Chapter 3
The Laplace Transform

A function $f(t)$ is called original function if [8, 9]:

1. $f(t) \equiv 0 \quad \text{for } t < 0$,
2. $|f(t)| < Me^{s_0 t}$ for $t > 0$ with $M > 0$, $s_0 \in \mathbb{R}$.
3. For every closed interval $[a, b]$, the function satisfies the Dirichlet conditions:
   (a) is bounded,
   (b) or is continuous, or has a finite number of discontinuities of first kind,
   (c) has a finite number of extremes.

We consider the complex variable $s = \alpha + i\beta$, where $\text{Re}(s) = \alpha \geq s_1 \geq s_0$. Then

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt, \tag{3.1}$$

is called the Laplace[1] integral, or Laplace transform (LT), or *image* of the original function $f(t)$. In the follow-up we denote by $L[f(t)] = F(s)$ or simply by Laplace transform (L) the Laplace transform. In Table 3.1 the LT of some elementary usual functions are listed.

The corresponding inverse Laplace transform is [1, 4]:

$$f(t) = \frac{1}{2\pi i} \lim_{t\to\infty} \int_{\gamma - it}^{\gamma + it} F(s)e^{st} dt = L^{-1}[F(s)], \tag{3.2}$$

where $i = \sqrt{-1}$ and $\gamma \in \mathbb{R}$, so that the contour path of integration is contained in the convergence region.

[1] Pierre Laplace (1749–1827).

C. Milici et al., *Introduction to Fractional Differential Equations*, Nonlinear Systems and Complexity 25, https://doi.org/10.1007/978-3-030-00895-6_3

Table 3.1 Images of basic elementary functions

Number	Original	Image	Number	Original	Image
1	1	$\frac{1}{s}$	7	$e^{\alpha t}\cos\beta t$	$\frac{s-\alpha}{(s-\alpha)^2+\beta^2}$
2	$\frac{t^n}{n!}$	$\frac{1}{s^{n+1}}$	8	$e^{\alpha t}\sin\beta t$	$\frac{\beta}{(s-\alpha)^2+\beta^2}$
3	$e^{\alpha t}$	$\frac{1}{s-\alpha}$	9	$\frac{t^n}{n!}e^{\alpha t}$	$\frac{1}{(s-\alpha)^{n+1}}$
4	$\cos\beta t$	$\frac{s}{s^2+\beta^2}$	10	$t\cos\beta t$	$\frac{s^2-\beta^2}{(s^2+\beta^2)^2}$
5	$\sin\beta t$	$\frac{\beta}{s^2+\beta^2}$	11	$t\sin\beta t$	$\frac{2s\beta}{(s^2+\beta^2)^2}$
6	$\cosh(\beta t)$	$\frac{s}{s^2-\beta^2}$	12	$\sinh(\beta t)$	$\frac{\beta}{s^2-\beta^2}$

3.1 Calculus of the Images

Example 1 Establish the image of $f(t) = t^\lambda$:

$$F(s) = \int_0^\infty e^{-ts} t^\lambda \, dt.$$

We introduce the change of variable $x = ts$. We have also $dx = s\,dt$. It results:

$$F(s) = \int_0^\infty e^{-x} \frac{x^\lambda}{s^\lambda} \frac{dx}{s},$$

$$F(s) = \frac{1}{s^{\lambda+1}} \int_0^\infty e^{-x} x^\lambda dx = \frac{\Gamma(\lambda+1)}{s^{\lambda+1}}.$$

The direct and inverse LT are:

$$L(t^\lambda) = \frac{\Gamma(\lambda+1)}{s^{\lambda+1}}, \qquad L^{-1}\left(\frac{1}{s^{\lambda+1}}\right) = \frac{t^\lambda}{\Gamma(\lambda+1)}.$$

Example 2 Find the image of: $f(t) = \sin^2(t)$.

