Chapter 2
Subitizing: The Neglected Quantifier

Douglas H. Clements, Julie Sarama, and Beth L. MacDonald

Abstract We define and describe how subitizing activity develops and relates to early quantifiers in mathematics. Subitizing is the direct perceptual apprehension and identification of the numerosity of a small group of items. Although subitizing is too often a neglected quantifier in educational practice, it has been extensively studied as a critical cognitive process. We believe that subitizing also helps explain early cognitive processes that relate to early number development and thus deserves more instructional attention. We also contend that integrating developmental/cognitive psychology and mathematics education research affords opportunities to develop learning trajectories for subitizing. A complete learning trajectory includes three components: *goal*, *developmental progression,* or learning path through which children move through levels of thinking, and *instruction*. Such a learning trajectory thus helps establish goals for educational purposes and frames instructional tasks and/or teaching practices. Through this chapter, it is our hope that early childhood educators and researchers begin to understand how to develop critical educational tools for early childhood mathematics instruction. Through this instruction, we believe that children will be able to use subitizing to discover critical properties of number and build on subitizing to develop capabilities such as unitizing, cardinality, and arithmetic capabilities.

Keywords Arithmetic · Early childhood education · Kindergarten · Learning trajectories · Mathematics education · Number · Preschool · Subitizing

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D. H. Clements (✉)
Morgridge College of Education, Marsico Institute for Early Learning and Literacy, University of Denver, Denver, CO, USA
e-mail: Douglas.Clements@du.edu

J. Sarama
Morgridge College of Education, University of Denver, Denver, CO, USA
e-mail: Julie.Sarama@du.edu

B. L. MacDonald
School of Teacher Education and Leadership, Utah State University, Logan, UT, USA
e-mail: beth.macdonald@usu.edu

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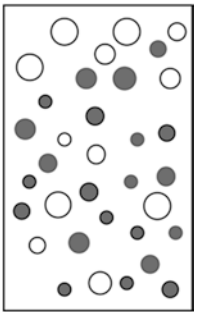

Fig. 2.1 A display shown to children that controls for area but represents a 1:2 ratio of unshaded to shaded circular regions

Children 6 months of age and younger appear to be sensitive to number. For example, they habituate to 1 versus 2 or 3 and 2 versus 3 objects (Antell & Keating, 1983; Starkey, Spelke, & Gelman, 1990). That is, they eventually “get used to” repeated sets of 3, even as color, size, and arrangements change, and become more attentive only when a set with a different number, such as 2, is shown. This indicates that infants are sensitive to the quantities in a small set of items before they are taught number words, counting, or finger patterns.

Children also are sensitive to displays with larger numbers of items. For example, they can habituate to *ratios* of 1:2 (Mazzocco, Feigenson, & Halberda, 2011, see Fig. 2.1 and Matthews, this volume). They also have a sense of the results when displays show a combination of large numbers of dots. Still, teachers may say that some much older, elementary school children cannot immediately name the number shown on dice. So, what is this ability to name exact numbers quickly? Is it a special way of counting or a separate way of acting on objects? Should we teach it, or is it simply innate? Does this ability develop as children learn more sophisticated understandings for number? How does it relate to other activities with number or quantity? As we shall see, although subitizing is too often a neglected quantifier in educational practice, it has been extensively studied as a critical cognitive process.

2.1 The Search for the Earliest Number Competencies

2.1.1 Subitizing: A Long History

Subitizing is “instantly seeing how many.” From a Latin word meaning *suddenly*, subitizing is the direct perceptual apprehension and identification of the numerosity of a small group of items. In the first half of the twentieth century, researchers believed counting did not imply a true understanding of number but subitizing did (e.g., Douglass, 1925). Some saw the role of subitizing as a developmental prerequisite to counting. Freeman (1912) suggested that whereas measurement focused on the whole and counting focused on the unit, only subitizing focused on both the whole and the unit—so, subitizing underlies number ideas. Carper (1942) agreed subitizing was more accurate than counting and more effective in abstract situations. Kaufman, Lord, Reese, and Volkmann (1949) initially named subitizing and distinguished this activity as very different from estimation activity. Individuals were relatively more accurate and experienced higher degrees of confidence in their enumeration when subitizing small sets of items (≤5) compared to when they were estimating larger sets of items (>5).

In the second half of the twentieth century, educators developed several models of subitizing and counting. *Subitizing* was initially defined in the field of psychology (Kaufman et al., 1949). Essentially, Kaufman et al. found that subitizing activity was quite different than estimation, as individuals drew from a unique form of visual number discrimination characterized by speed, accuracy, and degree of confidence (1949). More specifically, Kaufman et al. found that individuals numerically identifying sets of five or fewer objects were relatively faster (≤40 ms/item in a perceptual field) in their recall times, had higher levels of confidence, and had higher accuracy rates (1949). Klahr (1973a, 1973b) began discussing subitizing as a form of visual *information processing* and a type of *quantification operator* (e.g., counting, subitizing, estimating). Klahr posited that subitizing did not *rely* on an encoding process, but in fact *was* an encoding process, explaining such different recall times when individuals subitized items between one and five.

Based on the same notion that subitizing was a more "basic" skill than counting (Klahr & Wallace, 1976; Schaeffer, Eggleston, & Scott, 1974), Klahr (1973a) hypothesized that after items were encoded through subitizing activity, individuals stored matched patterned stimuli to numerical thinking structures in their long-term memory. This explained why children can subitize directly through interactions with the environment, without social interactions. Supporting this position, Fitzhugh (1978) found that some children could subitize sets of one or two but were not able to count them. None of these very young children were able to count any sets that they could not subitize. Fitzhugh concluded that subitizing is a necessary precursor to counting. This research also began to define subitizing, for the first time, as supported by pre-attentional mechanisms (Klahr, 1973b; Trick & Pylyshyn, 1994) and a form of numerical encoding system (Klahr, 1973a).

However, in 1924, Beckmann found that younger children used counting rather than subitizing (cited in Solter, 1976). Others agreed that children develop subitizing later, as a shortcut to counting (Beckwith & Restle, 1966; Brownell, 1928; Silverman & Rose, 1980). Developmental psychologists Gelman and Gallistel (1978) expressed this view, claiming that subitizing is simply a form of rapid counting.

Although debates continue, recent research has shown that—as the introduction shows—*some* sensitivity to very small numbers develops very early (we do not call this "subitizing" yet as children are not connecting an exact quantity to a number word). Further, that sensitivity exists for larger numbers in a different form. The latter has been termed the Approximate Number System and we turn to it next.

2.1.2 The Approximate Number System (ANS)

Figure 2.1 illustrates a situation revealing an ability to estimate that is shared across animals and people. For example, monkeys and birds can be trained to discriminate both large and small sets (of visual dots or sounds) that differ in a 1 to 2 (or greater) ratio (but not 2:3) (Starr, Libertus, & Brannon, 2013). Baby chicks, first imprinted with a set of three, shown 4 objects going behind a screen on the right, then 1 going beyond a screen on the left, then 1 moved from the right to the left, go immediately to the screen on the right (Vallortigara, 2012).

Neuroscience findings suggest that humans, like other animal species, encode approximate number (Piazza, Izard, Pinel, Le Bihan, & Dehaene, 2004). The IPS coding for number in humans is compatible with that observed in macaque monkeys, suggesting an evolutionary basis for human elementary arithmetic (Piazza et al., 2004). Most children without specific disabilities possess these competencies, which appear to form one of the innate, foundational abilities for all later numerical knowledge—the Approximate Number System (ANS). Six-month-old infants can discriminate the 1:2 ratio (as in Fig. 2.1) but by 9 months of age, they can also distinguish sets in a 2:3 ratio (e.g., 10 compared to 15). ANS correlates with mathematics competencies in preschoolers (Mazzocco et al., 2011; Soto-Calvo, Simmons, Willis, & Adams, 2015), even with age and verbal ability controlled (Libertus, Feigenson, & Halberda, 2011b), although these correlations are larger for children low in mathematical knowledge (Bonny & Lourenco, 2013). It may be that higher achievers have access to more and more sophisticated strategies that makes ANS precision less relevant. Further, lack of ANS proficiency may be one but only one of several sources of poor mathematics learning (Chu, vanMarle, & Geary, 2013).

