,$$

C. Milici et al., *Introduction to Fractional Differential Equations*, Nonlinear Systems and Complexity 25, https://doi.org/10.1007/978-3-030-00895-6_4

if and only if it satisfies the Voltera[1] *integral,*

$$y(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} f(u, y(u))\, du.$$

Proof Let $L[y(t] = Y$ be the LT of $y(t)$. We have

$$s^\alpha Y - s^{\alpha-1} y_0 = L[f(t, y(t))],$$

$$Y = \frac{y_0}{s} + \frac{1}{s^\alpha} L[f(t, y(t))],$$

$$y(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} f(u, y(u))\, du.$$

Definition 3 (Chebyshev[2] Norm) The Chebyshev norm on a set S is:

$$\|f\|_\infty = \sup\{|f(x)| : x \in S\},$$

where Supremum (sup) denotes the supremum.

Lemma 2 (The Weierstrass Test) *Suppose that* $\{f_n(t)\}$ *is a sequence of real functions defined on a set* A*, and there is a sequence of positive numbers* $\{R_n\}$ *satisfying:*

$$\forall n > 1, \quad \forall t \in A, \qquad |f_n(t)| \le R_n, \qquad \sum_{n=1}^{\infty} R_n < \infty.$$

Then the series $\sum_{n=1}^{\infty} f_n(t)$ *is convergent.*

Theorem 1 (Existence and Uniqueness for the Caputo Problem) *Let a Caputo FDE be*

$$D^\alpha y(t) = f(t, y(t)), \qquad 0 < \alpha \le 1, \qquad t > 0,$$

with the initial condition:

$$y(0) = y_0.$$

We consider the domain

$$D = [0, \eta] \times [y_0 - \eta, y_0 + \eta],$$

[1] V. Voltera (1860–1940).
[2] P.L. Chebyshev (1821–1894).

on which f satisfies:

- $f(t, y)$ *is continuous,*
- $|f(t, y)| < M$, *where* $M = \max_{(t,y)\in D} |f(t, y)|$, *and Maximum* (max) *denotes the maximum function*
- $f(t, y)$ *satisfy in D the Lipschitz*[3] *condition in y if there is a constant K such that:*

$$|f(t, y_2) - f(t, y_1)| \leq K|y_2 - y_1|.$$

Then it exists $\delta > 0$ *and a function* $y(t) \in C[0, \eta]$ *unique for*

$$\delta = \min\left\{\eta, \left(\frac{\eta\Gamma(\alpha+1)}{M}\right)^{1/\alpha}\right\},$$

where Minimum (min) *denotes the minimum function.*

Proof We consider the Voltera integral (see Lemma 1)

$$y(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} f(u, y(u))\, du,$$

and successive approximations:

$$y_n(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} f(u, y_{n-1}(u))\, du.$$

Using the method of successive approximations, on the basis of the Weierstrass test we prove the existence and the uniqueness of the solution of Caputo FDE.

For the sequence $\{y_n(t)\}$, we can prove that:

(i) the sequence $\{y_n(t)\}$ is well defined,
(ii) the sequence is uniformly continuous,
(iii) and its limit $y(t)$ is unique.

Proof

(i) We will use the induction method. In the case $n = 0$ it is obvious.

If $n = 1$, then we have:

$$|y_1(t) - y_0| = \left|\frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-i} f(u, y_0)\, du\right| < \left|\frac{M}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1}\, du\right|$$
$$= \left|\frac{Mt^\alpha}{\Gamma(\alpha+1)}\right| \leq \left|\frac{M\delta^\alpha}{\Gamma(\alpha+1)}\right| < \eta.$$

[3] R.O.S. Lipschitz (1832–1903).

If we assume

$$|y_{n-1} - y_0| \leq \eta,$$

then it follows that:

$$|y_n - y_0| = \left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1} f(u, y_{n-1})du\right| < \left|\frac{M}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}\,du\right|$$
$$= \left|\frac{Mt^\alpha}{\Gamma(\alpha+1)}\right| \leq \left|\frac{M\delta^\alpha}{\Gamma(\alpha+1)}\right| < \eta.$$

(ii) We consider the series

$$y_0 + \sum_{k=0}^{\infty}(y_{k+1}(t) - y(t)),$$

equal with:

$$y_0 + \sum_{k=0}^{n-1}(y_{k+1}(t) - y_k(t)) = y_{n+1}(t).$$

We have:

$$|y_2 - y_1| = \left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}[f(u, y_1(u)) - f(u, y_0)]\,du\right|$$
$$\leq \left|\frac{K}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}|y_1 - y_0|du\right| \leq \left|\frac{KM}{\Gamma(\alpha+1)}t^\alpha\right|$$
$$\leq \left|\frac{KM}{\Gamma(\alpha+1)}\delta^\alpha\right| = K\eta.$$

$$|y_3 - y_2| = \left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}[f(u, y_2(u)) - f(u, y_1)]du\right|$$
$$\leq \left|\frac{K}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}|y_2 - y_1|du\right| \leq \left|\frac{K^2\eta}{\Gamma(\alpha+1)}t^\alpha\right|$$
$$\leq \left|\frac{K^2\eta}{\Gamma(\alpha+1)}\delta^\alpha\right| = \frac{K^2\eta^2}{M},$$

$$|y_{n+1} - y_n| \leq \frac{K^n\eta^n}{M^{n-1}}.$$

Using the Weierstrass test [9, 10] we obtain:

$$\sum_{n=0}^{\infty} |\ldots| = y_0 + \eta + \frac{1}{M} \sum_{n=1}^{\infty} \left(\frac{K\eta}{M}\right)^n .$$

The series are convergent for $\eta < \frac{M}{K}$.

Thus the sequence $\{y_n(t)\}$ is uniform convergent on the compact $[0, \eta]$. Hence, $y_n(t)$ is convergent to a function $y(t)$ for $t \in [0, \eta]$.

$\forall \eta > 0$, $\exists N$ positive number so for $n > N$ we have:

$$|y_n(t) - y(t)| < \eta.$$

This limit is unique.

(iii) Let $x(t)$ be another limit for $\{y_n(t)\}$, then:

$$|x(t) - y(t)| = |x(t) - y_n(t) + y_n(t) - y(t)|$$
$$\leq |y_n(t) - x(t)| + |y_n(t) - y(t)| \leq \frac{\eta}{2} + \frac{\eta}{2} = \eta.$$

Remark (Another Solution) In order to prove the existence of the solution we can introduce the set

$$U = \{y \in C[0, \eta] : \|y - y_0\| \leq \eta\}$$

and an operator A:

$$Ay(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} f(u, y(u))\, du,$$

where A has a fixed point, and U is a closed and convex subset of all continuous functions on $[0, \eta]$ equipped with Chebyshev norm [12].

Generally:

$$y(t) = \sum_{j=0}^{n} \frac{b_j}{\Gamma(\alpha - j + 1)} (t-a)^{\alpha-j} + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(u, y(u))}{(t-u)^{1-\alpha}}\, du,$$

where $t > 0$, $n - 1 \leq \alpha < n$.

The technique used for proving the existence solution of the Voltera equation is often the successive approximation:

$$y_0(t) = \sum_{k=1}^{n} \frac{b_k}{\Gamma(\alpha - k + 1)} t^{\alpha-k},$$

$$y_i(t) = y_0(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, y_{i-1}(\tau))d\tau, \qquad i = 1, 2, \ldots$$

$$y(t) = \lim_{i\to\infty} y_i(t).$$

Example 1 Using method of successive approximation we solve the FDE:

$$D^\alpha y(t) = t^2 + y^2, \quad 0 < \alpha \le 1, \quad y(0) = 0, \quad (t, y) \in [-1, 1] \times [-1, 1].$$

Solution

– For $\alpha = 1$, we have method of successive approximation or the method of Picard.[4]

We construct a sequence $\{y_n(t)\}$ by the recurrence

$$y_n(t) = y_0 + \int_0^t f[u, y_{n-1}(u)]du, \quad n = 1, 2, \ldots$$

The $\{y_n(t)\}$ is convergent to an exact solution of the equation

$$y'(t) = f[t, y(t)] = t^2 + y^2, \quad y(0) = 0,$$

in some interval $0 - h < t < 0 + h$ in the rectangle $|t - t_0| \le a = 1$, $|y - y_0| \le b = 1$,

$$h = \min\left(a, \frac{b}{M}\right), \quad M = \max_{(t,y)\in D} |f(t, y)|,$$

$y_n(t)$ is given by the inequality

$$|y(t) - y_n(t)| \le \frac{MN^{n-1}}{n!} h^n, \quad N = \max_{(t,y)\in D} \left|\frac{\partial f}{\partial y}\right|.$$

For

$$M = 2, \quad a = 1, \quad h = \frac{1}{2},$$

it results:

$y_0(t) = 0,$

$y_1(t) = \int_0^t (u^2 + y_0^2)du = \frac{t^3}{3},$

[4] E. Picard (1856–1941).

$$y_2(t) = \int_0^t (u^2 + y_1^2)du = \frac{t^3}{3} + \frac{t^7}{63},$$
$$y_3(t) = \int_0^t (u^2 + y_1^2)du = \frac{t^3}{3} + \frac{t^7}{63} + \frac{2t^1 1}{2079} + \frac{t^1 5}{59535},$$

$$|y_3(t) - y(t)| \leq \frac{2}{3!}\left(\frac{1}{2}\right)^3 2^2 = \frac{1}{6}, \quad N = \max|2y| = 2.$$

