Chapter 7
Understanding Fractions: Integrating Results from Mathematics Education, Cognitive Psychology, and Neuroscience

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Abstract Many students face difficulties with fractions. Research in mathematics education and cognitive psychology aims at understanding where and why students struggle with fractions and how to make teaching of fractions more effective. Additionally, neuroscience research is beginning to explore how the human brain processes fractions. Yet, attempts to integrate research results from these disciplines are still scarce. Therefore, the aim of this chapter is to provide an integrated view on research from mathematics education, cognitive psychology, and neuroscience to better understand students' difficulties with fraction processing and fraction learning. We evaluate the difficulties students encounter with fractions on various levels, ranging from the brain level to the classroom level. Current research suggests that the human cognitive system is in principle prepared for processing natural numbers and fractions. Although proficiency with natural numbers is fundamental to learning fractions, the transition from natural numbers to fractions requires modifications of the initial concept of numbers, and natural number processing can interfere with

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fraction processing. Thus, when teaching fractions, it seems important to draw on students' fundamental abilities to process fractions, while explicating fraction properties that are conceptually different from those of natural numbers.

Keywords Rational numbers · Conceptual change · Natural number bias · Fraction processing · Numerical cognition

Students' difficulties with fractions have been studied for decades. Yet, research in cognitive psychology and neuroscience has only recently begun to unravel the underlying cognitive mechanisms of fraction processing, and this research has rarely been integrated with mathematics education. The aim of this chapter is, therefore, to make connections between these three disciplines to better understand the sources of difficulties students face with fractions.

In this chapter, we focus specifically on positive fractions, that is, positive rational numbers represented in the form $\frac{a}{b}$, where a and b are positive natural numbers. However, as fraction learning is an instance of learning about rational numbers more generally—which include negative fractions and numbers represented as decimals (e.g., 0.25)—we also consider core issues of the transition from natural number concepts to rational number concepts.

In the first section of the chapter, we review the importance of fraction learning, including arguments from mathematics education and cognitive psychology. The second section analyzes typical difficulties students encounter in fraction learning as documented by empirical research, as well as potential sources of these difficulties. We analyze difficulties on three different levels: (a) difficulties that may be inherent in the learning content, (b) difficulties that may arise from the way our cognitive system processes fractions, including the neural correlates of fraction processing, and (c) difficulties that may be due to common teaching practices. In the third section, we review experimental intervention studies aimed at supporting students' fraction learning to identify effective ways of instruction that may help students overcome difficulties with fractions. The fourth section includes recommendations for classroom practice and directions for further research. In the fifth section, we conclude the chapter with a suggestion for merging various research perspectives.

7.1 Importance of Fraction Learning

It is widely accepted that fractions are important to learn. A basic understanding of fractions is needed in daily life, for example, to understand information on street signs (e.g., $\frac{3}{4}$ mile), in cooking recipes (e.g., $\frac{1}{2}$ L), or regarding time ("quarter past five").

From a *mathematics education perspective*, fractions are important because they are an essential building block within the domain of numbers, one of the key domains of (school) mathematics (e.g., National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). Between primary school and the end of high school, students are supposed to learn about real numbers in a hierarchical manner. This hierarchy begins with natural numbers, and positive fractions are typically the first type of non-natural numbers students encounter[1]. One motivation for introducing rational numbers is that they allow for describing phenomena that cannot be described by natural numbers alone. For example, all arithmetic operations (addition, subtraction, multiplication, and division) can be performed within the set of rational numbers, which is not the case within natural numbers (e.g., $3-5$ and $1\div2$ are not defined within natural numbers). Moreover, rational numbers provide solutions to certain types of algebraic equations that do not have a solution within natural numbers, such as $2 \cdot x = 1$.

Fractions allow for a variety of interpretations in the domain of mathematics as well as the real world (Behr, Lesh, Post, & Silver, 1983; Ohlsson, 1988). For example, fractions (e.g., $\frac{3}{4}$) can be interpreted as parts of a whole (divide one whole into four parts and take three of these parts), as several parts of several wholes (take three out of four objects), as division (3 divided by 4), as operators (a function that produces three-fourths of any given input value), as measures of quantities (three quarters of a mile), or as solutions of algebraic equations (the number x that solves the equation $4 \cdot x = 3$). The variety of possible interpretations substantiates the complexity of the concept of fractions, and it suggests that the teaching and learning of fractions deserves careful attention.

From a *cognitive psychological perspective*, understanding fractions requires a higher level of abstraction than understanding natural numbers (DeWolf, Bassok, & Holyoak, 2016; Empson, Levi, & Carpenter, 2011). It may therefore facilitate the transition from concrete to formal operations (Inhelder & Piaget, 1958; Piaget & Inhelder, 1966). In this regard, understanding of fractions seems crucial for mathematical development. There is empirical evidence that fraction understanding is a unique predictor of later achievement in higher mathematics such as algebra. This holds true even when controlling for several other cognitive measures, including general cognitive ability and working memory (Bailey, Hoard, Nugent, & Geary, 2012; Booth & Newton, 2012; Siegler et al., 2012; Torbeyns, Schneider, Xin, & Siegler, 2015).

In sum, fractions are a key target for learning from both a mathematics education and a cognitive psychological perspective. Because fractions are a complex concept, it may not be surprising that learning and teaching fractions can pose special challenges. To analyze these challenges in more detail, the following section summarizes typical errors students make in fraction problems, as well as potential sources of these errors.

[1] There are also curricula in which negative integers are introduced earlier than fractions. This difference in sequencing is not essential for our analyses of difficulties with fraction learning, as we focus predominantly on issues related to the transition from integers to fractions rather than the transition from positive to negative numbers.

7.2 Solving Fraction Problems: Errors and Their Potential Sources

Numerous studies over several decades have documented typical errors students make when solving fraction problems (e.g., Aksu, 1997; Behr, Wachsmuth, & Post, 1985; Behr, Wachsmuth, Post, & Lesh, 1984; Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1981; Carraher, 1996; Hart, 1981; Hasemann, 1981). More recent studies suggest that there has not been significant progress, and that errors are invariant across many different countries and cultures (Bailey et al., 2015; Lortie-Forgues, Tian, & Siegler, 2015; Siegler & Pyke, 2013; Stafylidou & Vosniadou, 2004).