2.1.3 Is Subitizing Also an Approximate Estimator?

This raises the question of whether initial sensitivity to number is also based on approximate estimators, and only seems accurate early on in children's development because numbers are very small. Subitizing differs from the ANS in that the goal is to determine the *exact* number of items in a set and to connect the number to another representation, usually number words. Supporting the distinction, subitizing does not fit Weber's law for ANS and thus appears to be a distinct, dedicated method of quantification (Revkin, Piazza, Izard, Cohen, & Dehaene, 2008). Subitizing also appears distinct from counting. First, there is little or no relationship between children's performance on counting and subitizing tasks (Pepper & Hunting, 1998). Second, lesions that affect counting and subitizing appear to be in separate parts of the brain (Demeyere, Rotshtein, & Humphreys, 2012).

Still, questions remain about how subitizing operates. For example, some have questioned whether subitizing is really about number or a general sense of quantity. That is, some studies suggest that infants in "number" experiments may be responding to overall contour length, area, mass, or density rather than discrete number (Feigenson, Carey, & Spelke, 2002; Tan & Bryant, 2000). In one study, infants dishabituated to changes in contour length when the number of objects was held constant, but they did not dishabituate to changes in number when contour length was held constant (Clearfield & Mix, 1999), suggesting they may be more sensitive to continuous than discrete quantities. FMRI studies iterate these findings as they show 4-year-olds and adults exhibit a greater response in their IPS to visual arrays that change in the number of elements than to stimuli that change in shape (Cantlon, Brannon, Carter, & Pelphrey, 2006). Deaf people, who knew Japanese Sign Language but not American Sign Language, showed no activation in regions associated with numerical processing when taught ASL signs (but not their meanings) for numerals. However, when told what the signs represented, they showed just such activation—even when they could not accurately code those signs (Masataka, Ohnishi, Imabayashi, Hirakata, & Matsuda, 2006).

Models of subitizing There are then various empirical findings and theoretical models of subitizing (for reviews more detailed than this summary, see Butterworth, 2010; Hannula, Lepola, & Lehtinen, 2010; Sarama & Clements, 2009). Figure 2.2 illustrates several of them.

Some believe that recognition of patterns of movement (even eye movements), or *scan-paths* (Fig. 2.2), is the underlying *non*-numerical process that is then linked to specific numerosities (Chi & Klahr, 1975; Glasersfeld, 1982; Klahr & Wallace, 1976). Numerical subitizing requires a subsequent *reflective abstraction*, which occurs when the child abstracts the mental actions from the sensory-motor contexts and is capable of reflecting on these actions. Piaget (1977/2001) describes reflective abstraction as encapsulating two phases. The first phase is a “projection phase in

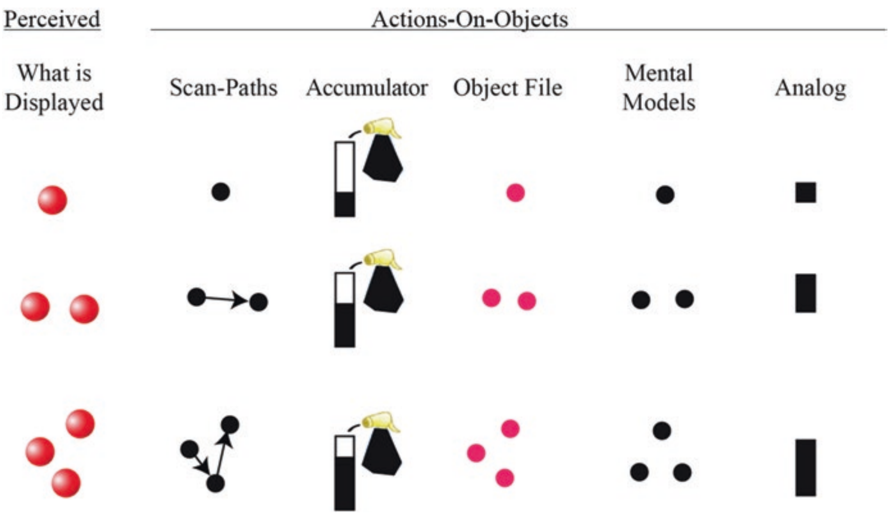

Fig. 2.2 Theories of subitizing

which the actions at one level become the objects of reflection at the next" (313). The second phase is a "reflection phase in which a reorganization takes place" (p. 313). Such abstractions can stem from temporal or rhythmic activity (Glasersfeld, 1982), are not grounded in perceptual pulsations (Glasersfeld, 1995), and may help to explain early number development (Steffe & Cobb, 1988).

Other models consider subitizing to be a numerical process. In the "accumulator" column of Fig. 2.2, subitizing is a numerical process enabled by the availability of the functional equivalent of a number line in the brain that operates on both simultaneous and sequential items (cf. Huntley-Fenner, 2001). There is a pacemaker that emits equivalent pulses at a constant rate. When a unitized item (e.g., a block taken as "one") is encountered, a pulse is allowed to pass through a gate, entering an *accumulator*—think of a squirt of water entering a tall glass. The gradations on the accumulator estimate the number in the collection of units, similar to height indicating the number of squirts in the glass (Meck & Church, 1983). This model does *not* require that the accumulator has an *exact* representation of number (see also Feigenson, Dehaene, & Spelke, 2004). The "squirts" and the amount in the "glass" are approximate. Support for this view comes from research indicating that children younger than 3 years tend not to represent any numbers except 1 and 2 precisely (Antell & Keating, 1983; Baroody, Lai, & Mix, 2005; Feigenson, Carey, & Hauser, 2002; Mix, Huttenlocher, & Levine, 2002).

The next theory holds that humans create "object files" that store data on each object's properties (Fig. 2.2). They can use these object files to respond differently to various situations. Thus, some situations can be addressed by using the objects' individuation or separateness as objects, and others can be addressed by using the analog properties of these objects, such as contour length (Feigenson, Carey, & Spelke, 2002). For example, children might use parallel-processed *individuation* for very small collections, but continuous extent when storage for individuation is exceeded. Individuation is the visual referencing of items as "that [which] refers to something we have picked out in our field of view without reference to what category it falls under or what properties it may have" (Pylyshyn, 2001, p. 129). Thus, processing is *preconceptual* prior to any entry into working memory (Pylyshyn, 2001). (From our perspective, even if such individuation is accepted as an early basis for number, it might not in itself constitute knowledge of *number, an issue to which we will return*.)

The mental models view (Fig. 2.2) postulates that children represent numbers nonverbally and approximately, then nonverbally but exactly, and eventually via verbal, counting-based processes (Huttenlocher, Jordan, & Levine, 1994; Mix et al., 2002). Children cannot initially differentiate between discrete and continuous quantities, but represent both approximately using one or more perceptual cues such as contour length (Mix et al., 2002). Children gradually develop the ability to individuate objects, providing the ability to build notions of discrete number. About the age of 2 years, they develop representational, or symbolic, competence, allowing them to create mental models of collections, which they can retain, manipulate (move), add to or subtract from, and so forth (although the model does not adequately describe how cardinality is ultimately cognized and how comparisons are made).

Creation of mental models with their added abstraction differentiates this view from the "object files" theory. Early nonverbal capabilities then provide a basis for the development of verbally based numerical and arithmetic knowledge (young children are more successful on nonverbal than verbal versions of number and arithmetic tasks, Huttenlocher et al., 1994; Jordan, Hanich, & Uberti, 2003; Jordan, Huttenlocher, & Levine, 1992; Jordan, Huttenlocher, & Levine, 1994; Levine, Jordan, & Huttenlocher, 1992). Meaningful learning of number words (in contrast to symbolic ability) may cause the transition to exact numerical representations (Baroody et al., 2005). This may provide the basis for understanding cardinality and other counting principles, as well as arithmetic ideas (Baroody, Lai, & Mix, 2006).

An alternative model postulates an innate abstract *module*. A module is a distinct mental component that is dedicated to a particular process or task and is unavailable for general processing. A number perception module would perceive numbers directly (Dehaene, 1997). This counting-like process is hypothesized to guide the development of whole number counting, hypothesized to be a privileged domain. Researchers use findings from both humans and non-human animals to support this position (Gallistel & Gelman, 2005).