Using the identity $\sin^2 t = \dfrac{1-\cos(2t)}{2}$, it results:

$$F(s) = \frac{1}{2s} - \frac{2s}{2(s^2+4)} = \frac{2}{s(s^2+4)}.$$

Example 3 Find the image of $f(t) = \frac{1}{2}te^{bt} + \frac{1}{2}te^{-bt}$.

$$F(s) = \frac{1}{2(s-b)^2} + \frac{1}{2(s+b)^2} = \frac{s^2+b^2}{(s^2-b^2)^2}.$$

3.2 Calculus of the Original Function

3.2.1 Calculus of Original Using Residues

If we denote by $L[f(t)] = F(s)$, then if we consider all residues of the function $F(s)e^{st}$, denoted by $r_1, r_2, \ldots r_n$ we can use the theorem:

$$f(t) = r_1 + r_2 + \ldots + r_n.$$

The residues, denoted by Residues (Res), can be calculated using the following procedure (theorem):

If a is a simple pole of the function, then:

$$\operatorname*{Res}_{a}[e^{st}F(s)] = \lim_{s\to a}[(s-a)e^{st}F(s)].$$

If a is a simple pole of order n of the function, then:

$$\operatorname*{Res}_{a}[e^{st}F(s)] = \frac{1}{(n-1)!}\lim_{s\to a}[(s-a)^n e^{st}F(s)]^{(n-1)}.$$

Example 1 Find the original function of the image:

$$F(s) = \frac{1}{(s-3)^2(s+1)}.$$

Solution The residue of the function $F(s)e^{st}$ is

$$r_1 = \lim_{t\to -1}(s+1)\frac{e^{st}}{(s-3)^2(s+1)} = \frac{e^{-t}}{16},$$

$$r_2 = \lim_{t\to 3}\frac{d}{ds}\left(\frac{e^{st}}{s+1}\right)$$

$$= \frac{te^{3t}-e^{3t}}{16},$$

resulting:

$$f(t) = r_1 + r_2 = \frac{e^{-t}}{16} + \frac{te^{3t}-e^{3t}}{16}.$$

Example 2 Find the original function of the following image:

$$F(s) = \frac{s^2}{s^4-1}.$$

Solution The function F has singularities: $1,\ -1,\ -i,\ i$. The residues will be:

$$r_1 = \operatorname*{Res}_{1} F(s)e^{st} = \lim_{s\to 1}(s-1)\frac{s^2 e^{st}}{(s-1)(s+1)(s^2+1)} = \frac{e^t}{4},$$

$$r_2 = \operatorname*{Res}_{-1} F(s)e^{st} = \lim_{s\to -1}(s+1)\frac{s^2 e^{st}}{(s-1)(s+1)(s^2+1)} = -\frac{e^{-t}}{4},$$

$$r_3 = \operatorname*{Res}_{-i} F(s)e^{st} = \lim_{s\to -i}(s+i)\frac{s^2 e^{st}}{(s^2-1)(s-i)(s+i)} = \frac{-e^{-it}}{4i},$$

$$r_4 = \operatorname*{Res}_{i} F(s)e^{st} = \lim_{s\to i}(s+i)\frac{s^2 e^{st}}{(s^2-1)(s-i)(s+i)} = \frac{e^{it}}{4i}.$$

It results finally

$$f(t) = \frac{1}{2}\left(\frac{e^t - e^{-t}}{2}\right) + \frac{1}{2}\left(\frac{e^{it} - e^{-it}}{2i}\right),$$

or:

$$f(t) = \frac{1}{2}(\sinh t + \sin t).$$

3.2.2 *Calculus of Original with Post's Inversion Formula*

E. Post[2] obtained the formula [7, 10]:

$$f(t) = \lim_{k\to\infty}\frac{(-1)^k}{k!}\left(\frac{k}{t}\right)^{k+1} F^{(k)}\left(\frac{k}{t}\right), \quad t > 0.$$

Example 3 Find the original function of the image:

$$F(s) = \frac{n!}{s^{n+1}} n! s^{-1-n}.$$

Solution With the aid of Post formula we have

$$f(t) = t^n \lim_{k\to\infty}\frac{k^{k+1}(n+k)!}{k!t^{k+1}}\left(\frac{k}{t}\right)^{-n-k-1}.$$

[2]E. Post (1897–1954).