– For $0 < \alpha \leq 1$, we obtain:

$$y_0 = 0,$$

$$y_n(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1}[u^2 + y_{n-1}^2(u)]du.$$

We can calculate $y_1(t)$:

$$y_1(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} u^2 du.$$

The LT of this convolution is

$$Y_1 = \frac{1}{\Gamma(\alpha)} L[u^{\alpha-1}]L[u^2] = \frac{1}{\Gamma(\alpha)} \frac{\Gamma(\alpha)}{s^\alpha} \frac{\Gamma(3)}{s^3} = \frac{\Gamma(3)}{s^{\alpha+3}},$$

from which, by inversion, we obtain:

$$y_1(t) = \frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}.$$

Similarly, we obtain also:

$$y_2(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} \left[u^2 + 4\frac{u^{2\alpha+4}}{\Gamma^2(\alpha+3)}\right] du,$$

with:

$$Y_2(s) = \frac{2}{s^{\alpha+3}} + \frac{4}{\Gamma^2(\alpha+3)} \frac{\Gamma(2\alpha+5)}{s^{3\alpha+5}},$$

and finally:

$$y_2(t) = \frac{2t^{\alpha+2}}{\Gamma(\alpha+3)} + \frac{4\Gamma(2\alpha+5)}{\Gamma^2(\alpha+3)} \frac{t^{3\alpha+4}}{\Gamma(3\alpha+5)},$$

For $\alpha = 1$, we obtain $y_2(t) = \frac{t^3}{3} + \frac{t^7}{63}$.

Example 2 Using method of successive approximation we solve the FDE:

$$D^\alpha y(t) = 1 + ty(t) + y^2(t), \qquad 0 < \alpha \leq 1, \qquad y(0) = 0.$$

Solution As in the previous example, we have:

$$y_0(t) = 0,$$
$$y_n(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} \left[1 + uy_{n-1}(u) + y_{n-1}^2(u)\right] du,$$
$$y_1(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1}\, du,$$
$$Y_1 = \frac{1}{\Gamma(\alpha)} L[u^{\alpha-1}]L[1] = \frac{1}{\Gamma(\alpha)} \frac{\Gamma(\alpha)}{s^\alpha} \frac{1}{s} = \frac{\Gamma(1)}{s^{\alpha+1}},$$
$$y_1(t) = \frac{t^\alpha}{\Gamma(\alpha+1)}$$

$$y_2(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} \left[1 + u \cdot y_1(u) + y_1^2(u)\right] du$$

$$Y_2 = \frac{1}{s^{\alpha+1}} \Gamma(\alpha+1) + \frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} \frac{1}{s^{3\alpha+1}} + \frac{\alpha+1}{s^{2\alpha+2}},$$

$$y_2 = \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + (\alpha+1) \frac{t^{2\alpha+1}}{\Gamma(2\alpha+1)}.$$

4.2 Linear Fractional Differential Equations

A **linear** FDE is an equation of form

$$(D^{\alpha_n} + a_{n-1} D^{\alpha_{n-1}} + \ldots a_1 D^{\alpha_1} + a_0) y(t) = f(t), \qquad \alpha \in \mathbb{R},$$

with the conditions:

$$y^{(k)}(0) = b_k, \qquad k = 0, 1, 2, \ldots, n-1.$$

An equation which is not linear is called nonlinear.

Theorem 2 (Existence and Uniqueness) *If $f(t)$ is bounded on $(0, T)$ and $a_k = a_k(t)$, $k \in \{0, 1, \ldots, n-1\}$ are continuous functions on $[0, T]$, the equation has a unique solution.*

Proof The proof used here will be based on the proof of the existence and uniqueness of the solution of Voltera integral equation.

Theorem 3 *The linear FDE:*

$$D^{\alpha} y(t) = f(t), \qquad \textit{where: } n-1 < \alpha < n,$$

and

$$y^{(k)}(0) = 0, \qquad k = 0, 1, 2, \ldots, n-1,$$

has the solution:

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} f(u)\, du.$$

Proof We apply the LT:

$$L[y(t)] = Y = Y(s),$$

$$L\left[D^{\alpha} y(t)\right] = L[f(t)] = F(s),$$

$$s^{\alpha} Y = F(s) \quad \Rightarrow \quad Y = \frac{F(s)}{s^{\alpha}}$$

and using the convolution theorem it results the assumption of this theorem.

Theorem 4 *The linear FDE:*

$$D^{\alpha} y(t) = \lambda y(t), \qquad \textit{where: } n-1 < \alpha < n,$$

with the initial condition

$$y^{(k)}(0) = b_k, \qquad b_k \in R, \qquad k = 0, 1, 2, \ldots, n-1,$$

has the solution:

$$y(t) = \sum_{k=0}^{n-1} b_k t^k E_{\alpha,k+1}(\lambda t^{\alpha}).$$

Proof We apply the LT method:

$$L[y(t)] = Y = Y(s),$$

$$L\left[D^{\alpha} y(t)\right] = \lambda L[y(t)],$$

$$s^{\alpha}Y - \sum_{k=0}^{n-1} s^{\alpha-k-1}y^{(k)}(0) - \lambda Y = 0,$$

$$Y = \sum_{k=0}^{n-1} \frac{s^{\alpha-k-1}}{s^{\alpha}-\lambda} b_k = \sum_{k=0}^{n-1} L\left[b_k t^k E_{\alpha,k+1}(\lambda t^{\alpha})\right] = L\left[\sum_{k=0}^{n-1} b_k t^k E_{\alpha,k+1}(\lambda t^{\alpha})\right].$$

Because $L[y(t)] = Y$ it results the statements of the theorem.

Theorem 5 *The linear FDE:*

$$D^{\alpha}y(t) = f(t), \qquad \textit{where: } 0 < \alpha < 1,$$

with the initial condition

$$y(0) = A,$$

and where:

$$f(t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} t^n,$$

has the solution:

$$y(t) = A + t^{\alpha} \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{\Gamma(n+\alpha+1)} t^n.$$

Proof We apply the LT:

$$L[y(t)] = Y = Y(s),$$

$$L\left[D^{\alpha}y(t)\right] = s^{\alpha}Y - As^{\alpha-1},$$

$$s^{\alpha}Y - As^{\alpha-1} = L[f(t)] = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} L[t^n],$$

$$y(t) = \sum_{n=0}^{\infty} y_n, \quad Y = L[y(t)] = \sum_{n=0}^{\infty} Y_n,$$

$$\sum_{n=0}^{\infty} Y_n = \frac{A}{s} + \frac{1}{s^{\alpha}} \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} L[t^n],$$

$$Y_0 = \frac{A}{s} \qquad \Rightarrow y_0 = A,$$

$$Y_n = \frac{f^{(n)}(0)}{\Gamma(n+1)} \frac{\Gamma(n+1)}{s^{n+\alpha+1}} \quad \Rightarrow y_n = \frac{f^{(n)}(0)}{\Gamma(n+\alpha+1)},$$

$$y(t) = A + \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{\Gamma(n+\alpha+1)} t^{n+\alpha}.$$

Theorem 6 *The linear FDE:*

$$aD^{\alpha} y(t) + by(t) = f(t), \qquad 0 < \alpha < 1,$$

$$y(0) = A,$$

where:

$$f(t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} t^n,$$

has the solution:

$$y(t) = AE_{\alpha,1}\left(-\frac{b}{a}t^{\alpha}\right) + \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{a} t^n \left[1 - E_{\alpha,n+1}(-\frac{b}{a}t^{\alpha})\right].$$

Proof Applying the LT, it results:

$$L[y(t)] = Y = Y(s),$$

$$L\left[D^{\alpha} y(t)\right] = s^{\alpha} Y - As^{\alpha-1},$$

$$s^{\alpha} Y - As^{\alpha-1} + bY = L[f(t)] = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} L[t^n],$$

$$y(t) = \sum_{n=0}^{\infty} y_n, \qquad Y = L[y(t)] = \sum_{n=0}^{\infty} Y_n,$$

$$\sum_{n=0}^{\infty} Y_n = \frac{As^{\alpha-1}}{as^{\alpha}+b} + \frac{1}{as^{\alpha}+b} \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} L[t^n],$$

$$Y_0 = \frac{As^{\alpha-1}}{as^\alpha + b} \quad \Rightarrow \quad Y_0 = \frac{A}{as}\frac{1}{1+\dfrac{b}{as^\alpha}},$$

$$Y_n = \frac{1}{as^\alpha + b}\frac{f^{(n)}(0)}{n!}L[t^n] \;\Rightarrow\; Y_n = \frac{f^{(n)}(0)}{as^{n+1+\alpha}}\frac{1}{1+\dfrac{b}{as^\alpha}},$$

$$Y_0 = A\left[\frac{1}{s} - \frac{b}{a}\frac{1}{s^{\alpha+1}} + \frac{b^2}{a^2}\frac{1}{s^{2\alpha+1}} + \dots\right],$$

$$y_0 = A\left[1 - \frac{b}{a}\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{b^2}{a^2}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \dots\right],$$

$$y_0 = AE_{\alpha,1}\left(-\frac{b}{a}t^\alpha\right),$$

$$Y_n = \frac{f^{(n)}(0)}{a}\left[\frac{1}{s^{n+1+\alpha}} - \frac{b}{a}\frac{1}{s^{n+1+2\alpha}} + \frac{b^2}{a^2}\frac{1}{s^{n+1+3\alpha}} + \dots\right],$$

$$y_n = \frac{f^{(n)}(0)}{a}\left[\frac{t^{n+\alpha}}{\Gamma(n+1+\alpha)} - \frac{b}{a}\frac{t^{n+2\alpha}}{\Gamma(n+1+2\alpha)} + \frac{b^2}{a^2}\frac{t^{n+3\alpha}}{\Gamma(n+1+3\alpha)} + \dots\right],$$

$$y_n = \frac{f^{(n)}(0)}{a}t^n\left[1 - E_{\alpha,n+1}\left(-\frac{b}{a}t^\alpha\right)\right].$$