A New Model for the Foundations of Subitizing A synthesis of these positions produces a model that we believe is most consistent with the research. What infants quantify are collections of rigid objects. Sequences of sounds and events, or materials that are non-rigid and non-cohesive (e.g., water), are not quantified (Huntley-Fenner, Carey, & Solimando, 2002). Quantifications of these collections begin as an undifferentiated, innate notion of the amount of objects. Object individuation, which occurs early in pre-attentive processing (and is a general, not quantitative, process, cf. Moore & Ashcraft, 2015), helps lay the groundwork for differentiating discrete from continuous quantity. That is, the object file system stores information about the objects, some or all of which is used depending on the situation.

Simultaneously, an estimator (accumulator) mechanism stores analog quantitative information (Feigenson, Carey, & Spelke, 2002; Gordon, 2004; Johnson-Pynn, Ready, & Beran, 2005). This estimator also includes a set of number filters, each tuned to an approximate very small number of objects (e.g., 2) although they overlap (Nieder, Freedman, & Miller, 2002). The child encountering small sets opens object files for each in parallel. By about a half-year of age, infants may represent very small numbers (1 or 2) as individuated objects (close to the "mental models" column of Fig. 2.2). However, larger numbers in which continuous extent varies or is otherwise not reliable (McCrink & Wynn, 2004) may be processed by the analog estimator as a collection of binary impulses (as are event sequences later in development, see the "analog" column of Fig. 2.2), but not by exact enumeration (Shuman & Spelke, 2005) by a brain region that processes quantity (size and number, undifferentiated, Pinel, Piazza, Le Bihan, & Dehaene, 2004). Without language support, these are inaccurate processes for numbers above two (Gordon, 2004).

To compare quantities, correspondences are processed. Initially, these are inexact estimates comparing the results of two estimators, depending on the ratio between the sets (Johnson-Pynn et al., 2005). Once the child can represent objects mentally, they can also make exact correspondences between these nonverbal representations, and eventually develop a quantitative notion of that comparison (e.g., not just that ••• is more than ••, but also that it contains one more •, Baroody et al., 2005).

Fully Functional Subitizing—Explicit Cardinality Even these correspondences, however, do not necessarily imply a cardinal representation of the collection (a representation of the collection qua a *numerosity* of a *group* of items). That is, our model distinguishes between noncardinal representations of a collection and *explicit* cardinal representations that is necessary to achieve fully functional subitizing competence. Indeed, a neuroimaging study found that brain regions that represent numerical magnitude also represent spatial magnitude, such as the relations between sizes of objects, and thus may not be numerical in function (Pinel et al., 2004). Only for numerical representations does the individual apply an integration operation (Steffe & Cobb, 1988) to create a composite with a numerical index. This integration operation uses present cognitive schemes to project and reorganize actions so they are considered mathematical objects. Some claim that the accumulator yields a cardinal output; however, it may be quantitative and still—because it indexes a collection using an abstract, cross-modality system for numerical magnitude (cf. Lourenco & Longo, 2011; Shuman & Spelke, 2005)—it may lack an explicit cardinality. For example, this system would not necessarily differentiate between ordinal and cardinal interpretations. Comparisons, such as correspondence mapping, might still be performed, but only at an implicit level (cf. Sandhofer & Smith, 1999). (It is possible to index a numerical label without attributing explicit cardinality. For example, lower animal species seem to have some perceptual number abilities, but only birds and primates also have shown the ability to connect a perceived quantity with a written mark or auditory label, Davis & Perusse, 1988.) In this view, only with experience representing and naming collections is an explicit cardinal representation created. This is a prolonged process. Children may initially make word-word mappings between requests for counting or numbers (e.g., "how many?") to number words until they have learned several (Sandhofer & Smith, 1999). Then they label some (small number) cardinal situations with the corresponding number word; that is, map the number word to the numerosity property of the collection. They begin this phase even before 2 years of age, but for some time, this applies mainly to the word "two," a bit less to "one," and with considerable less frequency, "three" and "four" (Fuson, 1992a; Wagner & Walters, 1982). We will discuss possible connections between subitizing and composite number understandings near the end of this chapter.

MacDonald and colleagues (MacDonald, 2015; MacDonald & Shumway, 2016; MacDonald & Wilkins, 2016, 2017) found that this early attention to 2 served preschool age children's ability to begin attending to subgroups of "two" when conceptually subitizing larger sets of items (e.g., four, five). Symmetrical orientations and orientations with a large space between subgroups of "two" seemed to afford these children's opportunities to attend to both subgroups. Symmetrical orientations freed children's working memory resources as they only needed to describe one 2 when building towards the total set of 4. Individuals' subitizing activity has been found to be affected by the space between the items in an orientation (Gebuis & Reynvoet, 2011) and was found to support young children's attention to the subgroups of the entire group of items (MacDonald & Wilkins, 2017).

Later Developments Only after many such experiences do children abstract the numerosities from the specific situations and begin to understand that the situations named by 3 correspond; that is, they begin to establish as what adults would term a numerical equivalence class. Counting-based verbal systems are then more heavily used and integrated, as described in the following section, eventually leading to explicit, verbal, mathematical abstractions. The construction of such schemes probably depends on guiding frameworks and principles developed from interactions with parents, teachers, and other knowledgeable people. Our model is supported by research on speakers of Mandrake in the Amazon, who lack number words for numbers above 5. They can compare and add large approximate numbers, but fail in exact arithmetic (Pica, Lemer, Izard, & Dehaene, 2004).

Nevertheless, it is significant that children discriminate exact collections on some quantitative bases from birth. Furthermore, most accounts suggest that these limited capabilities, with as yet undetermined contributions of maturation and experience, form a foundation for later learning. That is, they connect developmentally to culturally based cognitive tools such as number words and the number word sequence, to develop exact and extended concepts and skills in number.

Even though the shape of the items plays a secondary role in subitizing, particular orientations have been found to influence adults' degree of accuracy when subitizing larger sets of items (≤ 4). For instance, Logan and Zbrodoff (2003) found that the space between these groups of "twos" and "threes" afforded individuals more effective subitizing of four or more items. These findings suggest that individuals rely on patterned orientations of twos and threes (described as point-groupings) when subitizing. Thus, there is a special neural component of early numerical cognition present in the early years that may be the foundation for later symbolic numerical development. A language-independent ability to judge numerical values nonverbally appears to be important evolutionary precursor to later symbolic numerical abilities.

In summary, early quantitative abilities exist, but they may not initially constitute systems that can be said to have an explicit number concept. Instead, they may be pre-mathematical, foundational abilities (cf. Clements, Sarama, & DiBiase, 2004) that develop and integrate slowly, in a piecemeal fashion (Baroody, Benson, & Lai, 2003). For example, object individuation must be stripped of perceptual characteristics and understood as a perceptual unit item through abstracting and unitizing to be mathematical (Steffe & Cobb, 1988), and these items must be considered simultaneously as individual units and members of a collection whose numerosity has a cardinal representation to be numerical, even at the lowest levels.

2.1.4 Categories of Subitizing

Regardless of the precise mental processes in the earliest years, subitizing appears to be phenomenologically distinct from counting and other means of quantification and deserves differentiated educational consideration. Further, subitizing ability is not merely a low-level, innate process although it builds on innate sensitivity to number. As stated previously, in contrast to what might be expected from a view of innate ability, subitizing *develops* considerably and combines with other mental processes.

Types of Subitizing Early attention to numerosities reveals *preconcepts*, defined by Piaget are pre-operational, "action-ridden, imagistic, and concrete" early forms of concepts that young children depend on (1977/2001, p. 159–160). Children acting on static, concrete images that have yet to be unitized, operationalized, or abstracted are relying on preconcepts. Children engaging with preconcepts are not yet able to identify members as belonging to a given set (e.g., class identification within groups) necessary for unitizing. Preconcepts are the basis for early, perceptual subitizing activity. However, once early forms of perceptual subitizing develop, Clements (1999) posited that students developed and drew from this activity to develop conceptual activity for subitizing.