Using the Stirling[3] formula:

$$\lim_{k\to\infty} \frac{k!}{\sqrt{2\pi k}} k^k e^{-k} = 1,$$

it results:

$$f(t) = t^n e^{-n} \lim_{k\to\infty} \sqrt{1+\frac{n}{k}} \left(1+\frac{n}{k}\right)^k \left(1+\frac{n}{k}\right)^n = t^n.$$

3.3 The Properties of the Laplace Transform

In this section, we will use the notations $F(s) = L[f(t)]$ and $L(s) = G[g(t)]$. In the follow-up are discussed properties of the LT.

3.3.1 The Property of Linearity

$$L[af(t) + bg(t)] = aF(s) + bG(s), \quad a, b \in \mathbb{R}. \tag{3.3}$$

3.3.2 Similarity Theorem

$$L[f(\alpha t)] = \frac{1}{\alpha} F\left(\frac{s}{\alpha}\right), \qquad \alpha > 0. \tag{3.4}$$

3.3.3 The Differentiation and Integration Theorems

Theorem (Differentiation of an Original) *The LT of the derivative of order k from $f(t)$ gives:*

$$L[f^{(k)}(t)] = s^k F(s) - \left[s^{k-1} f(0) + s^{k-2} f'(0) + \ldots + f^{(k-1)}(0)\right]. \tag{3.5}$$

Proof For $k = 1$, using the definition and integrating by parts, we have:

$$\begin{aligned} L[f'(t)] &= \int_0^\infty e^{-st} f'(t)dt = e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)\,dt \\ &= -f(0) + sL[f(t)], \end{aligned}$$

[3]J. Stirling (1692–1770).

$$L[f'(t)] = sF(s) - f(0),$$

and for $k = 2$, using $L[f''(t)] = L[(f'(t))']$ we obtain:

$$L[f''(t)] = s^2F(s) - sf(0) - f'(0).$$

Using mathematical induction method, we have: $L[f^{(k)}(t)]$.

Example 1 Find the LT for original $f(t) = t^2$.

Solution

$f'(t) = 2t, \quad f''(t) = 2,$
$f(0) = 0, \quad f'(0) = 0, \quad f''(0) = 2,$
$L[f''(0)] = s^2F(s) - sf(0) - f'(0),$
$L[2] = \frac{2}{s} = s^2F(s) \quad \Rightarrow \quad F(s) = \frac{2}{s^3}.$

Example 2 Find the LT of following original $f(t) = \cos 2t$.

Solution $f'(t) = -2\sin(2t), \quad f''(t) = -4\cos(2t),$
$f(0) = 1, \quad f'(0) = 0, \quad f''(0) = -4,$
$L[f''(t)] = s^2F(s) - sf'(0) - f(0), \quad \Rightarrow \quad -4F(s) = s^2F(s) - s,$

$$F(s) = \frac{s}{s^2 + 4}.$$

Theorem (Integration of an Original) *It can be obtained:*

$$L\left[\int_0^t f(\tau)d\tau\right] = \frac{F(s)}{s},$$

Proof Let be $g(t) = \int_0^t f(\tau)d\tau$. Then:

$$L[g'(t)] = sL[g] - g(0).$$

Also, we have $g'(t) = f(t)$ and $g(0) = 0$. Then:

$$L[g] = \frac{F(s)}{s}.$$

Theorem (Differentiation of a Transform) *We have:*

$$F^{(n)}(s) = L[(-t)^n f(t)].$$

Proof The theorem can be proved by induction. For $n = 1$, we have, successively:

$$F'(s) = L[-tf(t)],$$

$$\frac{dF(s)}{ds} = \frac{d}{ds}\int_0^\infty e^{-st} f(t)dt = -\int_0^\infty e^{-st} tf(t)dt = -L[tf(t)],$$

and, finally:

$$F^{(n)}(s) = \frac{d}{ds}[F^{(n-1)}(s)].$$

3.3.4 Delay Theorem

For a positive number a we have:

$$L[f(t-a)] = e^{-as}F(s)$$

3.3.5 Displacement Theorem

It is valid the formula:

$$L[e^{\lambda t}f(t)] = F(s-\lambda).$$

3.3.6 Multiplication Theorem

The convolution product of two functions $f(t)$ and $g(t)$ is designated by the symbol $*$. We have:

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau,$$

$$L[(f * g)(t)] = F(s)G(s), \quad F(s) = L[f(t)], \quad G(s) = L[g(t)].$$

3.3.7 Properties of the Inverse Laplace Transform

The following formula is valid:

The inverse LT is not unique. We have:

$$L^{-1}\left[\frac{s^{-(\alpha-\beta)}}{s^\beta - a}\right] = t^{\alpha-1}E_{\beta,\alpha}(at^\beta),\ \alpha, \beta > 0, s^\alpha > |a|, \tag{3.6}$$

$$L^{-1}\left[\frac{s^{-(\alpha-1)}}{s-a}\right] = t^{\alpha-1}E_{1,\alpha}(at) = E(t,\alpha-1,a), \tag{3.7}$$

$$L^{-1}\left[\frac{s^{-\alpha}}{(s-a)^2}\right] = tE(t,\alpha,a) - \alpha E(t,\alpha+1,a), \tag{3.8}$$

$$L^{-1}\left[\frac{s^{-\alpha}}{(s-a)^3}\right] = \frac{1}{2}t^2E(t,\alpha,a) - \alpha tE(t,\alpha+1,a) + \frac{\alpha(\alpha+1)}{2}E(t,\alpha+2,a), \tag{3.9}$$

$$L^{-1}\left[\frac{1}{(s^\alpha + as^\beta)^{n+1}}\right] = t^{\alpha(n+1)-1}\sum_{k=0}^{\infty}\frac{(-a)^k\binom{n+k}{k}}{\Gamma[k(\alpha-\beta)+(n+1)\alpha]}t^{k(\alpha-\beta)}, \tag{3.10}$$

where $0 < \beta \leq \alpha$.

$$L^{-1}\left[\frac{s^\gamma}{s^\alpha + as^\beta + b}\right] = t^{\alpha-\gamma-1}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-b)^n(-a)^k\binom{n+k}{k}}{\Gamma[k(\alpha-\beta)+(n+1)\alpha-\gamma]}t^{k(\alpha-\beta)+n\alpha}, \tag{3.11}$$

where $\beta \leq \alpha,\ \ \gamma < \alpha,\ a \in \mathbb{R}$ or: $|a| < s^{\alpha-\beta},\ |b| < |s^\alpha + as^\beta|$.

Proof Proof of the identity (3.6):

$$\begin{aligned}
L\left[t^{\alpha-1}E_{\beta,\alpha}(at^\beta)\right] &= \int_0^\infty e^{-st}t^{\alpha-1}E_{\beta,\alpha}(at^\beta)dt \\
&= \sum_{k=0}^{\infty}\frac{a^k}{\Gamma(k\beta+\alpha)}\int_0^\infty e^{-st}t^{k\beta+\alpha-1}dt \\
&= \sum_{k=0}^{\infty}\frac{a^k}{\Gamma(k\beta+\alpha)}L[t^{k\beta+\alpha-1}] \\
&= \sum_{k=0}^{\infty}\frac{a^k}{\Gamma(k\beta+\alpha)}\frac{\Gamma(k\beta+\alpha)}{s^{k\beta+\alpha}} = \frac{1}{s^\alpha}\sum_{k=0}^{\infty}\left(\frac{a}{s^\beta}\right)^k = \frac{s^{-(\alpha-\beta)}}{s^\beta-a}.
\end{aligned}$$

