Chapter 4
Spontaneous Mathematical Focusing Tendencies in Mathematical Development and Education

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Abstract A growing body of evidence reveals the need for research on, and consideration for, children's and students' own—self-guided—spontaneous use of mathematical reasoning and knowledge in action. Spontaneous focusing on numerosity (SFON) and quantitative relations (SFOR) have been implicated as key components of mathematical development. In this chapter, we review existing research on SFON and SFOR tendencies in the broader context of the development of mathematical skills and knowledge and examine how the state-of-the-art evidence on SFON and SFOR is relevant for the field of mathematics education. We discuss individual differences in SFON and SFOR, associations between spontaneous focus on mathematical features and mathematics achievement, the contributions of situational contexts that implicitly prompt attention to number, and ways to increase children's focus on number regardless of their baseline level tendencies. We conclude that children's and students' tendencies to focus on number and quantitative relations–spontaneous or otherwise–are key components of mathematical development and education.

Keywords Spontaneous focusing on numerosity (SFON) · Spontaneous focusing on quantitative relations (SFOR) · Mathematical thinking · Numerical salience · Contextual influences · Individual differences · Early mathematics

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4.1 Spontaneous Mathematical Focusing Tendencies in Mathematical Development and Education

The everyday world is rich with mathematical features, and attending to these features is useful, often necessary, and can even be highly engaging. Indeed, mathematics educators emphasize the value of teaching mathematical modelling skills that allow students to apply their mathematical knowledge in everyday and work-life situations (Mullis, Martin, Goh, & Cotter, 2016), something that is highly relevant for most middle- and high-quality jobs (Advisory Committee on Mathematics Education, 2011). However, in order to make use of mathematical features in everyday situations, an individual—needs to recognize—often without external guidance—that mathematical aspects of a situation are present and relevant to begin with (Lehtinen & Hannula, 2006; Lobato, Rhodehamel, & Hohensee, 2012). Individuals with a higher tendency to recognize and use mathematical features of everyday situations may acquire more self-initiated practice. This is developmentally relevant because many opportunities to learn or practice mathematical behavior occur outside of formal mathematical learning contexts.

The tendency to recognize and focus on mathematical features when not explicitly guided to do so is not always automatic, even among individuals who possess the relevant underlying mathematical knowledge (Batchelor, Inglis, & Gilmore, 2015; Chan & Mazzocco, 2017; Hannula & Lehtinen, 2005; McMullen, Hannula-Sormunen, Laakkonen, & Lehtinen, 2016). Across a range of studies, researchers have shown that individuals differ in their tendency to focus on mathematical aspects of situations that are not *explicitly* mathematical, like copying a drawing of flowers or imitating someone else feeding a puppet. Not all individuals notice, or use, numerical information when, for instance, reproducing how many petals are on the flowers. Those individuals who are more likely to do so have been shown to have an advantage in learning formal mathematical skills and knowledge (e.g., Hannula & Lehtinen, 2005; Hannula-Sormunen, Lehtinen, & Räsänen, 2015). Mathematics educators in preschool and primary school, as well as their students, may benefit from considering these individual differences in spontaneous mathematical focusing tendencies. In this chapter, we argue that tendencies to spontaneously focus on mathematical features explain at least some of the individual differences observed in the development of mathematical thinking (e.g., Gray & Reeve, 2016; Hannula-Sormunen et al., 2015; McMullen, Hannula-Sormunen, & Lehtinen, 2017; Nanu, McMullen, Munck, Hannula-Sormunen, and Pipari Study Group, 2018; Van Hoof et al., 2016), and that promoting these tendencies across different contexts may improve specific aspects of mathematical learning and performance (Hannula, Mattinen, & Lehtinen, 2005; McMullen, Hannula-Sormunen, Kainulainen, Kiili, & Lehtinen, 2017).

4.2 What Are Spontaneous Mathematical Focusing Tendencies?

Thus far, most research examining spontaneous mathematical behavior in preschool and school-age children has focused primarily on how young children spontaneously focus on numerosity. The tendency of spontaneous focusing on numerosity (SFON) is defined as follows:

> a process of spontaneously (i.e., in a self-initiated way not prompted by others) focusing attention on the aspect of the exact number of a set of items or incidents and using of this information in one's action. SFON tendency indicates the amount of a child's spontaneous practice in using exact enumeration in her or his natural surroundings. (Hannula, Lepola, & Lehtinen, 2010, p. 395).

On a broad level, from early childhood through adulthood, substantial individual differences in SFON tendency have been differentiated from individual differences in related mathematical knowledge and skills (Gray & Reeve, 2016; Hannula-Sormunen, Nanu, et al., 2015, Hannula-Sormunen, Nanu, Laakkonen, Munck, Kiuru, Lehtonen, and Pipari Study Group, 2017; Hannula & Lehtinen, 2005; Hannula et al., 2010; Hannula, Räsänen, & Lehtinen, 2007; McMullen, Hannula-Sormunen, & Lehtinen, 2015; Rathé, Torbeyns, Hannula-Sormunen, & Verschaffel, 2016; Sella, Berteletti, Lucangeli, & Zorzi, 2016), despite their positive correlation with those skills (Hannula et al., 2010, 2007; Hannula & Lehtinen, 2005; Hannula-Sormunen et al., 2015; McMullen et al., 2015; Nanu et al., 2018). Although focusing on exact number is often relevant in a situation, there are also situations in which focusing on quantitative relations is more relevant than exact number (Singer-Freeman & Goswami, 2001; Sophian, 2000; Spinillo & Bryant, 1991). For example, a child might spontaneously notice there are two apples and four bananas in a bowl of fruit—a SFON behavior. However, that same child might then go on to notice that there are twice as many bananas as apples, or that one-third of the pieces of fruit are apples, exhibiting what could be described as spontaneous focusing on quantitative relations (SFOR). There are limitations to what natural numbers can represent in the real world, and those limitations underlie the need for rational numbers (Vamvakoussi, 2015). Focusing solely on numerosity may not be sufficient or appropriate in many such situations (Boyer, Levine, & Huttenlocher, 2008). For example, to equally divide two bananas among 3 persons, it is not possible to express the outcome with natural numbers. Thus, recent studies have examined the role of SFOR in mathematical development (e.g., McMullen et al., 2016; Van Hoof et al., 2016). Whereas SFON tendency reflects paying attention to a single quantity or numerosity and using it in action, SFOR tendency reflects recognizing and using mathematical relations between two or more quantities.[1]

[1] It should be noted that, at the moment, we do not distinguish between different aspects of quantitative relations, though most existing research examines either multiplicative relations with late primary school students or part-whole relations in preschoolers. SFOR tasks usually include discrete quantities and underlying exact numbers are a foil and/or a prerequisite for focusing on the

Importantly, the spontaneity indicated by SFON and SFOR tendencies does not refer to the spontaneous acquisition of skills or knowledge nor an innate nature to their origins (Hannula, 2005; Lehtinen & Hannula, 2006). Instead, the spontaneous nature of these tendencies refers to the unguided, self-initiated nature of the recognition and use of numerical features within a specific moment or situation (i.e., without external prompting). This means that some background skills and knowledge are requisites of SFON and SFOR tasks, and that these tendencies should respond to formal and explicit teaching of focusing on mathematical aspects across contexts (Hannula, 2005; McMullen, 2014).