Therefore, one major shift is the development from using only one, to using two types of subitizing. The first type, *perceptual subitizing* (Clements, 1999; see also theoretical justification in Karmiloff-Smith, 1992), is closest to the original definition of subitizing: Recognizing a number without consciously using other mental or mathematical processes and then naming it. Thus, perceptual subitizing employs a pre-attentional, encoding quantitative process but adds an intentional numerical process; that is, infant sensitivity to number is not (yet) perceptual subitizing. The term "perceptual" applies only to the quantification mechanism as phenomenologically experienced by the person; the intentional numerical labeling, of course, makes the complete cognitive act conceptual. A second type of subitizing (a distinction for which there is empirical evidence, Trick & Pylyshyn, 1994), *conceptual subitizing* (Clements, 1999), involves applying the perceptual subitizing processes repeatedly and quickly uniting those numbers. For example, one might recognize "10" on a pair of dice by recognizing the two collections (via perceptual subitizing) and composing them as units of units (Steffe & Cobb, 1988). Some research suggests that only the smallest numbers, perhaps up to 3, are actually perceptually recognized; thus, sets of 1 to 3 may be perceptually recognized, sets of 3 to about 6 may be and recomposed without the individual being aware of the subgroups. As we define it, conceptual subitizing refers to recognition in which the person uses such partitioning strategies and is aware of the parts and the whole. In the remainder of this section, we elaborate on each type.

Perceptual subitizing also plays the primitive role of *unitizing*, or making single "things" to count out of the stream of perceptual sensations (Glasersfeld, 1995). "Cutting out" pieces of experience, keeping them separate, and eventually coordinating them with number words are not trivial tasks for young children. For example, a toddler, to recognize the existence of a plurality, must focus on the items such as apples and repeatedly apply a template for an apple *and* attend to the *repetition* of the template application.

In an exploratory 22-session teaching experiment, MacDonald and Wilkins (2016) found that four preschool children (ages ranging from 4 years and 4 months to 5 years and 5 months) engaged in several types of perceptual subitizing that could explain early shifts in children's types of abstractions. Cross-case analyses determined similar activity children engaged in throughout the study. MacDonald and Wilkins (2016) developed a framework that explained types of activity that young children revisited when subitizing. In this framework, five sets of perceptual subitizing activity were found to explain how these children's perceptual subitizing activity changed. As shown in Table 2.1, these four preschool age children relied on perceptual figurative patterns when associating number with patterns when subitizing *(initial perceptual subitizing or IPS)*. Children were also found to subitize small subgroups, composed of two or three *(perceptual subgroup subitizing or PSS)*, but were not able to compose these subgroups. These activities, explained as a form of low-level processing, were purely associative and seemed to illustrate foundational operations of number in which children could project onto new schemes as early forms of mathematical objects. Further, when children's subitizing changed they began composing and decomposing subgroups of these total sets *(perceptual*

Table 2.1 Five different types of perceptual subitizing activity

Type	Description	Example
Initial Perceptual Subitizing (IPS)	• Children describe the visual motion or the shape of the dots	• Children will describe seeing "five" because it looks like a flower
Perceptual Subgroup Subitizing (PSS)	• Children numerically subitize small subgroups of two or three, but cannot subitize the entire composite group	• Children will state that they saw "two and three," or "two plus three," but do not use this to accurately describe the composite group
Perceptual Ascending Subitizing (PAS)	• Children describe the perceived cluster of items as subgroups and then the composite group	• Children will state that they saw "two and three," and then accurately describe the total composite group
Perceptual Descending Subitizing (PDS)	• Children describe the composite groups and then describe the perceived cluster of items as subgroups	• Children will state that they saw "five" because they saw "two and three"
Perceptual Counting Subitizing (PCS)	• Children initially describe seeing one more or one less than the composite group, and then counts down or up, respectively, to the composite group	• Children will state they saw "4 … 5" or "6 … 5" • Children will state they know it to be "5" because they saw "6 … 5"

Note. These five different types of perceptual subitizing activity categorically represent the observed child responses documented by MacDonald and Wilkins (2016) *ascending subitizing or PAS and perceptual descending subitizing or PDS)*, which was foundational for children's conceptual subitizing. PAS and PDS activity is similar to conceptual subitizing activity because the children are decomposing and composing units of units. However, PAS and PDS activity explained these children's reliance on perceptual material, spatial patterns, or finger patterns. Furthermore, when engaging in PAS and PDS activity these children acted on orientations in static means where subgroups did not have to be determined. For instance, they relied primarily upon the clustering of items or spatial arrangement of the items to determine operations that they would need to use when composing number. This means that children would not be required to partition the orientation into subgroups, but that they would operationalize the activity by partitioning and composing number in a more abstract manner. Thus, for these children to engage in conceptual subitizing, they would need to carry their (de)composition of number or their partitioning of orientations into the activity. Children also coordinated their counting with their perceptual subitizing. MacDonald and Wilkins found that all four children would subitize a set of items and then count up or down by one *(perceptual counting subitizing or PCS*). PCS activity was explained as a type of blend between both subitizing and counting activity.

These findings suggest that when children engage in perceptual subitizing, they are building initial schemes through a series of associations between orientations and early units of number. These schemes are foundational for (de)composition of number later, as these children begin developing conceptual processes of number in relation to their conceptual subitizing.

This takes us to the second type of subitizing, *conceptual subitizing* plays an advanced organizing role with the individual explicitly using partitioning, decomposing, and composing quickly to determine a number of items. Decomposing and composing are combining and separating operations that help children develop generalized part-whole relations, one of the most important accomplishments in arithmetic (National Research Council, 2001). The distinction between PDS activity and conceptual subitizing activity is that when children engage in PDS activity they are not able to numerically understand how these units relate to units because they are still relying on perceptual material, fingers, or spatial patterns. In PDS activity, young children are still dependent on the material shown to them when decomposing and composing number. In conceptual subitizing activity, children step away from the material and carry operations of number into the task. This distinction is explained further in a subsequent section where number and operations are explained as related to conceptual subitizing activity.

MacDonald and Wilkins (2016) also found two types of conceptual subitizing that describe how children's limited or flexible number understandings related to their subitizing activity (see Table 2.2). Children who have limited ability to draw from more than one set of subgroups when conceptually subitizing (evidenced through their description of exactly one set of subgroups) engage in *rigid conceptual subitizing (RCS)* (see Table 2.2). For instance, when children subitize "two, two, and one" each time they are shown a wide variety of "five" is evidence of their reliance on RCS. This activity indicates children's ability to see units of units when shown a wide variety of representations for “five.” However, the children engaging in RCS are limited because they cannot use flexible operations of number when conceptually subitizing. When children are capable of “seeing” two or more ways of composing items (e.g., two and two; three and one) when subitizing engages in *flexible conceptual subitizing (FCS)*. FCS activity evidences the multiple means in which children use operations when conceptual subitizing.

What is Subitized Another categorization involves the different types of things people can subitize. Spatial patterns such as those on dice are just one type. Other patterned modalities are temporal and kinesthetic, including finger patterns (motoric and visual/spatial), rhythmic patterns (e.g., 3 beats), and spatial-auditory patterns. Creating and using these patterns through conceptual subitizing helps children develop abstract number and arithmetic strategies. For example, children use temporal patterns when counting on. “I knew there were three more so I just said, nine … ten, eleven, twelve” (rhythmically gesturing three times, one “beat” with each count). They use finger patterns to figure out addition problems. For example, for 3 + 2, a child might put up a finger pattern they know as three, then put up two more (rhythmically—up, up) and then recognize the resulting finger pattern as “five.” Children who cannot subitize are handicapped in learning such arithmetic processes (Butterworth, 2010; Hannula et al., 2010). Children may be limited to subitize small numbers at first, but such actions are useful “stepping stones” to the construction of more sophisticated procedures with larger numbers, a point to which we return.