Proof of the identity (3.8):

$$\begin{aligned}
L[t\,E(t,\alpha,a) - \alpha E(t,\alpha+1,a)] &= -\frac{d}{ds}L[E(t,\alpha,a)] - \alpha L[E(t,\alpha+1,a)] \\
&= -\frac{d}{ds}\left[\frac{s^{-\alpha}}{s-a}\right] - \alpha\left[\frac{s^{-(\alpha+1)}}{s-a}\right] = \frac{1}{s^\alpha(s-a)^2}.
\end{aligned}$$

Proof of the identity (3.9):

$$L\left[\frac{1}{2}t^2E(t,\alpha,a)-\alpha tE(t,\alpha+1,a)+\frac{1}{2}\alpha(\alpha+1)E(t,\alpha+2,a)\right]$$

$$=\frac{1}{2}\frac{d^2}{ds^2}L\left[E(t,\alpha,a)\right]+\alpha\frac{d}{ds}L\left[E(t,\alpha+1,a)\right]+\frac{\alpha(\alpha+1)}{2}L\left[E(t,\alpha+2,a)\right]$$

$$=\frac{1}{2}\frac{d^2}{ds^2}\left(\frac{s^{-\alpha}}{s-a}\right)+\alpha\frac{d}{ds}\left(\frac{s^{-(\alpha+1)}}{s-a}\right)+\frac{\alpha(\alpha+1)}{2}\left(\frac{s^{-(\alpha+2)}}{s-a}\right)=\frac{1}{s^{\alpha}(s-a)^3}.$$

For the identities (3.10) and (3.11) the reader can use the reference [5].
Proof of the identity (3.10). We will apply the well-known identity [4]:

$$\frac{1}{(1+x)^n}=\sum_{k=0}^{\infty}\binom{n+k}{k}(-x)^k.$$

It follows:

$$\frac{1}{(s^{\alpha}+as^{\beta})^{n+1}}=\frac{1}{(s^{\alpha})^{n+1}}\frac{1}{\left(1+\frac{a}{s^{\alpha\beta}}\right)^{n+1}}$$

$$=\frac{1}{(s^{\alpha})^{n+1}}\sum_{k=0}^{\infty}\binom{n+k}{k}\left(\frac{-a}{s^{\alpha-\beta}}\right)^k.$$

Proof of the identity (3.11):

$$\frac{s^{\gamma}}{s^{\alpha}+as^{\beta}+b}=\frac{s^{\gamma}}{s^{\alpha}+as^{\beta}}\frac{1}{1+\dfrac{b}{s^{\alpha}+as^{\beta}}}=\sum_{n=0}^{\infty}\frac{s^{\gamma}(-b)^n}{(s^{\alpha}+as^{\beta})^{n+1}},$$

and for the case of (3.10) we obtain:

$$\frac{s^{\gamma}}{(s^{\alpha}+s^{\beta}a)^{n+1}}=\frac{s^{\gamma}}{s^{\alpha(n+1)}}\frac{1}{\left(1+\frac{a}{s^{\alpha-\beta}}\right)^{n+1}}$$

$$=\frac{1}{s^{\alpha(n+1)-\gamma}}\sum_{k=0}^{\infty}\binom{n+k}{k}\left(\frac{-a}{s^{\alpha-\beta}}\right)^k=\sum_{k=0}^{\infty}\binom{n+k}{k}\frac{(-a)^k}{s^{\alpha(n+1)+k(\alpha-\beta)-\gamma}},$$

$$\frac{s^{\gamma}}{s^{\alpha}+as^{\beta}+b}=\sum_{n=0}^{\infty}(-b)^n\sum_{k=0}^{\infty}\binom{n+k}{k}\frac{(-a)^k}{s^{\alpha(n+1)+k(\alpha-\beta)-\gamma}},$$

$$L^{-1}\left[\frac{s^{\gamma}}{s^{\alpha}+as^{\beta}+b}\right]=t^{\alpha-\gamma-1}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-b)^{n}(-a)^{k}\binom{n+k}{k}}{\Gamma[k(\alpha-\beta)+(n+1)\alpha-\gamma]}t^{k(\alpha-\beta)+n\alpha}.$$