These studies largely converge on a number of major findings. A general observation is that even students who are well able to carry out fraction arithmetic procedures may make errors when problems require fraction concepts (Hallett, Nunes, & Bryant, 2010; Hallett, Nunes, Bryant, & Thorpe, 2012; Siegler & Lortie-Forgues, 2015). One of the concepts that students often struggle with is that of fraction magnitude (Siegler, Thompson, & Schneider, 2011). Rather than seeing a fraction as representing a (rational) number, students tend to interpret a fraction as two separate whole numbers. For example, when a representative sample of eighth-graders in the United States were asked to choose the closest number to the result of $\frac{12}{13}+\frac{7}{8}$ with the options 1, 2, 19, and 21, only 24% chose the correct answer 2 (Carpenter et al., 1981). More than half of them chose 19 or 21, suggesting addition of the numerators (**12 + 7 = 19**) or the denominators (**13 + 8 = 21**) without considering each fractions' integrated magnitude (each being approximately 1). Lortie-Forgues et al. (2015) documented very similar results in a study conducted over 30 years later. Another example of limited understanding of fraction magnitudes is the finding that in fraction addition problems, students' most frequent error is adding the numerators and denominators separately, even though this produces unreasonable outcomes (e.g., $\frac{1}{2}+\frac{1}{2}=\frac{2}{4}$) (Behr et al., 1985; Brown & Quinn, 2006; Siegler & Pyke, 2013). Furthermore, students also struggle with understanding that different symbolic fractions can represent the same numerical magnitude. For example, in a study by Clarke and Roche (2009), more than a third of a sample of Australian sixth-graders did not consider $\frac{2}{4}$ and $\frac{4}{8}$ to be fractions of equal numerical magnitude.

Although many students have relative strength with *carrying out* fraction arithmetic procedures compared to their *understanding* of fraction concepts and procedures, this does not mean that students' performance on fraction arithmetic problems is overall high. Instead, Siegler and Pyke (2013) found that when US sixth- and eighth-graders solved a set of fraction arithmetic problems that included all four basic arithmetic operations (i.e., addition, subtraction, multiplication, and division), they were correct on only 41% (sixth-graders) and 57% (eighth-graders), respectively. They also found that accuracies varied substantially between the different arithmetic operations. While students were most accurate with addition and subtraction, they were less accurate with multiplication and division (Braithwaite, Pyke, & Siegler, 2017; Siegler & Lortie-Forgues, 2017). In addition to difficulties with carrying out arithmetic procedures, students often struggle with predicting the outcomes of arithmetic problems. For example, they are often reluctant to accept that the result of a multiplication problem involving fractions can be smaller than the initial number (Obersteiner, Van Hoof, Verschaffel, & Van Dooren, 2016; Siegler & Lortie-Forgues, 2015; Van Hoof, Vandewalle, Verschaffel, & Van Dooren, 2015). In line with this finding, some students tend to prefer division over multiplication to solve word problems with fractions for which they expect the result to be smaller than the initial number, even when the problem structure suggests multiplication (Swan, 2001).

Another notoriously difficult task for students is reasoning about the structure of the rational number domain as a whole. In a study by Vamvakoussi and Vosniadou (2010), about one-third of 11th-graders responded (incorrectly) that there was only a finite number of numbers between any two rational numbers. An especially common error is to think that increasing any given fraction's numerator by 1 generates the successor of that fraction (e.g., to think that $\frac{3}{5}$ is the successor of $\frac{2}{5}$) (Vamvakoussi & Vosniadou, 2004, 2010), although rational numbers, unlike natural numbers, do not have successors (see Sect. 7.2.1).

In sum, evidence for students' errors in fraction problems, which comes from a variety of studies collected over decades, suggests that difficulties are systematic, persistent over time, and exist in different learning environments. One may wonder what makes fractions so difficult to understand. Are fractions just a difficult mathematical concept? Is the human brain not well prepared to process fractions? Or are there limitations in the way fractions are commonly taught at school? In the following sections, we evaluate potential sources of difficulties with fractions on three different levels (see Lortie-Forgues et al., 2015, for a similar approach). First, we consider the learning content itself. We identify what aspects of fractions differ substantially from natural numbers because these aspects might be particularly challenging for learners. Second, we explore how psychological accounts conceive the mechanism of fraction learning, and—more fundamentally—how well the human cognitive architecture is prepared for processing fractions. Third, we review common teaching practices in mathematics classrooms, based on the available research on textbooks and surveys among teachers.

7.2.1 The Learning Content Itself

Fractions are symbolic representations of rational numbers. Mathematically speaking, rational numbers can be constructed as an extension of the set of integers, with rational numbers being defined as equivalence classes of pairs (a,b) of integers a and b, with $b \neq 0$. Two pairs (a,b) and (c,d) are considered equivalent if and only if $a \cdot d = b \cdot c$. After defining the operations of addition and multiplication, one gets to the field of rational numbers Q. These rational numbers are an extension of the set of natural numbers N in the sense that Q includes N, if one identifies natural numbers with the equivalence classes of those pairs in which the first component is positive and the second component is 1 (e.g., 2, with the equivalence class [2,1]). According to this definition, natural numbers and rational numbers have shared properties (because natural numbers are also rational numbers). For example, for rational and natural numbers, there is an order relation, meaning that for any two different numbers, it is possible to say which one is larger in numerical magnitude. Thus, rational and natural numbers can be represented on number lines.

However, despite shared properties, there are also important differences between the set of natural numbers and the set of rational numbers, and these differences may be stumbling blocks for learners when they have to make the transition from natural numbers to fractions (as representations of rational numbers). There are at least four important ways in which rational numbers—specifically in their representation as fractions—differ from natural numbers (see Obersteiner, Reiss, Van Dooren, & Van Hoof, in press; Prediger, 2008; Vamvakoussi & Vosniadou, 2004; Van Hoof, Vamvakoussi, Van Dooren, & Verschaffel, 2017). Table 7.1 provides an overview of these four differences.

One difference concerns the way natural numbers and fractions convey numerical magnitude. The symbolic representation of natural numbers complies to the base-10 place-value structure of our number system, which allows for straightforward strategies to identify numerical magnitude (see first row of Table 7.1). For instance, deciding which of two numbers is larger is simple because it can be done digit-by-digit from left to right (i.e., for three-digit numbers, comparing hundreds with hundreds, tens with tens, and units with units). Additionally, the number of digits is indicative of the magnitude of a number, with numbers consisting of more digits being larger in magnitude. Fractions, however, are composed of two integers, and only the numerator is positively related to overall fraction magnitude. Reasoning about fraction magnitude requires inferences about the ratio between numerators and denominators. As such, comparing the magnitudes of two fractions is less straightforward than comparing the magnitudes of natural numbers. Moreover,

Table 7.1 Examples of differences between natural numbers and fractions

	Natural numbers	Fractions
1. Representation of Magnitude	Base-10 place-value structure More digits—larger number $123 > 45$	Quotient of two numbers Neither number of digits nor natural number magnitudes as such determine fraction magnitudes $\frac{2}{3} > \frac{5}{19}$
2. Symbolic Representation	Unique for each number 2 as unique representation	Multiple (infinitely many) fractions can represent the same number $\frac{1}{2} = \frac{2}{4} = \frac{4}{8} =$ etc.
3. Density	Unique successors and predecessors Finite number of numbers between two natural numbers 1, 2, 3, 4, 5, etc.	No unique successors and predecessors Infinite number of numbers between two fractions $\frac{3}{5}$ is not the successor of $\frac{2}{5}$
4. Operation	Multiplication as repeated addition $3 \cdot 4 = 4 + 4 + 4$ Multiplication makes bigger, division smaller $2 \cdot 4 = 8,\ 15 \div 3 = 5$	Multiplication as repeated addition insufficient, more abstract definition required Multiplication and division can make bigger or smaller $\frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8},\ \frac{1}{2} \div \frac{1}{4} = 2$

comparing fractions may be counterintuitive because the larger fraction can be composed of the larger components (e.g., $\frac{4}{5} > \frac{1}{3}$), the smaller components (e.g., $\frac{1}{2} > \frac{3}{7}$), or one larger component (numerator) and one smaller component (denominator, e.g., $\frac{2}{3} > \frac{1}{5}$).