2.1.5 Possible Connections Between Unit Development and Subitizing Activity

Children’s subitizing activity changes over time that requires different types of actions that possibly relate to their ability to unitize members of a set. Thus, children’s perceptual subitizing activity may relate to *unit* development (Steffe & Cobb,

Table 2.2 Two different types of conceptual subitizing activity

Type	Description	Example
Rigid Conceptual Subitizer (RCS)	• Children describe seeing the composite unit and then one set of subgroups that always remain the same, regardless of the orientation or color of the items	• Children will always state they know a composite group to be four because they saw “two and two”
Flexible Conceptual Subitizing (FCS)	• Children describe seeing the composite unit and then two or more sets of subgroups in different tasks regardless of the orientation or color of the items	• Children will state that they know a composite group to be five because they saw “two and three,” but previously they explained the same orientation to be five because they saw “two, two, and one”

Note. These two different types of conceptual subitizing activity categorically represent the observed child responses documented by MacDonald and Wilkins (2016)

1988). Steffe and Cobb found that children engaged in counting developed *abstract singular units* (abstract units composed of 1) through their engagement with a variety of physical material of singular units (e.g., perceptual, figural patterns, motor activity, verbal utterances) (see Fig. 2.1). Children develop abstract singular units by first engaging with *perceptual singular units* as their unitization cuts away portions of "a specific experiential 'thing'" (Steffe & Cobb, 1988, p. 343). As children use *figural* (different representation), *motor* (motor pattern through activity), and *verbal* (utterance of a number word) singular units to represent perceptual material, they develop more abstract singular units (1988).

Steffe and Cobb found that first grade students "re-presented perceptual unit item[s]" (p. 342) when developing figural unit items. Further, children developed motor unit items by unitizing motor actions and associated them with isolated motor pattern (1988). Through development of these singular unit items, children re-presented singular activities and patterns through "an utterance of a number word that signifies a perceptual, figural, or motor unit item" (1988, p. 343). In developing and acting on these singular unit items, it should be noted that children may develop figural, verbal, and motor unit items concurrently or in one order versus another (i.e., figural, verbal units and then motor units, motor units, figural, and then verbal units). In re-presenting perceptual singular units with figural units, verbal utterances and motor patterns, children develop abstract singular unit items. Using abstract singular units, children can develop groups to engage in more sophisticated activity (e.g., partition, iterate) with number and develop *abstract composite units* (abstract units composed of more than one unit) that become countable units of units (Fig. 2.3).

One alternative manner that children may use to develop abstract composite units is their engagement with spatial patterns to develop templates or rules for *experiential composite units*. Essentially, Steffe (1994) posits that young children may initially rely on numerical patterns through their engagement with spatial patterns when developing figurative material (*figurative composite units*) and motor activity (*motor composite units*). Young children's activity with material with counting (and possibly subitizing) are foundational for experiential composite unit development. Steffe found that children constructed experiential composite units by attending to the *numerical rules* of a pattern. Through flexible engagement with numerical patterns, children develop experiential units as their development of "the records of a pattern do not take a picture of the pattern, but they constitute a program or recipe whose enactment constitutes a sensory pattern" (Steffe, 1994, p. 18). Steffe distinguishes these patterns as primarily numerical sequences, as subitizing was not considered in the framework of Steffe's research (cf. Glasersfeld, 1982). However, we posit that when considering multiple means in which patterns could be engaged, young children could construct experiential composite units based on subitizing, providing an alternative means to access abstract composite unit development.

For example, children rely on visual patterns when perceptually subitizing an orientation of "three and two." When children are then asked, "how many did you see?" they might need to "make it first." Here, children are primarily relying on the pattern and the figurative composite unit to engage with the numbers. Alternatively, if children represent the "two" and "three" with all fingers on the one hand and

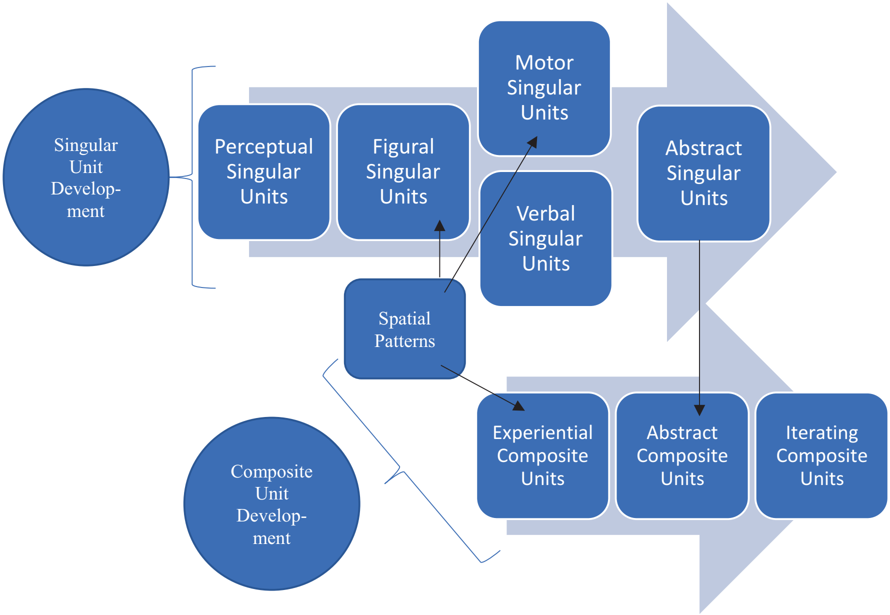

Fig. 2.3 MacDonald and Wilkins (under review) adapted conceptual framework from Steffe and Cobb (1988) and Steffe (1994)

partition an orientation when determining subgroups, then they may be relying on the rules of prior experience with patterns in which to conceptually develop composite units. More specifically, Steffe (1994) describes this transition as one of a "uniting operation" where the numerical pattern is used as one object "to instantiate the records that compose it" (p. 17). Thus, children's subitizing development and experiential unit development may be related.

2.2 Learning Trajectories: Integrating Developmental Psychology and Mathematics Education

Research identifying that subitizing is a distinct and central process has important ramifications for education. As we have seen, developmental psychology also helps us understand the natural *paths* of children's learning—invaluable for developing curriculum (the term's origins are a path for racing) and teaching strategies. Before we examine a course of development for subitizing relevant to educational practice, we briefly describe the theoretical and empirical foundations for our approach to learning and teaching.

We synthesized research and theories in developmental psychology and mathematics education from nativist and constructivist perspectives to form a theoretical framework called *hierarchic interactionalism* (Sarama & Clements, 2009). The term indicates the influence and interaction of global and local (domain specific) cognitive levels and the interactions of innate competencies, internal resources, and experience (e.g., cultural tools and teaching). Mathematical ideas are represented intuitively, then with language, then metacognitively, with the last indicating that the child possesses an understanding of the topic and can access and operate intentionally on those understandings. The theory has 12 tenets; several are particularly pertinent to this chapter (see Sarama & Clements, 2009, for a full discussion).

2.2.1 Selected Tenets of Hierarchic Interactionalism

Developmental Progression Most content knowledge is acquired along developmental progressions of levels of thinking. These play a special role in children's cognition and learning because they are particularly consistent with children's intuitive knowledge and patterns of thinking and learning at various levels of development (at least in a particular culture, but guided in all cultures by innate competencies), with each level characterized by specific mental objects (e.g., concepts) and actions (processes) (e.g., Clements, Wilson, & Sarama, 2004; Steffe & Cobb, 1988). These actions-on-objects are children's main way of operating on, knowing, and learning about, the world, including the world of mathematics.

Cyclic Concretization Development progressions often proceed from sensory-concrete and implicit levels at which perceptual concrete supports are necessary and reasoning is restricted to limited cases (such as small numbers) to more explicit, verbally based (or enhanced) generalizations and abstractions that are tenuous to integrated-concrete understandings relying on internalized mental representations that serve as mental models for operations *and abstractions* that are increasingly sophisticated and powerful. Again, such progressions can cycle within domains and contexts.

Different Developmental Courses Different developmental courses are possible within those constraints, depending on individual, environmental, and social confluences (Clements, Battista, & Sarama, 2001; Confrey & Kazak, 2006). At a group level, however, these variations are not so wide as to vitiate the theoretical or practical usefulness of the tenet of developmental progressions. The following tenet is closely related.

Environment and Culture Environment and culture affect the pace and direction of the developmental courses. Because environment, culture, and education affect developmental progressions, there is no single or "ideal" developmental progression, and thus learning trajectory, for a topic. Universal developmental factors interact with culture and mathematical content, so the number of paths is not unlimited, but, for example, educational innovations may establish new, potentially more advantageous, sequences, serving the goals of equity (Myers, Wilson, Sztajn, & Edgington, 2015). A latter section of this chapter deals explicitly with such differences.