Lemma *The following identities are valid:*

$$L^{-1}\left[\frac{1}{s^{\alpha}+as+b}\right]=t^{\alpha-1}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-b)^{n}(-a)^{k}\binom{n+k}{k}}{\Gamma[k(\alpha-1)+(n+1)\alpha]}t^{k(\alpha-1)+n\alpha}, \tag{3.12}$$

for $1\le\alpha$, $0<\alpha$, $a\in\mathbb{R}$, *and* $|a|<s^{\alpha-1}$, $|b|<|s^{\alpha}+as|$, *respectively:*

$$L^{-1}\left[\frac{s^{\alpha-1}}{s^{\alpha}+as+b}\right]=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-b)^{n}(-a)^{k}\binom{n+k}{k}}{\Gamma[k(\alpha-1)+n\alpha+1]}t^{k(\alpha-1)+n\alpha}, \tag{3.13}$$

for $1\le\alpha$, $a\in\mathbb{R}$ *and for* $|a|<s^{\alpha-1}$, $|b|<|s^{\alpha}+as|$.

Proof Proof of the identity (3.12). In (3.11) we take $\gamma=0$, $\beta=1$.
Proof of the identity (3.13). In (3.11) we take $\gamma=\alpha-1$, and $\beta=1$.

Example 1 Establish the LT of:

$$f(t)=y''(t)-2y'(t)-3y(t);\qquad \text{where: } y(0)=y'(0)=0.$$

Solution

$$F(s)=s^{2}Y(s)-sy(0)-y'(0)-2[sY(s)-y(0)]-3Y(s),$$

and finally:

$$F(s)=(s^{2}-2s-3)Y(s).$$

Example 2 Establish the LT of:

$$y=\int_{0}^{t}y\,dt+1.$$

Solution

$$Y(s)=\frac{Y(s)}{s}+\frac{1}{s}\quad\Rightarrow Y(s)=\frac{1}{s-1}.$$

Example 3 Establish the LT of:

$$\int_{0}^{t}y(\tau)\sin(t-\tau)d\tau=1-\cos t.$$

Solution

$$Y(s)\frac{1}{s^2+1} = \frac{1}{s} - \frac{s}{s^2+1} = \frac{1}{s(s^2+1)}, \qquad \Rightarrow Y(s) = \frac{1}{s}.$$

Example 4 Establish the LT of:

$$\int_0^t y(\tau)e^{t-\tau}d\tau = y(t) - e^t.$$

Solution

$$Y(s)\frac{1}{s-1} = Y(s) - \frac{1}{s-1}; \qquad \Rightarrow \qquad Y(s) = \frac{1}{s-2}.$$

3.4 Laplace Transform of the Fractional Integrals and Derivatives

3.4.1 Fractional Integrals

If $\alpha > 0$, the Riemann–Liouville and Caputo FI are the same for both cases:

$$I = I^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-y)^{\alpha-1} f(y)dy.$$

Using the LT of the convolution product formula, we have:

$$L[I] = \frac{1}{\Gamma(\alpha)}L[t^{\alpha-1}]L[f(t)] = \frac{F(s)}{s^\alpha}.$$

