Research in Mathematics Education
Series Editors: Jinfa Cai · James A. Middleton

Anderson Norton
Martha W. Alibali *Editors*

Constructing Number

Merging Perspectives from Psychology and Mathematics Education

Springer

Research in Mathematics Education

Series Editor:

Jinfa Cai
Newark, DE, USA
James A. Middleton
Tempe, AZ, USA

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Editors

Constructing Number

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Foreword

As the editors of this series, we both consider ourselves mathematics education researchers, but we have had significant training in cognitive psychology. Jinfa Cai majored in Cognitive Studies in Mathematics Education and served as a research assistant at the University of Pittsburgh's Learning Research and Development Center for 5 years. Jim Middleton is an educational psychologist whose doctoral work at the University of Wisconsin was focused at the interface between applied cognitive science and mathematics learning. For the past 25 years, our research has focused on cognitive studies in mathematics education: On learning mathematics and on designing curriculum, pedagogical strategies, and technology that supports learning mathematics. In our experiences over the years, we find it interesting (and a bit disappointing) that mathematics educators and cognitive psychologists—two groups of researchers interested in many of the same issues related to mathematics learning and teaching—have collaborated and interacted very little on the grand scale. It is true that their research on mathematics learning and teaching is typically conducted from different angles—each representing a different perspective on a common problem—but our experiences have shown us that these perspectives are complementary, not conflicting.

Because we both benefitted greatly from our interdisciplinary training, we have long worked to facilitate common dialogue in our respective research circles, at the National Science Foundation, and in our roles as leaders in mathematics education research, where such dialogue has been deep and genuine, mathematics education research has advanced both theoretically and pragmatically in ways that reflect the strengths of the two perspectives. Moreover, new theory and new approaches to teaching and learning have resulted from working in the interstices of our communities. This book is a product of such an effort. It critically examines research on the learning of number that combines cognitive, developmental psychology and mathematics education approaches. This is a sister book to a previous volume in this series on spatial visualization in mathematics (edited by Mix and Battista, 2019). Both volumes use the device of scholars reporting their own work, followed by critical commentary written by colleagues with complementary expertise.

In doing so, the editors of this book are developing mathematical epistemology, asking us, "what does it mean to learn, know, and understand mathematics?" The argument begins with mathematics itself—what it is as a field of knowledge and practice—and flows from neuropsychological foundations, through perception, through construction of number to the broadening of understanding that addresses the fields of rational and negative numbers. Thus, the book grounds learners' construction and conceptual development in fundamental understanding of learning processes, yet also reflects the important content which has puzzled students and researchers alike for centuries.

Finally, as series editors, we wish to thank the editors for this volume on numbers (Norton and Alibali) and the sister volume on spatial visualization (Mix and Battista), as well as authors for the quality of the chapters and commentaries they have provided.

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Preface

This book synergizes research across two disciplines—mathematics education and psychology—to address how children construct number. The opening chapter frames the problem in terms of children's activity, including mental and physical actions. Subsequent chapters are organized into sections that address specific domains of number: natural numbers, fractions, and integers. Chapters within each section address ways that children build upon biologically based foundational abilities (e.g., subitizing, the approximate number system) and prior constructs (e.g., counting sequences) to construct number. The chapters address a range of change mechanisms (e.g., reflective abstraction, analogy), a range of social contexts (e.g., informal interactions, formal educational settings), and a range of tools (e.g., curricular materials, technological tools). The book relies on co-authored chapters and commentaries at the end of each section to create dialogue among scholars from different disciplines. The final chapter brings this collective work together around the theme of children's activity and also considers additional themes that arise within the chapters.

We hope that this book will foster additional dialogue between psychologists and mathematics educators. As the chapters in the book demonstrate, mathematics educators can benefit from a better understanding of psychological constructs that they might leverage to support students' mathematical development. Conversely, psychologists can benefit from a better understanding of ways that students' activity supports that development.

Blacksburg, VA, USA — Anderson Norton
Madison, WI, USA — Martha W. Alibali

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Sybilla Beckmann is Josiah Meigs distinguished professor of mathematics at the University of Georgia, where she studies future teachers' mathematical cognition around ideas related to multiplication. She has a PhD in mathematics from the University of Pennsylvania, taught at Yale University, and did research in arithmetic geometry in the first part of her career. Beckmann is the author of *Mathematics for Elementary Teachers*, which is now in a fifth edition. She has served on a number of national and international committees and won a number of awards for her contributions to mathematics education.

Silke M. Bieck is a PhD student at the LEAD Graduate School & Research Network at the University of Tübingen. In her PhD project, she studies the development of children's and adolescents' fraction learning and understanding. Additionally, she investigates the neural underpinnings of fraction processing and fraction arithmetic, using functional magnetic resonance imaging (fMRI). Furthermore, she is interested in using neurostimulation methods (i.e., transcranial electric stimulation techniques) to facilitate fraction learning. Her aim is to identify and foster key factors of fraction understanding and processing on a behavioral as well as neural level to improve math education in schools.

Kristen P. Blair is a senior research scholar at Stanford University's Graduate School of Education. Her work focuses on improving STEM instruction and assessment in both formal and informal settings. Dr. Blair has collaborated with cross-institutional partners to develop new learning technologies and instructional materials. She also examines fundamental learning questions, such as how students learn from both positive and negative feedback and how the features of examples provided to students influence their effectiveness for learning. Dr. Blair holds a PhD in learning sciences and technology design and an undergraduate degree in mathematical and computational science, both from Stanford University.

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Jenny Yun-Chen Chan is a PhD candidate in developmental psychology at the University of Minnesota, where she studies mathematical learning and development. Her current research focuses on how language and executive function skills influence mathematical thinking and how contextual factors influence attention to mathematics. She contributed to the Development and Research in Early Math Education (DREME) Network projects and collaborated with Dr. von Baeyer on a review of cognitive developmental factors that influence children's ability to report pain intensity. She will join the Educational Psychology and Mathematics Learning Lab as a postdoctoral fellow at Worcester Polytechnic Institute in October 2018.

Douglas H. Clements is the Kennedy endowed chair in early childhood learning and distinguished university professor at the University of Denver, Colorado (USA). Dr. Clements has published over 135 refereed research studies, 22 books, 86 chapters, and 300 additional publications on the learning and teaching of early mathematics; creating, using, and evaluating research-based curricula; and taking interventions to scale. He served on the US President's National Mathematics Advisory Panel, the *Common Core State Standards* committee, and the National Research Council's Committee on early mathematics and is co-author on each of their reports.

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Chapter 1
Mathematics in Action

Anderson Norton and Martha W. Alibali

Abstract This opening chapter provides an introduction to the book. It also introduces a theme that integrates many of the contributions from the remaining chapters: we adopt Kant's perspective for merging rationalist and empiricist philosophies on the construction of knowledge. In particular, we focus attention on ways that biologically based abilities and experience in the world (coordinations of sensorimotor activity) each contribute to the construction of number. Additional themes arise within the content chapters and the commentaries on them.

Keywords Embodied cognition · Epistemology · Numerical development · Radical constructivism · Sensorimotor activity

What is mathematics, and how do humans come to know it? Once the domain of philosophy, the epistemology of mathematics now falls squarely within the purview of psychology and mathematics education. In this book, we address the cognitive roots of number, integrating theory and findings from both fields, along with strands from neuroscience. In focusing on number, we take on a relatively simple problem within the epistemology of mathematics. However, we find great complexity in children's construction of number, and tracing the complex roots of number illustrates the cognitive construction of mathematics in general.

The book contains three parts: one for whole (natural) numbers, one for fractions, and one for integers. Within each part, we draw upon the authors' collective expertise in mathematics education, cognitive and developmental psychology, and neuroscience. Their work elucidates the biological and experiential bases of number

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and its development. With whole numbers, biological bases include subitizing (discerning small collections of perceptually distinct items, the subject of Chap. 2). Critical experiences include the coordination of various mental actions, like unitizing (forming units within or from collections of items), which give rise to number sequences (the subject of Chap. 3).

With fractions, number becomes more complicated because each fraction is formally represented by a pair of whole numbers. As demonstrated by the chapters in the fractions part, some researchers have found evidence of whole number knowledge interfering with fractions knowledge (Chap. 7), while others see fractions knowledge as a reorganization of whole number knowledge (Chap. 8), and still others identify possible psychological roots for fractions independent of whole numbers (Chap. 10). Integers introduce additional complications with the inclusion of negative numbers (Chaps. 12–15).

In this opening chapter, we frame the book by considering the historical roots of mathematical epistemology. We also consider ways that recent contributions from psychology and mathematics education have begun to solve philosophical problems. In particular, we focus on the role of sensorimotor activity in constructing mathematical objects, such as number. We then preview how each chapter contributes to an action-based epistemology of mathematics in the context of constructing number.

1.1 The Nature of Mathematics

Mathematics constitutes a unique form of knowledge. "Mathematics, rightly viewed, possesses not only truth, but supreme beauty" (Bertrand Russell, as cited in Wigner, 1960). Mathematicians throughout history have praised the divinity of mathematics. Galileo went so far as to declare mathematics the language in which the universe is written (as cited in Drake, 1957). How else can we explain the unreasonable effectiveness of mathematics in the world, from counting sheep to predicting eclipses?

> So basic are numbers that a famous mathematician once said, "God made the integers, man did the rest" (Kronecker, 1634). The integers seem to us to be so fundamental that we expect to find them wherever we find intelligent life in the universe. I have tried, with little success, to get some of my friends to understand my amazement that the abstraction of integers for counting is both possible and useful. Is it not remarkable that 6 sheep plus 7 sheep make 13 sheep; that 6 stones plus 7 stones make 13 stones? Is it not a miracle that the universe is so constructed that such a simple abstraction as a number is possible? To me this is one of the strongest examples of the unreasonable effectiveness of mathematics. Indeed, I find it both strange and unexplainable. (Hamming, 1980, p. 84)

For centuries, the apparent truth and universal applicability of mathematics has presented a dilemma for Western philosophers. If mathematics is the language of God, or is otherwise written into fabric of the universe, how can we, as humans, ever acquire it? Rationalists (e.g., Descartes, 1628) respond that our intuitions, or other innate cognitive abilities, provide a basis for the logical deduction of mathematical truths. Empiricists (e.g., Locke, 1690) argue that our sensorimotor experiences provide that basis. Although remnants of these competing philosophies live on today in frameworks applied to study the construction of number, when we look across research from psychology and mathematics education, we can begin to settle much of the philosophical debate.

Kant (1934) was the first to break the impasse between rationalism and empiricism (Palmer, 2008). His *Critique of Pure Reason* admitted both innate cognitive structures and the role of experience in constructing new structures. In Kant's philosophy, humans assimilate experience into existing structures, beginning with innate concepts, such as time, space, causality, and quantity. Assimilation of experience into these innate structures explains the inevitability of mathematics in the world. Why should Galileo, Kronecker, or Hamming be surprised by the ubiquity of mathematical structure in the universe they experience when they are the ones doing the structuring?

Building on Kant's philosophy, Piaget was able to psychologically deconstruct time (Piaget, 1969), space (Piaget & Inhelder, 1967), causality (Piaget, 1974), and number (Piaget & Szeminska, 1952) into more primitive and general intuitions, such as internal regulations and groupings. As such, Piaget moved the epistemological investigation from the realm of philosophy to that of psychology. Since then, psychologists have identified particular psychological constructs that might undergird the construction of space and number. Many of these constructs are subjects of chapters in this book: subitizing (Chap. 2), the approximate number system (Chap. 3), and the ratio processing system (Chap. 10).

At the same time, mathematics education researchers have identified particular kinds of experiences that support the construction of new mathematical structures. They have emphasized the role of social interaction and the use of manipulatives in learning new concepts. For example, Simon (Chap. 9) illustrates a student's guided reinvention of fraction multiplication through interaction with a teacher mediated by tasks in a virtual manipulative environment. Note that our use of the term "guided reinvention" stands in recognition of the fact that mathematicians had invented fraction multiplication millennia before, but that the student's mathematics was nevertheless salient and foregrounded as a personal invention—not simply a conceptualization of mathematics that was already "out there."

While psychologists have leaned toward a rationalist perspective—relying on biologically based systems to explain mathematical development—mathematics educators have tended to adopt an empiricist perspective, focusing on experiences that lead to learning, especially in the classroom. Taking our cue from Kant's merged philosophy, we hope that mathematics educators will find ways to leverage the biological bases of number and that psychologists will investigate students' mathematical constructions as mathematics. For instance, in Chap. 2, Clements, Sarama, and MacDonald seek to accomplish the former by investigating ways that educators can take advantage of subitizing to support students' constructions of natural numbers.

1.2 Mathematical Actions

Sensorimotor experience plays a central role in several prominent theories in mathematics education research, including embodied cognition and radical constructivism. Likewise, psychologists generally recognize the control of sensorimotor activity as a primary purpose of cognition. Some have gone so far as to argue that "the brain evolved for the purpose of controlling action" (Koziol, Budding, & Chidekel, 2012, p. 505). We view sensorimotor activity as a meeting point for mathematics educators and psychologists, and as the psychological basis for mathematical objects, including number.

When considering neurological evidence of a connection between sensorimotor experience and mathematical development, particularly profound examples pertain to the hand. In a study of 5-year-old and fourth-grade children, Crollen and Noël (2015) found that unrelated hand movements interfered more than foot movements with counting and calculation. Several other studies demonstrate strong relationships between finger gnosis (spatial recognition of one's fingers) and mathematical development (e.g., Noël, 2005). Some studies even suggest a causal relationship between finger use and mathematical development (e.g., Kaufmann, 2008; Soylu, Lester, & Newman, 2018). Moreover, neuroimaging studies increasingly implicate a dual role of the intraparietal sulcus in mathematical development and hand-eye coordination—especially tool use (Penner-Wilger & Andrson, 2013).

Whether fingers, pencils, or manipulatives, tools are critical to mathematics education. From a Piagetian perspective, the actions performed with tools form the basis for constructing new mathematical objects because mathematical objects are coordinations of actions (Piaget, 1970). Similarly, proponents of embodied cognition argue that many concepts are embodied in sensorimotor experience, either via the body alone or via the body in interaction with external tools (Gallese & Lakoff, 2005). Research in embodied cognition has begun to make the link between sensorimotor action and mathematical concepts explicit.

One source of evidence for the embodied nature of mathematical thinking is the gestures that people produce when they reason and speak about mathematical ideas (see Alibali & Nathan, 2012). Research suggests that children often express mathematical ideas in gestures before they can express them in words (e.g., Church & Goldin-Meadow, 1986; Perry, Church, & Goldin-Meadow, 1988). Gestures are thought to manifest the simulated actions and perceptual states that are involved in reasoning and speaking (Hostetter & Alibali, 2008). From this perspective, the data on gesturing and mathematical reasoning imply that perception and action may be key sources of mathematical ideas—a view that aligns well with the Piagetian perspective on knowledge construction.

If the brain has evolved for controlling action, we should expect to find various biological systems, shared across species, that support the coordination of sensorimotor activity. Indeed, we do (Dominici et al., 2011; Flash & Hochner, 2005). If mathematics is a human construction, we should expect reflections upon—and abstractions from—coordinated sensorimotor activity to yield new mathematical concepts. Indeed, they do (Simon, Tzur, Heinz, & Kinzel, 2004; von Glasersfeld, 1987). Thus, sensorimotor activity might offer a productive meeting point between rationalism and empiricism, between biologically based systems and mathematical constructions, and between research from psychological and mathematics educational perspectives. The chapters in this volume begin to work at this meeting point by answering the following questions: What role do biologically based abilities play in assimilating mathematical experience, and what kinds of sensorimotor activity (and coordinations thereof) support mathematical development?

1.3 A Preview of the Construction of Number

As adults, natural numbers seem so intuitive and universal that we struggle to understand children's constructions of them. We might hold up five fingers to a child and say, "See, that's five," as if numbers were out there in the world to be taken in; as if anyone could see them.[1] The unreasonable effectiveness of mathematics buttresses such Platonist views of mathematics: that mathematics is "out there." The chapters in this book help explain the miracle of mathematics in terms of what is "in there"—in the minds of children—beginning with the construction of number. These chapters illustrate ways that the construction of number relies on both biologically based systems and sensorimotor experience and also illustrate how the construction of number structures the worlds we experience. Here, we introduce the chapters and frame them in terms of sensorimotor activity—the basis for mathematical construction.

As we have noted, Chap. 2 investigates ways that educators might leverage subitizing to promote the construction of number. Clements, Sarama, and MacDonald define subitizing as the "direct perceptual apprehension and identification of the numerosity of a small group of items." They describe this potentially innate ability as a sensitivity to number, but subitizing itself does not produce number. Rather, in its earliest form, it involves a network of pre-attentive, sensorimotor activity. The authors describe a prolonged process through which children progress toward number, as they begin to apply mental actions to perceptual material. Whereas subitizing individuates items in the perceptual field, the mental action of unitizing, applied to these individual items, overlooks their perceptual differences so that children can count them as identical units: "Some believe that recognition of patterns of movement… is the underlying non-numerical process that is then linked to specific numerosities… [number] occurs when the child abstracts the mental actions from the sensorimotor contexts and is capable of reflecting on these actions." As such, we find sensorimotor activity at the intersection of biologically based abilities, such as subitizing, and the construction of mathematical objects, such as whole numbers.

In Chap. 3, Ulrich and Norton focus on another meeting point in early number knowledge: between the approximate number system (ANS) and the onset of

[1] Thanks to Martin Simon for sharing this example.

counting. As a pre-linguistic construct for comparing collections of items, ANS provides for an early-developing (or potentially innate) sense of quantity, called gross quantity, that does not involve counting. However, gross quantity does not simply map onto the number names that children learn. The construction of number follows a progression of development abstracted from children's activity, including unitizing. Unitizing is critical to constructing numbers as exact measures because it provides the unit by which to measure. Ulrich and Norton argue that over-attribution of number to children (and animals) often causes psychologists to overlook such development. In particular, attributions often conflate number with magnitude or gross quantity. Nevertheless, mathematics educators can leverage psychological constructs like ANS to support that development.

In Chap. 4, McMullen, Chan, Mazzocco, and Hannula-Sormunen examine research on children's tendencies to project their mathematical knowledge on everyday experience. As they have argued, "spontaneous focusing on numerosity (SFON) and quantitative relations (SFOR) have been implicated as key components of mathematical development." Specifically, these authors report reciprocal relationships between measures of children's SFON and their knowledge of whole numbers, as well as between measures of SFOR and rational number knowledge. Path analyses indicate that SFON and SFOR predict later mathematical development, and they indicate even stronger predictive relationships in the other direction. This latter finding reinforces our claim that numbers do not exist in the world to be noticed or taken in but, rather, number is a construction that we can use to organize the worlds we experience. In further alignment with the framework, the authors suggest that SFON and SFOR can be strengthened by engaging students in embodied (sensorimotor) activities.

Mix, Smith, and Crespo conceptualize the learning of place value as a problem of relational learning. In Chap. 5, they describe two well-studied classes of psychological learning mechanisms that could support such relational learning: statistical learning and structure mapping. They then use this framework as a lens for analyzing specific instructional techniques and curricular materials that are widely used for teaching place value. Structure mapping involves identifying elements and relations and placing them in alignment, while statistical learning involves attending to statistical regularities in the material and using those regularities to focus attention for subsequent learning. Both processes are thought to occur automatically, without requiring explicit intervention or feedback; however, in both processes, learners' active role in engaging with the to-be-mapped relations is critical. Teachers and curriculum designers can create opportunities for students to apply these learning mechanisms by designing examples and activities that ensure sufficient opportunities to encounter basic structures, by using language and curricular materials that make critical elements of to-be-learned material particularly salient, and by using instructional practices that involve comparing instances and aligning elements and relations. Although statistical learning and structure mapping differ in the nature of learners' activity, both occur automatically when learners actively engage with relevant material.

In Chap. 7, Obersteiner, Dresner, Bieck, and Moeller focus on the challenges students face in making the transition from reasoning about natural numbers to reasoning about fractions. One persistent challenge is the tendency to overextend reasoning processes for natural numbers to fractions—termed the “natural number bias.” When fraction problems are congruent with natural number reasoning, these reasoning processes yield efficient and accurate performance; however, when fraction problems are incongruent with natural number reasoning, they yield slower responses and errors. From this perspective, the natural number bias arises from the application of action patterns appropriate for natural numbers to the components of symbolic fractions. For example, in adding fractions, learners sometimes err by adding numerators to yield a numerator and adding denominators to yield a denominator (i.e., $a/b + c/d = (a + c)/(b + d)$)—an overextension of the action pattern for natural number addition. Obersteiner and colleagues argue that learners may need to inhibit automatically activated natural number knowledge to reason appropriately about fractions. They describe instructional practices that prompt students to “stop and think” and to explicitly consider ways in which fraction concepts and procedures do and do not align with concepts and procedures for natural numbers.

In response to the natural number bias (or interference) hypothesis, in Chap. 8, Tzur presents a reorganization hypothesis. This reorganization hypothesis posits that children construct fractions as numbers by reorganizing the mental actions that undergird their whole number knowledge (actions like unitizing, mentioned previously). Tzur explains this reorganization as a process of reflecting on activity-effect relationships, and he describes how teachers can support this process. He also presents findings from an fMRI study that indicate neural correlates for solving whole number and fraction comparison tasks. Taken together, the findings highlight the importance of activity by demonstrating how coordinated actions might engender new mathematical structures.

In Chap. 9, Simon presents a similar process of reorganization, but one based on both cognitive and sociocultural perspectives on activity. Simon introduces *learning through activity* as a framework for intentionally supporting students’ constructions of mathematics. This framework integrates Piaget’s theory of reflective abstraction with Russian activity theory. The chapter describes how teachers can support students’ construction of a concept of multiplication that generalizes across whole numbers and fractions. Whereas concept formation is framed cognitively, as reorganization of the student’s activity, the support is framed socio-culturally, as the teacher guides the student’s activity through a sequence of tasks.

Adding to subitizing (the subject of Chap. 2) and the ANS (discussed in Chap. 3), Matthews and Ziols (Chap. 10) introduce another biologically based system that educators might leverage in supporting students’ construction of number. The ratio processing system (RPS) is analogous to the ANS, but applies to relative comparisons of continuous quantities, as well as discrete collections. Matthews and Ziols focus on the perceptual foundations of children’s developing concepts of fractions. As such, they argue that rational numbers are no less “natural” than natural numbers. However, just as exact measurements with natural numbers rely on the mental action of unitizing, precisely enumerating ratios from the RPS may require mental actions such as partitioning.

Bofferding (Chap. 12) characterizes the development of integer concepts as a process of conceptual change. From this perspective, children start with an initial mental model of whole numbers, and they apply this model to negative numbers, neglecting information that does not fit the model. Given experiences with negative numbers, children attempt to integrate new information, and this process yields a hybrid or synthetic model that contains both elements of their initial, whole-number-based model and elements of a more formal model. This synthetic model is unstable and "messy," so it is subject to continued reorganization and integration. With additional experience, children continue to reorganize their mental models, eventually reaching a formal model that reflects culturally accepted ways of conceiving and using negative numbers. In this process, experiences with negative numbers—in particular, activities that provoke reorganization—are critical, as such activities provide "raw materials," highlight inconsistencies, and trigger efforts to integrate and reorganize. Bofferding presents evidence that this process of conceptual change for integers takes places over a set of interrelated concepts, including numerical order, numerical values, addition and subtraction operations, and the interpretation of the minus sign. From this view, activities and experiences with a range of concepts are relevant to understanding the construction of the integers.

In Chap. 13, Enzinger argues that teachers need to support students' understanding of integers as directed numbers, but that it is futile to seek a perfect instructional model for doing so. Rather, she argues that educators need to provide students with opportunities to construct concepts of relativity (i.e., numbers as distances from an arbitrary referent, 0) and translation (i.e., moving from one number to another) from which students can build their own models of integers as directed numbers. Such constructions and models require reflection on students' own activity, including their embodied movement.

Varma, Blair, and Schwartz (Chap. 14) argue that one critical foundation for understanding integers is the additive inverse law, which holds that, for any integer x, there is an integer $-x$ such that $x + (-x) = 0$. They further propose that basic perceptual-motor mechanisms involved in processing visual symmetry can be recruited for processing numerical symmetry (i.e., the symmetry of x and $-x$). In their view, by integrating basic knowledge of magnitude and symmetry, the mental number line is transformed into a number line that is reflected around 0. They argue further that this integration can be accelerated via opportunities to practice applying the additive inverse law. Thus, in their view, integer understanding has a perceptual foundation in visual symmetry processing, and it can be supported by opportunities for appropriate action, specifically, actions that support applying symmetrical reasoning to magnitude representations. This perspective makes predictions about performance on a range of behavioral tasks involving integers, and it suggests that instruction that focuses on symmetry should be particularly effective at promoting integer understanding.

In addition to the chapters described so far, we invited three commentaries, one for each part. Collectively, the three commentaries complement the theme of constructing number from biologically based systems through the coordination of sensorimotor activity, while also introducing new themes. Bert de Smedt—associate professor of educational neuroscience at the University of Leuven (Belgium)—wrote the commentary for the part on natural numbers (Chap. 6). His commentary lends weight from emerging neuroimaging studies to inform our knowledge of how students construct number. Sybilla Beckman—professor of mathematics at the University of Georgia—wrote the commentary on fractions (Chap. 11). She applies her expertise in elementary mathematics teacher education to compare and contrast the four fractions chapters while considering ways that teachers might support students' constructions of fractions as measures. Guershon Harel is professor of mathematics at the University of California, San Diego. His commentary on integers (Chap. 15) elaborates on the role of formal structures in mathematical development.

1.4 Closing Remarks in Opening the Book

Like integers and fractions, natural numbers, too, are abstract. Although biologically based systems, such as subitizing and the ANS, may give children a head start, numbers do not exist in the world until we learn to act in the world. Thus, the construction of number relies, at least in part, on our own actions, such as unitizing, and their coordination. The chapters in Part I provide some indication for how whole numbers arise and how we use them to structure the worlds we experience. The chapters in Parts II and III demonstrate additional complexities in constructing fractions and integers, respectively.

We have framed the book within a Kantian perspective on ways that children and their teachers might leverage biological foundations to construct mathematical objects through particular kinds of sensorimotor and mental experiences. This perspective stands in stark contrast with Platonism, which takes mathematical objects, like number, for granted. It also introduces possibilities for collaborative research in psychology and mathematics education. Some of these possibilities are evident within the content chapters. In the concluding chapter, we highlight additional themes that arise from those chapters and the three commentaries on them.

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Part I
Natural Numbers and Operations on Natural Numbers

Chapter 2
Subitizing: The Neglected Quantifier

Douglas H. Clements, Julie Sarama, and Beth L. MacDonald

Abstract We define and describe how subitizing activity develops and relates to early quantifiers in mathematics. Subitizing is the direct perceptual apprehension and identification of the numerosity of a small group of items. Although subitizing is too often a neglected quantifier in educational practice, it has been extensively studied as a critical cognitive process. We believe that subitizing also helps explain early cognitive processes that relate to early number development and thus deserves more instructional attention. We also contend that integrating developmental/cognitive psychology and mathematics education research affords opportunities to develop learning trajectories for subitizing. A complete learning trajectory includes three components: *goal*, *developmental progression,* or learning path through which children move through levels of thinking, and *instruction*. Such a learning trajectory thus helps establish goals for educational purposes and frames instructional tasks and/or teaching practices. Through this chapter, it is our hope that early childhood educators and researchers begin to understand how to develop critical educational tools for early childhood mathematics instruction. Through this instruction, we believe that children will be able to use subitizing to discover critical properties of number and build on subitizing to develop capabilities such as unitizing, cardinality, and arithmetic capabilities.

Keywords Arithmetic · Early childhood education · Kindergarten · Learning trajectories · Mathematics education · Number · Preschool · Subitizing

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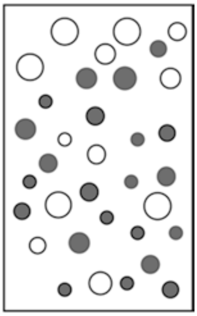

Fig. 2.1 A display shown to children that controls for area but represents a 1:2 ratio of unshaded to shaded circular regions

Children 6 months of age and younger appear to be sensitive to number. For example, they habituate to 1 versus 2 or 3 and 2 versus 3 objects (Antell & Keating, 1983; Starkey, Spelke, & Gelman, 1990). That is, they eventually "get used to" repeated sets of 3, even as color, size, and arrangements change, and become more attentive only when a set with a different number, such as 2, is shown. This indicates that infants are sensitive to the quantities in a small set of items before they are taught number words, counting, or finger patterns.

Children also are sensitive to displays with larger numbers of items. For example, they can habituate to *ratios* of 1:2 (Mazzocco, Feigenson, & Halberda, 2011, see Fig. 2.1 and Matthews, this volume). They also have a sense of the results when displays show a combination of large numbers of dots. Still, teachers may say that some much older, elementary school children cannot immediately name the number shown on dice. So, what is this ability to name exact numbers quickly? Is it a special way of counting or a separate way of acting on objects? Should we teach it, or is it simply innate? Does this ability develop as children learn more sophisticated understandings for number? How does it relate to other activities with number or quantity? As we shall see, although subitizing is too often a neglected quantifier in educational practice, it has been extensively studied as a critical cognitive process.

2.1 The Search for the Earliest Number Competencies

2.1.1 Subitizing: A Long History

Subitizing is "instantly seeing how many." From a Latin word meaning *suddenly*, subitizing is the direct perceptual apprehension and identification of the numerosity of a small group of items. In the first half of the twentieth century, researchers believed counting did not imply a true understanding of number but subitizing did (e.g., Douglass, 1925). Some saw the role of subitizing as a developmental prerequisite to counting. Freeman (1912) suggested that whereas measurement focused on the whole and counting focused on the unit, only subitizing focused on both the whole and the unit—so, subitizing underlies number ideas. Carper (1942) agreed subitizing was more accurate than counting and more effective in abstract situations. Kaufman, Lord, Reese, and Volkmann (1949) initially named subitizing and distinguished this activity as very different from estimation activity. Individuals were relatively more accurate and experienced higher degrees of confidence in their enumeration when subitizing small sets of items (≤5) compared to when they were estimating larger sets of items (>5).

In the second half of the twentieth century, educators developed several models of subitizing and counting. *Subitizing* was initially defined in the field of psychology (Kaufman et al., 1949). Essentially, Kaufman et al. found that subitizing activity was quite different than estimation, as individuals drew from a unique form of visual number discrimination characterized by speed, accuracy, and degree of confidence (1949). More specifically, Kaufman et al. found that individuals numerically identifying sets of five or fewer objects were relatively faster (≤40 ms/item in a perceptual field) in their recall times, had higher levels of confidence, and had higher accuracy rates (1949). Klahr (1973a, 1973b) began discussing subitizing as a form of visual *information processing* and a type of *quantification operator* (e.g., counting, subitizing, estimating). Klahr posited that subitizing did not *rely* on an encoding process, but in fact *was* an encoding process, explaining such different recall times when individuals subitized items between one and five.

Based on the same notion that subitizing was a more "basic" skill than counting (Klahr & Wallace, 1976; Schaeffer, Eggleston, & Scott, 1974), Klahr (1973a) hypothesized that after items were encoded through subitizing activity, individuals stored matched patterned stimuli to numerical thinking structures in their long-term memory. This explained why children can subitize directly through interactions with the environment, without social interactions. Supporting this position, Fitzhugh (1978) found that some children could subitize sets of one or two but were not able to count them. None of these very young children were able to count any sets that they could not subitize. Fitzhugh concluded that subitizing is a necessary precursor to counting. This research also began to define subitizing, for the first time, as supported by pre-attentional mechanisms (Klahr, 1973b; Trick & Pylyshyn, 1994) and a form of numerical encoding system (Klahr, 1973a).

However, in 1924, Beckmann found that younger children used counting rather than subitizing (cited in Solter, 1976). Others agreed that children develop subitizing later, as a shortcut to counting (Beckwith & Restle, 1966; Brownell, 1928; Silverman & Rose, 1980). Developmental psychologists Gelman and Gallistel (1978) expressed this view, claiming that subitizing is simply a form of rapid counting.

Although debates continue, recent research has shown that—as the introduction shows—*some* sensitivity to very small numbers develops very early (we do not call this "subitizing" yet as children are not connecting an exact quantity to a number word). Further, that sensitivity exists for larger numbers in a different form. The latter has been termed the Approximate Number System and we turn to it next.

2.1.2 The Approximate Number System (ANS)

Figure 2.1 illustrates a situation revealing an ability to estimate that is shared across animals and people. For example, monkeys and birds can be trained to discriminate both large and small sets (of visual dots or sounds) that differ in a 1 to 2 (or greater) ratio (but not 2:3) (Starr, Libertus, & Brannon, 2013). Baby chicks, first imprinted with a set of three, shown 4 objects going behind a screen on the right, then 1 going beyond a screen on the left, then 1 moved from the right to the left, go immediately to the screen on the right (Vallortigara, 2012).

Neuroscience findings suggest that humans, like other animal species, encode approximate number (Piazza, Izard, Pinel, Le Bihan, & Dehaene, 2004). The IPS coding for number in humans is compatible with that observed in macaque monkeys, suggesting an evolutionary basis for human elementary arithmetic (Piazza et al., 2004). Most children without specific disabilities possess these competencies, which appear to form one of the innate, foundational abilities for all later numerical knowledge—the Approximate Number System (ANS). Six-month-old infants can discriminate the 1:2 ratio (as in Fig. 2.1) but by 9 months of age, they can also distinguish sets in a 2:3 ratio (e.g., 10 compared to 15). ANS correlates with mathematics competencies in preschoolers (Mazzocco et al., 2011; Soto-Calvo, Simmons, Willis, & Adams, 2015), even with age and verbal ability controlled (Libertus, Feigenson, & Halberda, 2011b), although these correlations are larger for children low in mathematical knowledge (Bonny & Lourenco, 2013). It may be that higher achievers have access to more and more sophisticated strategies that makes ANS precision less relevant. Further, lack of ANS proficiency may be one but only one of several sources of poor mathematics learning (Chu, vanMarle, & Geary, 2013).

2.1.3 Is Subitizing Also an Approximate Estimator?

This raises the question of whether initial sensitivity to number is also based on approximate estimators, and only seems accurate early on in children's development because numbers are very small. Subitizing differs from the ANS in that the goal is to determine the *exact* number of items in a set and to connect the number to another representation, usually number words. Supporting the distinction, subitizing does not fit Weber's law for ANS and thus appears to be a distinct, dedicated method of quantification (Revkin, Piazza, Izard, Cohen, & Dehaene, 2008). Subitizing also appears distinct from counting. First, there is little or no relationship between children's performance on counting and subitizing tasks (Pepper & Hunting, 1998). Second, lesions that affect counting and subitizing appear to be in separate parts of the brain (Demeyere, Rotshtein, & Humphreys, 2012).

Still, questions remain about how subitizing operates. For example, some have questioned whether subitizing is really about number or a general sense of quantity. That is, some studies suggest that infants in "number" experiments may be responding to overall contour length, area, mass, or density rather than discrete number (Feigenson, Carey, & Spelke, 2002; Tan & Bryant, 2000). In one study, infants dishabituated to changes in contour length when the number of objects was held constant, but they did not dishabituate to changes in number when contour length was held constant (Clearfield & Mix, 1999), suggesting they may be more sensitive to continuous than discrete quantities. FMRI studies iterate these findings as they show 4-year-olds and adults exhibit a greater response in their IPS to visual arrays that change in the number of elements than to stimuli that change in shape (Cantlon, Brannon, Carter, & Pelphrey, 2006). Deaf people, who knew Japanese Sign Language but not American Sign Language, showed no activation in regions associated with numerical processing when taught ASL signs (but not their meanings) for numerals. However, when told what the signs represented, they showed just such activation—even when they could not accurately code those signs (Masataka, Ohnishi, Imabayashi, Hirakata, & Matsuda, 2006).

Models of subitizing There are then various empirical findings and theoretical models of subitizing (for reviews more detailed than this summary, see Butterworth, 2010; Hannula, Lepola, & Lehtinen, 2010; Sarama & Clements, 2009). Figure 2.2 illustrates several of them.

Some believe that recognition of patterns of movement (even eye movements), or *scan-paths* (Fig. 2.2), is the underlying *non*-numerical process that is then linked to specific numerosities (Chi & Klahr, 1975; Glasersfeld, 1982; Klahr & Wallace, 1976). Numerical subitizing requires a subsequent *reflective abstraction*, which occurs when the child abstracts the mental actions from the sensory-motor contexts and is capable of reflecting on these actions. Piaget (1977/2001) describes reflective abstraction as encapsulating two phases. The first phase is a "projection phase in

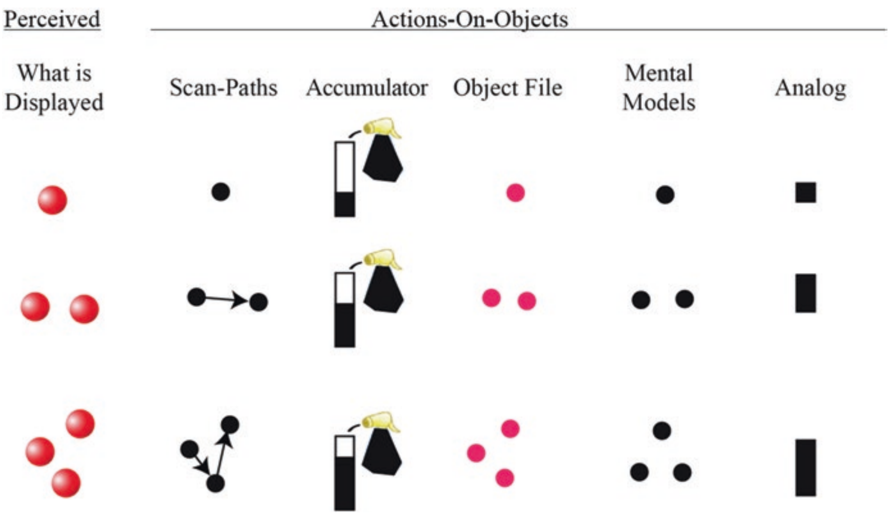

Fig. 2.2 Theories of subitizing

which the actions at one level become the objects of reflection at the next" (313). The second phase is a "reflection phase in which a reorganization takes place" (p. 313). Such abstractions can stem from temporal or rhythmic activity (Glasersfeld, 1982), are not grounded in perceptual pulsations (Glasersfeld, 1995), and may help to explain early number development (Steffe & Cobb, 1988).

Other models consider subitizing to be a numerical process. In the "accumulator" column of Fig. 2.2, subitizing is a numerical process enabled by the availability of the functional equivalent of a number line in the brain that operates on both simultaneous and sequential items (cf. Huntley-Fenner, 2001). There is a pacemaker that emits equivalent pulses at a constant rate. When a unitized item (e.g., a block taken as "one") is encountered, a pulse is allowed to pass through a gate, entering an *accumulator*—think of a squirt of water entering a tall glass. The gradations on the accumulator estimate the number in the collection of units, similar to height indicating the number of squirts in the glass (Meck & Church, 1983). This model does *not* require that the accumulator has an *exact* representation of number (see also Feigenson, Dehaene, & Spelke, 2004). The "squirts" and the amount in the "glass" are approximate. Support for this view comes from research indicating that children younger than 3 years tend not to represent any numbers except 1 and 2 precisely (Antell & Keating, 1983; Baroody, Lai, & Mix, 2005; Feigenson, Carey, & Hauser, 2002; Mix, Huttenlocher, & Levine, 2002).

The next theory holds that humans create "object files" that store data on each object's properties (Fig. 2.2). They can use these object files to respond differently to various situations. Thus, some situations can be addressed by using the objects' individuation or separateness as objects, and others can be addressed by using the analog properties of these objects, such as contour length (Feigenson, Carey, & Spelke, 2002). For example, children might use parallel-processed *individuation* for very small collections, but continuous extent when storage for individuation is exceeded. Individuation is the visual referencing of items as "that [which] refers to something we have picked out in our field of view without reference to what category it falls under or what properties it may have" (Pylyshyn, 2001, p. 129). Thus, processing is *preconceptual* prior to any entry into working memory (Pylyshyn, 2001). (From our perspective, even if such individuation is accepted as an early basis for number, it might not in itself constitute knowledge of *number, an issue to which we will return*.)

The mental models view (Fig. 2.2) postulates that children represent numbers nonverbally and approximately, then nonverbally but exactly, and eventually via verbal, counting-based processes (Huttenlocher, Jordan, & Levine, 1994; Mix et al., 2002). Children cannot initially differentiate between discrete and continuous quantities, but represent both approximately using one or more perceptual cues such as contour length (Mix et al., 2002). Children gradually develop the ability to individuate objects, providing the ability to build notions of discrete number. About the age of 2 years, they develop representational, or symbolic, competence, allowing them to create mental models of collections, which they can retain, manipulate (move), add to or subtract from, and so forth (although the model does not adequately describe how cardinality is ultimately cognized and how comparisons are made).

Creation of mental models with their added abstraction differentiates this view from the “object files” theory. Early nonverbal capabilities then provide a basis for the development of verbally based numerical and arithmetic knowledge (young children are more successful on nonverbal than verbal versions of number and arithmetic tasks, Huttenlocher et al., 1994; Jordan, Hanich, & Uberti, 2003; Jordan, Huttenlocher, & Levine, 1992; Jordan, Huttenlocher, & Levine, 1994; Levine, Jordan, & Huttenlocher, 1992). Meaningful learning of number words (in contrast to symbolic ability) may cause the transition to exact numerical representations (Baroody et al., 2005). This may provide the basis for understanding cardinality and other counting principles, as well as arithmetic ideas (Baroody, Lai, & Mix, 2006).

An alternative model postulates an innate abstract *module*. A module is a distinct mental component that is dedicated to a particular process or task and is unavailable for general processing. A number perception module would perceive numbers directly (Dehaene, 1997). This counting-like process is hypothesized to guide the development of whole number counting, hypothesized to be a privileged domain. Researchers use findings from both humans and non-human animals to support this position (Gallistel & Gelman, 2005).

A New Model for the Foundations of Subitizing A synthesis of these positions produces a model that we believe is most consistent with the research. What infants quantify are collections of rigid objects. Sequences of sounds and events, or materials that are non-rigid and non-cohesive (e.g., water), are not quantified (Huntley-Fenner, Carey, & Solimando, 2002). Quantifications of these collections begin as an undifferentiated, innate notion of the amount of objects. Object individuation, which occurs early in pre-attentive processing (and is a general, not quantitative, process, cf. Moore & Ashcraft, 2015), helps lay the groundwork for differentiating discrete from continuous quantity. That is, the object file system stores information about the objects, some or all of which is used depending on the situation.

Simultaneously, an estimator (accumulator) mechanism stores analog quantitative information (Feigenson, Carey, & Spelke, 2002; Gordon, 2004; Johnson-Pynn, Ready, & Beran, 2005). This estimator also includes a set of number filters, each tuned to an approximate very small number of objects (e.g., 2) although they overlap (Nieder, Freedman, & Miller, 2002). The child encountering small sets opens object files for each in parallel. By about a half-year of age, infants may represent very small numbers (1 or 2) as individuated objects (close to the “mental models” column of Fig. 2.2). However, larger numbers in which continuous extent varies or is otherwise not reliable (McCrink & Wynn, 2004) may be processed by the analog estimator as a collection of binary impulses (as are event sequences later in development, see the “analog” column of Fig. 2.2), but not by exact enumeration (Shuman & Spelke, 2005) by a brain region that processes quantity (size and number, undifferentiated, Pinel, Piazza, Le Bihan, & Dehaene, 2004). Without language support, these are inaccurate processes for numbers above two (Gordon, 2004).

To compare quantities, correspondences are processed. Initially, these are inexact estimates comparing the results of two estimators, depending on the ratio between the sets (Johnson-Pynn et al., 2005). Once the child can represent objects mentally, they can also make exact correspondences between these nonverbal representations, and eventually develop a quantitative notion of that comparison (e.g., not just that ••• is more than ••, but also that it contains one more •, Baroody et al., 2005).

Fully Functional Subitizing—Explicit Cardinality Even these correspondences, however, do not necessarily imply a cardinal representation of the collection (a representation of the collection qua a *numerosity* of a *group* of items). That is, our model distinguishes between noncardinal representations of a collection and *explicit* cardinal representations that is necessary to achieve fully functional subitizing competence. Indeed, a neuroimaging study found that brain regions that represent numerical magnitude also represent spatial magnitude, such as the relations between sizes of objects, and thus may not be numerical in function (Pinel et al., 2004). Only for numerical representations does the individual apply an integration operation (Steffe & Cobb, 1988) to create a composite with a numerical index. This integration operation uses present cognitive schemes to project and reorganize actions so they are considered mathematical objects. Some claim that the accumulator yields a cardinal output; however, it may be quantitative and still—because it indexes a collection using an abstract, cross-modality system for numerical magnitude (cf. Lourenco & Longo, 2011; Shuman & Spelke, 2005)—it may lack an explicit cardinality. For example, this system would not necessarily differentiate between ordinal and cardinal interpretations. Comparisons, such as correspondence mapping, might still be performed, but only at an implicit level (cf. Sandhofer & Smith, 1999). (It is possible to index a numerical label without attributing explicit cardinality. For example, lower animal species seem to have some perceptual number abilities, but only birds and primates also have shown the ability to connect a perceived quantity with a written mark or auditory label, Davis & Perusse, 1988.) In this view, only with experience representing and naming collections is an explicit cardinal representation created. This is a prolonged process. Children may initially make word-word mappings between requests for counting or numbers (e.g., "how many?") to number words until they have learned several (Sandhofer & Smith, 1999). Then they label some (small number) cardinal situations with the corresponding number word; that is, map the number word to the numerosity property of the collection. They begin this phase even before 2 years of age, but for some time, this applies mainly to the word "two," a bit less to "one," and with considerable less frequency, "three" and "four" (Fuson, 1992a; Wagner & Walters, 1982). We will discuss possible connections between subitizing and composite number understandings near the end of this chapter.

MacDonald and colleagues (MacDonald, 2015; MacDonald & Shumway, 2016; MacDonald & Wilkins, 2016, 2017) found that this early attention to 2 served preschool age children's ability to begin attending to subgroups of "two" when conceptually subitizing larger sets of items (e.g., four, five). Symmetrical orientations and orientations with a large space between subgroups of "two" seemed to afford these children's opportunities to attend to both subgroups. Symmetrical orientations freed children's working memory resources as they only needed to describe one 2 when building towards the total set of 4. Individuals' subitizing activity has been found to be affected by the space between the items in an orientation (Gebuis & Reynvoet, 2011) and was found to support young children's attention to the subgroups of the entire group of items (MacDonald & Wilkins, 2017).

Later Developments Only after many such experiences do children abstract the numerosities from the specific situations and begin to understand that the situations named by 3 correspond; that is, they begin to establish as what adults would term a numerical equivalence class. Counting-based verbal systems are then more heavily used and integrated, as described in the following section, eventually leading to explicit, verbal, mathematical abstractions. The construction of such schemes probably depends on guiding frameworks and principles developed from interactions with parents, teachers, and other knowledgeable people. Our model is supported by research on speakers of Mandrake in the Amazon, who lack number words for numbers above 5. They can compare and add large approximate numbers, but fail in exact arithmetic (Pica, Lemer, Izard, & Dehaene, 2004).

Nevertheless, it is significant that children discriminate exact collections on some quantitative bases from birth. Furthermore, most accounts suggest that these limited capabilities, with as yet undetermined contributions of maturation and experience, form a foundation for later learning. That is, they connect developmentally to culturally based cognitive tools such as number words and the number word sequence, to develop exact and extended concepts and skills in number.

Even though the shape of the items plays a secondary role in subitizing, particular orientations have been found to influence adults' degree of accuracy when subitizing larger sets of items (≤4). For instance, Logan and Zbrodoff (2003) found that the space between these groups of "twos" and "threes" afforded individuals more effective subitizing of four or more items. These findings suggest that individuals rely on patterned orientations of twos and threes (described as point-groupings) when subitizing. Thus, there is a special neural component of early numerical cognition present in the early years that may be the foundation for later symbolic numerical development. A language-independent ability to judge numerical values nonverbally appears to be important evolutionary precursor to later symbolic numerical abilities.

In summary, early quantitative abilities exist, but they may not initially constitute systems that can be said to have an explicit number concept. Instead, they may be pre-mathematical, foundational abilities (cf. Clements, Sarama, & DiBiase, 2004) that develop and integrate slowly, in a piecemeal fashion (Baroody, Benson, & Lai, 2003). For example, object individuation must be stripped of perceptual characteristics and understood as a perceptual unit item through abstracting and unitizing to be mathematical (Steffe & Cobb, 1988), and these items must be considered simultaneously as individual units and members of a collection whose numerosity has a cardinal representation to be numerical, even at the lowest levels.

2.1.4 Categories of Subitizing

Regardless of the precise mental processes in the earliest years, subitizing appears to be phenomenologically distinct from counting and other means of quantification and deserves differentiated educational consideration. Further, subitizing ability is not merely a low-level, innate process although it builds on innate sensitivity to number. As stated previously, in contrast to what might be expected from a view of innate ability, subitizing *develops* considerably and combines with other mental processes.

Types of Subitizing Early attention to numerosities reveals *preconcepts*, defined by Piaget are pre-operational, "action-ridden, imagistic, and concrete" early forms of concepts that young children depend on (1977/2001, p. 159–160). Children acting on static, concrete images that have yet to be unitized, operationalized, or abstracted are relying on preconcepts. Children engaging with preconcepts are not yet able to identify members as belonging to a given set (e.g., class identification within groups) necessary for unitizing. Preconcepts are the basis for early, perceptual subitizing activity. However, once early forms of perceptual subitizing develop, Clements (1999) posited that students developed and drew from this activity to develop conceptual activity for subitizing.

Therefore, one major shift is the development from using only one, to using two types of subitizing. The first type, *perceptual subitizing* (Clements, 1999; see also theoretical justification in Karmiloff-Smith, 1992), is closest to the original definition of subitizing: Recognizing a number without consciously using other mental or mathematical processes and then naming it. Thus, perceptual subitizing employs a pre-attentional, encoding quantitative process but adds an intentional numerical process; that is, infant sensitivity to number is not (yet) perceptual subitizing. The term "perceptual" applies only to the quantification mechanism as phenomenologically experienced by the person; the intentional numerical labeling, of course, makes the complete cognitive act conceptual. A second type of subitizing (a distinction for which there is empirical evidence, Trick & Pylyshyn, 1994), *conceptual subitizing* (Clements, 1999), involves applying the perceptual subitizing processes repeatedly and quickly uniting those numbers. For example, one might recognize "10" on a pair of dice by recognizing the two collections (via perceptual subitizing) and composing them as units of units (Steffe & Cobb, 1988). Some research suggests that only the smallest numbers, perhaps up to 3, are actually perceptually recognized; thus, sets of 1 to 3 may be perceptually recognized, sets of 3 to about 6 may be and recomposed without the individual being aware of the subgroups. As we define it, conceptual subitizing refers to recognition in which the person uses such partitioning strategies and is aware of the parts and the whole. In the remainder of this section, we elaborate on each type.

Perceptual subitizing also plays the primitive role of *unitizing*, or making single "things" to count out of the stream of perceptual sensations (Glasersfeld, 1995). "Cutting out" pieces of experience, keeping them separate, and eventually coordinating them with number words are not trivial tasks for young children. For example, a toddler, to recognize the existence of a plurality, must focus on the items such as apples and repeatedly apply a template for an apple *and* attend to the *repetition* of the template application.

In an exploratory 22-session teaching experiment, MacDonald and Wilkins (2016) found that four preschool children (ages ranging from 4 years and 4 months to 5 years and 5 months) engaged in several types of perceptual subitizing that could explain early shifts in children's types of abstractions. Cross-case analyses determined similar activity children engaged in throughout the study. MacDonald and Wilkins (2016) developed a framework that explained types of activity that young children revisited when subitizing. In this framework, five sets of perceptual subitizing activity were found to explain how these children's perceptual subitizing activity changed. As shown in Table 2.1, these four preschool age children relied on perceptual figurative patterns when associating number with patterns when subitizing *(initial perceptual subitizing or IPS)*. Children were also found to subitize small subgroups, composed of two or three *(perceptual subgroup subitizing or PSS)*, but were not able to compose these subgroups. These activities, explained as a form of low-level processing, were purely associative and seemed to illustrate foundational operations of number in which children could project onto new schemes as early forms of mathematical objects. Further, when children's subitizing changed they began composing and decomposing subgroups of these total sets *(perceptual*

Table 2.1 Five different types of perceptual subitizing activity

Type	Description	Example
Initial Perceptual Subitizing (IPS)	• Children describe the visual motion or the shape of the dots	• Children will describe seeing "five" because it looks like a flower
Perceptual Subgroup Subitizing (PSS)	• Children numerically subitize small subgroups of two or three, but cannot subitize the entire composite group	• Children will state that they saw "two and three," or "two plus three," but do not use this to accurately describe the composite group
Perceptual Ascending Subitizing (PAS)	• Children describe the perceived cluster of items as subgroups and then the composite group	• Children will state that they saw "two and three," and then accurately describe the total composite group
Perceptual Descending Subitizing (PDS)	• Children describe the composite groups and then describe the perceived cluster of items as subgroups	• Children will state that they saw "five" because they saw "two and three"
Perceptual Counting Subitizing (PCS)	• Children initially describe seeing one more or one less than the composite group, and then counts down or up, respectively, to the composite group	• Children will state they saw "4 … 5" or "6 … 5" • Children will state they know it to be "5" because they saw "6 … 5"

Note. These five different types of perceptual subitizing activity categorically represent the observed child responses documented by MacDonald and Wilkins (2016) *ascending subitizing or PAS and perceptual descending subitizing or PDS)*, which was foundational for children's conceptual subitizing. PAS and PDS activity is similar to conceptual subitizing activity because the children are decomposing and composing units of units. However, PAS and PDS activity explained these children's reliance on perceptual material, spatial patterns, or finger patterns. Furthermore, when engaging in PAS and PDS activity these children acted on orientations in static means where subgroups did not have to be determined. For instance, they relied primarily upon the clustering of items or spatial arrangement of the items to determine operations that they would need to use when composing number. This means that children would not be required to partition the orientation into subgroups, but that they would operationalize the activity by partitioning and composing number in a more abstract manner. Thus, for these children to engage in conceptual subitizing, they would need to carry their (de)composition of number or their partitioning of orientations into the activity. Children also coordinated their counting with their perceptual subitizing. MacDonald and Wilkins found that all four children would subitize a set of items and then count up or down by one *(perceptual counting subitizing or PCS*). PCS activity was explained as a type of blend between both subitizing and counting activity.

These findings suggest that when children engage in perceptual subitizing, they are building initial schemes through a series of associations between orientations and early units of number. These schemes are foundational for (de)composition of number later, as these children begin developing conceptual processes of number in relation to their conceptual subitizing.

This takes us to the second type of subitizing, *conceptual subitizing* plays an advanced organizing role with the individual explicitly using partitioning, decomposing, and composing quickly to determine a number of items. Decomposing and composing are combining and separating operations that help children develop generalized part-whole relations, one of the most important accomplishments in arithmetic (National Research Council, 2001). The distinction between PDS activity and conceptual subitizing activity is that when children engage in PDS activity they are not able to numerically understand how these units relate to units because they are still relying on perceptual material, fingers, or spatial patterns. In PDS activity, young children are still dependent on the material shown to them when decomposing and composing number. In conceptual subitizing activity, children step away from the material and carry operations of number into the task. This distinction is explained further in a subsequent section where number and operations are explained as related to conceptual subitizing activity.

MacDonald and Wilkins (2016) also found two types of conceptual subitizing that describe how children's limited or flexible number understandings related to their subitizing activity (see Table 2.2). Children who have limited ability to draw from more than one set of subgroups when conceptually subitizing (evidenced through their description of exactly one set of subgroups) engage in *rigid conceptual subitizing (RCS)* (see Table 2.2). For instance, when children subitize "two, two, and one" each time they are shown a wide variety of "five" is evidence of their reliance on RCS. This activity indicates children's ability to see units of units when shown a wide variety of representations for "five." However, the children engaging in RCS are limited because they cannot use flexible operations of number when conceptually subitizing. When children are capable of "seeing" two or more ways of composing items (e.g., two and two; three and one) when subitizing engages in *flexible conceptual subitizing (FCS)*. FCS activity evidences the multiple means in which children use operations when conceptual subitizing.

What is Subitized Another categorization involves the different types of things people can subitize. Spatial patterns such as those on dice are just one type. Other patterned modalities are temporal and kinesthetic, including finger patterns (motoric and visual/spatial), rhythmic patterns (e.g., 3 beats), and spatial-auditory patterns. Creating and using these patterns through conceptual subitizing helps children develop abstract number and arithmetic strategies. For example, children use temporal patterns when counting on. "I knew there were three more so I just said, nine … ten, eleven, twelve" (rhythmically gesturing three times, one "beat" with each count). They use finger patterns to figure out addition problems. For example, for 3 + 2, a child might put up a finger pattern they know as three, then put up two more (rhythmically—up, up) and then recognize the resulting finger pattern as "five." Children who cannot subitize are handicapped in learning such arithmetic processes (Butterworth, 2010; Hannula et al., 2010). Children may be limited to subitize small numbers at first, but such actions are useful "stepping stones" to the construction of more sophisticated procedures with larger numbers, a point to which we return.

2.1.5 Possible Connections Between Unit Development and Subitizing Activity

Children's subitizing activity changes over time that requires different types of actions that possibly relate to their ability to unitize members of a set. Thus, children's perceptual subitizing activity may relate to *unit* development (Steffe & Cobb,

Table 2.2 Two different types of conceptual subitizing activity

Type	Description	Example
Rigid Conceptual Subitizer (RCS)	• Children describe seeing the composite unit and then one set of subgroups that always remain the same, regardless of the orientation or color of the items	• Children will always state they know a composite group to be four because they saw "two and two"
Flexible Conceptual Subitizing (FCS)	• Children describe seeing the composite unit and then two or more sets of subgroups in different tasks regardless of the orientation or color of the items	• Children will state that they know a composite group to be five because they saw "two and three," but previously they explained the same orientation to be five because they saw "two, two, and one"

Note. These two different types of conceptual subitizing activity categorically represent the observed child responses documented by MacDonald and Wilkins (2016)

1988). Steffe and Cobb found that children engaged in counting developed *abstract singular units* (abstract units composed of 1) through their engagement with a variety of physical material of singular units (e.g., perceptual, figural patterns, motor activity, verbal utterances) (see Fig. 2.1). Children develop abstract singular units by first engaging with *perceptual singular units* as their unitization cuts away portions of "a specific experiential 'thing'" (Steffe & Cobb, 1988, p. 343). As children use *figural* (different representation), *motor* (motor pattern through activity), and *verbal* (utterance of a number word) singular units to represent perceptual material, they develop more abstract singular units (1988).

Steffe and Cobb found that first grade students "re-presented perceptual unit item[s]" (p. 342) when developing figural unit items. Further, children developed motor unit items by unitizing motor actions and associated them with isolated motor pattern (1988). Through development of these singular unit items, children re-presented singular activities and patterns through "an utterance of a number word that signifies a perceptual, figural, or motor unit item" (1988, p. 343). In developing and acting on these singular unit items, it should be noted that children may develop figural, verbal, and motor unit items concurrently or in one order versus another (i.e., figural, verbal units and then motor units, motor units, figural, and then verbal units). In re-presenting perceptual singular units with figural units, verbal utterances and motor patterns, children develop abstract singular unit items. Using abstract singular units, children can develop groups to engage in more sophisticated activity (e.g., partition, iterate) with number and develop *abstract composite units* (abstract units composed of more than one unit) that become countable units of units (Fig. 2.3).

One alternative manner that children may use to develop abstract composite units is their engagement with spatial patterns to develop templates or rules for *experiential composite units*. Essentially, Steffe (1994) posits that young children may initially rely on numerical patterns through their engagement with spatial patterns when developing figurative material (*figurative composite units*) and motor activity (*motor composite units*). Young children's activity with material with counting (and possibly subitizing) are foundational for experiential composite unit development. Steffe found that children constructed experiential composite units by attending to the *numerical rules* of a pattern. Through flexible engagement with numerical patterns, children develop experiential units as their development of "the records of a pattern do not take a picture of the pattern, but they constitute a program or recipe whose enactment constitutes a sensory pattern" (Steffe, 1994, p. 18). Steffe distinguishes these patterns as primarily numerical sequences, as subitizing was not considered in the framework of Steffe's research (cf. Glasersfeld, 1982). However, we posit that when considering multiple means in which patterns could be engaged, young children could construct experiential composite units based on subitizing, providing an alternative means to access abstract composite unit development.