Progressive Hierarchizing Within and across developmental progressions, children gradually make connections between various mathematically relevant concepts and procedures, weaving ever more robust understandings that are hierarchical in that they employ generalizations while maintaining differentiations. These generalizations, and the metacognitive abilities that engender them, eventually connect to form logical-mathematical structures. Children provided with high-quality educational experiences build similar structures across a wide variety of mathematical topics. For example, subitizing can have important interrelations with counting and arithmetic.

Consistency of Developmental Progressions and Instruction Instruction based on learning consistent with natural developmental progressions is more effective, efficient, and generative for the child than learning that does not follow these paths.

Learning Trajectories An implication of the tenets to this point is that a particularly fruitful instructional approach is based on hypothetical learning trajectories (Clements & Sarama, 2004b). Based on the hypothesized, specific, mental constructions (mental actions-on-objects), and patterns of thinking that constitute children's thinking, curriculum developers design instructional tasks that include external objects and actions that mirror the hypothesized mathematical activity of children as closely as possible. These tasks are sequenced, with each corresponding to a level of the developmental progressions, to complete the hypothesized learning trajectory. Specific learning trajectories are the main bridge that connects the "grand theory" of hierarchic interactionalism to particular theories and to educational practice.

Instantiation of Hypothetical Learning Trajectories Hypothetical learning trajectories must be interpreted by teachers and are only realized through the social interaction of teachers and children around instructional tasks (e.g., Wickstrom, 2015). Societally determined values, goals, and cultures are substantive components of any curriculum (Aguirre et al., 2017; Confrey, 1996; Hiebert, 1999; National Research Council, 2002; Tyler, 1949); research cannot ignore or determine these components (cf. Lester Jr. & Wiliam, 2002).

2.2.2 Hierarchic Interactionalism's Learning Trajectories

Learning trajectories, then, have three components: a goal (that is, an aspect of a mathematical domain children should learn), a developmental progression, or learning path through which children move through levels of thinking, and instruction that helps them move along that path. Formally, learning trajectories are descriptions of children's thinking as they learn to achieve specific goals in a mathematical domain, and a related, conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through a developmental progression of levels of thinking (Clements & Sarama, 2004b).

Learning trajectories are useful pedagogical, as well as theoretical, constructs (Clements & Sarama, 2004a; Simon, 1995; Smith, Wiser, Anderson, & Krajcik, 2006). Knowledge of developmental progressions—levels of understanding and skill, each more sophisticated than the last—is essential for high-quality teaching based on understanding both mathematics and children's thinking and learning. Early childhood teachers' knowledge of young children's mathematical development is related to their children's achievement (Fuson, Carroll, & Drueck, 2000; Kühne, Lombard, & Moodley, 2013; Peterson, Carpenter, & Fennema, 1989; Wright, Stanger, Stafford, & Martland, 2006).

2.3 A Developmental Progression for a Subitizing Learning Trajectory

2.3.1 Levels of Thinking

Research helps us describe the developmental progression for subitizing. Explicit naming of numbers begins early (because the task is not timed—displays are not shown and quickly hidden—we call this *recognition of number* rather than subitizing). In laboratory settings, children at about 33 months of age can initially name numbers that differentiate 1 from collections of more than 1 (Wynn, 1992). Between 35 and 37 months, they name 1 and 2, but not larger numbers. A few months later, at 38–40 months, they identify 3 as well. After about 42 months, they can identify all numbers that they can count, 4 and higher, at about the same time. However, research in natural, child-initiated settings shows that the development of these abilities can occur much earlier, with children working on 1 and 2 around their second birthdays or earlier (Mix, Sandhofer, & Baroody, 2005). Further, some children may begin saying "two" rather than "one." These studies suggest that language and social interactions interact with internal factors in development, as well as showing that number knowledge develops in levels, over time (see also Gordon, 2004). However, most studies suggest that children begin recognizing and saying "one," then "one" and "two," then "three" and then "four," whereupon they learn to count and know other numbers (see Gelman & Butterworth, 2005, for an opposing view concerning the role of language; Le Corre, Van de Walle, Brannon, & Carey, 2006).

Most Kindergartners appear to have good competence recognizing 2 and 3, with most recognizing 4 and some recognizing higher numbers (note that different tasks were used, some of which did not limit time, so wide ranges are expected). A recent study of low-income children beginning pre-K, using a short-exposure subitizing task, report 2–14% accuracy for 3, 0–5% for 4, and virtually no competence with 5, 8, or 10 (Sarama & Clements, 2011). Thus, children appear to be most confident with very small numbers, but those from less advantaged environments may not achieve the same skills levels as their more advantaged peers. Some special populations find subitizing particularly difficult. Only a minority (31%) of children with moderate mental handicaps (chronological ages 6–14 years) and a slight majority (59%) of children with mild mental handicaps (ages 6–13) successfully subitize sets of three and four (Baroody, 1986; see also Butterworth, 2010). Some children with learning disabilities could not subitize even at 10 years of age (Koontz & Berch, 1996). Early deficits in spatial pattern recognition may be the source of difficulty (Ashkenazi, Mark-Zigdon, & Henik, 2013). Subitizing in preschool is a better predictor of later mathematics success for children with ASD (autism spectrum disorder) than for typically developing children (Titeca, Roeyers, Josephy, Ceulemans, & Desoete, 2014).

2.3.2 Factors Affecting Difficulty of Subitizing Tasks

Several factors (spatial arrangement, physical size of the dots, and color of dots) affect subitizing ability. The spatial arrangement of sets, the size of the items, and the color of the items influences how difficult they are to subitize. Children usually find rectangular arrangements easiest, followed by linear, circular, and scrambled arrangements (Beckwith & Restle, 1966; Wang, Resnick, & Boozer, 1971). This is true for students from the primary grades to college in most cases. The only change across these ages is rectangular arrangements were much faster for the oldest students, who could multiply.

Certain arrangements are easier for specific numbers. Arrangements yielding a better "fit" for a given number are easier (Brownell, 1928). Further, when items are not arranged in rectangular or canonical arrangements, and the items increase in their relative size, children and adults have more difficulties subitizing these items accurately (Leibovich, Kadhim, & Ansari, 2017). More specifically, children make fewer errors for 10 dots than for eight with the "domino five" arrangement, but fewer errors for eight dots for the "domino four" arrangement. Of course, these are averages; experience with arrangements undoubtedly influences children's performances.

For young children, however, neither of these arrangements is easier for any number of dots. Indeed, children 2–4-years-old show no differences between any arrangements of four or fewer items (Potter & Levy, 1968). For larger numbers, the linear arrangements are easier than rectangular arrangements. It may be that many preschool children do not use decomposing (conceptual subitizing). Whelley (2002) also found that preschool children's subitizing is affected by color of the items shown to them. For instance, when children are shown clustered items of differing colors, Whelley found the colors should align with the clustering of the group of items for effective subitizing to occur (i.e., three items clustered are all red and two items clustered are all black). When children are shown different colored items where the colors do NOT align with the clustering of the groups (i.e., three items clustered have two red and one black item and four items clustered have two red and two black items) then their subitizing accuracy decreased. As preschool children's attentional mechanisms mature, they can learn to conceptually subitize where orientations, color of items, and size of items does not affect their subitizing accuracy in drastic means (Whelley, 2002) though older research indicated that children as old as first grade experienced subitizing limitations of about four or five scrambled arrangements (Dawson, 1953).

The spatial arrangement of sets influences how difficult they are to subitize. Children usually find rectangular arrangements easiest, followed by linear, circular, and scrambled arrangements (Beckwith & Restle, 1966; Wang et al., 1971). If the arrangement does not lend itself to grouping, people of any age have more difficulty with larger sets (Brownell, 1928). They also take more time with larger sets (Beckwith & Restle, 1966).

2.4 Education's First Concern: *Goals* for Subitizing

The ideas and skills of subitizing start developing very early, but they, as every other area of mathematics, are not just "simple, basic skills." Subitizing introduces basic ideas of cardinality—"how many," ideas of "more" and "less," ideas of parts and wholes and their relationships, beginning arithmetic, and, in general, ideas of quantity. Developed well, these are related, forming webs of connected ideas that are the building blocks of mathematics through elementary, middle, and high school and beyond.