A second difference is that symbolic representations for natural numbers are unique in the sense that there is only one way to write any given number using only natural number notations (e.g., there is only one way to notate the number "2"). In contrast, different fraction symbols can represent the same numerical value (see second row of Table 7.1).

A third aspect in which fractions differ from natural numbers is density (see third row of Table 7.1). While natural numbers have unique successors and predecessors (except for number 1), this is not the case for any rational number. Moreover, while within the natural number domain there is only a finite number of numbers between any two natural numbers, there are infinitely many other fractions between any two fractions.

Fourth, fractions differ from natural numbers with respect to arithmetic operations (see fourth row of Table 7.1). There is a difference in the way arithmetic operations are conceptualized. Whereas within natural numbers, multiplication is typically explained as repeated addition (i.e., $3 \cdot 4$ means to add the number 4 three times), this explanation is not generally meaningful for fractions. In the example of $\frac{2}{3} \cdot \frac{1}{2}$, it is hard to understand what adding $\frac{2}{3}$ times the number $\frac{1}{2}$ means. Furthermore, there is a difference in the effects that arithmetic operations have on numbers. While multiplication with natural numbers (other than 1) always yields a result that is numerically larger than the original operands, this is not always true for fractions. Instead, multiplying a positive number by a fraction smaller than one (e.g., $\frac{1}{4}$) makes the initial number smaller (e.g., $\frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$). Similarly, within natural numbers, division (by a number other than 1) always makes a number smaller, while within rational numbers, division can also make a number larger (e.g., $4 \div \frac{2}{3} = 6$).

Although the conceptual differences between natural numbers and fractions analyzed in this subsection are potential obstacles for learning, our analysis is not sufficient to identify learners' actual obstacles. The reason is that the analysis of the subject domain does not take into account the cognitive mechanisms underlying learning. Since learning does not necessarily follow the logic of the subject domain, insights into the cognitive mechanisms of learning can complement our search for difficulties with fractions.

7.2.2 The Human Cognitive System

Learning fractions may be influenced by the way our cognitive system processes new information, and more specifically by the way it processes numbers in general and fractions in particular. The following four subsections describe theoretical frameworks and empirical evidence that help in understanding the cognitive challenges of learning fractions. The first two subsections elaborate on theories of conceptual learning (conceptual change) and of the cognitive processes that occur during problem solving (dual processes) that may account for response biases (the natural number bias). The remaining two subsections then "zoom in" on the more fundamental ways our cognitive system processes fractions, and on the neural correlates of these processes.

7.2.2.1 Conceptual Change

Natural numbers are special cases of rational numbers (see Sect. 7.2.1), differing from other rational numbers in several respects. Therefore, learning fractions requires not only the extension but also the reorganization of existing knowledge about (natural) numbers. Accordingly, researchers have studied learning of rational numbers as an instance of conceptual change (Vamvakoussi, Van Dooren, & Verschaffel, 2012; Vamvakoussi & Vosniadou, 2004, 2010). The conceptual change approach was initially applied to the domain of science learning but was later transferred to mathematical learning as well (Merenluoto & Lehtinen, 2002).

In line with the conceptual change approach, there is broad evidence that students' errors in operating with fractions may be due to their reliance on natural number concepts in problems that require reasoning about rational number concepts. For example, when comparing the magnitudes of two fractions, children were found to rely on their natural number knowledge and treat fraction components as two separate natural numbers, rather than reasoning about the overall magnitudes of the respective fractions. Only 15% of more than 300 sixth-graders in the study by Clarke and Roche (2009) were able to correctly choose the larger fraction from the pair $\frac{5}{6}$ versus $\frac{7}{8}$ and provide an appropriate explanation for their choice. Almost 30% of all students in this study claimed that these fractions were the same because the difference between the numerator and the denominator was equal in both fractions. These students relied on reasoning about number magnitudes in ways that apply to natural numbers (each symbol represents a separate magnitude), although the problem required a conceptual change (quotients of *two* [natural] numbers represent *one* [rational] number magnitude). There is evidence that students also struggle with other concepts of fractions that differ from natural number concepts (i.e., those described in 2.1 and listed in Table 7.1), as predicted by the conceptual change approach (Merenluoto & Lehtinen, 2002; Vamvakoussi & Vosniadou, 2004, 2010; Van Hoof et al., 2017, see also the introduction to Sect. 7.2).

In contrast to such a focus on discontinuities in the learning process, other researchers have emphasized commonalities between natural and rational numbers and considered learning of numbers as a continuous learning path, rather than an instance of conceptual change. In their *integrative theory of numerical development*, Siegler and colleagues (Siegler et al., 2011; Siegler & Braithwaite, 2017; Siegler & Lortie-Forgues, 2014) emphasized that magnitude is the unifying idea between different kinds of numbers such as natural and rational numbers. As all real numbers (including natural and rational numbers) have magnitudes and can be represented on number lines, understanding these magnitudes may be particularly helpful for learners in extending their number knowledge to new number domains. Although there is initial evidence that understanding fraction magnitudes facilitates learning of fraction concepts more generally (see Sect. 7.3), the specific relation between understanding of fraction magnitudes and other fraction concepts remains to be understood.

Steffe and colleagues (e.g., Steffe, 2002; Steffe & Olive, 2010) proposed a constructivist account of fraction learning that also emphasizes the coherence between natural number knowledge and learning fractions. In their *reorganization hypothesis*, they consider how children's natural number knowledge may be modified in productive ways to construct fraction knowledge (for details, see also Tzur et al., this volume).

Note that these two theoretical accounts focus on the coherence between natural numbers and fractions, while our focus in this chapter is more strongly on the challenges (rather than the coherence) in students' transition from natural numbers to fractions. We relied on the conceptual change approach in this section because it connects these challenges to the conceptual differences between natural numbers and rational numbers. A systematic discussion of the various accounts proposed for learning fractions is, however, beyond the scope of this chapter.