Finally, we obtain the solution:

$$y(t) = y_0 + \sum_{n=0}^{\infty} y_n,$$

$$y(t) = AE_{\alpha,1}\left(-\frac{b}{a}t^\alpha\right) + \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{a}t^n\left[1 - E_{\alpha,n+1}\left(-\frac{b}{a}t^\alpha\right)\right].$$

Example 1 We will establish here the solution of the FDE:

$$D^\alpha y(t) + y(t) = 1,$$

with the initial condition:

$$y(0) = 0.$$

Solution We apply the LT:

$$L\left[D^\alpha y(t)\right] + L[y(t)] = L[1],$$

$$s^\alpha Y - s^{\alpha-1} y(0) + Y = \frac{1}{s},$$

$$Y = \frac{1}{s(s^\alpha + 1)},$$

$$Y = \frac{1}{s^{\alpha+1}} \frac{1}{1 + \frac{1}{s^\alpha}},$$

and using the identity

$$\frac{1}{1+u} = 1 - u + u^2 - u^3 + \dots \qquad |u| < 1,$$

we obtain:

$$Y = \frac{1}{s^{\alpha+1}} - \frac{1}{s^{2\alpha+1}} + \frac{1}{s^{3\alpha+1}} + \dots$$

Finally, it results:

$$y(t) = \frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \dots$$

For $\alpha = 1$, we can use the Maple or Mathematica commands in order to establish the solution:

MAPLE

```
ec: = diff(y(t),t) + y(t) = 1;
dsolve({ec,y(0) = 0},y(t),type = series);
```

MATHEMATICA

```
Clear["`*"]c
ec := y'[t] + y[t] == 1;
sol = DSolve[{ec, y[0] == 0}, y, t]
Series[y[t] /. sol, {t, 0, 10}]
```

It results the solution:

$$y(t) = t - \frac{t^2}{3} + \frac{t^3}{6} - \frac{t^4}{24} + \dots$$

Example 2 We consider the FDE

$$D^{\alpha} y(t) = y(t),$$

with the initial conditions:

$$y(0) = 0, \qquad y'(0) = 1.$$

We will establish the solution of this equation for the cases:

1. $1 < \alpha \leq 2$,
2. $2 < \alpha \leq 3$.

Solution

1. For the case $1 < \alpha \leq 2$, using the LT, we have:

$$L[D^{\alpha} y(t]) = L[y(t)],$$

$$s^{\alpha} Y - s^{\alpha-1} y(0) - s^{\alpha-2} y'(0) = Y,$$

$$Y = \frac{s^{\alpha-2}}{s^{\alpha} - 1},$$

$$Y = \frac{1}{s^2} \frac{1}{1 - \frac{1}{s^{\alpha}}},$$

and using the identity

$$\frac{1}{1-u} = 1 + u + u^2 + u^3 + \qquad |u| < 1,$$

we have

$$Y = \frac{1}{s^2} + \frac{1}{s^{\alpha+2}} + \frac{1}{s^{2\alpha+2}} + \ldots,$$

and the solution:

$$y(t) = \frac{t}{\Gamma(2)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \ldots$$

We can note that:

$$\lim_{\alpha \to 2} y(t) = \frac{t}{1!} + \frac{t^3}{3!} + \frac{t^5}{5!} \ldots = \sinh(t).$$

2. For the case $2 < \alpha \leq 3$, using same procedure, we have:

$$s^\alpha Y(s) - s^{\alpha-1}y(0) - s^{\alpha-2}y'(0) - s^{\alpha-3}y''(0) = Y(s).$$

For $y''(0) = b$, we obtain:

$$Y = \frac{s^{\alpha-2}}{s^\alpha - 1} + b\frac{s^{\alpha-3}}{s^\alpha - 1},$$

and using the residues theorem we have:

$$\lim_{\alpha \searrow 2} Y = \frac{s+b}{s(s^2-1)},$$

$$r_1 = \operatorname*{Res}_0(Ye^{st}) = \lim_{s\to 0} sYe^{st} = -b,$$

$$r_2 = \operatorname*{Res}_{-1}(Ye^{st}) = \lim_{s\to -1}(s+1)Ye^{st} = \frac{(b-1)e^{-t}}{2},$$

$$r_3 = \operatorname*{Res}_1(Ye^{st}) = \lim_{s\to 1}(s-1)Ye^{st} = \frac{(b+1)e^{t}}{2}.$$

The solution will be:

$$y(t) = r_1 + r_2 + r_3 = \sinh(t) + b\cosh(t) - b.$$

Observation

This equation can be solved also in terms of perturbation method. In this case we take

$$y(t) = t + c_1 t^{\alpha+1} + c_2 t^{2\alpha+1} + c_3 t^{3\alpha+1} + \ldots,$$

and using the formula:

$$D_*^\alpha t^\beta = \begin{cases} \dfrac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} t^{\beta-\alpha} & \beta > n-1 \\ 0 & \beta \leq n-1. \end{cases}$$

$$\begin{aligned} D_*^\alpha y(t) &= c_1 D_*^\alpha t^{\alpha+1} + c_2 D_*^\alpha t^{2\alpha+1} + c_3 D_*^\alpha t^{\alpha+1} + \ldots \\ &= t + c_1 t^{\alpha+1} + c_2 t^{2\alpha+1} + c_3 t^{3\alpha+1} + \ldots \end{aligned}$$

we have

$$c_1\frac{\Gamma(\alpha+2)}{\Gamma(2)}t+c_2\frac{\Gamma(2\alpha+2)}{\Gamma(\alpha+2)}t^{\alpha+1}+c_3\frac{\Gamma(3\alpha+2)}{\Gamma(2\alpha+2)}t^{2\alpha+1}+\dots$$
$$=t+c_1t^{\alpha+1}+c_2t^{2\alpha+1}+c_3t^{3\alpha+1}+\dots$$

and after the identification we obtain:

$$c_1\Gamma(\alpha+2)=1 \qquad \Rightarrow \qquad c_1=\frac{1}{\Gamma(\alpha+2)},$$

$$c_2\frac{\Gamma(2\alpha+2)}{\Gamma(\alpha+2)}=c_1 \qquad \Rightarrow \qquad c_2=\frac{1}{\Gamma(2\alpha+2)},$$

$$c_3\frac{\Gamma(3\alpha+2)}{\Gamma(2\alpha+2)}=c_2 \qquad \Rightarrow \qquad c_3=\frac{1}{\Gamma(3\alpha+2)},$$

$$\dots$$

For $\alpha=2$, we have:

$$y''(t)=y(t).$$

We apply the LT:

$$L[y''(t)]=L[y(t)], \qquad L[y(t)]=Y=Y(s),$$

$$s^2Y-sy'(0)-y(0)=Y,$$

$$Y=\frac{1}{s^2-1},$$

$$r_1=\operatorname*{Res}_{-1}Ye^{st}=\lim_{s\to-1}Ye^{st}=\lim_{s\to-1}(s+1)\frac{e^{st}}{(s-1)(s+1)}=-\frac{e^{-t}}{2},$$

$$r_2=\operatorname*{Res}_{1}Ye^{st}=\lim_{s\to1}Ye^{st}=\lim_{s\to1}(s-1)\frac{e^{st}}{(s-1)(s+1)}=\frac{e^{t}}{2},$$

$$f(t)=r_1+r_2=\frac{e^t-e^{-t}}{2}=\sinh(t).$$

We can use here the **MAPLE** commands:

```
with(inttrans);
ec:=diff(y(t),t$2) = y(t);
dsolve(ec,D(y)(0) = 1,y(0) = 0,y(t), method = laplace);
```

Example 3 Find the solution of the FDE:

$$D^2 y(t) - D^{3/2} y(t) - y(t) + t + 1 = 0,$$

with the initial conditions:

$$y(0) = y'(0) = 1.$$

Solution We apply the LT:

$$L\left[D^2 y(t)\right] - L\left[D^{\frac{3}{2}} y(t)\right] - L[y(t)] + L[t] + L[1] = 0,$$

$$L\left[D^2 y(t)\right] = s^2 Y - s y(0) - y'(0) = s^2 Y - s - 1,$$

$$L\left[D^{\frac{3}{2}} y(t)\right] = s^{\frac{3}{2}} Y - s^{\frac{1}{2}} y(0) - s^{-\frac{1}{2}} y'(0) = s^{\frac{1}{2}} Y - s^{\frac{1}{2}} - s^{-\frac{1}{2}} =$$

$$= \frac{s^2 - s - 1}{\sqrt{s}},$$

$$L[y(t)] = Y,$$

$$L[t] = \frac{1}{s^2},$$

$$L[1] = \frac{1}{s},$$

$$s^2 Y - s - 1 - \frac{s^2 Y - s - 1}{\sqrt{s}} - \left[s^2 Y - s - 1\right] = 0,$$

$$s^2 Y - s - 1 = 0, \qquad \Rightarrow \qquad Y = \frac{1}{s} + \frac{1}{s^2},$$

$$y(t) = L^{-1}[Y], \qquad \Rightarrow \qquad y(t) = 1 + t.$$

Example 4 We consider the problem [11], with the initial condition:

$$\frac{d^{1/2} y}{dx^{1/2}} = \frac{y}{x}, \qquad y(0) = 0.$$

which can be rewritten as:

$$t D^{1/2} y(t) - y(t) = 0, \qquad y(0) = 0.$$

We will establish the solution of this equation.