Young children may use perceptual subitizing to make units for counting and to build their initial ideas of cardinality (Slusser & Sarnecka, 2011). For example, their first cardinal meanings for number words may be labels for small sets of subitized objects, even if they determined the labels by counting (Fuson, 1992b; Steffe, Thompson, & Richards, 1982).

MacDonald and Wilkins (2017) found that one preschool child, Amy, used conceptual subitizing to develop early forms of units for counting. When given a counting task from Steffe and Cobb's (1988) counting scheme investigation, Amy drew from these same units and represented them with finger patterns, suggesting her ability to reorganize the patterns she may have relied on in earlier subitizing activity. Throughout sessions, Amy engaged in FCS for five and had constructed units "two and three" and "two, two, and one" when conceptually subitizing "five." At the end of the study, Amy was given a missing addend task to solve that required her to use "four" and "three." Essentially, in the task the teacher-researcher made a row of seven counters and covered four items in this row. Next, Amy was asked if there are "four" here, how many are there altogether? Amy showed with her fingers that she could make "three" (using three middle fingers from one hand) and then add "two" (using her pinky and thumb) and "two" more (using two fingers on her other hand) to discover there were "seven" altogether (MacDonald & Wilkins, 2017). This evidence seems to suggest relationships between early forms of number operations and conceptual subitizing activity.

This may be why subitizing predicts overall mathematics competencies of kindergartners (Yun et al., 2011, July). In another study, kindergartners' subitizing, but not the other early number skills, mediated the association between executive functioning and mathematics achievement in primary school (Fuhs, Hornburg, & McNeil, 2016). Executive function may help children quickly and accurately identify number sets as wholes instead of getting distracted by the individuals in the sets, and this focus on wholes may help develop advanced mathematics concepts.

As described earlier, counting and subitizing also interact to build arithmetic competencies. For example, consider how children progress from counting all to more sophisticated counting on strategies in solving arithmetic problems (Fuson et al., 2000; Peterson et al., 1989; Wright et al., 2006). As mentioned in discussing the types of subitizing, once their movement through the counting and subitizing learning trajectories has given them access to the notion that they can count up from a given quantity, they can solve 6 + 3 in a new way. They subitize the first addend (rather than counting it out one by one), and then count three more, using a *subitized rhythmic pattern* as an intuitive keeping track strategy: "Siiiix… *seven, eight, nine!*"

As another example, more advanced ability to quickly group and quantify sets in turn supports their development of number sense and arithmetic abilities. A first grader explains the process for us. Seeing a 3 by 3 pattern of dots, she says "Nine" immediately. Asked how she did it, she replies,

> When I was about four years old, I was in nursery school. All I had to do was count. And so, I just go like 1, 2, 3, 4, 5, 6, 7, 8, 9, and I just knew it by heart and I kept on doing it when I was five too. And then I kept knowing 9, you know. Exactly like this [she pointed to the array of nine dots]. (Ginsburg, 1977, p. 16)

As we discuss the details of teaching and learning of subitizing, let us not lose the whole—the big picture—of children's mathematical future. Let's not lose the wonderment that children so young can think, profoundly, about mathematics.

These foundations are significant beyond the earliest years. Subitizing in grades 3 and 4 significantly predicts of fluency in calculation and also general mathematics achievement a year later (Reigosa-Crespo et al., 2013). Starkey and McCandliss (2014) also found that kindergarten children's subitizing activity related to their "groupitizing" activity (a type of conceptual subitizing) and flexible operations on number when enrolled in third grade. Thus, as children develop more abstract means for number as a flexible set of units of units, they are capable of operating fluently on number more effectively in upper elementary school.

2.5 Instructional Tasks and Teaching Strategies

Although children are sensitive to quantity, interaction with others is essential to learning subitizing; it does not develop "on its own" (Baroody, Li, & Lai, 2008). Children who spontaneously focus on number and subitize number are more advanced in their number skills (Edens & Potter, 2013). This section describes the third part of a learning trajectory: Instructional activities and pedagogical strategies.

2.5.1 Developing Children's ANS System

For developing a sensitivity to quantity, research does suggest that making judgments of the number in sets of all sizes (including number of movements, tones, etc.) will help strengthen children's ANS systems (Libertus, Feigenson, & Halberda, 2013; Wang, Odic, Halberda, & Feigenson, 2016). These are usually not labeled with number words, but rather with vocabulary such as "more" and "fewer" (for dots) or "more" and "less." For the youngest children, intersensory redundancy—for example, you see a ball bouncing more times, it takes longer, you hear more noises—helps focus attention on number (Jordan, Suanda, & Brannon, 2008). Studies show these abilities can be developed, such as through special video games in which children make similar comparisons (Park, Bermudez, Roberts, & Brannon, 2016).

2.5.2 Mathematics Education: Supporting the Developmental Progression for Subitizing

For subitizing, or naming the exact number in sets, parents, teachers, and other caregivers might begin naming very small collections with numbers after children have established names and categories for some physical properties such as shape and color (Sandhofer & Smith, 1999). This section provides suggestions for helping children progress through the developmental progression for subitizing.

Everyday Number Recognition For everyone, but especially teachers of toddlers and 3-year olds, perhaps the easiest but most useful "activity" is simply to establish a habit of using small number words in everyday interactions frequently. They can replace, "Clear the cups off the table so we have room for this," with "We need more area on the table for this, would you please take those three cups off the table?" There is no need to be "artificial" in this kind of talk, just the use of small number words every time it makes sense. Teachers can give parents the same advice.

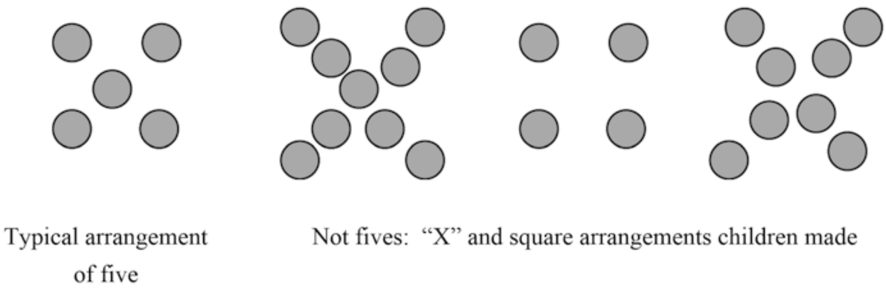

Fig. 2.4 Children had only seen a single pattern for 5—on the left. When asked to make a pattern of 5, some incorrectly produced arrangements like those on the right

Providing these types of repeated experiences naming collections help children build connections between quantity terms (number, how many) and number words, then build word-cardinality connections (• • is "two") and finally build connections among the representations of a given number. Non-examples are important, too, to clarify the boundaries of the number (Baroody et al., 2006). For instance, "Wow! That's not two horses. That's three horses!" For children who are less interested and competent in mathematics, it is especially important for caregivers and teachers to talk to them about number, for example, extending their interest in manipulating objects to include mathematical ideas such as number and shape (Edens & Potter, 2013). Research shows such experiences are helpful, especially for children who begin with lower abilities (Olkun & Özdem, 2015).

Practices to Avoid In contrast to these research-based practices, mis-educative experiences (Dewey, 1938/1997) may lead children to perceive collections as figural arrangements that are not exact. Richardson (2004) reported that for years she thought her children understood perceptual patterns, such as those on dice. However, when she finally asked them to reproduce the patterns, she was amazed that they did not use the same number of counters. For example, some drew an "X" with nine dots and called it "five" (see Fig. 2.4). Thus, without appropriate tasks and close observations, she had not seen that her children did not even accurately imagine patterns, and their patterns were certainly not numerical. Such insights are important in understanding and promoting children's mathematical thinking.

Textbooks and "math books" often present sets that discourage subitizing. Their pictures combine many inhibiting factors, including complex embedding, different units with poor form lack of symmetry, and irregular arrangements (Carper, 1942; Dawson, 1953). For example, they may show five birds, but have different types of birds spread out on a tree, with branches, leaves, flowers, a sun shining overhead—you get the idea. Such complexity hinders conceptual subitizing, increases errors, and encourages simple one-by-one counting.