In the following sections, we review the theory and methods around spontaneous mathematical focusing tendencies and their relation to requisite cognitive skills such as mathematical knowledge or attention, and to contextual factors such as social expectations or demands. In order to fully understand children's everyday mathematical behavior, it is crucial that all three factors are taken into account. We argue that there are complex concurrent and developmental relations among these three constructs, which we illustrate using a schematic representation (Fig. 4.1). Based on the literature, we argue that spontaneous mathematical focusing tendencies can be distinguished from other cognitive (Fig. 4.1b) or contextual (Fig. 4.1d) factors, and we depict the relation between these constructs with overlapping yet distinct circles. We summarize existing evidence for the iterative, developmental relations between spontaneous mathematical focusing tendencies and both the cognitive requisite skills (Fig. 4.1a), and contextual factors (Fig. 4.1e) that exist throughout mathematical development. Finally, we argue that in order to understand the full extent of mathematical behavior of preschool and school-age children, the intersection of all three circles (Fig. 4.1c) should be given serious consideration by researchers and educators.

4.3 Delineating SFON and SFOR from Requisite Skills

Any expression of SFON or SFOR tendency requires the use of the mathematical and domain general cognitive skills to solve a task (Fig. 4.1b). For instance, individuals need to fully attend to the situation or task at hand. Other factors, such as a disposition towards math (finding it useful, interesting, or important) may also affect when and how individuals spontaneously attend to mathematical features. Studies using the original SFON tasks have already shown that it is possible to reliably and uniquely measure the strength of children's SFON tendency (Hannula & Lehtinen, 2005; Hannula-Sormunen et al., 2015; Nanu et al., 2018), and several other measures have more recently been developed and thus contribute to the repertoire of SFON assessments (see Rathé, Torbeyns, Hannula-Sormunen, De Smedt, & Verschaffel, 2016 for an extensive review). Likewise, SFOR tendency can be reliably measured in a number of tasks in both early childhood and late primary school (McMullen et al., 2016; Van Hoof et al., 2016).

relational aspects of the task.

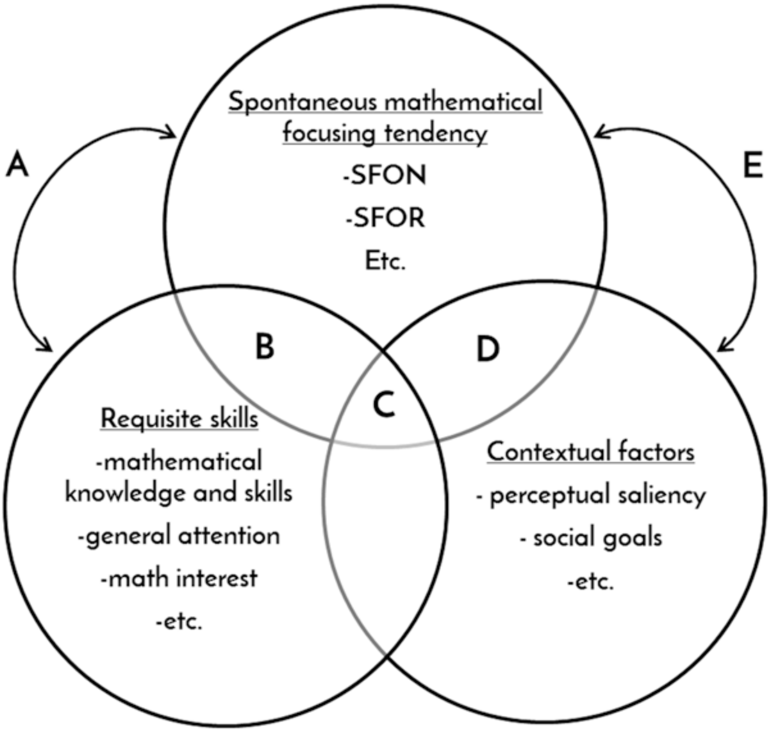

Fig. 4.1 Interrelations between spontaneous mathematical focusing tendencies, related skills, and general cognitive, meta-cognitive, and affective factors, which are present in any situation where a person uses or recognizes exact number or quantitative relations. This diagram is a schematic proposal, and the degree of overlap across these constructs is unknown and therefore should not be considered "to scale" with the figure

In an attempt to ensure that spontaneous mathematical focusing tendency measures truly capture this tendency independently from other factors, tasks must meet the following design principles: the task should be (1) mathematically unspecified, (2) open for multiple (mathematical and non-mathematical) possible interpretations, (3) fully engaging for all, and (4) within the range of competences (Hannula, 2005; Hannula & Lehtinen, 2005). The first two principles ensure that the context does not provide hints or constraints to numerical responses, so that the participants' focus on numerosity is spontaneous. There should be no hints that numerical responses are intended, and tasks and materials should not be associated with typical counting or numerical exercises. In this way, the probability of producing numerically accurate response without spontaneously focusing on numerosity is low. The last two principles diminish the likelihood that other potential factors (e.g., general attention or mathematical knowledge) explain the individual variation in SFON tendency. Specifically, the task needs to capture and maintain the child's attention. The numerical sets included in the tasks should be small enough for participants to reliably enumerate. Other cognitive and motor demands, such as verbal production, working memory, response inhibition, and motoric imitation, must be age-appropriate (Hannula, 2005). Satisfying these four conditions strengthens the validity of the SFON tasks and the interpretation that individual differences in SFON scores accurately reflect individual differences children's SFON tendency. It is important to note that additional numerical behaviors, such as counting or commenting on the set size, are indicators of SFON, even if the set produced by the child does not match numerically to the examiner's set.

A number of studies have shown that variation in students' performances on SFON and SFOR tasks is not entirely explained by the mathematical or other cognitive skills needed to solve the tasks (Hannula & Lehtinen, 2005; Hannula-Sormunen et al., 2015; McMullen et al., 2016; McMullen, Hannula-Sormunen, & Lehtinen, 2014). For example, many 6-year-old participants in early studies did not spontaneously focus on numerosity during SFON tasks, but almost all children were able to use the exact numbers in their actions when explicitly guided to do so (Hannula & Lehtinen, 2005). In a more recent study, there was clear discriminant validity separating SFON tendency and verbal counting skills although there was some overlap between 6-year-olds' performance on six tasks measuring these two constructs (Hannula-Sormunen et al., 2015). Nanu et al. (2018) showed that response patterns in the SFON tasks were significantly different from typical response patterns measuring enumeration skills. The findings from these studies support the claim that SFON tasks capture individual differences in SFON tendency rather than enumeration accuracy.