Solution We apply the LT method:

$$L\left[tD^{1/2}y(t)\right] - L[y(t)] = 0,$$

$$L\left[tD^{1/2}y(t)\right] = -\frac{d}{ds}\left[s^{1/2}Y - s^{-1/2}y(0)\right] - Y = 0,$$

$$\frac{dY}{ds} + \left(\frac{1}{2s} + \frac{1}{\sqrt{s}}\right)Y = 0.$$

We obtain:

$$Y(s) = C\frac{e^{-2\sqrt{s}}}{\sqrt{s}} \qquad \Rightarrow \qquad y(t) = \frac{Ce^{-1/t}}{\sqrt{\pi t}}.$$

The same solution can be found using the Maple program:

MAPLE

```
with(inttrans):
ec:= diff(Y(s),s) + (1/(2*s)+1/sqrt(s)) - Y(s) = 0;
F(s):= dsolve(ec,Y(s));
f(t):= invlaplace(F(s),s,t);
```

Other applications can be found in [5, 6].

4.3 Nonlinear Equations

4.3.1 The Adomian Decomposition Method

The Adomian[5] method [1–3], applied to the ordinary and partial differential equations of integer order was extended also to the case of FDE (for further details and examples see [7, 8]).

Adomian Polynomials

We will denote these polynomials by $A_0, A_1, \ldots A_n, \ldots$.

We consider a nonlinear analytic function $G(y(t), t)$ and that $y(t_0) = y_0$, in the D domain. The Adomian method consists in the decomposition the unknown function $y(t)$ in a series of form $y(t) = y_0 + y_1 + y_2 + \ldots + y_n + \ldots$, where y_n can be expressed in terms of Adomian polynomials A_n.

[5] G. Adomian (1922–1996).

The Adomian polynomials are defined [1, 2, 8]:

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[G(t, \sum_{j=0}^{n} y_j \lambda^j)\right]_{\lambda=0}.$$

These polynomials can be established algorithmically, using the symbolic programming packages, with the aid of **do ... end do** repetition statements:

Step 1: To calculate first the Adomian polynomial A_0, i.e., $A_0 = G(y_0, t)$.
Step 2: Iterative calculation of A_k using the **for ... do .. end do** loop:

> for k = 0 to n - 1 do
> $A_k = A_k(y_0 + \lambda * y_1, \ldots, y_k + (k+1) * \lambda * y_{k+1})$
> end do;

It must be underlined that in the A_k polynomial y_i is replaced with

$$y_i \to y_i + (i+1) * y_{i+1} * \lambda, \qquad \text{for: } i = 0, 1, \ldots, k.$$

Step 3:

$$\frac{d}{d\lambda}A_k\bigg|_{\lambda=0} = (k+1) * A_k.$$

Step 4: We obtain, finally $A_0, A_1, \ldots, A_n$ and $y_0, y_1, y_2, \ldots$ in terms of A_n.

A given $f(u)$ can be expressed as a series of A_n,

$$f(u) = \sum_{n=0}^{\infty} A_n,$$

and $u = \sum_{n=0}^{\infty} u_n$.

The series $\sum_{n=0}^{\infty} A_n$ can be rearranged as a generalized Taylor series:

$$\begin{aligned} f(u) = \sum_{n=0}^{\infty} A_n &= f(u_0) + (u_1 + u_2 + \ldots) f^{(1)}(u_0) \\ &+ \left[\frac{u_1^2}{2!} + u_1 u_2 + \ldots\right] f^{(2)}(u_0) + \ldots \\ &= \sum_{n=0}^{\infty} \left[(u - u_0)^n / n!\right] f^{(n)}(u_0) = \sum_{n=0}^{\infty} \left[(u_1 + u_2 + \ldots)^n / n!\right] f^{(n)}(u_0), \end{aligned}$$

so that:

$$A_0 = f(u_0),$$

$$A_1 = u_1 f^{(1)}(u_0),$$

$$A_2 = u_2 f^{(1)}(u_0) + (1/2!)u_1^2 f^{(2)}(u_0),$$

$$A_3 = u_3 f^{(1)}(u_0) + u_1 u_2 f^{(2)}(u_0) + (1/3!)u_1^3 f^{(3)}(u_0),$$

$$\ldots$$

Example 1 We will establish the Adomian polynomials for $G(y) = y^2$.
The Adomian polynomials will be:

$A_0 = y_0^2$, $A_1 = 2y_0y_1$, $A_2 = y_1^2 + 2y_0y_2$, $A_3 = 2y_1y_2 + 2y_0y_3$, etc.
For calculation you can use also the following Maple, or Mathematica sequences:

MAPLE

```
restart;
with(LinearAlgebra):
unassign('y,lambda'):
f:=y->y^2:
S:=lambda->sum(y[i]*lambda^i,i=0..4):
g:=lambda->(S(lambda))^2:
c:=Vector(4,n->diff(1/n!*g(lambda),lambda$n):
A:=<subs(lambda=0,g(lambda)),subs(lambda=0,c)>;
```

MATHEMATICA

```
Clear["`*"]
f[y_] := y^2;
S[\[Lambda]_] := Sum[y[i]*\[Lambda]^i, {i, 0, 5}];
g[\[Lambda]_] := f[S[\[Lambda]]];
ad = Table[
 1/n!*D[g[\[Lambda]], {\[Lambda],n}]/. \[Lambda]-> 0,
 {n, 0, 5}] // Simplify;
TableForm[ad, TableAlignments -> Left]
```

Example 2 Let us calculate the Adomian polynomials for $G = y^3$.

Solution

$A_0 = y_0^3$, $A_1 = 3y_1y_0^2$, $A_2 = 3y_0^2y_2 + 3y_1^2y_0$,
$A_3 = y_1^3 + 6y_0y_1y_2 + 3y_0^2y_1$, etc.

The basic principles of this algorithm remain unchanged for other definitions of the function G.

Example 3 Let us calculate the Adomian polynomials for $G = f(u)$.

We obtain successively:

$$A_0 = f(u_0),\, A_1 = u_1(d/du_0)f(u_0),$$

$$A_2 = u_2(d/du_0)f(u_0) + (u_1^2)/2!)(d^2/du_0^2)f(u_0),$$

$$A_3 = u_3(d/du_0)f(u_0) + (u_1u_2)(d^2/du_0^2)f(u_0) + (u_1^3/3!)(d^3/du_0^3)f(u_0), \text{etc.}$$

Example 4 Find the Adomian polynomials for $G = \sin\theta$. It results:

$$A_0 = \sin\theta_0,\, A_1 = \theta_1\cos\theta_0,\, A_2 = -(\theta_1^2/2)\sin\theta_0 + \theta_2\cos\theta_0, \text{etc.}$$

4.3.2 Decomposition of Nonlinear Equations

We consider the nonlinear FDE of type:

$$D^\alpha y(t) + Ry(t) + Ny(t) = f(t), \quad y^{(k)}(0) = c_k, k = 0, 1, \ldots, n-1, \quad \alpha > 0,$$

where N is a nonlinear operator, and Ry is a residual part of the equation.

We apply the LT to the equation. It follows:

$$L[D^\alpha y(t)] = s^\alpha Y - s^{\alpha-1}y(0) - s^{\alpha-2}y'(0) - \ldots - y^{(n-1)}(0) = s^\alpha Y - c,$$

$$c = s^{\alpha-1}y(0) + s^{\alpha-2}y'(0) - \ldots + y^{(n-1)}(0),$$

where c is a constant.