For example, children rely on visual patterns when perceptually subitizing an orientation of "three and two." When children are then asked, "how many did you see?" they might need to "make it first." Here, children are primarily relying on the pattern and the figurative composite unit to engage with the numbers. Alternatively, if children represent the "two" and "three" with all fingers on the one hand and

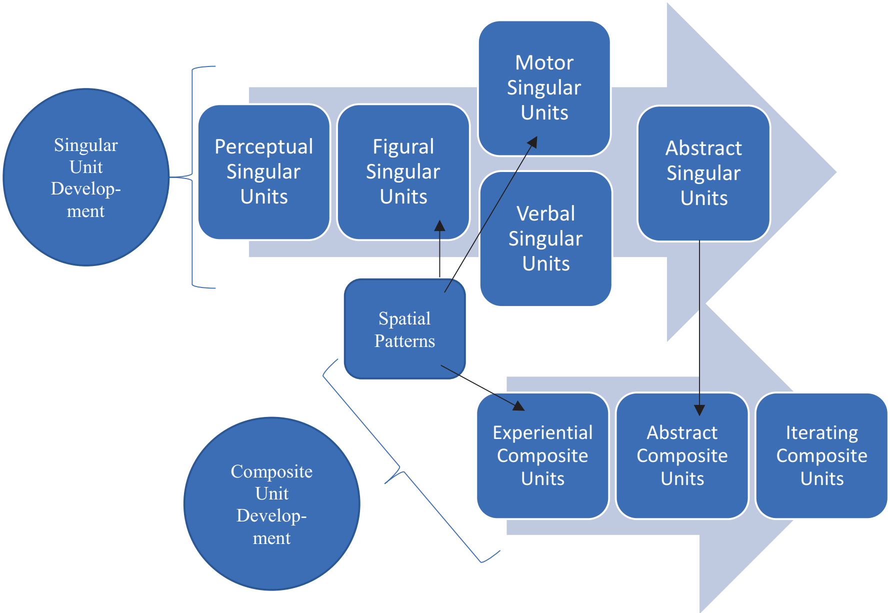

Fig. 2.3 MacDonald and Wilkins (under review) adapted conceptual framework from Steffe and Cobb (1988) and Steffe (1994)

partition an orientation when determining subgroups, then they may be relying on the rules of prior experience with patterns in which to conceptually develop composite units. More specifically, Steffe (1994) describes this transition as one of a "uniting operation" where the numerical pattern is used as one object "to instantiate the records that compose it" (p. 17). Thus, children's subitizing development and experiential unit development may be related.

2.2 Learning Trajectories: Integrating Developmental Psychology and Mathematics Education

Research identifying that subitizing is a distinct and central process has important ramifications for education. As we have seen, developmental psychology also helps us understand the natural *paths* of children's learning—invaluable for developing curriculum (the term's origins are a path for racing) and teaching strategies. Before we examine a course of development for subitizing relevant to educational practice, we briefly describe the theoretical and empirical foundations for our approach to learning and teaching.

We synthesized research and theories in developmental psychology and mathematics education from nativist and constructivist perspectives to form a theoretical framework called *hierarchic interactionalism* (Sarama & Clements, 2009). The term indicates the influence and interaction of global and local (domain specific) cognitive levels and the interactions of innate competencies, internal resources, and experience (e.g., cultural tools and teaching). Mathematical ideas are represented intuitively, then with language, then metacognitively, with the last indicating that the child possesses an understanding of the topic and can access and operate intentionally on those understandings. The theory has 12 tenets; several are particularly pertinent to this chapter (see Sarama & Clements, 2009, for a full discussion).

2.2.1 Selected Tenets of Hierarchic Interactionalism

Developmental Progression Most content knowledge is acquired along developmental progressions of levels of thinking. These play a special role in children's cognition and learning because they are particularly consistent with children's intuitive knowledge and patterns of thinking and learning at various levels of development (at least in a particular culture, but guided in all cultures by innate competencies), with each level characterized by specific mental objects (e.g., concepts) and actions (processes) (e.g., Clements, Wilson, & Sarama, 2004; Steffe & Cobb, 1988). These actions-on-objects are children's main way of operating on, knowing, and learning about, the world, including the world of mathematics.

Cyclic Concretization Development progressions often proceed from sensory-concrete and implicit levels at which perceptual concrete supports are necessary and reasoning is restricted to limited cases (such as small numbers) to more explicit, verbally based (or enhanced) generalizations and abstractions that are tenuous to integrated-concrete understandings relying on internalized mental representations that serve as mental models for operations *and abstractions* that are increasingly sophisticated and powerful. Again, such progressions can cycle within domains and contexts.

Different Developmental Courses Different developmental courses are possible within those constraints, depending on individual, environmental, and social confluences (Clements, Battista, & Sarama, 2001; Confrey & Kazak, 2006). At a group level, however, these variations are not so wide as to vitiate the theoretical or practical usefulness of the tenet of developmental progressions. The following tenet is closely related.

Environment and Culture Environment and culture affect the pace and direction of the developmental courses. Because environment, culture, and education affect developmental progressions, there is no single or "ideal" developmental progression, and thus learning trajectory, for a topic. Universal developmental factors interact with culture and mathematical content, so the number of paths is not unlimited, but, for example, educational innovations may establish new, potentially more advantageous, sequences, serving the goals of equity (Myers, Wilson, Sztajn, & Edgington, 2015). A latter section of this chapter deals explicitly with such differences.

Progressive Hierarchizing Within and across developmental progressions, children gradually make connections between various mathematically relevant concepts and procedures, weaving ever more robust understandings that are hierarchical in that they employ generalizations while maintaining differentiations. These generalizations, and the metacognitive abilities that engender them, eventually connect to form logical-mathematical structures. Children provided with high-quality educational experiences build similar structures across a wide variety of mathematical topics. For example, subitizing can have important interrelations with counting and arithmetic.

Consistency of Developmental Progressions and Instruction Instruction based on learning consistent with natural developmental progressions is more effective, efficient, and generative for the child than learning that does not follow these paths.

Learning Trajectories An implication of the tenets to this point is that a particularly fruitful instructional approach is based on hypothetical learning trajectories (Clements & Sarama, 2004b). Based on the hypothesized, specific, mental constructions (mental actions-on-objects), and patterns of thinking that constitute children's thinking, curriculum developers design instructional tasks that include external objects and actions that mirror the hypothesized mathematical activity of children as closely as possible. These tasks are sequenced, with each corresponding to a level of the developmental progressions, to complete the hypothesized learning trajectory. Specific learning trajectories are the main bridge that connects the "grand theory" of hierarchic interactionalism to particular theories and to educational practice.

Instantiation of Hypothetical Learning Trajectories Hypothetical learning trajectories must be interpreted by teachers and are only realized through the social interaction of teachers and children around instructional tasks (e.g., Wickstrom, 2015). Societally determined values, goals, and cultures are substantive components of any curriculum (Aguirre et al., 2017; Confrey, 1996; Hiebert, 1999; National Research Council, 2002; Tyler, 1949); research cannot ignore or determine these components (cf. Lester Jr. & Wiliam, 2002).

2.2.2 Hierarchic Interactionalism's Learning Trajectories

Learning trajectories, then, have three components: a goal (that is, an aspect of a mathematical domain children should learn), a developmental progression, or learning path through which children move through levels of thinking, and instruction that helps them move along that path. Formally, learning trajectories are descriptions of children's thinking as they learn to achieve specific goals in a mathematical domain, and a related, conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through a developmental progression of levels of thinking (Clements & Sarama, 2004b).

Learning trajectories are useful pedagogical, as well as theoretical, constructs (Clements & Sarama, 2004a; Simon, 1995; Smith, Wiser, Anderson, & Krajcik, 2006). Knowledge of developmental progressions—levels of understanding and skill, each more sophisticated than the last—is essential for high-quality teaching based on understanding both mathematics and children's thinking and learning. Early childhood teachers' knowledge of young children's mathematical development is related to their children's achievement (Fuson, Carroll, & Drueck, 2000; Kühne, Lombard, & Moodley, 2013; Peterson, Carpenter, & Fennema, 1989; Wright, Stanger, Stafford, & Martland, 2006).

2.3 A Developmental Progression for a Subitizing Learning Trajectory

2.3.1 Levels of Thinking

Research helps us describe the developmental progression for subitizing. Explicit naming of numbers begins early (because the task is not timed—displays are not shown and quickly hidden—we call this *recognition of number* rather than subitizing). In laboratory settings, children at about 33 months of age can initially name numbers that differentiate 1 from collections of more than 1 (Wynn, 1992). Between 35 and 37 months, they name 1 and 2, but not larger numbers. A few months later, at 38–40 months, they identify 3 as well. After about 42 months, they can identify all numbers that they can count, 4 and higher, at about the same time. However, research in natural, child-initiated settings shows that the development of these abilities can occur much earlier, with children working on 1 and 2 around their second birthdays or earlier (Mix, Sandhofer, & Baroody, 2005). Further, some children may begin saying "two" rather than "one." These studies suggest that language and social interactions interact with internal factors in development, as well as showing that number knowledge develops in levels, over time (see also Gordon, 2004). However, most studies suggest that children begin recognizing and saying "one," then "one" and "two," then "three" and then "four," whereupon they learn to count and know other numbers (see Gelman & Butterworth, 2005, for an opposing view concerning the role of language; Le Corre, Van de Walle, Brannon, & Carey, 2006).

Most Kindergartners appear to have good competence recognizing 2 and 3, with most recognizing 4 and some recognizing higher numbers (note that different tasks were used, some of which did not limit time, so wide ranges are expected). A recent study of low-income children beginning pre-K, using a short-exposure subitizing task, report 2–14% accuracy for 3, 0–5% for 4, and virtually no competence with 5, 8, or 10 (Sarama & Clements, 2011). Thus, children appear to be most confident with very small numbers, but those from less advantaged environments may not achieve the same skills levels as their more advantaged peers. Some special populations find subitizing particularly difficult. Only a minority (31%) of children with moderate mental handicaps (chronological ages 6–14 years) and a slight majority (59%) of children with mild mental handicaps (ages 6–13) successfully subitize sets of three and four (Baroody, 1986; see also Butterworth, 2010). Some children with learning disabilities could not subitize even at 10 years of age (Koontz & Berch, 1996). Early deficits in spatial pattern recognition may be the source of difficulty (Ashkenazi, Mark-Zigdon, & Henik, 2013). Subitizing in preschool is a better predictor of later mathematics success for children with ASD (autism spectrum disorder) than for typically developing children (Titeca, Roeyers, Josephy, Ceulemans, & Desoete, 2014).

2.3.2 Factors Affecting Difficulty of Subitizing Tasks

Several factors (spatial arrangement, physical size of the dots, and color of dots) affect subitizing ability. The spatial arrangement of sets, the size of the items, and the color of the items influences how difficult they are to subitize. Children usually find rectangular arrangements easiest, followed by linear, circular, and scrambled arrangements (Beckwith & Restle, 1966; Wang, Resnick, & Boozer, 1971). This is true for students from the primary grades to college in most cases. The only change across these ages is rectangular arrangements were much faster for the oldest students, who could multiply.

Certain arrangements are easier for specific numbers. Arrangements yielding a better “fit” for a given number are easier (Brownell, 1928). Further, when items are not arranged in rectangular or canonical arrangements, and the items increase in their relative size, children and adults have more difficulties subitizing these items accurately (Leibovich, Kadhim, & Ansari, 2017). More specifically, children make fewer errors for 10 dots than for eight with the “domino five” arrangement, but fewer errors for eight dots for the “domino four” arrangement. Of course, these are averages; experience with arrangements undoubtedly influences children’s performances.

For young children, however, neither of these arrangements is easier for any number of dots. Indeed, children 2–4-years-old show no differences between any arrangements of four or fewer items (Potter & Levy, 1968). For larger numbers, the linear arrangements are easier than rectangular arrangements. It may be that many preschool children do not use decomposing (conceptual subitizing). Whelley (2002) also found that preschool children’s subitizing is affected by color of the items shown to them. For instance, when children are shown clustered items of differing colors, Whelley found the colors should align with the clustering of the group of items for effective subitizing to occur (i.e., three items clustered are all red and two items clustered are all black). When children are shown different colored items where the colors do NOT align with the clustering of the groups (i.e., three items clustered have two red and one black item and four items clustered have two red and two black items) then their subitizing accuracy decreased. As preschool children's attentional mechanisms mature, they can learn to conceptually subitize where orientations, color of items, and size of items does not affect their subitizing accuracy in drastic means (Whelley, 2002) though older research indicated that children as old as first grade experienced subitizing limitations of about four or five scrambled arrangements (Dawson, 1953).

The spatial arrangement of sets influences how difficult they are to subitize. Children usually find rectangular arrangements easiest, followed by linear, circular, and scrambled arrangements (Beckwith & Restle, 1966; Wang et al., 1971). If the arrangement does not lend itself to grouping, people of any age have more difficulty with larger sets (Brownell, 1928). They also take more time with larger sets (Beckwith & Restle, 1966).

2.4 Education's First Concern: *Goals* for Subitizing

The ideas and skills of subitizing start developing very early, but they, as every other area of mathematics, are not just "simple, basic skills." Subitizing introduces basic ideas of cardinality—"how many," ideas of "more" and "less," ideas of parts and wholes and their relationships, beginning arithmetic, and, in general, ideas of quantity. Developed well, these are related, forming webs of connected ideas that are the building blocks of mathematics through elementary, middle, and high school and beyond.

Young children may use perceptual subitizing to make units for counting and to build their initial ideas of cardinality (Slusser & Sarnecka, 2011). For example, their first cardinal meanings for number words may be labels for small sets of subitized objects, even if they determined the labels by counting (Fuson, 1992b; Steffe, Thompson, & Richards, 1982).

MacDonald and Wilkins (2017) found that one preschool child, Amy, used conceptual subitizing to develop early forms of units for counting. When given a counting task from Steffe and Cobb's (1988) counting scheme investigation, Amy drew from these same units and represented them with finger patterns, suggesting her ability to reorganize the patterns she may have relied on in earlier subitizing activity. Throughout sessions, Amy engaged in FCS for five and had constructed units "two and three" and "two, two, and one" when conceptually subitizing "five." At the end of the study, Amy was given a missing addend task to solve that required her to use "four" and "three." Essentially, in the task the teacher-researcher made a row of seven counters and covered four items in this row. Next, Amy was asked if there are "four" here, how many are there altogether? Amy showed with her fingers that she could make "three" (using three middle fingers from one hand) and then add "two" (using her pinky and thumb) and "two" more (using two fingers on her other hand) to discover there were "seven" altogether (MacDonald & Wilkins, 2017). This evidence seems to suggest relationships between early forms of number operations and conceptual subitizing activity.

This may be why subitizing predicts overall mathematics competencies of kindergartners (Yun et al., 2011, July). In another study, kindergartners' subitizing, but not the other early number skills, mediated the association between executive functioning and mathematics achievement in primary school (Fuhs, Hornburg, & McNeil, 2016). Executive function may help children quickly and accurately identify number sets as wholes instead of getting distracted by the individuals in the sets, and this focus on wholes may help develop advanced mathematics concepts.

As described earlier, counting and subitizing also interact to build arithmetic competencies. For example, consider how children progress from counting all to more sophisticated counting on strategies in solving arithmetic problems (Fuson et al., 2000; Peterson et al., 1989; Wright et al., 2006). As mentioned in discussing the types of subitizing, once their movement through the counting and subitizing learning trajectories has given them access to the notion that they can count up from a given quantity, they can solve 6 + 3 in a new way. They subitize the first addend (rather than counting it out one by one), and then count three more, using a *subitized rhythmic pattern* as an intuitive keeping track strategy: "Siiiix… *seven, eight, nine!*"

As another example, more advanced ability to quickly group and quantify sets in turn supports their development of number sense and arithmetic abilities. A first grader explains the process for us. Seeing a 3 by 3 pattern of dots, she says "Nine" immediately. Asked how she did it, she replies,

> When I was about four years old, I was in nursery school. All I had to do was count. And so, I just go like 1, 2, 3, 4, 5, 6, 7, 8, 9, and I just knew it by heart and I kept on doing it when I was five too. And then I kept knowing 9, you know. Exactly like this [she pointed to the array of nine dots]. (Ginsburg, 1977, p. 16)

As we discuss the details of teaching and learning of subitizing, let us not lose the whole—the big picture—of children's mathematical future. Let's not lose the wonderment that children so young can think, profoundly, about mathematics.

These foundations are significant beyond the earliest years. Subitizing in grades 3 and 4 significantly predicts of fluency in calculation and also general mathematics achievement a year later (Reigosa-Crespo et al., 2013). Starkey and McCandliss (2014) also found that kindergarten children's subitizing activity related to their "groupitizing" activity (a type of conceptual subitizing) and flexible operations on number when enrolled in third grade. Thus, as children develop more abstract means for number as a flexible set of units of units, they are capable of operating fluently on number more effectively in upper elementary school.

2.5 Instructional Tasks and Teaching Strategies

Although children are sensitive to quantity, interaction with others is essential to learning subitizing; it does not develop "on its own" (Baroody, Li, & Lai, 2008). Children who spontaneously focus on number and subitize number are more advanced in their number skills (Edens & Potter, 2013). This section describes the third part of a learning trajectory: Instructional activities and pedagogical strategies.

2.5.1 Developing Children's ANS System

For developing a sensitivity to quantity, research does suggest that making judgments of the number in sets of all sizes (including number of movements, tones, etc.) will help strengthen children's ANS systems (Libertus, Feigenson, & Halberda, 2013; Wang, Odic, Halberda, & Feigenson, 2016). These are usually not labeled with number words, but rather with vocabulary such as "more" and "fewer" (for dots) or "more" and "less." For the youngest children, intersensory redundancy—for example, you see a ball bouncing more times, it takes longer, you hear more noises—helps focus attention on number (Jordan, Suanda, & Brannon, 2008). Studies show these abilities can be developed, such as through special video games in which children make similar comparisons (Park, Bermudez, Roberts, & Brannon, 2016).

2.5.2 Mathematics Education: Supporting the Developmental Progression for Subitizing

For subitizing, or naming the exact number in sets, parents, teachers, and other caregivers might begin naming very small collections with numbers after children have established names and categories for some physical properties such as shape and color (Sandhofer & Smith, 1999). This section provides suggestions for helping children progress through the developmental progression for subitizing.

Everyday Number Recognition For everyone, but especially teachers of toddlers and 3-year olds, perhaps the easiest but most useful "activity" is simply to establish a habit of using small number words in everyday interactions frequently. They can replace, "Clear the cups off the table so we have room for this," with "We need more area on the table for this, would you please take those three cups off the table?" There is no need to be "artificial" in this kind of talk, just the use of small number words every time it makes sense. Teachers can give parents the same advice.

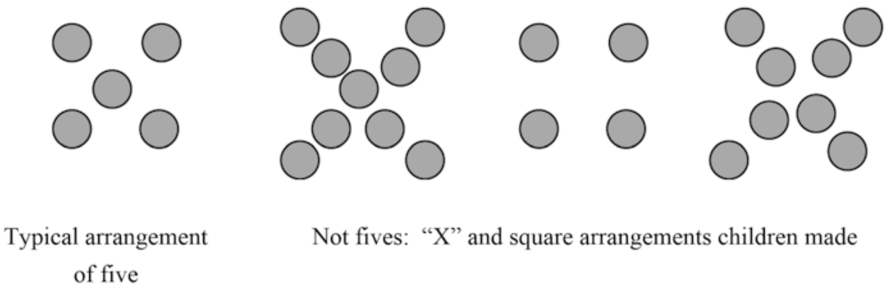

Fig. 2.4 Children had only seen a single pattern for 5—on the left. When asked to make a pattern of 5, some incorrectly produced arrangements like those on the right

Providing these types of repeated experiences naming collections help children build connections between quantity terms (number, how many) and number words, then build word-cardinality connections (• • is "two") and finally build connections among the representations of a given number. Non-examples are important, too, to clarify the boundaries of the number (Baroody et al., 2006). For instance, "Wow! That's not two horses. That's three horses!" For children who are less interested and competent in mathematics, it is especially important for caregivers and teachers to talk to them about number, for example, extending their interest in manipulating objects to include mathematical ideas such as number and shape (Edens & Potter, 2013). Research shows such experiences are helpful, especially for children who begin with lower abilities (Olkun & Özdem, 2015).

Practices to Avoid In contrast to these research-based practices, mis-educative experiences (Dewey, 1938/1997) may lead children to perceive collections as figural arrangements that are not exact. Richardson (2004) reported that for years she thought her children understood perceptual patterns, such as those on dice. However, when she finally asked them to reproduce the patterns, she was amazed that they did not use the same number of counters. For example, some drew an "X" with nine dots and called it "five" (see Fig. 2.4). Thus, without appropriate tasks and close observations, she had not seen that her children did not even accurately imagine patterns, and their patterns were certainly not numerical. Such insights are important in understanding and promoting children's mathematical thinking.

Textbooks and "math books" often present sets that discourage subitizing. Their pictures combine many inhibiting factors, including complex embedding, different units with poor form lack of symmetry, and irregular arrangements (Carper, 1942; Dawson, 1953). For example, they may show five birds, but have different types of birds spread out on a tree, with branches, leaves, flowers, a sun shining overhead—you get the idea. Such complexity hinders conceptual subitizing, increases errors, and encourages simple one-by-one counting.

Due to their curriculum, or perhaps a lack of training in subitizing, teachers may not pay proper attention to subitizing. For example, one study showed that children regressed in subitizing from the beginning to the end of kindergarten (Wright, Stanger, Cowper, & Dyson, 1996). How could that be? The following type of interaction might help explain. A child rolls a die and says "five." Looking on, the teacher says, "Count them!" The child counts them by ones. What has happened? The teacher thought her job was to teach counting. But the child was using subitizing—which is more appropriate and better in this situation. However, the teacher is unintentionally telling the child that her way is not good, that one must always count. Further, always telling children to count may actually hurt their development of counting and number sense. Naming small groups with numbers, *before* counting, helps children understand number words and their cardinal meaning ("how many") without having to shift between ordinal (counting items in order) and cardinal uses of number words inherent in counting (Baroody et al., 2005). These can be used to help infuse early counting with meaning.

Specific Subitizing Activities Many number activities can promote perceptual, and then conceptual subitizing (Sayers, Andrews, & Boistrup, 2016). Perhaps the most direct activity simply challenging children to subitize, an activity called "Quickdraw" (Wheatley, 1996), "Snapshots" (Clements & Sarama, 1998, 2007), and "Draw what you see" (MacDonald & Wilkins, 2016). As an example, children are told that they have to quickly take a "snapshot" of how many they see—their minds have to take a "fast picture." They are shown a collection of counters for 2 s only, then asked to construct, draw, or say the number. Consistent with research, arrangements may be straight lines of objects, then rectangular shapes, and then dice arrangements, all with small numbers. Separating these typical dice arrangements with a large space promotes children's attention to subgroups for Perceptual Subgroup Subitizing (MacDonald & Wilkins, 2016). As children learn, they use different arrangements and larger numbers. See the Box, Variations of the "Snapshots" Activity, for many engaging modifications.

Variations of the "Snapshots" Activity

- Have children construct a quick image arrangement with manipulatives (and watch for any misconceptions such as shown in Fig. 2.4).
- Play Snapshots on educational technology platforms (e.g., www.learning-trajectories.org/activity/subitize-planets-perceptual-subitizer-4).
- Play finger-placement game on computer. In Fingu, pieces of fruit are shown briefly and the child has to place that many fingers on the screen with one or two hands (Barendregt, LindstrÖm, Rietz-Leppänen, Holgersson, & Ottosson, 2012).
- Play a matching game. Show several cards, all but one of which have the same number. Ask children which does not belong (this also teaches early classification).
- Play concentration-type matching games (we call them "memory" games) with cards that have different arrangements for each number and a rule that you can only "peek" for 2 s.
- Give each child cards with 0–10 dots in different arrangements. Have children spread the cards in front of them. Then announce a number. Children

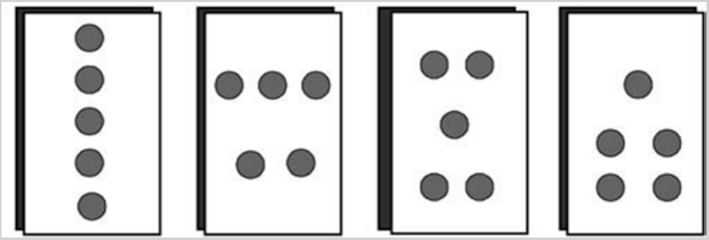

Fig. 2.5 Arrangements for conceptual subitizing that may suggest 5 as 5, 3 + 2, 2 + 1 + 2, or 1 + 4 (or other interpretations)

find the matching card as fast as possible and hold it up. Have them use different sets of cards, with different arrangements, on different days. Later, hold up a written numeral as their cue. Adapt other card games for use with these card sets.

- Emphasize conceptual subitizing as soon as possible. Use different arrangements that suggest different partitions of a number (see Fig. 2.5).
- Place various arrangements of dots on a large sheet of poster board. With children gathered around you, point to one of the groups as children say its number as fast as possible. Rotate the poster board on different sessions.
- Challenge children to say the number that is one (later, two) more than the number on the quick image. They might also respond by showing a numeral card or writing the numeral. Or, they can find the arrangement that matches the numeral you show.
- Remember that patterns can also be temporal and kinesthetic, including rhythmic and spatial-auditory patterns. A motivating subitizing and numeral writing activity involves auditory rhythms. Scatter children around the room on the floor with individual chalkboards. Walk around the room, then stop and make a number of sounds, such as ringing three times. Children should write the numeral 3 (or hold up three fingers) on their chalkboards and hold it up. These can also develop conceptual subitizing.

Across many types of activities, from class discussions to textbooks, children can be shown displays of numbers that encourage conceptual subitizing. Guidelines to make groups for this purpose include the following: (a) avoid embedding groups in pictorial context; (b) use simple forms such as homogeneous groups of circles or squares (rather than pictures of animals or mixtures or any shapes) for the units; (c) emphasize regular arrangements (most including symmetry, with linear arrangements for preschoolers and rectangular arrangements for older children being easiest); and (d) provide good figure-ground contrast.

To develop strong conceptual subitizing, children should experience many real-life situations such as finger patterns, arrangements on dice and dominoes, egg cartons (for "double-structures"), and arrays that separate two subgroups. To extend conceptual subitizing, teachers might discuss and especially cooperatively build arrangements to "make it easy to see how many." Such thoughtful, interactive, constructive experiences are effective ways of building spatial sense and connect it to number sense (Nes, 2009). For example, children might draw flowers with a given number of petals or draw or build pictures with manipulatives of houses with a certain number of windows so that they and others can subitize the number.

Such conceptual subitizing provides a direct phenomenological experience with additive situations, as children conceptualize two parts and the whole. Having both parts and whole in working memory builds a foundation for "knowing addition facts." Indeed, this is arguably better than emphasizing only counting-based solutions. Consider children using the initial "counting all" strategy for 3 + 2: counting out 3 objects, then counting out 2 objects, then starting over at "one" and counting all 5. The children answer correctly, but it is likely that *only the 5 is retained in working memory.* In comparison, the two addends may not be, and so it is unlikely that a connection is made between the addends and the sum. In subitizing, the addends and the same are retained in working memory in the same time period.

Subitizing is not only a separate complement to counting-based approaches to arithmetic but a valuable process to integrate with counting. That is, children can use subitizing in concert with counting to advance to more sophisticated addition and subtraction. As one example, children who are encouraged to subitize 3 in the previous example may move from counting all to early counting on, recognizing the set of 3, and counting only, "4, 5!". As another example, a child may be unable to count on keeping track, as in solving 4 + 5 by counting "4…5 is 1 more, 6 is 2 more…9 is 5 more." However, counting on two using rhythmic subitizing—for 5 + 2, saying "five…*six, seven!*" matching the counting to a "tah-dum" beat of two—gives them a way to figure out how counting on work. Later they can learn to count on with larger numbers, by developing their conceptual subitizing or by learning different ways of "keeping track." Eventually, children come to recognize number patterns as both a whole (as a unit itself) and a composite of parts (individual units).

2.6 Final Words

Across development, numerical knowledge initially develops qualitatively and becomes increasingly mathematical. In subitizing, children's ability to "see small collections" grows from pre-attentive but quantitative, to attentive perceptual subitizing, to imagery-based subitizing, and to conceptual subitizing (Clements, 1999; Steffe, 1992). Perceptual patterns are those the child can, and must, immediately see or hear, such as domino patterns, finger patterns, or auditory patterns (e.g., three beats). A significant advance is a child's focus on the exact number in these patterns, attending to the cardinality. Finally, children develop conceptual patterns, which they can operate on, as and when they can mentally decompose a five pattern into 2 and 3 and then put them back together to make five again. These types of patterns may “look the same” on the surface, but are qualitatively different. All can support mathematical growth and thinking, but conceptual patterns are the most powerful.

Subitizing small numbers appears to precede and support the development of counting ability (Le Corre et al., 2006). Thus, it appears to form a foundation for all learning of numbers. Indeed, a language-independent ability to judge numerical values nonverbally appears to be an important evolutionary precursor to adult symbolic numerical abilities. Children can use subitizing to discover critical properties of number, such as conservation and compensation. They can build on subitizing to develop capabilities such as unitizing as well as arithmetic capabilities. Thus, subitizing is a critical competence in children’s number development.

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Chapter 3
Discerning a Progression in Conceptions of Magnitude During Children's Construction of Number

Catherine Ulrich and Anderson Norton

Abstract Psychological studies of early numerical development fill a void in mathematics education research. However, conflations between magnitude awareness and number, and over-attributions of researcher conceptions to children, have led to psychological models that are at odds with findings from mathematics educators on later numerical development. In this chapter, we use the approximate number system as an example of a psychological construct that would benefit the mathematics education community if reframed to account for distinctions between number and magnitude. We provide such a reframing that also accounts for the role of children's sensorimotor activity in the construction of number.

Keywords Approximate number system · Counting · Early number · Magnitude · Students' mathematics

In the 1980s, an interdisciplinary group of researchers used qualitative methods to investigate how students construct numerical operations (e.g., Steffe, Glasersfeld, von Richards, & Cobb, 1983). They worked from a neo-Piagetian, radical constructivist perspective, and their work yielded a hierarchy of numerical stages. In the decades since, their research program has been elaborated and extended to include students' constructions of fractions and integers as reorganizations of those early numerical schemes (e.g., Steffe & Olive, 2010). More specifically, Steffe and colleagues describe how numerical operations arise from students' coordinations of sensorimotor activity and how students reorganize their numerical operations to form more advanced conceptions of number.

In the meantime, psychologists have been conducting quantitative studies on early number knowledge. This work has led to theoretical constructs such as

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subitizing and the approximate number system—constructs that are presumed innate and that allow for non-verbal magnitude representations in humans and animals (e.g., Dehaene, 1992; Gallistel & Gelman, 2000). Thus, Gallistel and Gelman (2000), among others, view with suspicion the idea that the verbal number word sequence is critical in the development of numerical thinking, as proposed by the neo-Piagetian theory underlying numerical stages.

We argue in this chapter that some important distinctions grew up out of Piagetian and neo-Piagetian work. We further argue that these distinctions could help reframe results from recent psychological studies into how subjects view and operate with numbers. One distinction of interest is between the learner's emerging concepts of a quantity's magnitude and the learner's ability to measure the magnitude. A second distinction of interest is between the learner's assimilation of a situation and the observer's assimilation of the situation. The following questions relate to those distinctions, respectively:

- What are magnitude and number?
- Where do numbers reside?

We introduce and reframe the research around the approximate number system as one example of the potential utility of our framework. This example also illustrates ways that mathematics education researchers may benefit from utilizing the results of psychological studies to inform their theories of number development.

3.1 What Are Magnitude and Number?

In reading reports of studies on early number development from the field of psychology, we quickly noticed that the terms *magnitude* and *number* are used differently than they are used in mathematics education. In fact, these terms are often ill-defined in both mathematics education and psychological research. For example, *number* can refer to a conception that includes the real numbers or other extensions of the natural numbers. In this chapter, we use the term *number* to refer to the numerical operations used to determine the numerosity of a collection of objects. In the following subsections, we will expand on what we mean by numerical operations as well as describe the subtle, but important, distinction we draw between magnitude and number. In particular, we lay out Piaget's use of the terms *gross* and *extensive quantity* to illustrate a fundamental distinction in the ways people can understand a given magnitude; we will additionally draw on Thompson, Carlson, Byerley, and Hatfield (2014) to tease out the role of measurement in the development of any *magnitude* awareness; and we will finally draw on Glasersfeld's (1981) construct of a *unitizing operation*, as well as Steffe and colleagues' (e.g., Steffe et al., 1983; Ulrich, 2015) theory on the development of numerical stages, in order to better characterize magnitude and number.

3.1.1 *Gross Quantity Versus Extensive Quantity*

> The most fundamental idea I have derived … is the idea that mathematical thinking begins, not with counting, but with comparisons between quantities, in particular the identification of equality and inequality relationship. (Sophian, 2007, p. xiv).

Psychologists and mathematics education researchers alike agree that mathematical thinking is fundamentally based in actions other than counting. However, the question remains of when an awareness of magnitudes and number begins and how they develop (or whether they are innate). Piaget (1970) believed that all mathematical operations can be traced back to abstractions from sensorimotor activity. In particular, Piaget and Szeminska (1952) and Steffe (1991) hold that even our earliest awareness of magnitude has to be built up out of our sensorimotor experiences and, therefore, is not innate. For example, consider the development of an awareness of length as a magnitude. One way that a sense of length can develop is through an awareness of duration of eye motion when traveling across perceptual material (or the amount of time an object is in the child's attentional field when sweeping an eye across the room). One can also imagine how other sensorimotor activity, such as walking along an object or moving a hand along an object, could similarly lead to a sense of length. In these scenarios, our awareness of length is that of a quality of the object that is delimited by some kind of visual or action-based cue. We can think of Piaget's *gross quantity* as exactly this kind of unspecified extension of a quality. This characterization aligns with a distinction made separately by McLellan and Dewey (1895): "Quantity is a limited quality, *and there is no quantity save where there is a certain qualitative whole or limitation*" (p. 57, emphasis in original).

A gross quantity does not include the type of measurable attribute that psychologists would normally associate with the term *magnitude*. Piaget documents how particular visual features, such as the extension of one item beyond another, are the main features of comparison of length when length is still a gross quantity. This leads to orderings inconsistent with adult notions of length. From the perspective of this chapter, one of the most important features of a gross quantity is that it includes a size awareness in some sense. In fact, "size awareness" is the name of this first level of magnitude awareness in the finer-grained framework of magnitude awareness offered by Thompson, Carlson, Byerley, and Hatfield (2014). There is no sense of being able to measure that size. For example, a student at this level of angle measure awareness may talk about "how far apart" the sides of an angle are. Such a student might view the angle in Fig. 3.1a or b as bigger than that of Fig. 3.1c or even

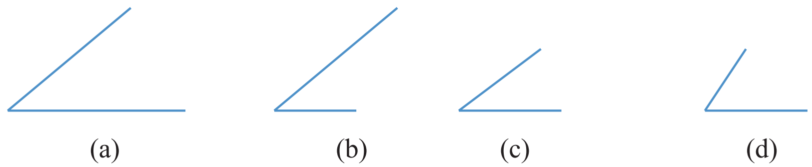

Fig. 3.1 Angles for comparison

Fig. 3.1d by attending to perceptual cues based on the distance between endpoints or the extension of one/both sides.

We should also note that any attempts to help children use numbers to describe their magnitude awareness are fraught with difficulties when children are still constrained to operating with gross quantities. For example, if blocks are linked together to make towers, the child's assessment of whether one of these towers is longer than another will be unaffected by the numbers reached by counting the blocks (e.g., Piaget, Inhelder, & Szeminksa, 1960; Tzur et al., 2013). This is one indication that gross quantity is not numerical from the child's perspective.

Eventually, the child will understand length in a way that allows for the use of measurement units, and subordinates comparisons made on the basis of visual or action-based cues to the results of subdividing quantities with these measurement units. This is referred to as *extensive quantity*. To better understand this distinction, imagine qualities you can compare the extension of, such as pain or even something like brightness of a light, but that you may not have a good way of explicitly measuring. You have an awareness of the magnitude of these qualities and may be able to directly compare two of them, but you probably lack a standard way to compare them in a consistent manner across different situations. This is the same level of magnitude awareness a child may have with respect to a gross quantity.

3.1.2 The Role of Measurement in the Progression of Magnitude Constructions

Piaget hypothesized that the major shift in thinking that takes place between understanding of gross quantities and the understanding of explicit measurement is the ability to imagine subdividing a quantity into measurement units that allows a standardized comparison between quantities (Piaget & Szeminska, 1952). For example, the child may use a finger or smaller rope to segment two ropes to decide which is longer. The construction of a unit of measurement implies that the child has delimited a portion of the (potentially mental) activity they use to construct the quantity in question to produce a measurement unit that they can use to make definite the sense of large or small magnitude associated with perceptual cues. The perceptual cues, such as right-hand extension or straightness, are aspects of situations that are usually correlated with larger or smaller magnitudes, but that are not sufficient alone to make consistent comparisons of magnitude over disparate situations.

In fact, once measurement units come into play, there are still many different levels of magnitude awareness that students can construct that would all correspond to an extensive quantity because students must still figure out how to interpret different numerical outcomes from measuring when using different measurement units, such as inches and feet or degrees and radians. Thompson et al. (2014) outline four such levels of magnitude awareness. *Measure magnitude* includes an ability to construct measurement units, but lacks an awareness that the numerical outcome of measurement depends on which measurement unit they are using. So, for example, a child might measure one angle in terms of a 1-degree unit angle and another angle in terms of a 1-radian unit angle and not realize that these two numerical measures are not useful for comparison without adjusting for the differing measurement units. The construction of a *Steffe magnitude* implies that "a person conceives of the size of Quantity A relative to a unit B and they are both measured in a common unit" (p. 2). So, for example, a child can measure in feet, knowing that a foot can be measured in terms of inches, so they could just as well measure in inches. The last two levels of magnitude awareness involve awareness of invariant relationships between the magnitude of the measurement unit and the numerical outcome of measurement (a *Wildi magnitude)* and the invariant relationship between the numerical outcomes of measurement using two different units (a *relative magnitude).*

Note that in this framework, "the idea of magnitude, at all levels, is grounded in the idea of a quantity's size" (Thompson et al., 2014, p. 1). Thus "a quantity…is not something in the world. It is a person's conception of an object and an attribute of it, and [at more advanced levels] a means by which to measure that attribute" (p. 1). Therefore, an increasingly sophisticated understanding of measuring a magnitude corresponds to a more sophisticated awareness of the magnitude itself. In fact, one can imagine how a more sophisticated awareness of a magnitude could allow for more sophisticated use of measurement, but also how increased experience with measurement could lead to a more sophisticated awareness of magnitude. That is, measurement is inextricably intertwined with our sense of magnitude.

Piaget and Szeminska (1952) identify two operations that are coordinated to lead to measurement schemes. The first is subdivision and the second is change of position. Subdivision refers to the child's construction of a small instantiation of the magnitude that can be used as the measurement unit. Change of position refers to the translations of that measurement unit in order to subdivide other instantiations of the magnitude. Often our measurement instruments, such as a ruler, are already subdivided, but the child still must be able to interpret the subdivisions as translations of the measurement unit in order to understand the count of these subdivisions as a quantitative measure of the size of the quality in question, such as length. In the next section, we will see how similar operations are important in the theory of numerical stages.

3.1.3 Unitizing and the Construction of Number

Like Piaget, Humberto Maturana was trained as a biologist. He has articulated in several places (e.g., Maturana, 1988) the idea that we are biological beings with specific physical (including neural) attributes that allow us to organize our experience in particular ways. This leads to commonalities across individuals. From this perspective, human beings do not organize the world in similar ways because we are able to directly experience the world, but because we are running up against the same constraints in the world with similar tools for organization. Ernst von

Glasersfeld (1995) codified these ideas of the construction of knowledge in the theory of radical constructivism.

While a radical constructivist would not see any reason to claim that number is inherent to the world, much less directly perceivable by humans or other animals, we could agree with cognitive psychologists that there is an innate foundation for certain later constructions in the way we are physically organized. Glasersfeld has hypothesized a particular innate ability—attentional bounding—as leading to the foundational operation of unitizing. *Unitizing* is reflectively abstracting an attentional bounding of experience as unitary in some way.

In a 1981 article in the flagship mathematics education journal, *Journal for Research in Mathematics Education,* Glasersfeld laid out a theoretical hierarchy of applications of the unitizing operation that eventually lead to the construction of what he and others call *arithmetic units*. Early applications of the unitizing operation construct objects. Glasersfeld (1981) theorizes that the child initially experiences the world through an undifferentiated mass of sensory inputs. If we imagine just the visual inputs, the child must attentionally bound and unitize constant patterns in the undifferentiated mass of visual inputs in order to construct objects. We can imagine, for example, a child unitizing a dark picture against a white wall by unitizing attentional pulses starting from the sharp transition in color (light wave frequencies of the visual input) at the border of the picture, continuing while scanning across the picture, and ending at the attentional pulse associated with another marked change in color on the other side. In fact, a child might experience attentional pulses associated with these kinds of transitions many times before applying the unitizing operation to the intermediate visual inputs to form an early visual object. The child has, in essence, utilized the visually similar aspects of the dark picture (in contrast with the white wall) in order to unitize these dark visual inputs.

After discussing the construction of objects, Glasersfeld (1981) discusses the application of the unitizing operation when constructing collections of like objects. The determination of what constitutes like objects that serve in the construction of categories of objects is influenced by socially negotiated constructs such as the colors that are categorized as “red,” and more abstract conceptual categories (dog, food, etc.). The quantitative property of these categories, that is, their numerosity, can be made specific by *counting activity*. At first, what we call counting activity does not necessarily have a numerical character for the child in that the categories and the items belonging to them are experiential so they are available to the child only in experience. The activity involves pointing at each element while simultaneously uttering a word in the number word sequence, which need be no more numerical for the child than the alphabet song. The output of the counting activity is repeating the last word uttered during the pointing activity.

There are then a series of abstractions that are made related to counting activity that lead to the construction of the numerosity of a collection. First, the child may reflectively abstract “things that can be counted” as a conceptual category and use that to form what are, to the observer, more general collections. Second, the child can modify the counting activity in various ways. For example, the child can substitute focused attention for pointing and mentally represented objects for visible objects, allowing counting in a wider variety of situations. Third, the child can recognize the counting activity as a way to make definite their indefinite gross quantity corresponding to the numerosity of a collection. These three types of abstractions allow the child to count his/her own counting acts, treat number word sequences as sets, and recognize how different number word sequences are related to each other: "1, 2,…, 8" is smaller than "1, 2, …, 12" not just because 8 comes before 12, but also because there are fewer things represented by 8 (the result of counting the first sequence) than by 12 (the result of counting the second sequence). This is the point at which we would attribute a *numerical* counting scheme to the child because there is an awareness of both order and numerosity inherent to number words and counting activity. Note that well before this point, children would appear to have constructed the numerosity of a collection by correctly answering questions such as, "How many apples are in the bowl?", when in fact they are carrying out a choreographed counting activity that does not contain an awareness of magnitude.

Although students at this point have constructed the first numerical stage, there are still three more stages that students must construct to develop fluency with the arithmetic operations and to construct fraction concepts (Steffe, 2010; Ulrich, 2015, 2016). In the second stage, students develop the ability to construct numerical units greater than 1 to allow them, for example, to make sense of the decimal number system and its reliance on groups of 10. In the third stage, students apply the unitizing operation to their acts of counting to abstract out the "plus one-ness" of the act. This creates a new kind of unit of one, an iterable unit, that allows students to think about how many times bigger a number is than 1. So instead of thinking of 9 as the set of nine numbers in the sequence from 1 to 9, the student would think of 9 as 9 iterations of a single unit, or as 1 nine times. This leads to an early understanding of multiplicative relationships that is completed in the fourth numerical stage, when students can form iterable units greater than 1, to think of one number, such as 28, as made up of iterations of a second number, such as 7.

Part of the reason we have laid out this construction of number in so much detail is to note that number is dependent on many previous constructions that take place over a long period of time and include reflections on an individual's own counting activity. Hence, it is a socially occasioned category, albeit different from many other linguistic or written signs in its reliance on abstractions of mental activity itself. Therefore, we reject the idea that number would be a construct available to infants or is somehow an intuitively obvious aspect of the environment though we recognize the importance of biological primitives as a starting point. Among research in developmental psychology, this view most closely aligns with that summarized by Sarnecka, Goldman, and Slusser (2015): "[Natural number concepts] do not appear to be innate—rather, they are constructed… Cardinality seems to be the marker of a profound conceptual achievement, involving an implicit understanding of the successor function and of equinumerosity, as well as of how counting works" (p. 307).

3.1.4 Tying Together Number and Measurement

Before leaving our discussion of number and magnitude, we will briefly note the interdependence of number and the measurement operations that undergird magnitude awareness.

> There is no necessity for children to measure or to count to introduce units. Segmenting sensory experience into units is the results of a unitizing activity prior to measuring or to counting that makes these activities possible. For example, …if the 'height' of the shorter of the two children is abstracted and then projected into the height of the taller child, this would be a segmenting activity (Steffe, 1991, p. 63).

In this sense, "children who are gross quantitative comparers are not without units" (p. 63). However, these units are not yet numerical (what Steffe and Piaget call *arithmetic*). Therefore, while unitizing is involved in any magnitude awareness, a child cannot construct an extensive quantity without the ability to interpret their measurement units as arithmetic units that can be counted and/or compared. Therefore, an extensive quantity requires the psychological equivalent of a numerical scheme. Conversely, the development of a numerical counting sequence is actually the development of an extensive quantity representing the numerosity of a collection. Therefore, students may have an awareness of the numerosity of a collection (a gross quantity) before they are numerical, but the ability to measure the numerosity of a collection (an extensive quantity) corresponds with the ability to think numerically. In the end, a number is not the same as any particular magnitude. It is an abstraction of measurement activity that that has led to a set of generalized magnitude measures.

3.2 Where Do Numbers Reside?

In characterizing magnitude and number, we hope it is clear that neither magnitude nor number exist in the situation. They are constructed and exist in the mind of the observer, be it an adult or child. Much of the psychological research on early numerical representation and comparison seems to operate with a very different understanding of number and magnitude. In particular, statements such as, "number, like color or movement, is a basic property of the environment" (Piazza, Izard, Pinel, Le Bihan, & Dehaene, 2004, p. 547) directly counter epistemological assumptions and empirically based theory in constructivist research. Similarly, arithmetic operations are seen to exist in the situation as opposed to the mind of the observer.

For example, consider Wynn's (1992) experiment involving infants. When a researcher put a new doll behind a screen with a doll that was already observed by the infant, Wynn infers from the infant's reaction that "infants could clearly see the *nature of the arithmetical operations being performed*" (emphasis added, p. 749). While one doll is being *added* to what is behind the screen in a general sense of putting a new object with an existing collection of objects, to claim this is addition would imply that anytime we increase the size of something we are carrying out addition. In contrast, we reserve more formal mathematical language, such as the terms *addition* or *number*, for numerical operations, and we note that different observers could view the same situation as involving different mathematics.

Consider a situation in which someone removes some cookies from a row of cookies, from three different perspectives: (1) A toddler may not utilize any mathematical operations and just attend to who is getting the cookies; (2) another child might utilize a number sequence to interpret the row of cookies as a countable collection with an unknown number of cookies, but not attend to the increasing or decreasing actions; (3) and an elementary school teacher might use a subtraction scheme to model the situation as a subtraction problem. From our perspective, subtraction does not exist *in* the situation, waiting to be "seen" by the observer. Rather, the situation is one in which an observer could use a subtraction scheme to make sense of the situation. Going back to the Wynn example, there is no reason to think that an infant would interpret a general situation of "increase" or even "new object," such as that in Wynn's study, as a numerical situation, and therefore we would not refer to the mental actions of the infant as *addition*. On the other hand, it is clearly an arithmetic situation from the researcher's point of view.

In our view, researchers of early number construction are particularly susceptible to over-attribution of our ways of operating because the concepts seem so basic and obvious to us. It is hard to imagine a world in which we are unaware of numbers or lack the ability to measure size in general. However, the inanity of attributing a mathematical model to the situation instead of the thinker becomes clearer if we consider more complicated mathematical models than numbers.

Consider an outfielder catching a fly ball. From a mathematical perspective, we could model the situation with a system of linear and quadratic motion equations and use that model to determine where the outfielder needs to stand in order to catch the ball. Therefore, we could attribute to the outfielder the ability to solve a system of linear and quadratic equations if they are consistently able to move to the correct location to catch a ball. However, from our own experiences catching a ball, it is clear that the outfielder is doing no such thing. Not only does the outfielder have access to the numerical measures of things like location and speed that would be necessary to do these calculations, but researchers have found that the outfielder does not even have a good intuitive awareness of these magnitudes (Rushton & Wann, 1999). The most plausible hypothesis seems to be that outfielders carry out a running pattern that linearizes the ball's trajectory so that they are not attending to a covariation of time and distance or speed, but only attending to spatial position (McBeath, Shaffer, & Kaiser, 1995). So not only would it be a major over-attribution to imagine the outfielder using our normal mathematical model of the situation, but it could lead us to focus on the wrong variables for research. Similarly, we must be careful to avoid attributing our models of situations to situations themselves and to the research subjects in these situations. Doing so may blind us to completely different explanations that would fit the observed behaviors equally well or even more closely.

Think now about how a similar issue arises in the attribution of zero as a number to bees. In a recent study, Howard, Avarguès-Weber, Garcia, Greentree, and Dyer (2018) found that bees could be trained to fly to one of two cards—the one with the fewest symbols on it—under which they would find sugar water. When given the option of a card with no symbols on it, versus a card with one to six symbols on it, the bees chose the card with no symbols at a rate statistically better than chance (significantly better than 50%). From this, the researchers inferred that bees understand the concept of zero as a number. There are numerous possible interpretations of the bees' behavior that do not involve attributing the construct of number (let alone zero) to the bees. Just because we interpret the cards in terms of number, a construct humans took centuries to develop, does not mean that bees have a number sequence or a concept for zero.

We see the distinction between the research subject's perspective and the researcher's perspective as one of the keys to reframing findings on early number in psychological research. Over-attribution leads to a lack of respect for children's rational ways of mathematically operating that might differ from our adult mathematics. In mathematics education, this can lead to missing out on the opportunity to affirm and build on students' ways of operating as opposed to dismissing the validity of whatever does not fit with adults' ways of operating.

The question remains of what proto-numerical understandings we can attribute to non-numerical children. Just like there are gross quantities that underlie our later, numerical awareness of magnitude, we can have intuitive and non-numerical understandings of actions that underlie arithmetic operations. For example, I may have an experience of *adding* (in the general sense) blocks to a basket of blocks and link that to a sense of *greater*, whether that is a greater height of blocks in the basket, more fullness of the basket, a greater amount of blocks, and so forth. Or I may have an experience of *dividing* up (in the general sense) food when I share it with someone else, and I may want us both to have the same amount of food. However, I would not say I am carrying out addition or division in those situations if I am not framing the situation as involving adding or dividing up numerical quantities.

3.3 The Approximate Number System

Psychologists have long noted the apparent ability of humans and non-humans to perform above chance when comparing the sizes of two collections. Dehaene (1992) proposed a construct, called the *approximate number system* (ANS), to explain this phenomenon. Many researchers have since attempted to measure the acuity of the approximate number system and investigate its relationship to, among other things, the development of numerical thinking.

ANS serves as the semantic component within a triple-code model for numbers (Dehaene, 1992). The model includes a visual component for interpreting Arabic number forms, an auditory component for interpreting verbal word names, and an "analogical magnitude representation." ANS refers to this analogical magnitude representation. Although it relates to the auditory and visual components within the triple-code model, the ANS construct is supposed to function independent of language and modality (auditory or visual).

> Exact arithmetic is acquired in a language-specific format, transfers poorly to a different language or to novel facts, and recruits networks involved in word-association processes. In contrast, approximate arithmetic shows language independence, relies on a sense of numerical magnitudes, and recruits bilateral areas of the parietal lobes involved in visuo-spatial processing. (Dehaene, Spelke, Pinel, Stanescu, & Tsivkin, 1999, p. 284)

ANS can be modeled in at least two ways: with the mental number line (Dehaene, 2003) or the accumulator model (Gallistel & Gelman, 2000; Gibbon, 1977; Meck & Church, 1983). Here, we critically examine the construct itself and those two models of it. Existing critiques mostly concern the validity of measures of ANS acuity (e.g., Inglis & Gilmore, 2014) and the relationship between ANS and other psychological constructs, such as subitizing (e.g., Cutini, Scatturin, Moro, & Zorzi, 2014). Our own critique recognizes the importance of early magnitude comparisons but questions its supposed relation to number. We emphasize the role of sensorimotor activity in building number from early (or innate) constructs, like the ANS and subitizing.

3.3.1 Critical Analysis of the ANS

Researchers typically assess ANS acuity by displaying two collections of dots and asking subjects to determine which collection is larger. Researchers can then measure acuity in terms of accuracy (percentage of correct responses in identifying the larger collection), but theories about ANS suggest at least two additional measures: a subject's Weber fraction and their numerical ratio. The Weber fraction is based on the idea that estimates of sizes of collections follow a normal distribution, and a subject's Weber fraction is the standard deviation of the normal curve that best fits the subject's responses (Barth et al., 2006). The numerical ratio is based on the idea that comparisons between collections worsen as their sizes become closer (the numerical distance effect; Sekuler & Mierkiewicz, 1977), and a subject's numerical ratio is the largest fraction m/n for which a subject can reliably discern a collection of m dots from a larger collection with n dots (the closer to 1, the stronger the subject's acuity).

Although the ANS explains some phenomena in comparing and estimating sizes of collections, various characterizations and measures of ANS do not hang together coherently. For example, it appears that the Weber fraction and the numerical ratio effect measure different, but related, constructs. In a study of 49 adults and 56 children, Inglis and Gilmore (2014) found weak correlations between measures of numerical ratio and measures of both Weber fraction and accuracy. They also found unacceptably low test-retest reliability in measures of subjects' Weber fractions and numerical ratios. On the other hand, test-retest reliability of accuracy was acceptable, and accuracy was strongly associated with Weber fraction.

Questions about the ANS construct persist, even when reliable measures of accuracy are used. For example, results from Rousselle, Palmers, and Noël (2004) suggest that preschool children rely on perceptual cues (e.g., the surface area taken up by the collections of dots), rather than numerosity, to solve ANS tasks. Furthermore, if ANS were the basis for number, we would expect a neat relationship between numerical/arithmetic development and ANS acuity, but results are mixed (e.g., De Smedt, Noël, Gilmore, & Ansari, 2013; Libertus, Feigenson, & Halberda, 2011; Lyons, Ansari, & Beilock, 2012). For example, in a study of children between 4 and 7 years old, Soltész, Szűcs, and Szűcs (2010) found that "verbal counting knowledge and performance on simple arithmetic tests did not correlate with non-symbolic magnitude comparison [ANS task performance] at any age" (p. 12).

Collectively, results suggest that ANS does describe a valid psychological construct but that researchers may need to re-conceptualize models of it. We offer one possible re-conceptualization. First, we examine assumptions about how the ANS—however conceptualized—supports the construction of number.

3.3.2 ANS and Number

Like subitizing (the focus of Chap. 2), the ANS involves an apprehension of magnitude that does not rely upon counting. Magnitude is rarely explicitly defined in ANS research (though we discuss an exception with regard to the mental number line, below). In the context of dot comparison tasks, it appears to refer to numerosity, or "the number of objects in a set" (Dehaene, Molko, Cohen, & Wilson, 2004, p. 218). We agree that, from the researchers' perspective, ANS tasks involve number. However, we question the assumption that infants (and non-human animals) share that experience. Mathematics educators, in general, have learned to be vigilant against the fallacy of attributing their own mathematics to their students (e.g., Steffe & Tzur, 1994).

A related assumption about the ANS is that it maps onto an exact knowledge of number, as represented by number names and Arabic symbols (Dehaene, 1992). This assumption does not account for the role of children's activity in constructing number (Simon, Placa, & Avitzur, 2016). Nor does it explain why the Intraparietal Sulcus (IPS) is prominently implicated in neural correlates of both ANS (Ansari, 2008) and manual tool use (see Norton, Ulrich, Bell, & Cate, 2018, for a summary of research connecting the hands with mathematical development). Furthermore, the assumption runs counter to evidence that comparing symbolic and non-symbolic magnitudes is a non-trivial process, even for adults: "Data suggest instead that a numeral does not provide direct access to an approximate sense of the quantity it represents" (Lyons et al., 2012, p. 639).

Dehaene (1992) has described the ANS as a language-independent system for developing an understanding of number. As mathematics educators, we agree that children develop a sense of number that does not necessarily rely on language or symbols. Rather, language and symbols serve to support mathematical development by referencing existing concepts (Gravemeijer, Lehrer, van Oers, & Verschaffel, 2013). In fact, the mathematics education community at large warns against an educational focus on rote, or procedural knowledge, wherein students might memorize multiplication tables or manipulate algebraic symbols with no reference to underlying concepts (e.g., Hiebert & Lefevre, 1986). However, the concepts that number words and Arabic symbols refer to are not innate; they require years of mathematical experience to develop. As reported by Laski and Siegler (2007), experience supports even the development of numerical estimation skills that ANS tasks assess.

In addition to eschewing language and symbols, models of the ANS tend to dismiss the role of sensorimotor activity in constructing number: "This mechanism departs significantly from Piaget's (1952, 1954) sensorimotor scheme in allowing for internal learning to take place purely by mental experiment, without any overt action of the organism on the external world" (Dehaene & Changeux, 1993, p. 404). However, research in mathematics education consistently demonstrates the critical role of sensorimotor activity in children's development of number, and in mathematics in general (e.g., Baroody, Lai, & Mix, 2006; Sarama & Clements, 2009). As illustrated in the opening sections of this chapter, counting is much more than a language game, requiring, for example, that children create unit items and establish a one-to-one correspondence between actions of pointing to those items and reciting number words. Such findings are the basis for educators' use of manipulatives, such as two-color counters and base-10 blocks, in elementary school classrooms (Clements, 2000).

Research in mathematics education is buttressed by studies in cognitive psychology and neuroscience demonstrating close connections between sensorimotor activity and mathematical development. In particular, several studies indicate that finger gnosis—a sense of location of one's fingers—is an early predictor of mathematical achievement (Noël, 2005; Rusconi, Walsh, & Butterworth, 2005; Soylu, Lester, & Newman, 2018), and the relationship between fingers and number knowledge persists into adulthood, well beyond finger counting (Sato, Cattaneo, Rizzolatti, & Gallese, 2007). Penner-Wilger and Anderson (2013) point to such findings in positing that areas of the brain evolved for manual dexterity have been repurposed to support numerical development, suggesting a strong evolutionary link between manual and numerical digits. Researchers of embodied cognition have taken advantage of these connections to design sensorimotor interventions in support of children's numerical development, even coining the term "manumerical cognition" (Wood & Fischer, 2008). One particularly relevant intervention study engaged children in full-body motion in response to numerical comparison and estimation tasks (Fischer, Moeller, Bientzle, Cress, & Nuerk, 2011). The researchers found that this sensorimotor-trained group made substantial gains on paper-and-pencil assessments of ANS acuity and mathematical achievement, significantly beyond that of a control group, which received training on a computer.

These studies suggest that sensorimotor activity supports the development of ANS acuity and mediates its role in supporting the construction of number. Although research on the ANS provides evidence for pre-linguistic knowledge of magnitude, the preponderance of evidence from mathematics education, cognitive psychology, and educational neuroscience suggests that sensorimotor activity is critical for numerical development. Before reframing ANS within our own model of numerical development, which accounts for sensorimotor activity, we consider two existing models of ANS itself: the accumulator model and the mental number line.

3.3.3 The Accumulator Model

Gallistel and Gelman (2000) describe the ANS as a non-verbal counting process. They use the metaphor of a fluid beaker with discrete rulings to describe this process. The beaker accumulates a unit of volume (a cup) for each instance of a perceived object, so that each counted item is coded as another step on the beaker. The total volume of the beaker is presumed exact, but memory introduces a "sloshing" effect. The signals that encode and later retrieve the magnitude representation lead to recalled representations that are distributed around the true numerical description of the magnitude.