SFOR tendency has also been examined in relation to the requisite skills needed to solve tasks using exact quantitative relations (McMullen et al., 2014; McMullen et al., 2016). In a sample of US students in kindergarten to third grade (McMullen et al., 2014), substantial individual differences in SFOR tendency, both within and across grade levels, were not entirely explained by the requisite mathematical skills needed to complete the tasks. In a more recent study, third to fifth graders in Finland completed three paper-and-pencil measures of SFOR tendency, and then completed one item from each task in guided format (McMullen et al., 2016). Since all participants completed the guided versions of these tasks, it was possible to statistically account for the students' guided performance. A "pure" SFOR tendency variable was calculated using residualized scores for SFOR responses adjusted for performance on the guided versions of the tasks. This statistical procedure effectively removes the overlap between SFOR and requisite skills (Fig. 4.1b). Even after taking into account students' guided performance, substantial individual differences in SFOR tendency remained, within and across grade levels.

Although previous studies have directly juxtaposed SFON and SFOR tendencies with the task-relevant requisite mathematical skills and knowledge, fewer studies have explicitly focused on how other cognitive, meta-cognitive, and affective aspects of mathematical development are related to SFON and SFOR tendencies (e.g., Hannula et al., 2010; Van Hoof et al., 2016). Instead, these aspects of mathematics development are more often used as control measures in SFON and SFOR studies to examine whether they explain associations between SFON and SFOR tendencies and mathematical skills. Across several such studies, the relation between SFON and mathematical skills remained significant, even after controlling for age and cognitive skills including full scale IQ (Nanu et al., 2018), verbal IQ (Poltz et al., 2013), or non-verbal IQ (Hannula et al., 2010; Hannula & Lehtinen, 2005; Hannula-Sormunen et al., 2015; Poltz et al., 2013); rapid serial naming (Hannula et al., 2010), working memory (Batchelor et al., 2015; Nanu et al., 2018; Poltz et al., 2013), inhibition (Poltz et al., 2013), executive function skills and vocabulary (Gray & Reeve, 2016), verbal comprehension (Hannula et al., 2010; Hannula & Lehtinen, 2005), verbal production skills (Batchelor et al., 2015), and spatial location detection (Hannula et al., 2010).

Fewer studies have focused on the relation between SFOR tendency and other cognitive factors related to mathematical development, but the evidence to emerge thus far implicates that SFOR tendency does overlap, to some extent, with students' mathematical skills and knowledge, along with other related cognitive skills. SFOR tendency does appear to be a unique component of mathematical cognition, and it remains a significant predictor of rational number knowledge and development when controlling for non-verbal intelligence (McMullen et al., 2016; McMullen, Hannula-Sormunen, & Lehtinen, 2017; Van Hoof et al., 2016). This relation between SFOR tendency and rational number knowledge and development is not explained by grade level, arithmetic fluency, whole number estimation, guided focusing on quantitative relations, mathematical achievement, spatial reasoning, or interest in mathematics (McMullen, Hannula-Sormunen, Lehtinen, & Siegler, submitted; McMullen et al., 2016; Van Hoof et al., 2016).

To summarize, as represented in Fig. 4.1, SFON and SFOR tendencies overlap with mathematical or other cognitive skills required in a given situation, but neither SFON nor SFOR is entirely explained by these requisite skills. In short, there is substantial evidence supporting that both SFON and SFOR tendencies are unique aspects of mathematical cognition.

4.4 The Relation Between SFON/SFOR and Mathematical Development

We now review studies showing how SFON and SFOR are related to mathematical development (Fig. 4.1a). We propose that SFON and SFOR tendencies are indicators preschool and school-age children's spontaneous mathematical activities in and out of the classroom (Hannula et al., 2005). We hypothesize that preschool and school-age children who have higher SFON and SFOR tendencies more readily recognize the mathematics embedded in everyday life, compared to children with low SFON and SFOR tendencies, and that through this increased awareness they gain more opportunities to practice their mathematical skills. This increased self-initiated practice helps students deepen their mathematical knowledge, and the deeper mathematical knowledge subsequently supports further development of spontaneous mathematical focusing tendencies. Thus, SFON and SFOR have a bi-directional and iterative developmental relation with related mathematical skills and knowledge.

Several studies demonstrate the relation between SFON or SFOR tendencies and the development of mathematical skills. Of the two, SFON tendency has a stronger evidence base, because across studies SFON has been linked to a broader range of mathematical abilities (Batchelor et al., 2015; Edens & Potter, 2013; Gray & Reeve, 2016; Hannula et al., 2010; Hannula & Lehtinen, 2005; Hannula-Sormunen et al., 2015; Kucian et al., 2012; McMullen et al., 2015; Nanu et al., 2018; Poltz et al., 2013). Still, there is a growing evidence base for the relation between SFOR tendency and mathematical development (McMullen et al., submitted; McMullen et al., 2014; McMullen et al., 2016; McMullen, Hannula-Sormunen, & Lehtinen, 2017; Van Hoof et al., 2016). There are differences and similarities in SFON and SFOR associations with domain-specific correlates of mathematical development (McMullen et al., submitted; Hannula et al., 2010). Relative to SFOR, SFON has been more consistently and more strongly associated with whole number enumeration and arithmetic skills (e.g., Hannula et al., 2010), whereas SFOR tendency has been more closely associated with rational number knowledge (e.g., Van Hoof et al., 2016). These results suggest that SFON and SFOR tendencies may each play a specific role in mathematical development.

The effects of prior knowledge on mathematical development are well acknowledged throughout topics, from early counting development to rational numbers and algebra (Siegler et al., 2012; Siegler, Thompson, & Schneider, 2011; Sophian, 1988). In order to pay attention to the mathematical aspects in and out of the classroom, students need to have at least some knowledge about where and when to apply their formal knowledge (Lehtinen & Hannula, 2006; Lobato et al., 2012). In early childhood, SFON tendency has been found to be supported by earlier enumeration and subitizing skills (Hannula & Lehtinen, 2005; Hannula-Sormunen et al., 2015). In fact, SFON tendency and enumeration skills were found to be in a reciprocal relation, with each predicting the other over time (Hannula & Lehtinen, 2005). SFOR tendency and rational number knowledge were found to follow a similar pattern of reciprocity, in which early SFOR tendency predicted later rational number knowledge, and vice versa (McMullen, Hannula-Sormunen, & Lehtinen, 2017). These iterative processes (Fig. 4.2) suggest that there is a strong link between formal and typically examined mathematical skills and knowledge and children's and students' spontaneous mathematical behavior.