We use the following decomposition of $y(t)$

$$y(t) = \sum_{n=0}^{\infty} y_n(t),$$

with

$$Ny(t) = \sum_{n=0}^{\infty} A_n,$$

where A_n are Adomian polynomials:

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty}\lambda^i y_i\right)\right],$$

$$L\left[\sum_{n=0}^{\infty} y_n\right] = \frac{c}{s^\alpha}Y - \frac{1}{s^\alpha}L\left[R\sum_{n=0}^{\infty} y_n\right] - \frac{1}{s^\alpha}L\left[\sum_{n=0}^{\infty} A_n\right].$$

We have after calculations:

$$Y_0 = L[y_0] = \frac{c}{s^\alpha} + \frac{1}{s^\alpha}L\left[f(t)\right],$$

$$Y_1 = L[y_1] = -\frac{1}{s^\alpha}L[Ry_0] - \frac{1}{s^\alpha}L[A_0],$$

$$Y_2 = L[y_2] = -\frac{1}{s^\alpha}L[Ry_1] - \frac{1}{s^\alpha}L[A_1],$$

$$\ldots$$

$$Y_n = L[y_n] = -\frac{1}{s^\alpha}L[Ry_{n-1}] - \frac{1}{s^\alpha}L[A_{n-1}].$$

Example 1 Solve the nonlinear FDE using the Adomian decomposition method:

$$D^\alpha y(t) = t + y^2, \qquad 1 < \alpha \le 2,$$

$$y(0) = 0, \qquad y'(0) = 1.$$

Solution In order to solve the equation we apply the LT:

$$L[D^\alpha y(t)] = L\left[t + y^2\right],$$

$$L[y(t)] = Y,$$

$$L[D^\alpha y(t)] = s^\alpha Y - s^{\alpha-1}y(0) - s^{\alpha-2}y'(0) = s^\alpha Y - s^{\alpha-2},$$

$$Y = \frac{1}{s^2} + \frac{1}{s^\alpha}L\left[t + y^2\right],$$

so that, for the decomposition

$$y(t) = \sum_{n=0}^{\infty} y_n(t),$$

we obtain:

$$Y = \sum_{n=0}^{\infty} Y_n, \qquad t + y^2 = \sum_{n=0}^{\infty} A_n,$$

where A_n are Adomian polynomials. We obtain:

$$A_0 = t + y_0^2, \quad A_1 = 2y_0y_1, \quad A_2 = y_1^2 + 2y_0y_2, \quad A_3 = 2y_1y_2 + 2y_0y_2, \ldots$$

$$\sum_{n=0}^{\infty} Y_n = \frac{1}{s^2} + \frac{1}{s^\alpha} \sum_{n=0}^{\infty} A_n,$$

$$Y_0 = \frac{1}{s^2} \quad \Rightarrow \quad y_0 = t,$$

$$A_0 = t + t^2,$$

$$Y_1 = \frac{1}{s^\alpha} L[A_0] \qquad \Rightarrow \qquad Y_1 = \frac{1}{s^\alpha} L\left[t + t^2\right],$$

$$Y_1 = \frac{1}{s^{\alpha+2}} + 2\frac{1}{s^{\alpha+3}} \qquad \Rightarrow \qquad y_1 = \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + 2\frac{t^{\alpha+2}}{\Gamma(\alpha+3)},$$

$$A_1 = 2y_0y_1,$$

$$A_1 = 2\frac{t^{\alpha+2}}{\Gamma(\alpha+2)} + 4\frac{t^{\alpha+3}}{\Gamma(\alpha+3)},$$

$$Y_2 = \frac{1}{s^\alpha} L[A_1] \quad \Rightarrow \quad Y_2 = \frac{2(\alpha+2)}{s^{2\alpha+3}} + \frac{4(\alpha+3)}{s^{2\alpha+4}},$$

$$y_2 = \frac{2(\alpha+2)t^{2\alpha+2}}{\Gamma(2\alpha+3)} + \frac{4(\alpha+3)t^{2\alpha+3}}{\Gamma(2\alpha+4)},$$

$$y(t) = y_0(t) + y_1(t) + y_2(t) + \ldots,$$

and finally, it yields:

$$y(t) = t + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + 2\frac{t^{\alpha+2}}{\Gamma(\alpha+3)} + \frac{2(\alpha+2)t^{2\alpha+2}}{\Gamma(2\alpha+3)} + \frac{4(\alpha+3)t^{2\alpha+3}}{\Gamma(2\alpha+4)} + \ldots$$

For $\alpha = 2$, we obtain:

$$y''(t) = t^2 + y^2(t), \qquad y(0) = 0, \qquad y'(0) = 1.$$

The result

$$y(t) = t + \frac{t^3}{6} + \frac{t^4}{12} + \ldots,$$

can be obtained also on computer, using the sequences:

MAPLE

```
ec: = diff(y(t), t$2) = t + (y(t))^2;
dsolve({ec,y(0) = 0,D(y)(0) = 1},y(t),type = series);
```

MATHEMATICA

```
ec:=y"[t] == t + y[t]*y[t];
sol=DSolve[{ec,y[0]==0,y'[0]==1},y,t];
Series[y[t]/.sol,{t,0,10}]
```

MATHEMATICA

```
Clear["`*"]
Manipulate[
 f[t_] := t + t^3/6 + t^4/12 + t^5/120;
 y[t_] :=
  t + t^(a + 1)/Gamma[a + 2] + 2*t^(a + 2)/Gamma
   [a + 3] + 2*(a + 2)*t^(2*a + 2)/Gamma[2*a + 3] +
   4*(a + 3)*t^(2*a + 3)/Gamma[2*a + 4];
Plot[{f[t], y[t]}, {t, 0, 1}, ImageSize -> 300,
  Frame -> True], {{a, 1/2}, 0, 1}]
```

The functions $f(t)$ and $y(t)$ calculated here are plotted in Fig. 4.1.

Example 2 Let us solve the FDE:

$$D^\alpha y(t) = 1 + y^2(t), \qquad 0 < \alpha \leq 1,$$

where:

$$y(0) = 0,$$

using the Adomian decomposition method.

Solution To solve this problem we apply the LT:

$$L[D^\alpha y(t)] = L[1] + L\left[y^2\right],$$

$$L[y(t)] = Y,$$

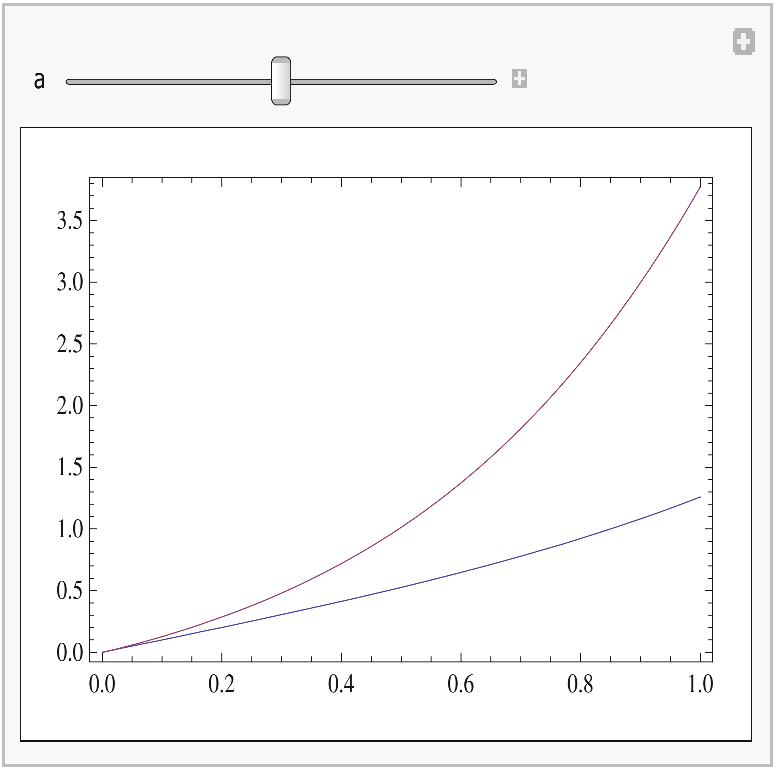

Fig. 4.1 Plots of the functions $f(t)$ and $y(t)$ from the Example 1

$$L[D^\alpha y(t)] = s^\alpha Y - s^{\alpha-1} y(0) = s^\alpha Y,$$

$$Y = \frac{1}{s^{\alpha+1}} + \frac{1}{s^\alpha} L\left[y^2\right].$$

For the decomposition

$$y(t) = \sum_{n=0}^{\infty} y_n(t),$$

we obtain

$$Y = \sum_{n=0}^{\infty} Y_n, \qquad y^2 = \sum_{n=0}^{\infty} A_n,$$

where A_n are the Adomian polynomials:

$$A_0 = y_0^2, \qquad A_1 = 2y_0 y_1, \qquad A_2 = y_1^2 + 2y_0 y_2, \qquad A_3 = 2y_1 y_2 + 2y_0 y_2, \ldots$$

$$\sum_{n=0}^{\infty} Y_n = \frac{1}{s^{\alpha+1}} + \frac{1}{s^{\alpha}} \sum_{n=0}^{\infty} A_n,$$

$$Y_0 = \frac{1}{s^{\alpha+1}} \quad \Rightarrow \quad y_0 = \frac{t^{\alpha}}{\Gamma(\alpha+1)},$$

$$A_0 = y_0^2 = \frac{t^{2\alpha}}{\Gamma^2(\alpha+1)},$$

$$Y_1 = \frac{1}{s^{\alpha}} L[A_0] \quad \Rightarrow \qquad Y_1 = \frac{1}{s^{\alpha}} L\left[\frac{t^{2\alpha}}{\Gamma^2(\alpha+1)}\right],$$

$$Y_1 = \frac{1}{s^{3\alpha+1}} \frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} \qquad \Rightarrow \quad y_1 = L^{-1} Y_1,$$

$$y_1 = \frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} \frac{t^{3\alpha}}{\Gamma(3\alpha+1)},$$

$$A_1 = 2 y_0 y_1,$$

$$Y_2 = \frac{1}{s^{\alpha}} L[A_1],$$

$$Y_2 = 2 \frac{\Gamma(2\alpha+1)}{\Gamma^3(\alpha+1)} \frac{\Gamma(4\alpha+1)}{\Gamma(3\alpha+1)} \frac{1}{s^{5\alpha+1}},$$

$$y_2 = L^{-1}[Y_2],$$

$$y_2 = 2 \frac{\Gamma(2\alpha+1)}{\Gamma^3(\alpha+1)} \frac{\Gamma(4\alpha+1)}{\Gamma(3\alpha+1)\Gamma(5\alpha+1)} t^{5\alpha},$$

$$y(t) = y_0(t) + y_1(t) + y_2(t) + \ldots$$

$$y(t) = \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + 2 \frac{\Gamma(2\alpha+1)}{\Gamma^3(\alpha+1)} \frac{\Gamma(4\alpha+1)}{\Gamma(3\alpha+1)\Gamma(5\alpha+1)} t^{5\alpha} + \ldots$$