> The non-verbal representations of number are mental magnitudes (real numbers) with scalar variability. Scalar variability means that the signals encoding these magnitudes are 'noisy'; they vary from trial to trial, with the width of the signal distribution increasing in proportion to its mean. (Gallistel & Gelman, 2000, p. 59)

The accumulator model fits the original intent of the ANS—"[to describe] how a continuum of sensation, such as loudness or duration, is represented in the mind" (Dehaene, 2003, p. 145). However, because the continuum of sensation includes experiences with no given unit (such as loudness), we should not expect the accumulator to produce exact integer values in the first place. Inglis and Gilmore (2013) present results suggesting that it does not.

In a study of 12 adults who performed 400 trials of dot comparisons, Inglis and Gilmore (2013) tested the implicit assumption that the non-verbal counts in the accumulator model do not depend on duration of exposure to the stimuli. Contrary to this assumption, the researchers found that accuracy in dot comparison tasks improved as exposure increased. "We propose that when an individual observes a numerical stimuli, rather than taking a single sample from this [normal] distribution, they actually take many (the number determined by a function of display time) and use the mean as the resultant ANS representation" (Inglis & Gilmore, 2013, p. 67). A normal curve based on multiple samples fit the data better than the single-sample model and would not require any of the samples to be exact.

As particularly striking examples of the ANS phenomenon, Dehaene (1997) summarized results from a sequence of studies on rats (e.g, Mechner, 1958). These rats were conditioned to press a lever a specific number of times before pressing a second lever to receive food. The rats' performances roughly fit a normal curve centered on the accurate response, as the accumulator model would predict. Moreover, while increased hunger affected the rate of lever presses, it did not affect the number of lever presses. In a related study in which rats chose between two levers in response to an external stimulus—sound or light pulses—rats tended to choose the correct corresponding lever, analogous to the way humans solve dot comparison tasks (Church, 1984). The rats even seemed to collectively accumulate units for sounds and light: When two pairs of simultaneous light and sound pulses were presented, the rats preferred the lever corresponding to 4. This latter finding came as a shock to the researchers who took it as evidence for rat addition, but just as plausibly, it could be taken as evidence that the rats were neither adding nor counting.

Any concept of count or number relies on identifying a unit and treating perceived items as if they were identical instantiations of that unit. The rats were conditioned to attend to certain stimuli—pulses of light and pulses of sound—but what constituted the unit in the paired light/sound pulses? It is clear that the rats were not counting the two pairs of stimuli as units, as the researchers had expected. Instead, it would appear they were responding to an experience of intensity from the collective stimuli (light and sound), separate from but similar to their experience of hunger. As Dehaene (1997) noted, the rats pressed levers more quickly with increased hunger, but we would not take this as evidence of proportional reasoning—only as an association. Likewise, the rats seemed to associate the intensity of their experience of stimuli with an intensive response in lever presses—an association that achieved less than 30% accuracy, even for small numbers like 4 (Mechner, 1958).

As we have noted, over-attribution is a common concern in mathematics education research. Our own mathematical knowledge is vital to understanding the mathematics of our students, but we cannot assume that students structure experience the same way that we do or that students' mathematics is a subset of our own (Hackenberg, 2005; Steffe & Tzur, 1994; Ulrich, Tillema, Hackenberg, & Norton, 2014). We turn next to the mental number line, for which assumptions highlight the potential fallacy of ignoring this caveat.

3.3.4 The Mental Number Line

In describing the mental number line, Gallistel and Gelman (2000) define mental magnitudes as real numbers. Conflations between number and magnitude, along with assumptions about innate numerical ability, lead to claims that obscure students' active role in constructing number. For example, if we take mental magnitudes as real numbers on the mental number line, we get claims like the following:

> Irrational numbers can only be defined rigorously as the limits of infinite series of rational numbers, a definition so elusive and abstract that it took more than two thousand years to achieve—an arduously reached pinnacle of mathematical thought. We suggest that the scaling of this pinnacle was a Platonic rediscovery of what the non-verbal brain was doing all along—using arithmetically processed magnitudes to represent both countable and uncountable quantities. These noisy mental magnitudes are arithmetically processed—added, subtracted, multiplied, divided and ordered. (Gallistel & Gelman, 2000, p. 60)

When Euclid, Plato, and other ancient Greeks looked at the continuum, they did not see an uncountable collection of limits of sequences of rational numbers. The continuum was given, as primary, and its quantification was secondary, through geometric construction (Brouwer, 1998). Plato developed simple rules for geometric construction using a compass and a straight edge: swinging circles centered at one point and passing through another, drawing points through pairs of lines, and marking points of intersection. Euclid explicitly built on Platonic rules for construction, including them as the first three postulates (axioms) in Euclidean geometry. He defined a line as "a breadthless length" (as cited in Joyce, 1996).

The line was a gapless continuum on which numbers could be constructed, beginning from a unit of 1. Constructible numbers include positive integers, positive rational numbers, and even some irrational numbers like the square root of 2. Note that, because numbers were lengths of segments, negative numbers did not exist. Although pi can be constructed as the length of a circumference, it is not constructible (from a unit of 1) as a segment length and is therefore not a constructible number. Ancient Greek mathematicians (and later Western mathematicians) spent centuries attempting to construct numbers like pi and the cubed root of 2, not knowing (until Galois) that their efforts were futile.

If children have an innate mental number line, there is no reason to believe that it comes complete as an uncountable collection of Dedekind cuts or "limits of infinite series of rational numbers." Rather, it might appear as a breadthless length and a medium for constructing number, as it did for the ancient Greeks. In his own critique, Núñez (2011) argued against the existence of an innate number line by noting that the ancient Babylonians had developed a particularly sophisticated number system (a base-60 system that included fractions) with no appeal to a number line. "In the absence of a clear number line depiction or narrative, simply because we see numbers, magnitudes, and lines on clay tablets we cannot anachronistically infer that Old Babylonians operated with a number line mapping or with a mental number line representation" (p. 655). Likewise, we cannot impose our mathematical structures on the conceptions of children whose mathematical development will depend on their own experience.

3.4 Reframing the ANS

The accumulator model assumes that numbers exist as innate psychological constructs, shared by humans and other animals. The mental number line model equates the continuum with real numbers. Critiques of these models and the ANS construct itself (e.g., Inglis & Gilmore, 2014) suggest that researchers have attributed their own mathematical structures to their subjects while conflating magnitude and number. In our own framing, we argue that the ANS points to an important pre-linguistic psychological construct undergirding magnitude but that numerical development depends upon many layers of abstraction of sensorimotor experience.

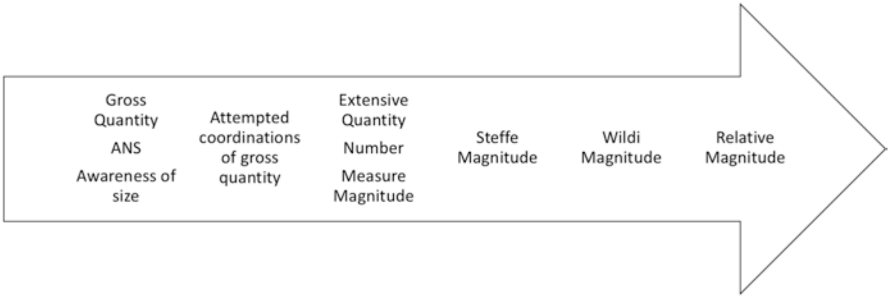

Fig. 3.2 Continuum of quantity/magnitude awareness

We reframe ANS and number as representing different levels of magnitude awareness (see Fig. 3.2) of the numerosity of a collection. We refer to the associated gross quantity as *gross numerosity* and the associated extensive quantity as number. ANS describes representations of gross numerosity. As such, both ANS and number are related to measures of the size of a collection, but ANS develops very early (or is innate; e.g., Xu & Spelke, 2000) and may rely primarily on perception. In contrast, the construction of number requires numerous abstractions from sensorimotor activity in coordination with perceptual cues from the environment.

In his framework for explaining general mathematical development, David Tall (2004) described three interrelated worlds of mathematics: one based on perception, another based on action, and a third based on formal properties (such as definitions and symbol systems). Within the triple-code model for number, ANS might rely on cues from the perceptual world, onto which number words/symbols from the formal world might later map. The world of action is missing from this model. In contrast, our framing casts ANS as a possible starting point for sensorimotor activity, which can move learners along the continuum of magnitude/quantity awareness.

This reframing of ANS as gross numerosity explains why differing methods for measuring ANS might lead to conflicting results. For example, when numerals are used in ANS studies (e.g., Lyons et al., 2012), number is presumably already constructed by the subject. Such tasks would therefore measure a different construct than tasks that only rely on comparisons of collections of dots, which would only require the construction of gross numerosity.

Additionally, acknowledging the laborious process of unitizing, as well as other abstractions from sensorimotor activity that lead to the construction of countable units by children, supports the suggestions by some psychological researchers that infants in ANS studies are not attending to numerosity at all, but rather to a continuous quantity, such as area or density (Rousselle et al., 2004). Even gross numerosity would require the construction of countable units in sets, however gross continuous quantities, such as area and density, do not require the same unitizing operations. If we adopt the view that children's mathematics might differ from our mathematics, we can accept constructs like gross quantity that explain infant's behavior without attributing real numbers to infants.

Mathematics educators rarely focus on pre-numerical ways of operating, and so psychological research on the ANS and other pre-numerical constructs complement the work of mathematics educators nicely if reframed in terms of how children develop gross quantities and other pre-numerical ways of operating. For example, studies on acuity of the ANS could be reframed to answer the question of how the ability to compare gross quantities can improve without a measurement system. Studies that investigate the relationship between the ANS and formal uses of number (e.g., De Smedt et al., 2013; Libertus et al., 2011) could be reframed to answer the question of how our awareness of magnitude and our ability to measure magnitude are interrelated: Does acuity in comparisons of gross quantities (ANS) predict a quicker development of a measurement system (number)? Does experience with a measurement system improve acuity with the underlying gross quantity?

3.5 Concluding Remarks

In this chapter, we have highlighted two key issues: (1) how our sense of a quantity is intertwined with, but distinct from, our sense of number and (2) the over-attribution of our own mathematical models of situations to the research subjects with whom we work. We have argued that ANS research often conflates an emergent sense of a quantity—be it of area, numerosity of a collection, or some other emergent quantity—with our advanced sense of that quantity as something measurable with number. We have also acknowledged that our own work, as mathematics educators, has neglected the study of the underlying non-numerical operations that contribute to the development of numerical operations.

A closer attention to these two key issues, which could be reframed as disambiguating measurement from quantity and disambiguating our own mathematical thought from the way the research subject experiences a situation, could help our two fields in communicating about research on numerical and pre-numerical operations in other situations. These issues are quite difficult and only become magnified when moving to understanding rational numbers and integers. Consider the development of counting laid out in this chapter and how intertwined it is with a child's goal of making definite the indefinite numerosity of a collection. What types of measurement goals would provoke the development of rational numbers or the real numbers?

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Chapter 4
Spontaneous Mathematical Focusing Tendencies in Mathematical Development and Education

Jake McMullen, Jenny Yun-Chen Chan, Michèle M. M. Mazzocco, and Minna M. Hannula-Sormunen

Abstract A growing body of evidence reveals the need for research on, and consideration for, children's and students' own—self-guided—spontaneous use of mathematical reasoning and knowledge in action. Spontaneous focusing on numerosity (SFON) and quantitative relations (SFOR) have been implicated as key components of mathematical development. In this chapter, we review existing research on SFON and SFOR tendencies in the broader context of the development of mathematical skills and knowledge and examine how the state-of-the-art evidence on SFON and SFOR is relevant for the field of mathematics education. We discuss individual differences in SFON and SFOR, associations between spontaneous focus on mathematical features and mathematics achievement, the contributions of situational contexts that implicitly prompt attention to number, and ways to increase children's focus on number regardless of their baseline level tendencies. We conclude that children's and students' tendencies to focus on number and quantitative relations–spontaneous or otherwise–are key components of mathematical development and education.

Keywords Spontaneous focusing on numerosity (SFON) · Spontaneous focusing on quantitative relations (SFOR) · Mathematical thinking · Numerical salience · Contextual influences · Individual differences · Early mathematics

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4.1 Spontaneous Mathematical Focusing Tendencies in Mathematical Development and Education

The everyday world is rich with mathematical features, and attending to these features is useful, often necessary, and can even be highly engaging. Indeed, mathematics educators emphasize the value of teaching mathematical modelling skills that allow students to apply their mathematical knowledge in everyday and work-life situations (Mullis, Martin, Goh, & Cotter, 2016), something that is highly relevant for most middle- and high-quality jobs (Advisory Committee on Mathematics Education, 2011). However, in order to make use of mathematical features in everyday situations, an individual—needs to recognize—often without external guidance—that mathematical aspects of a situation are present and relevant to begin with (Lehtinen & Hannula, 2006; Lobato, Rhodehamel, & Hohensee, 2012). Individuals with a higher tendency to recognize and use mathematical features of everyday situations may acquire more self-initiated practice. This is developmentally relevant because many opportunities to learn or practice mathematical behavior occur outside of formal mathematical learning contexts.

The tendency to recognize and focus on mathematical features when not explicitly guided to do so is not always automatic, even among individuals who possess the relevant underlying mathematical knowledge (Batchelor, Inglis, & Gilmore, 2015; Chan & Mazzocco, 2017; Hannula & Lehtinen, 2005; McMullen, Hannula-Sormunen, Laakkonen, & Lehtinen, 2016). Across a range of studies, researchers have shown that individuals differ in their tendency to focus on mathematical aspects of situations that are not *explicitly* mathematical, like copying a drawing of flowers or imitating someone else feeding a puppet. Not all individuals notice, or use, numerical information when, for instance, reproducing how many petals are on the flowers. Those individuals who are more likely to do so have been shown to have an advantage in learning formal mathematical skills and knowledge (e.g., Hannula & Lehtinen, 2005; Hannula-Sormunen, Lehtinen, & Räsänen, 2015). Mathematics educators in preschool and primary school, as well as their students, may benefit from considering these individual differences in spontaneous mathematical focusing tendencies. In this chapter, we argue that tendencies to spontaneously focus on mathematical features explain at least some of the individual differences observed in the development of mathematical thinking (e.g., Gray & Reeve, 2016; Hannula-Sormunen et al., 2015; McMullen, Hannula-Sormunen, & Lehtinen, 2017; Nanu, McMullen, Munck, Hannula-Sormunen, and Pipari Study Group, 2018; Van Hoof et al., 2016), and that promoting these tendencies across different contexts may improve specific aspects of mathematical learning and performance (Hannula, Mattinen, & Lehtinen, 2005; McMullen, Hannula-Sormunen, Kainulainen, Kiili, & Lehtinen, 2017).

4.2 What Are Spontaneous Mathematical Focusing Tendencies?

Thus far, most research examining spontaneous mathematical behavior in preschool and school-age children has focused primarily on how young children spontaneously focus on numerosity. The tendency of spontaneous focusing on numerosity (SFON) is defined as follows:

> a process of spontaneously (i.e., in a self-initiated way not prompted by others) focusing attention on the aspect of the exact number of a set of items or incidents and using of this information in one's action. SFON tendency indicates the amount of a child's spontaneous practice in using exact enumeration in her or his natural surroundings. (Hannula, Lepola, & Lehtinen, 2010, p. 395).

On a broad level, from early childhood through adulthood, substantial individual differences in SFON tendency have been differentiated from individual differences in related mathematical knowledge and skills (Gray & Reeve, 2016; Hannula-Sormunen, Nanu, et al., 2015, Hannula-Sormunen, Nanu, Laakkonen, Munck, Kiuru, Lehtonen, and Pipari Study Group, 2017; Hannula & Lehtinen, 2005; Hannula et al., 2010; Hannula, Räsänen, & Lehtinen, 2007; McMullen, Hannula-Sormunen, & Lehtinen, 2015; Rathé, Torbeyns, Hannula-Sormunen, & Verschaffel, 2016; Sella, Berteletti, Lucangeli, & Zorzi, 2016), despite their positive correlation with those skills (Hannula et al., 2010, 2007; Hannula & Lehtinen, 2005; Hannula-Sormunen et al., 2015; McMullen et al., 2015; Nanu et al., 2018). Although focusing on exact number is often relevant in a situation, there are also situations in which focusing on quantitative relations is more relevant than exact number (Singer-Freeman & Goswami, 2001; Sophian, 2000; Spinillo & Bryant, 1991). For example, a child might spontaneously notice there are two apples and four bananas in a bowl of fruit—a SFON behavior. However, that same child might then go on to notice that there are twice as many bananas as apples, or that one-third of the pieces of fruit are apples, exhibiting what could be described as spontaneous focusing on quantitative relations (SFOR). There are limitations to what natural numbers can represent in the real world, and those limitations underlie the need for rational numbers (Vamvakoussi, 2015). Focusing solely on numerosity may not be sufficient or appropriate in many such situations (Boyer, Levine, & Huttenlocher, 2008). For example, to equally divide two bananas among 3 persons, it is not possible to express the outcome with natural numbers. Thus, recent studies have examined the role of SFOR in mathematical development (e.g., McMullen et al., 2016; Van Hoof et al., 2016). Whereas SFON tendency reflects paying attention to a single quantity or numerosity and using it in action, SFOR tendency reflects recognizing and using mathematical relations between two or more quantities.[1]

[1] It should be noted that, at the moment, we do not distinguish between different aspects of quantitative relations, though most existing research examines either multiplicative relations with late primary school students or part-whole relations in preschoolers. SFOR tasks usually include discrete quantities and underlying exact numbers are a foil and/or a prerequisite for focusing on the

Importantly, the spontaneity indicated by SFON and SFOR tendencies does not refer to the spontaneous acquisition of skills or knowledge nor an innate nature to their origins (Hannula, 2005; Lehtinen & Hannula, 2006). Instead, the spontaneous nature of these tendencies refers to the unguided, self-initiated nature of the recognition and use of numerical features within a specific moment or situation (i.e., without external prompting). This means that some background skills and knowledge are requisites of SFON and SFOR tasks, and that these tendencies should respond to formal and explicit teaching of focusing on mathematical aspects across contexts (Hannula, 2005; McMullen, 2014).

In the following sections, we review the theory and methods around spontaneous mathematical focusing tendencies and their relation to requisite cognitive skills such as mathematical knowledge or attention, and to contextual factors such as social expectations or demands. In order to fully understand children's everyday mathematical behavior, it is crucial that all three factors are taken into account. We argue that there are complex concurrent and developmental relations among these three constructs, which we illustrate using a schematic representation (Fig. 4.1). Based on the literature, we argue that spontaneous mathematical focusing tendencies can be distinguished from other cognitive (Fig. 4.1b) or contextual (Fig. 4.1d) factors, and we depict the relation between these constructs with overlapping yet distinct circles. We summarize existing evidence for the iterative, developmental relations between spontaneous mathematical focusing tendencies and both the cognitive requisite skills (Fig. 4.1a), and contextual factors (Fig. 4.1e) that exist throughout mathematical development. Finally, we argue that in order to understand the full extent of mathematical behavior of preschool and school-age children, the intersection of all three circles (Fig. 4.1c) should be given serious consideration by researchers and educators.

4.3 Delineating SFON and SFOR from Requisite Skills

Any expression of SFON or SFOR tendency requires the use of the mathematical and domain general cognitive skills to solve a task (Fig. 4.1b). For instance, individuals need to fully attend to the situation or task at hand. Other factors, such as a disposition towards math (finding it useful, interesting, or important) may also affect when and how individuals spontaneously attend to mathematical features. Studies using the original SFON tasks have already shown that it is possible to reliably and uniquely measure the strength of children's SFON tendency (Hannula & Lehtinen, 2005; Hannula-Sormunen et al., 2015; Nanu et al., 2018), and several other measures have more recently been developed and thus contribute to the repertoire of SFON assessments (see Rathé, Torbeyns, Hannula-Sormunen, De Smedt, & Verschaffel, 2016 for an extensive review). Likewise, SFOR tendency can be reliably measured in a number of tasks in both early childhood and late primary school (McMullen et al., 2016; Van Hoof et al., 2016).

relational aspects of the task.

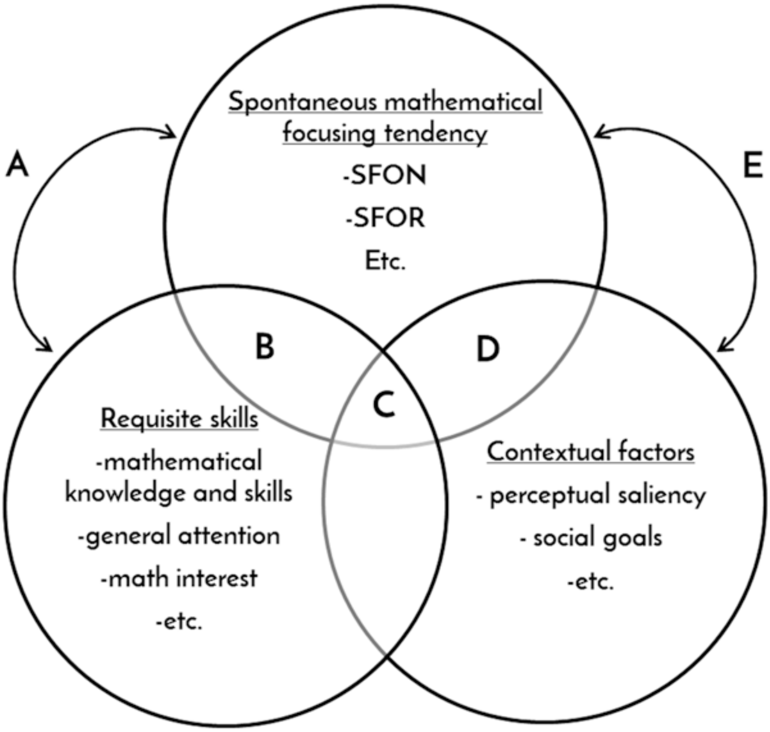

Fig. 4.1 Interrelations between spontaneous mathematical focusing tendencies, related skills, and general cognitive, meta-cognitive, and affective factors, which are present in any situation where a person uses or recognizes exact number or quantitative relations. This diagram is a schematic proposal, and the degree of overlap across these constructs is unknown and therefore should not be considered "to scale" with the figure

In an attempt to ensure that spontaneous mathematical focusing tendency measures truly capture this tendency independently from other factors, tasks must meet the following design principles: the task should be (1) mathematically unspecified, (2) open for multiple (mathematical and non-mathematical) possible interpretations, (3) fully engaging for all, and (4) within the range of competences (Hannula, 2005; Hannula & Lehtinen, 2005). The first two principles ensure that the context does not provide hints or constraints to numerical responses, so that the participants' focus on numerosity is spontaneous. There should be no hints that numerical responses are intended, and tasks and materials should not be associated with typical counting or numerical exercises. In this way, the probability of producing numerically accurate response without spontaneously focusing on numerosity is low. The last two principles diminish the likelihood that other potential factors (e.g., general attention or mathematical knowledge) explain the individual variation in SFON tendency. Specifically, the task needs to capture and maintain the child's attention. The numerical sets included in the tasks should be small enough for participants to reliably enumerate. Other cognitive and motor demands, such as verbal production, working memory, response inhibition, and motoric imitation, must be age-appropriate (Hannula, 2005). Satisfying these four conditions strengthens the validity of the SFON tasks and the interpretation that individual differences in SFON scores accurately reflect individual differences children's SFON tendency. It is important to note that additional numerical behaviors, such as counting or commenting on the set size, are indicators of SFON, even if the set produced by the child does not match numerically to the examiner's set.

A number of studies have shown that variation in students' performances on SFON and SFOR tasks is not entirely explained by the mathematical or other cognitive skills needed to solve the tasks (Hannula & Lehtinen, 2005; Hannula-Sormunen et al., 2015; McMullen et al., 2016; McMullen, Hannula-Sormunen, & Lehtinen, 2014). For example, many 6-year-old participants in early studies did not spontaneously focus on numerosity during SFON tasks, but almost all children were able to use the exact numbers in their actions when explicitly guided to do so (Hannula & Lehtinen, 2005). In a more recent study, there was clear discriminant validity separating SFON tendency and verbal counting skills although there was some overlap between 6-year-olds' performance on six tasks measuring these two constructs (Hannula-Sormunen et al., 2015). Nanu et al. (2018) showed that response patterns in the SFON tasks were significantly different from typical response patterns measuring enumeration skills. The findings from these studies support the claim that SFON tasks capture individual differences in SFON tendency rather than enumeration accuracy.

SFOR tendency has also been examined in relation to the requisite skills needed to solve tasks using exact quantitative relations (McMullen et al., 2014; McMullen et al., 2016). In a sample of US students in kindergarten to third grade (McMullen et al., 2014), substantial individual differences in SFOR tendency, both within and across grade levels, were not entirely explained by the requisite mathematical skills needed to complete the tasks. In a more recent study, third to fifth graders in Finland completed three paper-and-pencil measures of SFOR tendency, and then completed one item from each task in guided format (McMullen et al., 2016). Since all participants completed the guided versions of these tasks, it was possible to statistically account for the students' guided performance. A "pure" SFOR tendency variable was calculated using residualized scores for SFOR responses adjusted for performance on the guided versions of the tasks. This statistical procedure effectively removes the overlap between SFOR and requisite skills (Fig. 4.1b). Even after taking into account students' guided performance, substantial individual differences in SFOR tendency remained, within and across grade levels.

Although previous studies have directly juxtaposed SFON and SFOR tendencies with the task-relevant requisite mathematical skills and knowledge, fewer studies have explicitly focused on how other cognitive, meta-cognitive, and affective aspects of mathematical development are related to SFON and SFOR tendencies (e.g., Hannula et al., 2010; Van Hoof et al., 2016). Instead, these aspects of mathematics development are more often used as control measures in SFON and SFOR studies to examine whether they explain associations between SFON and SFOR tendencies and mathematical skills. Across several such studies, the relation between SFON and mathematical skills remained significant, even after controlling for age and cognitive skills including full scale IQ (Nanu et al., 2018), verbal IQ (Poltz et al., 2013), or non-verbal IQ (Hannula et al., 2010; Hannula & Lehtinen, 2005; Hannula-Sormunen et al., 2015; Poltz et al., 2013); rapid serial naming (Hannula et al., 2010), working memory (Batchelor et al., 2015; Nanu et al., 2018; Poltz et al., 2013), inhibition (Poltz et al., 2013), executive function skills and vocabulary (Gray & Reeve, 2016), verbal comprehension (Hannula et al., 2010; Hannula & Lehtinen, 2005), verbal production skills (Batchelor et al., 2015), and spatial location detection (Hannula et al., 2010).

Fewer studies have focused on the relation between SFOR tendency and other cognitive factors related to mathematical development, but the evidence to emerge thus far implicates that SFOR tendency does overlap, to some extent, with students' mathematical skills and knowledge, along with other related cognitive skills. SFOR tendency does appear to be a unique component of mathematical cognition, and it remains a significant predictor of rational number knowledge and development when controlling for non-verbal intelligence (McMullen et al., 2016; McMullen, Hannula-Sormunen, & Lehtinen, 2017; Van Hoof et al., 2016). This relation between SFOR tendency and rational number knowledge and development is not explained by grade level, arithmetic fluency, whole number estimation, guided focusing on quantitative relations, mathematical achievement, spatial reasoning, or interest in mathematics (McMullen, Hannula-Sormunen, Lehtinen, & Siegler, submitted; McMullen et al., 2016; Van Hoof et al., 2016).

To summarize, as represented in Fig. 4.1, SFON and SFOR tendencies overlap with mathematical or other cognitive skills required in a given situation, but neither SFON nor SFOR is entirely explained by these requisite skills. In short, there is substantial evidence supporting that both SFON and SFOR tendencies are unique aspects of mathematical cognition.

4.4 The Relation Between SFON/SFOR and Mathematical Development

We now review studies showing how SFON and SFOR are related to mathematical development (Fig. 4.1a). We propose that SFON and SFOR tendencies are indicators preschool and school-age children's spontaneous mathematical activities in and out of the classroom (Hannula et al., 2005). We hypothesize that preschool and school-age children who have higher SFON and SFOR tendencies more readily recognize the mathematics embedded in everyday life, compared to children with low SFON and SFOR tendencies, and that through this increased awareness they gain more opportunities to practice their mathematical skills. This increased self-initiated practice helps students deepen their mathematical knowledge, and the deeper mathematical knowledge subsequently supports further development of spontaneous mathematical focusing tendencies. Thus, SFON and SFOR have a bi-directional and iterative developmental relation with related mathematical skills and knowledge.

Several studies demonstrate the relation between SFON or SFOR tendencies and the development of mathematical skills. Of the two, SFON tendency has a stronger evidence base, because across studies SFON has been linked to a broader range of mathematical abilities (Batchelor et al., 2015; Edens & Potter, 2013; Gray & Reeve, 2016; Hannula et al., 2010; Hannula & Lehtinen, 2005; Hannula-Sormunen et al., 2015; Kucian et al., 2012; McMullen et al., 2015; Nanu et al., 2018; Poltz et al., 2013). Still, there is a growing evidence base for the relation between SFOR tendency and mathematical development (McMullen et al., submitted; McMullen et al., 2014; McMullen et al., 2016; McMullen, Hannula-Sormunen, & Lehtinen, 2017; Van Hoof et al., 2016). There are differences and similarities in SFON and SFOR associations with domain-specific correlates of mathematical development (McMullen et al., submitted; Hannula et al., 2010). Relative to SFOR, SFON has been more consistently and more strongly associated with whole number enumeration and arithmetic skills (e.g., Hannula et al., 2010), whereas SFOR tendency has been more closely associated with rational number knowledge (e.g., Van Hoof et al., 2016). These results suggest that SFON and SFOR tendencies may each play a specific role in mathematical development.

The effects of prior knowledge on mathematical development are well acknowledged throughout topics, from early counting development to rational numbers and algebra (Siegler et al., 2012; Siegler, Thompson, & Schneider, 2011; Sophian, 1988). In order to pay attention to the mathematical aspects in and out of the classroom, students need to have at least some knowledge about where and when to apply their formal knowledge (Lehtinen & Hannula, 2006; Lobato et al., 2012). In early childhood, SFON tendency has been found to be supported by earlier enumeration and subitizing skills (Hannula & Lehtinen, 2005; Hannula-Sormunen et al., 2015). In fact, SFON tendency and enumeration skills were found to be in a reciprocal relation, with each predicting the other over time (Hannula & Lehtinen, 2005). SFOR tendency and rational number knowledge were found to follow a similar pattern of reciprocity, in which early SFOR tendency predicted later rational number knowledge, and vice versa (McMullen, Hannula-Sormunen, & Lehtinen, 2017). These iterative processes (Fig. 4.2) suggest that there is a strong link between formal and typically examined mathematical skills and knowledge and children's and students' spontaneous mathematical behavior.

The above described results all are important indicators of a potential link between SFON and SFOR tendencies and mathematical knowledge. Nonetheless, a causal link has only been tested in a few limited quasi-experimental studies of SFON tendency (Hannula et al., 2005; Hannula-Sormunen, Alanen, McMullen, & Lehtinen, 2016. One of the first such studies showed that 3-year-olds who participated in a training program that aimed to increase their SFON tendency had long-term gains in their enumeration skills (Hannula et al., 2005). In that study, an increase in SFON tendency led to improvements in later counting skills in these children. More recently, we found that 5-year-olds' arithmetic skills and SFON tendency developed as a result of playing the iPad game Fingu integrated with SFON-based everyday activities (Hannula-Sormunen, Alanen, et al., 2017). The

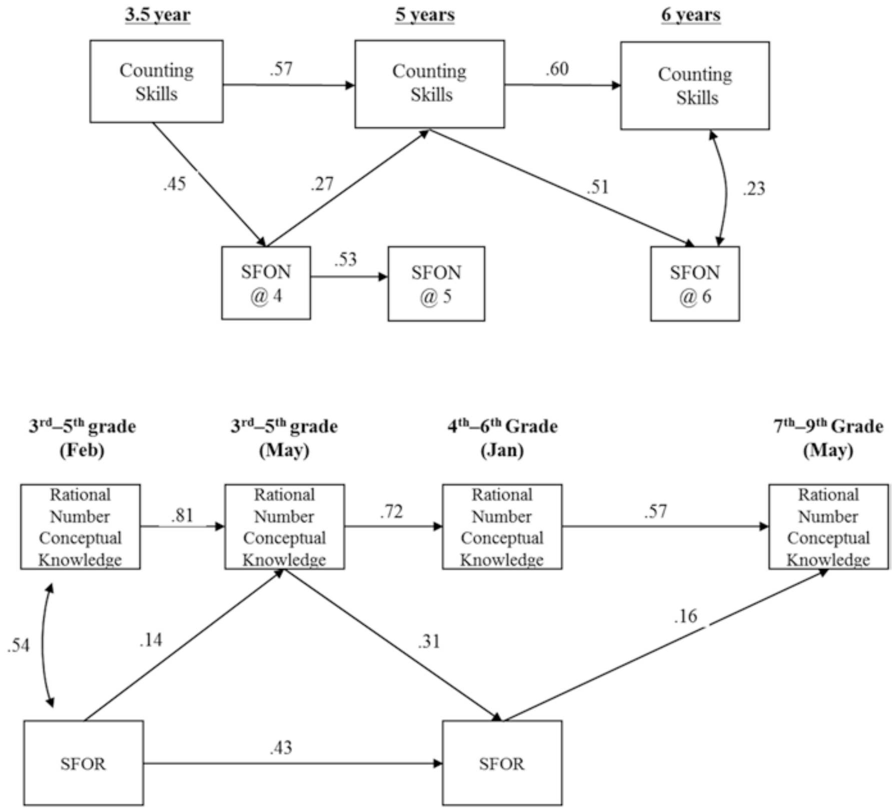

Fig. 4.2 Reciprocal relations between SFON tendency and counting skills (top; modified from Hannula & Lehtinen, 2005) and SFOR tendency and rational number knowledge (bottom; modified from McMullen, Hannula-Sormunen, & Lehtinen, 2017). For all paths, $p < 0.05$

results show a clear developmental advantage for the training group over the control group in arithmetic skills.

Collectively, these studies suggest a causal link between SFON tendency and early numerical skills, with SFON tendency having a positive impact on the development of early counting and enumeration skills. SFOR tendency has been identified as a unique predictor of mathematical development in late primary and early secondary school years. Overcoming minimal transfer effects of easily isolated drill-and-practice kinds of mathematical activities is an important goal for future investigations in mathematics education (e.g., Lehtinen, Hannula-Sormunen, McMullen, & Gruber, 2017). Promoting and supporting students' and children's self-initiated practice of newly learnt mathematical skills may help them start using these skills in their own activities, in addition to adult-guided mathematical exercises. In this way, the SFON and SFOR concepts, assessments, and training activities are of great educational relevance.

4.5 Attentional Considerations in SFON (and SFOR) Research

An essential feature of SFON and SFOR is the unprompted nature of the tendencies. It is important, both theoretically and educationally, to determine what other form "prompts" may take. In other words, to what extent do spontaneous mathematical focusing tendencies and contextual factors overlap (Fig. 4.1d)? Identifying explicit prompts to focus on number is fairly straightforward. Such prompts would involve number or exact quantity in instructions, such as "put the *same number* of cookies on this plate as I have," or "*how many* cookies are there?" or "bring *just enough* socks for Mr. Caterpillar" (Shusterman et al., 2017). These types of prompts are intentionally avoided in SFON and SFOR measures. But what if implicit prompts from *nonverbal* features in instructional materials promote SFON tendencies? For example, what if numerical (or other mathematical) features are more perceptually salient under some conditions, such as crowded versus uncrowded arrangements of item sets or arrays of colorful vs. monochromatic sets (e.g., Chan & Mazzocco, 2017)? The answers to such questions have implications for measuring SFON and SFOR tendencies and for intentionally promoting attention to mathematical features through instruction or the design and use of materials.

Prompts to attend to number or quantitative relations may exist throughout daily routines, but to different degrees depending on the nature of the task at hand. For instance, block play or meal preparation may elicit more attention to and discussion of numbers and mathematics (e.g., to determine the number of plates, forks, napkins, and cups needed for all persons who will be seated at the table) than dramatic play or free form painting at an easel (Chan, Mazzocco, & Praus-Singh, under review; Ferrara, Hirsh-Pasek, Newcombe, Golinkoff, & Lam, 2011; Susperreguy & Davis-Kean, 2016). Likewise, the arrangement of items in SFOR measures may make multiplicative relations more salient than additive relations, even when both would be mathematically correct (Degrande, Verschaffel, & Van Dooren, 2017). Multiple studies suggest that there are individual differences in the use of additive versus proportional reasoning that shift with age, suggesting that the use of one type of relation over the other may develop in concert with other mathematical skills (Van Dooren, Bock, & Verschaffel, 2010).

Although these considerations might be interpreted as challenging the notion of context-independent SFON or SFOR, an alternative perspective is that the relative degree to which these tendencies manifest across children simply interacts with such external influences (Fig. 4.1e). This bi-directional relation would lead to interactions, for example, between SFON tendency and perceptual salience like those demonstrated in earlier studies, and shown in Fig. 4.3. Hannula et al. (2005) found that an intervention based on caregivers' number-focused activities with preschoolers led to greater gains in the preschoolers' SFON tendencies, relative to a control group in which no such number-focused interactions were promoted. Importantly, the effect of this intervention was apparent for only those preschoolers in the experimental group who had at least some measurable SFON tendency at baseline.

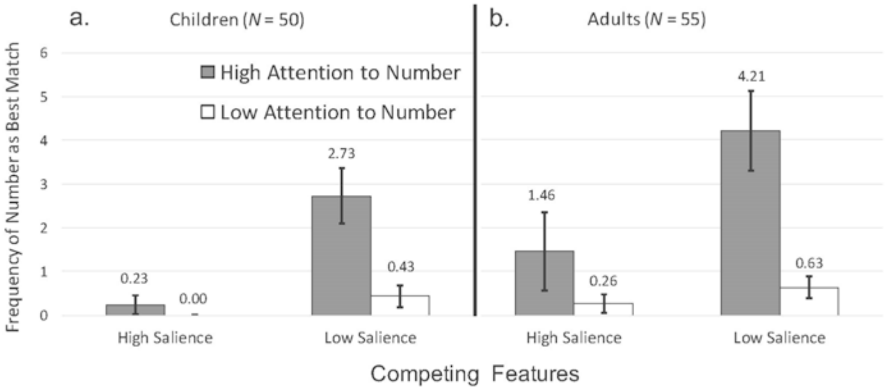

Fig. 4.3 Interactions between the salience of competing features and high versus low attention to number tendencies for children (**a**) and adults (**b**), based on data reported by Chan and Mazzocco (2017). Error bars represent standard deviations. Each salience condition included eight trials on which number was a possible matching feature. Alternative possible matching features were object color or shape (high salience) or object pattern or location (low salience)

Although Chan & Mazzocco (2017) did not measure *baseline* SFON tendencies in their study of picture matching, they found that by manipulating the relative salience of number (as a visual feature of the match options), they could also manipulate the frequency of number-based matches in children and adults during the task. Still, some children and adults *never* matched on number during the task, and in these potentially "low SFON" matchers, there were different effects of perceptual salience across individuals. This general lack of SFON may have inoculated children (and adults) from the main effect of feature salience, as illustrated by the significant interactions shown in Fig. 4.3. This suggests that eliciting mathematical behavior may have promise for promoting children's SFON behavior.

In another study of eliciting SFON behavior, the use of SFON "baits" was the focus of an intervention, in which 2.5–3-year-old children's SFON and small number recognition skills were supported (Hannula-Sormunen, Nanu, Södervik, & Mattinen, in preparation). The program aimed to promote noticing numerical features by embedding SFON baits around the daycare environment. These SFON baits were similar toys and everyday life materials arranged in a manner that made the numerical features very salient (e.g., Fig. 4.4). This often involved using several identical objects arranged in close proximity (e.g., two identical toy cars side-by-side), which increased the likelihood of counting behavior as the items were more likely to be perceived as a set to be counted. If the child did not focus on the numerosity of the items in the SFON bait, the early educators were asked to explicitly guide the child's attention by asking how many items there are, or, by taking away or adding items. Deliberate manipulation of numerosity has proven to be an efficient way of attracting children's attention towards numerosity of items in a set (Hannula et al., 2005). In contrast with previous SFON interventions that were effective only in children with some initial SFON tendency (Hannula et al., 2005),

Fig. 4.4 Examples of SFON-bait used at daycare in the SFON intervention study. (**a**) Two identical trucks arranged side by side on the yard. (**b**) "SFON slippers" with two similar slippers each with three decorating flowers. (**c**) multiple sets of two items (e.g., frying pans, chairs, dolls) arranged at a kitchen play area

the intervention with SFON baits led to significant improvement in SFON tendency and cardinality recognition and production skills particularly among even those with the weakest SFON tendency and cardinality recognition skills at the start of the intervention, in comparison to a control group where the participants received special training in listening comprehension skills (Hannula-Sormunen et al., in preparation).

The notion of contextual influences on SFON, through implicit manipulation of the environment or more explicit verbal prompting, generates testable hypotheses we believe are worthy of empirical pursuit and which we and others have begun to test. Numerous studies have provided evidence for individual stability in SFON and SFOR tendencies across tasks and time (Hannula & Lehtinen, 2005; McMullen, Hannula-Sormunen, & Lehtinen, 2017). Specific investigation into the effects of varying contexts on the expression of spontaneous focusing tendencies within and across tasks would clarify the nature of SFON and SFOR tendencies and provide valuable information about the nature of interventions that can enhance spontaneous or implicitly prompted mathematical focusing tendencies, including additional mathematical features such as spatial characteristics (Chan et al., under review; Degrande et al., 2017).

4.6 Implications for Classroom Practices

Ultimately, SFON and SFOR tendencies are not the end that is sought. Rather, they are a means for understanding individual differences in mathematical behavior in everyday situations, and are argued to be a key component of early mathematics education. Most studies examining mathematical development, teaching, and learning focus on the bottom two circles in Fig. 4.1, namely explicit skills and knowledge and contextual factors contributing to individual differences in these skills and knowledge. However, in order to fully understand the nature of children's and students' mathematical behavior and development, we must look at the intersection of all three circles (Fig. 4.1c), by also taking into consideration children's and students' own spontaneous mathematical activities. Supporting the mathematical behavior situated in this three-way intersection may lead to improvements in spontaneous mathematical focusing tendencies and mathematical skills and concepts.

There is consistent evidence that targeted interventions aimed at enhancing SFON and SFOR tendencies can be successful with both young children and older students (Hannula et al., 2005; McMullen, Hannula-Sormunen, Kainulainen, et al., 2017). In children as young as the age of 3 years old (Hannula et al., 2005), and in interventions as short as 20 min (Braham, Libertus, & McCrink 2018), evidence suggests that it is possible to increase SFON tendency among children. Just a few hours spent with a combination of student- and teacher-led activities over the course of a few weeks led to increases in SFOR tendency in sixth grade students (McMullen, Hannula-Sormunen, Kainulainen, et al., 2017). These results suggest that SFON and SFOR tendencies are malleable, despite the relative consistency in students' and children's performance on SFON and SFOR tasks over time when no intervention has occurred (e.g., Hannula & Lehtinen, 2005).

A key component of applying relevant mathematical concepts in formal and informal settings is recognizing exactly when mathematical aspects are present and useful in reasoning (Lobato, 2012; Lobato et al., 2012). In order to model the world mathematically, a child must first recognize that this can be done (McMullen & Resnick, 2018). In previous training studies aimed at supporting SFON and SFOR tendencies, the main goal was to make number and quantitative relations more explicit targets of focus in students' eyes (e.g., Mattinen, 2006). These programs explicitly highlighted and modelled when and how number and quantitative relations can be used in reasoning in and out of the classroom.

A working assumption regarding the development of SFON and SFOR tendencies is that they are a dimension of the advantages of social norms and practices offered by a rich mathematical home environment on performance in the mathematics classroom (e.g., Skwarchuk, Sowinski, & LeFevre, 2014). Equipping early childhood professionals with knowledge and skills to recognize and support SFON tendency (e.g., Mattinen, 2006) and facilitating peer interaction in small group activities (McMullen, Hannula-Sormunen, Kainulainen, et al., 2017) were effect means to increase SFON and SFOR tendencies. An in-depth analysis of behaviors among groups of students suggested that interaction between individuals can create mutual targets of focusing and mathematizing everyday objects or situations into abstract mathematical entities (Hilppö & Rajala, 2017). In general, in the case of both SFON and SFOR tendency, social interaction proved valuable for supporting SFON and SFOR tendencies.

Along with social interaction, multiple interventions aimed at improving SFON and SFOR tendencies also relied on embodied activities to reinforce the mathematical nature of everyday situations. These activities may include having the individuals enact the mathematical features or move within the space in which the mathematical aspects are embedded and are proven valuable for a variety of formal skills (Link, Moeller, Huber, Fischer, & Nuerk, 2013; Mix & Cheng, 2012). With a SFON intervention among preschool children, the mobile game "Fingu" (Holgersson et al., 2016) involved children recognizing numerosities as quickly as possible and assigning a cardinal value to them using both spoken words and finger touches. These activities were then extended outside of the digital learning environment, as the children were asked to use their virtual avatar in their everyday surroundings to find sets of objects and assign cardinal values to these objects (Hannula-Sormunen et al., 2016). The SFOR intervention also had students assign mathematical relations to everyday locations and distances (McMullen, Hannula-Sormunen, Kainulainen, et al., 2017). Students were sent on a mathematical treasure hunt, in which they needed to follow relational directions in order to find checkpoints. For example, starting at their classroom door, students were sent down the corridor to the library door, at which point they were asked to find the half-way point between their classroom door and the library door (or, e.g., three times this distance). These embodied activities, supported by digital tools that allow for highlighting the mathematical aspects of everyday spaces and objects, may have proved crucial for supporting SFON and SFOR tendencies among a wide range of individuals.

This is not to say that mathematical instruction should always and intensively involve promoting SFON and SFOR tendencies. As can be seen in Fig. 4.1, mathematical knowledge and skills are necessary conditions for focusing on aspects of number and relations in everyday situations (Hannula & Lehtinen, 2005; McMullen, Hannula-Sormunen, & Lehtinen, 2017), and contextual factors, including social interactions, also play a role (Chan & Mazzocco, 2017). Even so, it is expected that training SFON and SFOR tendencies could have fairly long-term effects and wide-ranging impact on related aspects of mathematical development (e.g., McMullen, Hannula-Sormunen, & Lehtinen, 2017). A potential boon for more long-lasting impact is possible through working with teachers in examining their beliefs and attitudes about the nature of mathematics and its role in everyday reasoning. Providing teachers with the tools to integrate activities and routines that promote SFON and SFOR tendencies into their everyday instruction may go a long way to offering students authentic experiences with mathematical reasoning (Verschaffel, Greer, & De Corte, 2000) that are not too burdensome in terms of their cognitive load (Kirschner, Sweller, & Clark, 2006), nor too loaded with extraneous details that do not support the mathematical meaning making process.

4.7 Conclusions and Future Directions

Research on spontaneous focusing on number and numerical relations, SFON and SFOR, opens our eyes to the broader possibilities of examining students' own spontaneous, self-initiated mathematical activities, the role of contextually-bound implicit prompts to attend to mathematical features, and the impact of these activities on students' success with mathematics. The state-of-the-science on spontaneous mathematical focusing tendencies indicates that there is much theoretical and educational value in examining and promoting young children's SFON and SFOR tendencies. Although more research is needed to determine the specific pathways between SFON or SFOR and mathematical development and the causal pathways implicated as potential underlying sources of variations in the mathematical thinking and learning, there is strong empirical evidence that these attentional processes are, at a minimum, highly relevant to early mathematics education. As reviewed in this chapter, the overlap, distinctions, and relations between (a) spontaneous mathematical focusing tendencies, (b) requisite skills (i.e., mathematical, motivational, and cognitive factors), and (c) contextual factors appears crucial for understanding exactly how children recognize and use mathematical features of everyday situations. These situations are ripe with opportunities to acquire lots of practice with mathematical skills. In addition to SFON and SFOR tendencies, other spontaneous mathematical focusing tendencies may interact with the requisite skills and contextual factors to influence mathematical development (e.g., spatial reasoning, Chan et al., under review). Nevertheless, in view of the potential power of bootstrapping informal activities and reasoning onto formal mathematical thinking (Resnick, 1987), educational practices and routines that promote mathematical focusing tendencies, including SFON, SFOR, and contextually-based prompts, may be an essential, foundational step in many mathematical activities. We believe such practices are also a fruitful target of inquiry into effective ways to support mathematical development for children who do not seem to "get" math. Their success may be the eventual outcome of a cascading set of developments that begins with children simply starting to notice the numbers and quantitative relations that surround them in their everyday lives. For all of these reasons, we conclude that promoting children's focusing on number and quantitative relations is a key component of early mathematics education.

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Chapter 5
Leveraging Relational Learning Mechanisms to Improve Place Value Instruction

Kelly S. Mix, Linda B. Smith, and Sandra Crespo

Abstract In this chapter, we focus on the difficulties children face when learning place value and how current psychological theories of relational learning may be leveraged by teachers. We discuss two major psychological mechanisms known to support relational learning—statistical learning and structure mapping—and review the evidence showing how these mechanisms are implicated in place value learning. We further identify a set of four specific instructional elements teachers could use to engage and support these learning mechanisms. We also review three major curricula for teaching place value, including *Developmentally Appropriate Mathematics*, *Number Talks*, and *the Montessori Method*, in light of this conceptual framework. Our review highlights both strengths of these current curricula and ways they might be modified to more fully leverage relational learning mechanisms and increase student learning.

Keywords Place value · Relational learning · Statistical learning · Structure mapping · Number Talks · Montessori · Developmentally Appropriate Mathematics

Mathematics uses a complex system of written and spoken symbols to represent quantities and operations on quantities. It is inherently relational, involving many interconnected layers of mapping from symbol-to-referent, quantity-to-quantity, and symbol-to-symbol. Teachers and mathematics educators have acknowledged the need for children to form associations among various components of this symbol system and have designed activities and materials consistent with this aim. There is growing convergence around the basic idea that to comprehend mathematics, children must

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recognize shared relational structures across these layers and to successfully map among its various components. However, extensive psychological research also tells us that some instructional conditions support learning about relational structures more than others. An important question is whether these new insights are reflected in existing instructional approaches. In this chapter, we examine three promising approaches to elementary mathematics instruction to see what additional leverage might be gained by incorporating these learning mechanisms more fully and explicitly.

We focus our review on children's learning of place value and multi-digit number meanings, for natural numbers in particular (vs. integers or rational numbers), as this is the first opportunity children have to "crack the code" of multi-digit representations. Place value presents an interesting relational learning problem, as children must interpret and align multiple components of the written and spoken symbol system with each other and with their referents. Given this complexity, it is not surprising that place value is notoriously difficult for children to master (Anderson, 2013; Moeller, Klein, Fischer, Nuerk, & Willmes, 2011). Indeed, many children struggle with weak place value concepts throughout elementary and middle school (e.g., Booth & Siegler, 2008; Chan, Au, & Tang, 2014; Fuson, 1990; Geary, Hoard, & Hamson, 1999; Gervasoni, Hadden, & Turkenburg, 2007; Hanich, Jordan, Kaplan, & Dick, 2001; Hieber & Wearne, 1996; Kamii, 1986; Moeller, Pixner, Zuber, Kaufmann, & Nuerk, 2011; Ross & Sunflower, 1995). Thus, place value presents an interesting test case as it is both a complex relational learning problem and one with far-reaching educational implications.

We review three instructional approaches: (1) *Developmental Appropriate Mathematics* (Van de Walle, Karp, & Bay-Williams, 2010); (2) *Number Talks* (Parrish, 2014); and (3) *The Montessori Method* (Montessori, 1934). These approaches are in widespread use and represent state-of-the-art elementary mathematics instruction. Many of their core elements already align with relational learning theory. Our goal is to point out where these alignments already exist and where others might be added to boost their positive effects. But first, we begin with a survey of psychological research on acquisition of relational structures to establish a firm theoretical framework. Note that the examples we present include but are not limited to mathematics, as relational learning principles have been identified across a broad range of topics, such as word learning and categorization.

5.1 Relational Learning

Extensive research in cognitive development (Aslin, Saffran, & Newport, 1998; Colunga & Smith, 2003; Namy & Gentner, 2002; Son, Smith, & Goldstone, 2008; Yu & Smith, 2007), and adult learning (Gentner, 2010; Goldstone & Byrge, 2014), has identified mechanisms through which learners access relational structures. By viewing place value as a syntactic notational system within a complex set of symbol-to-meaning mappings, we can build on these advances, thereby leveraging the general-purpose instructional techniques that are effective in other domains. In the following section, we review two of the major relational learning mechanisms—statistical learning and structure mapping. Later, we will identify specific instructional elements based on these mechanisms, and in those sections, we will present detailed evidence related to each element.

5.1.1 Statistical Learning

Statistical learning involves tabulating co-occurrences and using the resulting probabilities to focus attention in subsequent learning. This process is thought to be implicit and automatic. For example, to explain how infants pick out the referents for new words in a complex environmental scene, Yu and Smith (2007) demonstrated that infants gradually shift their attention based on the frequency with which words and objects co-occur in different situations. Statistical learners may also—with sufficient learning—discover systems of co-occurrences, or second-order correlations (i.e., "over-hypotheses"), that yield general rule-like performances (Colunga & Smith, 2005; Xu & Tenenbaum, 2007). In many domains, this knowledge of higher order relations is also implicit. That is, it is not readily explainable by those who possess the knowledge, but discoverable in the latent structure of large systems of mappings. To illustrate, consider English grammar. Few speakers of English can describe the rules for the sound of "s" when adding the plural to nouns (i.e., use the *z*-sound for words ending in voiced consonants such as "pigs" but the *s*-sound for words ending in unvoiced consonants such as "pits."), yet 5-year-olds show this knowledge in their production of novel plurals (Berko, 1958; Ettlinger & Zapf, 2011).

By many accounts, this implicit knowledge emerges given massive experience in a domain—the kind of experience accrued from everyday encounters with a highly structured world of symbols and things (Hockema, 2006; McClelland & Rogers, 2003; Recchia & Jones, 2009; Treiman & Kessler, 2006). Recent research suggests such encounters with multi-digit numbers and number names may be sufficient to improve children's understanding of place value structure (Yuan & Smith, under review); however, it is not known how much exposure to multi-digit numbers children would need to learn the entire place value system via statistical learning or whether it is even possible. Although the mechanism of statistical learning may be potent enough to yield a structural understanding of place value, the typical number of everyday encounters children have with large numbers is likely insufficient. Thus, place value learning probably occurs through a combination of processes that includes but is not limited to statistical learning. Still, place value instruction should feature dense co-occurrences that highlight statistical regularities so as to take advantage of this automatic learning process.

5.1.2 Structure Mapping

Structure mapping is the process theorized to underlie learning about relations by way of making comparisons (see Gentner, 2010). Comparison itself involves the simultaneous experience of multiple examples with the same structure. For example, children might abstract the meaning of "bird" by comparing a duck and a robin.

Similarly, they could abstract base-10 relations by aligning the count+unit structures in the numeral *23* with the same structures in the spoken number name "twenty-three." The idea is that by mapping the analogous components from one example to the other, learners isolate what is common about their shared structure and thereby discover the structure at the same time. Like statistical learning, this process is assumed to occur implicitly and automatically.

To illustrate how structure mapping works, consider again the case of comparing a duck to a robin. Children might start by noticing that the duck and the robin are moving, animate creatures. This is one point of alignment. Noticing this feature initiates a broader search for other points of alignment. Children might discover that these creatures also have wings and stand on two feet, and that they are covered with feathers. If children are told that the duck is a bird and the robin is a bird, the outline of a category could emerge—a category named "bird" that includes animals with feathers, wings, and two feet. Critically, when children encounter new things called birds, or other animals that are not birds, partial knowledge of birds from prior experience will highlight (in memory, in attention) features that have been relevant in the past. In this way, children gradually access deeper and more abstract relational structures. Co-occurrences, shared surface features, shared labels—anything that stimulates aligning the appropriate compositional units in comparison—can direct learners' attention toward the elements to be compared and initiate the alignment process (Gentner, 2010).

Theories of both statistical learning and structure mapping assume that the discovery of latent syntactic structures proceeds automatically and without supervision. This automatic learning is probably what allows preschoolers to learn something about the syntax of multi-digit numbers from mere exposure (Mix, Prather, Smith, & Stockton, 2014; Yuan & Smith, under review). However, from both perspectives what children learn depends on the structure in the learning materials. Thus, what children learn about place value on their own through casual encounters could be wrong or, even if it is correct in part, it could be too weak to support reasoning in more complicated tasks (e.g., multi-digit calculation). Moreover, place value poses special problems to learning, particularly when viewed as a structure-mapping problem. Without carefully chosen instructional materials and explicit scaffolding for comparisons, young learners are unlikely to decipher its underlying structure.

5.2 Barriers to Understanding Place Value

Statistical learning and structure mapping operate best when structures are regular. Of course, English grammar is not regular, and neither is the syntax for place value. One problem is that the elements of place value have an ambiguous correlational structure—an attribute that has contributed to mapping failures in other domains (Uttal, O'Doherty, Newland, Hand, & DeLoache, 2009). Numbers are used in many ways to mean many things (e.g., whole number cardinality, ordinality in counting, ordinality in dates and addresses, arbitrary tags in phone numbers, account numbers), so the correlation between symbol and meaning for any numeral is not perfect. In a multi-digit numeral, the correlations are even weaker because the same digits have different meanings based on their spatial positions (e.g., 14 vs. 41). In this way, place value notation is at odds with previous learning about numbers because the direct one-to-one mappings that allowed children to interpret the number words from 1 to 9 are pitted against the new relational mappings for which the numerals seem to mean something else. Of course, this is not true in reality because even numerals in the ones place represent counts and units (i.e., counts of the unit "one"), but from the perspective of a new learner, the "1" in 14 and 41 may seem like homonyms (i.e., identical symbols that mean different things). This example highlights the potential value of a statistical learning or structure-mapping framework: namely, that abstract relational structures are not obvious in single examples.

Another problem with place value is that, as in most symbol-referent mappings, the overall similarity among elements is low—a second reason mappings fail in other domains (Bassok & Medin, 1997; Gentner & Markman, 1994; Gick & Holyoak, 1983). The written numeral "42" has no perceptual similarity linking it to a pile of 42 rocks. Without counting the rocks, there is literally nothing connecting the two, and even then, there are layers of symbolic meaning to coordinate, including the counting sequence up to 42 and the spoken number name "forty-two." In some cases, the available surface similarity may initially be misleading. For example, the spoken number word, "fourteen," sounds like it should map onto the written symbol, "40," considering only the temporal sequence of phonemes. It is an open question whether the multi-digit numbers children haphazardly encounter outside of school, and perhaps in school as well, present enough systematic regularities to permit discovery of the relational structure that underlies place value notation.

Despite these challenges, children clearly attempt to find such regularities, and with some success (Byrge, Smith, & Mix, 2014; Mix et al., 2014; Yuan & Smith, under review). A striking example comes from number transcoding studies, in which children are asked to write multi-digit numerals from dictation (e.g., "Write the number 'three hundred twenty-six'."). Four- to 8-year-olds perform poorly at these tasks, particularly for three- and four-digit numerals (Barrouillet, Camos, Perruchet, & Seron, 2004; Moura et al., 2013; Zuber, Pixner, Moeller, & Nuerk, 2009). A frequent error, called "additive composition," resembles expanded notation (e.g., 300+20+6) and seems telling. When asked to write "three hundred twenty-six," for example, an additive composition response would look like "300206." This error has been interpreted to mean that young children do not know the syntactic rules for place value, they lack the working memory to keep track of base-10 syntax, they misunderstand the meaning of zero, or the queried numbers are relatively unfamiliar (Barrouillet et al., 2004; Geary et al., 1999; Moura et al., 2013; Zuber et al., 2009). However, these errors, which are evident in preschool children before instruction, may well reflect something else; that is, an attempt to find order in a symbol system that is not completely regular (Byrge et al., 2014).

Indeed, the additive composition error is much like the well-known over-regularization errors children make when they encounter grammatical exceptions (e.g., "goed" instead of "went," Berko, 1958). Such errors indicate that young children are attuned to the underlying structures, albeit perhaps subconsciously, and are attempting to align them. Regularization errors might even be construed as signatures of relational learning. The main idea of this chapter is that by fully supporting the learning mechanisms children naturally engage to make sense of place value (i.e., statistical learning and structure mapping), educators can help them discover the deep relational structure of multi-digit numbers more quickly, and avoid reliance on shallow heuristics that can interfere with subsequent learning.