Due to their curriculum, or perhaps a lack of training in subitizing, teachers may not pay proper attention to subitizing. For example, one study showed that children regressed in subitizing from the beginning to the end of kindergarten (Wright, Stanger, Cowper, & Dyson, 1996). How could that be? The following type of interaction might help explain. A child rolls a die and says "five." Looking on, the teacher says, "Count them!" The child counts them by ones. What has happened? The teacher thought her job was to teach counting. But the child was using subitizing—which is more appropriate and better in this situation. However, the teacher is unintentionally telling the child that her way is not good, that one must always count. Further, always telling children to count may actually hurt their development of counting and number sense. Naming small groups with numbers, *before* counting, helps children understand number words and their cardinal meaning ("how many") without having to shift between ordinal (counting items in order) and cardinal uses of number words inherent in counting (Baroody et al., 2005). These can be used to help infuse early counting with meaning.

Specific Subitizing Activities Many number activities can promote perceptual, and then conceptual subitizing (Sayers, Andrews, & Boistrup, 2016). Perhaps the most direct activity simply challenging children to subitize, an activity called "Quickdraw" (Wheatley, 1996), "Snapshots" (Clements & Sarama, 1998, 2007), and "Draw what you see" (MacDonald & Wilkins, 2016). As an example, children are told that they have to quickly take a "snapshot" of how many they see—their minds have to take a "fast picture." They are shown a collection of counters for 2 s only, then asked to construct, draw, or say the number. Consistent with research, arrangements may be straight lines of objects, then rectangular shapes, and then dice arrangements, all with small numbers. Separating these typical dice arrangements with a large space promotes children's attention to subgroups for Perceptual Subgroup Subitizing (MacDonald & Wilkins, 2016). As children learn, they use different arrangements and larger numbers. See the Box, Variations of the "Snapshots" Activity, for many engaging modifications.

Variations of the "Snapshots" Activity

- Have children construct a quick image arrangement with manipulatives (and watch for any misconceptions such as shown in Fig. 2.4).
- Play Snapshots on educational technology platforms (e.g., www.learning-trajectories.org/activity/subitize-planets-perceptual-subitizer-4).
- Play finger-placement game on computer. In Fingu, pieces of fruit are shown briefly and the child has to place that many fingers on the screen with one or two hands (Barendregt, LindstrÖm, Rietz-Leppänen, Holgersson, & Ottosson, 2012).
- Play a matching game. Show several cards, all but one of which have the same number. Ask children which does not belong (this also teaches early classification).
- Play concentration-type matching games (we call them "memory" games) with cards that have different arrangements for each number and a rule that you can only "peek" for 2 s.
- Give each child cards with 0–10 dots in different arrangements. Have children spread the cards in front of them. Then announce a number. Children

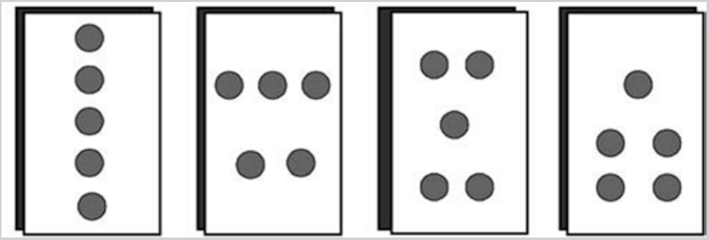

Fig. 2.5 Arrangements for conceptual subitizing that may suggest 5 as 5, 3 + 2, 2 + 1 + 2, or 1 + 4 (or other interpretations)

find the matching card as fast as possible and hold it up. Have them use different sets of cards, with different arrangements, on different days. Later, hold up a written numeral as their cue. Adapt other card games for use with these card sets.

- Emphasize conceptual subitizing as soon as possible. Use different arrangements that suggest different partitions of a number (see Fig. 2.5).
- Place various arrangements of dots on a large sheet of poster board. With children gathered around you, point to one of the groups as children say its number as fast as possible. Rotate the poster board on different sessions.
- Challenge children to say the number that is one (later, two) more than the number on the quick image. They might also respond by showing a numeral card or writing the numeral. Or, they can find the arrangement that matches the numeral you show.
- Remember that patterns can also be temporal and kinesthetic, including rhythmic and spatial-auditory patterns. A motivating subitizing and numeral writing activity involves auditory rhythms. Scatter children around the room on the floor with individual chalkboards. Walk around the room, then stop and make a number of sounds, such as ringing three times. Children should write the numeral 3 (or hold up three fingers) on their chalkboards and hold it up. These can also develop conceptual subitizing.

Across many types of activities, from class discussions to textbooks, children can be shown displays of numbers that encourage conceptual subitizing. Guidelines to make groups for this purpose include the following: (a) avoid embedding groups in pictorial context; (b) use simple forms such as homogeneous groups of circles or squares (rather than pictures of animals or mixtures or any shapes) for the units; (c) emphasize regular arrangements (most including symmetry, with linear arrangements for preschoolers and rectangular arrangements for older children being easiest); and (d) provide good figure-ground contrast.

To develop strong conceptual subitizing, children should experience many real-life situations such as finger patterns, arrangements on dice and dominoes, egg cartons (for "double-structures"), and arrays that separate two subgroups. To extend conceptual subitizing, teachers might discuss and especially cooperatively build arrangements to "make it easy to see how many." Such thoughtful, interactive, constructive experiences are effective ways of building spatial sense and connect it to number sense (Nes, 2009). For example, children might draw flowers with a given number of petals or draw or build pictures with manipulatives of houses with a certain number of windows so that they and others can subitize the number.

Such conceptual subitizing provides a direct phenomenological experience with additive situations, as children conceptualize two parts and the whole. Having both parts and whole in working memory builds a foundation for "knowing addition facts." Indeed, this is arguably better than emphasizing only counting-based solutions. Consider children using the initial "counting all" strategy for 3 + 2: counting out 3 objects, then counting out 2 objects, then starting over at "one" and counting all 5. The children answer correctly, but it is likely that *only the 5 is retained in working memory.* In comparison, the two addends may not be, and so it is unlikely that a connection is made between the addends and the sum. In subitizing, the addends and the same are retained in working memory in the same time period.

Subitizing is not only a separate complement to counting-based approaches to arithmetic but a valuable process to integrate with counting. That is, children can use subitizing in concert with counting to advance to more sophisticated addition and subtraction. As one example, children who are encouraged to subitize 3 in the previous example may move from counting all to early counting on, recognizing the set of 3, and counting only, "4, 5!". As another example, a child may be unable to count on keeping track, as in solving 4 + 5 by counting "4…5 is 1 more, 6 is 2 more…9 is 5 more." However, counting on two using rhythmic subitizing—for 5 + 2, saying "five…*six, seven!*" matching the counting to a "tah-dum" beat of two—gives them a way to figure out how counting on work. Later they can learn to count on with larger numbers, by developing their conceptual subitizing or by learning different ways of "keeping track." Eventually, children come to recognize number patterns as both a whole (as a unit itself) and a composite of parts (individual units).

2.6 Final Words

Across development, numerical knowledge initially develops qualitatively and becomes increasingly mathematical. In subitizing, children's ability to "see small collections" grows from pre-attentive but quantitative, to attentive perceptual subitizing, to imagery-based subitizing, and to conceptual subitizing (Clements, 1999; Steffe, 1992). Perceptual patterns are those the child can, and must, immediately see or hear, such as domino patterns, finger patterns, or auditory patterns (e.g., three beats). A significant advance is a child's focus on the exact number in these patterns, attending to the cardinality. Finally, children develop conceptual patterns, which they can operate on, as and when they can mentally decompose a five pattern into 2 and 3 and then put them back together to make five again. These types of patterns may "look the same" on the surface, but are qualitatively different. All can support mathematical growth and thinking, but conceptual patterns are the most powerful.

Subitizing small numbers appears to precede and support the development of counting ability (Le Corre et al., 2006). Thus, it appears to form a foundation for all learning of numbers. Indeed, a language-independent ability to judge numerical values nonverbally appears to be an important evolutionary precursor to adult symbolic numerical abilities. Children can use subitizing to discover critical properties of number, such as conservation and compensation. They can build on subitizing to develop capabilities such as unitizing as well as arithmetic capabilities. Thus, subitizing is a critical competence in children's number development.

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