You can also use the programs:

MAPLE

```
ec:=diff(y(t),t) = 1 + y(t))^2;
dsolve({ec,y(0) = 0},y(t),type = series);
```

MATHEMATICA

```
ec:=y'[t] == 1 + y[t]*y[t];
sol=DSolve[{ec,y[0]==0},y,t];
Series[y[t]/.sol,{t,0,10}]
```

Finally, we have:

$$y(t) = t + \frac{t^3}{3} + \frac{2t^5}{15} + \dots$$

Example 3 Solve the Ghelfand's[6] FDE:

$$D^{2\alpha} y(t) = 2e^{y(t)}, \qquad 0 < \alpha \leq 1,$$

where:

$$y(0) = y^{\alpha}(0) = 0,$$

using the Adomian decomposition method.

Solution To solve this problem we apply the LT:

$$L[D^{2\alpha} y(t)] = 2L\left[e^{y(t)}\right],$$

$$L[y(t)] = Y, \qquad y_0(t) = y(0) + \frac{y^{\alpha}(0)}{\Gamma(\alpha+1)} t^{\alpha} = 0,$$

$$L[D^{2\alpha} y(t)] = s^{2\alpha} Y - s^{2\alpha-1} y(0) = s^{\alpha} Y,$$

$$L\left[e^{y(t)}\right] = L\left[\sum_{n=0}^{\infty} A_n\right].$$

For the decomposition

$$y(t) = \sum_{n=0}^{\infty} y_n(t),$$

we obtain

$$Y = \sum_{n=0}^{\infty} Y_n, \qquad y^2 = \sum_{n=0}^{\infty} A_n,$$

[6] I.M. Ghelfand (1913–2009).

where A_n are the Adomian polynomials:

$$A_0 = e^{y_0}, \qquad A_1 = y_1 e^{y_0}, \ldots$$

$$\sum_{n=0}^{\infty} Y_n = \frac{2}{s^{2\alpha}} L\left[\sum_{n=0}^{\infty} A_n\right],$$

$$y_0 = 0,$$

$$A_0 = e^{y_0} = 1,$$

$$Y_1 = \frac{1}{s^{2\alpha}} L[A_0] \qquad \Rightarrow \qquad Y_1 = \frac{2}{s^{2\alpha+1}},$$

$$y_1 = \frac{2t^{2\alpha}}{\Gamma(2\alpha+1)},$$

$$A_1 = y_1 e^{y_0},$$

$$Y_2 = \frac{4}{s^{4\alpha+1}},$$

$$y_2 = L^{-1}[Y_2],$$

$$y_2 = \frac{4t^{4\alpha}}{\Gamma(4\alpha+1)},$$

$$y(t) = y_0(t) + y_1(t) + y_2(t) + \ldots$$

$$y(t) = 0 + \frac{2t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{4t^{4\alpha}}{\Gamma(4\alpha+1)} + \ldots$$

It can be used also the program:

MATHEMATICA

```
Clear["`*"]
Manipulate[
 f[t_] := t^2 + t^4/6;
 y[t_] := 2*t^(2*a)/Gamma[2*a + 1] + $*t^(4*a)/Gamma
 [4*a + 1]; Plot[{f[t], y[t]}, {t, 0, 1},
 ImageSize -> 300, Frame -> True], {{a, 1/2}, 0, 1}]
```

Figure 4.2 shows the plots of the functions $f(t)$ and $y(t)$.

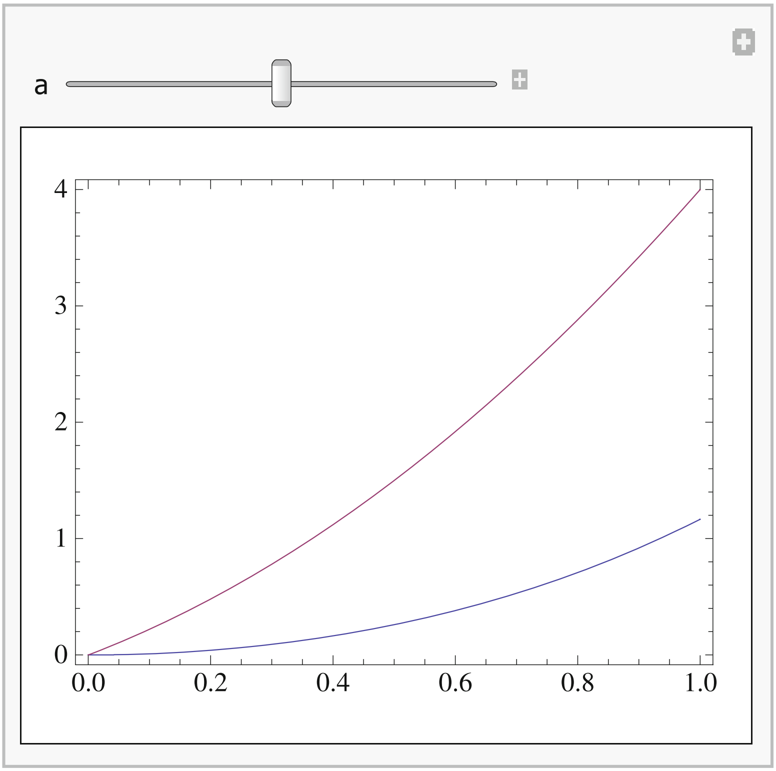

Fig. 4.2 Plots of the functions $f(t)$ and $y(t)$ from the Example 3

4.3.3 Perturbation Method

Here, we will extend the perturbation method for the case of FDE with the aid of some examples.

Example 1 Find the solution of the FDE:

$$D^{\alpha} y(t) = 1 + y^2(t), \qquad 0 < \alpha \leq 1,$$

for the initial condition:

$$y(0) = 0,$$

using the small parameter (perturbation) method, $0 < \epsilon \ll 1$.

Solution We consider a solution of form:

$$y(t) = y_0(t) + \epsilon y_1(t) + \epsilon^2 y_2(t) + \epsilon^3 y_3(t) + \ldots,$$

which, replaced in the equation, gives:

$$D^\alpha y_0(t) = 1, \qquad y_0(0) = 0,$$

$$D^\alpha y_1(t) = y_0^2, \qquad y_1(0) = 0,$$

$$D^\alpha y_2(t) = 2y_0 y_1, \qquad y_2(0) = 0,$$

$$\dots$$

We apply the LT:

$$L[D^\alpha y_0(t)] = L[1], \qquad y_0(0) = 0,$$

$$s^\alpha Y_0 - y_0(0) = \frac{1}{s}, \qquad Y_0 = \frac{1}{s^{\alpha+1}} \qquad \Rightarrow \qquad y_0(t) = \frac{t^\alpha}{\Gamma(\alpha+1)},$$

$$L[D^\alpha y_1)] = L[y_0^2], \qquad y_1(0) = 0,$$

$$s^\alpha Y_1 = \frac{1}{\Gamma^2(\alpha+1)} \frac{\Gamma(2\alpha+1)}{s^{2\alpha+1}}, \qquad Y_1 = \frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} \frac{1}{s^{3\alpha+1}},$$

and by inverse LT, we obtain:

$$y_1(t) = \frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}.$$

It results also:

$$L[D^\alpha y_2] = L[2y_0 y_1], \qquad y_2(0) = 0,$$

$$s^\alpha Y_2 = 2\left[\frac{t^\alpha}{\Gamma(\alpha+1)}\right] L\left[\frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\right],$$

$$s^\alpha Y_2 = 2\frac{\Gamma(2\alpha+1)}{\Gamma^3(\alpha+1)\Gamma(3\alpha+1)} L\left[t^{4\alpha}\right],$$

$$Y_2 = 2\frac{\Gamma(2\alpha+1)}{\Gamma^3(\alpha+1)\Gamma(3\alpha+1)} \frac{\Gamma(4\alpha+1)}{s^{5\alpha+1}},$$

$$y_2 = 2\frac{\Gamma(2\alpha+1)}{\Gamma^3(\alpha+1)\Gamma(3\alpha+1)} \frac{\Gamma(4\alpha+1)}{\Gamma(5\alpha+1)} t^{5\alpha},$$

The solution for this example:

$$y(t) = y_0(t) + y_1(t) + y_2(t) + \dots,$$

will be:

$$y(t) = \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} +$$

$$+2\frac{\Gamma(2\alpha+1)}{\Gamma^3(\alpha+1)\Gamma(3\alpha+1)} \frac{\Gamma(4\alpha+1)t^{5\alpha}}{\Gamma(5\alpha+1)} + \dots$$

The plot of $f(t)$ and $y(t)$ is done with the following program and presented in Fig. 4.3.