5.3 Instructional Elements Based on Relational Learning

To our knowledge, there are currently no mathematics curricula that explicitly build on the learning mechanisms of statistical learning and structure mapping. However, research has revealed enough about how these mechanisms work for us to describe how these curricula might look. Below, we identify four instructional elements that could improve place value instruction by harnessing these learning mechanisms, and explain the empirical and theoretical basis for each one.

5.3.1 Co-Occurrence

The more learners experience co-occurring elements, the more likely they are to notice and remember them (see Smith, Colunga, & Yoshida, 2010, for a review). At the most basic level, this type of learning is evident when animals learn simple associations, such as Pavlov's dog learning to associate food with the sound of a bell. However, complex variants of associative learning characterize many forms of human learning. When learners are given extensive exposure to highly regular, complex systems, their cognitive systems—through the basic operations of memory and attention—instantiate the statistical probabilities of various pairings, and these probabilities serve to isolate conceptual units and direct attention to future input.

For example, a vexing problem in child development has been explaining how infants isolate and identify words in continuous natural speech, for which word boundaries are often unmarked by pauses or prosodic cues. A landmark study by Aslin et al. (1998) showed that 8-month-olds could detect novel word boundaries in a continuous speech stream of nonsense words, using only the transitional probabilities for various syllables. No prosodic or rate cues were given. The only cue available to infants was the regularity of certain syllables following others (e.g., in the speech stream "bidakupadotigolabubidaku…" the syllable "pa" was more likely to follow "ku" than "bi"). Remarkably, infants required only a single, 3-min session to detect these transitional probabilities and learn specific words (e.g., "kupa"). Bear in mind that at a rate of 4.5 syllables per second, infants were exposed to hundreds of pairwise syllable sequences, and the target nonsense words themselves appeared at least 45 times each. This study demonstrates that dense (albeit brief) exposure to a statistically regular input stream is sufficient for learners to abstract its structure.

Another word learning challenge is determining to which, of all the objects in a given scene, a new word refers. How, in a room full of toys, are there sufficient co-occurrences between balls and the word, "ball," to make the correct mapping? Abundant research indicates that these situations are more constrained than they may at first seem, and that in fact, there is enough statistical regularity across situations for infants to use patterns of co-occurrence to induce word meanings (Smith & Yu, 2008). For example, a recent study used head-mounted cameras to determine what objects were most frequent in a typical toddler's visual field (Clerkin, Hart, Rehg, Yu, & Smith, 2017). The results indicated that certain objects, such as *table, bowl,* and *cup,* were much more likely to be seen by toddlers, thus limiting the number of possible referents and increasing the density of co-occurrences between word and referent for these objects.

Research using connectionist models also demonstrates that the co-occurrences among elements are sufficient to induce new ways of categorizing the environment (Colunga & Smith, 2005; Kruschke, 1992; McClelland & Rogers, 2003; Siskind, 1996). Connectionist networks are simple computational models made of interconnected processing units that function like individual neurons. Each unit can be more or less activated. As the network accepts input, the activation levels of the processing units adjust and over time, the network responds, or generates output, that reflects this learning. By exposing such units to various patterns of input, but not building any particular structure into the network *a priori*, scientists can test the conditions that support various kinds of learning. Colunga and Smith (2005) fed a connectionist network perceptual information about various noun categories for which shape and material (i.e., solid vs. non-solid) were correlated. At test, the network was asked to categorize new examples that matched on shape and were either solid or non-solid materials. After 100 training epochs, the network learned to ignore shape when an item consisted of non-solid material. This generalization arose purely from tabulating the statistical probability that shape was predictive of word meaning for solid, but not non-solid materials.

Findings such as these suggest that dense co-occurrences of structural elements in the domain of mathematics (and more specifically, experiences related to place value notation) will lead to recognition and abstraction of its underlying patterns, even when those regularities are imperfect and not obvious in the surface forms. Indeed, recent research has demonstrated this to be true (Yuan & Smith, under review). Thus, there is good reason to think that place value learning requires extended sequences of mappings in the same session. Instructionally potent activities likely incorporate multiple mappings within each task—number names to written numbers, written and spoken numbers to sets of things, one number to the next in the count sequence—so that they are dense with co-occurrences.

5.3.2 Alignable Elements

Recall that there are two common barriers to relational learning; one is weak internal consistency within a relational system and the other is low surface similarity among elements. To overcome these obstacles, it would be beneficial to use instructional content that provides greater internal consistency within elements and greater surface similarity among elements. Teachers cannot reinvent the conventional symbol systems used to represent place value, but they can select language and materials that make the mappings within this system more predictable and relevant to finding the underlying base-10 structure.

Consistent with this idea, we know that Asian languages convey base-10 structure more transparently than many European languages, including English, and this difference may support place value learning. Fuson (1990) referred to these languages as “named value” numbering systems because the base-10 units and their counts are explicitly labeled. For example, the Chinese word for 23 is translated, *two-ten, three*. Asian children fare better on place value tasks than their US peers, and their transparent language may be one reason (Geary, Bow-Thomas, Liu, & Siegler, 1996; Ho & Fuson, 1998; Laski & Yu, 2014; Miura et al., 1994) (although cross-cultural differences in place value instruction is another, see Laski & Yu, 2014). We also know that using transparent language improves student performance in other mathematical content areas, such as fractions, even when children are from the same cultural group (Paik & Mix, 2003). Thus, using alignable language may be one way teachers can scaffold relational mappings.

Another way to incorporate alignable materials is through the use of concrete manipulatives that make base-10 structure more explicit and regular. Such manipulatives have a long tradition in mathematics education, having been developed to help children ground their understanding of abstract mathematical concepts in concrete experiences (see Mix, 2010, for a review). This grounding is thought possible because these concrete objects are structurally isomorphic to the written symbols (Post, 1988). In the realm of place value, many classrooms already use commercially produced base-10 blocks for this purpose (see Fig. 5.1). As we will see, materials such as this featured prominently in Montessori’s mathematics curriculum as well (Montessori, 1934).

Research further suggests that it is easier to abstract relations from simple examples than it is from rich, complex examples (Son et al., 2008). For example, when toddlers were taught a novel word for one of two toy vehicles, the words were learned and generalized more readily if the toys were solid colored, geometric shapes rather than richly detailed versions (see Fig. 5.2) (Son et al., 2008). Whereas adults can use prior learning to direct their attention optimally in complex learning situations (Nosofsky, 1984), children are less capable of screening out distracting information (Gentner, 1988; Keil & Batterman, 1984). Thus, access to clear, straightforward examples may be particularly important for children, and particularly important when structures are complex and non-obvious.

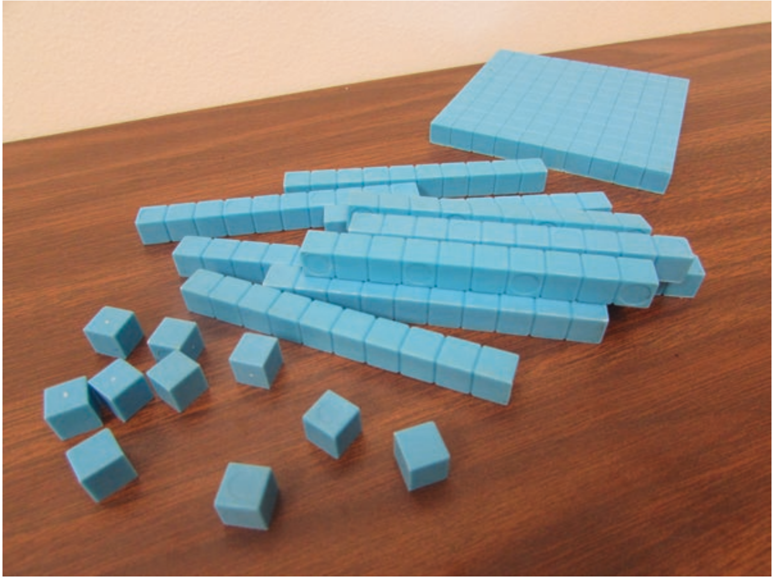


Fig. 5.1 Base-10 blocks (Source: Kelly S. Mix)

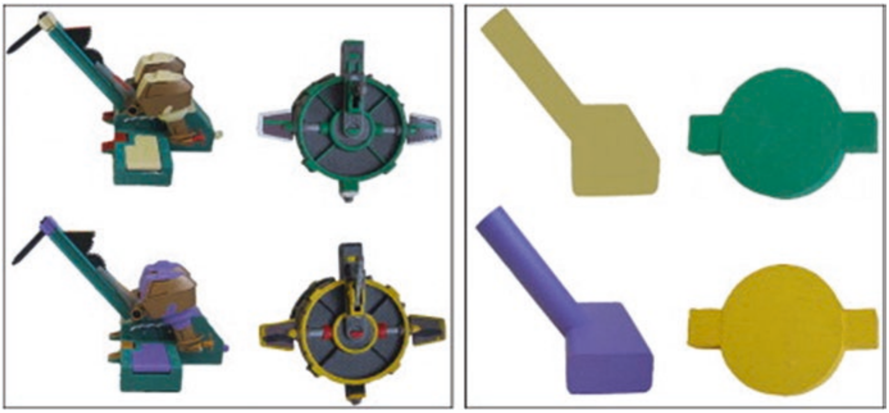


Fig. 5.2 Stimuli used in Son et al.'s (2008) word learning study (Reprinted by permission of Elsevier)

These findings suggest that children learning place value would benefit from exposure to instructional elements (i.e., words and materials) that are structurally transparent, simple, and regular. As noted above, the idea of structural isomorphism has a long tradition in mathematics education and these aims have motivated the use of concrete models to represent base-10 relations. There is evidence that such materials are effective (e.g., Fuson, 1990), though when stringent controls are used, the evidence favoring base-10 blocks is limited to certain mathematical outcomes

(Mix, Smith, Stockton, Cheng & Barterian, 2017). Perhaps the full benefit of these materials is not realized unless other elements are included, such as dense co-occurrences and those additional elements described below. In short, isomorphism alone may not be enough.

5.3.3 Multiple Examples

It would be possible to focus instruction on a single example of a concept and teachers often do so. They might, for example, present a wolf as an example of a carnivore in science class, or use a short story to illustrate a literary genre. However, research indicates that comparison is the engine that drives discovery of structural similarity (Gentner, 2010), so there is great instructional value in presenting multiple examples or analogies to stimulate these comparisons. Multiple examples may be particularly crucial for learning relational concepts, such as place value.

Promoting comparisons also engages partial knowledge, which is crucial to acquiring new mental structures. Indeed, a fundamental question in developmental psychology is how new concepts or behaviors emerge. How does one discover what a dog is, if one does not already recognize what a dog is? How can you understand algebra, without already knowing algebra? To solve this chicken-egg problem, psychologists have invoked the notion of bootstrapping, or the idea that partial knowledge allows learners to focus attention in ways that promote acquisition of additional knowledge (Carey, 2004; Gentner, 2010).

To illustrate, consider how bootstrapping has been demonstrated in research on language acquisition (e.g., Piantadosi, Tenenbaum, & Goodman, 2012; Werker & Yeung, 2005; Yurovsky, Fricker, Yu, & Smith, 2014). Yurovsky et al. (2014), for example, manipulated the completeness of information available to adults learning novel words. They showed that (1) learners encoded partial information about the novel words prior to acquiring the full meaning of these words and (2) having partial knowledge of a subset of words served to speed up learning of the rest. Thus, learners neither acquire complex systems in their totality nor acquire elements of complex systems in isolation from each other, but rather, they use partial knowledge of system elements to direct attention and make reasonable inferences (i.e., bootstrap) to acquire the rest.

The hypothesis that comparing multiple examples drives learning has been supported repeatedly in word learning experiments. Children learn new words faster when they are given two referents and a label versus a single referent paired with a label (Childers, 2011; Gentner, Anggoro, & Klibanoff, 2011; Liu, Golinkoff, & Sak, 2001; Namy & Gentner, 2002; Pruden, Hirsh-Pasek, Maguire, & Meyer, 2004; Son, Smith, & Goldstone, 2011). For example, Gentner et al. (2011) demonstrated that simply presenting 4- to 6-year-olds with two examples of a relation, such as a knife cutting a watermelon and an ax cutting a tree, was sufficient for children to induce the common relation “for cutting” between tool and object.

In place value learning, activities that promote comparison are likely to be highly effective. For example, rather than simply showing children a physical set and calling it "twenty-eight," children may need to see and compare multiple physical sets (e.g., sets constructed of different materials, materials grouped in different ways) to extract the common properties to which the verbal labels refer.

5.3.4 *Scaffolding Alignment*

Simply presenting multiple examples to students is a good start, but may not be enough, particularly for a complex representational system like place value. Children may need additional scaffolding to align elements of this system and recognize similarities. Although presenting children with two examples of a new category is sometimes sufficient to induce comparison and abstraction (Gentner et al., 2011), that is not always the case (Christie & Gentner, 2010). How can teachers more actively scaffold the alignment process?

5.3.4.1 Temporal Contiguity

It helps learners make comparisons if all elements are presented in close temporal contiguity, if not simultaneously. When elements are separated too far in time, comparison is hindered. For example, if children represent place value problems with one material on Monday and another material on Tuesday, they likely fail to benefit from having multiple examples because the examples are too far apart in time to be aligned. In contrast, if children constructed a representation with one material and then immediately constructed the same representation with a different material, the comparison across representations would be more straightforward and easy to perceive.

5.3.4.2 Surface Similarity

The features and physical presentation of instructional materials can be manipulated to induce comparison. Recall that when learners recognize any point of alignment, it is sufficient to initiate the structure-mapping process by which new points of alignment and deep structures are discovered (e.g., Gentner, 2010). Thus, even unrelated "surface" similarities can stimulate comparison. These similarities may be inherent to the objects themselves (e.g., color-coding) or conveyed via spatial organization (e.g., arranging two lines of objects across from each other in small-to-large order) (Kosslyn, 2006; Matlen, Gentner, & Franconeri, 2014; Novick & Bassok, 2005).

Distinctiveness, or alignable differences, can also support structure mapping (Markman & Gentner, 1996). In one study, children were most likely to match objects across two arrays that were the same relative size (small, medium, large) when the objects themselves were not only highly similar across arrays, but also highly distinctive within arrays (i.e., different from each other) (Paik & Mix, 2006). Base-10 materials could be structured using such cues. For example, when presenting different concrete models, such as stick bundles and base-10 blocks, teachers could color code the place values so that in both models, the ones are red, tens are blue, hundreds are yellow.

5.3.4.3 Gesture

A third way to scaffold alignment is to indicate which specific elements of one example align with specific elements of the other using gestures (Alibali et al., 2013; Alibali & Nathan, 2007; Richland, 2015; Vendetti, Matlen, Richland, & Bunge, 2015). For example, Richland and McDonough (2010) demonstrated that when teachers use gesture to scaffold comparisons in mathematics lessons, student learning is enhanced. Consistent with this, Alibali and colleagues have reported a link between teachers' use of gestures to scaffold comparisons in mathematics and student learning outcomes in several case studies (Alibali et al., 2013; Alibali & Nathan, 2007). Indeed, in countries with the strongest mathematics performance, teachers regularly reinforce relational alignment through gestures, whereas US teachers do so more rarely (Hiebert et al., 2005; Richland, Zur, & Holyoak, 2007). This instructional difference might explain why US children fare better in classrooms with more teacher-directed instruction and fewer manipulatives (Morgan, Farkas, & Maczuga, 2015) because when US teachers use concrete models in mathematics instruction, they may allow too much unguided exploration and fail to provide sufficient scaffolding for aligned comparisons.

5.3.4.4 Shared Labels

When two objects or events have the same verbal name, or label, it is a strong signal to learners of an underlying commonality (Gentner & Christie, 2010; Gentner, Özyürek, Gürcanli, & Goldin-Meadow, 2013; Loewenstein & Gentner, 2005). The shared label may point out the relation directly. For example, when preschool children were asked to find an object on a three-tiered display after seeing one hidden at an analogous location in a separate display, performance was significantly better when spatial labels such as "on," "in," or "under" were used (Loewenstein & Gentner, 2005). However, the word need not describe the relation to be helpful. Simply naming a noun category signals similarity, even if the words are novel (Golinkoff, Hirsh-Pasek, Cauley, & Gordon, 1987; Smith, Jones, Landau, Gershkoff-Stowe, & Samuelson, 2002; Waxman & Gelman, 1986).

One way to use shared labels is to name one instance and then, when another instance is encountered later, give it the same name. However, this kind of shared labeling may lack potency if the instances are spaced too far apart in time. If a child sees a bear at the zoo on Sunday, for example, and then sees one in a book three months later, the shared label may not be recognized as such. This problem is even more acute for abstract concepts, for which there are few shared surface features. However, when shared labels are used in the presence of multiple examples, it maximizes comparison and alignment by clearly signaling there is a commonality and providing at least two examples to compare. In place value learning, triangulating co-occurrences in this way may be quite important. For example, rather than simply showing children a physical set and calling it “twenty-eight” (or even “two tens and eight ones”), children will likely learn faster if they experience multiple mappings in the same instance (e.g., two sets of 28 that both receive the same label).

5.3.4.5 Progressive Alignment

When a relation is not easily observed, it may be helpful to lead learners through a carefully constructed sequence of comparisons (i.e., progressive alignment; Kotovsky & Gentner, 1996; Goldstone & Son, 2005; Thompson & Opfer, 2010). The sequence need not be long. In fact, studies have shown results with only one intermediary comparison (Kotovsky & Gentner, 1996). The key is to introduce overlapping instances that connect instances at one extreme to another. For example, children who recognize the relation of symmetry should be able to match completely different sets of objects that are symmetrical (see Fig. 5.3). However, when

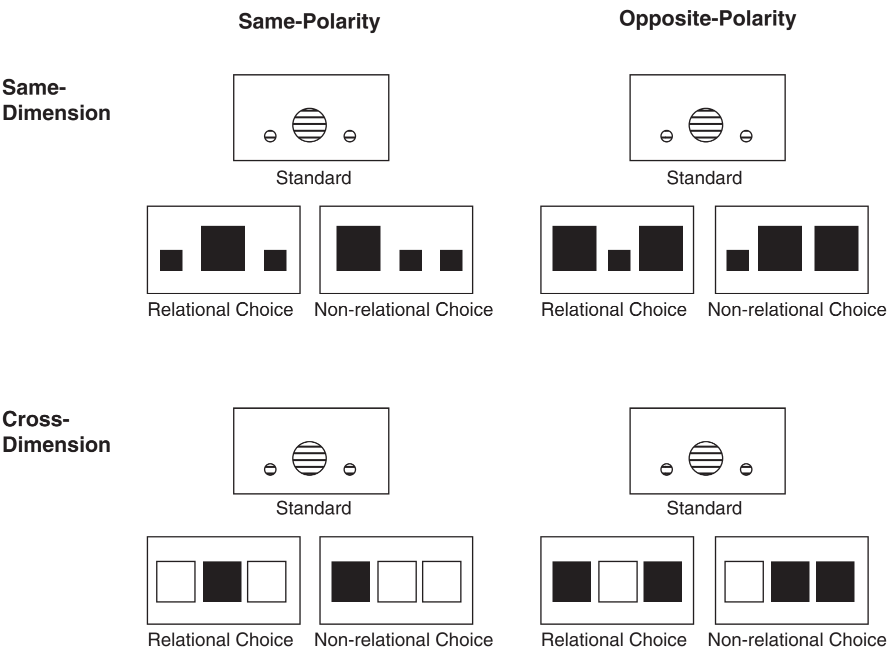


Fig. 5.3 Stimuli used in Kotovsky and Genter’s (1996) progressive alignment study (Reprinted by permission of Wiley)

the object sets are so disparate that all they have in common is symmetry, there may not be enough similarity to engage structural alignment. To bridge this gap, it helps to have a highly similar pairing (i.e., one that elicits alignment), followed by the dissimilar pairing (i.e., one that isolates the relation).

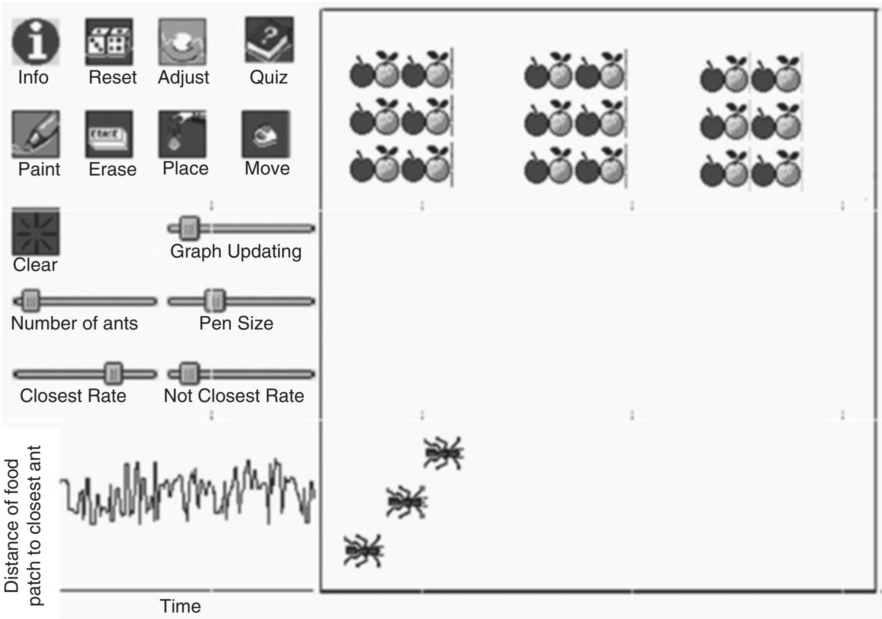
One well-studied version of progressive alignment has used chains of comparisons that move from concrete to abstract examples (i.e., concreteness fading; see Fig. 5.4). Goldstone and Son (2005) showed that adults learned abstract scientific concepts such as patterns in the way animals distribute themselves while foraging, if the first example is highly concrete and subsequent examples gradually remove detail until only the relation itself is represented. Similar effects have been demonstrated in mathematics learning, as well (McNeil & Fyfe, 2012; Fyfe, McNeil & Borjas, 2014). Specifically, adults and children learned new mathematics concepts best when examples transitioned gradually from concrete to abstract, compared to either a concrete- or abstract-only approach.

5.3.4.6 Summary

As we have seen, the choice of materials used to instantiate relations (i.e., objects, words, and symbols) and the specific way these materials are presented can determine whether learners will engage statistical learning and structure mapping—the engines of relational learning. Extensive research on concept acquisition and word learning has established that learners benefit from dense co-occurrences, multiple examples that have alignable elements, and explicit scaffolding for comparisons, including temporal contiguity, surface similarity, shared labels, gestures, and progressive alignment. Although few studies have applied these ideas to place value learning in particular, it is reasonable to expect the same patterns to hold. In the sections that follow, we review three mathematics curricula in light of these learning mechanisms and instructional elements. As we will see, there are many ways these curricula provide strong support for relational learning, but may not go far enough. Small but important modifications could optimize relational learning and lead to significant student gains.

5.4 A Brief Review of Three Mathematics Curricula in the Context of Relational Learning Mechanisms

We selected three highly influential approaches to elementary level mathematics instruction, that include (1) *Developmentally Appropriate Mathematics* (Van de Walle et al., 2010), (2) *Number Talks* (Parrish, 2014), and (3) the *Montessori Method* (Montessori, 1917). As noted previously, we restrict our focus to instruction on place value and multi-digit notation, as this topic is a persistent stumbling block for elementary students (Booth & Siegler, 2008; Gervasoni et al., 2007), as well as a



A screen-dump of an initial configuration for the "ants and food" simulation. At each time step, a patch of food is randomly selected, and the ant closest to the patch moves to-ward the patch with one speed (specified by the slider "closest rate") and the other ants move to-ward the patch with another speed ("not closest rate").

An example of the idealized version of the ants and food simulation.

Fig. 5.4 Stimuli used in Goldstone and Son's (2005) concreteness fading study (Reprinted by permission of Taylor & Francis) complex relational learning challenge. Our aim is not to evaluate each curriculum in its entirety, but rather to survey its conceptual framework and closely analyze a few specific activities. Our hope is that a narrow focus will generate specific recommendations that might be useful by way of example. The materials we reviewed were designed for second- and third-grade students, and all aim to teach the meaning of multi-digit numbers. For each curriculum, we consider separately the four instructional elements identified above; that is (1) Co-Occurrence, (2) Alignable Elements, (3) Multiple Examples, and (4) Scaffolding Alignment.

5.4.1 Developmentally Appropriate Mathematics (Van de Walle et al., 2010)

5.4.1.1 Overview

The first curriculum, *Developmentally Appropriate Mathematics* suggests activities based on the notion that children of various ages can learn mathematics deeply when tasks are designed so as to provide the most opportunity for drawing mathematical connections. Its theoretical framework is essentially Piagetian constructivism, which holds that children construct and adapt schemas in response to various experiences. The overarching instructional goal of *Developmentally Appropriate Mathematics* is to help children construct new ideas through the Piagetian mechanisms of assimilation and accommodation, and connect these ideas through guided exploration in student-centered activities.

The literature on relational learning is not cited, but the authors draw a pertinent distinction between what they call "instrumental understanding" and "relational understanding." *Instrumental understanding* is considered rote application of isolated procedures, and *relational understanding* is considered a set of connected ideas that help students know "what to do and why." These connected ideas are thought to arise through relational activities, such as "explaining, providing evidence or justification, finding or creating examples, generalizing, analyzing, making predictions, applying concepts, representing ideas in different ways, and articulating connections or relationships between the given topic and other ideas." (p. 6).

These ideas are broadly consistent with current psychological theories of relational learning but there is a crucial difference. *Relational understanding* in *Developmentally Appropriate Mathematics* means achieving a complete and coherent structure for a set of concepts, as well as a meta-awareness of this structure that permits explicit strategy choices and justifications. *Relational learning* in psychology means seeing how two things are related (e.g., similar). In mathematics, seeing similarity could be something as simple as seeing that the teacher's handwritten number 24 is the same as another student's handwritten number 24, or something as complex as seeing how commutativity in solving equations is the same as weights arranged a certain way on a balance beam. To use a non-mathematics example, relational learning is also how we understand metaphoric comparisons, such as "time is like a river" (Bowdle & Gentner, 2005). Although relational understanding is probably built via relational learning mechanisms, they are not the same thing. This distinction is critical to the present review because, even though the importance of forming connections is certainly acknowledged in *Developmentally Appropriate Mathematics*, the psychological processes by which children make these connections are not centered in this curriculum.

5.4.1.2 Learning Activities

The specific activities we reviewed are presented in Chap. 10, "Developing Whole-Number Place-Value Concepts" (Van de Walle et al., 2010). The overall gist of the chapter is that elementary children are developmentally working to acquire number sense, or a relational sense of number concepts and skills. The introduction to the chapter offers a mechanism based on Howden (1989), which holds that children will gradually discover these relations through "exploring numbers, visualizing them in a variety of contexts, and relating them in ways that are not limited by traditional algorithms" (p. 11). The introduction further notes that "being able to recognize and generate equivalent representations of the same number is the component of number sense that will serve students well …" and that this awareness of equivalent representations linked to place value understanding will lead to greater flexibility. As noted above, whereas these claims are consistent with relational learning theory, they seem more focused on the results of relational learning than relational learning processes themselves.

In the *Building It in Parts* activity (10.1, p. 153), children are asked to decompose various quantities, such as 40, into two sets using ten-frames. Children are instructed to record their solutions both by displaying the ten-frames used on a small mat, and writing an addition equation (e.g., 25 + 15 = 40; 20 + 20 = 40) (see Fig. 5.5). Teachers are told that writing the equations serves to focus children's attention on the relevant aspects of this activity, and "make apparent the clear connection between part-whole concepts and addition."

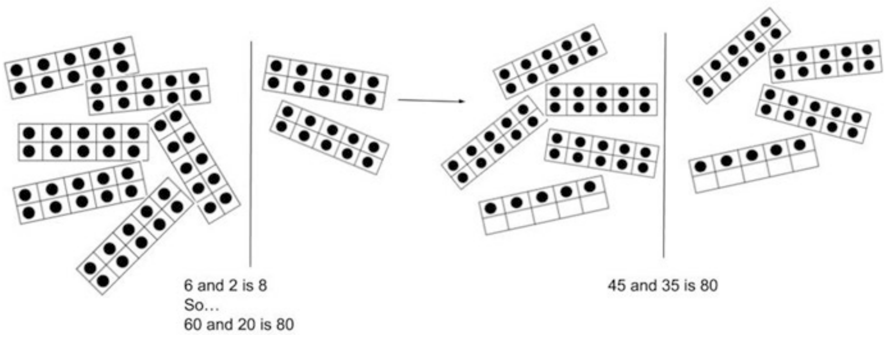


Fig. 5.5 Tens frames used in the *Building in Parts* activity (adapted from Van de Walle et al., 2014)

Several relations are involved in this activity. The main relation involves noticing that different addends can reach the same total (e.g., 60 + 23 = 83 and 50 + 33 = 83), and further, that one addend offsets the other (i.e., “If I align the 60 and 50 on the left, and the 23 and 33 on the right, I can see that when the leftmost addend increases, the rightmost number must decrease by the same quantity.”). However, to see these more complex relations, children must also understand the mapping from written addition problems to ten-frames representations.

In support of relational learning, the *Building It in Parts* activity incorporates several of the instructional elements we identified earlier. First, it includes co-occurrences because children are asked to find as many different combinations as possible. Given enough time, children could generate dozens of combinations for a given quantity. That said, the activity description does not suggest a length of time or target number of combinations, so it is up to teachers’ discretion when enough combinations have been generated. If teachers do not understand the benefit of dense co-occurrences, they may stop children too soon.

The *Building It in Parts* activity also generates multiple examples with alignable elements, and there is some scaffolding for noticing these elements. Specifically, the sum operates as a shared label that indicates the various equations refer to the same total and are, thus, somehow related. However, the shared label alone may not be sufficient. It may be important for the various combinations to remain visible using both the materials (e.g., tens frames) and for the equations to be spatially aligned so children can see how the addends in various equations map onto the materials and one another. Children may also benefit from gestures that highlight these components moving back and forth from one equation to another. None of this scaffolding is mentioned in the chapter or the activity description, so teachers would need to know how and why to provide it.

There was less evidence of the other instructional elements we identified. Because there is only one referent to map to each written addition problem, this activity does not incorporate multiple examples and triangulation. A second referent would be needed to engage that process. It should be noted, too, that although the elements are alignable, these relations may be difficult to perceive because of the low surface similarity between ten-frames and written numerals. Some children may need extra scaffolding to remember how the two are related. For example, it may help to count the dots in the ten-frames to show that they represent the cardinal number in the place value notation (e.g., “Let’s see. Two tens—one, two–and we write that ‘2’ here, to show it means tens and not ones.”).

In another activity (*Too Many to Count*, 10.4, p. 160), teachers are directed to present a container filled with at least 1000 objects, such as straws or packing peanuts, and then ask students to estimate how many objects there are. Students next bundle the objects into tens and hundreds so as to facilitate an accurate count. The main aim is for students to “see how the 10 groups of 100 are the same as the 1000 individual items,” a connection that the author rightly points out is often obscured when pre-grouped materials such as base-10 blocks are used. This is an important activity because facility with base-10 decomposition is highly predictive of later mathematics achievement and appears to be the main stumbling block in children’s developing understanding of place value (Chan et al., 2014; Laski, Ermakova, & Vasilyeva, 2014).

It is possible that some children will discover these underlying relations through the activity of estimating one large set size and decomposing the same set into units; however, there are reasons to think most children will need greater support.

First, the activity provides multiple representations of the same quantity, but there is no way to make an explicit comparison between the starting set and the ending, decomposed set, because they are never present at the same time. What children would be doing is either comparing their memory of the starting set to the decomposed set, using logical inference (e.g., Piagetian conservation) to know the sets must be equivalent, or both. A simple way to allow children to make a direct comparison would be to have children photograph the starting set and print the photo on a large sheet of paper that could then be used in point-by-point comparisons.

Second, there is not much scaffolding for comparison. The initial written numeral is not the same as the final count because it is an estimate, so children do not have a shared label to connect the initial and decomposed end state. Nor do they see both states in spatial contiguity, unless a photograph is provided. Even with the photograph, it may help to provide a shared label by writing the final count on both the picture of the initial state and the end state.

Third, the activity as described does not provide dense co-occurrences. It provides two mappable examples of a single quantity that are separated by the time it would take to bundle all the objects. For children to benefit from the activity, they likely need to repeat it several more times with the same quantities going in both directions (i.e., from composed to decomposed and then recomposed). Obviously, this back and forth would be difficult and time-consuming for quantities in the 1000s, so perhaps it is preferable to illustrate the bridge counting by ones to other places only up to 100s, and then bridge 100s to 1000s by providing objects that are pre-grouped, but only into 10s or 100s. This modification leverages the notion of progressive alignment we described above, wherein children are scaffolded toward more abstract correspondences in stages that move from more concrete to less concrete examples. One could consider counting the objects by ones to be more concrete, partial bundling to be less concrete, and pre-grouped 1000s materials to be the most abstract (short of the written notation).

The idea of partial grouping is also suggested, but as a separate activity (*Can You Make the Link?* 10.6, p. 161). Here, children group and regroup the same quantity ($\leq$ 150) by ones, tens, and hundreds. As in the *Too Many to Count* activity, the various states are not simultaneously available, but shared labels (i.e., written totals) signal to learners that there is something to compare. What may be important to convey to teachers is that children likely need these partial groupings to achieve adequate temporal contiguity and dense enough co-occurrences to recognize the same principles for set sizes in the 1000s.

5.4.1.3 Summary

The *Developmentally Appropriate Mathematics* activities are geared toward helping children discover the underlying relational structure of place value by juxtaposing various representations (see Fig. 5.6). The curriculum explicitly identifies alignable

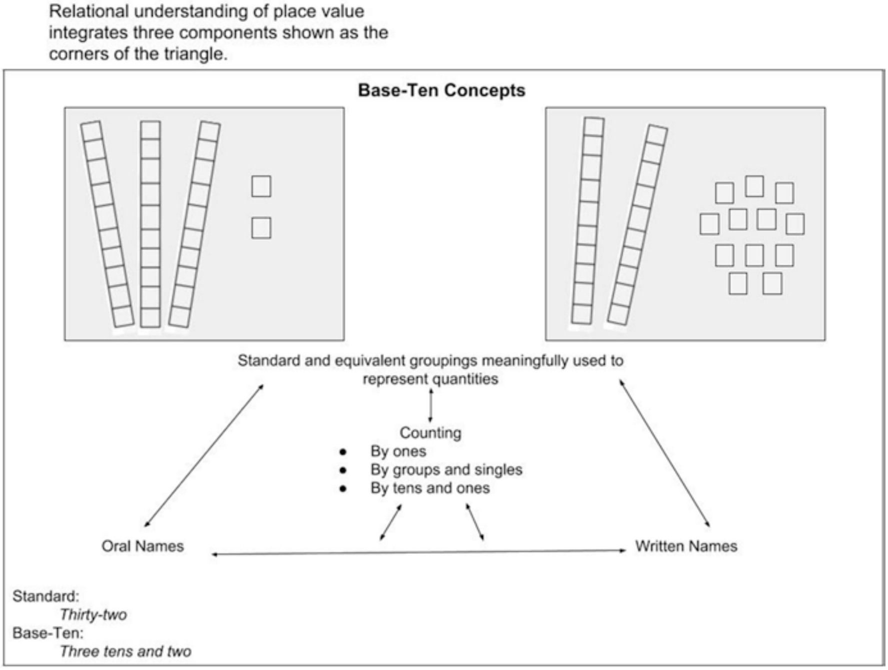


Fig. 5.6 Graphic depiction of base-10 relations presented in *Developmentally Appropriate Mathematics* (adapted from Van de Walle et al., 2014)

elements and the activities are clearly intended to promote comparison. However, the parameters of these comparisons (How many? How dense? Scaffolded how?) are not specified, leaving it to the teacher to determine. We know from research on relational learning in both children and adults that simply providing a single pair of examples without additional scaffolding is not usually sufficient to induce deep relational learning (Bransford & Schwartz, 1999; Gick & Holyoak, 1983; Goldstone & Wilensky, 2008). The presented activities promote relational learning by providing varied examples and setting up situations ripe for structural alignment and statistical learning, but without sufficient scaffolding, children may not reap all the potential benefits.

5.4.2 Number Talks: Whole Number Computation (Parrish, 2014)

5.4.2.1 Overview

Number Talks are short instructional routines designed to elicit strategies for representing numbers and number operations. The activities are a starting point for teacher-led discussions that highlight the mathematical principles underlying various strategies (Boaler, 2015; Humphreys & Parker, 2015; Parrish, 2011). *Number Talks* are a relatively new instructional method in the United States but it is commonly used in other parts of the world, particularly those countries whose students have high achievement in international comparative studies such as TIMSS (Stevenson & Stigler, 1994; Stigler & Hiebert, 2009; TIMSS, 2007).

The activities we will review are drawn from a resource book for teachers. This book does not provide a strong theoretical framework, but does identify several key components of number talks, including (1) fostering a supportive classroom environment that permits free exchange of ideas; (2) a structured discussion format that features the search for multiple solutions to the same problem; (3) the teacher as facilitator rather than expert; (4) mental computation and problem visualization to limit application of rote procedures; and (5) carefully chosen problems that highlight or at least permit critical structures to be discovered. The fifth component—structuring the problem set—may offer the clearest connection to relational learning. However, the idea that children will be comparing and contrasting solutions permeates the curriculum and appears to be the engine that drives learning in this approach. The question is whether the recommended activities do all that they can to drive this engine.

5.4.2.2 Learning Activities

The lesson we selected, "Breaking Each Number into Its Place Value" (p. 133), aimed at conveying the strategy of breaking addends into their expanded forms, and then calculating the sum of each place before recombining the totals (see Fig. 5.7). Each of the 27 second-grade problem sets contain four problems and build in complexity from simple problems that elicit the strategy through discovery (e.g., 10 + 10, 10 + 11, 12 + 13, 14 + 15) to problems that offer practice applying the strategy (e.g., 15 + 27, 23 + 18, 17 + 25, 16 + 27), to problems that extend to larger numbers and more complicated combinations (e.g., 38 + 58, 67 + 17, 44 + 38, 25 + 66).

Video clips of a few lessons are provided online. Although a second-grade version of the above lesson is not offered, there was a video for a more general lesson with third-grade students, in which children discuss solutions to the problem "38 + 37." The video first shows the teacher writing the problem on a whiteboard. Next, children are asked to solve it mentally (i.e., without paper and pencil) and indicate when they have reached a solution by giving a thumbs-up. Then children describe their procedures, one by one, while the teacher writes the equations on the whiteboard. The first three solutions are shown in Fig. 5.8. The third child offered a "Breaking Each Number into Its Place Value" solution, written here in the lower right corner of the whiteboard.

Of interest here is whether this lesson includes instructional elements that can promote relational learning. Unlike the *Developmentally Appropriate Mathematics* activities, number talks center on the symbols themselves and do not include mappings to concrete models (other than ten frames). Instead, the objects in this particular lesson are the symbolic quantities, so the relevant mappings are among the equations representing these quantities. The point of the lesson is to show that the

Fig. 5.7 Example of a *Number Talks* strategy for place value understanding (Reprinted by permission of Math Solutions)

(A-7) Addition Strategy: Breaking Each Number into its Place Value

Once students begin to understand plane value, this is one of the first strategies they utilize. Each addend is broken into expanded form and like place value amounts are combined. When combining quantities, children typically work left to right because it maintains the magnitude of the numbers.

24 + 38 (24 + 38) + (30 + 8) 20 + 30 = 50 4 + 8 = 12 50 + 12 = 62	Each addend is broken into its place value. Tens are combined. Ones are combined. Totals are added from the previous sums.

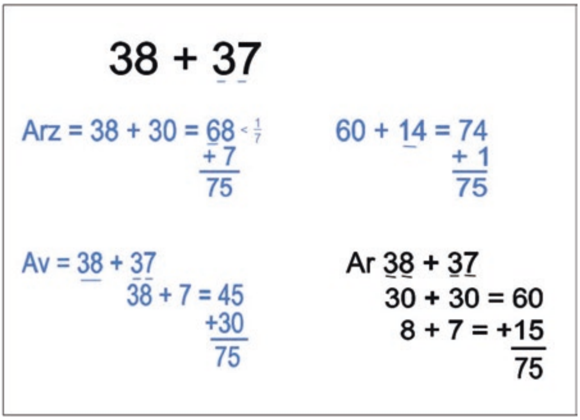


Fig. 5.8 Equations used to record students' problem-solving approaches in a Number Talks video lesson (adapted from Parrish, 2014 by permission of Math Solutions)

same quantities may be manipulated in various ways that all reach the same answer. Thus, for children to benefit from the activity, they have to recognize how the equations map onto one another.

At one level, the mapping in this video lesson is straightforward. The problem itself is the same and all the presented solutions arrive at the correct answer of 75. These points of alignment should signal to children that the problems are similar and might initiate a comparison process. It is less clear whether children can independently make the more intricate mappings among various methods for decomposition. For example, in the second solution, a child suggests splitting the ones place away from the rest of the second addend (7), adding this number to the first addend (38 + 7 = 45), and then picking up the remainder of the second addend (45 + 30 = 75). Are there any indications to children that the "30" in this approach is similar to the "30"s in the third solution? Unfortunately, the teacher did not explicitly compare the equations, so such commonalities were likely difficult for children to discern.

One strength of the lesson was that the equation sets for each strategy were spatially aligned, with the expanded versions neatly presented underneath the original version. This spatial alignment likely cued children as to which parts of the expanded equations mapped to their analogues in the original equation. However, there were few other scaffolds provided for these mappings. In the third solution, for example, the teacher underlined the tens places in the original equation while writing the expanded version beneath it. This use of surface features to guide attention probably helped somewhat, but these markings lost their meaning as soon as the ones places were added and also underlined. A better choice might have been to use arrows, connecting lines, or color-coding so that children could see the separate place relations after the whole equation was written out.

The *Number Talks* book focuses on six possible strategies for multi-digit addition, all of which children discover through interactions with various problem sets. In this scenario, co-occurrences are the number of practice problems for which the same patterns (i.e., solution strategies) are discovered and applied. This means that to have dense co-occurrences, children would need to solve many problems in each of the six ways. It is not clear from the lessons as presented how teachers can ensure all children are getting repeated exposure to each of the six structures (i.e., solutions). For example, one child could offer the “Compensation” strategy on every addition problem and essentially zone out of the discussion when other strategies are offered by classmates. Also, recall that some lessons are designed to elicit multiple strategies for the same problem rather than multiple problems with the same strategy. As we noted, alignable differences can be highly informative, so there is potential value in comparing different solutions to the same problem (provided these differences are explicitly aligned). However, if every problem is solved multiple ways, there may not be dense enough co-occurrences for any particular strategy to be abstracted as a pattern. Thus, the balance of multiple strategies within problem and same strategy across problems would need to be carefully managed by teachers for students to benefit optimally. Indeed, the right balance may depend on individual differences in learners, such as prior knowledge, working memory capacity, and so forth, so the task left to teachers is quite complicated.

5.4.2.3 Summary

The *Number Talks* problem sets are carefully constructed to afford certain solutions. In psychological terms, we might think of these solutions as patterns or structures that children can recognize across problems. If the problems are chosen well, as they appear to be in the lesson we sampled, the materials themselves will elicit these structures and generate the alignable elements needed for relational learning. We might also think about a particular problem as having a structure that is exposed by deconstructing and solving it in various ways.

However, whereas *Number Talks* activities provide opportunities for children to learn both of these relations, as relations go, these are fairly intricate and the activities as described leave most of the psychological meat in teachers’ on-the-fly decision-making process. In terms of statistical learning, one could argue there are dense co-occurrences as long as (1) students provide multiple strategies for each problem and (2) teachers present multiple problems that elicit the same strategy in close temporal contiguity (i.e., in the same lesson). It is an open question how many exposures students need to abstract these structures, though we know from extensive research in mathematics education that children readily acquire procedures and apply them in a rote way, suggesting that most children passively watching a lesson will achieve at least that much.

The question of structure mapping is thornier. The lessons do not build in explicit comparisons across solutions, so children will only align and map them if they are able to do so independently. The teacher in the video modeled a few techniques for scaffolding comparison, but these did not provide enough lasting cues (i.e., structural marks that could be inspected over time) and were added only to indicate mappings from one step to another within the same solution to the same problem. Scaffolding for mappings across solutions or across problems was not evident, but could be added.

5.4.3 The Montessori Method

5.4.3.1 Overview

Montessori's place value activities were developed in the early twentieth century as part of a comprehensive elementary curriculum that encompassed mathematics, reading, language arts, science, and social science (Hainstock, 1978/1997; Lillard, 1980/1997). Montessori was a physician who took an interest in the education of children with intellectual and developmental disabilities and eventually opened a school for low income, urban children in Rome. Through close observation and short teaching experiments, Montessori eventually built a full-blown theory of children's learning that included carefully designed and implemented developmentally appropriate pedagogical activities (Hainstock, 1978/1997; Lillard, 1980/1997; Lillard, 2005). Her mathematics activities and materials have been adopted widely and are commonly used in both traditional Montessori schools and in mainstream public schools to varying degrees of adherence to her prescriptions for implementation. Relatively few studies have examined the effectiveness of the Montessori approach due, in part, to difficulty choosing appropriate non-Montessori comparison groups; however, there is an emerging literature reporting strong positive effects of the Montessori Method on mathematics in particular (e.g., Lillard & Else-Quest, 2006; Mix et al., 2017).

In terms of relational learning, there are several notable components of Montessori's overall approach. First, concrete models for symbol grounding feature prominently. These materials were carefully constructed to embody as fully as possible the abstract relations they are meant to represent. The idea was for the objects themselves to be designed well enough that meaning was obtainable simply by interacting with them (P. Lillard, 1980/1997) (though as we will see, the curriculum did not stop there). Second, Montessori emphasized child-centered learning, meaning that children learned through their independent activities with prepared materials and not following a teacher-led discussion or lesson. That is not to say that children could interact with materials wildly. Each activity had a strictly prescribed sequence of movements that was demonstrated for children (A. Lillard, 2005). Rather than showing or explaining concepts, the teacher's main role was to prepare the materials, demonstrate the procedure for interacting with the materials, and redirect children if they veered off-script. As we will see, however, there was a great deal of scaffolding for relational learning built into these scripts.

5.4.3.2 Learning Activities

Montessori's approach to teaching place value centered on two constructs: (1) counting units up to 9 and (2) making a transition to counting larger units at the boundaries (10, 100, 1000, etc.). To facilitate these counting operations, Montessori designed physical and symbolic materials that embodied differences in magnitude across units (see Fig. 5.9). Golden beads represented physical sets of ones, tens, hundreds, and thousands, with each unit being comprised of ten smaller units (e.g., a 100 square consists of 10 tens chains). Layered cards represented these relations symbolically, with each unit of magnitude being one zero longer (e.g., the card for 100 was one place longer than the card for 10, such that the smaller magnitude could be laid atop the larger to make a multi-digit numeral).

Montessori's representational system was intentionally devised to promote structure mapping and statistical learning. Although the modern theoretical vocabulary for these processes was not available in 1917, the gist is the same. Montessori wrote, "The two materials—the decimal system beads and cards—lend themselves to clear, easy combinations offering opportunities for a very large number of exercises and therefore, ample practice." She went on to describe numerous mapping activities, such as laying out tens combinations in order from 10 to 90, with the corresponding layered cards underneath. The aim was for children to use 1-to-9 counting within each unit type so as to (1) encounter the need to shift units after 9 time and again, and (2) recognize this counting structure as a pattern that repeats across all place value units.

The Montessori activities build on each other sequentially and are presented in a prescribed sequence. So, following practice at counting within units up to the unit boundary, children are introduced to representing multi-digit numbers. The initial introduction is via counting layered cards alone, without mapping to the beads, but eventually, children construct representations using both (see Fig. 5.10). These representations are constructed in stages, with the layered cards used to label each group of beads (1s, 10s, 100s, etc.) in expanded form and then combined to show the final, conventional multi-digit numeral. Montessori explicitly recommended composing and decomposing the represented numerals, back and forth, so children could see how the two are related. Interestingly, Montessori did not recommend a

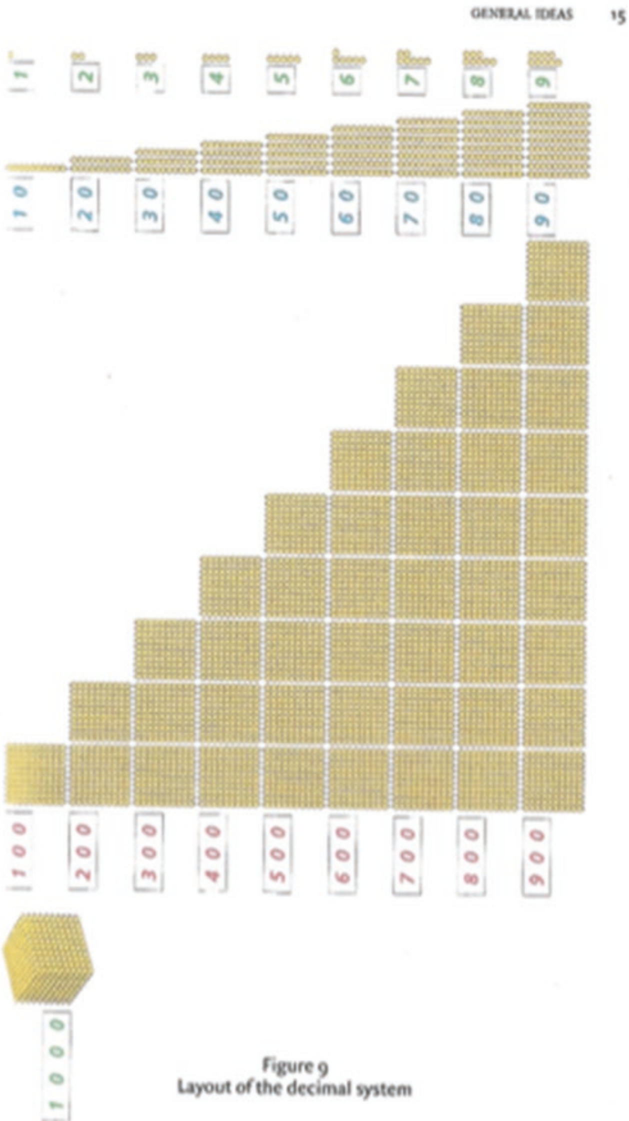
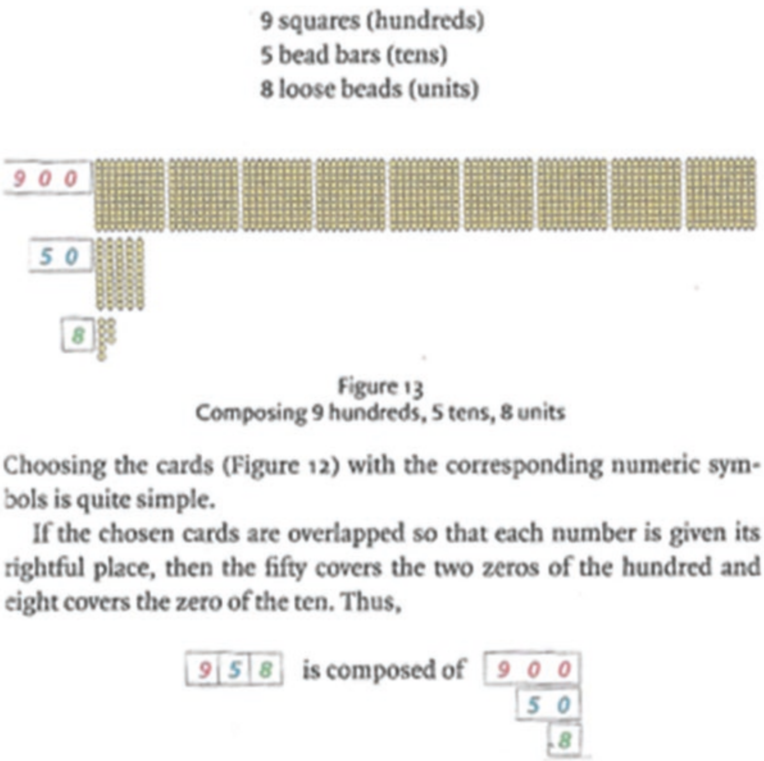


Fig. 5.9 Montessori's Golden Beads and Layered Cards (Reprinted by permission of Montessori-Pierson Publishing Company)

gradual progression from smaller to larger units. Rather, she presented all units simultaneously and perhaps even starting with 4-digit numerals. Her reasoning was that the repeating patterns within the system were only discernible when it is presented in its entirety and children were guided to discover these larger shared structures first, while filling in the details of specific places and their symbols later.


9 squares (hundreds)
5 bead bars (tens)
8 loose beads (units)

900
50
8

Figure 13
Composing 9 hundreds, 5 tens, 8 units

Choosing the cards (Figure 12) with the corresponding numeric symbols is quite simple.

If the chosen cards are overlapped so that each number is given its rightful place, then the fifty covers the two zeros of the hundred and eight covers the zero of the ten. Thus,

958 is composed of 900 50 8

Fig. 5.10 Montessori's representation of expanded notation (Reprinted by permission of Montessori-Pierson Publishing Company)

Action on materials and specific sequences of movement play a strong role in the *Montessori Method*. The use of perception-action is exemplified in the Snake Game (see Fig. 5.11). As noted above, the main structure children were encouraged to recognize was how counting beyond 9 of any unit necessitated a shift to a new unit. In the Snake Game, children use counting and movement to repeatedly experience these shifts. It begins with children creating a snake from beads of various lengths, from 1 to 9 single beads. These shorter bead chains are color coded by numerosity and would be familiar to children from their previous experiences counting and grounding cardinality up to 10. As the exercise progresses, children count the individual beads one by one, stopping when they get to 10, and replacing the multi-colored beads with a tens chain of golden beads. Importantly, the two chains (multi-colored and golden) remain visible throughout the exercise and are aligned side by side. Special, black beads are used to represent leftover ones that must be carried from one count to the next (see Fig. 5.11). The goal is for children to see how a randomly constructed line of chunks can be regrouped into tens and rapidly counted to determine its cardinal number.

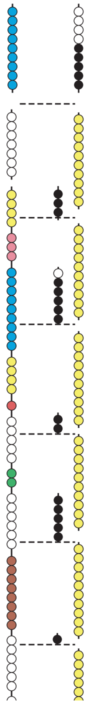


Fig. 5.11 Montessori's "Snake Game" (Reprinted by permission of Montessori-Pierson Publishing Company)

The *Montessori Method* incorporates many elements of structure mapping and statistical learning. First, all of the materials and activities involve direct comparisons between multiple representations. These representations included symbol-symbol mappings, as well as symbol-referent mappings, and a variety of triangulations such as multi-digit numeral-expanded numeral-concrete materials. The comparisons are carefully scaffolded using a variety of techniques that are consistent with relational learning theory, including spatial alignment of exemplars, color-coding, proportion and size coding, and close temporal contiguity of alternating states (e.g., expanded and composed). Montessori's exercises also broke larger systems of mapping into local, one-to-one mappings, so that children's attention was directed to specific points of alignment within and across larger mappable systems. Finally, the express aim is to provide dense co-occurrences. For example, the structure Montessori identified as crucial—counting to 9 and one step beyond—was built in multiple times to each exercise. Also, whereas Montessori did not prescribe a certain number of exposures per activity, she did provide guidelines to teachers based on children's engagement with a material. In short, Montessori teachers are instructed to present the same materials and exercises as long as children display intense concentration and interest in the materials (Hainstock, 1978/1997; Lillard, 1980/1997). This rule of thumb ensures children receive exposure to enough mappings to extract the critical relations without reaching a point of tedium. Thus, in many important ways, the Montessori Method encompasses the active ingredients of modern relational learning theory.

One might question why the *Montessori Method* appears more potent than *Developmentally Appropriate Mathematics*, given that the latter curriculum uses quite a few of Montessori's materials and activities, and also adopts the same general theoretical framework based on forming associations across representations. The key difference we observed is that the *Montessori Method* is presented to teachers as a coherent, step-by-step process with a highly prescribed sequence of activities that are each presented in a highly prescribed way. Other curricula, including but not limited to *Developmentally Appropriate Mathematics*, use potentially beneficial activities but leave much of the implementation unspecified. In terms of relational learning, the devil is in these implementation details. One or two explanations—even brilliant explanations—are not enough. Relational learning requires repeated exposure to well-structured examples that permit regularities to be discovered. The *Montessori Method* differs in this critical way, by explaining specifically how to use these materials so as to harness children's relational learning, rather than relying on the materials to spontaneously transmit these relations by their physical properties alone.

5.4.3.3 Summary

Despite being developed 100 years ago, long before contemporary relational learning theories were articulated, all four of the instructional elements we identified are quite evident in the *Montessori Method*. The materials, both concrete and symbolic, are designed to facilitate alignment. The activities themselves center on repeated exposure to critical structures and direct, scaffolded comparisons among elements. The express aim of the activities is to help children isolate place value structure via rich and dense co-occurrences. It seems that although many of Montessori's materials have been adopted in mainstream education, her prescriptions for presenting the materials have not. This may be a serious oversight because these implementation guidelines are where the active ingredients of relational learning likely lie.

5.5 Conclusion

In this chapter, we asked whether current psychological theories that explain how people learn relations can improve our educational approaches to teaching place value to elementary children. We began by reviewing these theories, focusing on statistical learning and structure-mapping theory. The underlying driver of relational learning in these theories is comparison. We identified four instructional elements that facilitate learning from comparisons, including dense co-occurrences, multiple examples, alignable elements, and various forms of scaffolding (e.g., gestures, color-coding, spatial arrangement). Finally, we applied this conceptual framework to three current instructional approaches to see whether relational learning is supported.

The three curricula we reviewed varied in this regard. They all appeal to the notion of “forming associations” and seeing similarities across instances, at least implicitly, and in this way, all are aimed at engaging relational learning. Furthermore, all three included elements such as shared labels, multiple examples, and spatial alignment that should promote learning by comparison. However, simply presenting two examples with a shared label may not be enough instructional support for children to grasp a set of complex relations like those that make up the place value system. Only the *Montessori Method* broke the comparison process down and provided consistent scaffolding for point-by-point alignments. Also, whereas all three curricula included multiple co-occurrences, only the *Montessori Method* was repetitive enough to be considered “dense.” It is possible that the other curricula could be enacted in a way that provided dense exposures to the same structures; however, unless the means of achieving these co-occurrences are communicated clearly to teachers, as well as providing mechanistic explanations for what counts as dense co-occurrences and why the frequency and timing of these exposures are important, too much is left to chance.

The psychology literature offers insight into the mechanisms of relational learning that may be applied in many educational contexts, including mathematics instruction. In this chapter, we focused on the example of place value learning, but the same principles and instructional elements could be useful across a range of mathematics topics and age levels. It is hoped that this chapter provides a framework that can support ongoing efforts to improve children’s learning, within the domain of mathematics and beyond.

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Chapter 6
The Complexity of Basic Number Processing: A Commentary from a Neurocognitive Perspective

Bert De Smedt

Abstract In this commentary, I reflect from a neurocognitive perspective on the four chapters on natural number development included in this section. These chapters show that the development of seemingly basic number processing is much more complex than is often portrayed in neurocognitive research. The chapters collectively illustrate that children's development of natural number cannot be reduced to one basic neurocognitive factor, but instead requires a multitude of skills with different developmental trajectories. Specifically, these contributions highlight that there is much more than the processing of magnitude, or the so-called Approximate Number System, and they elaborate on the roles of subitizing, place value understanding, and children's spontaneous attention to number and relations. They also point out that number is something that needs to be constructed and that number processing is in essence a symbolic activity, which requires the integration of multiple symbolic representations, a focus that has been increasingly emphasized in more recent neurocognitive research. The contributions in this volume provide fresh perspectives that will help to further our understanding of children's natural number development and how it should be fostered. They also offer novel avenues for investigating the origins of atypical mathematical development or dyscalculia.