7.2.2.2 Dual-Process Theories and the Natural Number Bias

Some researchers have focused more strongly on the cognitive processes involved in fraction problem solving rather than on an understanding of fraction concepts. *Dual-process theories* assume that problem solving includes two types of processes: Processes that are fast, largely automatic, and intuitive ("System 1 processes") and processes that are analytic and time-consuming ("System 2 processes") (Gillard, Van Dooren, Schaeken, & Verschaffel, 2009; Kahneman, 2000). When people solve rational number problems, their strongly internalized knowledge of natural numbers might trigger intuitive System 1 processes, while analytic System 2 processes are particularly important when problems require reasoning about novel and less automatized features of rational numbers.

The overreliance on natural number knowledge even in problems that require rational number reasoning has been referred to as the "whole number bias" or "natural number bias" (Alibali & Sidney, 2015; Ni & Zhou, 2005; Van Hoof et al., 2017). To investigate the natural number bias, researchers have compared performance on problems that are either *congruent* or *incongruent* with natural number reasoning. Problems are congruent when reasoning about natural numbers (rather than rational numbers) yields the correct response, and they are incongruent when this is not the case. For example, in fraction comparison, the two to-be-compared fractions of a pair can be classified as congruent when comparing denominators and numerators separately yields the correct result (e.g., $\frac{4}{5} > \frac{1}{3}$ with $4 > 1$ and $5 > 3$) but incongruent when doing so leads to an incorrect result (e.g., $\frac{1}{2} > \frac{3}{7}$ although $1 < 3$ and $2 < 7$). In the case of arithmetic operations with fractions, the intuition that multiplication makes numbers bigger may lead to a correct response in problems congruent with natural number characteristics (e.g., "Is it possible that $4 \cdot x$ is larger than 4?", where considering x a natural number will lead to a correct response) but to an incorrect response in incongruent problems (e.g., "Is it possible that $4 \cdot x$ is smaller than 4?").

Importantly, numerous studies have documented this bias, not only in primary and lower secondary school students, but also in upper secondary students and adu lts (Byrnes & Wasik, 1991; Obersteiner, Van Dooren, Van Hoof, & Verschaffel, 2013; Siegler & Lortie-Forgues, 2015; Vamvakoussi et al., 2012; Van Hoof et al., 2015; Van Hoof, Lijnen, Verschaffel, & Van Dooren, 2013). These findings suggest that the natural number bias in fraction problems can persist even after people have acquired sound conceptual knowledge of fractions. This implies that solving fraction problems requires—in addition to conceptual understanding of fractions—some inhibition of intuitive knowledge about natural numbers.

7.2.2.3 Processing of Fraction Magnitudes

Research suggests that our cognitive system is well prepared for processing natural numbers (see Feigenson, Dehaene, & Spelke, 2004, for a review). There is, however, more controversy about how well our cognitive system is prepared for processing fractions. A central question is whether people can mentally process fractions *holistically* by their integrated fraction magnitudes (e.g., $\frac{2}{5}$ as one numerical value), or whether they can only process fractions *componentially* by their components (e.g., $\frac{2}{5}$ as two separate numbers, 2 and 5). Numerous studies have used fraction comparison tasks and evaluated whether participants' comparison performance depended on the numerical distance between fractions or on the distances between fraction components. When comparing natural numbers, a typical finding is that responses become faster and less error prone, as the numerical distance between to-be-compared numbers gets larger (e.g., 1 vs. 9 is easier than 4 vs. 5). This finding is often referred to as the *numerical distance effect*. The distance effect is considered evidence that people actually rely on number magnitude information when comparing two numbers (Moyer & Landauer, 1967).

Initial studies found no such distance effect for fractions and concluded that people mentally represent fractions predominantly in a componential way, that is, they represent each component separately rather than represent the fraction as an integrated entity (Bonato, Fabbri, Umiltà, & Zorzi, 2007; Ganor-Stern, Karasik-Rivkin, & Tzelgov, 2011). However, later studies revealed that the way participants process fractions depended on the type of fraction comparison and on the strategies they use to solve these problems (Faulkenberry & Pierce, 2011; Ganor-Stern, 2012; Meert, Grégoire, & Noël, 2010a, b; Obersteiner et al., 2013; Schneider & Siegler, 2010). For instance, Obersteiner et al. (2013) found that when academic mathematicians solved fraction comparisons, there was a distance effect of overall fraction magnitude only for fraction pairs that did not have common components (e.g., $\frac{11}{18}$ vs. $\frac{19}{24}$). Additionally, they observed no natural number bias for these problems. However, when fraction pairs did have common components (e.g., $\frac{17}{23}$ vs. $\frac{20}{23}$, or $\frac{12}{13}$ vs. $\frac{12}{19}$), there was no effect of overall distance and a clear natural number bias, which was reflected by lower performance on incongruent rather than congruent problems. Together, this line of research suggests that adults rely more strongly on componential comparison strategies in comparison problems with common components (with less activation of holistic overall fraction magnitudes). Such a strategy is more prone to natural number bias. In contrast, adults seem to rely more strongly on holistic magnitudes in problems without common components, a strategy that discourages natural number bias. Recent eye-tracking research substantiated the claim that adults use different strategies depending on problem types (Huber, Moeller, & Nuerk, 2014; Ischebeck, Weilharter, & Korner, 2016; Obersteiner & Tumpek, 2016).

Research also suggests that the way people process fractions depends on how familiar they are with specific fractions. Liu (2018) found that when participants compared symbolic fractions to values marked on a number line, their performance depended on how close fractions were to familiar fractions (e.g., $\frac{1}{2}$ or $\frac{3}{4}$) that people used as benchmarks. Thus, whether fractions are processed holistically may also be a question of practice and familiarity rather than of cognitive ability alone.

These studies provide evidence that adults' cognitive architecture allows them to process symbolic fractions in a holistic manner. Further research suggests that the ability to process fractions and ratios may be traced back to very fundamental abilities for processing non-symbolic ratios, and that humans are equipped with a perceptually based ratio processing system (Boyer & Levine, 2015; Lewis, Matthews, & Hubbard, 2016; Matthews & Chesney, 2015; Matthews, Lewis, & Hubbard, 2016). As such, this processing system might be predisposed for developing magnitude representations of fractions (see Matthews et al., this volume).