MATHEMATICA

```
Clear["`*"]
Manipulate[
 f[t_] := t + t^3/3 + 2/15*t^5;
 y[t_] :=
  t^a/Gamma[a + 1] +
```

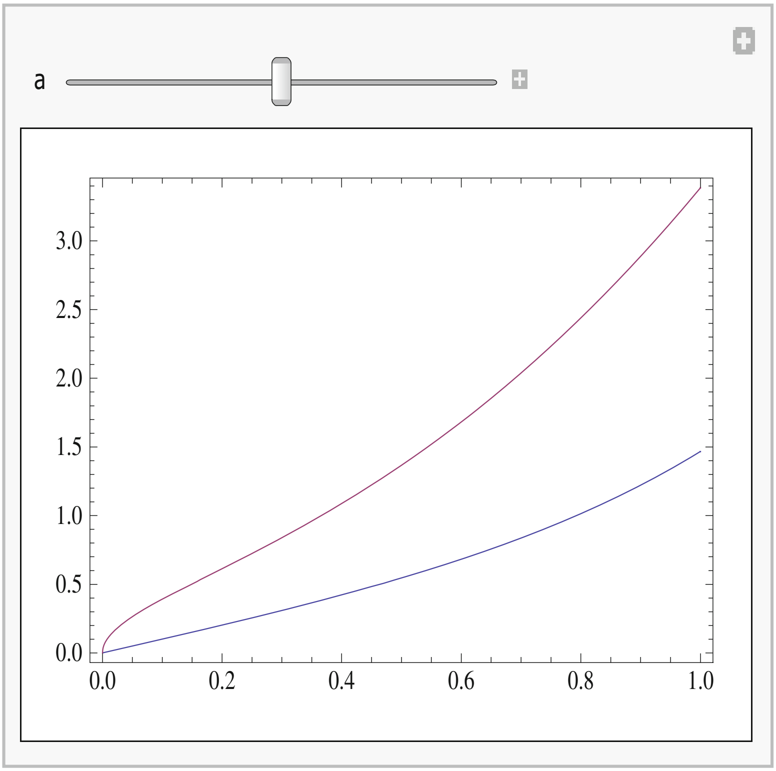

Fig. 4.3 Plots of the functions $f(t)$ and $y(t)$ from the above Example 1

```
    Gamma[2*a + 1]/(Gamma[a + 1])^2*t^(3*a)/Gamma
    [3*a + 1] + 2 Gamma[2*a + 1]/((Gamma[a + 1])^3*Gamma
    [3*a + 1])* Gamma[4*a + 1]/Gamma[5*a + 1]*t^(5*a);
  Plot[{f[t], y[t]}, {t, 0, 1}, ImageSize -> 300,
    Frame -> True], {{a, 1/2}, 0, 1}]
```

Example 2 Find the solution of the FDE:

$$D^{\alpha}y(t) = \frac{-y(t)}{1+\epsilon}, \qquad 0 < \alpha \le 1,$$

for the initial condition:

$$y(0) = \cos(\epsilon), \qquad 0 < \epsilon \ll 1,$$

using the small parameter (perturbation) method.

Solution We consider a solution of form:

$$y(t) = y_0(t) + \epsilon y_1(t) + \epsilon^2 y_2(t) + \epsilon^3 y_3(t) + \ldots,$$

$$\frac{1}{1+\epsilon} = 1 - \epsilon + \epsilon^2 - \epsilon^3 + \ldots$$

$$\cos(\epsilon) = 1 - \frac{1}{2!}\epsilon^2 + \frac{1}{4!}\epsilon^4 + \ldots,$$

which, replaced in the equation, gives:

$$D^{\alpha}(y_0 + \epsilon y_1 + \epsilon^2 y_2 + \ldots) = -(y_0 + \epsilon y_1 + \epsilon^2 y_2 + \ldots)(1 - \epsilon + \epsilon^2 + \ldots)$$

$$D^{\alpha}y_0 = -y_0, \qquad y_0(0) = 1,$$

$$D^{\alpha}y_1 = y_0 - y_1, \qquad y_1(0) = 0,$$

$$D^{\alpha}y_2 = -y_2 + y_1 + y_0, \qquad y_2(0) = -\frac{1}{2},$$

$$\ldots$$

We apply the LT:

$$L[D^{\alpha}y_0(t)] = -L[y_0], \qquad y_0(0) = 1,$$

$$s^{\alpha}Y_0 - y_0(0)s^{\alpha-1} = -Y_0, \qquad Y_0 = \frac{s^{\alpha-1}}{s^{\alpha}+1}$$

$$L[D^{\alpha}y_1] = L[y_0] - L[y_1], \qquad y_1(0) = 0,$$

$$L[D^{\alpha}y_2] = -L[y_2] + L[y_1] + L[y_0], \qquad y_2(0) = -\frac{1}{2},$$

$$(s^{\alpha}+1)Y_1 = Y_0, \qquad Y_1 = \frac{s^{\alpha}}{(s^{\alpha}+1)^2},$$

$$Y_2 = \frac{s^{\alpha-1}}{(s^{\alpha}+1)^3} + \frac{s^{\alpha-1}}{(s^{\alpha}+1)^2} - \frac{1}{2}\frac{s^{\alpha}}{s^{\alpha}+1},$$

$$y_0(t) = 1 - \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \ldots,$$

$$Y_1 = \frac{1}{s^{\alpha+1}}\frac{1}{\left(1+\frac{1}{s^{\alpha}}\right)^2},$$

$$Y_1 = \frac{1}{s^{\alpha+1}} - 2\frac{1}{s^{2\alpha+1}} + 3\frac{1}{s^{4\alpha+1}} + \ldots,$$

and by inverse LT, we obtain:

$$y_1(t) = \frac{t^{\alpha}}{\Gamma(\alpha+1)} - 2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + 3\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + \ldots$$

But:

$$\frac{s^{\alpha-1}}{s^{3\alpha}\left(1+\frac{1}{s^{\alpha}}\right)^3} = \frac{1}{s^{2\alpha+1}}\frac{1}{2}\left[2 - 2\cdot 3\frac{1}{s^{\alpha}} + 4\cdot 3\frac{1}{s^{2\alpha}} + \ldots\right].$$

It results also:

$$\begin{aligned} y_2(t) = {} & \frac{1}{2}\left[2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - 2\cdot 3\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \ldots\right] \\ & + \left[\frac{t^{\alpha}}{\Gamma(\alpha+1)} - 2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + 3\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + \ldots\right] \\ & - \frac{1}{2}\left[1 - \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \ldots\right]. \end{aligned}$$

4.4 Fractional Systems of Differential Equations

4.4.1 Linear Systems

Examples Solve the system of FDE:

$$\begin{cases} D^{\alpha}x(t) = D^{\beta}y(t) + 1, & x(0) = 1, \quad 0 < \alpha \le 1 \\ D^{\beta}y(t) = 2D^{\alpha}x(t) - 1, & y(0) = 1, \quad 0 < \beta \le 1. \end{cases},$$

Solution We apply the LT method:

$$L[x(t)] = X, \quad L[y(t)] = Y,$$

$$L[D^{\alpha}x(t)] = s^{\alpha}X - s^{\alpha-1}x(0) = s^{\alpha}X - s^{\alpha-1},$$

$$L[D^{\beta}y(t)] = s^{\beta}Y - s^{\beta-1}y(0) = s^{\beta}Y - s^{\beta-1}.$$

We obtain the system

$$\begin{cases} X = \dfrac{1}{s}, \\ Y = \dfrac{1}{s} - \dfrac{1}{s^{\beta+!}}, \end{cases}$$

with the solution:

$$\begin{cases} x(t) = 1, \\ y(t) = 1 - \dfrac{t^{\beta}}{\Gamma(\beta+1)}. \end{cases}$$

4.4.2 Nonlinear Systems

(A) Method of Successive Approximations

For the system of FDE:

$$\begin{cases} D^{\alpha}x(t) = f(t, y(t)), & x(0) = x_0, \\ D^{\alpha}y(t) = g(t, x(t)), & y(0) = y_0, \end{cases}$$

we can use the following successive approximations [4]:

$$\begin{cases} x_n(t) = x_0 + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t f(u, y_{n-1}(u))(t-u)^{\alpha-1} du, \\ y_n(t) = y_0 + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t g(u, x_{n-1}(u))(t-u)^{\alpha-1} du. \end{cases}$$

Example We apply the successive approximation method for the system of FDE with initial conditions:

$$\begin{cases} D^\alpha x(t) = 3.5y(t)(1-y(t)), \qquad x(0) = 0.2, \\ D^\alpha y(t) = 4x(t)(1-x(t)), \qquad y(0) = 0.2. \end{cases}$$

We have:

$$\begin{cases} x_n(t) = 0.2 + \dfrac{3.5}{\Gamma(\alpha)} \displaystyle\int_0^t \left(y_{n-1}(u) - y_{n-1}^2(u)\right)(t-u)^{\alpha-1} du, \\ y_n(t) = 0.2 + \dfrac{4}{\Gamma(\alpha)} \displaystyle\int_0^t \left(x_{n-1}(u) - x_{n-1}^2(u)\right)(t-u)^{\alpha-1} du, \end{cases}$$

$$\begin{cases} x_1 = 0.2 + \dfrac{3.5}{\Gamma(\alpha)} \displaystyle\int_0^t \left(0.2 - (0.2)^2\right)(t-u)^{\alpha-1} du, \\ y_1 = 0.2 + \dfrac{4}{\Gamma(\alpha)} \displaystyle\int_0^t \left(0.2 - (0.2)^2\right)(t-u)^{\alpha-1} du. \end{cases}$$