Keywords Number processing · Neurocognitive factors · Approximate number system · Dyscalculia · Symbolic representations

6.1 Introduction

The four contributions in this section on natural number development in children highlight that the development of seemingly basic number processing is much more complex than is often portrayed in neurocognitive studies in numerical cognition. The section illustrates that basic number processing cannot be reduced to just one core cognitive system or one brain area, such as the intraparietal sulcus (see also

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Vanbinst & De Smedt, 2016). This collection of chapters on natural number development highlights that there is more than the so-called Approximate Number System (ANS) and emphasizes the critical roles of subitizing (Clements, Sarama, & MacDonald, Chap. 2), place value understanding (Mix, Smith, & Crespo, Chap. 5), and children's spontaneous attention to number and relations (McMullen, Chan, Mazzocco, & Hannula-Sormunen, Chap. 4). These chapters point out that number is something that needs to be constructed and that number processing is in essence a symbolic activity, which requires the integration of multiple symbolic representations (Ulrich & Norton, Chap. 3). Interestingly, this focus on symbolic representations has also been emphasized in more recent neurocognitive research (Merkley & Ansari, 2016; Schneider et al., 2017; Vanbinst & De Smedt, 2016).

After a very brief sketch of the neurocognitive approach to number processing, I discuss, against the background of the chapters in this section, the relevance of the ANS (Ulrich & Norton, Chap. 3) and I illustrate that there is more than the processing of magnitude, by pointing to the roles of subitizing (Clements et al., Chap. 2), place value understanding (Mix et al., Chap. 5), and spontaneous focusing on number and relations (McMullen et al., Chap. 4). I end this commentary with some concluding thoughts and avenues for future research, inspired by the four contributions in the current section on natural number development.

6.2 A Neurocognitive Perspective on Number Processing

Neurocognitive research on number processing in children is a young but rapidly expanding field of inquiry, with nearly all studies published in the last decade (Schneider et al., 2017). A key aim in these neurocognitive studies has been to understand why there are large individual differences in the way children acquire mathematical skills (Dowker, 2005) and why learning mathematics is so easy for some but so difficult for others (Berch, Geary, & Mann-Koepke, 2016). It is assumed that by understanding the very basic cognitive processes that underlie these individual differences, learners' profiles can be identified. These profiles then allow one to develop educational interventions and diagnostic approaches that are optimally tailored to the needs of the individual learner. A particular focus in this research has been the study of children with atypical mathematical development, a condition also known as dyscalculia or mathematical learning disability, which is a persistent and specific disorder in learning mathematics that is not explained merely by uncorrected sensory problems, intellectual disabilities, other mental disorders or inadequate instruction (American Psychiatric Association, 2013).

Dyscalculia has been categorized as a *neuro*developmental disorder (American Psychiatric Association, 2013), suggesting that the origin of these difficulties lies at the neurobiological level (De Smedt, Peters, & Ghesquiere, in press). It is important to emphasize that only a handful of brain imaging studies have investigated these neurobiological factors, i.e., brain function and/or structure (De Smedt et al., in press; Peters & De Smedt, 2018) and the same applies to the study of typical development (Merkley & Ansari, 2016). Most research has focused on the study of neurocognitive variables, as these are on a theoretical level closer to the study of neurobiological factors (Hulme & Snowling, 2009). These behavioral studies do not involve collecting neurobiological data, but rather they consist of investigating cognitive variables whose roles can be predicted on the basis of (developmental) brain imaging data on the processing of number and arithmetic (Arsalidou, Pawliw-Levac, Sadeghi, & Pascual-Leone, 2018; Peters & De Smedt, 2018). These cognitive variables can be characterized as domain-specific skills, i.e., skills that are exclusively relevant for learning mathematics (e.g., numerical magnitude processing, De Smedt, Noel, Gilmore, & Ansari, 2013), or domain-general skills that are also relevant for learning in other academic domains, of which working memory has been the most extensively studied (Peng, Namkung, Barnes, & Sun, 2016).

This neurocognitive body of evidence originally focused on non-symbolic numerical magnitude processing as a domain-specific core factor of individual differences in mathematics (e.g., Piazza, 2010) and of dyscalculia (Wilson & Dehaene, 2007). Likewise, neuroimaging studies have narrowed their focus to activity in the intraparietal sulcus (IPS) during mathematical tasks, highlighting it as a key and specific area for processing number (e.g., Nieder & Dehaene, 2009). This narrow focus on one core factor has been seriously criticized and challenged by both behavioral and neuroimaging data.

Several studies have failed to observe an association between non-symbolic number processing and mathematics achievement (De Smedt et al., 2013) and meta-analytic data indicate that this association is small ($r = 0.24$, Schneider et al., 2017). Neuroimaging studies have revealed that many more brain regions other than the IPS show specific increases in activity when children engage in processing number (Arsalidou et al., 2018; Peters & De Smedt, 2018). The increases in brain activity in the IPS during the processing of number have been interpreted to reflect not only numerical processing but also other general cognitive functions, such as spatial working memory, serial order processing, or visual attention (see Fias, 2016, for a discussion). The chapters in this volume collectively align with these criticisms, as they indicate that the understanding of natural number represents a much more complex endeavor that cannot be reduced to one factor. Instead, this understanding builds on a variety of learning mechanisms that are domain-specific as well as domain-general.

6.3 The Approximate Number System: Is It Relevant for Understanding Number Development?

One central concept in many neurocognitive studies on children's number development has been the so-called ANS, or the ability to process non-symbolically presented numerical magnitudes (Dehaene, 1997; Gebuis, Kadosh, & Gevers, 2016; Leibovich, Katzin, Harel, & Henik, 2017). This system has been suggested to be innate as well as to be the foundation of understanding symbolic number and mathematical development (Feigenson, Dehaene, & Spelke, 2004; Piazza et al., 2010) and individual differences therein (Halberda, Mazzocco, & Feigenson, 2008). It has been proposed that the etiology of dyscalculia is best explained by a deficit in this ANS (Wilson & Dehaene, 2007). The existence of an ANS and its role in mathematical development continues to be the most debated topic in the field of numerical cognition (Ansari, 2016; Gebuis et al., 2016; Leibovich et al., 2017). Increasing evidence suggests that the ANS might not be numerical (Gebuis et al., 2016; Leibovich et al., 2017) and that it even may not be the ground onto which the understanding of number, which is in essence a symbolic activity, is built (Leibovich & Ansari, 2016). These neurocognitive studies have been executed without much contact with the relevant work in mathematics education research. The contribution by Ulrich and Norton (Chap. 3), focusing on children's construction of number, nicely illustrates how mathematics education research might help to constrain theories of the ANS and its role in children's understanding of natural number.

Ulrich and Norton (Chap. 3) aptly point to the critical difference between magnitude and number. They indicate that the ANS deals with magnitude but not with number. Number entails the measurement of a magnitude; it needs to be constructed and it necessitates the understanding of a countable unit (see also Clements et al., Chap. 2). This points to the critical role of understanding counting, which requires learning number words and symbolic representations and which takes years of mathematical experience to develop.

The contribution of Ulrich and Norton (Chap. 3) nicely echoes recent discussions in the neurocognitive field on the extent to which the ANS is numerical (Gebuis et al., 2016; Leibovich et al., 2017) and to which it provides a ground for learning symbolic number (Leibovich & Ansari, 2016). For example, Gebuis et al. (2016) contend that the ANS merely reflects the integration of different sensory cues, such as area and/or density, rather than something numerical. These authors argue that a sense of magnitude, based on area or density, rather than a sense of number, enables the discrimination between two magnitudes, as is also suggested by Ulrich and Norton (Chap. 3). Lyons, Bugden, Zheng, De Jesus, and Ansari (2018) recently coined the term Approximate Magnitude System (AMS), as an alternative to ANS. In line with the reasoning of Ulrich and Norton (Chap. 3), AMS might be a better term to denote this cognitive ability.

Another important conundrum in neurocognitive research, touched upon by Ulrich and Norton (Chap. 3), is the extent to which the ANS provides a ground for learning symbolic number (Leibovich & Ansari, 2016). While the dominant theory assumes that the ANS provides the ground for children's symbolic representations of number (Piazza, 2010), this has been seriously challenged by developmental and brain imaging data (Ansari, 2016; Leibovich & Ansari, 2016). For example, Lyons et al. (2018) showed that in kindergartners, symbolic comparison abilities predicted subsequent non-symbolic comparison but not vice versa. This suggests that it is the acquisition of exact number that facilitates growth in the ANS, rather than vice versa. This aligns with the critical role of unitizing and measurement in the development of number, as discussed by Ulrich and Norton (Chap. 3; see also Clements et al., Chap. 2).

The contribution by Ulrich and Norton (Chap. 3) provides new avenues for further study that can benefit from collaborations between researchers in mathematics education and cognitive psychology. These studies should clarify how the awareness of magnitude and the development of number are related. Even though infants may have a sense of magnitude, it may not be critical to learning number. On the other hand, we need to understand how the development of number affects the awareness of magnitude (see Lyons et al., 2018).

This view of the ANS as relevant to magnitude rather than number also offers a fresh perspective on understanding dyscalculia. De Smedt et al. (2013) observed in their review of the literature that impairments on ANS-tasks were only observed in older (starting from age 10) children with dyscalculia (when compared to typically developing children). It might be that children with dyscalculia have a preserved awareness of magnitude, but that they do not benefit as much as typically developing children from their understanding of number or their ability to measure magnitude that allows them to fluently execute the dot comparison task. As suggested by Ulrich and Norton (Chap. 3), children could use different strategies to solve a seemingly basic dot comparison task. Children with dyscalculia might rely more on their perceptual sense of magnitude to perform this task, while typically developing children might rely more on their understanding of (symbolic) number and quantity, leading to differences in performance. Future studies are needed to verify this conjecture. They will require the consideration of different strategies that children use during comparison tasks, and these are not necessarily the same as the ones used by adults, as pointed out by Ulrich and Norton (Chap. 3).

6.4 More than Magnitude: The Roles of Subitizing, Place Value, and Spontaneous Focusing

Subitizing—the immediate apprehension and identification of the exact number of items in small sets up to four items—has been studied for a long time in cognitive psychological research, yet it has been relatively neglected in mathematics education research (Clements et al., Chap. 2). It needs to be emphasized that it is not so easy to measure subitizing reliably, as subitizing is typically a very accurate process that occurs within a timeframe of less than 1 s. Clements et al. (Chap. 2) aptly point out that the basic process of subitizing has a much more complex and protracted developmental course than is assumed in cognitive psychological research. They argue that, during this development, a perceptual process that is in essence non-numerical has to be linked with an exact (symbolic) concept of number, echoing Ulrich and Norton's (Chap. 3) discussion of the ANS. Fully functional subitizing requires the understanding of a countable unit as well as the number words to construct an exact cardinal representation of a collection (Clements et al., Chap. 2), but the critical question remains when in development this happens. This again emphasizes that number processing is in essence a symbolic activity, which requires the integration of multiple symbolic representations (Merkley & Ansari, 2016), the developmental trajectories of which remain to be further understood.

Clements et al. (Chap. 2) also discuss a more complex type of subitizing, *conceptual subitizing,* which has a high educational relevance. Conceptual subitizing refers to the child's ability to organize a set of items via partitioning, decomposing, and composing to quickly determine its number. This conceptual subitizing provides children experiences with additive situations, and it fosters their understanding of part-whole relations, which are a critical scaffold for learning arithmetic operations. This discussion of Clements et al. (Chap. 2) provides a nice example of how elementary numerical activities can act as a stepping-stone for learning more complex arithmetic and mathematics. This type of theorizing on the mechanisms of why basic number processing correlates with more advanced mathematical achievement has been somewhat lacking in neurocognitive studies. These latter studies have typically focused on what predicts mathematics achievement but not on why it predicts this achievement (De Smedt et al., 2013). The combination of perspectives from mathematics education with psychological research might be a fruitful avenue to further understand these mechanisms. Such research is needed to further elucidate when conceptual subitizing develops and how it is related to children's learning of arithmetic and its individual differences.

The large majority of neurocognitive studies on (symbolic) number processing have narrowed their focus to single-digit numbers, but to fully "crack the code" of Arabic numerals, children need to learn place value and multi-digit number meanings, which are concepts that are difficult to master for many of them (Mix et al., Chap. 5). Mix et al. elaborate on this learning of place value and how it can be fostered, through the domain-general lens of relational learning mechanisms, such as statistical learning and structure mapping. Their chapter nicely illustrates that the development of symbolic number is much more complex than the simple mapping between a symbol and the quantity it represents, as has often been assumed in neurocognitive studies. Their chapter offers a key to the solution of the symbol-grounding problem in numerical cognition (Leibovich & Ansari, 2016). More specifically, Mix et al. (Chap. 5) highlight that, in addition to domain-specific numerical mechanisms, domain-general relational learning mechanisms, which play a role in the acquisition of language, particularly the learning of syntax (Ullman, 2004), also need to be investigated. These investigations have the potential to further explain the strong associations between measures of language and mathematics (LeFevre et al., 2010) and to elucidate the comorbidity of dyscalculia with language disorders (Evans & Ullman, 2016).

It is important to emphasize that the learning of place value depends on the transparency of the language in which children learn number. Some languages, such as Chinese, have a very regular alignment between the structure of their number words and their numerals (23 = two times ten and three) whereas other languages, such as Dutch, do not (23 = three-and-twenty). It is evident that the learning of place value will be much harder in the latter languages than in the former and that different types of instruction might be needed in these different languages. In all, this highlights that contextual factors moderate children's understanding of number (see also Clements et al., Chap. 2, and McMullen et al., Chap. 4), an issue that has been central in educational research but that has been often ignored in neurocognitive studies.

McMullen et al. (Chap. 4) guide our attention to children's spontaneous focusing tendencies on number (SFON) and relations (SFOR) and highlight that these are key elements of children's understanding of number and its individual differences. McMullen et al. emphasize that children differ in their attention to mathematical elements of everyday situations outside the formal learning context. Children who are more attentive to the numerical and mathematical aspects of an everyday situation will have more (self-initiated) practice with it and, consequently, develop better mathematical skills. This again points to the critical role of the environment, including both the home and school environment, and the contexts in which children are confronted with number as powerful moderators of children's numerical development. It remains, however, as yet unclear what aspects of the environment trigger children's attention to number. On the other hand, it is clear that children's understanding of number and numerical relations and their spontaneous focus on it develop in an iterative way (McMullen et al., Chap. 4).

6.5 Concluding Thoughts

The chapters in this volume collectively indicate that children's development of natural number cannot be reduced to one basic neurocognitive ability but instead requires a multitude of skills that have different developmental trajectories. These chapters also suggest that these skills develop in a bidirectional way although their precise interactions and their developmental timing need further investigation.

It is also important to point out that the use of the term "neurocognitive" sometimes mistakenly suggests a direction of associations, such that neurocognitive variables are more easily perceived as predictive or causal in learning, in this case, natural number. However, it also might be that learning natural number itself changes related neurocognitive processes. It is the research design and not the type of data (i.e., either neurocognitive or brain imaging data) that determines predictive value or causality. This should be kept in mind when evaluating the existing neurocognitive data. Intervention studies that manipulate a given factor are needed to further determine which factors are causal and which are not. Carefully controlled longitudinal studies (i.e., cross-lagged designs) can also test the directions of associations between these skills (see McMullen et al., Chap. 4, for an example).

The idea that the so-called basic processing of number consists of a multitude of skills also opens opportunities for understanding the origins of dyscalculia, which has been characterized in neurocognitive studies as a disorder that originates from a deficit in processing number (De Smedt et al., in press). Against the background of the chapters in this volume, it seems unlikely that such a deficit in number processing can be reduced to one single deficit in one numerical ability. This echoes recent models of other neurodevelopmental disorders, such as dyslexia or ADHD, which have posited that multiple deficits rather than one single deficit account for their emergence (Peterson & Pennington, 2015). The numerical abilities highlighted in this section might all constitute risk factors for developing deficits in learning to calculate and consequently, future studies on atypical development should consider the relative contribution of each of these risk factors. As has been illustrated throughout the chapters in this volume, these numerical skills are also related to domain-general learning mechanisms, such as statistical learning (Mix et al., Chap. 5), perceptual abilities (Clements et al., Chap. 2), or sensorimotor abilities (Ulrich & Norton, Chap. 3), which also require additional consideration when studying the origins of atypical mathematical development.

The current collection of chapters also reveals that children's learning of natural number will require specific instruction. Clements et al. (Chap. 2) and Mix et al. (Chap. 5) nicely illustrate that cognitive models of different types of numerical skills can help to inform the design of educational programs. Outlining the developmental trajectories of a given numerical ability, such as subitizing, provides a ground for designing activities that can be optimally tailored to support students at various points in these different trajectories (Clements et al., Chap. 2). Similarly, general psychological learning mechanisms, such as statistical learning or structure mapping (Mix et al., Chap. 5) can provide insight into ways to improve educational programs. It needs to be acknowledged that there will be individual differences in both these domain-specific and domain-general components that are critical to understanding number (Vanbinst & De Smedt, 2016). A cognitive analysis of these components will allow educators to verify which abilities require more scaffolding (weaknesses) and which abilities can be used as compensatory factors (strengths) (see Mix et al., Chap. 5). For example, children who are less likely to spontaneously attend to number and relations might require more guided instruction compared with others (McMullen et al., Chap. 4).

To conclude, the contributions in the current volume clearly show that children's understanding of number cannot be reduced to one neurocognitive factor, such as the ANS, but instead represents a complex development of different types of abilities that become gradually connected over development. It is clear that this development involves domain-specific as well as domain-general learning mechanisms. The contributions in this volume provide fresh perspectives that will help to further our understanding of children's natural number development in both the mathematics education and neurocognitive research communities. It is clear that both disciplines can learn from each other and that these chapters are a starting point for further inquiry on the cognitive mechanisms of children's understanding of number as well as on the design and evaluation of educational interventions that aim to support this understanding.

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Part II
Fractions and Operations on Fractions

Chapter 7
Understanding Fractions: Integrating Results from Mathematics Education, Cognitive Psychology, and Neuroscience

Andreas Obersteiner, Thomas Dresler, Silke M. Bieck, and Korbinian Moeller

Abstract Many students face difficulties with fractions. Research in mathematics education and cognitive psychology aims at understanding where and why students struggle with fractions and how to make teaching of fractions more effective. Additionally, neuroscience research is beginning to explore how the human brain processes fractions. Yet, attempts to integrate research results from these disciplines are still scarce. Therefore, the aim of this chapter is to provide an integrated view on research from mathematics education, cognitive psychology, and neuroscience to better understand students' difficulties with fraction processing and fraction learning. We evaluate the difficulties students encounter with fractions on various levels, ranging from the brain level to the classroom level. Current research suggests that the human cognitive system is in principle prepared for processing natural numbers and fractions. Although proficiency with natural numbers is fundamental to learning fractions, the transition from natural numbers to fractions requires modifications of the initial concept of numbers, and natural number processing can interfere with

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fraction processing. Thus, when teaching fractions, it seems important to draw on students' fundamental abilities to process fractions, while explicating fraction properties that are conceptually different from those of natural numbers.

Keywords Rational numbers · Conceptual change · Natural number bias · Fraction processing · Numerical cognition

Students' difficulties with fractions have been studied for decades. Yet, research in cognitive psychology and neuroscience has only recently begun to unravel the underlying cognitive mechanisms of fraction processing, and this research has rarely been integrated with mathematics education. The aim of this chapter is, therefore, to make connections between these three disciplines to better understand the sources of difficulties students face with fractions.

In this chapter, we focus specifically on positive fractions, that is, positive rational numbers represented in the form $\frac{a}{b}$, where a and b are positive natural numbers. However, as fraction learning is an instance of learning about rational numbers more generally—which include negative fractions and numbers represented as decimals (e.g., 0.25)—we also consider core issues of the transition from natural number concepts to rational number concepts.

In the first section of the chapter, we review the importance of fraction learning, including arguments from mathematics education and cognitive psychology. The second section analyzes typical difficulties students encounter in fraction learning as documented by empirical research, as well as potential sources of these difficulties. We analyze difficulties on three different levels: (a) difficulties that may be inherent in the learning content, (b) difficulties that may arise from the way our cognitive system processes fractions, including the neural correlates of fraction processing, and (c) difficulties that may be due to common teaching practices. In the third section, we review experimental intervention studies aimed at supporting students' fraction learning to identify effective ways of instruction that may help students overcome difficulties with fractions. The fourth section includes recommendations for classroom practice and directions for further research. In the fifth section, we conclude the chapter with a suggestion for merging various research perspectives.

7.1 Importance of Fraction Learning

It is widely accepted that fractions are important to learn. A basic understanding of fractions is needed in daily life, for example, to understand information on street signs (e.g., $\frac{3}{4}$ mile), in cooking recipes (e.g., $\frac{1}{2}$ L), or regarding time ("quarter past five").

From a *mathematics education perspective*, fractions are important because they are an essential building block within the domain of numbers, one of the key domains of (school) mathematics (e.g., National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). Between primary school and the end of high school, students are supposed to learn about real numbers in a hierarchical manner. This hierarchy begins with natural numbers, and positive fractions are typically the first type of non-natural numbers students encounter[1]. One motivation for introducing rational numbers is that they allow for describing phenomena that cannot be described by natural numbers alone. For example, all arithmetic operations (addition, subtraction, multiplication, and division) can be performed within the set of rational numbers, which is not the case within natural numbers (e.g., $3-5$ and $1\div 2$ are not defined within natural numbers). Moreover, rational numbers provide solutions to certain types of algebraic equations that do not have a solution within natural numbers, such as $2 \cdot x = 1$.

Fractions allow for a variety of interpretations in the domain of mathematics as well as the real world (Behr, Lesh, Post, & Silver, 1983; Ohlsson, 1988). For example, fractions (e.g., $\frac{3}{4}$) can be interpreted as parts of a whole (divide one whole into four parts and take three of these parts), as several parts of several wholes (take three out of four objects), as division (3 divided by 4), as operators (a function that produces three-fourths of any given input value), as measures of quantities (three quarters of a mile), or as solutions of algebraic equations (the number x that solves the equation $4 \cdot x = 3$). The variety of possible interpretations substantiates the complexity of the concept of fractions, and it suggests that the teaching and learning of fractions deserves careful attention.

From a *cognitive psychological perspective*, understanding fractions requires a higher level of abstraction than understanding natural numbers (DeWolf, Bassok, & Holyoak, 2016; Empson, Levi, & Carpenter, 2011). It may therefore facilitate the transition from concrete to formal operations (Inhelder & Piaget, 1958; Piaget & Inhelder, 1966). In this regard, understanding of fractions seems crucial for mathematical development. There is empirical evidence that fraction understanding is a unique predictor of later achievement in higher mathematics such as algebra. This holds true even when controlling for several other cognitive measures, including general cognitive ability and working memory (Bailey, Hoard, Nugent, & Geary, 2012; Booth & Newton, 2012; Siegler et al., 2012; Torbeyns, Schneider, Xin, & Siegler, 2015).

In sum, fractions are a key target for learning from both a mathematics education and a cognitive psychological perspective. Because fractions are a complex concept, it may not be surprising that learning and teaching fractions can pose special challenges. To analyze these challenges in more detail, the following section summarizes typical errors students make in fraction problems, as well as potential sources of these errors.

[1] There are also curricula in which negative integers are introduced earlier than fractions. This difference in sequencing is not essential for our analyses of difficulties with fraction learning, as we focus predominantly on issues related to the transition from integers to fractions rather than the transition from positive to negative numbers.

7.2 Solving Fraction Problems: Errors and Their Potential Sources

Numerous studies over several decades have documented typical errors students make when solving fraction problems (e.g., Aksu, 1997; Behr, Wachsmuth, & Post, 1985; Behr, Wachsmuth, Post, & Lesh, 1984; Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1981; Carraher, 1996; Hart, 1981; Hasemann, 1981). More recent studies suggest that there has not been significant progress, and that errors are invariant across many different countries and cultures (Bailey et al., 2015; Lortie-Forgues, Tian, & Siegler, 2015; Siegler & Pyke, 2013; Stafylidou & Vosniadou, 2004).

These studies largely converge on a number of major findings. A general observation is that even students who are well able to carry out fraction arithmetic procedures may make errors when problems require fraction concepts (Hallett, Nunes, & Bryant, 2010; Hallett, Nunes, Bryant, & Thorpe, 2012; Siegler & Lortie-Forgues, 2015). One of the concepts that students often struggle with is that of fraction magnitude (Siegler, Thompson, & Schneider, 2011). Rather than seeing a fraction as representing a (rational) number, students tend to interpret a fraction as two separate whole numbers. For example, when a representative sample of eighth-graders in the United States were asked to choose the closest number to the result of $\frac{12}{13}+\frac{7}{8}$ with the options 1, 2, 19, and 21, only 24% chose the correct answer 2 (Carpenter et al., 1981). More than half of them chose 19 or 21, suggesting addition of the numerators (**12 + 7 = 19**) or the denominators (**13 + 8 = 21**) without considering each fractions' integrated magnitude (each being approximately 1). Lortie-Forgues et al. (2015) documented very similar results in a study conducted over 30 years later. Another example of limited understanding of fraction magnitudes is the finding that in fraction addition problems, students' most frequent error is adding the numerators and denominators separately, even though this produces unreasonable outcomes (e.g., $\frac{1}{2}+\frac{1}{2}=\frac{2}{4}$) (Behr et al., 1985; Brown & Quinn, 2006; Siegler & Pyke, 2013). Furthermore, students also struggle with understanding that different symbolic fractions can represent the same numerical magnitude. For example, in a study by Clarke and Roche (2009), more than a third of a sample of Australian sixth-graders did not consider $\frac{2}{4}$ and $\frac{4}{8}$ to be fractions of equal numerical magnitude.

Although many students have relative strength with *carrying out* fraction arithmetic procedures compared to their *understanding* of fraction concepts and procedures, this does not mean that students' performance on fraction arithmetic problems is overall high. Instead, Siegler and Pyke (2013) found that when US sixth- and eighth-graders solved a set of fraction arithmetic problems that included all four basic arithmetic operations (i.e., addition, subtraction, multiplication, and division), they were correct on only 41% (sixth-graders) and 57% (eighth-graders), respectively. They also found that accuracies varied substantially between the different arithmetic operations. While students were most accurate with addition and subtraction, they were less accurate with multiplication and division (Braithwaite, Pyke, & Siegler, 2017; Siegler & Lortie-Forgues, 2017). In addition to difficulties with carrying out arithmetic procedures, students often struggle with predicting the outcomes of arithmetic problems. For example, they are often reluctant to accept that the result of a multiplication problem involving fractions can be smaller than the initial number (Obersteiner, Van Hoof, Verschaffel, & Van Dooren, 2016; Siegler & Lortie-Forgues, 2015; Van Hoof, Vandewalle, Verschaffel, & Van Dooren, 2015). In line with this finding, some students tend to prefer division over multiplication to solve word problems with fractions for which they expect the result to be smaller than the initial number, even when the problem structure suggests multiplication (Swan, 2001).

Another notoriously difficult task for students is reasoning about the structure of the rational number domain as a whole. In a study by Vamvakoussi and Vosniadou (2010), about one-third of 11th-graders responded (incorrectly) that there was only a finite number of numbers between any two rational numbers. An especially common error is to think that increasing any given fraction's numerator by 1 generates the successor of that fraction (e.g., to think that $\frac{3}{5}$ is the successor of $\frac{2}{5}$) (Vamvakoussi & Vosniadou, 2004, 2010), although rational numbers, unlike natural numbers, do not have successors (see Sect. 7.2.1).

In sum, evidence for students' errors in fraction problems, which comes from a variety of studies collected over decades, suggests that difficulties are systematic, persistent over time, and exist in different learning environments. One may wonder what makes fractions so difficult to understand. Are fractions just a difficult mathematical concept? Is the human brain not well prepared to process fractions? Or are there limitations in the way fractions are commonly taught at school? In the following sections, we evaluate potential sources of difficulties with fractions on three different levels (see Lortie-Forgues et al., 2015, for a similar approach). First, we consider the learning content itself. We identify what aspects of fractions differ substantially from natural numbers because these aspects might be particularly challenging for learners. Second, we explore how psychological accounts conceive the mechanism of fraction learning, and—more fundamentally—how well the human cognitive architecture is prepared for processing fractions. Third, we review common teaching practices in mathematics classrooms, based on the available research on textbooks and surveys among teachers.

7.2.1 The Learning Content Itself

Fractions are symbolic representations of rational numbers. Mathematically speaking, rational numbers can be constructed as an extension of the set of integers, with rational numbers being defined as equivalence classes of pairs *(a,b)* of integers a and b, with $b \neq 0$. Two pairs *(a,b)* and *(c,d)* are considered equivalent if and only if $a \cdot d = b \cdot c$. After defining the operations of addition and multiplication, one gets to the field of rational numbers Q. These rational numbers are an extension of the set of natural numbers N in the sense that Q includes N, if one identifies natural numbers with the equivalence classes of those pairs in which the first component is positive and the second component is 1 (e.g., 2, with the equivalence class [2,1]). According to this definition, natural numbers and rational numbers have shared properties (because natural numbers are also rational numbers). For example, for rational and natural numbers, there is an order relation, meaning that for any two different numbers, it is possible to say which one is larger in numerical magnitude. Thus, rational and natural numbers can be represented on number lines.

However, despite shared properties, there are also important differences between the set of natural numbers and the set of rational numbers, and these differences may be stumbling blocks for learners when they have to make the transition from natural numbers to fractions (as representations of rational numbers). There are at least four important ways in which rational numbers—specifically in their representation as fractions—differ from natural numbers (see Obersteiner, Reiss, Van Dooren, & Van Hoof, in press; Prediger, 2008; Vamvakoussi & Vosniadou, 2004; Van Hoof, Vamvakoussi, Van Dooren, & Verschaffel, 2017). Table 7.1 provides an overview of these four differences.

One difference concerns the way natural numbers and fractions convey numerical magnitude. The symbolic representation of natural numbers complies to the base-10 place-value structure of our number system, which allows for straightforward strategies to identify numerical magnitude (see first row of Table 7.1). For instance, deciding which of two numbers is larger is simple because it can be done digit-by-digit from left to right (i.e., for three-digit numbers, comparing hundreds with hundreds, tens with tens, and units with units). Additionally, the number of digits is indicative of the magnitude of a number, with numbers consisting of more digits being larger in magnitude. Fractions, however, are composed of two integers, and only the numerator is positively related to overall fraction magnitude. Reasoning about fraction magnitude requires inferences about the ratio between numerators and denominators. As such, comparing the magnitudes of two fractions is less straightforward than comparing the magnitudes of natural numbers. Moreover,

Table 7.1 Examples of differences between natural numbers and fractions

	Natural numbers	Fractions
1. Representation of Magnitude	Base-10 place-value structure More digits—larger number $123 > 45$	Quotient of two numbers Neither number of digits nor natural number magnitudes as such determine fraction magnitudes $\frac{2}{3} > \frac{5}{19}$
2. Symbolic Representation	Unique for each number 2 as unique representation	Multiple (infinitely many) fractions can represent the same number $\frac{1}{2} = \frac{2}{4} = \frac{4}{8} =$ etc.
3. Density	Unique successors and predecessors Finite number of numbers between two natural numbers 1, 2, 3, 4, 5, etc.	No unique successors and predecessors Infinite number of numbers between two fractions $\frac{3}{5}$ is not the successor of $\frac{2}{5}$
4. Operation	Multiplication as repeated addition $3 \cdot 4 = 4 + 4 + 4$ Multiplication makes bigger, division smaller $2 \cdot 4 = 8,\ 15 \div 3 = 5$	Multiplication as repeated addition insufficient, more abstract definition required Multiplication and division can make bigger or smaller $\frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8},\ \frac{1}{2} \div \frac{1}{4} = 2$

comparing fractions may be counterintuitive because the larger fraction can be composed of the larger components (e.g., $\frac{4}{5} > \frac{1}{3}$), the smaller components (e.g., $\frac{1}{2} > \frac{3}{7}$), or one larger component (numerator) and one smaller component (denominator, e.g., $\frac{2}{3} > \frac{1}{5}$).

A second difference is that symbolic representations for natural numbers are unique in the sense that there is only one way to write any given number using only natural number notations (e.g., there is only one way to notate the number “2”). In contrast, different fraction symbols can represent the same numerical value (see second row of Table 7.1).

A third aspect in which fractions differ from natural numbers is density (see third row of Table 7.1). While natural numbers have unique successors and predecessors (except for number 1), this is not the case for any rational number. Moreover, while within the natural number domain there is only a finite number of numbers between any two natural numbers, there are infinitely many other fractions between any two fractions.

Fourth, fractions differ from natural numbers with respect to arithmetic operations (see fourth row of Table 7.1). There is a difference in the way arithmetic operations are conceptualized. Whereas within natural numbers, multiplication is typically explained as repeated addition (i.e., $3 \cdot 4$ means to add the number 4 three times), this explanation is not generally meaningful for fractions. In the example of $\frac{2}{3} \cdot \frac{1}{2}$, it is hard to understand what adding $\frac{2}{3}$ times the number $\frac{1}{2}$ means. Furthermore, there is a difference in the effects that arithmetic operations have on numbers. While multiplication with natural numbers (other than 1) always yields a result that is numerically larger than the original operands, this is not always true for fractions. Instead, multiplying a positive number by a fraction smaller than one (e.g., $\frac{1}{4}$) makes the initial number smaller (e.g., $\frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$). Similarly, within natural numbers, division (by a number other than 1) always makes a number smaller, while within rational numbers, division can also make a number larger (e.g., $4 \div \frac{2}{3} = 6$).

Although the conceptual differences between natural numbers and fractions analyzed in this subsection are potential obstacles for learning, our analysis is not sufficient to identify learners’ actual obstacles. The reason is that the analysis of the subject domain does not take into account the cognitive mechanisms underlying learning. Since learning does not necessarily follow the logic of the subject domain, insights into the cognitive mechanisms of learning can complement our search for difficulties with fractions.

7.2.2 The Human Cognitive System

Learning fractions may be influenced by the way our cognitive system processes new information, and more specifically by the way it processes numbers in general and fractions in particular. The following four subsections describe theoretical frameworks and empirical evidence that help in understanding the cognitive challenges of learning fractions. The first two subsections elaborate on theories of conceptual learning (conceptual change) and of the cognitive processes that occur during problem solving (dual processes) that may account for response biases (the natural number bias). The remaining two subsections then "zoom in" on the more fundamental ways our cognitive system processes fractions, and on the neural correlates of these processes.

7.2.2.1 Conceptual Change

Natural numbers are special cases of rational numbers (see Sect. 7.2.1), differing from other rational numbers in several respects. Therefore, learning fractions requires not only the extension but also the reorganization of existing knowledge about (natural) numbers. Accordingly, researchers have studied learning of rational numbers as an instance of conceptual change (Vamvakoussi, Van Dooren, & Verschaffel, 2012; Vamvakoussi & Vosniadou, 2004, 2010). The conceptual change approach was initially applied to the domain of science learning but was later transferred to mathematical learning as well (Merenluoto & Lehtinen, 2002).

In line with the conceptual change approach, there is broad evidence that students' errors in operating with fractions may be due to their reliance on natural number concepts in problems that require reasoning about rational number concepts. For example, when comparing the magnitudes of two fractions, children were found to rely on their natural number knowledge and treat fraction components as two separate natural numbers, rather than reasoning about the overall magnitudes of the respective fractions. Only 15% of more than 300 sixth-graders in the study by Clarke and Roche (2009) were able to correctly choose the larger fraction from the pair $\frac{5}{6}$ versus $\frac{7}{8}$ and provide an appropriate explanation for their choice. Almost 30% of all students in this study claimed that these fractions were the same because the difference between the numerator and the denominator was equal in both fractions. These students relied on reasoning about number magnitudes in ways that apply to natural numbers (each symbol represents a separate magnitude), although the problem required a conceptual change (quotients of *two* [natural] numbers represent *one* [rational] number magnitude). There is evidence that students also struggle with other concepts of fractions that differ from natural number concepts (i.e., those described in 2.1 and listed in Table 7.1), as predicted by the conceptual change approach (Merenluoto & Lehtinen, 2002; Vamvakoussi & Vosniadou, 2004, 2010; Van Hoof et al., 2017, see also the introduction to Sect. 7.2).

In contrast to such a focus on discontinuities in the learning process, other researchers have emphasized commonalities between natural and rational numbers and considered learning of numbers as a continuous learning path, rather than an instance of conceptual change. In their *integrative theory of numerical development*, Siegler and colleagues (Siegler et al., 2011; Siegler & Braithwaite, 2017; Siegler & Lortie-Forgues, 2014) emphasized that magnitude is the unifying idea between different kinds of numbers such as natural and rational numbers. As all real numbers (including natural and rational numbers) have magnitudes and can be represented on number lines, understanding these magnitudes may be particularly helpful for learners in extending their number knowledge to new number domains. Although there is initial evidence that understanding fraction magnitudes facilitates learning of fraction concepts more generally (see Sect. 7.3), the specific relation between understanding of fraction magnitudes and other fraction concepts remains to be understood.

Steffe and colleagues (e.g., Steffe, 2002; Steffe & Olive, 2010) proposed a constructivist account of fraction learning that also emphasizes the coherence between natural number knowledge and learning fractions. In their *reorganization hypothesis*, they consider how children's natural number knowledge may be modified in productive ways to construct fraction knowledge (for details, see also Tzur et al., this volume).

Note that these two theoretical accounts focus on the coherence between natural numbers and fractions, while our focus in this chapter is more strongly on the challenges (rather than the coherence) in students' transition from natural numbers to fractions. We relied on the conceptual change approach in this section because it connects these challenges to the conceptual differences between natural numbers and rational numbers. A systematic discussion of the various accounts proposed for learning fractions is, however, beyond the scope of this chapter.

7.2.2.2 Dual-Process Theories and the Natural Number Bias

Some researchers have focused more strongly on the cognitive processes involved in fraction problem solving rather than on an understanding of fraction concepts. *Dual-process theories* assume that problem solving includes two types of processes: Processes that are fast, largely automatic, and intuitive ("System 1 processes") and processes that are analytic and time-consuming ("System 2 processes") (Gillard, Van Dooren, Schaeken, & Verschaffel, 2009; Kahneman, 2000). When people solve rational number problems, their strongly internalized knowledge of natural numbers might trigger intuitive System 1 processes, while analytic System 2 processes are particularly important when problems require reasoning about novel and less automatized features of rational numbers.

The overreliance on natural number knowledge even in problems that require rational number reasoning has been referred to as the "whole number bias" or "natural number bias" (Alibali & Sidney, 2015; Ni & Zhou, 2005; Van Hoof et al., 2017). To investigate the natural number bias, researchers have compared performance on problems that are either *congruent* or *incongruent* with natural number reasoning. Problems are congruent when reasoning about natural numbers (rather than rational numbers) yields the correct response, and they are incongruent when this is not the case. For example, in fraction comparison, the two to-be-compared fractions of a pair can be classified as congruent when comparing denominators and numerators separately yields the correct result (e.g., $\frac{4}{5} > \frac{1}{3}$ with $4 > 1$ and $5 > 3$) but incongruent when doing so leads to an incorrect result (e.g., $\frac{1}{2} > \frac{3}{7}$ although $1 < 3$ and $2 < 7$). In the case of arithmetic operations with fractions, the intuition that multiplication makes numbers bigger may lead to a correct response in problems congruent with natural number characteristics (e.g., "Is it possible that $4 \cdot x$ is larger than 4?", where considering x a natural number will lead to a correct response) but to an incorrect response in incongruent problems (e.g., "Is it possible that $4 \cdot x$ is smaller than 4?").

Importantly, numerous studies have documented this bias, not only in primary and lower secondary school students, but also in upper secondary students and adu lts (Byrnes & Wasik, 1991; Obersteiner, Van Dooren, Van Hoof, & Verschaffel, 2013; Siegler & Lortie-Forgues, 2015; Vamvakoussi et al., 2012; Van Hoof et al., 2015; Van Hoof, Lijnen, Verschaffel, & Van Dooren, 2013). These findings suggest that the natural number bias in fraction problems can persist even after people have acquired sound conceptual knowledge of fractions. This implies that solving fraction problems requires—in addition to conceptual understanding of fractions—some inhibition of intuitive knowledge about natural numbers.

7.2.2.3 Processing of Fraction Magnitudes

Research suggests that our cognitive system is well prepared for processing natural numbers (see Feigenson, Dehaene, & Spelke, 2004, for a review). There is, however, more controversy about how well our cognitive system is prepared for processing fractions. A central question is whether people can mentally process fractions *holistically* by their integrated fraction magnitudes (e.g., $\frac{2}{5}$ as one numerical value), or whether they can only process fractions *componentially* by their components (e.g., $\frac{2}{5}$ as two separate numbers, 2 and 5). Numerous studies have used fraction comparison tasks and evaluated whether participants' comparison performance depended on the numerical distance between fractions or on the distances between fraction components. When comparing natural numbers, a typical finding is that responses become faster and less error prone, as the numerical distance between to-be-compared numbers gets larger (e.g., 1 vs. 9 is easier than 4 vs. 5). This finding is often referred to as the *numerical distance effect*. The distance effect is considered evidence that people actually rely on number magnitude information when comparing two numbers (Moyer & Landauer, 1967).

Initial studies found no such distance effect for fractions and concluded that people mentally represent fractions predominantly in a componential way, that is, they represent each component separately rather than represent the fraction as an integrated entity (Bonato, Fabbri, Umiltà, & Zorzi, 2007; Ganor-Stern, Karasik-Rivkin, & Tzelgov, 2011). However, later studies revealed that the way participants process fractions depended on the type of fraction comparison and on the strategies they use to solve these problems (Faulkenberry & Pierce, 2011; Ganor-Stern, 2012; Meert, Grégoire, & Noël, 2010a, b; Obersteiner et al., 2013; Schneider & Siegler, 2010). For instance, Obersteiner et al. (2013) found that when academic mathematicians solved fraction comparisons, there was a distance effect of overall fraction magnitude only for fraction pairs that did not have common components (e.g., $\frac{11}{18}$ vs. $\frac{19}{24}$). Additionally, they observed no natural number bias for these problems. However, when fraction pairs did have common components (e.g., $\frac{17}{23}$ vs. $\frac{20}{23}$, or $\frac{12}{13}$ vs. $\frac{12}{19}$), there was no effect of overall distance and a clear natural number bias, which was reflected by lower performance on incongruent rather than congruent problems. Together, this line of research suggests that adults rely more strongly on componential comparison strategies in comparison problems with common components (with less activation of holistic overall fraction magnitudes). Such a strategy is more prone to natural number bias. In contrast, adults seem to rely more strongly on holistic magnitudes in problems without common components, a strategy that discourages natural number bias. Recent eye-tracking research substantiated the claim that adults use different strategies depending on problem types (Huber, Moeller, & Nuerk, 2014; Ischebeck, Weilharter, & Korner, 2016; Obersteiner & Tumpek, 2016).

Research also suggests that the way people process fractions depends on how familiar they are with specific fractions. Liu (2018) found that when participants compared symbolic fractions to values marked on a number line, their performance depended on how close fractions were to familiar fractions (e.g., $\frac{1}{2}$ or $\frac{3}{4}$) that people used as benchmarks. Thus, whether fractions are processed holistically may also be a question of practice and familiarity rather than of cognitive ability alone.

These studies provide evidence that adults' cognitive architecture allows them to process symbolic fractions in a holistic manner. Further research suggests that the ability to process fractions and ratios may be traced back to very fundamental abilities for processing non-symbolic ratios, and that humans are equipped with a perceptually based ratio processing system (Boyer & Levine, 2015; Lewis, Matthews, & Hubbard, 2016; Matthews & Chesney, 2015; Matthews, Lewis, & Hubbard, 2016). As such, this processing system might be predisposed for developing magnitude representations of fractions (see Matthews et al., this volume).

7.2.2.4 Neural Correlates of Fraction Processing

In recent years, researchers have begun to evaluate the neurocognitive foundations of numerical cognition using neuroimaging (Arsalidou & Taylor, 2011; Dehaene, Piazza, Pinel, & Cohen, 2003). An increasing number of studies on adults and children revealed that the intraparietal sulcus (IPS; see Fig. 7.1) seems to be the central area for representing symbolic and non-symbolic numerical magnitudes (Nieder & Dehaene, 2009; Piazza, Pinel, Le Bihan, & Dehaene, 2007; Pinel, Dehaene, Rivière, & LeBihan, 2001). A major finding that led to this conclusion was that neural activation within the IPS is inversely related to the numerical distance between two to-be-compared numbers in number comparison tasks (Cohen Kadosh et al., 2005; Kaufmann et al., 2005), reflecting the neural instantiation of the behavioral distance effect (see Sect. 7.2.2.3). Studies also report activation of frontal brain areas during number processing, resulting in the notion of a fronto-parietal network underlying numerical processing (Ansari, Garcia, Lucas, Hamon, & Dhital, 2005; Pesenti, Thioux, Seron, & Volder, 2000).

Concerning rational numbers and fractions in particular, some researchers initially suggested that fraction concepts were fundamentally incompatible with the neurocognitive architectures underlying numerical cognition (Dehaene, 1998; Feigenson et al., 2004; Geary, Hoard, Byrd-Craven, Nugent, & Numtee, 2007).

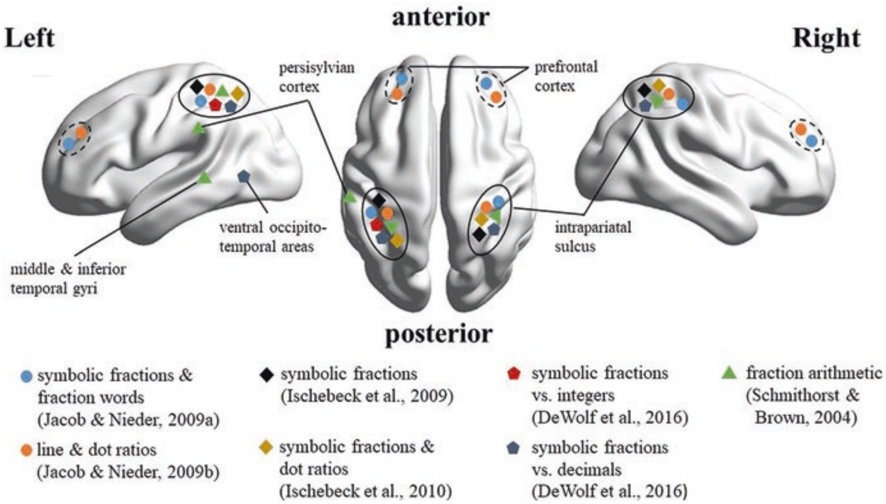


Fig. 7.1 Schematic overview of brain areas involved in fraction processing during (i) magnitude comparison of different notations (i.e., symbolic fractions, fraction words, line and dot ratios; Jacob & Nieder, 2009a, 2009b; Ischebeck et al., 2009, 2010); (ii) magnitude comparisons between fractions, decimals, and integers (DeWolf et al., 2016); and (iii) fraction arithmetic (i.e., addition and subtraction; Schmithorst & Brown, 2004). Most of the studies consistently observed activation in the bilateral intraparietal sulcus (IPS, full line ellipses). However, there are also studies showing additional bilateral prefrontal cortex activation (PFC, dashed ellipses) and left-lateralized activation for specific tasks (e.g., left inferior and middle temporal gyrus for the comparison of fractions and decimals, DeWolf et al. (2016); left ventral occipitotemporal and perisylvian areas were activated in fraction arithmetic, Schmithorst & Brown, 2004). In general, activation patterns observed for fraction processing are very similar to those found for natural number processing. This figure was adapted from Lewis et al. (2016).

Therefore, the question remains what the underlying neural mechanisms for fraction processing are. At the moment, there exist only a few studies in adults investigating the neural underpinnings of processing proportions (i.e., dot ratios, line ratios), symbolic fractions, or fraction number words (Ischebeck et al., 2010; Ischebeck, Schocke, & Delazer, 2009; Jacob & Nieder, 2009a, 2009b; Schmithorst & Brown, 2004). For instance, during fraction magnitude comparison, Ischebeck and colleagues (2009) observed that IPS activation was modulated by the overall numerical distance between the to-be-compared fractions, but not by the numerical distance between numerators or denominators. Moreover, Ischebeck et al. (2010) observed the same results during proportion comparison (involving symbolic fractions and dot patterns as non-symbolic proportions), with stronger right IPS activation for dot patterns and stronger left IPS activation for symbolic fractions. Jacob and Nieder (2009a, 2009b) adapted participants to a certain fraction magnitude (e.g., $\frac{1}{6}$) by showing different fractions reflecting this magnitude (e.g., $\frac{1}{6}$, $\frac{2}{12}$, $\frac{5}{30}$) with interspersed deviants differing in magnitude (e.g., $\frac{2}{6}$, $\frac{3}{6}$, $\frac{4}{6}$, $\frac{5}{6}$) presented in the same or a different notation (i.e., symbolic fractions and fraction words or dots and triangles). The authors observed that the activation in parietal cortex was specifically tuned to the overall magnitudes of fractions rather than to the magnitudes of their components, indicating that fraction magnitude is represented holistically in the same brain areas as natural numbers. Moreover, Jacob and Nieder (2009a) provided evidence for a notation-independent activation patterns. In particular, they reported that the same cortical areas were activated to a similar extent regardless of whether a fraction magnitude was presented as a symbolic fraction (i.e., $\frac{1}{4}$) or written as a number word (i.e., “one-fourth”).

Overall, these studies indicated that the IPS plays a crucial role in the processing of proportion and fraction magnitude, similar to the processing of natural numbers. In contrast to behavioral studies on fraction magnitude comparison, which showed that holistic versus componential processing of fractions depended on the respective fraction type (i.e., with vs. without common components, see Sect. 7.2.2.3), the existing neuroimaging data suggest that fraction magnitudes are represented holistically on the neural level.

Furthermore, fraction arithmetic also seems to elicit patterns of neuronal activation similar to those observed for natural number arithmetic. Schmithorst and Brown (2004) studied adult participants solving fraction addition or subtraction problems. Their analyses again revealed activation in bilateral inferior parietal areas (including the IPS) with additional activation in left-hemispheric perisylvian areas (associated with verbal processing), and ventral occipitotemporal areas (often associated with more perceptual aspects, i.e., ventral visual pathway, see Fig. 7.1).

In spite of the generally large overlap in the neural networks for natural numbers and fractions documented in these studies, DeWolf, Chiang, Bassok, Holyoak, and Monti (2016) found differences in activation patterns within the IPS for fractions as compared to whole numbers and decimals. The authors argue that these differences in activation patterns may be due to the differences in the symbolic notations we use for natural numbers and decimals (both base-10 representations) on the one hand and fractions on the other (two natural numbers). Presumably, our brain needs more resources to get access to the magnitudes of fractions than those of natural numbers or decimals. This assumption is in line with evidence from behavioral research (DeWolf, Grounds, Bassok, & Holyoak, 2014).

Finally, based on theoretical considerations and initial empirical evidence, Lewis et al. (2016) recently argued that there exists a neural circuitry specifically dedicated to represent non-symbolic proportions comprising a fronto-parietal network. According to these authors, this system is also recruited when representing fractions as it provides a non-symbolic foundation for understanding fraction concepts. In particular, the authors proposed that both formal and informal learning experiences help to generate links between perceptually based representations of non-symbolic ratios and fraction symbols (i.e., verbal fraction labels and symbolic-digital fraction symbols). This non-symbolic-to-symbolic link may be an important basis for the understanding of fraction magnitudes.

Taken together, these studies suggest that the human brain is able to process holistic fraction magnitude. The IPS, which has long been known to be the key area for the representation of natural number magnitude, also seems crucial for processing fraction magnitudes. However, strong conclusions seem premature, due to the limited number of available studies. Moreover, all existing studies examined the neural correlates of fraction processing in adults, and studies on the neural correlates of how fraction processing develops and shapes the brain are completely lacking. An important external factor that may shape the way students think about fractions is the way they encounter fractions in the classroom.

7.2.3 *Current Classroom Teaching Practices*

Fractions are complex constructs, and there are many ways to interpret and represent fractions (see Sect. 7.1). There may be considerable variation in the ways students encounter fractions in the classroom, and varying classroom experiences may affect fraction learning. To date, there is little empirical evidence about how fractions are actually taught in classrooms. Much of the existing research into teaching of fractions has focused on teachers' competence with fractions and on the instructional materials teachers use. In the following, we first review general characteristics of common classroom teaching of fractions that might contribute to students' difficulties. We then focus on the quality of instructional materials, and finally on teachers' competence with fractions.

7.2.3.1 Characteristics of Classroom Teaching

One characteristic of current classroom teaching of fractions—at least in many Western countries—is a strong focus on memorization of procedures rather than on understanding of fraction concepts (Lortie-Forgues et al., 2015; National Mathematics Advisory Panel, 2008). Such a focus may have benefits in the short run: procedures are probably easier to teach, easier to test, and they may promise quicker success (and thus motivation). However, important disadvantages are that procedures are remembered less well if they are not connected to conceptual understanding, and that they may lead to inert knowledge that cannot be adapted for novel contexts (Swan, 2001). Moreover, given the sheer number and the relative complexity of fraction arithmetic procedures, students may confuse fraction procedures or parts of them. Finally, the omnipresence of electronic computing devices in our modern society raises fundamental questions about the importance of learning arithmetic procedures.

Another characteristic of current fraction teaching is the dominance of interpreting fractions as discrete and countable parts of a whole (e.g., pieces of a pizza). Teachers often also use this approach to introduce fraction procedures. For example, when learning about fraction multiplication, the first type of problems is often of the form "natural number × fraction" (e.g., $3 \cdot \frac{1}{4}$), which can be explained by repeated addition (take three quarters of a pizza, or $3 \cdot \frac{1}{4} = \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$). Such a strong emphasis on the part-whole relation of fractions as countable objects could be problematic because it could raise the expectation in students that fractions are not very different from natural numbers. Such an expectation may lead to overgeneralizations of natural number properties to fractions, as discussed above (see Sect. 7.2.2), including natural number bias and componential processing of fractions.

More generally, in many current classrooms, particularly in Western countries, there seems to be strong emphasis on the commonalities between natural numbers and fractions (as rational numbers), and only little emphasis on the differences between these types of numbers (Van Hoof et al., 2017). This seems problematic from the perspective of learning psychology, which would recommend fostering both generalization learning (emphasizing similarities between natural and rational numbers) and discrimination learning (explicating differences between natural and rational numbers) (see Sect. 7.4.1 for a discussion).

7.2.3.2 Instructional Material

Teaching practices may be influenced by the available instructional materials such as textbooks. There are a few systematic analyses of instructional materials on fractions (Alajmi, 2012; Braithwaite et al., 2017; Shin & Lee, 2017; Son & Senk, 2010; Watanabe, 2007). Their findings suggest that the majority of fraction problems in textbooks require procedural rather than conceptual knowledge (Son & Senk, 2010), and that textbooks often focus on standard algorithms for solving these problems (Alajmi, 2012). Moreover, there are large variations in the frequency with which textbooks present different types of fraction problems. For example, fraction division problems—the most challenging type of fraction problems for most students (see the introduction to Sect. 7.2)—are much less frequent than multiplication problems (Siegler & Lortie-Forgues, 2017; Son & Senk, 2010). Braithwaite et al. (2017) extended this finding by developing a computational model of fraction arithmetic that simulated students' most frequent errors in fraction arithmetic procedures as documented in empirical studies. Using problems from common US mathematics textbooks as input, the model predicted students' typical errors fairly well. Thus, the type of problems and the frequency with which these types appear in textbooks may to some extent explain students' difficulties with fractions.

7.2.3.3 Teacher Competence

Classroom materials do not entirely determine how the content is taught in the classroom. Rather, it is the role of the teacher to use instructional materials in a specific way. Teachers thus need to be competent with fractions in order to teach fractions appropriately. Unfortunately, research suggests that not all teachers have sufficient competence with fractions (Ball, 1990; Depaepe et al., 2015; Ma, 1999; Newton, 2008; Siegler & Lortie-Forgues, 2015; Simon & Blume, 1994). For example, Depaepe et al. (2015) found that, on average, prospective teachers were correct on only 75% of items that assessed conceptual fraction knowledge—even after having taken a course on teaching rational numbers. Siegler and Lortie-Forgues (2015) found that when pre-service teachers were asked to predict in which direction fraction arithmetic operations would change an initial number (e.g., whether $\frac{31}{56} \cdot \frac{17}{42} > \frac{31}{56}$ was true or false), they performed significantly lower (in some cases as low as about 30% correct) when these predictions were not in line with natural number reasoning (i.e., when the result suggested that multiplication makes the original operand smaller) than when they were. Thus, these prospective teachers showed response biases similar to those documented in students (see Sect. 7.2.2.2). Additionally, Ball (1990) and Ma (1999) found that teachers had particular difficulties with generating appropriate stories or situations for a given fraction division problem. In conclusion, limitations in teachers' understanding of fractions may aggravate the limitations of classroom materials discussed above.

7.3 Improvements: Evidence from Intervention Studies

While there are various intervention studies in the literature, few studies have evaluated the effectiveness of interventions in controlled experimental designs (for an overview of intervention studies especially for students with math difficulties, see Shin & Bryant, 2015). In the following, we elaborate on selected studies that focused on fraction magnitude understanding and used highly controlled experimental designs with control or comparison groups.

In a study by Gabriel et al. (2012), Belgian fourth- and fifth-graders played games that involved cards with different representations of fractions as well as wooden disks that children used to represent and manipulate fractions. Using these representations, children worked on comparisons of fraction magnitudes and on matching symbolic fractions with non-symbolic fraction representations. There were two 30-min intervention sessions per week, over a period of ten weeks. Results showed significantly greater improvements in conceptual understanding of fractions in children in the experimental group compared to children in a control group who received regular classroom instruction but no intervention. Instead, children in the control group showed significantly higher gains in procedural fraction arithmetic skills, substantiating that typical classroom teaching focuses more strongly on procedures than on concepts (see Sect. 7.2.3.1).

Fuchs et al. (2013) designed an intervention that also included training of general cognitive abilities.[2] Participants were US-American fourth-graders who performed below the 35th percentile on an arithmetic test and who were therefore considered to be at risk of low mathematical achievement. The study contrasted two different instructional approaches. The more conventional approach focused on part-whole aspects of fractions and on procedural aspects of fraction arithmetic, whereas the other, more innovative approach emphasized the measurement aspect of fractions and focused on fraction magnitudes. Each session lasted 30 min, with three sessions per week over a period of twelve weeks. The results showed that children who were

[2] See Lamon (2007) and Fazio, Kennedy, and Siegler (2016) for intervention studies with similar approaches.

taught with a focus on the measurement aspect of fractions showed greater gains in conceptual and procedural understanding of fractions than children who received conventional teaching. In another study, Fuchs et al. (2016) replicated these positive effects with a similar version of the measurement-based intervention program in another sample of at-risk fourth-graders.

Finally, Hamdan and Gunderson (2017) compared an intervention based on the use of number lines with an intervention that focused on the area model of fractions. The area model represents fractions as parts of two-dimensional shapes such as circles. Participants were children in grades two and three in the USA. The intervention occurred on one day within a 30-min time period. From pretest to posttest, relative to a control group, both interventions led to improvements on problems that required using the respective representation (number line or area) that children used during the intervention phase. However, children who used number lines were better able to transfer their knowledge to novel problems than were learners who used area models during the intervention phase. This suggests that the use of number lines may be particularly beneficial for fraction learning.

These examples illustrate how studies implemented interventions on enhancing understanding of fraction magnitudes (rather than on other aspects such as fraction arithmetic procedures). Overall, such a focus seems to be effective as it allows transfer to other fraction concepts. Evidence from broader intervention studies not reported here (e.g., Butler, Miller, Crehan, Babbitt, & Pierce, 2003; Cramer, Post, & delMas, 2002; Moss & Case, 1999) largely supports this conclusion. On the other hand, most controlled intervention studies contrasted only one or two different teaching approaches against control conditions, making it difficult to identify which of the large variety of teaching approaches is the most effective one.

7.4 Recommendations and Future Directions

In this section, we first draw conclusions that are relevant for the teaching and learning of fractions in the classroom, and then discuss directions for future research.

7.4.1 Recommendations for Classroom Practice

Rather than providing a comprehensive overview of recommendations about fraction teaching in general (for plenty of valuable recommendations, see, for example, Carraher, 1996; Moss & Case, 1999; National Mathematics Advisory Panel, 2008; Steffe & Olive, 2010), we restrict our discussion to six recommendations that follow from the different perspectives discussed in the previous sections.