7.2.2.4 Neural Correlates of Fraction Processing

In recent years, researchers have begun to evaluate the neurocognitive foundations of numerical cognition using neuroimaging (Arsalidou & Taylor, 2011; Dehaene, Piazza, Pinel, & Cohen, 2003). An increasing number of studies on adults and children revealed that the intraparietal sulcus (IPS; see Fig. 7.1) seems to be the central area for representing symbolic and non-symbolic numerical magnitudes (Nieder & Dehaene, 2009; Piazza, Pinel, Le Bihan, & Dehaene, 2007; Pinel, Dehaene, Rivière, & LeBihan, 2001). A major finding that led to this conclusion was that neural activation within the IPS is inversely related to the numerical distance between two to-be-compared numbers in number comparison tasks (Cohen Kadosh et al., 2005; Kaufmann et al., 2005), reflecting the neural instantiation of the behavioral distance effect (see Sect. 7.2.2.3). Studies also report activation of frontal brain areas during number processing, resulting in the notion of a fronto-parietal network underlying numerical processing (Ansari, Garcia, Lucas, Hamon, & Dhital, 2005; Pesenti, Thioux, Seron, & Volder, 2000).

Concerning rational numbers and fractions in particular, some researchers initially suggested that fraction concepts were fundamentally incompatible with the neurocognitive architectures underlying numerical cognition (Dehaene, 1998; Feigenson et al., 2004; Geary, Hoard, Byrd-Craven, Nugent, & Numtee, 2007).

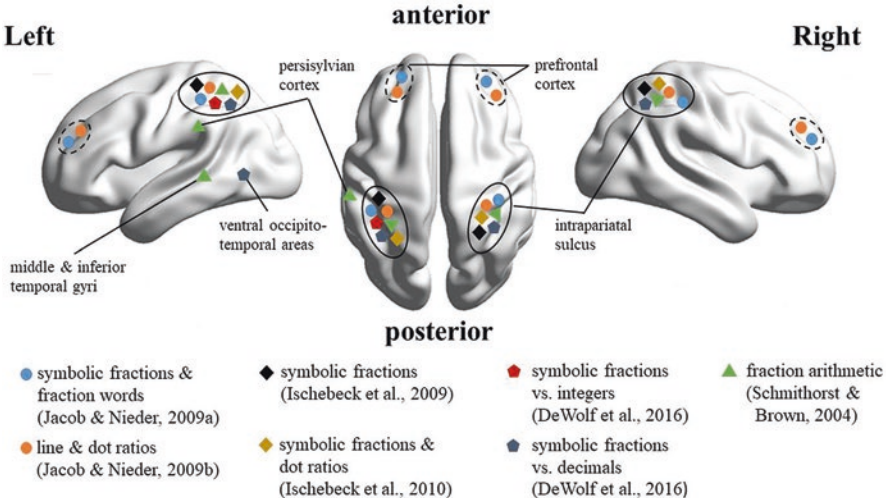

Fig. 7.1 Schematic overview of brain areas involved in fraction processing during (i) magnitude comparison of different notations (i.e., symbolic fractions, fraction words, line and dot ratios; Jacob & Nieder, 2009a, 2009b; Ischebeck et al., 2009, 2010); (ii) magnitude comparisons between fractions, decimals, and integers (DeWolf et al., 2016); and (iii) fraction arithmetic (i.e., addition and subtraction; Schmithorst & Brown, 2004). Most of the studies consistently observed activation in the bilateral intraparietal sulcus (IPS, full line ellipses). However, there are also studies showing additional bilateral prefrontal cortex activation (PFC, dashed ellipses) and left-lateralized activation for specific tasks (e.g., left inferior and middle temporal gyrus for the comparison of fractions and decimals, DeWolf et al. (2016); left ventral occipitotemporal and perisylvian areas were activated in fraction arithmetic, Schmithorst & Brown, 2004). In general, activation patterns observed for fraction processing are very similar to those found for natural number processing. This figure was adapted from Lewis et al. (2016).

Therefore, the question remains what the underlying neural mechanisms for fraction processing are. At the moment, there exist only a few studies in adults investigating the neural underpinnings of processing proportions (i.e., dot ratios, line ratios), symbolic fractions, or fraction number words (Ischebeck et al., 2010; Ischebeck, Schocke, & Delazer, 2009; Jacob & Nieder, 2009a, 2009b; Schmithorst & Brown, 2004). For instance, during fraction magnitude comparison, Ischebeck and colleagues (2009) observed that IPS activation was modulated by the overall numerical distance between the to-be-compared fractions, but not by the numerical distance between numerators or denominators. Moreover, Ischebeck et al. (2010) observed the same results during proportion comparison (involving symbolic fractions and dot patterns as non-symbolic proportions), with stronger right IPS activation for dot patterns and stronger left IPS activation for symbolic fractions. Jacob and Nieder (2009a, 2009b) adapted participants to a certain fraction magnitude (e.g., $\frac{1}{6}$) by showing different fractions reflecting this magnitude (e.g., $\frac{1}{6}, \frac{2}{12}, \frac{5}{30}$) with interspersed deviants differing in magnitude (e.g., $\frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}$) presented in the same or a different notation (i.e., symbolic fractions and fraction words or dots and triangles). The authors observed that the activation in parietal cortex was specifically tuned to the overall magnitudes of fractions rather than to the magnitudes of their components, indicating that fraction magnitude is represented holistically in the same brain areas as natural numbers. Moreover, Jacob and Nieder (2009a) provided evidence for a notation-independent activation patterns. In particular, they reported that the same cortical areas were activated to a similar extent regardless of whether a fraction magnitude was presented as a symbolic fraction (i.e., $\frac{1}{4}$) or written as a number word (i.e., "one-fourth").

Overall, these studies indicated that the IPS plays a crucial role in the processing of proportion and fraction magnitude, similar to the processing of natural numbers. In contrast to behavioral studies on fraction magnitude comparison, which showed that holistic versus componential processing of fractions depended on the respective fraction type (i.e., with vs. without common components, see Sect. 7.2.2.3), the existing neuroimaging data suggest that fraction magnitudes are represented holistically on the neural level.

Furthermore, fraction arithmetic also seems to elicit patterns of neuronal activation similar to those observed for natural number arithmetic. Schmithorst and Brown (2004) studied adult participants solving fraction addition or subtraction problems. Their analyses again revealed activation in bilateral inferior parietal areas (including the IPS) with additional activation in left-hemispheric perisylvian areas (associated with verbal processing), and ventral occipitotemporal areas (often associated with more perceptual aspects, i.e., ventral visual pathway, see Fig. 7.1).

In spite of the generally large overlap in the neural networks for natural numbers and fractions documented in these studies, DeWolf, Chiang, Bassok, Holyoak, and Monti (2016) found differences in activation patterns within the IPS for fractions as compared to whole numbers and decimals. The authors argue that these differences in activation patterns may be due to the differences in the symbolic notations we use for natural numbers and decimals (both base-10 representations) on the one hand and fractions on the other (two natural numbers). Presumably, our brain needs more resources to get access to the magnitudes of fractions than those of natural numbers or decimals. This assumption is in line with evidence from behavioral research (DeWolf, Grounds, Bassok, & Holyoak, 2014).