Using the theorem regarding the product of convolution, we obtain the following three iterations:

$$\begin{cases} x_1(t) = 0.2 + 0.56 \dfrac{t^\alpha}{\Gamma(\alpha+1)}, \\ y_1(t) = 0.2 + 0.56 \dfrac{t^\alpha}{\Gamma(\alpha+1)}. \end{cases}$$

$$\begin{cases} x_2(t) = 0.2 + \dfrac{3.5}{\Gamma(\alpha)} \displaystyle\int_0^t \left(y_1(u) - y_2^1(u)\right)(t-u)^{\alpha-1} du, \\ y_2(t) = 0.2 + \dfrac{4}{\Gamma(\alpha)} \displaystyle\int_0^t \left(x_1(u) - x_2^1(u)\right)(t-u)^{\alpha-1} du, \end{cases}$$

$$\begin{cases} x_2(t) = 0.2 \\ +0.56\dfrac{t^{\alpha}}{\Gamma(\alpha+1)} + 1.344\dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} - 0.808\dfrac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}\dfrac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \\ y_2(t) = 0.2 \\ +0.64\dfrac{t^{\alpha}}{\Gamma(\alpha+1)} + 1.344\dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} - 0.7077\dfrac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}\dfrac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \end{cases}$$

$$\begin{cases} x_3(t) = 0.2 + \dfrac{3.5}{\Gamma(\alpha)}\displaystyle\int_0^t \left(y_2(u) - y_2^2(u)\right)(t-u)^{\alpha-1}du \\ y_3(t) = 0.2 + \dfrac{4}{\Gamma(\alpha)}\displaystyle\int_0^t \left(x_2(u) - x_2^2(u)\right)(t-u)^{\alpha-1}du \end{cases}$$

$$\cdots$$

For $\alpha = 0.9$ we can apply the Maple and Mathematica programs:

MAPLE

```
> restart;
> with(inttrans):
> Digits:=5:
> x:=array(0..10):
> y:=array(0..10):
> x[0]:=0.2:
> y[0]:=0.2:
> for k from 1 to 5 do
> x[k]:=evalf(0.2+3.5*invlaplace(1/s^0.9*laplace
                                   (y[k-1]-(y[k-1])^2,
                                   t,s),s,t));
> y[k]:=evalf(0.2+4*invlaplace(1/s^0.9*laplace
                               (x[k-1]-(y[k-1])^2,
                               t,s),s,t));
> od:
> for k from 0 to 4 do
> print([x[k],y[k]]) od:
 x                        y
 0.2                       0.2
 0.2 + 0.58230 t^(9/10)    0.2 + 0.66548t^(9/10)
 0.2 + 0.58230 t^(9/10) + 0.80171 t^(9/5) - 0.62304
        t^(27/10),
 0.2 + 0.66548 t^(9/10) + 0.72536 t^(9/5) - 0.71204
         t^(27/10)
                  . . . . .
```

MATHEMATICA

```
Clear["`*"]
Array[x, 10]
Array[y, 10]
For[{n = 0, x[0] = 0.2, y[0] = 0.2}, n < 4,
 n++, {x[n + 1] =
   0.2 + 3.5 InverseLaplaceTransform[
   1/s^0.9 LaplaceTransform[y[n] - (y[n])^2, t, s],
    s, t]// FullSimplify,
y[n + 1] =
 0.2 + 4 InverseLaplaceTransform[
 1/s^0.9 LaplaceTransform[(x[n] - (x[n])^2), t, s],
     s, t]//
 FullSimplify, Print["x=", x[n], " , ", "y= ", y[n]]}]
  x = 0.2,                    y=0.2
  x = 0.2+0.582262 t^0.9 , y= 0.2+0.665443 t^0.9
```

(B) Method of Laplace's Transform

We will illustrate this method on the function:

$$F(y) = y - y^2.$$

First, we will decompose F in terms of Adomian's polynomials

$$F = \sum_{n=0}^{\infty} A_n,$$

where $A_0 = y_0 - y_0^2$ and

$$\phi_1(\lambda) = (y_0 + \lambda y_1) - (y_0 + \lambda y_1)^2,$$

$$\phi_1' = y_1 - 2y_1(y_0 + \lambda y_1),$$

$$A_1 = \frac{1}{1!}\phi_1(0),$$

from which we obtain: $A_1 = y_1 - 2y_0 y_1$.

In the case of next step we have:

$$\phi_2(\lambda) = (y_1 + 2\lambda y_2) - 2(y_0 + \lambda y_1)(y_1 + 2\lambda y_2),$$

$$\phi_2' = 2y_2 - 2y_1(y_1 + 2\lambda y_2) - 2y_2(y_0 + \lambda y_1),$$

$$A_2 = \frac{1}{2!}\phi_2'(0),$$

or finally: $A_2 = y_2 - y_1^2 - y_2 y_0$.

We have also:

$$\phi_3(\lambda) = (y_2 + 3\lambda y_3) - (y_1 + 2\lambda y_2)^2 - (y_2 + 3\lambda y_3)(y_0 + \lambda y_1),$$

$$A_3 = \frac{1}{3!}\phi_3'(0),$$

$$\dots$$

We consider now a system described by the equations [8]:

$$\begin{cases} L[D^\alpha x(t)] = 3.5L[y(t)(1-y(t))], & x(0) = 0.2, \\ L[D^\alpha y(t)] = 4L[x(t)(1-x(t))], & y(0) = 0.2, \end{cases}$$

with initial conditions and we apply the LT to this system. We have:

$$L[x(t)] = X, \qquad L[y(t)] = Y,$$

$$L[D^\alpha x(t)] = s^\alpha - x(0)s^{\alpha-1},$$

$$L[D^\alpha y(t)] = s^\alpha - y(0)s^{\alpha-1},$$

We consider the solutions:

$$X = \sum_{n=0}^{\infty} X_n, \qquad Y = \sum_{n=0}^{\infty} Y_n.$$

After calculations we have:

$$L\left[x(t)(1-x(t))\right] = L\left[\sum_{n=0}^{\infty} A_n\right], \qquad L\left[y(t)(1-y(t))\right] = L\left[\sum_{n=0}^{\infty} B_n\right],$$

where A_n and B_n are Adomian's polynomials.

$$\sum_{n=0}^{\infty} X_n = \frac{0.2}{s} + \frac{3.5}{s^\alpha} L\left[\sum_{n=0}^{\infty} B_n\right]$$

$$\sum_{n=0}^{\infty} Y_n = \frac{0.2}{s} + \frac{4}{s^\alpha} L\left[\sum_{n=0}^{\infty} A_n\right]$$

$$X_0 = \frac{0.2}{s} \quad \Rightarrow \quad x_0 = 0.2$$

$$Y_0 = \frac{0.2}{s} \quad \Rightarrow \quad x_0 = 0.2$$

$$\begin{cases} X_1 = \dfrac{3.5}{s^\alpha} L[B_0] = \dfrac{0.56}{s^{\alpha+1}}, \quad \Rightarrow \quad x_1(t) = 0.56 \dfrac{t^\alpha}{\Gamma(\alpha+1)} \\ \\ Y_1 = \dfrac{4}{s^\alpha} L[A_0] = \dfrac{0.64}{s^{\alpha+1}}, \quad \Rightarrow \quad y_1(t) = 0.64 \dfrac{t^\alpha}{\Gamma(\alpha+1)} \end{cases}$$

$$\begin{aligned} B_1 &= y_1 - 2y_0 y_1 = y_1(1 - 2y_0) = 0.64 \frac{t^\alpha}{\Gamma(\alpha+1)} (1 - 2 \cdot 0.2) \\ &= 0.64 \cdot 0.6 \cdot \frac{t^\alpha}{\Gamma(\alpha+1)} = 0.384 \frac{t^\alpha}{\Gamma(\alpha+1)} \end{aligned}$$

$$\begin{aligned} A_1 &= x_1 - 2x_0 x_1 = x_1(1 - 2x_0) = 0.56 \frac{t^\alpha}{\Gamma(\alpha+1)} (1 - 2 \cdot 0.2) \\ &= 0.56 \cdot 0.6 \cdot \frac{t^\alpha}{\Gamma(\alpha+1)} = 0.336 \frac{t^\alpha}{\Gamma(\alpha+1)} \end{aligned}$$

$$X_2 = \frac{3.5}{s^\alpha} L[B_1] = \frac{1.344}{s^{2\alpha}} \quad \Rightarrow x_2(t) = 1.344 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

$$Y_2 = \frac{4}{s^\alpha} L[A_1] = \frac{1.344}{s^{2\alpha}} \quad \Rightarrow y_2(t) = 1.344 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

Finally, the solution is:

$$\begin{cases} x(t) = x_0(t) + x_1(t) + x_2(t) + \ldots = 0.2 + 0.56 \dfrac{t^\alpha}{\Gamma(\alpha+1)} + 1.344 \dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \ldots \\ \\ y(t) = y_0(t) + y_1(t) + y_2(t) + \ldots = 0.2 + 0.64 \dfrac{t^\alpha}{\Gamma(\alpha+1)} + 1.344 \dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \ldots \end{cases}$$

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