Our first recommendation is that fraction teaching may benefit from *drawing more strongly on fundamental cognitive abilities for processing fractions and ratios*. There are plenty of psychological studies that suggest that our cognitive system is readily able to process magnitudes of symbolic fractions. The available neuroscience studies corroborate this conclusion and suggest that processing fractions activates a similar neural network as processing natural numbers—although caution is required due to the lack of evidence in young learners. Importantly, though, behavioral studies including young children suggest that the ability to process fraction magnitudes may be rooted in fundamental abilities to process non-symbolic ratios (see Sect. 7.2.2.3). Thus, instruction may draw on these fundamental abilities even before introducing symbolic fractions, for example by using appropriate visual representations such as bar representations. In particular, continuous rather than discrete bar representations may have the advantage that they encourage students to focus on holistic magnitudes of fractions rather than on countable segments (Boyer, Levine, & Huttenlocher, 2008; Huttenlocher, Duffy, & Levine, 2002). Connecting these representations with symbolic fractions later on may then help prevent the common overreliance on natural number concepts when working with fractions (i.e., the natural number bias). This conclusion remains tentative because there is limited empirical evidence for the effectiveness of specific visual representations of fractions (Rau & Matthews, 2017).

Our second recommendation is that instruction on fractions may benefit from a *stronger focus on fraction magnitudes and the use of number lines*. This is related to the previous recommendation (in that fraction magnitudes should be linked to students' early abilities) but it is more general. Research shows that students have difficulties with understanding how fraction symbols represent numerical magnitudes. Therefore, as supported by results from controlled intervention studies (see Sect. 7.3), students may benefit from a stronger focus on fraction magnitudes (e.g., the measurement aspect of fractions) rather than on the part-whole aspect of fractions. In particular, notwithstanding the limited empirical evidence for visual representations of fractions, number lines have proven to be effective for representing magnitudes of symbolic fractions. A unique advantage of the number line representation is that all real numbers can be represented on the same line, so that this representation may foster students' ability to integrate their concept of numbers across number domains (Booth & Newton, 2012; Common Core State Standards Initiative, 2010; Gersten, Schumacher, & Jordan, 2017; Hamdan & Gunderson, 2017; National Mathematics Advisory Panel, 2008; Siegler et al., 2011).

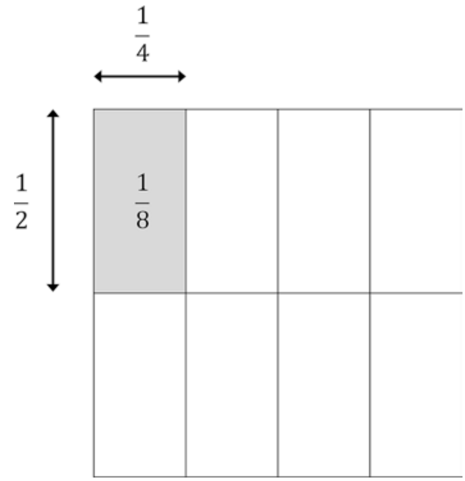
Our third recommendation is that students may benefit from *meta-level prompts to "stop and think"* in order to inhibit potentially misleading intuitive or "System 1" thinking. Although reasoning about fractions necessarily requires knowledge of natural numbers, intuitive knowledge of natural numbers can interfere with processing fractions, resulting in the natural number bias (see Sect. 7.2.2.2). Research suggests that this bias is very persistent, and that it influences performance on fraction problems even in individuals who have acquired sound conceptual understanding of fractions. Thus, students may make errors because they do not engage in analytical thinking regardless of their level of conceptual knowledge. Therefore, it seems advisable to encourage students to think about the reasonableness of their responses, especially for fraction arithmetic. Two concrete instructional approaches to address that goal are prompting students to self-explain their reasoning and refutation texts that require students to argue why a given solution is wrong (Tippett, 2010; Van Hoof et al., 2017).

Our fourth recommendation is that teachers should offer their students *sufficient opportunities to acquire concepts of fractions and fraction operations*. The conceptual change approach (see Sect. 7.2.2.1) suggests that changes in learners' initial concepts may be challenging, and a content analysis (see Sect. 7.2.1) can identify those concepts that learners need to change in order to fully understand fractions. Importantly, several concepts can be relevant at the same time for understanding any one problem situation. To illustrate this, consider the common erroneous expectation that multiplication always makes a number bigger (see Prediger, 2008, for more details on that example). This expectation may be due to three misconceptions. First, students may have internalized as a "rule" the regularity that multiplication makes bigger because they have never experienced a situation in which this was not true. In that case, students may benefit from applying the multiplication algorithm to fraction multiplication problems in which they then discover that multiplication can actually make a number smaller. Second, students may be unable to conceptualize what multiplying two fractions means because repeated addition does not offer a meaningful interpretation (see Sect. 7.2.1). The third scenario is related: Students may be unable to conceptualize what multiplying two fraction means because their concept of fractions is limited to the part-whole aspect. It is in fact difficult to understand what multiplying $\frac{1}{4}$ of a pizza with $\frac{2}{3}$ of a pizza should mean. In the latter two scenarios, students need to acquire appropriate concepts of fractions and fraction operations (see also Simon et al., this volume, for an elaboration on fraction multiplication). For example, the multiplication problem $\frac{1}{2}\cdot\frac{1}{4}$ may be explained as "$\frac{1}{2}$ *of* $\frac{1}{4}$," where $\frac{1}{2}$ is an operator that operates on $\frac{1}{4}$. Alternatively, $\frac{1}{2}\cdot\frac{1}{4}$ may be explained using the area model, in which both fractions are interpreted as measures of length, while the resulting fraction represents the area (see Fig. 7.2).

This example illustrates that learning fractions necessarily includes learning of new concepts, which is an unavoidable obstacle—whether big or small—for learners. Mathematics educators have used the term "epistemological obstacles" to refer to those obstacles that are inherent in the content structure (Broussou, 1983; Prediger, 2006, 2008; Schneider, 2014). Notably, epistemological obstacles are considered an opportunity for learning in themselves. Thus, these obstacles can and should not be avoided during the learning process.

Our fifth recommendation is that students may benefit from *explicating which aspects of fractions are in line with natural number concepts and which are not*. The above content analysis showed that there are important differences between natural numbers and fractions (see Sect. 7.2.1), and students need to understand these differences. At the same time, in order to build on students' existing knowledge of natural numbers, and to illustrate continuities in the number concept, teachers should highlight similarities between natural numbers and fractions. Current classroom practices seem to put more emphasis on the similarities rather than the differences between natural numbers and fractions. As a consequence, students may get too little support in distinguishing between aspects of rational numbers that are conceptually aligned with natural numbers and those that are conceptually different.

Fig. 7.2 Area model for fraction multiplication



Empirical evidence shows that learning of fraction division concepts can be more or less successful depending on whether the activated previous knowledge of natural numbers is helpful (conceptual similarity) or not (superficial similarity). In this context, Sidney and Alibali (2015, 2017) found that when learning about fraction division, students benefited more from practicing division of natural numbers (similar concept but different numbers) rather than fraction problems without division (similar numbers but different concept) immediately before engaging with fraction division. This suggested sequencing of fraction problems (fraction division directly preceded by natural number division) differs from common mathematics textbooks, where fraction division typically follows fraction multiplication. It is eventually up to the teacher to present fraction division in a way that students can make appropriate links to natural number division.

The important role of teachers leads to our sixth and final recommendation: More effort is needed to *provide teachers with the knowledge they need to teach fractions effectively*. Empirical studies have documented teachers' limitations predominantly with respect to fraction concepts (see Sect. 7.2.3). Thus, it seems imperative for teacher education to enhance teachers' content knowledge. A particular focus should be on fraction concepts that are counterintuitive and therefore prone to biased reasoning.

7.4.2 Future Directions

There are currently only a few studies on how fractions are commonly taught in classrooms. This shortage concerns at least three aspects, namely teachers' behavior, classroom materials, and—more specifically—visual representations of fractions used in teaching. Concerning teachers' behavior, classroom observation studies are needed to find out which teaching approaches teachers actually use in classrooms. With respect to classroom materials, further analyses of textbooks should focus more systematically on the fraction concepts and the types and nature of fraction problems and the visualizations that occur in textbooks. Concerning these visualizations, controlled intervention studies may investigate the specific effects of individual representations, as well as how multiple representations should be combined so that they are effective for students (Rau, 2017).

Although proficiency with natural numbers is a prerequisite for learning about rational numbers, overgeneralizations of natural number principles can actually cause difficulties with learning fractions. Further research is needed to better understand the specific relations between previously acquired knowledge of natural numbers and fraction learning. Studies to date have addressed the development of natural and rational number knowledge in longitudinal designs (e.g., Braithwaite & Siegler, 2017; Mou et al., 2016; Resnick et al., 2016; Rinne, Ye, & Jordan, 2017). However, these studies have focused on very specific aspects of numerical development (e.g., number magnitudes), and they have not included many external variables that may contribute to this development. To better understand the relative contributions of various factors, studies should consider taking into account both cognitive variables (e.g., general cognitive abilities, working memory) and also non-cognitive variables (e.g., mathematics self-concept, mathematics anxiety) as well as school-related factors (e.g., classroom teaching, textbooks), and socio-economic factors (e.g., learning opportunities at home).

7.5 Conclusion

In this chapter, we aimed to make connections between research on fractions from mathematics education and cognitive psychology, and also to include neuroscience evidence. We note that studies from different perspectives address issues on very different levels of explanation, such as the level of classrooms, of student behavior, or of brain activations. Integrating studies with such different perspectives is a challenge for many reasons (De Smedt et al., 2010; Nathan & Alibali, 2010; Schumacher, 2007). For example, learning processes at different levels occur on completely different time scales, ranging from milliseconds (neural activation) to days or weeks (learning across classroom sessions). More fundamentally, authors from different fields do not always speak the same language or address the same questions. For example, mathematics educators often ask what students *should* learn and how they *could* learn best, whereas psychologists are more used to ask what students *are able to* learn, or *when* in their development they learn certain things.

On a meta-level, research should strive for a shared theoretical framework that provides guidance for researchers from different perspectives as to how an integration may be made most fruitful. Our attempts to make connections between various perspectives in this chapter may spark further discussions across disciplines. In spite of apparent challenges, such cross-disciplinary discussions are necessary to improve teaching and learning of fractions in the best possible way.

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Chapter 8
Developing Fractions as Multiplicative Relations: A Model of Cognitive Reorganization

Ron Tzur

Abstract In this chapter, I propose a stance on learning fractions as multiplicative relations through reorganizing knowledge of whole numbers as a viable alternative to the Natural Number Bias (NNB) stance. Such an alternative, rooted in the constructivist theory of knowing and learning, provides a way forward in thinking about and carrying out teaching-learning of fractions, while eschewing a deficit view that seems to underlie the ongoing impasse in this area. I begin with a brief presentation of key aspects of NNB. Then, I discuss key components of the alternative framework, called reflection on activity-effect relationship, which articulates the cognitive process of reorganizing one's anticipations as two types of reflection that give rise to two stages in constructing fractions as numbers. Capitalizing on this framework, I then delineate cognitive progressions of nine fractional schemes, the first five drawing on operations of iterating units and the last four on recursive partitioning operations. To illustrate the benefits of the alternative, conceptually driven stance, I link it to findings from a recent brain study, which includes significant gains for adult participants and provides a glance (fMRI) into circuitry recruited to process whole number and fraction comparisons.

Keywords Cognitive reorganization · Anticipation · Fractions · Numerical comparisons · fMRI

The purpose of this chapter is to articulate the reorganization hypothesis (Olive, 1999; Steffe, 2010a) and use this articulation to depict a progression in constructing fractions as multiplicative relations. The work draws on epistemological and psychological underpinnings of Piaget's (1985) theory, while delving into research on mathematics in the brain (e.g., Dehaene, 1997). Various components of this work have appeared separately. In this chapter, I integrate them as a means to further theorize the knowing, learning, and teaching of fractions.

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I begin with a well-known stance—Natural Number Bias (NNB)—with which I shall contrast my twofold stance. Then, I describe a specific model of reorganization—*Reflection on Activity-Effect Relationship (Ref*AER)*—and a corresponding lens on fractions as multiplicative relations. Next, I present a reorganization progression of nine fractional schemes. I follow this with findings from a study on numerical comparisons in adults' brains, to illustrate benefits of reorganizing natural (whole) numbers into unit fractions with multiplicative relations.

8.1 Contrasting Background: A Natural Number Bias (NNB) Stance

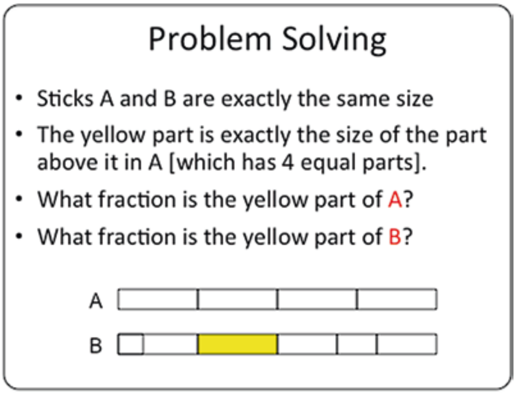
Properly reasoning with fractional quantities is difficult (Lamon, 2007; Verschaffel, Greer, & DeCorte, 2007). Behr, Wachsmuth, Post, and Lesh (1984) pointed out how students' knowledge of natural numbers alters their understanding of fractions. Streefland (1991) portrayed this as *interference*; in recent years, the notions of *whole number bias* (Ni & Zhou, 2005) and *natural number bias* (NNB; Obersteiner et al., this volume; Van Hoof, Janssen, Verschaffel, & Van Dooren, 2015) have gained traction.

Van Hoof, Verschaffel, and Van Dooren (2016) provided an excellent example of empirically grounded data that, they claimed, supported their NNB stance. They specified three main aspects of NNB. The first, *size,* refers to the ordering of fractions. The second, *operations*, refers to carrying out and determining results of arithmetical operations. The third, *density*, refers to placing more fractions between any two given fractions.

To me, this three-aspect NNB stance manifests the researchers' *first-order model* of the mathematics involved, a claim that draws on Steffe's (1995) distinction. One's own mathematics serves as her or his *first-order model*. A *second-order model* refers to how one makes sense of someone else's mathematics, that is, an observer's inferences about an observed person's first-order model.

Mathematics educators' mature, first-order models of natural numbers and fractions allow them to make sophisticated distinctions. For example, they may see differences between density of natural and fractional numbers as "painfully obvious." However, at issue is not how, from one's mature frame of reference, students inappropriately solve such tasks while using what the researchers consider whole number knowledge. Rather, the challenge is to create a second-order model that explains how, from a student's conceptual frame of reference as inferred by the researchers, her or his solutions do make sense. I contend that the notion of "bias" manifests (a) researcher interpretations based on their first-order models and (b) overlooking second-order models that articulate students' conceptualization of natural and fractional numbers. Thus, what looks like NNB to observers may not be so from the student's frame of reference.

Fig. 8.1 A task for perturbing the limited concept of fractions as "parts-of-whole"



The NNB stance seems to also reflect two prevalent aspects of teaching fractions. The first is a reliance on an "equal-parts-of-whole" conception of fractions (Simon, Placa, Kara, & Avitzur, in press). This reliance seems to (a) limit students' reasoning and (b) underlie much of their regress to natural number reasoning (e.g., adding $a/n + b/n = (a+b)/(n+n)$, or saying that 1/6>1/5 because 6>5). This reliance seems to underlie adults' inability to solve a non-routine, two-part task seen in Fig. 8.1 (Simon, personal communication), because the yellow piece is *not* part of Stick A and is part of six, *unequal* parts in Stick B. I return to this task in Sect. 8.4, as it was used in the brain study.

The second aspect pertains to teachers' understanding of fractions, which by-and-large seems limited to meaningless execution of procedures and algorithms (Izsák, Jacobson, de Araujo, & Orrill, 2012). Thus, it is no wonder research studies repeatedly demonstrated faulty use of taught procedures—including those they use for natural numbers. Erlwanger's (1973) seminal study of "Benny" provided evidence that a major shift is needed for teachers and students alike to construct adequate meanings for fractions that help diminish what some call NNB (Behr, Harel, Post, & Lesh, 1992; Lamon, 2007).

8.2 Reorganization of Anticipation of Activity-Effect Relationship

In this section, I briefly describe key components of a model I find useful for explaining the knowing, learning, and teaching of fractions. At the core of this model is the constructivist hypothesis that learning is a cognitive process of reorganizing what one already knows (Piaget, 1985). To explain such reorganization, I first situate mathematical knowing within the constructivist scheme theory and the notion of anticipation (von Glasersfeld, 1995), which I explain in more detail below. I illustrate this theory with an example of a natural number scheme that, *if available to students*, can become a conceptual basis for constructing fractions as multiplicative relations. I then depict conceptual learning as a *reorganization of anticipation of activity-effect relationship*. To this end, I describe the mechanism of *reflection on activity-effect relationships* (Ref*AER, see Simon, Tzur, Heinz, & Kinzel, 2004), which was postulated as an elaboration on Piaget's (1985) notion of reflective abstraction. I illustrate this mechanism with two examples. One example illustrates the transition from additive to multiplicative reasoning; the other example illustrates how the mental system may reorganize a scheme of whole numbers into a scheme of unit fractions (*1/n*), which enables solving both tasks shown in Fig. 8.1. Scheme theory and Ref*AER can serve as tools for both developing second-order models (i.e., inferences about others' mathematics) and articulating hypothetical learning trajectories (HLT; see Simon, 1995, this volume) to explain, and foster, others' learning of particular mathematics.

8.2.1 Scheme: An Anticipation-Centered Constituent of Mathematical Cognition

Building on and further interpreting Piaget's (1985) constructivist theory, von Glasersfeld (1995) postulated "*scheme*" as a foundational, three-part constituent of cognition. The first part of a scheme is a *situation*—a sort of "recognition template" that frames one's experience and sets a goal for the actions to follow. Setting such a goal typically involves experiencing a discrepancy between a current and a desired state—a perturbation that instigates activity of the mental system (Piaget, 1985; Skemp, 1979). The second part of a scheme is *activity directed to accomplish the goal*—typically constituted by sequencing several goal-directed (mental) actions. The third part of a scheme is a *result* yielded by the goal-directed activity. Simon et al. (2004) used [Goal→Action→Effect], signified as [*G*→*A*→*E*], to depict scheme as a singular mental structure consisting of a twofold anticipation (indicated by the arrows)—anticipation linking situation/goal with activity, and activity with its effect. To clarify, I use "anticipation" as a shorthand for the rather complex notion of *relationship the mental system creates between a scheme's goal, activity, and effect(s), which a person isolates as ensuing from the activity—before, during or after carrying it out*.

As an example, I present the foundational construct of number as a composite unit (Steffe & von Glasersfeld, 1985), which underlies a scheme Steffe (1992) called the *Explicitly Nested Number Sequence* (ENS, see more below). Consider a child who is shown two opaque bags, one labeled "8 marbles" and the other "7 marbles." The child is asked: "If all marbles from both bags would be placed in a single box, how many marbles will be in that box?" After a brief silence, not using any observable action or language, the child responds, "there will be 15 marbles in the box." To explain her solution, she says: "I gave 2 from the 7 to the 8, to make 10; then I added the 5 [remaining from 7] to 10." This additive strategy was termed *Break-Apart-Make-Ten* (BAMT; see Murata & Fuson, 2006).

For a second-order model, we can infer that to perform such a complex calculation, the child has used a scheme—explicitly nested number sequence—for operating symbolically and abstractly with numbers as composite units. Specifically, the child assimilated the task as a situation (recognition template) that consists of two quantities (8 and 7). The child could anticipate, before/without any action, that each of those quantities, indicated by a numerical symbol (word, or numeral), could be produced if the unit of 1 were iterated (while counting) up to the given number of marbles in each bag. This situation, along with the question in the task, sets the child's goal—figure out the total of 1s that would constitute a quantity produced from combining the two given quantities (e.g., adding 8+7). This goal triggers a mental activity inferred to comprise the following sequence of five, [A➔E] anticipations:

1. Call up the number 10 (not given in the task!), which embeds at least one of the addend(s).
2. Select a number (here, 2) that, with 8, would be nested within 10[1].
3. Strategically decompose the unit of 7 into two sub-units that are nested within it (here, 2 and 5), so one of those units fits the number needed to complement 8.
4. Compose a unit of 10 from the given unit of 8 and the unit of 2 that was disembedded from 7, never losing track of the unit of 5 that was also disembedded from 7.
5. Compose a unit of 15 from the intermediate unit of 10 she has just composed and the previously disembedded unit of 5.

The example of adding 8 marbles and 7 marbles illustrates how the two anticipations that constitute the three-part notion of a scheme may be inferred from observable behaviors (actions and/or language). Furthermore, it illustrates that units (e.g., 1s, composite units) are themselves mental structures reflecting anticipation of an activity-effect relationship. Specifically, "8" for the child was a symbol for an anticipatory effect of iterating 1 eight times; "15" was a symbol for an anticipatory effect of composing eight 1s and seven 1s while decomposing 7 to first capitalize on the nesting of 8 and 2 within 10.

This example also highlights a core distinction (see Tzur et al., 2013), between the child's performance (e.g., "she got the correct answer," "she used break-apart-make-ten," "she did not use fingers," "she knows addition of 1-digit numbers") and conceptual analysis—observer's inferences—about the child's mental operations and units on which she operated (Steffe, Thompson, & von Glasersfeld, 2000). A focus on performance quite often manifests the observer's use of a first-order model (i.e., the observer's own model of the mathematics), whereas conceptual analysis of units and anticipation of relationships between goal-directed activities and their effects is at the heart of using a second-order model (i.e., inferences about the other's model of the mathematics). All in all, this stance on anticipation, which I consider the mental core of schemes as cognitive "building blocks," seems compatible with the following:

[1] The first and second activities differ, as we often see children selecting 3 to complement 7 into 10.

> Success is not preceded by trial-and-error and is not a matter of luck but is assured by operational anticipation. (Piaget, Inhelder, & Szeminska, 1960, p. 319)

> Immediately before taking an action, the child begins to formulate in words a *pattern, a plan of action* that thereby anticipates the further course of action. (Vygotsky, 1986, p. 35; cited in Zinchenko, 2002, p. 21)

> So to us, the term "a quantity's magnitude" means the scheme of meanings and operations that allows one to anticipate, and therefore to operate under the constraint, that a quantity's size is invariant under change of unit. (Thompson, Carlson, Byerley, & Hatfield, 2013)

8.2.2 Reorganization of Anticipation: Reflection on Activity-Effect Relationships

Building on and further specifying Piaget's (1985) central construct of reflective abstraction, the mechanism postulated to bring about conceptual learning as reorganization of available schemes—*reflection on activity-effect relationship (Ref*AER)*—has been detailed in several papers (Simon et al., 2004; Simon & Tzur, 2004; Tzur & Simon, 2004). I have also proposed a preliminary set of conjectures that may link this mechanism to brain systems and functioning (Tzur, 2011). In this section, I thus provide a brief summary of Ref*AER, which essentially consists of two types of reflection and two stages in the construction of a new scheme through reorganizing previously constructed schemes (Piaget, 1985).

8.2.2.1 Ref*AER: Two Types of Reflection

The mental mechanism of reflection on activity-effect relationships (Ref*AER) consists of two types of reflections, that is, mental comparisons. *Reflection Type-1* involves comparing an anticipated and an actual effect of one's goal-directed activity. For example, when beginning to learn to reason multiplicatively, a child may respond "6 + 3 = 9" to the task, "How many marbles are there in 6 bags, with 3 marbles in each bag?" When asked to check that answer, the child may draw 6 circles for the bags, 3 dots in each circle for the marbles, then count each and every dot from 1 through 18. After counting the dots in the first 3 circles, and clearly from that point to the end of her counting activity, the child's conception of numbers enables her to notice that the number of 1s she counted (18) does not match her initial, anticipated effect (9). Her mental system may then record an experience in which the effect of counting composite units, which she distributed equally (marble triplets) over other units (bags), differs from the effect anticipated to ensue from adding the two units.

To explain the difference between the anticipated and actual effects of her activity, the child may begin attending to the different units operated on. This is possible because her assimilatory scheme (e.g., explicitly nested number sequence) already includes a distinction of units she used (as input) for the activity of counting eighteen 1s: composite units (e.g., 6 circles) and triplets of 1s within each of them (e.g., 3 dots). The latter inference is supported by the child's independent and spontaneous production of figural items—6 circles (for bags) and dot-triplets (for marbles) as a sub-goal for counting all 1s. Thus, for a goal compatible with the goal in adding (figure out the total of 1s), a distinction between two types of units that are being counted may begin to emerge for the child. These two types of units emerge through producing an effect that, for all numbers except for 2×2, differs from what the child anticipates would ensue from an activity of adding the two given units.

Another example of reflection type-I can be identified in children's construction of unit fractions as multiplicative relations through reorganization of their conception of number as composite unit. As Simon et al. (2004) and Hunt, Tzur, and Westenskow (2016) described in detail, this can occur when children are engaged in solving equal-sharing tasks involving paper strips (and/or computer bars) referred to as "French Fries" (Tzur & Hunt, 2015). To promote the intended reorganization, two specific constraints require the children use neither folding nor rulers. For example, a child may be asked to equally share a given paper strip among 5 people. Assimilating this task into the aforesaid explicitly nested number sequence scheme, she can set a goal of decomposing the given whole into 5 pieces. To accomplish this goal, she could then bring forth iterating of just one person's share, which is available through her conception of iterable 1s. She can thus estimate the size of one person's share, repeat it while marking the accruing length of the repeated piece, and notice whether or not the resulting, 5-piece whole is equal to the given whole. If not, the effect she would notice ensuing from her activity is a need to adjust the estimated size of the single person's share. Simon et al. (2004) referred to this sequence of goal-directed activities for equal sharing as the *Repeat Strategy*. Tzur and Hunt (2015) detailed two kinds of anticipations fostered by the child's repeating experiences of adjusting the size of one person's share and iterating it the given number of times. The first anticipation is for the direction of change—make the next piece shorter/longer than the previous pieces. The second anticipation is for the amount of change needed to one piece so, when iterated, it produces an iterated whole equal in size to the given whole.

Once successful in equally partitioning the given whole (e.g., among 5 people), the child can be engaged in solving a different task (e.g., sharing among 6 people). Often, children's initial estimate of one person's share indicates their established scheme for whole numbers: because 6 is larger than 5, their first estimate of one person's share for 6 people may, incorrectly, be longer than the piece used for sharing among 5 people. Once iterating that piece, they would notice a twofold, actual effect of this activity that is inconsistent with their anticipated effect: the iterated whole would exceed the given whole, and fewer than 5 pieces would actually fit within the given whole. The child may also call upon and link the inverse adjustment needed with another scheme of equal sharing she possibly has constructed; namely, when the same quantity is shared among more people, each person gets a smaller share. The child may then be engaged in equally sharing the same paper strip ("French fry") among 4 people, again possibly predicting that the share of one person has to be shorter than the piece used for sharing among 5 or 6 people, then realizing the actual effect of iterating the initial estimate is inconsistent with this whole-number based anticipation.

With those two examples in mind, reflection type-I leads to the production and recording of *newly noticed activity-effect dyads* by the mental system (Simon et al., 2004). A scrutiny of brain research literature (Tzur, 2011) led me to postulate that, for properly functioning human brains, reflection type-I occurs automatically. Not surprisingly, interactions with others (peers, teacher, parents) whose responses differ from one's own quite often become a source for orienting reflection type-I.

Reflection Type-II involves comparing across mental records (instances) of recurring activity-effect dyads and thus, possibly, constructing a novel, invariant activity-effect relationship. The key here is that the mental system has the capacity to distinguish, and later justify, a new regularity of what is anticipated to remain the same in one's goal-directed activities (Huang, Miller, & Tzur, 2015; Simon et al., 2004). For example, after solving a variation of tasks like the aforesaid 6 bags with 3 marbles each, which can differ in numerical values and/or contexts, the mental system may isolate the invariant activity of simultaneously counting the accrual of composite units and of 1s that constitute each. Eventually, the activity of simultaneously counting 1s and composite units remains the same in the sense that it reliably yields the effect of the total number of 1s. That is, "1-is-3, 2-is-6, 3-is-9, 4-is-12, 5-is-15, 6-is-18" becomes an instance of the abstracted, invariant anticipation for any two numbers, "1-is-k, 2-is-2k, 3-is-3k, 4-is-4k, …, n-is-n*k." Furthermore, within the effect of such an activity, one may distinguish two types of units—a total (composite unit) of 1s, such as 18, as well as a compilation of composite units of 1s, such as "eighteen 1s are also 6 units of 3 units of 1" (Hackenberg & Tillema, 2009; Norton, Boyce, Ulrich, & Phillips, 2015; Tzur et al., 2013).

In the repeat strategy for equal sharing problems example, reflection type-II can focus on two recurring regularities. One regularity arises out of comparing across repeated instances of anticipating and adjusting the size of an estimated piece (one person's share) in comparison to the given whole. For the child, a successful solution for each equal-sharing task is marked by an effect that there is a unique piece that could precisely fit (via iteration) the given number of times within the whole and, correspondingly, the whole is so many times as much as that piece. A second regularity arises out of comparing across instances of sharing the same whole among a larger/smaller number of people. For the child, successful use of previously estimated shares while varying the number of shares yields an anticipation of the inverse relationship between the size and number of unit fractions (e.g., for the goal of sharing a whole among 6 people the estimated piece must necessarily be smaller than the piece used for sharing it among 5 people). The first regularity can then be linked to the symbolic convention of unit fractions: $1/n$ (for any natural number) is a unit fraction in the sense that it constitutes a *1-to-n relationship with the whole.* Simply put, what defines any particular unit fraction ($1/n$) is that the whole is n times as much as it. This regularity enables a single and rather straightforward solution to the tasks in Fig. 8.1: The yellow piece is 1/4 of Stick A, and 1/4 of Stick B, because each of those wholes is 4 times as much as the yellow piece, which one could justify/confirm by iterating the yellow piece. The second regularity can then be linked to the symbolic convention of inverse relations among unit fractions: $1/m < 1/n$ for any two natural numbers $m > n$, regardless of the size of the whole, because $1/m$ has to fit more times into its relative whole than the $1/n$ into its relative whole. When both regularities are fully coordinated for the child, they constitute the first fractional scheme, called equi-partitioning (Steffe & Ulrich, 2010; Tzur, 1996).

The two examples of reorganization, from additive to multiplicative reasoning and from whole-number to the equi-partitioning scheme, illustrate that reflection type-II can lead to constructing a new scheme. Specifically, this second type of reflection enables abstraction of the linkage between the anticipated [A→E] dyad and the situation/goal in which the mental system would anticipate and trigger it, that is, [*G*→[*A*→*E*]] (Simon et al., 2004).

It should be noted that a newly constructed scheme, which comprises two anticipations (the signified "arrows"), may differ from previous schemes in the goal, and/or the activity the goal would trigger, and/or the effect anticipated to ensue from that activity. In the example of shifting from adding to multiplying the number of marbles, this could be signified as $[G_0 \rightarrow [A_0 \rightarrow E_0]]$ reorganized into $[G_0 \rightarrow [A_1 \rightarrow E_0]]$: the goal (figuring out the total) and the effect (anticipated total of 1s) remained the same whereas the activity (e.g., decomposing units vs. double-counting units) was linked anew. In this example, if the [A→E] dyad is also constructed as units-of-units-of-units (e.g., 18 = 6 units of 3 units of 1), signifying this will also include the effect: $[G_0 \rightarrow [A_1 \rightarrow E_1]]$. Empirical examples of cognitive reorganization in which one's goal changes can be found in Simon et al.'s (Simon, 2012, 2015; Simon, Placa, & Avitzur, 2016) recent work.[2] The example of constructing unit fractions through the repeat strategy, in which the goal, the activity, and the anticipated effects were available through the conception of number as a composite unit (the ENS scheme) all changed into the equi-partitioning scheme, can be signified as the change from $[G_0 \rightarrow [A_0 \rightarrow E_0]]$ to $[G_1 \rightarrow [A_1 \rightarrow E_1]]$.

A crucial element of the equi-partitioning example is that it did not depict the construction of unit fractions based on the *number of parts within a given whole*. Rather, the reorganization of the child's conception of whole numbers as composite units was based on the *mental activity of unit iteration* postulated to give rise to that conception of number in the first place (Olive, 1999; Steffe, 2010b; Tzur, 1996). This is a subtle but important distinction. If cognitive reorganization is considered to evolve through one's reflection on relationships between an activity and its effects, a new scheme is more likely to arise from the goal-directed activity available to the child through established schemes—than from the conceptual product (here, number) of that activity. As shown in the repeat strategy example, a reorganization stance presumes an incorrect use of previously established schemes (e.g., initially making one person's share for 6 people larger than the share for 5 people). However, this stance differs markedly from the NNB stance in how the

[2] Their work did not include E for effect; in my analysis, such a reorganization will be signified as the change from $[G_0 \rightarrow [A_0 \rightarrow E_0]]$ to $[G_1 \rightarrow [A_0 \rightarrow E_1]]$.

observer's sense of preference for whole numbers is explained, to which I return in the next section after explaining the two stages in the reorganization of a new scheme.

My scrutiny of brain research literature (see Tzur, 2011) led me to conjecture that, for properly functioning human brains, reflection type-II is a capacity of the mental system, but for many people it may not occur automatically. Accordingly, interactions with others—particularly intentional teacher–learner interactions (e.g., "How were your solutions to the last three tasks similar/different?")—may be required to bring forth and sustain reflection type-II (for more, see Hunt & Tzur, 2017). This is why, for students whose schemes are inferred to bring forth noticing and recording of novel [A→E] dyads, teaching with variation (Gu, Huang, & Marton, 2006; Jin & Tzur, 2011b) seems a powerful pedagogical tool to promote their reflection type-II. In the next section, I elaborate on the two stages in the reorganization of assimilatory schemes into a new, more advanced scheme and link those stages to the two types of reflection that constitute the reflection on activity-effect relationship (Ref*AER) mechanism.

8.2.2.2 Ref*AER: Two Stages in Constructing a New Scheme

Anticipation-centered explanations of schemes and reorganization open the way to explaining a renowned phenomenon observed when learners construct new schemes. Tzur and Simon (2004) termed it the "next day phenomenon" to capture a behavior sequence familiar to most teachers: (a) periods in which a learner behaves as if she constructed a new scheme, are followed by (b) periods in which the learner behaves as if she lost that scheme and regressed to previously constructed schemes, and then, (c) somehow resumes the use of the newly constructed scheme. A key characteristic of such a behavior sequence is that some sort of prompting, external and/or internal, allows the learner to re-enact the novel scheme, access her newly constructed activity-effect dyad ([A→E], conception), and explain why this relationship is anticipated. Yet, independently and spontaneously (without prompting) the learner seems to only have access to previously constructed schemes. In the example of using the repeat strategy, many children would initially demonstrate an anticipation that 1/5 > 1/6, then go back to saying "1/5 < 1/6, because 5 < 6," and then later change again once engaged in the repeat strategy.

Tzur and Simon (2004) and Simon, Placa, and Avitzur (in press) proposed that provisional, prompt-dependent access to a newly forming scheme is a hallmark of a stage in the construction process, which they termed the *participatory* stage. This provisional stage is explained in that the activity-effect [A→E] dyad, while being abstracted, is yet to be linked with the situation/goal part of a new scheme. That is, the learner does not lose the new [A→E] anticipation, which has been abstracted at a prompt-dependent, provisional stage, as indicated by her ability to regenerate and use it once engaged in the activity that brings forth the newly noticed effect. However, what to a knowledgeable observer appears like a similar (invariant) task is yet to become so in the learner's frame of reference.

Tzur and Lambert (2011) proposed a theoretical status for the prompting experience, as a link between assimilation and anticipation. They further postulated that the participatory stage is a cognitive correlate of Vygotsky's (1986) Zone of Proximal Development (ZPD). Tzur and Lambert (2011) emphasized that the participatory stage is not an "on-off," toggle-like marker within the construction process. Specifically, they postulated gradations within the participatory stage, indicated through three dimensions of prompting as it is experienced by a learner. The first dimension of prompting is its *locus*, either solely mental (often indicated by a self-generated "oops" experience) or including external "stimuli" that bring forth the novel [A→E]. The second dimension of prompting is its *focus*, ranging on a continuum between indirect and general (e.g., "What if a friend joined your group and you want to give everyone an equal share of the pizza?") to direct and specific ("Yesterday, after you successfully shared the 'fry' among 4 people, you made one person's share for 5 people shorter; does this change your thinking to make the share for 8 people larger than for 5?"). The third dimension of prompting is the *number (amount) of prompting experiences*, ranging on a continuum between just one, through a few (2–3), to many (4 and more).

Eventually, this provisional stage (participatory) *may* give rise to a rather stable stage, termed *anticipatory*, in which a learner consistently calls upon and meaningfully uses the new scheme without prompting (Simon, Placa, & Avitzur, in press; Tzur & Simon, 2004). At this stage, a learner does not need to implement the scheme's activity to produce the scheme's result, whether with or without awareness, because the activity has been stably tied to its effect(s) and both (as a single dyad) have been stably tied to the scheme's situation/goal. This is signified as [G→[A→E]], with one or more of the three components being at a higher level than previous schemes.

Hunt et al. (2016) provided a detailed analysis of how two fifth graders, labeled by their school system as "Learning Disabled," capably constructed a participatory stage and then an anticipatory stage of the equi-partitioning scheme as reorganization in their numerical schemes. As the stage distinction predicts, once the two girls reached the anticipatory stage, including anticipation of both the direction and amount of adjustment needed to one person's share, they could correctly and reliably solve any task of ordering unit fractions—whether in symbolic form (e.g., "Which fraction is larger, 1/8 or 1/11?") or in a realistic word problem (e.g., "If you are hungry, and can have 1/13 or 1/10 of the same-size pizza, which piece would you choose and why?").

The participatory-anticipatory stage distinction provides a plausible reason for expecting learners to use previously constructed schemes. This distinction implies that, at the participatory stage of constructing a new scheme, learners are bound to *fold back* (Pirie & Kieren, 1994) to previously established schemes. The reason I use the term "folding back" is to eschew a connotation of "bias" (or "interference") that seems to me rooted in interpreting the learner's work through the lens of the observer's first-order model. "Bias" connotes preference for one of two ways of reasoning that are both available to the learner, while folding back embraces the use of schemes available at a more established stage of construction (for a similar view in science learning, see Shtulman & Valcarcel, 2012). By default, test items not sensitive to measuring participatory stages in fraction schemes, such as those used in NNB research, are likely to manifest folding back to reasoning based on natural numbers (Van Hoof et al., 2015; Van Hoof, Lijnen, Verschaffel, & Van Dooren, 2013). Instead of calling it "bias," I embrace bringing forth natural number reasoning, which learners do have available, as a starting point to a reorganization process where *their* anticipated effect might prove inadequate to them (Sidney & Alibali, 2015), and thus lead to constructing the equi-partitioning scheme. Generally, for learners at the participatory stage of constructing *any* new scheme by reorganizing previous, anticipatory schemes, folding back to anticipatory schemes seems a developmental necessity—not a "bias."

8.2.2.3 Ref*AER: Linking Reflection Types with Stage Distinction

I conclude the articulation of reflection on activity-effect relationship (Ref*AER) by postulating links between the two types of reflection and the two stages in this constructive process. To recap, in reflection type-I the goal-directed modus of cognitive processes provides a basis for noticing and linking novel effects to the activity that produced them, signified as: $[G_0 \rightarrow A_0 \rightarrow E_0] \rightarrow$ notice $E_1 \rightarrow$ link $[A_0\text{-}E_1]$. It is thus postulated that reflection type-I is necessary for initiating the transition to a participatory (prompt-dependent) stage of a new scheme. In this regard, the instructional practice of bridging (Huang et al., 2015; Jin & Tzur, 2011a) seems highly effective, as it focuses on orienting students to call upon their available schemes in service of advancing to new ones.

Reflection type-II involves comparisons a learner's mental system may make across instances in which novel activity-effect dyads were used to accomplish goal(s) set by a recognition template of available schemes (Skemp, 1979; Tzur & Simon, 2004). Such a comparison seems compatible with Piaget's (1971) notion of reflective abstraction as involving projection onto a higher plane of operation in which the mental system further coordinates previously coordinated actions. This second type of reflection is postulated to be necessary for abstracting an anticipatory stage of a new scheme. Simply put, to construct a novel scheme (that is, $[G \rightarrow A \rightarrow E]$), students need to compare across multiple, related $[A \rightarrow E]$ dyads.

As noted above, for many learners reflection type-II may not occur automatically (Tzur, 2011). However, reflection type-II may be purposely engendered by others (e.g., teachers) through interactions that "elevate" the learner's attention—from focusing mainly on separate $[A \rightarrow E]$ dyadic instances to similarities she notices across those instances. In particular, proactively fostering reflection type-II seems crucial to cope with the constant risk faced by learners at the participatory stage—being left behind conceptually. Questions such as "How is this solution related (similar, different) to previous solutions?" or "What if instead of … you'd be doing …?" are but two examples of pedagogical interventions that can promote this type of reflection (Mason, 2008). In this regard, the practice of teaching with variation

(Gu et al., 2006), which is commonly used by Chinese teachers to foster students' progress from available to the new, intended mathematics, seems highly effective.

8.3 Developing Fractions as Multiplicative Relations: A Progression of Reorganizations

Having explained the equi-partitioning scheme, in this section I briefly summarize the subsequent reorganization of eight additional schemes for fractions as multiplicative relations. This summary is inspired by Steffe and Olive's (2010) meticulous conceptual analyses, as well as research related to their seminal work (Hackenberg, 2007, 2013; Hackenberg & Lee, 2015; Norton & Boyce, 2013; Norton & Hackenberg, 2010; Norton & Wilkins, 2010, 2012; Tzur, 1999, 2000, 2004). My summary adapts the description of fraction schemes found in a recent paper (Tzur, 2014) that synthesized Steffe, Liss, and Lee's (2014) study on how algebraic schemes may be constructed as a reorganization of fractional schemes. Specifically, I focus on two pivotal activities inferred to underlie the progression in reorganizing fractional schemes (Tzur, 1996), *iteration* and *recursive partitioning*. While this conceptual progression arose out of empirical studies with children, my work with adults in general and K-12 teachers in particular (Tzur, Hodkowski, & Uribe, 2016) showed it can inform their learning, too.

8.3.1 Developing Fraction Schemes Based on Unit Iteration

Table 8.1 presents succinct descriptions of four, iteration-based fraction schemes that further reorganize the equi-partitioning scheme by operating on unit fractions as input for further coordination of units (Norton & Boyce, 2013). As Sáenz-Ludlow (1994) and Tzur (1996) explained, a focus on iteration-based schemes embodies a measurement approach to fractional units and operations—an approach extended substantially in recent work of Simon and his colleagues (Dougherty & Simon, 2014; Simon, Placa, Kara, & Avitzur, in press). The equi-partitioning scheme is included, with index "0," to indicate its commencing role in this progression. Said differently, an iterable unit fraction ($1/n$) serves as the "building block" to other fractions, similarly to how an iterable unit of 1 serves as the "building block" to other whole numbers. For example, the partitive fraction scheme is marked by reasoning about non-unit fractions (m/n, $m>1$) in terms of anticipating the effect of iterating unit fractions a number of times that does not exceed the n/n whole (e.g., 3/7 is the anticipated effect of iterating 1/7 three times). Such an anticipation reflects constituting non-unit fractions as *double-multiplicative relations*, which corresponds to Behr et al.'s (1992) and Tzur's (1999, 2000) depiction of non-unit fractions: $m/n = m * 1/n$ (e.g., $3/7 = 3 * 1/7$).

Table 8.1 Progression in reorganization of iteration-based fractional schemes

Scheme	Description	Reorganized Anticipation
0. Equi-Partitioning (EPS)	Using *her concept of number* as a template for a partitioning operation, a learner can disembed a part and anticipate that iterating that part would confirm if it is (or not) an equal share for N people (2-level unit coordination). Enacting the iteration involves the learner in progressive integration (uniting) operations, but the learner may not be explicitly aware of the iterated segments being nested	The learner applies her ENS-based, simultaneous anticipation of both partitioning a composite unit and iterating a unit of 1 to produce a composite unit of size n to equally partition a continuous whole and give meaning to each iterated part as *1/n* of that whole (a multiplicative relation to the whole and inverse relation among unit fractions)
1. Partitive Fraction (PFS)	Using her numerically based partitioning operation, a learner anticipates that iteration of a disembedded part (e.g., 1/8) can produce another unit, which she regards as a measured length (e.g., if 1/8 is iterated three times it would produce a non-unit fraction of length 3/8; here, 1/8 is not yet iterable). This coordinated operation (disembedding + iterating) can be linked to part-whole (measuring) comparisons. The learner cannot yet reverse this operation to reproduce the unit fraction that generated it (e.g., partition a 3/8 into three parts of 1/8 each), which entails that the unit fraction is not yet iterable	The learner's anticipated effect of equi-partitioning (*1/n*) serves as input for further activity of iteration, leading to possibly anticipating production of a composed fractional unit (*m/n*) via the coordinated operation of disembedding *1/n* and iterating it m times
2. Splitting	Fraction operation with simultaneous anticipation of a hypothetical effect (before enacting)—both of partitioning a whole into n parts of $1/n$ each, and of each part being iterable in the sense the whole is n times as many of it ($n/n = n*1/n$). When anticipatory, the result of a splitting scheme is an inverse multiplicative relation in the sense Gauss specified for extensive quantitative relations (see Steffe, Liss, & Lee, 2014)	The learner's anticipation of partitioning + disembedding + iterating becomes an operation in the splitting scheme, and thus enables anticipating equi-partitioning and partitive scheme in one fell swoop

(continued)

Table 8.1 (continued)

Scheme	Description	Reorganized Anticipation
3. Reversible Fraction (RFS)	The anticipatory effect, of composing non-unit fractions from an iterable unit fraction, can be reversed in the sense of decomposing (partitioning) a non-unit fraction (e.g., 3/7 or 8/7) into the *m* unit fractions (e.g., 3 or 8 units of 1/7 each) that constituted it in the first place, for a goal of reproducing the original whole (e.g., 7/7) for which the given fraction is *m*/*n*, or for producing other non-unit fractions (e.g., 6/7, or 18/7)	The learner reorganizes anticipation of iterative and splitting by "undoing the iteration" to produce the building block unit fraction (1/*n*) of which the given non-unit fraction could be created as a means to create other non-unit fractions (whole included)
4. Iterative Fraction (IFS)	Iterable unit fraction (e.g., 1/7) resulting from partitioning is "freed" from the whole; it can be disembedded and iterated as a "thing" in and of itself. The learner can anticipate composing it with the whole (e.g., 7/7) to produce, say, 8/7, or 12/7, or 14/7 as two wholes. The learner is aware that the composed unit is also a potential result of iterating the unit fraction so many times (i.e., 8/7 = 8*1/7). For the learner, then, any fraction *m/n* is an anticipated effect of integrating m units of *1/n*	The learner reorganizes anticipation of partitive and splitting by extending iteration of the unit fraction (1/*n*) to any number of multiples of that unit—not depending on the partitioned whole of which, initially, it was a part

Combined, the five schemes depicted in Table 8.1 tell a story of conceptual reorganization that revolves around noticing and linking effects of iteration as a core, goal-directed, mental activity. Because iteration is inferred to underlie learners' numerical schemes (Steffe, 1992), this story helps in eschewing the NNB stance, which seems to entail some conceptual deficit in need of fixing (Van Hoof et al., 2016). Eschewing this stance does *not* mean dismissal of the recurring observation that (while reorganizing each scheme) a learner may fold back to anticipatory schemes of operating with/on natural numbers. Quite the contrary, such folding back is predicted by and has been grounded empirically through identification of the participatory stage in constructing each of those schemes (Simon et al., 2016; Tzur, 1999, 2000, 2004).

Eschewing the NNB stance means positioning learners' numerical schemes at the core of a measurement approach to fractions (Dougherty & Simon, 2014; Simon, Placa, Kara, & Avitzur, in press). In particular, the iterative fraction scheme culminates this progression in the sense of relating anticipatory effects of multiplying unit fractions (m times $1/n = m/n$) with division of non-unit fractions (m/n divided by $m = 1/n$). Thus, the reorganization stance, in contrast to the NNB deficit view, embraces, and builds on, numerical schemes (anticipatory goal-directed activities and their effects) that *are available* to the learner as a means to foster the five, iteration-based fractional schemes. In turn, those iteration-based schemes open the way for constructing higher-level schemes in which unit and non-unit fractions can be partitioned further.

8.3.2 Developing Fractional Schemes Based on Recursive Partitioning

Table 8.2 presents succinct descriptions of four fractional schemes that arise out of the goal-directed activity of *recursive partitioning*, that is, applying partitioning operations to fractional quantities that, themselves, are the results of previous partitions (Steffe, 2010c). My review of the literature suggests that Sáenz-Ludlow (1994) provided the first conceptual analysis of a child's construction of recursive partitioning, which she coined *part-partitioning*. Capitalizing on her work, I studied fractional schemes rooted in the same operation, but changed the term to recursive partitioning (Tzur, 1996). While distinguishing it from Confrey's (1994) notion of splitting, I suggested to link the recursive partitioning operation with a related one, termed *distributive partitioning*. The following two quotes explicate both operations, which underlie all four schemes in Table 8.2 below:

> The first step in the child's establishment of the distributive partitioning scheme consists of the child partitioning a part that was produced by partitioning a given whole into a specific number of sub-parts. Partitioning a part of the whole is regarded as a recursive operation, because the child partitions the result of a previous partitioning. To find the relationships between the whole, the parts of the first partitioning, and the parts of the second partitioning, the child distributes (first actually, then mentally) the second partitioning across each part of the first partitioning. (Tzur, 1996, p. 214)

> The scheme of operation the children seemed to be using was indicated by their words "fit into." Later (episode 14), we termed the children's way of operating "distributive partitioning," because the children distributed the sub-parts of the second partition across each of the "slots" of the first partition. (Tzur, 1996, p. 232)

Combined, the four schemes depicted in Table 8.2 tell a story of conceptual reorganization that revolves around noticing and linking effects of recursive and distributive partitioning as a core, goal-directed mental activity. This story helps to further eschew the NNB stance. Recently, while substantially extending previous research on those schemes (Sáenz-Ludlow, 1994; Steffe & Olive, 2010; Tzur, 1996), Simon, Kara, Norton, and Placa (in press) demonstrated that schemes rooted in recursive partitioning provide a common conceptual root for multiplication of whole numbers and of fractions, by explicating the crucial role played by an intermediate unit (e.g., "1/3" in the multiplicative sequence, 1/5 of 1/3 of the whole). I

Table 8.2 Progression in reorganization of recursive partitioning based fractional schemes

5. Recursive Partitioning (RPS; 1/*m* of 1/*n*)	Operating mentally to partition unit fraction (e.g., 1/4 of 1/7) allows the learner to anticipate the result as if—but without or before—the second partitioning would have been applied to each and every part of the first partition (e.g., 1/4 of a single 1/7 would be as if each 1/7 is partitioned into 4/4 parts and thus could potentially result in partitioning the original 7/7 whole into 28/28 segments, making the 1/4 of a part (1/7) to be 1/28 of the whole (which marks this scheme as an anticipatory 3-level unit coordination that is multiplicative)	The learner reorganizes anticipation of splitting and IFS to apply it for partitioning a unit fraction as both a "thing" and a part of a whole. This composition allows linking the result of the second partition to the whole, through an imagined (but not enacted) full split of each of the original unit fractions
6. Unit-Fraction Composition (UFCS; *k/m of 1/n*, *k>1*)	A recursive partitioning of unit fraction (e.g., 1/4) can include linking the result of the second partition back to the whole and to reversing the operation of composing a non-unit fraction of the initial segment (e.g., 3/5 of 1/4 can be accomplished mentally by reversing the iteration of three 1/4s to isolate a single 1/4, apply it to the first partition to find 1/20 of the whole, and then recompose with the three units sought, that is, 3/20). This is considered a first scheme for multiplying fractions, as eloquently articulated in Simon, Kara, Norton, and Placa (in press)	The learner reorganizes her anticipation of the effect of recursive partitioning by coordinating it with the reversible operation of de-iterating unit fractions (i.e., with reversed IFS)
7. Distributive Partitioning (DPS; *1/n of k*, *k>1*)	Anticipating effects of splitting operations allows the learner to recompose partitions of *each of n items* (e.g., 3 pizzas) so they can be shared equally into *m* shares (e.g., 7 people). The child distributes the results of splitting each item (e.g., 1/7 of a pizza) across all *m* shares (e.g., giving each person 1/7 from the first pizza, 1/7 from the second pizza, and 1/7 from the third pizza). Key here is the learner's anticipation that replicating, *m* times, the combined share of one person (e.g., shares of all 7 people) would necessarily reproduce the original whole (e.g., the given 3 pizzas), which implies the child must operate multiplicatively in the mental realm. This is compatible with the work of Streefland (1991), in the context of Realistic Mathematics Education	The learner coordinates anticipation of splitting with result of ENS (i.e., a composite unit produced from iterable 1s), and thus reorganizes partitioning to distribute its results onto separate items (each item is simultaneously a discrete unit within a composite whole and a continuous whole of its own right)

(continued)

Table 8.2 (continued)

8. General Fraction Composition (*1/n* of *k/m* and later also *b/n* of *k/n*, with *b>1* and *k>1*)	Applying recursive partitioning (e.g., find 1/7 of __ or 2/7 of __), in anticipation, to a composed fractional unit (e.g., 5/9). The learner's situation includes reversing the IFS for composing 5*1/9 and disembedding these 5 pieces, then splitting each of them (as in DPS above) into the given number of mini-parts (e.g., 1/7 of each 1/9) and then composing the sought-after result by integrating/iterating these (1/63rds) five times. Using this scheme, the learner operates on two different composite fractional units (e.g., 5/9 split into 7 parts and the whole containing of the 9/9), which is compatible to the operations that constitute the General Number Sequence. An HLT for this scheme has been articulated by Simon, Kara, Norton, and Placa (Simon, Kara, Norton, & Placa, in press), including the crucial realization of the change in unit involved in such a multiplicative operation	The learner coordinates and generalizes the anticipation of two previous schemes—DPS and the reversible scheme. The newly linked anticipation allows operating on disembedded non-unit fractions (5/9) similarly to how the learner operated on 1s that constituted a whole number (e.g., 3 pizzas) without losing sight of the whole (9/9)

consider the empirically grounded HLTs they proposed for fostering all four schemes as strong support for a reorganization stance that embraces and builds on whole number and fractional schemes available to the learner.

In summary, the nine fractional schemes depicted in Sects. 8.3.1 and 8.3.2 provide a progression, grounded in numerous studies, of the development of fractions as multiplicative relations. Articulating each scheme as changes in the learner's previous anticipations substantiates the reorganization hypothesis of whole numbers into fractional knowledge (Olive, 1999; Steffe, 2010a). In the next section, I use results from a brain study to illustrate the benefits of a reorganization stance.

8.4 Illustrating Benefits of the Reorganization Stance: Brain Study Findings

My purpose in this section is twofold. First, I attempt to illustrate the benefits of a reorganization stance for promoting the construction of the building block for fractional reasoning—unit fractions (1/*n*) as multiplicative relations. Second, I attempt to support the argument that this stance greatly benefits the growing awareness of mathematics educators for the need to explaining knowing and learning in linkage with the organ (brain) that gives rise to such mental phenomena (De Smedt & Verschaffel, 2010; Leikin & Tzur, 2015). To this twofold purpose, I review findings from a study on how adults' brains process numerical (symbolic) comparisons—between whole numbers (e.g., "7 > 5 ?") or between unit fractions (e.g., "1/7 > 1/5 ?").

This is not a research report. Detailed reports of those findings can be found elsewhere (Tzur & Depue, 2014a, 2014b). Nevertheless, to make sense of the findings it seems helpful to first give a brief background about the study. Then, I present findings that highlight the beneficial impact of a purely *conceptual* intervention on the performance of numerical comparisons by adults whose pre-intervention concept of unit fractions was limited to the typically taught-and-learned, part-of-whole conception. I culminate this section with analysis of fMRI data that indicate similarities and differences in activation of brain regions when it processes those numerical comparisons, and lends further support to claims about reorganization.

8.4.1 Brain Study Background

The study addressed the problem of how task design for brain research and for teaching unit fractions, rooted in the reorganization stance, might impact brain processing when adults (mostly teachers) compare pairs of numbers. It focused on a milestone shift in reasoning—from direct comparison of whole numbers (e.g., 8>3) to the inverse relationship among unit fractions (e.g., 1/3>1/8 although 8>3). Specifically, we wanted to know more about how a purely conceptual intervention, which included no repeated practice of numerical comparisons, would impact performance of adults who already knew the “inverse rule” for unit fractions. This focus arose out of a review of brain research, which at the time of designing our study revealed only a few studies on how the brain processes fractions (Bonato, Fabbri, Umiltà, & Zorzi, 2007; Ischebeck, Schocke, & Delazer, 2009; Jacob & Nieder, 2009), with inconclusive findings, and none that included comparison of both direct and inverse relationships. The study addressed the following two research questions:

1. What impact would a short, conceptually driven instruction (using the Repeat Strategy and the “French Fry” activity) have on adults’: (1a) responses to the tasks in Fig. 8.1 (Stick A, Stick B) and (1b) response time and accuracy when comparing whole numbers (WN) or unit fractions (FR)?
2. What common or different brain circuitry is activated for comparing whole numbers (WN) vs. for comparing unit fractions (FR)?

To address these questions, the study used mixed methods (Creswell, 2009). First, for pretest, each individual participant ($N = 20$; ages 23–36; 14 females) took a pre-intervention, computerized test (ePrime) comprised of 4 task sets, each set including 90, four-step number comparisons (randomized). In Step A of each task (1 s), the symbol of a number or an operation appeared (e.g., 7, 1/7, >, or =). In Step B (1 s), another symbol appeared along with the first (e.g., 7>, 1/7=). In Step C, the comparison task appeared fully (e.g., 7>8?, 1/7>1/8?). Answering the given comparison could be done within 2.5 s, by pressing a key on the right of the keyboard for “true” or the left for “false.” Step D showed three dots as a break between tasks (ITI, 0.5 s).

A video recorded teaching session (~50 min), in which the researcher taught each participant individually, immediately followed the pretest. It commenced with me asking the participant to provide, with some drawing, her or his definition of fractions (using 1/4 as an example). Then, I posed the Stick A–B problem (Fig. 8.1) to foster the participant's perturbation, expected to arise because an expected part-of-whole definition would be inadequate. I then engaged the participant in the challenging task (novel to him/her) of equally sharing unmarked paper strips among 7 people, and then among 11, without folding the paper or using a ruler. I encouraged them to use the repeat strategy: estimating one person's share, repeating that estimated piece 7 times, comparing the resulting whole to the given one, adjusting the estimate, etc. To promote both types of reflection described above, I probed them to reason about adjustments they should make to one person's share. When they seemed to realize the uniqueness of a piece that fits exactly the given number of times and the inverse relationship among pieces used for different number of people, we discussed the generalization of why a larger denominator implied a smaller unit fraction. It should be noted that during the entire teaching session no practice of unit fraction comparisons (or whole numbers) was used.

Two posttests were conducted, the first immediately after the intervention, using the same (randomized) 4 task sets of 90 number comparisons each described above, and the second a few months later during fMRI scanning sessions. To increase fMRI signal, sets were altered to include 140 three-step tasks (eliminating Step B above), and further organized in a hybrid-block design that included random-length sequences of same type comparisons (e.g., 1/3>1/8, 1/7>1/2, 8=8, 5>3, 9>7, 4>3, 6>4, etc.).

8.4.2 Brain Study Results

I first consider research question #1, "What impact would a short, conceptually driven instruction (using the Repeat Strategy and the 'French Fry' activity) have on adults': (1a) responses to the tasks in Fig. 8.1 (Stick A, Stick B) and (1b) response time and accuracy when comparing whole numbers (WN) or unit fractions (FR)?" To address Question 1a, at the teaching session start, as expected, all 20 participants (100%) provided the limiting definition of fractions as equal-parts-of-wholes. They explained that a unit fraction is, "One out of so many equal parts of a whole," drew a circular figure ("pizza") partitioned into 4 parts, and shaded one to show 1/4. Critically, none of them were able to successfully answer both tasks about the yellow part on Stick B. Particularly prevalent (>50%) were responses such as, "The yellow part cannot be a fraction of Stick *A* because it is not a part of *A*," and "I cannot determine what fraction is the yellow part of Stick *B* because there are six unequal pieces on it."