Finally, based on theoretical considerations and initial empirical evidence, Lewis et al. (2016) recently argued that there exists a neural circuitry specifically dedicated to represent non-symbolic proportions comprising a fronto-parietal network. According to these authors, this system is also recruited when representing fractions as it provides a non-symbolic foundation for understanding fraction concepts. In particular, the authors proposed that both formal and informal learning experiences help to generate links between perceptually based representations of non-symbolic ratios and fraction symbols (i.e., verbal fraction labels and symbolic-digital fraction symbols). This non-symbolic-to-symbolic link may be an important basis for the understanding of fraction magnitudes.

Taken together, these studies suggest that the human brain is able to process holistic fraction magnitude. The IPS, which has long been known to be the key area for the representation of natural number magnitude, also seems crucial for processing fraction magnitudes. However, strong conclusions seem premature, due to the limited number of available studies. Moreover, all existing studies examined the neural correlates of fraction processing in adults, and studies on the neural correlates of how fraction processing develops and shapes the brain are completely lacking. An important external factor that may shape the way students think about fractions is the way they encounter fractions in the classroom.

7.2.3 Current Classroom Teaching Practices

Fractions are complex constructs, and there are many ways to interpret and represent fractions (see Sect. 7.1). There may be considerable variation in the ways students encounter fractions in the classroom, and varying classroom experiences may affect fraction learning. To date, there is little empirical evidence about how fractions are actually taught in classrooms. Much of the existing research into teaching of fractions has focused on teachers' competence with fractions and on the instructional materials teachers use. In the following, we first review general characteristics of common classroom teaching of fractions that might contribute to students' difficulties. We then focus on the quality of instructional materials, and finally on teachers' competence with fractions.

7.2.3.1 Characteristics of Classroom Teaching

One characteristic of current classroom teaching of fractions—at least in many Western countries—is a strong focus on memorization of procedures rather than on understanding of fraction concepts (Lortie-Forgues et al., 2015; National Mathematics Advisory Panel, 2008). Such a focus may have benefits in the short run: procedures are probably easier to teach, easier to test, and they may promise quicker success (and thus motivation). However, important disadvantages are that procedures are remembered less well if they are not connected to conceptual understanding, and that they may lead to inert knowledge that cannot be adapted for novel contexts (Swan, 2001). Moreover, given the sheer number and the relative complexity of fraction arithmetic procedures, students may confuse fraction procedures or parts of them. Finally, the omnipresence of electronic computing devices in our modern society raises fundamental questions about the importance of learning arithmetic procedures.

Another characteristic of current fraction teaching is the dominance of interpreting fractions as discrete and countable parts of a whole (e.g., pieces of a pizza). Teachers often also use this approach to introduce fraction procedures. For example, when learning about fraction multiplication, the first type of problems is often of the form "natural number × fraction" (e.g., $3 \cdot \frac{1}{4}$), which can be explained by repeated addition (take three quarters of a pizza, or $3 \cdot \frac{1}{4} = \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$). Such a strong emphasis on the part-whole relation of fractions as countable objects could be problematic because it could raise the expectation in students that fractions are not very different from natural numbers. Such an expectation may lead to overgeneralizations of natural number properties to fractions, as discussed above (see Sect. 7.2.2), including natural number bias and componential processing of fractions.

More generally, in many current classrooms, particularly in Western countries, there seems to be strong emphasis on the commonalities between natural numbers and fractions (as rational numbers), and only little emphasis on the differences between these types of numbers (Van Hoof et al., 2017). This seems problematic from the perspective of learning psychology, which would recommend fostering both generalization learning (emphasizing similarities between natural and rational numbers) and discrimination learning (explicating differences between natural and rational numbers) (see Sect. 7.4.1 for a discussion).

7.2.3.2 Instructional Material

Teaching practices may be influenced by the available instructional materials such as textbooks. There are a few systematic analyses of instructional materials on fractions (Alajmi, 2012; Braithwaite et al., 2017; Shin & Lee, 2017; Son & Senk, 2010; Watanabe, 2007). Their findings suggest that the majority of fraction problems in textbooks require procedural rather than conceptual knowledge (Son & Senk, 2010), and that textbooks often focus on standard algorithms for solving these problems (Alajmi, 2012). Moreover, there are large variations in the frequency with which textbooks present different types of fraction problems. For example, fraction division problems—the most challenging type of fraction problems for most students (see the introduction to Sect. 7.2)—are much less frequent than multiplication problems (Siegler & Lortie-Forgues, 2017; Son & Senk, 2010). Braithwaite et al. (2017) extended this finding by developing a computational model of fraction arithmetic that simulated students' most frequent errors in fraction arithmetic procedures as documented in empirical studies. Using problems from common US mathematics textbooks as input, the model predicted students' typical errors fairly well. Thus, the type of problems and the frequency with which these types appear in textbooks may to some extent explain students' difficulties with fractions.

7.2.3.3 Teacher Competence

Classroom materials do not entirely determine how the content is taught in the classroom. Rather, it is the role of the teacher to use instructional materials in a specific way. Teachers thus need to be competent with fractions in order to teach fractions appropriately. Unfortunately, research suggests that not all teachers have sufficient competence with fractions (Ball, 1990; Depaepe et al., 2015; Ma, 1999; Newton, 2008; Siegler & Lortie-Forgues, 2015; Simon & Blume, 1994). For example, Depaepe et al. (2015) found that, on average, prospective teachers were correct on only 75% of items that assessed conceptual fraction knowledge—even after having taken a course on teaching rational numbers. Siegler and Lortie-Forgues (2015) found that when pre-service teachers were asked to predict in which direction fraction arithmetic operations would change an initial number (e.g., whether $\frac{31}{56} \cdot \frac{17}{42} > \frac{31}{56}$ was true or false), they performed significantly lower (in some cases as low as about 30% correct) when these predictions were not in line with natural number reasoning (i.e., when the result suggested that multiplication makes the original operand smaller) than when they were. Thus, these prospective teachers showed response biases similar to those documented in students (see Sect. 7.2.2.2). Additionally, Ball (1990) and Ma (1999) found that teachers had particular difficulties with generating appropriate stories or situations for a given fraction division problem. In conclusion, limitations in teachers' understanding of fractions may aggravate the limitations of classroom materials discussed above.

7.3 Improvements: Evidence from Intervention Studies

While there are various intervention studies in the literature, few studies have evaluated the effectiveness of interventions in controlled experimental designs (for an overview of intervention studies especially for students with math difficulties, see Shin & Bryant, 2015). In the following, we elaborate on selected studies that focused on fraction magnitude understanding and used highly controlled experimental designs with control or comparison groups.