The above described results all are important indicators of a potential link between SFON and SFOR tendencies and mathematical knowledge. Nonetheless, a causal link has only been tested in a few limited quasi-experimental studies of SFON tendency (Hannula et al., 2005; Hannula-Sormunen, Alanen, McMullen, & Lehtinen, 2016. One of the first such studies showed that 3-year-olds who participated in a training program that aimed to increase their SFON tendency had long-term gains in their enumeration skills (Hannula et al., 2005). In that study, an increase in SFON tendency led to improvements in later counting skills in these children. More recently, we found that 5-year-olds' arithmetic skills and SFON tendency developed as a result of playing the iPad game Fingu integrated with SFON-based everyday activities (Hannula-Sormunen, Alanen, et al., 2017). The

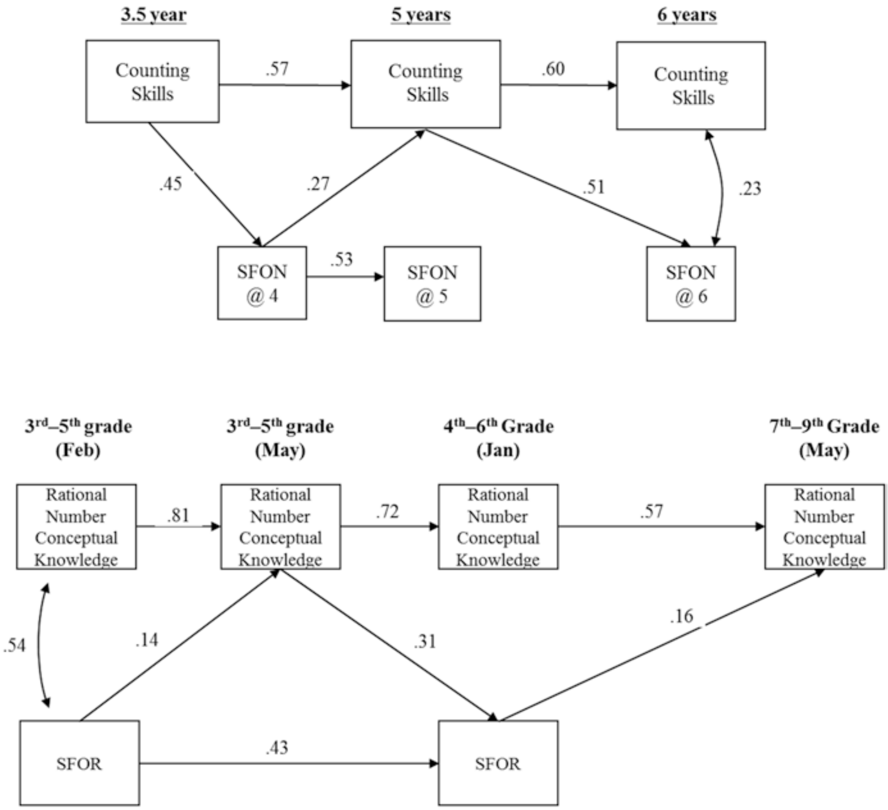

Fig. 4.2 Reciprocal relations between SFON tendency and counting skills (top; modified from Hannula & Lehtinen, 2005) and SFOR tendency and rational number knowledge (bottom; modified from McMullen, Hannula-Sormunen, & Lehtinen, 2017). For all paths, $p < 0.05$

results show a clear developmental advantage for the training group over the control group in arithmetic skills.

Collectively, these studies suggest a causal link between SFON tendency and early numerical skills, with SFON tendency having a positive impact on the development of early counting and enumeration skills. SFOR tendency has been identified as a unique predictor of mathematical development in late primary and early secondary school years. Overcoming minimal transfer effects of easily isolated drill-and-practice kinds of mathematical activities is an important goal for future investigations in mathematics education (e.g., Lehtinen, Hannula-Sormunen, McMullen, & Gruber, 2017). Promoting and supporting students' and children's self-initiated practice of newly learnt mathematical skills may help them start using these skills in their own activities, in addition to adult-guided mathematical exercises. In this way, the SFON and SFOR concepts, assessments, and training activities are of great educational relevance.

4.5 Attentional Considerations in SFON (and SFOR) Research

An essential feature of SFON and SFOR is the unprompted nature of the tendencies. It is important, both theoretically and educationally, to determine what other form "prompts" may take. In other words, to what extent do spontaneous mathematical focusing tendencies and contextual factors overlap (Fig. 4.1d)? Identifying explicit prompts to focus on number is fairly straightforward. Such prompts would involve number or exact quantity in instructions, such as "put the *same number* of cookies on this plate as I have," or "*how many* cookies are there?" or "bring *just enough* socks for Mr. Caterpillar" (Shusterman et al., 2017). These types of prompts are intentionally avoided in SFON and SFOR measures. But what if implicit prompts from *nonverbal* features in instructional materials promote SFON tendencies? For example, what if numerical (or other mathematical) features are more perceptually salient under some conditions, such as crowded versus uncrowded arrangements of item sets or arrays of colorful vs. monochromatic sets (e.g., Chan & Mazzocco, 2017)? The answers to such questions have implications for measuring SFON and SFOR tendencies and for intentionally promoting attention to mathematical features through instruction or the design and use of materials.

Prompts to attend to number or quantitative relations may exist throughout daily routines, but to different degrees depending on the nature of the task at hand. For instance, block play or meal preparation may elicit more attention to and discussion of numbers and mathematics (e.g., to determine the number of plates, forks, napkins, and cups needed for all persons who will be seated at the table) than dramatic play or free form painting at an easel (Chan, Mazzocco, & Praus-Singh, under review; Ferrara, Hirsh-Pasek, Newcombe, Golinkoff, & Lam, 2011; Susperreguy & Davis-Kean, 2016). Likewise, the arrangement of items in SFOR measures may make multiplicative relations more salient than additive relations, even when both would be mathematically correct (Degrande, Verschaffel, & Van Dooren, 2017). Multiple studies suggest that there are individual differences in the use of additive versus proportional reasoning that shift with age, suggesting that the use of one type of relation over the other may develop in concert with other mathematical skills (Van Dooren, Bock, & Verschaffel, 2010).

Although these considerations might be interpreted as challenging the notion of context-independent SFON or SFOR, an alternative perspective is that the relative degree to which these tendencies manifest across children simply interacts with such external influences (Fig. 4.1e). This bi-directional relation would lead to interactions, for example, between SFON tendency and perceptual salience like those demonstrated in earlier studies, and shown in Fig. 4.3. Hannula et al. (2005) found that an intervention based on caregivers' number-focused activities with preschoolers led to greater gains in the preschoolers' SFON tendencies, relative to a control group in which no such number-focused interactions were promoted. Importantly, the effect of this intervention was apparent for only those preschoolers in the experimental group who had at least some measurable SFON tendency at baseline.