During the teaching session, when asked to equally share a given paper strip among 7 people without folding it or using a ruler, they initially had no solution. When I prompted, "Could you estimate the share of one person and then find out?"

some could generate the repeat strategy independently; for the others, I suggested it explicitly by asking, “Could you estimate the share of just one person and use it for the whole ‘fry’?" When, after a few attempts, they correctly adjusted one person’s share for 7 people and successfully solved this task, I asked them about the share for 11; all (100%) knew to make it shorter, “because I have to squeeze even more parts into the same whole.” At this point, in reference to their activity, I provided a definition (while they wrote it): “A unit fraction is a multiplicative relation to the whole; what makes 1/n what it is has to do with how many times it fits in the whole, or that the whole is *n* times as much of it. For example, your first estimated piece was 1/7 because the whole is 7 times as much of it.”

Returning each of the participants to the stick task, all (100%) then correctly explained that the yellow part is 1/4 of Stick *A* and 1/4 of Stick *B* for one and the same reason, namely, “the length of the whole is 4 times as much as the shaded piece’s length.” These data indicate that the conceptually based intervention, via the repeat strategy, fostered every participant’s reconceptualization of what a unit fraction is—no longer conceiving of it solely or mainly as a part of a whole but rather as a multiplicative relation between two magnitudes. They could thus “see” the shaded part on B as 1/4 in spite of the whole being marked into 6 unequal pieces, or as 1/4 of A although not part of A.

To address Question 1b, I conducted quantitative analysis of the accuracy rate (AR) and the response time (RT). On the average, the analysis of accuracy rate showed no significant difference across all conditions, with rates ranging from 95% (WN before operation in posttest) to 99% (FR before operation in pretest). This result deviates from previous studies (Moyer & Landauer, 1967). It is plausible this deviation was due to the combination between having no time pressure (2.5 s allotted for each task whereas only about 1 s was actually needed) and the directions given to subjects: “Focus on the correct answer and only then on quickly pressing the button.”

Unlike error rates, participants’ response times (RT) were substantially improved (decreased) due to the conceptually driven intervention. To avoid possible “learning curves” for responding, we eliminated responses to the first set (of four) in all 3 testing events of pre-, immediate-post, and remote-post. Data and analysis in Table 8.3

Table 8.3 Impact of teaching intervention on RT in milliseconds (Stdv) from pretest to posttest

	Operation first		
	Pre	Post	Difference
Fractions (FR)	1158 (249)	903 (244)	−255 ($t = 5.9$, df = 19, $p < 0.0005$, ES = 1.06)
Whole numbers (WN)	915 (176)	741 (194)	−174 ($t = 5.5$, df = 19, $p < 0.0005$, ES = 0.96)
	Number first		
Fractions (FR)	1112 (278)	872 (244)	−240 ($t = 5.7$, df = 19, $p < 0.0005$, ES = 0.94)
Whole number (WN)	941 (209)	748 (186)	−194 ($t = 5.1$, df = 19, $p < 0.0005$, ES = 1.00)

confirm the hypothesis that in each of four cue conditions, comparing whole numbers would take significantly less time than fractions (paired-sample *t*-tests ranging between $t = 4.6$ and $t = 8.8$, with df = 19 and $p < 0.0005$ in each). It also supports the hypothesis that the conceptually driven intervention (no practice) would significantly decrease RT for comparing fractions in both cue conditions. In the operation-first condition, participants' comparisons of unit fractions in posttest took, on average, 255 ms less than in pretest (paired-sample $t = 5.9$, df = 19, $p < 0.0005$), with a large Cohen's-*d* effect size (ES = 1.06). In the number-first condition, participants' comparisons of unit fractions in posttest took, on average, 240 ms less than in pretest (paired-sample $t = 5.7$, df = 19, $p < 0.0005$), with a large Cohen's-*d* effect size (ES = 0.94). Those posttest results essentially reached a level comparable with pretest RT for whole numbers. This finding is highly encouraging, because it suggests performance benefits for adults who already knew the "rule," not due to more practice but rather to reconceptualization of unit fractions through reorganization of their WN schemes.

Unexpectedly, the conceptually driven intervention also improved the response time for whole numbers. In the operation-first condition, participants' comparisons of whole numbers in posttest took, on average, 174 ms less than in pretest (paired-sample $t = 5.5$, df = 19, $p < 0.0005$), with a large Cohen's-*d* effect size (ES = 0.96). In the number-first condition, participants' comparisons of unit fractions in posttest took, on average, 194 ms less than in pretest (paired-sample $t = 5.1$, df = 19, $p < 0.0005$), with a large Cohen's-*d* effect size (ES = 1.00). This finding is also encouraging, because it suggests performance benefits for adults who already mastered whole number comparisons, not due to more practice but rather due to reconceptualization of unit fractions through reorganization of their WN schemes. My hypothesis, in need of further study, is that the iteration of one piece to form a whole (e.g., a "fry" composed of exactly 7 or 11 equal shares) also strengthened their conception of number as composite unit.

I now turn to research question #2, "What common or different brain circuitry is activated for comparing whole numbers (WN) vs. for comparing unit fractions (FR)?" The study indicated overlapping and differentiated brain regions recruited for comparing WN and FR. Figure 8.2 shows activation of adult brain circuitry that indicates numerical comparisons (WN in red/yellow, FR in blue) recruit largely overlapping regions as compared to activation for an operation symbol. These fMRI

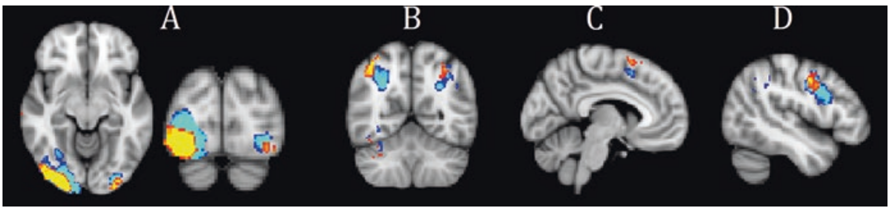


Fig. 8.2 Four overlapping regions (A – Ventral Visual, B – IPS, C – Supplemental Motor Area, D – Posterior Dorsolateral PFC) recruited for numbers more than for operations (yellow/red and blue colors indicate WN and FR, respectively)

simulations show four regions (A, B, C, and D as explained here) known to be recruited for numerical processing as indicated below (Mohamed & Faro, 2010; Talairach & Tournoux, 1988):

(A) the ventral visual processing stream, which is typically activated during object-based, visual processing—mostly in the right hemisphere.
(B) the intraparietal sulcus (IPS) and angular gyrus, which are typically activated during numerical judgments.
(C) the supplemental motor area, which is typically activated when the brain is preparing for a response.
(D) the posterior dorsolateral prefrontal cortex, which is typically activated when the brain is attending to demanding tasks.

As for differentiation of brain regions, Fig. 8.3 shows adult brain circuitry activated more during WN than during FR comparisons, and more during FR than during WN comparisons. Specifically, WN comparisons (Fig. 8.3a) were implicated more than FR comparisons in: (A) the hippocampus and (B) the medial frontal and anterior pole, which are typically activated during abstract retrieval from long-term memory. This finding is consistent with the expectation that long-mastered facts about whole number ordering would be accessed through such retrieval.

In contrast, substantially greater activation for FR than for WN comparisons were implicated in (Fig. 8.3b):

(A) the bilateral IPS and angular gyrus, which indicate the stronger activation needed to process numerical judgments for ordering unit fractions; and the ventral visual processing stream, which indicates object-based visual processing of unit fractions—possibly due to the written format and/or to the way participants learned about them in school and during the conceptually driven teaching session.
(B) the dorsal fronto-parietal control network, which is typically engaged in attention-demanding tasks and is hypothesized to be recruited here more for unit fractions than for whole numbers due to the order inversion required.
(C) the ventral-frontal working memory network and pulvinars, which (as noted above) are likely recruited due to the processing of unit fractions as visually conceptualized objects and thus also involving attention to and selection of those objects.

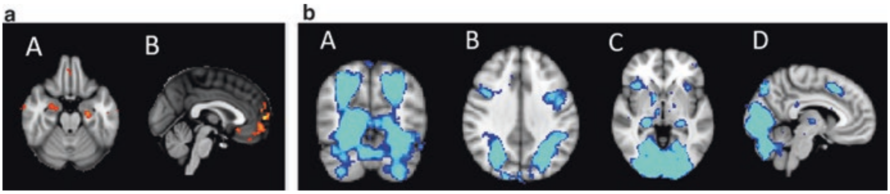


Fig. 8.3 (**a**) WN > FR. (**b**) FR > WN

(D) the supplementary motor area, which (as noted above) would be recruited for preparing the participant's response—likely requiring further processing due to less certainty about the comparison than for whole numbers.

Combined, these analyses suggest that brain circuitry used by adults to compare FR involves higher activation in some areas used also for WN (e.g., IPS), along with a more widespread use of brain regions.

8.4.3 Brain Study Significance

The findings of this brain study contribute to the growing collaborative efforts among mathematics educators and brain researchers (De Smedt & Verschaffel, 2010; Thomas, Wilson, Corballis, Lim, & Yoon, 2010) to better understand mathematical processes in the human brain. For mathematics education, it demonstrates how task design for teaching unit fractions, rooted in the reorganization stance, promotes adults' desired combination of conceptual understanding (clearly shifting from part-of-whole to multiplicative relations) and computational fluency (~20% decrease in RT). Simply put, conceptualizing fractions as multiplicative relations supports mastery, while serving as a basis for learning higher-level mathematics.

The findings of differentiated brain circuitry recruited to compare WN and FR were not identified in previous studies. The limited scope of this study, including the lack of a control group, precludes determining when and how regions that process FR evolved as the participants were learning, and what impact, if any, the constructivist-based intervention had on adults' brains. Nevertheless, distinguishing those regions can inform: (a) studying such changes in brain circuitry, (b) figuring out if they depend on the nature of instructional methods, and (c) appreciating the implied, greater cognitive load involved in making sense of and solving FR comparison tasks.

8.5 Concluding Remarks

In this chapter, I presented the model of reflection on activity-effect relationships (Ref*AER) as a specification of the constructivist reorganization stance and used it to explain the development of fractions as multiplicative relations. The Ref*AER model draws on von Glasersfeld's (1995) depiction of schemes as a three-part building block of cognition, comprised of a situation/goal, an activity, and an effect, signified as [G→[A→E]]. Ref*AER extends Piaget's (1971, 1985) core notion of reflective abstraction, by articulating two types of reflections (mental comparisons) that bring forth available schemes and, through the participatory (prompt-dependent) and anticipatory (spontaneous) stages, reorganize them into a new scheme (Simon, Placa, & Avitzur, in press; Tzur & Simon, 2004).

I applied this model to articulate a process by which a learner's conception of natural numbers as composite units (the Explicitly Nested Number Sequence) is reorganized into the foundational conception of unit fractions ($1/n$) as a multiplicative relation to a given whole with inverse relations among those units ($1/n>1/m$ if $m>n$), called the equi-partitioning scheme. I then described a progression of 8 additional schemes, four rooted in unit iteration and four in recursive partitioning, each being constructed through reorganization of previous schemes—fractions as well as natural numbers. Finally, I provided evidence from a recent study that involved both behavioral tasks and neuroimaging to illustrate the impact that an intervention rooted in the reorganization stance could have on shifting adults' part-of-whole limiting conception of unit fractions into the more powerful equi-partitioning scheme.

Throughout this chapter, I contrasted the reorganization stance with the natural number bias (NNB) or interference stance (Ni & Zhou, 2005; Obersteiner, Van Dooren, Van Hoof, & Verschaffel, 2013; Streefland, 1991; Vamvakoussi, Van Dooren, & Verschaffel, 2012; Van Hoof et al., 2016). This contrast emphasizes the need to interpret research findings not through the lens of the researchers' first-order models, but rather through second-order models that explain how, for the learners, their ways of reasoning about both number types do make sense. To me, what seems at stake with NNB is the focus on demonstrating that inappropriate use of natural number knowledge exists—but not on explaining why, when, and for whom such use occurs. Thus, a "bias" stance seems to also underlie recommendations for making teachers aware of the "bias" (see Van Hoof et al., 2016). As I contended in this chapter, the reorganization stance seems better slanted to explain (a) plausible causes (e.g., participatory stage) for people's folding back from fractional to natural number reasoning and (b) teaching-learning processes conducive to diminishing such folding back (e.g., fostering reflection type-II as a means to support construction of the anticipatory stage of a new scheme).

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Chapter 9
Developing a Concept of Multiplication of Fractions: Building on Constructivist and Sociocultural Theory

Martin A. Simon

Abstract Promoting an understanding of multiplication of fractions has proved difficult for mathematics educators. I discuss a research study aimed at developing a concept of multiplication that supports both multiplication of whole numbers and multiplication of fractions. The study demonstrates how domain-specific theories grounded in two major psychological theories contribute to the development of an empirically based approach to developing this concept. Specifically, the researchers used Learning Through Activity, grounded in constructivism, and aspects of the Elkonin-Davydov Curriculum, grounded in Russian activity theory (sociocultural theory).

Keywords Reflective abstraction · guided reinvention · Elkonin-Davydov Curriculum · Mathematical operations · Hypothetical learning trajectories

In keeping with the book's focus on the interface of psychology and mathematics education, I will discuss work that grew out of a confluence of Vygotskian and Piagetian lines of research. Vygotsky and Piaget are two of the most significant figures in developmental psychology; their influence persists today. The work on conceptualization of fraction multiplication, discussed here, was situated within the domain of mathematics education research. However, it was influenced by lines of research deriving from the work of these giants of psychology.

This is not a research report. (For the research report, see Simon, Kara, Norton, & Placa, in press) In this chapter, I focus on the relationship of the research to the two streams of theoretical work, referring to the data at times to contextualize our efforts. The research referred to was conducted using an adaptation (Simon, in press) of a single-subject teaching experiment (Steffe & Thompson, 2000). The subject, Kylie, was in Grade 5 (11 years old).

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9.1 Background

9.1.1 The Relationship of Constructivism and Sociocultural Theory

In this brief section (with an inappropriately ambitious title), I discuss how we brought together an important principle of each theory. This is not a philosophical discussion, but rather the practical utilization of theories by mathematics education researchers. “There is nothing so practical as a good theory” (Lewin, 1951).

Consistent with the constructivist principles of Piaget (1971), we take conceptual learning to be a process of individual construction. This in no way is in conflict with the sociocultural view of learning as a social process. These two theoretical perspectives describe learning at different grain sizes and can be understood as complementary (Cobb, 2007; Cobb & Yackel, 1996; Simon, 2009).

The claim that learning is a social process recognizes the extent to which learning is socially mediated. That is, learning is afforded and constrained by interaction with others, language, social practices, and the use of artifacts. Thus, what is learned is largely determined by the cultural and historical context in which the learning takes place. One question that is not satisfactorily answered by sociocultural theory alone is how individuals learn concepts that have been developed in their culture. Vygotsky (1978) stressed the importance of revealing “how developmental processes stimulated by the course of school learning are carried through inside the head of each individual child” (p. 91). He argued that knowledge first exists on an *intermental* level and later becomes knowledge on an *intramental* level through a process of *internalization* (sometimes referred to as “interiorization”). “It is through this interiorization of historically determined and culturally organized ways of operating on information that the social nature of people comes to be their psychological nature as well” (Luria, 1979, p. 45, quoted in Lantolf, 2003, p. 350). Bereiter (1985) observed, “How does internalization take place? It is evident from Luria’s first-hand account (1979) of Vygotsky and his group that they recognized this as a problem yet to be solved” (p. 206). In the Learning Through Activity (LTA) research program, we consider that constructivism provides a theoretical foundation for addressing this important issue.

A second related question that is not satisfactorily answered by sociocultural theory alone is what types of mediation (specifically pedagogy) effectively promote the internalization of (mathematical) concepts. Our Learning Through Activity (LTA) research program works to build answers to this question on the basis of emerging answers to the first question.

9.1.2 A Priori Theoretical Commitments of the LTA Research Program

The LTA empirical and theoretical work has been developed based on the following two theoretical commitments.

Guided reinvention. *Guided reinvention* is a construct proposed by Freudenthal (1991). It has three parts. First is the idea that mathematics is invented, not discovered. It is a human construct, not a pre-existing reality discovered by humans. Second, the "re" in reinvention signals that students are inventing knowledge that was previously invented. "The learner shall invent something that is new to him but well-known to the guide" (p. 48). Third, "guided" suggests that pedagogical strategies are needed to allow students to reinvent mathematics that resulted from thousands of years of human invention.

Freudenthal (1991) justified the use of guided reinvention:

> There are sound pedagogical arguments in favour of this policy. First knowledge and ability, when acquired by one's own activity, stick better and are more readily available then when imposed by others. Second, the discovery can be enjoyable and so learning by reinvention may be motivating. Third it fosters the attitude of experiencing mathematics as a human activity. (p. 47).

In the LTA research program, we are committed to the principle of guided reinvention.[1] In particular, we attempt to foster conceptual learning through the learners' activity.

Learning through activity. "Piaget explains that, in his view, knowledge arises from the active subject's activity, either physical or mental" (von Glasersfeld, 1995, p. 56). This tenet of Piaget's constructivism was a foundational idea for LTA research. We reasoned that if learners develop mathematical concepts through their activity, we should be able to find ways to study learning by focusing on learners' activity. Further, if we can identify activity through which learners can learn (reinvent) a new concept, we should be able to design tasks that elicit that activity, thus promoting the learning.

9.2 Multiplication of Fractions

Research has demonstrated that conceptualizing multiplication of fractions is often difficult for students (Kennedy & Tipps, 1997). This difficulty has been observed in their lack of success in distinguishing fraction word problems that are appropriately solved using multiplication (Mack, 2000; Prediger, 2008). Students also often

[1] See Simon, Kara et al. (in press) for the additional reasons that guided reinvention is a core principle of LTA.

struggle determining the appropriate reference unit for each number in a multiplication-of-fractions word problem (Hackenberg & Tillema, 2009; Izsák, 2008; Olive, 1999; Webel & Deleeuw, 2016). The Learning Through Activity (LTA) research program took on the challenge of developing an empirically based hypothetical learning trajectory (HLT, Simon, 1995) for promoting a unified conception of multiplication for whole numbers and fractions.

9.2.1 An Attempt to Use Generalizing Assimilation: A Constructivist Construct

Generalizing assimilation. A key theoretical construct in constructivist theory deriving from Piaget is the concept of *assimilation*. Assimilation is a complex concept. For the discussion here, we focus on one aspect of assimilation. Assimilation is the process by which the individual recognizes that that which is perceived, imagined, or thought about is an instantiation of something for which she has prior knowledge. Generalizing assimilation is the process by which a concept is broadened by the assimilation of additional examples. Assimilation of new examples causes at least a minor *accommodation* in the concept. For example, consider someone whose concept of apples includes the idea that they are red. If he encounters a golden delicious apple, and if he assimilates it as an apple, his concept of apple will be changed. Apples are now red or yellow.

Application of the construct. We had determined that Kylie, as is typical among US students (age 10), had an equal-groups concept of multiplication (i.e., that multiplication is the totaling of equal-sized groups of units). This concept does not effectively support multiplication of fractions, particularly when the *multiplier (number of groups) is a fraction*. Having a fraction as the multiplicand and a whole number as the multiplier posed no problem. For example, Kylie had no trouble thinking about multiplication that indicated 5 groups of 2/3. Her equal-groups concept could assimilate that without difficulty. Our hypothesis was that we could promote a concept of multiplication that would support both whole-number and fraction multiplication in two steps, each involving generalizing assimilation.

The basic design involved the use of the computer application JavaBars (Biddlecomb & Olive, 2000). Quantities were represented by the lengths of bars (rectangles). In our modified version of *JavaBars*, the bars could be partitioned, a part of the bar pulled out, and a bar or a part of a bar iterated,[2] resulting in an equally partitioned bar.

[2] The MARN researchers modified *JavaBars* by creating an "iterate" button. The iterate button takes a bar and a counting number (we will call it "*n*") as inputs and creates a new bar that is *n* times as long as the original bar. We are grateful to Frank Iannucci for programming this modification.

Our initial HLT was based on the hypothesis that students who have a multiple-groups concept of multiplication could expand their concept of multiplication, through generalizing assimilation, to include multiplying by a mixed number and subsequently expand their concept to include multiplication by a fraction.[3] The HLT began by eliciting the activity of representing multiplication as iteration of a bar that had a length specified by the multiplicand. For example, 5×4 (5 is the multiplicand in our teaching experiments[4]) was represented by creating a bar that was 5 units long and iterating it 4 times. This representation was chosen because we envisioned that it could be used by students to represent their whole-number multiplication concept and come to be used for reasoning about multiplication by a mixed number and by a fraction.

The first two conjectures were solid. Kylie had no difficulty representing whole number multiplication as an iteration of a bar representing a composite unit. Second, she had no difficulty moving to multiplication by a mixed number (generalizing assimilation). The first mixed-number-multiplier task I posed was "Here is a bar that is 6 units long. Can you make me a bar that is 6 times 3 1/2?" Kylie iterated the bar 3 times, partitioned the original bar into two parts, pulled out one part, and attached it to the bar she had created through iteration. In subsequent tasks, she moved from solutions of this type to narrating how she would solve the task without actually doing it. As predicted, the nature of the multiplicand (whole number or fraction) had no effect on Kylie's ability to represent multiplication by a whole or mixed number.

The third conjecture was that, once she had expanded her concept of multiplication to include multiplication by a mixed number, she would be able, through generalizing assimilation, to expand her concept to include multiplication by a fraction. The rationale was that multiplication by a mixed number already involved multiplication by a fraction as the second step. For example, in the task described above, Kylie seemed to have created a bar to represent 6 multiplied by 3 and a bar to represent 6 multiplied by 1/2 (before joining them). However, this conjecture was not borne out. When given the task "Here is a bar that is 1/3. Can you make me a bar that is 1/3 times 1/5?" Kylie said that she had no idea how to carry out the task. I modified the task to "Make me a bar that's 1/3 times 4 1/5." She solved the modified task without difficulty. Although Kylie had been able to expand her concept of multiplication, through generalizing assimilation, to include mixed-number multipliers, she was unable to do so with fraction multipliers.

Our explanation for the data was that multiplication for Kylie involved making multiple copies of the multiplicand. She was able to assimilate into her conception making multiple copies including a partial copy. However, Kylie did not recognize

[3] Whether the multiplicand is a fraction or a whole number was not a problem for Kylie and tends not to be a problem for students with a multiple-groups concept of multiplication. They can easily think about iterating a fractional quantity a whole number of times.

[4] This convention was used in the Elkonin-Davydov curriculum. It was useful to us because the order of factors was consistent with the student activity (i.e., creating a bar to represent the multiplicand and using the multiplier to act on the multiplicand).

multiplying by a fraction multiplier as an instance of making multiple copies, so it was not assimilated into her multiplication concept. Generalizing assimilation did not work. We needed a different approach. Rather than trying to promote expansion of her multiple groups conception, we needed to foster the construction of a *new* concept. For the construction of a new concept, we build on the constructivist concept of reflective abstraction (Piaget, 2001; Simon, Placa, & Avitzur, 2016).

9.2.2 Reflective Abstraction: A Constructivist Construct

Piaget's construct of reflective abstraction has been foundational to the empirical and theoretical work of the LTA research program. One focus of the work has been an elaboration of reflective abstraction for the purpose of mathematics pedagogy. This elaboration is discussed in detail in Simon, Kara, Placa, and Avitzur (in press) and Simon et al. (2016). Piaget (1980) wrote:

> All new knowledge presupposes an abstraction, since, despite the reorganization it involves, new knowledge draws its elements from some pre-existing reality, and thus never constitutes an absolute beginning. Two kinds of abstraction are distinguishable …. In the first place, there is a kind of abstraction then that we can refer to as empirical, because its information is drawn directly from external objects themselves. A second form also exists which is fundamental in that it includes all cases of logic-mathematical abstraction. We can call it "reflecting abstraction", because it is drawn not from objects, but from the coordination of actions or operations, (in other words from the subjects own activities). (p. 89–90, cited in Montangero & Maurice-Naville, 1997, p. 57).

I underscore several of the points made by Piaget in this paragraph and add a few points from our elaboration of the construct.

1. All new knowledge involves an abstraction.
2. Reflective abstraction is a process by which individuals construct mathematical concepts.
3. Reflective abstraction describes the making of an abstraction through learners' activity.
4. An activity is a sequence of available goal-directed actions (Simon et al., 2016).
5. Reflective abstraction involves a coordination of actions to create a higher-level action (Piaget, 2001).
6. Reflective abstraction results in learners no longer needing to carry out the activity (sequence of actions) through which the abstraction was made. The abstraction results in an ability to anticipate the result of the activity without carrying it out (Simon et al., 2016).

9.2.3 Building Design Principles on the LTA Elaboration of Reflective Abstraction

LTA focuses on the learning of and promoting the learning of mathematical concepts. The LTA instructional design theory (see Simon, Kara et al., in press, for a detailed discussion) originated with the following conjecture. If reflective abstraction involves a coordination of actions deriving from activity, educators should be able to elicit an activity that can be the basis of the intended abstraction and foster a coordination of actions that produces the abstraction. LTA instructional design begins with two ubiquitous steps: identification of the students' extant knowledge and setting of conceptual goals for the instruction. Following these steps, the two steps that characterize the LTA approach to instruction ensue. First, the designers identify an activity available to the students. Recall that an activity is a sequence of goal-directed actions. "Available" means that students can call on these actions without additional learning (i.e., based on their current conceptual knowledge). Not only must the activity identified be available to the students, but the designers must have a clear hypothesis as to how the activity can be the basis of the intended concept. Once the activity has been selected, a task sequence is designed, as part of an HLT, to elicit the activity and promote reflective abstraction from the activity (coming to anticipate the result of the activity without carrying out the activity).

9.2.4 An Attempt at Reflective Abstraction: Using the MULTIPLY Button

In order to promote reflective abstraction, we sought an activity that when applied to the multiplicand would be consistent across diverse types of multipliers (i.e., whole-number, mixed-number, and fraction multipliers). Our hypothesis was that if Kylie could engage in mentally using the multiplier as the number of times the multiplicand measures the quantity, she would be engaged in an activity through which she could abstract the intended (unified) concept. We began this trajectory by introducing an imaginary JavaBars function, the "MULTIPLY button." We demonstrated its use with whole numbers. One starts with a bar of a certain length (e.g., 1/3), clicks on the bar, puts a number into the little window that opens (e.g., 4), and the result is a new bar (e.g., a bar that is 4 of the 1/3-unit bars or a 4/3-unit bar). Kylie was able to use the imaginary MULTIPLY button (i.e., create the product bar) without difficulty for whole-number and mixed-number multipliers. After four mixed-number-multiplier tasks, I posed a task with a fraction multiplier: "This is 1/3. I click on [the bar] and click on the MULTIPLY button and put in 1/2. Can you show me what it makes?" Kylie was unable to do it. I then spontaneously modified the tasks.

R: Here's a bar. I'm not going to tell you how big it is. If I click on it, click on MULTIPLY and put in one third, do you know what it's going to do?
S: ...Um...No, I don't know.
R: Okay, how 'bout if I click on it. And I click on MULTIPLY and I put in one and one-third.
S: Well then it would make one and one-third.
R: Go ahead.
S: [Kylie partitions the bar into three parts, pulls out one part, and joins it to the original bar.]
R: [On this one] you don't have to do it, just tell me. If I click on this [bar], and then I hit MULTIPLY, and I put in two and one-fourth, what will it do?...
S: It would make two and one-fourth.
R: What do you mean, "It will make two and one-fourth"?
S: Well, if I took it and I repeated it two times, and I took that [original bar] and broke it up into four pieces and pulled out one of them and then attached it to the other pieces, and it would be two and one-fourth.
R: Okay. What if I click on this bar, click on the MULTIPLY [button] and put in one-half?
S: I don't know what to do then.
R: Why?
S: Oh, I could break it into one-half and then pull out one of them and then iterate it once. And that would be one-half.

In Simon, Placa et al. (in press), we attributed what transpired in this excerpt to two changes in Kylie's activity:

> The first change was prompted by the researcher posing tasks that gave no value for the length of the bar used as input to the MULTIPLY button. Because there was no pair of numbers to multiply—there was only a bar and a number applied to the bar—Kylie focused on her activity that represented what the MULTIPLY button would do. That is, she focused on the direct effect of the number input on the bar input rather than thinking about showing multiplication of the numbers. The second change (perhaps also prompted by the change in tasks) was that she began quickly summarizing the effect of the MULTIPLY button, for example, "It would make two and one-fourth." These statements provided a consolidation of what the MULTIPLY button did, allowing her to see the commonality, which was less accessible when she was describing a whole set of actions. Through her work with these tasks (working with a multiplicand of unspecified length) and using these consolidated descriptions, she came to see a commonality in her activity—she was just making a multiplier-sized bar using the inputted bar as her unit.

We initially questioned whether Kylie's activity was useful. After all, she was using the input bar as a unit bar, so we questioned whether she was really building a concept of multiplication. Subsequently, we came to see potential in Kylie's activity (discussed in the next section).

9.2.5 *Reasons for Enlisting Another Approach*

Our analysis of our attempt to promote reflective abstraction using the MULTIPLY button produced three results, which contributed to our decision to build on the Elkonin-Davydov (E-D, Davydov, Gorbov, Mukulina, Savelyeva, & Tabachnikova, 1999, discussed below) approach to teaching a concept of multiplication. Because this is not a research report, but rather an account of the confluence of two scholarly traditions in developmental psychology, we summarize briefly the three relevant conclusions, describing only briefly the data[5] on which they were based.

Lack of a unified concept of multiplication. Kylie's work with the imaginary MULTIPLY button did lead to her being able to anticipate what the MULTIPLY button would do if a fraction multiplier were inputted. However, when questioned, she indicated that she had assimilated the effect of the MULTIPLY button with whole-number and mixed-number multipliers to her concept of multiplication, but considered the effect with fraction multipliers to not be multiplication. Thus, her work with the MULTIPLY button did not provide the means to a unified concept of multiplication.

Difficulty with units. In this section, I skip ahead chronologically to discuss an additional pedagogical problem we faced. After Kylie seemed to have developed a concept of multiplication by a fraction (although not unified with her whole-number concept) and had reinvented a multiplication-of-fractions algorithm,[6] it became clear to us that Kylie could not identify the units of the product of a multiplication-of-fractions word problem and even struggled to identify the unit of the product in a task limited to the JavaBars context. For example, I posed the task, "A recipe calls for 2/3 of a cup of sugar, I want to make 3/4 of a recipe. How much sugar do I need?" Kylie correctly determined the answer as 6/12. However, when I asked her, "Six-12ths what?" she responded, "Six-twelfths of two-thirds." If she did not understand the units of the product, Kylie did not have a useful concept of multiplication of fractions.

A promising activity. Although Kylie's work with the MULTIPLY button was unsuccessful in the ways discussed, the progress she made when applying the MULTIPLY button to bars of unspecified size and its relationship to the E-D approach (discussed below) prompted an exploration of how we might adapt the approach for our revised trajectory.

[5] The analyses of data can be found in Simon, Placa et al. (in press).

[6] See in press, Placa for a detailed account.

9.2.6 The E-D Approach: A Product of Sociocultural Theory

One aspect of Russian activity theory. Elkonin and Davydov were major figures in the development of Russian activity theory (e.g., Davydov, 1990; Elkonin, 1972), which had its roots in the sociocultural theory of Vygotsky. In this discussion, I will refer to only one aspect of this theory, an aspect that was the basis of the E-D approach to multiplication. In Russian activity theory, and particularly the work of Davydov, generalization was an important focus. Davydov took generalization to be a key aspect of a concept and wrote, "One usually means by a 'concept' not just a group of common attributes, but a group of essential common attributes" (p. 7). Davydov (1966) embraced the idea of promoting learning from the general to the specific. By this he meant that initial experiences should be aimed at developing the general theoretical idea. This is followed by elaborating the idea in specific domains. For example, in science, one might promote understanding of the idea of an *ecosystem*. This understanding would be deepened by exploring particular ecosystems. Elkonin and Davydov, in designing their elementary mathematics curriculum, attempted to specify as goals of instruction the most encompassing mathematical generalizations (concepts), generalizations that do not need to be modified as students expand the types of numbers with which they are working (e.g., from whole numbers to fractions). This was true for their approach to teaching multiplication.

The E-D approach to multiplication. As might be suspected, the E-D approach to multiplication, which starts when students are working only with whole numbers, was not directed towards developing a multiple-groups concept. As we have seen, a multiple-groups concept is not very useful for conceptualizing multiplication by a fraction. Instead, the E-D curriculum aims towards a concept of multiplication as a change in units. Towards this end they introduce the idea of an "intermediate unit" (Ji Yeong & Dougherty, 2013). The idea is that if one measures a quantity with an intermediate unit and measures the intermediate unit with the unit, one can find the measurement of the quantity by the unit through multiplication. For example, if I know that a piece of material is 4 yards long (a yard is the intermediate unit) and that a yard is 36 in., I can determine that the length of the piece of material measured in inches is 36×4. Similarly, if I know that the distance between two points is 25 km (a kilometer is the intermediate unit) and I know that a kilometer is approximately 3/5 of a mile, I can determine that the distance measured in miles is $3/5 \times 25$. In both cases, there is a change in units from measuring with the intermediate unit to measuring with the unit. This conception of multiplication applies equally well to multiplying by whole numbers and by fractions.

This relationship is represented in the E-D curriculum by an arrow diagram. Figure 9.1 gives an example of the arrow diagram representing the distance task discussed above. Each arrow goes from the unit that is being measured with to the quantity being measured. The number at the middle of the arrow specifies how many times that unit measures that quantity.

9.2.7 The Affordances of the E-D Approach for the LTA HLT

The work with the imaginary MULTIPLY button showed some potential, but it also left deficiencies in promoting an adequate concept of multiplication. The series of tasks involving the MULTIPLY button elicited an activity associated with the E-D approach. That is, the inputted bar can be understood as an intermediate unit, the inputted number as the measure of the quantity by the intermediate unit. We can say that Kylie was imagining creating a quantity measured in intermediate units. However, our approach did not create a unified concept of multiplication and did not promote a clear understanding of the units involved.

We realized that an important difference between our approach and the E-D approach was the E-D emphasis on multiplication as a change in units. Our approach focused on the generation of the quantity using the intermediate unit, which had been generated from or measured by the unit. A focus on a change in units has the potential to remediate the two problems observed. First, as discussed, the idea of a change in units can apply to multipliers of all types. In contrast, Kylie's work with the MULTIPLY-button tasks resulted in her focusing on *how* the quantity was generated from the intermediate unit, creating copies or taking a part. Second, a focus on a change in units has the potential to promote an understanding of the units of the product of a multiplication of fractions task because the focus is on relating measurement by two different units.

9.2.8 Combining the E-D Approach and the LTA Design Methodology

The challenge. It is here that we explicate the confluence of aspects of the two streams of theoretical and empirical research. In generating a revised HLT, we adapted the E-D approach to multiplication of fractions, derived from a sociocultural tradition, and made use of the LTA design approach, derived from a constructivist tradition. (We note that the elements combined here represent neither the vast expanse of sociocultural-based work nor the vast expanse of constructivist-based work.) The goal of combining these two approaches was to promote reflective abstraction of multiplication as a change in units.

One might ask, "Why was the E-D approach not sufficient on its own?" There are several reasons. First, the students who participate in the E-D curriculum have

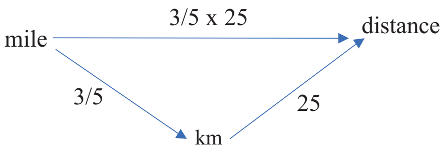

Fig. 9.1 E-D arrow diagram for multiplication

approximately 2.5 years of intensive work with measurement, units, and related representations. Second, the E-D curriculum is not particularly focused on promoting reflective abstraction or guided reinvention. Third, the E-D curriculum has its own potential limitation on creating an understanding of multiplication of fractions. Following is an explanation of this last point.

In the E-D curriculum, multiplication as a change of units is the focus of the students' first experience with multiplication. They learn multiplication when the only numbers available to them are whole numbers. Therefore, the only examples they encounter when developing a concept of multiplication are examples involving whole numbers. Based on the constructivist construct of assimilation, discussed earlier, we might conjecture that, even though multiplication as a change in units is a general concept of multiplication (from the designers' perspective), students are likely to develop a limited concept of multiplication as a result of the limited set of examples they are able to consider. This conjecture has some empirical support. Dougherty (2016) reported that, in her Hawaiian adaptation of the E-D curriculum (Dougherty, 2008), students who had learned multiplication as a change in units did not recognize multiplication-of-fraction word problems as involving multiplication. Instead of calling on multiplication or multiplication arrow diagrams, they drew rectangular area diagrams to figure out a fraction of the multiplicand.

Let us look more in depth at the origin of this shortcoming as it informed our generation of a remedy. The work with multiplication in the E-D curriculum was initiated by the problem of measuring a very large quantity with a very small unit (e.g., the volume of water contained in a large fish aquarium measured with a drinking cup). Students are encouraged to measure with an intermediate unit (e.g., a bucket for measuring the water in the aquarium), to make the job more efficient, and then figure out the measure of the aquarium in cups. Problems of this type become the model for the E-D students' conception of multiplication. Based on this model, intermediate units are made by iterating units and quantities are made by iterating intermediate units. This leads to a conception that applies to natural numbers but does not tend to promote a concept that can assimilate multiplying by a fraction. In contrast, the revised HLT that follows was aimed at students who already have a concept of fraction, specifically fraction-as-measure (Simon, Placa, et al. in press). In designing the HLT, we needed to identify an activity that was not specific to a particular type of multiplier (e.g., whole number or fraction).

The basis of the activity. The activity that we identified was based on two Fraction Bars[7] functions, one that we created and one that we adapted. Fraction Bars (Kaput Center for Research and Innovation in STEM Education, 2015) is a more user-friendly application based on JavaBars. The first function is a MAKE button that takes two inputs, a bar and a number (whole, mixed, or fractional). The bar is treated as a unit and the number as the number of those units in the bar produced. This function reduces to a single action an activity that students have ready at hand—the

[7] We have switched from JavaBars to Fraction Bars (Kaput Center for Research and Innovation in STEM Education, 2015). The latter is a more user-friendly application based on JavaBars

ability to make a bar of any length from a unit bar. The idea is that there is a built-in recursive relationship among the unit, intermediate unit, and quantity. The make button can make the intermediate unit from the unit and (recursively) make the quantity from the intermediate unit. For example, the student can create the product 9/7 × 5/4 by first inputting a unit bar to the MAKE button along with the number 9/7. This produces the intermediate unit. By inputting the intermediate unit and the number 5/4, she can make the product (quantity).[8]

The second activity that is automated in our revised HLT is measuring. The MEASURE button takes two bars as inputs. The first is treated as the measurement unit and the second as the quantity to be measured. The output is the number that is the result of that measurement. Once again, the button is meant to reduce to a single action an activity that is ready at hand for the students. These two buttons allow students to perform activities as single actions. As single actions, they can be called on more easily as elements of the activity through which students come to abstract multiplication as a change in units.

9.2.9 The Revised HLT

9.2.9.1 Purpose

1 Develop a concept of multiplication that affords reasoning about multiplication of fractions and subsumes a prior multiple-groups concept of whole-number multiplication.
2 Develop a concept of multiplication as a *change in units*. Multiplication determines the number of units in a quantity, given b intermediate units in the quantity and a units in the intermediate unit. Multiplication can be expressed as a x b.

9.2.9.2 Prerequisites

1. Concept of fraction-as-measure.
2. Multiple-groups concept of whole-number multiplication.

9.2.9.3 Prior knowledge

Our revised HLT was designed to fit into a sequence of HLTs. We list here other knowledge that we make use of. The knowledge listed here, in contrast with the "prerequisites," was not necessary for carrying out the basic idea of the HLT.

[8] In this discussion, we focus only on multiplication involving two factors. The MAKE button could be used to produce a product of three or more factors.

- Anticipation of the result of taking a fraction of a composite unit, when the result is a whole number (often referred to as “fraction of a set”).
- Anticipation of the result of multiplying a fraction multiplicand by a whole-number multiplier.
- Anticipation of the result of taking a unit fraction of a unit fraction.

9.2.9.4 Task Sequence

In this section, I share an abridged version of the task sequence—only those parts that are germane to the focus of this chapter. The original task numbering has been retained, so a jump in step numbers or task numbers indicates that steps or tasks have been deleted from this version. The full task sequence is available in Simon, Kara, et al. (in press). Each task listed below represents a *set* of similar tasks. Tasks are in light (not bold) italics. Only the tasks themselves are given to students.

Step 2: Measuring on the screen and introduction of the MEASURE button and arrow diagram.

Task 5. Estimating and measuring
[Introduce the MEASURE button and demonstrate how to use it.]

(a) *Measure the blue bar [quantity] with the yellow bar [unit]. Estimate the result if the yellow bar does not measure the blue bar exactly.*
(b) *Measure the blue bar with the yellow bar precisely by using the MEASURE button.*

Task 6. Recording measurement
[Introduce arrow diagram for representing measurement.] (See Fig. 9.2 for example of arrow diagram.) *Measure the blue bar with the yellow bar using the MEASURE button. Record the result in an arrow diagram.*

Task 7. Recording measurement of real-world quantities
[Introduce arrow diagram for real-world quantities.] (See Fig. 9.3 for example of arrow diagram.) *Ellie measured the width of her desk with her hand. The result was 4 1/2. Create an arrow diagram to represent Ellie’s measurement of the desk. (Be specific about the unit and the quantity.)*

Step 3: Introduction of MAKE button and MAKE-MEASURE tasks

Task 9. Make and measure
The bar on the screen is one unit.

(a) *Make a bar that is 5/7 of a unit long.*
(b) *If you measure the bar with the unit, what result will the MEASURE button give?*
(c) *Use the MEASURE button to verify your prediction.*

Fig. 9.2 Arrow diagram for measurement of quantity (Q) by unit (U)

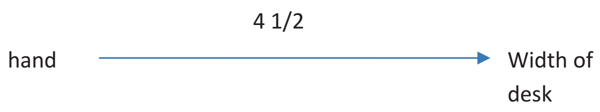

Fig. 9.3 Arrow diagram for measurement of real-world quantities

Task 10. Make (using MAKE) and measure
[Introduce the MAKE button]

(a) *Use the MAKE button and the unit bar on the screen to make a bar that is 9/8 of a unit.*
(b) *Use the MEASURE button to verify what you made.*

Step 4: MAKE-MAKE-MEASURE tasks.

Task 11: Prediction with computable numbers.
[Products that students can compute in their heads, whole number x whole number, whole number x fraction with whole number product, whole number x mixed number with whole number product, fraction x whole number (second number is the multiplier). After a few tasks, we drop parts c and d.]
The bar on the screen is one unit long.

(a) *Make a bar that is 3 units long using the MAKE button.*
(b) *Use the MAKE button to make a bar that is 5 of those bars long. Because we unitized the 3-unit bar to make another bar (i.e., used something that was made from a unit as a unit), we call it an "intermediate unit." We will call the bar you made from the intermediate unit the "quantity."*
(c) *If you unitize the intermediate unit and measure the quantity bar, predict what number you would get.*
(d) *Use the MEASURE button to verify your prediction.*
(e) *If you unitize the unit bar and measure the quantity bar, predict what number you would get.*
(f) *Use the MEASURE button to verify your prediction.*

Task 12: No predictions.
[Products that students are unable to compute mentally]
The bar on the screen is one unit.

(a) *Make a bar that is 5 units long. This is the intermediate unit.*
(b) *Make a quantity bar that is 3 4/7 intermediate units long.*
(c) *Use MEASURE. How many units long is the quantity bar you just made?*

Step 5: Word problems

Task 13: Word problems modeled by MAKE-MAKE-MEASURE activity in Fraction Bars.

Jose walked 8 miles. His sister, Kelly, walked 1 3/5 of the distance Jose walked. How many miles did Kelly walk?

(a) *Make a bar to represent one mile. Use the mile bar to make a bar for the distance Jose walked.*
(b) *Make a bar for the distance Kelly walked.*
(c) *Use the MEASURE button to solve the word problem.*

[Eventually, no direction will be given on what quantities to represent with bars or how to solve the task.]

Step 6: Recording non-context task solutions

Task 14: Using arrow diagram (see Fig. 9.4)

[Introduce diagram as a way to record the work of creating an intermediate unit, creating a quantity, and determining how many units in the quantity. We modified the E-D arrow diagram to emphasize the activity of making the intermediate unit from the unit and the quantity from the intermediate unit (straight across top of diagram). We believed this would be a more accessible representation than the triangle arrow diagram used by the E-D curriculum.]

(a) *The bar on the screen is one unit in length. Make a bar that is 5 1/3 units long. The bar you just made is the intermediate unit.*
(b) *Make a quantity bar that is 4/5 intermediate units long.*
(c) *How many units long is the quantity bar?*
(d) *Record your work on an arrow diagram*

Step 7: Recording word-problem solutions

Task 15: Using Arrow Diagram (see Fig. 9.5 for an example)

Solve the following word problems using Fraction Bars. Record your work on an arrow diagram. Use labels that describe what each bar represents in the word problem.

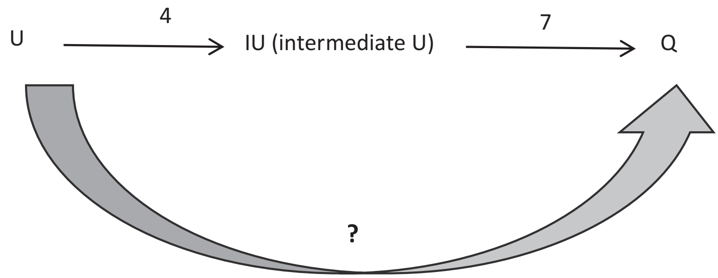

Fig. 9.4 Arrow diagram for a non-context task

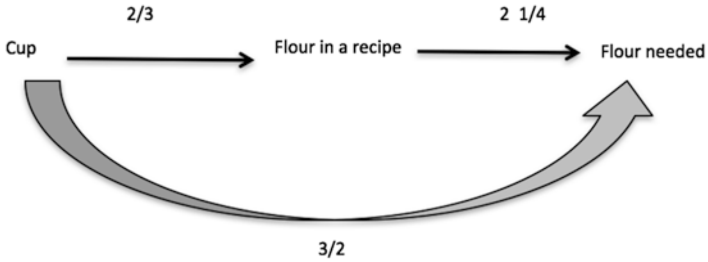

Fig. 9.5 Arrow diagram for a context task (word problem)

Jose walked 8 miles. His sister, Kelly, walked 1 3/5 of the distance Jose walked. How many miles did Kelly walk?

Step 8: Creating arrow diagrams directly from word problem

Task 16: Examples and non-examples

For the following word problems, create an arrow diagram as you read the task. (Do not solve the tasks.) Put a question mark for the number that needs to be found. Some of the word problems that follow cannot be represented by an arrow diagram. If the word problem cannot be represented with an arrow diagram, put "N/A" (not applicable) next to the task, and go on to the next one.

(a) *Melissa has 4/5 of a yard of material. She gave 3/10 of it to Nora. How much material did Nora receive from Melissa?*
(b) *A traveler walked 2/3 of the distance to her destination. During the second day, she walked 1/15 of the distance to her destination. What fraction of the trip had she walked by the end of the second day?*

Step 9: Solution of word problems with emphasis on the referents for the numbers.

Task 17: Referents based on arrow diagrams

(a) *Represent each word problem with an arrow diagram.*
(b) *Solve for the missing number in the arrow diagram mentally or using Fraction Bars if needed.*
(c) *Write down each of the numbers with a label describing what the number refers to in the word problem (e.g., 3 ft in a yard, 2 1/2 kg of sand in the bag, 3/4 of a cup of flour in a recipe).*

Step 10: Definition and symbolization of multiplication

Task 18: Connecting to multiplication

[In this set of tasks, we use whole-number and mixed-number multipliers, so students can recognize multiplication.]

(a) *Create an arrow diagram for each word problem.*
(b) *Calculate the missing number.*
(c) *Write an equation to represent the word problem.*

There are 5 1/3 boxes of candies. In each box, there are 6 candies. How many candies are there?

Task 19: Using new definition of multiplication

[Define multiplication: Multiplication is a change in units from measurement by the intermediate unit to measurement by the unit. We know the number of intermediate units in the quantity and the number of units in the intermediate unit. We need to find how many units in the quantity. Symbolization *m* x *n* means there are *m* units in the intermediate unit, *n* intermediate-units in the quantity, and *m* x *n* units in the quantity.]

(a) *Create arrow diagrams for the word problems. Put N/A for the word problems that cannot be represented by the arrow diagrams we have been using.*
(b) *Write expressions for the word problems represented by the arrow diagrams.*

[Word problems are similar to those exemplified in Task 16]

Step 11: From symbolic expression to word problems

Task 20: Reverse tasks

(a) *Create arrow diagrams for the multiplication expression 3/4 × 5/8.*
(b) *Write a word problem that can be represented by the multiplication expression.*

[After a few tasks, we eliminate the arrow diagram step]

9.3 Discussion

Our research on developing a unified concept of multiplication demonstrates the role that major theories of learning from developmental psychology can play in mathematics education. These major theories, such as constructivism and sociocultural theory (or Russian activity theory), cannot in themselves inform most areas of mathematics education; more domain-specific theories must be derived from them. We see such domain-specific theory in the work of Elkonin and Davydov (Davydov, 1990; Davydov & Tsvetkovich, 1991) and in the LTA work on reflective abstraction of mathematical concepts (Simon et al., 2016; Simon, Kara et al., in press).

We also see in this example how domain-specific theories from more than one major theory can contribute to addressing a single research goal. Whereas most of the theoretical work of the LTA research program derives from constructivism, the E-D approach to multiplication, which was developed through a lens of Russian activity theory, made an essential contribution to this work. The E-D approach, which included mediation by the arrow diagrams, allowed us to develop a way to deal with the shortcomings of the LTA approach, lack of a unified concept, and lack of understanding of the units of the product. The LTA design approach allowed us to develop a way to deal with the shortcoming of the E-D approach, the lack of a methodology for promoting students' construction of the unified concept that was the potential of the approach. We emphasize that it was the domain-specific theories derived from the two major theories, constructivism and socio-cultural theory, that guided the research discussed here. It is only through the development and use of useful domain-specific theories that these major psychological theories contribute in important ways to mathematics education.

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Chapter 10
What's Perception Got To Do with It? Re-framing Foundations for Rational Number Concepts

Percival G. Matthews and Ryan Ziols

Abstract Rational number knowledge is critical for mathematical literacy and academic success. However, despite considerable research efforts, rational numbers present perennial difficulties for a large number of learners. These difficulties have led some to posit that rational numbers are not a natural fit for human cognition. In this chapter, we challenge this assumption, describing recent research into intuitive routes to understanding rational number concepts that diverge from those popular in current curricular recommendations. Namely, we develop the claim that humans are perceptually sensitive to nonsymbolic ratio magnitudes, and that this sensitivity is an early developing, robust and abstract aspect of cognition. We suggest that attending to this perceptually based sensitivity can inform existing theory and help provide a basis for the design of more effective instruction on rational number concepts.

Keywords Rational numbers · Perceptual learning · Numerical cognition · Perception · Fractions

10.1 Introduction

Might fractions be "natural" numbers, too?[1] This question might initially appear facetious. After all, learners' difficulties understanding rational number-related concepts are pervasive and persistent—so pervasive that both mathematics education

[1] With this play on words we challenge the long-held assumption that the counting numbers are "natural" to cognition whereas fractions are not. To avoid confusion, we use the term "whole numbers" throughout and reserve use of the word *natural* in the everyday sense of "existing in nature and not made or caused by people."

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researchers and cognitive developmental psychologists often assume learning to think with and about rational numbers is neither natural to cognition nor easy to learn (e.g., Feigenson, Dehaene, & Spelke, 2004; Gelman & Meck, 1983; Kieren, 1980; Wilkins & Norton, 2011; but see Boyer & Levine, 2012; Davydov & Tsvetkovich, 1991 for counterexamples). Indeed, many researchers converge on the view that whole number concepts provide the foundational tools for the cognitive work needed to build rational number concepts (e.g., Kieren, 1980; Siegler, Thompson, & Schneider, 2011; Steffe, 2002). This chapter provides an alternative to this influential perspective.

We argue that an emerging body of research suggests that human learners are equipped with visually based proto-numerical intuitions that map neatly onto rational number concepts. We further argue that these intuitions seem to be largely perceptual in nature and that they are robust and abstract at an early age. Elsewhere, we have referred to this perceptual ability as the Ratio Processing System (or RPS), championing a *cognitive primitives approach* to constructing rational number (see Lewis, Matthews, & Hubbard, 2016 for a review). The RPS parallels the approximate number system (or ANS; e.g., Halberda & Feigenson, 2008) in that it is presumed to be a phylogenetically ancient system that is tuned to analogs of numerical quantities. It is distinct from the ANS in that it processes ratios, which are inherently relational quantities. We address some of these similarities and differences in Sect. 10.3.2 below.

The cognitive primitives approach differs in two key ways from most of the currently popular curricular approaches to the development of rational number concepts. First, we suggest that perceptual learning can be conceptualized in ways that extend rather than contend with much existing theory about the nature of symbolic number. Second, we take the position that leveraging these proto-numerical intuitions to formalize a "sense" or "feel" for proportion may provide an alternate route to building rational number concepts (see also Abrahamson, 2014; Matthews & Ellis, 2018). This account is quite different from approaches positing that rational number concepts most naturally emerge from processes such as equipartitioning or learning to coordinate units (e.g., Hackenberg, 2007; Olive & Lobato, 2008; Pothier & Sawada, 1983; Steffe, 2002). Rather, our conception foregrounds ratio perception as a primitive antecedent ability that is sensitive to relations between magnitudes or quantities. Ultimately, we argue that this primitive cognitive ability may provide learners with a substantial but currently under-appreciated foundation for building rational number concepts and suggest it may provide fertile ground for math education researchers to investigate further.

10.1.1 The Hegemony of Whole Numbers

The claim that rational numbers are in some way less "natural" than whole numbers permeates a surprisingly large portion of theories from a number of different research traditions (for a lengthier discussion, see Schmittau, 2003). The mathematician Kronecker famously asserted that "God created the natural numbers; all else is the work of man" (Bell, 1986, p. 477), and many—either implicitly or explicitly—seem to agree. For instance, Clements and Del Campo (1990) reflected on the history of the development of rational numbers to argue that they are "not natural, and [are] needed only for the purpose of studying more mathematics" (p. 188). In a less value-laden judgment, cognitive psychologists Feigenson, Dehaene and Spelke (2004) suggested that rational number concepts are difficult because they are not particularly compatible with the human cognitive architecture, which they posit evolved to deal with natural numbers and their analogs (i.e., countable groups of objects) via subitizing and the ANS.

For many math education researchers, such assumptions are much more subtle and implicit but share key similarities. For example, Steffe (2002) hypothesized that rational number schemes are built largely through accommodations of whole number schemes related to partitioning and/or iterating. Indeed, Steffe has noted etymological connections between fraction concepts and the experience of breaking, such as the breaking of a plate into (countable) pieces—an idea put forward in 1896 by John Dewey and James McClellan and implicit in the etymological roots of the word "fraction" (from the Latin *franger* or "to break"; see also Norton & Hackenberg, 2010). From a distinct, but somewhat related position,[2] Confrey's (1994) "splitting" model argued that rational number concepts emerge in part from a psychological ability to "split" a physical object into many equal-sized pieces without counting. In these theories, the relational aspects of rational numbers are subsidiary to processes of coordinating increasingly complex and layered piece-wise or unit schemes. In other words, the core of such theories remains focused on creating fractions from discrete pieces that correspond alternately to an equipartitioned whole or a "disembedded" (and countable) reference unit.

In sum, multiple prominent research perspectives heavily privilege whole number intuitions and their analogs—through the use of countable, discrete parts and pieces—as the cornerstone for developing rational number concepts (for discussion, see Davydov & Tsvetkovich, 1991; Schmittau, 2003). Our purpose here is neither to exhaustively catalog the prevalence of whole number based approaches to learning rational numbers nor to argue against the pedagogical power of such approaches. Instead, we want to convey the general dominance of views that presume a primacy for whole numbers in building fractions and other rational number-related concepts, whether via counting-first, fair-sharing, or equipartitioning, as a backdrop against which we present our current perceptually based framework.

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[2] For one perspective on how "splitting" is related to aspects of Steffe's "reorganization hypothesis," see Steffe and Olive (2010). On our reading, the *n-split* of Confrey's model (e.g., Confrey & Smith, 1995) is clearly a case of equipartitioning that primarily maps to whole numbers. However, the similarity-based aspects of splitting discussed by Confrey seem to be a case of the RPS in action.

10.1.2 Nonsymbolic Ratio as Relational Magnitude

Turning to our theory, the central argument we present is that perception can and does provide intuitive access to *ratio* (i.e., relationally defined) magnitudes (e.g., Abrahamson, 2012; Bonn & Cantlon, 2017; Carraher, 1993; Jacob, Vallentin, & Nieder, 2012; Lewis et al., 2016; Matthews & Chesney, 2015). By relational, we refer to the fact that ratio quantities–such as that denoted by a:b or *a/b*–are defined based on the relative magnitudes of components *a* and *b* rather than by the magnitudes of either component considered in isolation. Moreover, we argue that this sensitivity to ratio is rooted in the basic psychophysics that governs human perception of magnitudes in multiple modalities and formats (Bonn & Cantlon, 2017; Matthews & Chesney, 2015; Sidney, Thompson, Matthews, & Hubbard, 2017). We further suggest that perceptual intuitions about ratios may provide an important foundation for building formal understandings of rational number concepts—including fractions—that are distinct from and complementary to understandings built on whole number knowledge.

A vulgar view of perception might cast it as primarily concerned with "lower-order" phenomena as opposed to higher-order relational concepts. Accordingly, in Sect. 10.2, we begin by reviewing two arguments that clarify what we mean by perception and the potential it may have for generating "boundary work" between mathematics education researchers and cognitive psychologists.[3] First, we argue that perception is not merely some "low-level" activity; it often involves extracting higher-order relations from the environment. In other words, perception as we conceptualize it involves robust forms of abstraction that can support meaningful engagement in mathematical thinking and learning (see also de Freitas, 2016; Howison, Trninic, Reinholz, & Abrahamson, 2011; Stroup, 2002). Indeed, we suggest that perceptual learning is actually compatible with several aspects of social and radical constructivisms in this regard (e.g., Ernest, 1998; von Glasersfeld, 1984).

In Sect. 10.2, we review empirical work that has marshaled considerable evidence that humans and even nonhuman primates have perceptual sensitivity for nonsymbolic ratio magnitudes (see Fig. 10.1). We cite this not to argue about whether monkeys and apes do math but rather to suggest that aspects of formal mathematics can be accessed in part by leveraging phylogenetically ancient perceptual sensitivities. Finally, in Sect. 10.4 we consider possible implications for research and pedagogy concerning the development of rational number concepts. Specifically, we highlight the need for recruiting math education researchers' expertise if we are to successfully formulate approaches for using nonsymbolic ratios in ways that can support mathematical thinking.

[3] We wish to recognize that a good number of math education researchers, particularly those who employ an embodied perspective, already conceive of perception as concerned with higher-order thinking (e.g., Abrahamson, 2014; de Freitas, 2016).

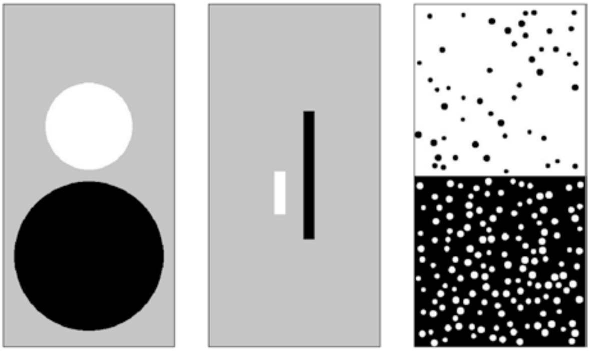

Fig. 10.1 Sample nonsymbolic ratios made from paired circle areas, line segment lengths, and dot arrays. All three represent a ratio of 1/3 instantiated in different graphical formats. These three formats are just a small sample of the space of possible nonsymbolic ratios that can be represented graphically (adapted from Matthews, Meng, Toomarian, & Hubbard, 2016)

10.2 Perception as Selective Extraction of Relevant Information

10.2.1 Perception vs. "Raw Sensation"

Researchers concerned with perceptual learning have long argued that perception should not be conceived of as only concerned with low-level sensorimotor representations of the external world (e.g., Barsalou, 2008). Rather, perception is highly selective and can be a source of complex and abstract understandings (Gibson, 1979; Goldstone & Barsalou, 1998; Goldstone, Landy, & Son, 2010; Kellman, Massey, & Son, 2010). From a Gibsonian perspective, perception is fundamentally about selectively attending to relevant information and relations from the vast amount of information taken in by the senses (e.g., Gibson & Gibson, 1955). Here, it is critical to underscore the selective nature of perception versus the indiscriminate nature of what we call "raw sensation." Perception sifts through the vast amount of information from the entirety of a visual display and registers the presence of simple features. This can be profoundly more powerful than verbally mediated thought for detecting meaningful features in the environment—largely due to the prodigious volume of things that can be visually processed in parallel compared to the limited serial processing of verbally mediated logic (e.g., Goldstone & Barsalou, 1998).