In a study by Gabriel et al. (2012), Belgian fourth- and fifth-graders played games that involved cards with different representations of fractions as well as wooden disks that children used to represent and manipulate fractions. Using these representations, children worked on comparisons of fraction magnitudes and on matching symbolic fractions with non-symbolic fraction representations. There were two 30-min intervention sessions per week, over a period of ten weeks. Results showed significantly greater improvements in conceptual understanding of fractions in children in the experimental group compared to children in a control group who received regular classroom instruction but no intervention. Instead, children in the control group showed significantly higher gains in procedural fraction arithmetic skills, substantiating that typical classroom teaching focuses more strongly on procedures than on concepts (see Sect. 7.2.3.1).

Fuchs et al. (2013) designed an intervention that also included training of general cognitive abilities.[2] Participants were US-American fourth-graders who performed below the 35th percentile on an arithmetic test and who were therefore considered to be at risk of low mathematical achievement. The study contrasted two different instructional approaches. The more conventional approach focused on part-whole aspects of fractions and on procedural aspects of fraction arithmetic, whereas the other, more innovative approach emphasized the measurement aspect of fractions and focused on fraction magnitudes. Each session lasted 30 min, with three sessions per week over a period of twelve weeks. The results showed that children who were

[2] See Lamon (2007) and Fazio, Kennedy, and Siegler (2016) for intervention studies with similar approaches.

taught with a focus on the measurement aspect of fractions showed greater gains in conceptual and procedural understanding of fractions than children who received conventional teaching. In another study, Fuchs et al. (2016) replicated these positive effects with a similar version of the measurement-based intervention program in another sample of at-risk fourth-graders.

Finally, Hamdan and Gunderson (2017) compared an intervention based on the use of number lines with an intervention that focused on the area model of fractions. The area model represents fractions as parts of two-dimensional shapes such as circles. Participants were children in grades two and three in the USA. The intervention occurred on one day within a 30-min time period. From pretest to posttest, relative to a control group, both interventions led to improvements on problems that required using the respective representation (number line or area) that children used during the intervention phase. However, children who used number lines were better able to transfer their knowledge to novel problems than were learners who used area models during the intervention phase. This suggests that the use of number lines may be particularly beneficial for fraction learning.

These examples illustrate how studies implemented interventions on enhancing understanding of fraction magnitudes (rather than on other aspects such as fraction arithmetic procedures). Overall, such a focus seems to be effective as it allows transfer to other fraction concepts. Evidence from broader intervention studies not reported here (e.g., Butler, Miller, Crehan, Babbitt, & Pierce, 2003; Cramer, Post, & delMas, 2002; Moss & Case, 1999) largely supports this conclusion. On the other hand, most controlled intervention studies contrasted only one or two different teaching approaches against control conditions, making it difficult to identify which of the large variety of teaching approaches is the most effective one.

7.4 Recommendations and Future Directions

In this section, we first draw conclusions that are relevant for the teaching and learning of fractions in the classroom, and then discuss directions for future research.

7.4.1 Recommendations for Classroom Practice

Rather than providing a comprehensive overview of recommendations about fraction teaching in general (for plenty of valuable recommendations, see, for example, Carraher, 1996; Moss & Case, 1999; National Mathematics Advisory Panel, 2008; Steffe & Olive, 2010), we restrict our discussion to six recommendations that follow from the different perspectives discussed in the previous sections.

Our first recommendation is that fraction teaching may benefit from *drawing more strongly on fundamental cognitive abilities for processing fractions and ratios*. There are plenty of psychological studies that suggest that our cognitive system is readily able to process magnitudes of symbolic fractions. The available neuroscience studies corroborate this conclusion and suggest that processing fractions activates a similar neural network as processing natural numbers—although caution is required due to the lack of evidence in young learners. Importantly, though, behavioral studies including young children suggest that the ability to process fraction magnitudes may be rooted in fundamental abilities to process non-symbolic ratios (see Sect. 7.2.2.3). Thus, instruction may draw on these fundamental abilities even before introducing symbolic fractions, for example by using appropriate visual representations such as bar representations. In particular, continuous rather than discrete bar representations may have the advantage that they encourage students to focus on holistic magnitudes of fractions rather than on countable segments (Boyer, Levine, & Huttenlocher, 2008; Huttenlocher, Duffy, & Levine, 2002). Connecting these representations with symbolic fractions later on may then help prevent the common overreliance on natural number concepts when working with fractions (i.e., the natural number bias). This conclusion remains tentative because there is limited empirical evidence for the effectiveness of specific visual representations of fractions (Rau & Matthews, 2017).

Our second recommendation is that instruction on fractions may benefit from a *stronger focus on fraction magnitudes and the use of number lines*. This is related to the previous recommendation (in that fraction magnitudes should be linked to students' early abilities) but it is more general. Research shows that students have difficulties with understanding how fraction symbols represent numerical magnitudes. Therefore, as supported by results from controlled intervention studies (see Sect. 7.3), students may benefit from a stronger focus on fraction magnitudes (e.g., the measurement aspect of fractions) rather than on the part-whole aspect of fractions. In particular, notwithstanding the limited empirical evidence for visual representations of fractions, number lines have proven to be effective for representing magnitudes of symbolic fractions. A unique advantage of the number line representation is that all real numbers can be represented on the same line, so that this representation may foster students' ability to integrate their concept of numbers across number domains (Booth & Newton, 2012; Common Core State Standards Initiative, 2010; Gersten, Schumacher, & Jordan, 2017; Hamdan & Gunderson, 2017; National Mathematics Advisory Panel, 2008; Siegler et al., 2011).

Our third recommendation is that students may benefit from *meta-level prompts to "stop and think"* in order to inhibit potentially misleading intuitive or "System 1" thinking. Although reasoning about fractions necessarily requires knowledge of natural numbers, intuitive knowledge of natural numbers can interfere with processing fractions, resulting in the natural number bias (see Sect. 7.2.2.2). Research suggests that this bias is very persistent, and that it influences performance on fraction problems even in individuals who have acquired sound conceptual understanding of fractions. Thus, students may make errors because they do not engage in analytical thinking regardless of their level of conceptual knowledge. Therefore, it seems advisable to encourage students to think about the reasonableness of their responses, especially for fraction arithmetic. Two concrete instructional approaches to address that goal are prompting students to self-explain their reasoning and refutation texts that require students to argue why a given solution is wrong (Tippett, 2010; Van Hoof et al., 2017).