3.4.2 Fractional Derivatives

– The Riemann–Liouville FD is

$$L\left[D_t^\alpha f(t)\right] = L\left[\frac{1}{\Gamma(n-\alpha)}\left(\frac{dt^n}{d^n t}\right)\int_0^t (t-u)^{n-\alpha-1} f(u)du\right]$$
$$= L\left[\left(\frac{dt^n}{d^n t}\right) I^{n-\alpha} f(t)\right],$$

where we can apply the classical formula:

$$L[f^{(n)}(t)] = s^n F(s) - s^{n-1} f'(0) - \ldots - f^{(n-1)}(0),$$

$$L[D_t^\alpha f(t)] = s^n \frac{F(s)}{s^{n-\alpha}} - \sum_{k=0}^{n-1} s^{n-\alpha-1} [D^k I^{n-\alpha} f(t)]_{t=0},$$

$$L[D_t^\alpha f(t)] = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{n-\alpha-1} [D^k I^{n-\alpha} f(t)]_{t=0}.$$

– The Caputo [2, 3] FD is

$$L[D^\alpha f(t)] = L\left[\frac{1}{\Gamma(n-\alpha)} \int_0^t (t-u)^{n-\alpha-1} f^{(n)}(u)du\right] = L[I^{n-\alpha} f^{(n)}(t)],$$

where we can apply the classical formula:

$$L[f^{(n)}(t)] = s^n F(s) - s^{n-1} f'(0) - \ldots - f^{(n-1)}(0),$$

and

$$L[D^\alpha f(t)] = s^n \frac{F(s)}{s^{n-\alpha}} - \sum_{k=0}^{n-1} s^{n-\alpha-1} [I^{n-\alpha} f^{(k)}(t)]_{t=0}.$$

$$L[D^\alpha f(t)] = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{n-\alpha-1} f^{(k)}(0).$$

Exercise 5 For the function $f(t) = t^2$, calculate the Caputo $L[D^\alpha]$. It results:

1. $\alpha = \frac{1}{2}$,
2. $\alpha = -\frac{1}{2}$.

Solution

1.

$$L[D^{1/2} t^2] = \frac{1}{s^{1-\frac{1}{2}}} L[2t],$$

$$L[D^{1/2} t^2] = \frac{2}{s^{\frac{5}{2}}},$$

$$D^{1/2} t^2 = L^{-1}\left[\frac{2}{s^{\frac{5}{2}}}\right] = \frac{2t^{\frac{3}{2}}}{\Gamma\left(\frac{5}{2}\right)} = \frac{8t^{\frac{3}{2}}}{3\sqrt{\pi}}.$$

In **MAPLE**, the FD of order $1/2$ of t from t^2 can be evaluated using the command: **fracdiff(t^2,t,1/2)**.

For $0 < \alpha < 1$, $f^{\alpha}(0) = 0$, and

$$F(t) = \int_0^t f(u)(du)^{\alpha} = \alpha \int_0^t (t-u)^{\alpha-1} f(u)\, du$$

we define:

$$L_{\alpha}[f(t)] = F_{\alpha}(s) = \int_0^{\infty} E_{\alpha}(-s^{\alpha} t^{\alpha}) f(t)\, (dt)^{\alpha}.$$

The following formulae can be obtained without difficulty [6]:

1. $L_{\alpha}[t^{\alpha} f(t)] = -D^{\alpha} L_{\alpha}[f(t)]$.
2. $L_{\alpha}[f(at)] = \dfrac{1}{a^{\alpha}} L_{\alpha}[f(t)]$.
3. $L_{\alpha}[f(t-b)] = E_{\alpha}(-s^{\alpha} b^{\alpha}) L_{\alpha}[f(t)]$.
4. $L_{\alpha}\left[\int_0^t f(u)\, (du)^{\alpha}\right] = \dfrac{1}{a^{\alpha}\Gamma(\alpha+1)} L_{\alpha}[f(t)]$.
5. $L_{\alpha}[g^{(\alpha)}(t)] = s^{\alpha} L_{\alpha}[g(t)] - \Gamma(\alpha+1) g(0)$.

For

$$(f(t) * g(t))_{\alpha} = \int_0^t f(t-u) g(u) (du)^{\alpha},$$

we have:

$$L_{\alpha}\left[(f(t) * g(t))_{\alpha}\right] = L_{\alpha}[f(t)]\, L_{\alpha}[g(t)].$$

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