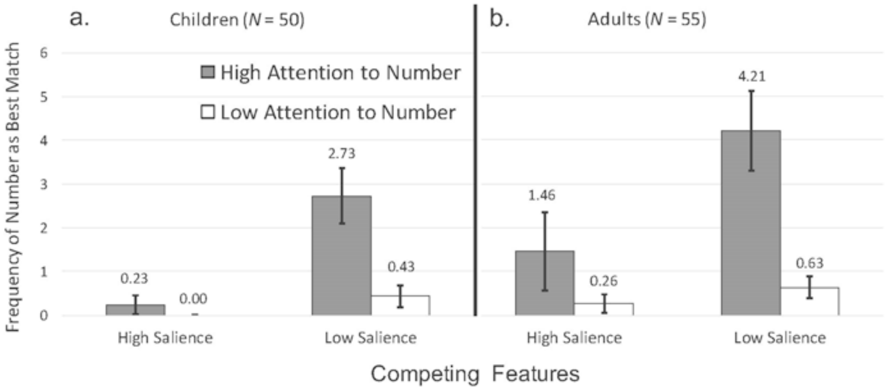

Fig. 4.3 Interactions between the salience of competing features and high versus low attention to number tendencies for children (**a**) and adults (**b**), based on data reported by Chan and Mazzocco (2017). Error bars represent standard deviations. Each salience condition included eight trials on which number was a possible matching feature. Alternative possible matching features were object color or shape (high salience) or object pattern or location (low salience)

Although Chan & Mazzocco (2017) did not measure *baseline* SFON tendencies in their study of picture matching, they found that by manipulating the relative salience of number (as a visual feature of the match options), they could also manipulate the frequency of number-based matches in children and adults during the task. Still, some children and adults *never* matched on number during the task, and in these potentially "low SFON" matchers, there were different effects of perceptual salience across individuals. This general lack of SFON may have inoculated children (and adults) from the main effect of feature salience, as illustrated by the significant interactions shown in Fig. 4.3. This suggests that eliciting mathematical behavior may have promise for promoting children's SFON behavior.

In another study of eliciting SFON behavior, the use of SFON "baits" was the focus of an intervention, in which 2.5–3-year-old children's SFON and small number recognition skills were supported (Hannula-Sormunen, Nanu, Södervik, & Mattinen, in preparation). The program aimed to promote noticing numerical features by embedding SFON baits around the daycare environment. These SFON baits were similar toys and everyday life materials arranged in a manner that made the numerical features very salient (e.g., Fig. 4.4). This often involved using several identical objects arranged in close proximity (e.g., two identical toy cars side-by-side), which increased the likelihood of counting behavior as the items were more likely to be perceived as a set to be counted. If the child did not focus on the numerosity of the items in the SFON bait, the early educators were asked to explicitly guide the child's attention by asking how many items there are, or, by taking away or adding items. Deliberate manipulation of numerosity has proven to be an efficient way of attracting children's attention towards numerosity of items in a set (Hannula et al., 2005). In contrast with previous SFON interventions that were effective only in children with some initial SFON tendency (Hannula et al., 2005),

Fig. 4.4 Examples of SFON-bait used at daycare in the SFON intervention study. (**a**) Two identical trucks arranged side by side on the yard. (**b**) "SFON slippers" with two similar slippers each with three decorating flowers. (**c**) multiple sets of two items (e.g., frying pans, chairs, dolls) arranged at a kitchen play area

the intervention with SFON baits led to significant improvement in SFON tendency and cardinality recognition and production skills particularly among even those with the weakest SFON tendency and cardinality recognition skills at the start of the intervention, in comparison to a control group where the participants received special training in listening comprehension skills (Hannula-Sormunen et al., in preparation).

The notion of contextual influences on SFON, through implicit manipulation of the environment or more explicit verbal prompting, generates testable hypotheses we believe are worthy of empirical pursuit and which we and others have begun to test. Numerous studies have provided evidence for individual stability in SFON and SFOR tendencies across tasks and time (Hannula & Lehtinen, 2005; McMullen, Hannula-Sormunen, & Lehtinen, 2017). Specific investigation into the effects of varying contexts on the expression of spontaneous focusing tendencies within and across tasks would clarify the nature of SFON and SFOR tendencies and provide valuable information about the nature of interventions that can enhance spontaneous or implicitly prompted mathematical focusing tendencies, including additional mathematical features such as spatial characteristics (Chan et al., under review; Degrande et al., 2017).

4.6 Implications for Classroom Practices

Ultimately, SFON and SFOR tendencies are not the end that is sought. Rather, they are a means for understanding individual differences in mathematical behavior in everyday situations, and are argued to be a key component of early mathematics education. Most studies examining mathematical development, teaching, and learning focus on the bottom two circles in Fig. 4.1, namely explicit skills and knowledge and contextual factors contributing to individual differences in these skills and knowledge. However, in order to fully understand the nature of children's and students' mathematical behavior and development, we must look at the intersection of all three circles (Fig. 4.1c), by also taking into consideration children's and students' own spontaneous mathematical activities. Supporting the mathematical behavior situated in this three-way intersection may lead to improvements in spontaneous mathematical focusing tendencies and mathematical skills and concepts.

There is consistent evidence that targeted interventions aimed at enhancing SFON and SFOR tendencies can be successful with both young children and older students (Hannula et al., 2005; McMullen, Hannula-Sormunen, Kainulainen, et al., 2017). In children as young as the age of 3 years old (Hannula et al., 2005), and in interventions as short as 20 min (Braham, Libertus, & McCrink 2018), evidence suggests that it is possible to increase SFON tendency among children. Just a few hours spent with a combination of student- and teacher-led activities over the course of a few weeks led to increases in SFOR tendency in sixth grade students (McMullen, Hannula-Sormunen, Kainulainen, et al., 2017). These results suggest that SFON and SFOR tendencies are malleable, despite the relative consistency in students' and children's performance on SFON and SFOR tasks over time when no intervention has occurred (e.g., Hannula & Lehtinen, 2005).

A key component of applying relevant mathematical concepts in formal and informal settings is recognizing exactly when mathematical aspects are present and useful in reasoning (Lobato, 2012; Lobato et al., 2012). In order to model the world mathematically, a child must first recognize that this can be done (McMullen & Resnick, 2018). In previous training studies aimed at supporting SFON and SFOR tendencies, the main goal was to make number and quantitative relations more explicit targets of focus in students' eyes (e.g., Mattinen, 2006). These programs explicitly highlighted and modelled when and how number and quantitative relations can be used in reasoning in and out of the classroom.

A working assumption regarding the development of SFON and SFOR tendencies is that they are a dimension of the advantages of social norms and practices offered by a rich mathematical home environment on performance in the mathematics classroom (e.g., Skwarchuk, Sowinski, & LeFevre, 2014). Equipping early childhood professionals with knowledge and skills to recognize and support SFON tendency (e.g., Mattinen, 2006) and facilitating peer interaction in small group activities (McMullen, Hannula-Sormunen, Kainulainen, et al., 2017) were effect means to increase SFON and SFOR tendencies. An in-depth analysis of behaviors among groups of students suggested that interaction between individuals can create mutual targets of focusing and mathematizing everyday objects or situations into abstract mathematical entities (Hilppö & Rajala, 2017). In general, in the case of both SFON and SFOR tendency, social interaction proved valuable for supporting SFON and SFOR tendencies.