Our fourth recommendation is that teachers should offer their students *sufficient opportunities to acquire concepts of fractions and fraction operations*. The conceptual change approach (see Sect. 7.2.2.1) suggests that changes in learners' initial concepts may be challenging, and a content analysis (see Sect. 7.2.1) can identify those concepts that learners need to change in order to fully understand fractions. Importantly, several concepts can be relevant at the same time for understanding any one problem situation. To illustrate this, consider the common erroneous expectation that multiplication always makes a number bigger (see Prediger, 2008, for more details on that example). This expectation may be due to three misconceptions. First, students may have internalized as a "rule" the regularity that multiplication makes bigger because they have never experienced a situation in which this was not true. In that case, students may benefit from applying the multiplication algorithm to fraction multiplication problems in which they then discover that multiplication can actually make a number smaller. Second, students may be unable to conceptualize what multiplying two fractions means because repeated addition does not offer a meaningful interpretation (see Sect. 7.2.1). The third scenario is related: Students may be unable to conceptualize what multiplying two fraction means because their concept of fractions is limited to the part-whole aspect. It is in fact difficult to understand what multiplying $\frac{1}{4}$ of a pizza with $\frac{2}{3}$ of a pizza should mean. In the latter two scenarios, students need to acquire appropriate concepts of fractions and fraction operations (see also Simon et al., this volume, for an elaboration on fraction multiplication). For example, the multiplication problem $\frac{1}{2}\cdot\frac{1}{4}$ may be explained as "$\frac{1}{2}$ *of* $\frac{1}{4}$," where $\frac{1}{2}$ is an operator that operates on $\frac{1}{4}$. Alternatively, $\frac{1}{2}\cdot\frac{1}{4}$ may be explained using the area model, in which both fractions are interpreted as measures of length, while the resulting fraction represents the area (see Fig. 7.2).

This example illustrates that learning fractions necessarily includes learning of new concepts, which is an unavoidable obstacle—whether big or small—for learners. Mathematics educators have used the term "epistemological obstacles" to refer to those obstacles that are inherent in the content structure (Broussou, 1983; Prediger, 2006, 2008; Schneider, 2014). Notably, epistemological obstacles are considered an opportunity for learning in themselves. Thus, these obstacles can and should not be avoided during the learning process.

Our fifth recommendation is that students may benefit from *explicating which aspects of fractions are in line with natural number concepts and which are not*. The above content analysis showed that there are important differences between natural numbers and fractions (see Sect. 7.2.1), and students need to understand these differences. At the same time, in order to build on students' existing knowledge of natural numbers, and to illustrate continuities in the number concept, teachers should highlight similarities between natural numbers and fractions. Current classroom practices seem to put more emphasis on the similarities rather than the differences between natural numbers and fractions. As a consequence, students may get too little support in distinguishing between aspects of rational numbers that are conceptually aligned with natural numbers and those that are conceptually different.

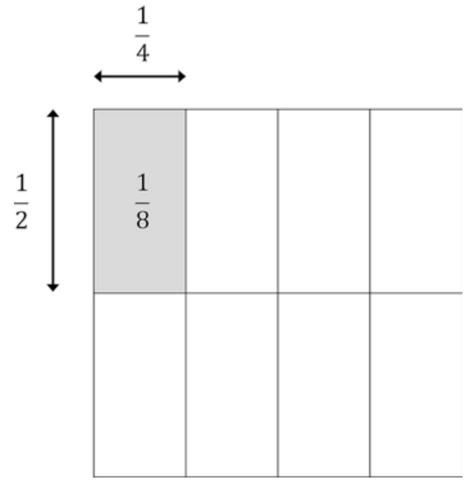

Fig. 7.2 Area model for fraction multiplication

Empirical evidence shows that learning of fraction division concepts can be more or less successful depending on whether the activated previous knowledge of natural numbers is helpful (conceptual similarity) or not (superficial similarity). In this context, Sidney and Alibali (2015, 2017) found that when learning about fraction division, students benefited more from practicing division of natural numbers (similar concept but different numbers) rather than fraction problems without division (similar numbers but different concept) immediately before engaging with fraction division. This suggested sequencing of fraction problems (fraction division directly preceded by natural number division) differs from common mathematics textbooks, where fraction division typically follows fraction multiplication. It is eventually up to the teacher to present fraction division in a way that students can make appropriate links to natural number division.

The important role of teachers leads to our sixth and final recommendation: More effort is needed to *provide teachers with the knowledge they need to teach fractions effectively*. Empirical studies have documented teachers' limitations predominantly with respect to fraction concepts (see Sect. 7.2.3). Thus, it seems imperative for teacher education to enhance teachers' content knowledge. A particular focus should be on fraction concepts that are counterintuitive and therefore prone to biased reasoning.

7.4.2 Future Directions

There are currently only a few studies on how fractions are commonly taught in classrooms. This shortage concerns at least three aspects, namely teachers' behavior, classroom materials, and—more specifically—visual representations of fractions used in teaching. Concerning teachers' behavior, classroom observation studies are needed to find out which teaching approaches teachers actually use in classrooms. With respect to classroom materials, further analyses of textbooks should focus more systematically on the fraction concepts and the types and nature of fraction problems and the visualizations that occur in textbooks. Concerning these visualizations, controlled intervention studies may investigate the specific effects of individual representations, as well as how multiple representations should be combined so that they are effective for students (Rau, 2017).

Although proficiency with natural numbers is a prerequisite for learning about rational numbers, overgeneralizations of natural number principles can actually cause difficulties with learning fractions. Further research is needed to better understand the specific relations between previously acquired knowledge of natural numbers and fraction learning. Studies to date have addressed the development of natural and rational number knowledge in longitudinal designs (e.g., Braithwaite & Siegler, 2017; Mou et al., 2016; Resnick et al., 2016; Rinne, Ye, & Jordan, 2017). However, these studies have focused on very specific aspects of numerical development (e.g., number magnitudes), and they have not included many external variables that may contribute to this development. To better understand the relative contributions of various factors, studies should consider taking into account both cognitive variables (e.g., general cognitive abilities, working memory) and also non-cognitive variables (e.g., mathematics self-concept, mathematics anxiety) as well as school-related factors (e.g., classroom teaching, textbooks), and socio-economic factors (e.g., learning opportunities at home).

7.5 Conclusion

In this chapter, we aimed to make connections between research on fractions from mathematics education and cognitive psychology, and also to include neuroscience evidence. We note that studies from different perspectives address issues on very different levels of explanation, such as the level of classrooms, of student behavior, or of brain activations. Integrating studies with such different perspectives is a challenge for many reasons (De Smedt et al., 2010; Nathan & Alibali, 2010; Schumacher, 2007). For example, learning processes at different levels occur on completely different time scales, ranging from milliseconds (neural activation) to days or weeks (learning across classroom sessions). More fundamentally, authors from different fields do not always speak the same language or address the same questions. For example, mathematics educators often ask what students *should* learn and how they *could* learn best, whereas psychologists are more used to ask what students *are able to* learn, or *when* in their development they learn certain things.

On a meta-level, research should strive for a shared theoretical framework that provides guidance for researchers from different perspectives as to how an integration may be made most fruitful. Our attempts to make connections between various perspectives in this chapter may spark further discussions across disciplines. In spite of apparent challenges, such cross-disciplinary discussions are necessary to improve teaching and learning of fractions in the best possible way.

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