Along with social interaction, multiple interventions aimed at improving SFON and SFOR tendencies also relied on embodied activities to reinforce the mathematical nature of everyday situations. These activities may include having the individuals enact the mathematical features or move within the space in which the mathematical aspects are embedded and are proven valuable for a variety of formal skills (Link, Moeller, Huber, Fischer, & Nuerk, 2013; Mix & Cheng, 2012). With a SFON intervention among preschool children, the mobile game "Fingu" (Holgersson et al., 2016) involved children recognizing numerosities as quickly as possible and assigning a cardinal value to them using both spoken words and finger touches. These activities were then extended outside of the digital learning environment, as the children were asked to use their virtual avatar in their everyday surroundings to find sets of objects and assign cardinal values to these objects (Hannula-Sormunen et al., 2016). The SFOR intervention also had students assign mathematical relations to everyday locations and distances (McMullen, Hannula-Sormunen, Kainulainen, et al., 2017). Students were sent on a mathematical treasure hunt, in which they needed to follow relational directions in order to find checkpoints. For example, starting at their classroom door, students were sent down the corridor to the library door, at which point they were asked to find the half-way point between their classroom door and the library door (or, e.g., three times this distance). These embodied activities, supported by digital tools that allow for highlighting the mathematical aspects of everyday spaces and objects, may have proved crucial for supporting SFON and SFOR tendencies among a wide range of individuals.

This is not to say that mathematical instruction should always and intensively involve promoting SFON and SFOR tendencies. As can be seen in Fig. 4.1, mathematical knowledge and skills are necessary conditions for focusing on aspects of number and relations in everyday situations (Hannula & Lehtinen, 2005; McMullen, Hannula-Sormunen, & Lehtinen, 2017), and contextual factors, including social interactions, also play a role (Chan & Mazzocco, 2017). Even so, it is expected that training SFON and SFOR tendencies could have fairly long-term effects and wide-ranging impact on related aspects of mathematical development (e.g., McMullen, Hannula-Sormunen, & Lehtinen, 2017). A potential boon for more long-lasting impact is possible through working with teachers in examining their beliefs and attitudes about the nature of mathematics and its role in everyday reasoning. Providing teachers with the tools to integrate activities and routines that promote SFON and SFOR tendencies into their everyday instruction may go a long way to offering students authentic experiences with mathematical reasoning (Verschaffel, Greer, & De Corte, 2000) that are not too burdensome in terms of their cognitive load (Kirschner, Sweller, & Clark, 2006), nor too loaded with extraneous details that do not support the mathematical meaning making process.

4.7 Conclusions and Future Directions

Research on spontaneous focusing on number and numerical relations, SFON and SFOR, opens our eyes to the broader possibilities of examining students' own spontaneous, self-initiated mathematical activities, the role of contextually-bound implicit prompts to attend to mathematical features, and the impact of these activities on students' success with mathematics. The state-of-the-science on spontaneous mathematical focusing tendencies indicates that there is much theoretical and educational value in examining and promoting young children's SFON and SFOR tendencies. Although more research is needed to determine the specific pathways between SFON or SFOR and mathematical development and the causal pathways implicated as potential underlying sources of variations in the mathematical thinking and learning, there is strong empirical evidence that these attentional processes are, at a minimum, highly relevant to early mathematics education. As reviewed in this chapter, the overlap, distinctions, and relations between (a) spontaneous mathematical focusing tendencies, (b) requisite skills (i.e., mathematical, motivational, and cognitive factors), and (c) contextual factors appears crucial for understanding exactly how children recognize and use mathematical features of everyday situations. These situations are ripe with opportunities to acquire lots of practice with mathematical skills. In addition to SFON and SFOR tendencies, other spontaneous mathematical focusing tendencies may interact with the requisite skills and contextual factors to influence mathematical development (e.g., spatial reasoning, Chan et al., under review). Nevertheless, in view of the potential power of bootstrapping informal activities and reasoning onto formal mathematical thinking (Resnick, 1987), educational practices and routines that promote mathematical focusing tendencies, including SFON, SFOR, and contextually-based prompts, may be an essential, foundational step in many mathematical activities. We believe such practices are also a fruitful target of inquiry into effective ways to support mathematical development for children who do not seem to "get" math. Their success may be the eventual outcome of a cascading set of developments that begins with children simply starting to notice the numbers and quantitative relations that surround them in their everyday lives. For all of these reasons, we conclude that promoting children's focusing on number and quantitative relations is a key component of early mathematics education.

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References

Advisory Committee on Mathematics Education. (2011). *Mathematical needs: Mathematics in the workplace and in higher education*. *Report of the ACME Mathematical Needs*. Retrieved from http://www.acme-uk.org/media/7624/acme_theme_a_final%20(2).pdf

Batchelor, S., Inglis, M., & Gilmore, C. (2015). Spontaneous focusing on numerosity and the arithmetic advantage. *Learning and Instruction, 40*, 79–88. https://doi.org/10.1016/j.learninstruc.2015.09.005

Boyer, T. W., Levine, S. C., & Huttenlocher, J. (2008). Development of proportional reasoning: Where young children go wrong. *Developmental Psychology, 44*(5), 1478–1490. https://doi.org/10.1037/a0013110

Braham, E., Libertus, M., & McCrink, K. (2018). Children's spontaneous focus on number before and after guided parent–child interactions in a children's museum. Developmental Psychology. http://dx.doi.org/10.1037/dev0000534

Chan, J. Y. C., Mazzocco, M. M., & Praus-Singh, T. L. (under review). *Parents and young children attend to numerical and mathematical features during play*.

Chan, J. Y.-C., & Mazzocco, M. M. M. (2017). Competing features influence children's attention to number. *Journal of Experimental Child Psychology, 156*, 62–81. https://doi.org/10.1016/j.jecp.2016.11.008

Degrande, T., Verschaffel, L., & Van Dooren, W. (2017). Spontaneous focusing on quantitative relations: Towards a characterisation. *Mathematical Thinking and Learning, 19*(4), 260–275. https://doi.org/10.1080/10986065.2017.1365223

Edens, K. M., & Potter, E. F. (2013). An exploratory look at the relationships among math skills, motivational factors and activity choice. *Early Childhood Education Journal, 41*(3), 235–243. https://doi.org/10.1007/s10643-012-0540-y

Ferrara, K., Hirsh-Pasek, K., Newcombe, N. S., Golinkoff, R. M., & Lam, W. S. (2011). Block talk: Spatial language during block play. *Mind, Brain, and Education, 5*(3), 143–151. https://doi.org/10.1111/j.1751-228X.2011.01122.x

Gray, S. A., & Reeve, R. A. (2016). Number-specific and general cognitive markers of preschoolers' math ability profiles. *Journal of Experimental Child Psychology, 147*, 1–21. https://doi.org/10.1016/j.jecp.2016.02.004

Hannula, M. M. (2005). *Spontaneous focusing on numerosity in the development of early mathematical skills* (Doctoral dissertation). Painosalama, University of Turku, Finland.

Hannula, M. M., & Lehtinen, E. (2005). Spontaneous focusing on numerosity and mathematical skills of young children. *Learning and Instruction, 15*(3), 237–256. https://doi.org/10.1016/j.learninstruc.2005.04.005

Hannula, M. M., Lepola, J., & Lehtinen, E. (2010). Spontaneous focusing on numerosity as a domain-specific predictor of arithmetical skills. *Journal of Experimental Child Psychology, 107*(4), 394–406. https://doi.org/10.1016/j.jecp.2010.06.004

Hannula, M. M., Mattinen, A., & Lehtinen, E. (2005). Does social interaction influence 3-year-old children's tendency to focus on numerosity? A quasi-experimental study in day care. In L. Verschaffel, E. De Corte, G. Kanselaar, & M. Valcke (Eds.), *Powerful environments for promoting deep conceptual and strategic learning* (pp. 63–80). Leuven: Leuven University Press.

Hannula, M. M., Räsänen, P., & Lehtinen, E. (2007). Development of counting skills: Role of spontaneous focusing on numerosity and subitizing-based enumeration. *Mathematical Thinking and Learning, 9*, 51–57. https://doi.org/10.1207/s15327833mtl0901_4

Hannula-Sormunen, M., Alanen, A., McMullen, J., & Lehtinen, E. (2016). *Integrating SFON enhancement with computerized arithmetical training—A pilot study*. Poster presented in Workshop Domain-General and Domain-Specific Foundations of Numerical and Arithmetic Processing. University of Tuebingen, Germany, 28th–30th September.

Hannula-Sormunen, M. M., Lehtinen, E., & Räsänen, P. (2015). Preschool children's spontaneous focusing on numerosity, subitizing, and counting skills as predictors of their mathematical performance seven years later at school. *Mathematical Thinking and Learning, 17*, 155–177. https://doi.org/10.1080/10986065.2015.1016814

Hannula-Sormunen, M. M., Nanu, C. E., Laakkonen, E., Munck, P., Kiuru, N., Lehtonen, L., & Pipari Study Group. (2017). Early mathematical skill profiles of prematurely and full-term born children. *Learning and Individual Differences, 55*, 108–119. https://doi.org/10.1016/j.lindif.2017.03.004

Hannula-Sormunen, M. M., Nanu, C. E., Södervik, I., & Mattinen, A. (in preparation). *Enhancement of SFON tendency and cardinality-related skills in daycare*.

Hilppö, J., & Rajala, A. (2017, November). *Sharing revelations: How quantitative relations are noticed in joint mathematical activities*. Paper presented at the meeting of the European Association for Practitioner Research on improving learning (EAPRIL), Hämmenlinnä, Finland.

Holgersson, I., Barendregt, W., Emanuelsson, J., Ottosson, T., Rietz, E., & Lindstrom, B. (2016). Fingu—A game to support children's development of arithmetic competence: Theory, design and empirical research. In P. S. Moyer-Packenham (Ed.), *Mathematis education in the digital era: Vol. 7. International perspectives on teaching and learning mathematics with virtual manipulatives* (pp. 123–145). https://doi.org/10.1007/978-3-319-32718-1_6

Kirschner, P. A., Sweller, J., & Clark, R. E. (2006). Why minimal guidance during instruction does not work: An analysis of the failure of constructivist, discovery, problem-based, experiential, and inquiry-based teaching. *Educational Psychologist, 41*(2), 75–86. https://doi.org/10.1207/s15326985ep4102_1

Kucian, K., Kohn, J., Hannula-Sormunen, M. M., Richtmann, V., Grond, U., Käser, T., & von Aster, M. (2012). Kinder mit Dyskalkulie fokussieren spontan weniger auf Anzahligkeit [Children with developmental dyscalculia focus spontaneously less on numerosities]. *Lernen Und Lernstörungen, 1*(4), 241–253. https://doi.org/10.1024/2235-0977/a000024

Lehtinen, E., & Hannula, M. M. (2006). Attentional processes, abstraction and transfer. In L. Verschaffel, F. Dochy, M. Boekaerts, & S. Vosniadou (Eds.), *Instructional psychology: Past, present and future trends* (pp. 39–55). Oxford: Elsevier.

Lehtinen, E., Hannula-Sormunen, M. M., McMullen, J., & Gruber, H. (2017). Cultivating mathematical skills: From drill-and-practice to deliberate practice. *ZDM Mathematics Education, 49*(4), 625–636. https://doi.org/10.1007/s11858-017-0856-6

Link, T., Moeller, K., Huber, S., Fischer, U., & Nuerk, H. C. (2013). Walk the number line—An embodied training of numerical concepts. *Trends in Neuroscience and Education, 2*(2), 74–84. https://doi.org/10.1016/j.tine.2013.06.005

Lobato, J. (2012). The actor-oriented transfer perspective and its contributions to educational research and practice. *Educational Psychologist, 47*(3), 232–247. https://doi.org/10.1080/00461520.2012.693353

Lobato, J., Rhodehamel, B., & Hohensee, C. (2012). "Noticing" as an alternative transfer of learning process. *Journal of the Learning Sciences, 21*(3), 433–482. https://doi.org/10.1080/10508406.2012.682189

Mattinen, A. (2006). *Huomio lukumääriin: Tutkimus 3-vuotiaiden lasten matemaattisten taitojen tukemisesta päiväkodissa [Focus on numerosities: A study on supporting 3 year-old children's mathematical development in day care]* (Doctoral dissertation). University of Turku, Turku, Finland.

McMullen, J. (2014). *Spontaneous focusing on quantitative relations and the development of rational number conceptual knowledge* (Doctoral dissertation). University of Turku, Turku.

McMullen, J., Hannula-Sormunen, M. M., Kainulainen, M., Kiili, K., & Lehtinen, E. (2017). Moving mathematics out of the classroom: Using mobile technology to enhance spontaneous focusing on quantitative relations. *British Journal of Educational Technology*. Advance online publication. https://doi.org/10.1111/bjet.12601

McMullen, J., Hannula-Sormunen, M. M., Laakkonen, E., & Lehtinen, E. (2016). Spontaneous focusing on quantitative relations as a predictor of the development of rational number conceptual knowledge. *Journal of Educational Psychology, 108*(6), 857–868. https://doi.org/10.1037/edu0000094

McMullen, J., Hannula-Sormunen, M. M., & Lehtinen, E. (2014). Spontaneous focusing on quantitative relations in the development of children's fraction knowledge. *Cognition and Instruction, 32*(2), 198–218. https://doi.org/10.1080/07370008.2014.887085

McMullen, J., Hannula-Sormunen, M. M., & Lehtinen, E. (2015). Preschool spontaneous focusing on numerosity predicts rational number knowledge six years later. *ZDM Mathematics Education, 47*(5), 813–824. https://doi.org/10.1007/s11858-015-0669-4

McMullen, J., Hannula-Sormunen, M. M., Lehtinen, E., & Siegler, R. S. (submitted). Adaptive rational number knowledge and spontaneous focusing on multiplicative relations. https://doi.org/10.17605/OSF.IO/XWGPQ

McMullen, J., Hannula-Sormunen, M. M. M., & Lehtinen, E. (2017). Spontaneous focusing on quantitative relations as a predictor of rational number and algebra knowledge. *Contemporary Educational Psychology, 51*(6), 356–365. https://doi.org/10.1016/j.cedpsych.2017.09.007

McMullen, J., & Resnick, L. B. (2018). Linking informal and formal mathematical reasoning: Two directions across the same bridge?. In Bergqvist, E., Österholm, M., Granberg, C., and Sumpter, L. (eds.), *Proceedings of the 42nd conference of the international group for the psychology of mathematics education (July 3–8, Umeå, Sweden*).

Mix, K. S., & Cheng, Y. L. (2012). The relation between space and math: Developmental and educational implications. In J. B. Benson (Ed.), *Advances in child development and behavior* (Vol. 42, pp. 197–243). https://doi.org/10.1016/B978-0-12-394388-0.00006-X

Mullis, I. V. S., Martin, M. O., Goh, S., & Cotter, K. (Eds.). (2016). *TIMSS 2015 encyclopedia: Education policy and curriculum in mathematics and science*. Retrieved from http://timssandpirls.bc.edu/timss2015/encyclopedia/

Nanu, C. E., McMullen, J., Munck, P., Hannula-Sormunen, M. M., & Pipari Study Group. (2018). Spontaneous focusing on numerosity in preschool as a predictor of mathematical skills and knowledge in the fifth grade. *Journal of Experimental Child Psychology, 169*, 42–58. https://doi.org/10.1016/j.jecp.2017.12.011

Poltz, N., Quandte, S., Wyschkon, A., Hannula-Sormunen, M. M., von Aster, M., & Esser, G. (2013). *Spotaneous focusing on numerosity and the development of mathematical skills in kindergarten.* Paper presented at the meeting of ECDP, Lausanne, Switzerland.

Rathé, S., Torbeyns, J., Hannula-Sormunen, M. M., De Smedt, B., & Verschaffel, L. (2016). Spontaneous focusing on numerosity: A review of recent research. *Mediterrean Journal for Research on Mathematics Education, 15*, 1–25.

Rathé, S., Torbeyns, J., Hannula-Sormunen, M. M., & Verschaffel, L. (2016). Kindergartners' spontaneous focusing on numerosity in relation to their number-related utterances during numerical picture book reading. *Mathematical Thinking and Learning, 18*(2), 125–141. https://doi.org/10.1080/10986065.2016.1148531

Resnick, L. B. (1987). The 1987 presidential address: Learning in school and out. *Educational Researcher, 16*(9), 13–20.; 54. https://doi.org/10.3102/0013189X016009013

Sella, F., Berteletti, I., Lucangeli, D., & Zorzi, M. (2016). Spontaneous non-verbal counting in toddlers. *Developmental Science, 19*(2), 329–337. https://doi.org/10.1111/desc.12299

Shusterman, A., Cheung, P., Taggart, J., Bass, I., Berkowitz, T., Leonard, J. A., & Schwartz, A. (2017). Conceptual correlates of counting: Children's spontaneous matching and tracking of large sets reflects their knowledge of the cardinal principle. *Journal of Numerical Cognition, 3*(1), 1–30. https://doi.org/10.5964/jnc.v3i1.65

Siegler, R. S., Duncan, G. J., Davis-Kean, P. E., Duckworth, K., Claessens, A., Engel, M., … Chen, M. (2012). Early predictors of high school mathematics achievement. *Psychological Science, 23*(7), 691–697. https://doi.org/10.1177/0956797612440101

Siegler, R. S., Thompson, C., & Schneider, M. (2011). An integrated theory of whole number and fractions development. *Cognitive Psychology, 62*(4), 273–296. https://doi.org/10.1016/j.cogpsych.2011.03.001

Singer-Freeman, K. E., & Goswami, U. (2001). Does half a pizza equal half a box of chocolates? Proportional matching in an analogy task. *Cognitive Development, 16*, 811–829.

Skwarchuk, S. L., Sowinski, C., & LeFevre, J. A. (2014). Formal and informal home learning activities in relation to children's early numeracy and literacy skills: The development of a home numeracy model. *Journal of Experimental Child Psychology, 121*(1), 63–84. https://doi.org/10.1016/j.jecp.2013.11.006

Sophian, C. (1988). Limitations on preschool children's knowledge about counting: Using counting to compare two sets. *Developmental Psychology, 24*(5), 634–640. https://doi.org/10.1037/0012-1649.24.5.634

Sophian, C. (2000). Perceptions of proportionality in young children: Matching spatial ratios. *Cognition, 75*(2), 145–170. https://doi.org/10.1016/S0010-0277(00)00062-7

Spinillo, A. G., & Bryant, P. (1991). Children's proportional judgments: The importance of "half". *Child Development, 62*(3), 427–440. https://doi.org/10.1111/j.1467-8624.1991.tb01542.x

Susperreguy, M. I., & Davis-Kean, P. E. (2016). Maternal math talk in the home and math skills in preschool children. *Early Education and Development, 27*(6), 841–857. https://doi.org/10.1080/10409289.2016.1148480

Vamvakoussi, X. (2015). The development of rational number knowledge: Old topic, new insights. *Learning and Instruction, 37*, 50–55. https://doi.org/10.1016/j.learninstruc.2015.01.002

Van Dooren, W., Bock, D. D., & Verschaffel, L. (2010). From addition to multiplication … and back: The development of students' additive and multiplicative reasoning skills. *Cognition and Instruction, 28*(3), 360–381. https://doi.org/10.1080/07370008.2010.488306

Van Hoof, J. J., Degrande, T., McMullen, J., Hannula-Sormunen, M. M., Lehtinen, E., Verschaffel, L., & Van Dooren, W. (2016). The relation between learners' spontaneous focusing on quantitative relations and their rational number knowledge. *Studia Psychologica, 58*(2), 156–170.

Verschaffel, L., Greer, B., & De Corte, E. (2000). *Making sense of word problems*. Lisse: Swets & Zeitlinger.