For instance, we can effortlessly recognize a good friend's face, but our ability to verbally describe the similarities and differences between her face and another is much more limited. There is a reason that a picture is sometimes worth more than a thousand words, and that reason lies in the sheer computational power of our perceptual processing abilities (Larkin & Simon, 1987). As Kellman and Massey (2013) have argued:

> "Sensorimotor knowledge" does not convey the scope and power of what perceptual mechanisms deliver. Not only is explicit abstract thinking a newer evolutionary acquisition, but the work of abstraction is not exclusively the province of [conscious] thinking processes alone. Much of thinking turns out to be seeing, if seeing is properly understood. (p. 120)

This notion of "seeing…properly understood" is an example of how current psychological notions of perception go well beyond a simple sight-as-sensation concept. Elements of abstraction are deeply entangled with perception, and with experience, perception can be trained to select socially designated relevant information from a "background" of mostly unconscious sensory information (cf. de Freitas, 2016; Gal & Linchevski, 2010).

10.2.2 Perceptual Learning and Concept Formation

It is no small feat that even infants can rapidly learn to identify dogs as members of a category distinct from other animals (Quinn, Eimas, & Rosenkrantz, 1993). From experience with only a few exemplars, children are able to extract relevant features that allow them to identify previously encountered instances and to generalize and recognize new, quite different instances—while also excluding superficially similar exemplars that belong to another category (Kellman & Massey, 2013). For instance, Quinn et al. (1993) documented the ability of 3- and 4-month-old infants to exclude cat photos from the dog category, even though they share a large number of surface features (coloring, fur, number of legs, shape, tails, number of eyes, etc.). This is because perception filters sensory inputs so that we extract increasingly complex and abstract relations relevant to detecting ecologically important properties of objects (like dogs and cats) and events (Gibson, 1969, 1979; Goldstone, 1998; Kellman & Massey, 2013).

The relevance of perceptual learning for concept acquisition is not restricted to such simple examples. Because of the power of perceptual learning, humans are able to become chess experts, Olympic judges, radiologists, and air traffic controllers (Gauthier, Tarr, & Bubb, 2009; Kellman et al., 2010). In each of these professions, people must develop perceptual tools for rapidly categorizing their worlds and making complex conceptual distinctions that cannot be separated from perception (Gauthier et al., 2009; Goldstone et al., 2010). That is, these experts' perceptual processes have been tailored by experience such that there are experience-induced changes in the ways these experts extract information about the world (Kellman & Garrigan, 2009). It is in part because we can make such perceptually based conceptual distinctions over and above ever-emerging novelty and variation that we can develop the abstracting powers needed to solve problems, reason, and create (Goldstone and Barsalou, 1998; Kellman & Massey, 2013). Indeed, as we argue in the next section, perceptual learning also plays a critical role in mathematical learning and mastery.

```
5  +  2*3  +  7 =
  5+2  *  3+7 =
```

Fig. 10.2 When asked to evaluate the top expression, participants often will answer 18. When presented with the lower expression, they are more likely to answer 70. The difference in the solutions offered is primarily due to perceptual groupings, as they are the same according to the formal order of operations (adapted from Landy & Goldstone, 2007)

10.2.3 Perception and Mathematical Notation

Even efficiently reading mathematical notation involves training the perceptual system to isolate certain regularities. As Landy and Goldstone (2007) have noted, "although notational mathematics is typically treated as a particularly abstract symbol system, it is nevertheless the case that these notations are visually distinctive forms that occur in particular spatial arrangements and physical contexts" (p. 720).

It may seem counterintuitive to some, but this same sort of perceptual learning helps provide a foundation for much mathematical thinking involving symbols (Kellman et al., 2010; Landy, Allen, & Zednik, 2014; Landy & Goldstone, 2007). This is because with practice, we can train our perceptual systems to tune into the particular features of a symbolic representation that matter and ignore the rest. For example, Landy and Goldstone (2007) found that the same arithmetic expressions written with somewhat different spacing consistently led their participants to generate different solutions, despite the fact that the order of operations dictates that they should be equivalent (Fig. 10.2). Thus, even our processing of symbolic systems is quite often largely perceptual in nature (Landy et al., 2014).

10.3 Ratios as Percepts: The Ratio Processing System

10.3.1 Perceptual Access to Rational Number Concepts

How could something as mathematically complicated as ratio possibly be an object of perception? Why would sensitivity to ratios of things in the world be picked by natural selection? If one begins with a formal mathematical analysis of rational numbers, then it may be hard to see how perception might apply in any more than a trivial sense (see Matthews and Ellis, 2018 for a distinction between "perceived ratio" and "mathematical ratio"). To illustrate, we provide an intuitive example of the relevant properties of visual perception that will be familiar to the reader. Consider the following:

1. Figure 10.3a shows a cup pictured on a table.
2. The same cup is shown in **b** and **c** at different distances along the table.
3. Consider this question: Could the cup in panel **d** be the same cup? Why or why not?

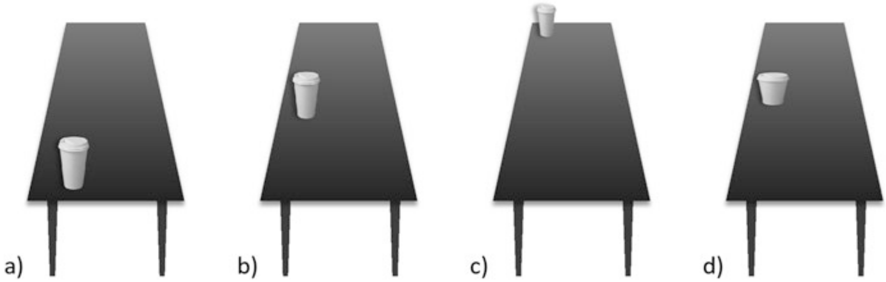

Fig. 10.3 The set of panels (**a**–**d**) show a set of coffee cups in different positions on a table. Perceptual sensitivity to ratio allows us to see that the cup in **d** is different from the other cups

The reader will recognize the cups in panels **a**, **b**, and **c** as possibly the same because they share the same aspect ratio of width:height. Even though the same cup casts images of different sizes on the retina as it is moved closer to or farther from us, the aspect ratio of those images is conserved. In contrast, we know that the cup in panel **d** cannot be the same because it has a different aspect ratio. It is easy to generate examples like this, because perceptual sensitivity to ratio is ecologically important (Jacob et al., 2012). Basic visual sensitivity to ratio helps us to perceive object identity and to discriminate between different objects (cf. Confrey & Smith, 1995; Piaget & Inhelder, 1969/2008).

This sensitivity to ratio is important for a wide range of organisms and presumably has been for a large swath of evolutionary time. Jacob et al. (2012) provide a brief but informative review of examples ratio sensitivity across species. To list a few, chimpanzees attack rival groups only when they outnumber their rivals by a ratio of 3:2 (Wilson, Britton, & Franks, 2002); lions avoid battle when the proportion of roars in a rival group passes a certain threshold (McComb, Packer, & Pusey, 1994); and feeding mallards distribute themselves according to ratios of foraging patch richness (Harper, 1982). All told, there is considerable evidence that multiple species exhibit a primitive sensitivity to nonsymbolic ratios. Thus, from an evolutionary perspective, it is reasonable to suspect that humans may also have evolved to be sensitive to ratios composed of physical features in the environment.

10.3.2 The Ratio Processing System vs. the Approximate Number System

To be clear, the ratio processing system should not be confused with the approximate number system (ANS)—the perceptual system thought by many to allow the rapid enumeration of discrete sets such as dot arrays (e.g., Dehaene,

Dehaene-Lambertz, & Cohen, 1998; Feigenson et al., 2004; Meck & Church, 1983; Nieder, 2005; Piazza, 2010). There are several clear distinctions between the two systems. First, the ANS is conceived of as processing individual dot arrays (or other discrete quantities) that correspond to whole numbers, whereas the RPS processes magnitudes that emerge relationally from pairs of magnitudes. Thus, a typical ANS-based task might involve estimating the numerosity of a single dot array or judging the larger of two such arrays. In contrast, an RPS-based estimation task requires concatenating two arrays, treating the separate numerosities such as those on the right in Fig. 10.1 as a *single relational magnitude*. Second, the ANS is limited to processing discrete stimuli, whereas the RPS can process a broader set of stimuli, including ratios composed of line segments (Jacob et al., 2012; Matthews, Lewis, & Hubbard, 2016), circle areas (Matthews & Chesney, 2015), the implied areas of numerical fonts (Matthews & Lewis, 2017), and areas of squares (Bonn & Cantlon, 2017). Finally, the RPS covers a much broader set of magnitudes than the ANS can. The pairing of discrete components confines the RPS to rational numbers. However, when ratios are constructed as continuous stimuli each component can be made arbitrarily large or small, which means the RPS can process magnitudes that correspond to real numbers. For instance, the ratio made by juxtaposing the diagonal of a square with one of its sides corresponds to $\sqrt{2}$.

These differences noted, the exact relations between the ANS and the RPS have yet to be fully determined. There are obvious reasons to suspect that there may be connections. For example, a task comparing the larger of two dot arrays is usually conceived of as an ANS-based task governed by Weber's law. However, the classic formulation of Weber's law holds that discriminability between two stimulus magnitudes depends not upon the absolute magnitudes of the stimuli, but upon the ratio between them (Fechner, Howes, & Boring, 1966; Piazza, 2010). Thus, it is reasonable to ask whether the typical ANS discrimination task might be reframed as a ratio perception task if we foreground ratio by rotating the stimuli and considering them as a single ratio whose magnitude is to be estimated as closer to one (more difficult to discriminate) or farther from one (easier to judge). Although this alternative framing seems reasonable when considering ratios made from pairs of dot arrays, it is clear that the ANS as typically conceived cannot be responsible for RPS processing of circle stimuli, because the ANS cannot process that class of stimuli.

Ultimately, answers regarding possible connections between the ANS and the RPS will have to wait for more basic research on the psychophysics of quantity perception. Several authors have pointed to similarities in the developmental trajectories of discrimination acuities for quantities in multiple formats (e.g., Feigenson, 2007; Odic, Libertus, Feigenson, & Halberda, 2013; Walsh, 2003). To the extent that processing different classes of quantities relies upon shared neurocognitive architectures, the more likely it is that a given person's ANS acuity can predict that person's discrimination abilities for other types of quantities, including RPS acuities for ratios constructed from these stimuli. To the extent that quantity processing among different stimulus types is dissociated, the more separate the ANS and the more general RPS will seem. For the moment, we adopt an agnostic view on the possible connections between the ANS and RPS and focus on the demonstrated abilities of the RPS.

10.3.3 Ratio Sensitivity and Math Education Research

Turning to mathematics education research, it is this type of perceptually accessible, nonsymbolic ratio that Carraher (1996) highlighted when he pointed to the distinction between a ratio of quantities and a ratio of numbers. According to his definition, a ratio of quantities concerns the *relationship* between two nonsymbolic magnitudes, such as the ratio of the lengths of two line segments considered in tandem (e.g., "1/2" instantiated as ). In this case, each of the two component line segments can be conceptualized as an extensive quantity. That is, their magnitudes can be defined by their lengths considered in isolation from each other. In contrast, the ratio of these nonsymbolic magnitudes is an intensive quantity determined by the ratio between them. Carraher further suggested that nonsymbolic ratios could serve as a proto-numerical foundation for reasoning about ratio concepts since number knowledge is "developed through acting and reflecting upon physical quantities" (p. 241).

In parallel arguments, Abrahamson (2012) argued that educators should attempt to leverage what he called *perceptually privileged intensive quantities*—that is, holistic or intuitive understandings children have for ratio-based magnitudes such as slope, velocity, or likelihood (see also Abrahamson, Shayan, Bakker, & Van Der Schaaf, 2015). Importantly, much research has demonstrated that very young children show considerable—albeit naïve and difficult to articulate—sensitivity to the magnitudes of ratio-based quantities (Duffy, Huttenlocher, & Levine, 2005; McCrink & Wynn, 2007; Sophian & Wood, 1997; Spinillo & Bryant, 1991). As intensive quantities, these are all fundamentally based on ratios between and among elements of a system; they are *not* a summary catalog of the overall amount of "stuff" in a system. Both Abrahamson's and Carraher's arguments presuppose (1) that ratios are perceptually accessible on some level and (2) that this perceptual access can help lead to conceptual understandings when leveraged via socially mediated practice.

It is important to underscore that what we mean by ratio sensitivity here is a primitive perceptual competence and not an explicitly constructed, verbally mediated scheme. This differs substantially from accounts such as the incrementally acquired protoquantitative ratio sensitivity described by Resnick and Singer (1993). On Resnick and Singer's formulation, ratio sensitivity is a sort of scheme built slowly over time that combines (a) explicitly reasoning about "fittingness" of objects (i.e., the intuitive understanding that some things go together, such as a round peg and a round hole) and (b) noticing co-variation of "size-ordered series" (see also Singer & Resnick, 1992). By contrast, our concept of perceptual sensitivity to ratio magnitudes is not so particular or developed so gradually. Instead, as we describe below, it emerges at a very young age and is fairly abstract.

10.3.4 Considering the Empirical Evidence for Ratio Sensitivity

So far in this chapter, we have largely appealed to intuitive examples and analogies to illustrate the nature and extent of humans' perceptual access to ratio. However, recent research has begun to more systematically examine the nature and scope of human sensitivity to ratio properties, and the findings are compelling. Although formal instruction on rational numbers typically does not begin until grades 3 and 4 (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010), children far younger than this demonstrate some understanding of nonsymbolic ratios. For instance, McCrink and Wynn (2007) demonstrated that even 6-month-old infants are sensitive to nonsymbolic ratios. Infants who were habituated to specific nonsymbolic ratios subsequently looked longer at novel ratio stimuli that differed by a factor of 2. For example, infants perceived a 2:1 ratio composed of yellow Pacmen intermixed with blue pellets as different from a 4:1 ratio. By 4 years of age, children can order pictures of part-whole ratios based on ratio magnitudes (Goswami, 1995) and perform above chance on tasks that require the addition and subtraction of nonsymbolic ratios (Mix, Levine, & Huttenlocher, 1999).

Moreover, this sensitivity to nonsymbolic ratios is flexible and abstract: Singer-Freeman and Goswami (2001) showed that children can draw proportional analogies among pizzas, chocolates, and lemonade even though the materials equated are visually dissimilar. Matthews and Chesney (2015) recently demonstrated the abstract nature of ratio magnitude perception by having participants make cross-format ratio comparisons (Fig. 10.4). Adult participants could quickly and accurately judge which was larger between a ratio composed of circle areas and a ratio composed of dot arrays or between symbolic numerical ratios and nonsymbolic ones.

Here we highlight several important aspects of these tasks and the observed pattern of results: First, they were created so that participants had to rely on perception to feel out the ratios represented in these different formats. The comparison tasks involved ratio magnitudes presented in nonsymbolic formats that were not amena-

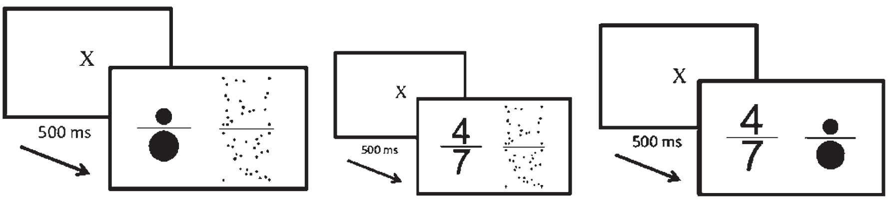

Fig. 10.4 Examples of comparison tasks used with adult participants. Adults were able to easily make accurate and fast comparisons across both numeric and non-numeric ratios when controlling for dot numerosity and scalar differences in sizes of component parts (adapted from Matthews & Chesney, 2015)

ble to counting or to calculation via symbolic algorithms. Circle ratios were composed of pairs of continuous quantities that cannot easily be partitioned into uniformly regular parts to facilitate coordination among counted units. Moreover, dot ratios were composed of a minimum of 46 dots, far too many to count in the approximately 1100–1400 ms on average it took participants took to make their decisions. Second, participants compared nonsymbolic ratios without first translating them into symbolic form. The clearest evidence for this is that participants made nonsymbolic ratio comparisons across formats (i.e., circles versus dot arrays) faster than they made symbolic comparisons within the same format (e.g., 3/5 vs. 2/3). If participants had first converted nonsymbolic ratios into symbolic form, there would be a cost for translation added to the time for symbolic comparison. Finally, participants responded as though the ratio stimuli were perceived as intensive quantities. Any given nonsymbolic ratio magnitude could be instantiated in a multitude of ways, because it was not the size of any individual component that determined the ratio magnitude. Still, participants were proficient making these cross-format comparisons despite the variability of representations.

Additionally, in a recent study we found that preschool-age children show a sensitivity to nonsymbolic ratios that parallels that of adults with similar tasks (Matthews, 2015). To avoid the linguistic difficulties involved with explaining ratio comparison tasks with young children, we used match-to-sample tasks with nonsymbolic images like those in Fig. 10.4. Preschool children, kindergartners, fifth-grade students, and college undergraduates from a selective university were presented with target nonsymbolic ratios corresponding to specific ratio magnitudes and asked to indicate which of two stimuli matched each target magnitude. Nonsymbolic ratios took two forms—ratios of circle areas and ratios of line segment lengths (Fig. 10.5). The ratios between matching and distractor stimuli were presented at each of five levels to assess discrimination acuity (3:1; 2:1, 3:2, 4:3, and 6:5 in order of increasing difficulty). Note that each of these levels indicates a ratio of ratios. For instance, pairing a nonsymbolic 1/2 with a nonsymbolic 1/6 corresponds to a ratio of 3:1.

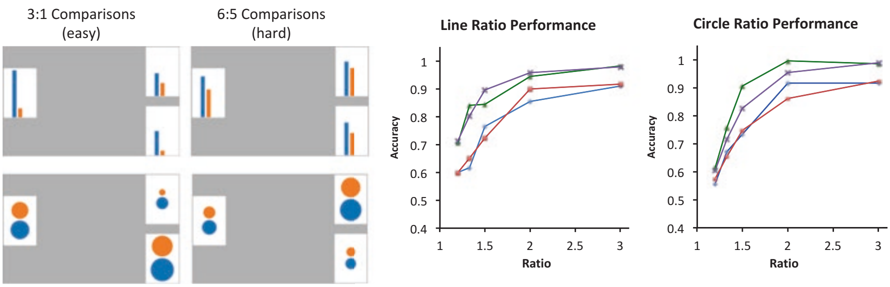

Fig. 10.5 Stimuli (*left*) and comparisons of accuracy on match-to-sample tasks (*right*) for young children in preschool (*blue line*) and kindergarten (*red line*), fifth graders (*green line*), and adults (*purple line*). Note that fifth graders performed better than adults in some instances (*green vs. purple lines*) and accuracy shows similar patterns across all groups (adapted from Matthews, 2015)

We found that children's performance followed the same pattern as college students' performance, with accuracy well above chance and increasing as the ratio between matching and distractor stimuli approached one (Fig. 10.5). Fifth-grade participants performed at least as well as college undergraduates, despite obvious differences in mathematical experience. Moreover, the top half of kindergarten and preschool children performed at near-adult levels, despite not having formal instruction with rational number concepts. Considering the tasks involved, and the ages at which participants demonstrated competence, these results present a clear case in which perception is sensitive to ratio, a higher-order relation. In the parlance of cognitive psychology, these results imply that people can encode nonsymbolic ratio magnitudes in a relatively specific analog format (i.e., in an intuitive, nonverbal, and approximate form). Stated more plainly, these results suggest that perceptual sensitivity to ratio emerges early and across different nonsymbolic formats.

All told, sensitivity to the magnitudes of nonsymbolic ratios has been demonstrated for multiple animal species (Drucker, Rossa, & Brannon, 2016; Jacob et al., 2012; Rugani, McCrink, de Hevia, Vallortigara, & Regolin, 2016) and among pre-verbal infants (McCrink & Wynn, 2007), elementary school-aged children (Boyer, Levine, & Huttenlocher, 2008; Duffy et al., 2005; Jeong, Levine, & Huttenlocher, 2007; Meert, Grégoire, Seron, & Noël, 2013; Sophian, 2000; Spinillo & Bryant, 1999), typically developing adults (Hollands & Dyre, 2000; Matthews & Lewis, 2017; Meert, Grégoire, Seron, & Noël, 2012; Stevens & Galanter, 1957), and individuals with limited number vocabularies and formal arithmetic skills (McCrink, Spelke, Dehaene, & Pica, 2013). This widespread competence—even among nonhuman animals, infants, and societies without formal number concepts—indicates that these abilities are present even in the absence of formal education (Lewis et al., 2016; Matthews & Chesney, 2015; Matthews, et al., 2016). Such empirical evidence raises important questions about the assumption that rational number pedagogy must or should begin with an established whole number schema as a sole or primary foundation.

10.4 Implications for Pedagogy

In this section, we explore some potential directions and implications for education. On the front end, we wish to note that research on the RPS is in its infancy. As such, much more research is required before we would feel comfortable making any strong statements about the adaptation of basic research in this domain to practical pedagogy. With this caveat in mind, we offer a few speculative points below.

10.4.1 Exploring with Continuous Roots

Some research suggests that attending to countable elements in a ratio display can actually interfere with children's tendencies to rely on their perceptual sensitivity to ratio (e.g., Boyer & Levine, 2012; Jeong et al., 2007). The ratio similarity between

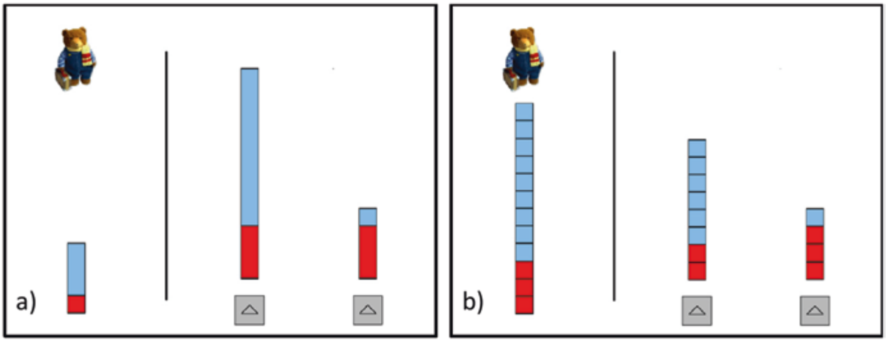

Fig. 10.6 When asked to choose which drink has the same juice to water mixture that Wally Bear wanted, children were much more successful when partitions were removed (**a**) than when partitions were present (**b**) Reprinted from Boyer, T. W., & Levine, S. C. (2012). Child proportional scaling: Is 1/3 = 2/6 = 3/9 = 4/12? *Journal of Experimental Child Psychology, 111*, 516–533, with permission from Elsevier

the images on the left of the two portions of Fig. 10.6 and those on the right should be easy for the reader to see. The only difference is that the left panels are presented in unpartitioned, continuous form whereas the right panels are partitioned. However, whether figures are partitioned mattered for young children. In experiments involving stimuli from Fig. 10.6, young children (6–8 years) were told that the left-most of the three images in each box corresponds to the way Wally Bear prefers to mix his juice and asked which of the two alternatives on the right would taste the same as Wally's mixture. Young children were far more likely to pick the accurate solution when presented in unpartitioned form (Fig. 10.6b) vs. the partitioned form (Fig. 10.6b). It seems that the unpartitioned form allowed children to rely on their perceptual abilities, whereas the partitioned form led them to use count-based schemas that did not generalize to the ratio context.

We should pay close attention to the contrast in children's competence with these two types of representation given currently dominant instructional practices. Because conventional instruction on rational numbers typically begins by leaning heavily on discrete and/or partitioned representations and engages count-based logic, it may be that conventional instruction misses opportunities to support children's perceptually based sensitivity to nonsymbolic ratio. To the extent that these perceptual abilities can foster meaningful mathematical thinking about rational number concepts, it is worthwhile to think about the potential gains to be realized by designing instruction to more directly build on children's intuitive ratio sensitivity. Our point mirrors that made by Lewis et al. (2016):

> …if the brain of the elementary school child - like that of the human adult or the nonhuman primate - is able to represent the holistic magnitudes of these nonsymbolic ratios, pedagogies based on this capacity may also help children build an intuitive understanding of [rational number] that serves as a generative foundation for future learning…[i]n comparison to count-based methods, which may bind thought in terms of whole numbers, proper engage-

> ment of [nonsymbolic ratio sensitivity] can potentially help students develop a clearer understanding of the relational properties of numbers (p. 156).

10.4.2 The Potential of Perceptual Training

As mentioned above, one potential route for leveraging the RPS is to employ various perceptual learning modules (e.g., Gibson, 2000; Goldstone et al., 2010; Kellman et al., 2010). In contrast to top-down or discourse-based teaching methods typically thought of as encouraging "mathematical thinking," perceptual learning modules can at first blush appear relatively simplistic. As succinctly described by Kellman et al. (2010), perceptual learning modules involve (1) tasks requiring discrimination or classification (much like the ratio comparison and match-to-sample tasks described above); (2) many short trials in which a learner makes a decision and receives feedback; and (3) minimal emphasis on explicit instruction.

Although such rapid-fire training may evoke images of "drill and kill," principled use of perceptual training is far from mindless. Indeed, studies have shown the critical role that perceptual learning plays in complex domains including symbolic mathematics (Goldstone et al., 2010) and language learning (Saffran, Aslin, & Newport, 1996). At the heart of perceptual training techniques is creating a scenario in which a learner is continually confronted with to-be-learned structures in multiple instantiations so that commonalities or variations in structures begin to "pop out."

To be more concrete, research reviewed above demonstrates that humans are readily able to extract important aspects of ratio magnitudes from visual displays made from paired nonsymbolic stimuli. At the same time, it is abundantly clear that learners face difficulties extracting the same structure from symbolically presented rational numbers, despite being quite knowledgeable about the whole number components that comprise them (e.g., Lipkus, Samsa, & Rimer, 2001; National Mathematics Advisory Panel, 2008; Newton, 2008; Post, Harel, Behr, & Lesh, 1991; Stigler, Givvin, & Thompson, 2010). It may be that using symbolic ratios and fractions alongside nonsymbolic ratios in a perceptual learning intervention can help learners map the relational quantities they see in nonsymbolic ratios onto rational number symbols.

Of course, as Erlwanger's (1973) well-circulated case of Benny has shown, targeted interventions are limited by how well they are integrated into the sociocultural milieu. Thus, we wish to make clear that we do not necessarily advocate for a contemporary version of plugging children into more sophisticated and data-driven "workbooks." Rather, we wish to suggest that "practice" as repetition is necessary at some level for developing mathematical thinking and that modules can be designed in ways that align well with perception. When employed as part of a larger and comprehensive set of efforts, perceptual learning modules may offer fruitful paths for mathematics learning that are as yet unappreciated in popular theory.

10.4.3 Further Considerations and Implications for Further Research

Our aim in this chapter so far has been to describe with cautious optimism the potential benefits of aligning perceptual sensitivity to nonsymbolic ratio magnitudes with instructional approaches that incorporate this sensitivity. We have focused on magnitudes in particular because a raft of recent studies have demonstrated that understanding rational number magnitude is an important piece of conceptual knowledge that predicts general mathematics achievement (Bailey, Hoard, Nugent, & Geary, 2012; Fazio, Bailey, Thompson, & Siegler, 2014; Mazzocco & Devlin, 2008; Siegler et al., 2011; Torbeyns, Schneider, Xin, & Siegler, 2015). Nonsymbolic ratios thus have the potential to serve as a powerful pedagogical tool for learning about this aspect of rational number knowledge.

That said, we are less sanguine about assuming that nonsymbolic ratios will prove very useful in promoting other important aspects of rational number knowledge. It is particularly difficult for us to see how nonsymbolic ratios can be used to teach arithmetic procedures with rational numbers. The composed units reasoning at the heart of most research (and referenced above) in rational number pedagogy is certainly a critical component for developing competence with rational numbers. Our perspective is based on a deep and abiding respect for Kieren's (Kieren, 1980) position that rational numbers should be seen as a mega-concept involving many interwoven strands (or a series of subconstructs); rational numbers thus can be seen as representing part-whole relations, as ratios, as quotients, as measures, and as operators (see also Behr, Lesh, Post, & Silver, 1983). Yet as powerful as the dominant composed-units approach may be, it is clear that (a) its reliance on whole number knowledge to support knowledge about rational numbers can be difficult for young learners and (b) it does not address every strand of this mega-concept. We thus propose that pre-existing intuitions in the form of nonsymbolic ratios may both be more accessible to learners and help supplement existing approaches educators may take in building rational number sense (Abrahamson, 2012; Abrahamson & Sánchez-García, 2016; Matthews & Ellis, 2018).

Of course, this view must be balanced by two very important caveats addressed to some extent above: First, representations do not teach on their own. Even if humans have some native sensitivity to visually presented nonsymbolic ratios, the mapping from nonsymbolic sensation to symbolic representations and formal concepts is neither automatic nor simple (Rau & Matthews, 2017). Additionally, psychophysics is a normative science. Learners may need to approach rational number learning in a variety of ways that do not align well with a visually mediated approach to learning. However, to underscore with redundancy, we are not advocating for a universal approach to ratio learning. Rather, for ratio sensitivity to be a productive tool in education much additional work must be dedicated to investigating how we might use privileged nonsymbolic representations as didactic objects within classroom discourses (e.g., Sfard, 2007; Thompson, 2002). That is, how might we deploy these and other related representations and intuitions as things that support reflective mathematical discourse about rational numbers? This work is the clear province of mathematics education researchers, and indeed a number of math education researchers have written explicitly about the use of perception to construct rational number (e.g., Abrahamson et al., 2015; Carraher, 1993; Confrey & Smith, 1995; Davydov & Tsvetkovich, 1991; Kaput & Maxwell-West, 1994; Resnick & Singer, 1993; Stroup, 2002).

10.5 Concluding Remarks

A growing body of research suggests that human beings are sensitive to nonsymbolic ratio magnitudes composed of pairs of objects in multiple formats. This perceptually based sensitivity provides intuitive access to intensive quantities that correspond to rational numbers. Nonsymbolic ratios may not be numbers *per se*, but they most certainly correspond to rational numbers and illustrate important qualities of rational numbers—most directly those involving size or magnitude. In this sense, our ability to perceive ratios parallels abilities like the ANS or subitizing; it is a basic ability that can be co-opted to support mathematical thinking (Feigenson et al., 2004; Piazza, 2010).

This nonsymbolic ability appears to develop early and may even be phylogenetically ancient. Why should this concern us? It should concern us because teaching and learning are fundamentally tied up in the work of trying to get evolutionarily ancient brains to become adept at engaging in and transforming culturally constructed activities. This often involves complex coordination of primitive abilities. On this point, we find Dehaene and Cohen's (2007) neuronal recycling hypothesis to be compelling: it essentially says that we will be proficient at culturally constructed activities to the extent that we succeed at leveraging ancient abilities in support of (new) cultural modes of thought. It is plausible to suspect that sensitivity to nonsymbolic ratio is in fact a basic property of human cognition—one that has proved important enough to persist through millennia of human evolution. This primitive intuition about ratio is something that we should seriously consider when thinking about teaching and learning.

To date, educational researchers seeking to leverage perception for training about rational number have done so without recent insights into the psychophysics of ratio perception that have emerged over the past half-decade. This newer work (a) provides explicit evidence about the ability to process specific ratio magnitudes in analog form (e.g., Matthews & Chesney, 2015; Matthews & Lewis, 2017); (b) demonstrates that these perceptually processed ratio magnitudes automatically affect symbolic processing in certain contexts (e.g., Matthews & Lewis, 2017); and (c) explicates neuropsychological theories regarding the potential for linking the general human psychophysical apparatus to process specific ratio magnitudes (e.g., Bonn & Cantlon, 2017; Chen & Verguts, 2017; Jacob et al., 2012; Lewis et al., 2016; Sidney et al., 2017). Many of these new insights come from cognitive psychology, but we openly admit the limits of the discipline. Thus, we are convinced that the pedagogical potential of the psychophysically based gaze on nonsymbolic ratio sensitivity will not be fully actualized without considerable effort to determine how this ability may be harnessed by educators and learners alike.

Many important open questions remain regarding whether and how we might adapt (or *exapt*)[4] this ratio sensitivity for pedagogical purposes. At the moment, cognitive psychologists have made considerable headway in showing that this intuitive sense of proportion exists, but they have made very little progress in translating this research so that it applies to educational concepts. We thus ask: Might nonsymbolic ratio images provide more intuitive access to certain subconstructs of the rational number mega-concept? If so, how can these intuitions be brought to bear on mathematical thinking? That is, how can nonsymbolic images that engage these intuitions best be made use of as didactic objects? Finally, how might new pedagogies that engage ratio perception be integrated with existing pedagogies that focus on units-coordination logic?

At this point, the questions currently outnumber the answers. What we do know is that ratio perception is a basic capacity that clearly aligns with important aspects of rational number concepts that are not given much play in current school curricula. Given learners' perennial difficulties with rational number concepts, it seems prudent to consider how these basic abilities might help us better understand and supplement existing approaches.

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[4] Exaptation is a phrase coined by Stephen J. Gould to refer to traits that were not built by natural selection for their current use, but which have affordances that organisms leverage for new functions (Gould & Vrba, 1982).

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Chapter 11
Commentary on Fractions

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Abstract This commentary raises and discusses questions based on some of the agreements, disagreements, and themes found in the four chapters on fractions. It considers (1) the importance of tasks that are based in perception and readily available activity in light of an emphasis on problem solving in mathematics education, and the role that theories about thinking and learning play in designing such tasks; (2) some potential connections among various theories about thinking and learning as they relate to fractions; (3) the natural number bias and how ideas about natural numbers could serve as a foundation for fractions; and (4) the roles that magnitude, measurement, and linear representations of number play for fractions.

Keywords Fractions · Magnitude · Measurement · Multiplication · Ratio · Rational numbers

Fractions are one of the most difficult topics in school mathematics. Much research has investigated why this is so and how we might help students make better sense of fractions and reason productively with fractions. Several themes are discussed across the four chapters on fractions in this volume: considering fraction thinking from different angles and different research domains, attending to components and foundations of fraction thinking, investigating into the nature of fraction thinking as it develops, and comparing student thinking about fractions with expert thinking.

In keeping with the goals of this book, the authors of all four chapters express a desire to work across research traditions, to find productive ways to join modes of inquiry and theoretical approaches, or to make connections and integrate findings produced by different approaches. Because fractions are such a complex topic, it makes sense to consider issues involved in their learning from many different angles, and the four chapters foreground and highlight different aspects of fraction learning. Obsersteiner, Dresler, Bieck, and Moeller (this volume, Chap. 7) focus on what makes fractions so difficult for students. They locate sources of difficulty in the transition from natural numbers to fractions. Students sometimes apply

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knowledge and reasoning about natural numbers to fractions even when these do not apply, and brain studies indicate that natural number knowledge can interfere even with experts' performance on fraction tasks. On the other hand, Obersteiner, Dresler, Bieck, and Moeller note that a focus on fraction magnitudes seems to be promising for helping students learn fraction concepts. Tzur (this volume, Chap. 8) discusses how an understanding of fractions could arise from a reorganization of natural number conceptions. Tzur foregrounds the actions of iterating and (recursive) partitioning and the role that reflection on such activities plays in helping students reorganize what they anticipate the effect of these activities will be in situations that lead to fractions. Similarly, Simon (this volume, Chap. 9) foregrounds activity and reflective abstraction as means for internalizing mathematical concepts. He develops empirically based hypothetical learning trajectories for promoting a conception of multiplication that spans whole numbers and fractions. Matthews and Ziols (this volume, Chap. 10) do not take as given that the natural numbers must be a sole or primary foundation for learning fractions. Instead, they argue that humans have an inherent perceptual ability to process non-symbolic ratios and that perceptual learning that leverages this ability might provide an alternate route into fractions.

In the remainder of this commentary I will discuss some commonalities, differences, ideas, and themes that stand out to me among the issues and approaches discussed in the four fraction chapters. These commonalities, differences, ideas, and themes raise questions for research on mathematics education, for combining research across different fields, and for teaching mathematics.

11.1 Tasks Based in Perception and Activity Versus Problem Solving

In mathematics education, we often place a high value on solving challenging problems. We expect students to work on problems that they do not immediately know how to solve, to persevere even when their initial attempts fail, and to monitor their thinking and progress during the process (e.g., National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). Yet the fraction chapters in this volume point to the importance of mathematical engagement of a very different sort—based in perception and in readily available actions—even though these would seem to involve a much lower form of cognitive engagement than mathematical problem solving. As Matthews and Ziols (this volume, Chap. 10) put it "a vulgar view of perception might cast it as primarily concerned with 'lower order' phenomena as opposed to higher-order relational concepts" (p. 6). Has the importance of engaging in perception and other activities that are readily available to students been somewhat overlooked in mathematics education today? Why might tasks based in perception and activity be important?

Matthews and Ziols (this volume, Chap. 10) suggest that perceptual intuitions about non-symbolic ratios that humans are naturally endowed with—what they term a Ratio Processing System (RPS)—could be an important foundation for constructing rational number concepts. They argue that perceptual intuitions about non-symbolic ratios are robust and abstract at an early age, and that leveraging these proto-numerical intuitions might provide an alternate route to building rational number concepts. On their proposal, perceptual learning, based in selective attention to relevant information and relations, might help students develop fraction ideas. Matthews and Ziols are cautious to distinguish the perceptual training they have in mind from "drill and kill." But what are characteristics of well-designed tasks based in perception and activity?

Although a worksheet of fraction comparison problems could be an example of a readily available activity for a student, the kinds of tasks that Tzur (this volume, Chap. 8) and Simon (this volume, Chap. 9) present in their chapters are very different. For one, almost all of them ask students to engage with quantities that are presented either physically (e.g., a strip of paper representing a "French Fry") or on a computer screen. But also, Tzur and Simon developed their tasks based on their theories of learning. Tzur's tasks are designed so that even though students can readily engage with the activities of the task, they may notice a difference between anticipated and actual effects of their activities. For example, when a student who has just shared a French Fry equally among five people is asked to estimate one share if six people share such a French Fry, the student might initially make a larger piece than before (because six is greater than five). Upon iterating that piece, the student finds that the actual effect of their activity—six copies of the piece is longer than the original Fry—is different from their anticipated effect—they expected six copies of the piece to be the same length as the original Fry. This feature of Tzur's task is linked directly to his theory, Reflection on Activity-Effects Relationships (Ref*AER), which he posits as a mechanism for learning. Similarly, Simon formulates hypothetical learning trajectories based directly on his theory of learning. Simon posits that some learning can occur by broadening the examples that a learner recognizes as an instance of something already known, but new mathematical concepts arise from coordinating available goal-directed actions so as to create a higher-level action. Thus the tasks in Simon's hypothetical learning trajectories intend to elicit activities that will either be recognized as a new kind of example for a known activity or will foster a coordination of actions.

Carefully designed activities that draw on perception and other readily available actions and that are crafted to develop mathematical concepts based on our understanding of how people learn are very different from routine drill worksheets. But they are also different from some descriptions of problem solving as a challenge that involves looking for ways to approach the problem. It seems likely that we need both of these experiences in mathematics teaching and learning. The four fraction chapters in this volume make a persuasive case that perceptual training or activity together with socially mediated reflection on it is likely to be valuable for helping students learn fraction ideas. This leads to several questions. When and for what purpose should we seek to develop mathematical ideas and skills through problem solving and when and for what purpose might it be better to draw on perception and readily available actions? Will students learn different things from these different kinds of experiences? Are some ideas and skills better learned one way than the other?

11.2 Potential Connections Across Theories About Thinking and Learning

The theories about thinking and learning that the four fraction chapters in this volume draw from concern very different aspects of thinking, such as how and in what region the brain processes numerical information versus sociocultural influences. These theories might therefore not be directly related, even though they could be connected at some level—for example if human cognition is founded on a handful of core systems that include one for numbers and one for social partners (Spelke & Kinzler, 2007). So studies that investigate how the brain processes numerical information may not need to consider sociocultural theory, and conversely. Other theories or parts of theories that are discussed in these chapters seem like they could be closely related and invite the question of whether they are different ways of describing similar phenomena or similar ways of describing different phenomena. I will discuss a few such examples that stand out to me.

To discuss perceptual learning, Matthews and Ziols (this volume, Chap. 10) cite (among others) Goldstone, Landy, and Son (2010), who describe "rigged up perceptual systems," in which scientific and mathematical reasoning is grounded in perceptual processing by co-opting natural perceptual processes for tasks requiring abstract or analytic reasoning. Goldstone et al. argue that systematically training perception and action systems is a highly effective way of facilitating sophisticated responses and that deep conceptual understanding requires the support of perceptual-motor grounding. They further claim that making and perceptually interpreting diagrams can be methods for changing perceptions and that diagramming processes are not just a means to explicate abstractions but can substitute for abstraction. Is this "rigged up perceptual systems" view of systematically training perception and action systems related to Tzur's (this volume, Chap. 8) theory of Reflection on Activity-Effects Relationships and Simon's (this volume, Chap. 9) theory in his Learning Through Activity research program?

Based on their theories, both Tzur and Simon have students repeatedly carry out carefully selected actions designed to foster specific understandings, and they include built-in feedback. For example, Tzur describes tasks that have students estimate and then check the size of one person's share when a given whole is to be shared equally among some number of people. Simon describes tasks that ask students to measure one bar with another bar and to make another bar of a given numerical (including mixed number and fractional) length, given a bar that is 1 unit long. Are such tasks systematically training perception and action systems a la Goldstone et al.? Are Tzur and Simon in essence co-opting natural perceptual processes for tasks that involve abstract reasoning about fractions? On the other hand, the tasks that Tzur and Simon use may be significantly different from tasks in the kinds of "perceptual learning modules" that Matthews and Ziols discuss. These perceptual learning modules have the feature that their tasks require discrimination or classification, which the tasks that Tzur and Simon describe do not require, or at least not directly.

At the theory level, both Tzur (this volume, Chap. 8) and Simon (this volume, Chap. 9) use elaborations of Piaget's notion of reflective abstraction. Tzur considers two types of reflection. The first type comes from making predictions and seeing that the outcome is inconsistent with what was expected. Tzur posits that this first type of reflection occurs automatically and leads to newly noticed activity-effect dyads. Tzur's second type of reflection generally needs to be facilitated by interaction with others. Tzur posits that this second type of reflection enables the abstraction of a linkage between an anticipated activity-effect dyad and the situation or goal in which the mental system would anticipate and trigger it. Tzur then links these two types of reflection to two stages of construction of a scheme, and he uses these two stages to explain why students are sometimes capable of carrying out an activity but cannot yet see that such an activity would help them solve a problem they are given. Simon cites Piaget who described two types of reflection, empirical reflection, which draws information directly from external objects, and reflecting abstraction, which is drawn from coordination of subjects' own mental activities. Simon explains that his elaboration of reflective abstraction is for the purpose of mathematics pedagogy. But might Simon's and Tzur's theories inform theories about training perception and action systems, and conversely?

There are other recent theories that concern perception and action, which were not discussed in the four fraction chapters, but might be relevant to the discussion above. These come from the idea that the brain functions by prediction. Brain cells "support perception and action by constantly attempting to match incoming sensory inputs with top-down expectations or predictions" (Clark, 2013, p. 181). And "research and theory are converging on the idea of the brain as an active inference generator that functions according to a Bayesian approach to probability: sensory inputs constrain estimates of prior probability (from past experience) to create the posterior probabilities that serve as beliefs about the causes of such inputs in the present" (Barrett & Simmons, 2015, p. 419). Do these Bayesian theories of prediction and revision apply only to lower-level sensory inputs, or might they be like a fractal and also apply to higher-order thinking, such as reasoning about a fraction task? Are these Bayesian theories related to theories that Tzur and Simon draw on? For example, in discussing scheme theory, Tzur highlights anticipation. Anticipation links a situation or goal with an activity and it links an activity with its effect. Is the "anticipation" of scheme theory related to the "prediction" of Bayesian theories about the brain? Is there a relationship between viewing learning as involving the construction of an invariant activity-effect relationship (Tzur) or a coordination of actions to create a higher-level action (Simon) and Bayesian views?

Ultimately, of course, theories must be useful for the specific purpose at hand. In his concluding comments, Simon notes the importance of domain-specific theories. Although he drew from both constructivism and socio-cultural theory, it was domain-specific theories derived from those broader theories that guided his research and that he believes can contribute in important ways to mathematics education.

11.3 The Natural Number Bias and Natural Numbers as a Foundation for Fractions

Obersteiner et al. (this volume, Chap. 7) discuss a number of persistent difficulties that students tend to have with fractions and they locate the source of these difficulties in the mathematical content itself and in the human cognitive system. Mathematically, there are significant differences in the systems of fractions and whole numbers. Obersteiner et al. discuss differences related to how fractions and whole numbers are represented, issues of successors and density, and how the operation of multiplication behaves. However, they caution us that even though these differences pose potential obstacles for learning, learners' actual obstacles may be different because learning does not necessarily follow the logic of a subject domain. Thus we should seek insights from looking into the cognitive mechanisms of learning. Obsersteiner et al. consider a number of theories on cognition and learning, some of which concern thinking at the neuro-cognitive level. Overall, the theories and the empirical evidence that Obersteiner et al. review lead to a mixed picture of both coherence and discontinuity across whole numbers and fractions.

A disagreement across the four fraction chapters of this volume is in whether or not there is a natural number bias. For Obersteiner et al. (this volume, Chap. 7), the natural number bias is "the over-reliance on natural number knowledge even in problems that require rational number reasoning" (p. 15). They cite research that shows differences in performance on fraction tasks according as these tasks are congruent or incongruent with reasoning one would use with whole numbers. These differences occur even for people who have acquired sound conceptual knowledge of fractions. Obsersteiner et al. conclude that solving fraction problems requires inhibition of intuitive knowledge about natural numbers. In contrast, Tzur (this volume, Chap. 8) eschews the natural number bias, which he sees as entailing a deficit view of children and as a manifestation of researchers' own sophisticated distinctions between natural numbers and fractions. For Tzur, the researchers' distinctions do not take into account students' conceptual frames of reference, from which students' solutions do make sense. I will make an overlapping point in the next paragraph.

One facet of the natural number bias that may not have been adequately explored is whether some of the reasoning that is attributed to over-reliance on natural number knowledge has been accurately characterized. For example, given the task of comparing 5/6 and 7/8, suppose a student imagines a set of six marbles, five of which are blue and another set of eight marbles, seven of which are blue. The student may see a sameness in the two sets—in both cases, all but one marble is blue—and may therefore say that the fractions 5/6 and 7/8 are the same. Similarly, a student who is asked to add 5/6 and 7/8 might imagine combining those two sets of marbles and therefore write 5/6 + 7/8 = 12/14 because 12 out of the 14 total marbles are blue. I do not see these examples as misapplications of natural number concepts to cases where rational number concepts are needed. Instead, I see the issue as one of the norms of mathematical communication and the very precise ways that we use mathematical ideas and notation to express particular interpretations of situations. From the point of view of the student, the eqs. 5/6 = 7/8 and 5/6 + 7/8 = 12/14 may appear to express their ideas about the sets of marbles. In fact, the fractions 5/6 and 7/8 *do* legitimately apply to the sets of marbles, an equation *is* a statement about sameness, and addition *does* model combining things. It is only by appealing to our very precise conventions about what exactly equations and addition mean in the mathematics community that we can explain why the mathematics community does not use 5/6 = 7/8 and 5/6 + 7/8 = 12/14 to model the two marble situations.

As an alternative to the natural number bias, Tzur (this volume, Chap. 8) takes the stance that fractions can be learned by reorganizing whole number knowledge. Tzur describes a progression of fraction schemes that students could develop, starting from whole number schemes. In recent years there has been much research on how children learn natural number concepts, with ongoing inquiry and debate (for example, see Spelke, 2017). If ideas about fractions are grounded in ideas about natural numbers, then it seems natural to ask whether the natural number schemes on which fraction schemes are grounded fit with current conceptualizations of how children understand natural numbers. For example, in discussing how a child might think about combining eight marbles and seven marbles, Tzur describes a very specific way of thinking about the numbers in this context, namely as symbols for the anticipatory effect of iterating 1 a certain number of times, e.g., "'8' for the child was a symbol for an anticipatory effect of iterating 1 eight times" (p. 7). Viewing whole numbers in terms of iterating is important in Tzur's theory because the fraction schemes he presents are based on iteration. According to Tzur, "an iterable unit fraction (1/n) serves as the 'building block' to other fractions, similar to how an iterable unit of 1 serves as the 'building block' to other whole numbers" (p. 20). But how faithful is Tzur's iteration model to the way children think about cardinalities of sets in situations like the marbles? It seems unlikely that the child would think of the set of marbles itself as formed by iterating one marble eight times—for one, the marbles might have different colors and sizes. Instead, the child might think of each marble as an object that falls under a "marble" category. Tzur's assumption thus seems to be that the child sees the cardinality of the set of marbles through the idea of counting the marbles one by one, viewed as measuring the set by one marble. Do children think about cardinality that way?

There is evidence that humans, including babies who cannot count and adults who are well versed at counting, automatically process set size through the approximate number system (e.g., see Dehaene, 2011 and Chap. 4 of this volume). One current proposal is that children's natural number understanding builds on innate systems, including the approximate number system and a collection of systems that serve to represent objects as members of kinds (Spelke, 2017, p. 148). Thus, it seems possible that even children who understand how to determine the cardinality of a set by counting might still view cardinality primarily in a way that is different from counting and different from iterating 1 s. In other words, it seems possible that cardinality is not usually viewed as the result of measurement by a unit, a point to which I will return in the next section.

11.4 Magnitude, Measurement, and Linear Representation of Numbers

The idea that numbers, including fractions, can be viewed in terms of magnitude or lengths, or located on number lines, is recognized as important across all four fraction chapters in this volume. Obersteiner et al. (this volume, Chap. 7) discuss work of Siegler et al. (2012), who have proposed an integrated theory of whole number and fraction development, with magnitude as a unifying idea. Based on findings from a number of studies, Obsersteiner et al. conclude that overall, a focus on fraction magnitudes seems to be effective and transfers to other fraction concepts. As a consequence, one of their recommendations for classroom practice is that instruction should focus on fraction magnitudes and the use of number lines.

The central argument that Matthews and Ziols (this volume, Chap. 10) make is that perception can and does provide intuitive access to relationally defined magnitude and that these perceptual intuitions may provide an important foundation for building formal understanding of rational number concepts. They further suggest that a foundation built on perceptual intuitions may be distinct from and complementary to understandings built on whole number knowledge. How might perceptual intuitions be leveraged to build fraction ideas? One type of example that could be useful is two adjacent strips or line segments. As Matthews and Ziols show (see their Figs. 1 and 6), this kind of example has been used in empirical work that shows evidence of perceptual sensitivity for non-symbolic ratio magnitudes. If humans have an intuitive sense that given pair of strips are in some specific relationship, one can imagine leveraging this intuitive sense to define numbers as the result of measuring one strip by another strip. In fact, this is exactly the approach that Davydov and Tsvetkovich (1991) take, which Simon (this volume, Chap. 9) adapts for his revised hypothetical learning trajectory to develop a unified sense of multiplication across whole numbers and fractions.

In my own work with Andrew Izsák and our research group, we have also emphasized a measurement sense of number in our courses for future middle grades and secondary mathematics teachers. We discuss (positive) whole numbers and fractions as the result of questions of the form “how many of this strip does it take to make that strip exactly?” and we use this measurement view of numbers to define multiplication as coordinated measurement. Our definition applies across whole numbers and fractions and across every type of multiplication word problem (see Izsák & Beckmann, 2018). Our approach to multiplication is similar to (but more general than) Simon’s (this volume, Chap. 9) adaptation of the Elkonin-Davydov approach, which Simon proposes to use in his revised hypothetical learning trajectory. However Izsák and I treat unit fractions ($1/n$) differently from Simon (Simon, Placa, Avitzur, & Kara, in press) or Tzur (this volume, Chap. 8), even though all of us see measurement as important to understanding fractions. Izsák and I emphasize that a unit fraction ($1/n$) can be viewed not only as a unit, which is “stuff” that can be counted (e.g., 9/7 is 9 one-sevenths), but *also* as the result of measurement, i.e., as the answer to a “how many of this strip does it take to make that strip exactly?”

question. Neither Tzur nor Simon et al. emphasize that unit fractions can be viewed as the result of measurement. We think that viewing unit fractions primarily as units with which to measure and not also as *results* of measurement by a larger unit could be a significant shortcoming because we believe this will not adequately support understanding unit fractions as multipliers, as I will explain.

Izsák and I have found that the future teachers in our courses are generally successful at using our coordinated measurement definition of multiplication. Most of the future teachers can use the definition to explin why fraction multiplication or division (which we view as multiplication with an unknown factor) model given word problems. They are also able to develop and explain why fraction multiplication and division procedures work and to generate fraction multiplication and division word problems for a given equation. However, a sticky point occurs for virtually all of our research participants across multiple years and multiple iterations of our courses: the future teachers do not always maintain a measurement view of fractions in places where, from our point of view, it would be helpful. In particular, they sometimes interpret a fraction as a unit—stuff—instead of as *how much* of a unit there is. For example, suppose a future teacher intends to interpret the fraction 1/8 as a multiplier. According to our definition of multiplication, the 1/8 should be interpreted in a measurement sense, as how many groups one is considering. With our coordinated measurement view of multiplication, there is a quantity called 1 group that functions as a measurement unit, and when we use this unit to measure another quantity of interest in this situation (the product amount), the result is 1/8. In such situations, future teachers sometimes seem to interpret 1/8 as standing for stuff, and in particular as standing for the group itself, not as *how many* groups one is considering. To interpret 1/8 • X, the future teacher might draw a strip, partition it into 8 equal parts, and consider one of those parts to be 1 group that contains X instead of understanding 1/8 as how much of X one is considering. In many cases, such interpretations are fleeting and the future teachers go on to revise their thinking. However, we conjecture that an important part of expertise might be knowing when it is productive to think of fractions (and even whole numbers) as "stuff" that can be partitioned, iterated, and counted and when it is better to view fractions as results of measurement, as how many or much of 1 unit there is in the given stuff. For further discussion of these issues, see Beckmann & Izsák (2018a, b).

Turning now to Simon's (this volume, Chap. 9) research participant, Kylie, I wonder whether her difficulties with extending multiplication of whole numbers to fractions could be due to differences in how Kylie thinks about whole numbers and fractions. Does Kylie know that fractions, as well as whole numbers, can be viewed as a result of measurement by a unit, namely as how many or how much of 1 unit there is in some given stuff? Could it be that Kylie views whole number multipliers as *how many* of the multiplicand she is considering but does not think of fractional multipliers in that same way? For example, when Kylie models 5 times 4 by creating a bar that is 5 units long and iterating it 4 times and when she models 6 times 3 ½ by creating a bar that is 6 units long, iterating it 3 times, and then partitioning the original bar into 2 parts, pulling out one, and attaching it to the bar she had created through iteration, perhaps she interprets the 4 and the 3 as *how many* of her initial bar she should make. But does Kylie think of the ½ in 3 ½ in the same way, or does she think of the ½ as operating on the 6 units to create 3 units? More generally, we know that Kylie is able to interpret fractions as operating on stuff to produce stuff because she is able to take a part of a part and a fraction of a set (Simon, Kara, Norton, & Placa, in press, p. 18). But viewing a fraction as *operating* on stuff—taking a part of something—is perhaps different for students from viewing the fraction as *how much* of the stuff one is considering. So even though Kylie understands how to take a part of a part, when Kylie is unable to make a bar that is 1/3 times 1/5, could it be because she does not see 1/5 as *how much* of the 1/3 she should consider? When Kylie works on the task, "A recipe calls for 2/3 of a cup of sugar, I want to make ¾ of a recipe, how much sugar do I need?" (Simon, p. 14) and is asked "Six-12ths what?" and responds "Six-twelfths of two-thirds," could it be because Kylie is viewing the 6/12 as stuff that came from taking a fraction of some stuff (the 2/3), and therefore sees the 6/12 as "of" the 2/3 in that way? Could Kylie be thinking of "6/12" as a name for the stuff she is considering instead of seeing it as the result of measurement by a unit? These seem like sensible ways for Kylie to be thinking, even though they are not the precise ways sanctioned by the mathematics community.

Experts may have internalized the mathematics community's conventions and rules about how to connect notation to quantities. Experts may also be able to move easily, flexibly, and even subconsciously between viewing fractions as stuff, fractions as operating on stuff to produce stuff, and fractions as how much of 1 unit it takes to make some stuff, selecting a view that is appropriate for the purpose at hand. But for Kylie and for the future teachers that Izsák and I teach, knowing the exact conventions of mathematical notation and knowing that many different views can apply to fractions—and when to use which view—might be pieces of expertise they are still developing.

Returning to the findings that Obersteiner et al. (this volume, Chap. 7) discuss about magnitude, they point out that although there is evidence that understanding fraction magnitudes is helpful, the specific relationship between understanding fraction magnitudes and other fraction concepts remains to be understood. Even more mysterious is the link between understanding fraction magnitudes and other topics, such as algebra. For example, Siegler et al. (2012) suggest that fraction magnitudes should be crucial for algebra because students who do not understand fraction magnitudes would not know that "for the equation ¾ $X = 6$, the value of X must be somewhat, but not greatly, larger than 6" (p. 692). But another possibility is that students' understanding of fraction magnitudes and fraction equations *both* benefit from a measurement sense of fractions, i.e., an understanding of fractions as answers to how much of a unit there is in something. To plot numbers (including fractions) accurately on a number line it should help to understand the numbers as how many of 1 unit it takes to make an interval from 0. To understand an equation such as ¾ $X = 6$ it should help to interpret ¾ in a measurement sense: the ¾ in the equation ¾ $X = 6$ is how much of X is being considered and that portion of X is 6. A student who thinks of ¾ as "stuff" might equate the ¾ with X, just as in the case of the $1/8 \bullet X = 6$ example I described above.

11.5 Conclusion

Fractions are a surprisingly deep and subtle topic in mathematics, which are difficult yet essential for students to learn. This combination makes fractions an important and intriguing topic in mathematics education. Just what is it about fractions that makes them so difficult? How does fraction thinking fit within cognition more generally? What ideas do we draw on when we reason with fractions? What ideas need more attention to help students learn to reason proficiently with fractions? Which teaching-learning paths through fraction ideas can help students make sense of fractions? The fraction chapters in this volume make the case that we should consider strands of research from within mathematics education as well as from psychology when we grapple with questions like these. In this commentary I have used some of the themes and points that were raised in the four fraction chapters in this volume to generate a number of speculative questions and directions that might be considered in future research. I hope readers will find these interesting and worth considering.

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Part III
Integers and Operations on Integers

Chapter 12
Understanding Negative Numbers

Laura Bofferding

Abstract This chapter focuses on the development of concepts that children draw on as they work toward understanding negative numbers. Framed from a conceptual change lens, I discuss different interpretations children have of minus signs, numerical order, numerical values, and addition and subtraction operations and how children draw on these varied conceptions to solve addition and subtraction problems involving negative numbers. Children's unconventional attempts at solving these problems reflect their efforts to make sense of negatives in light of their whole number understanding.

Keywords Negative numbers · Conceptual change · Mental models · Meanings of the minus sign · Linear value · Absolute value

Negative numbers have an unfortunate name, unfairly characterizing them as something to be avoided. Their name reflects their challenging history in being accepted by mathematicians (e.g., Bishop et al., 2014; Vlassis, 2008), which some use as an excuse for delaying their introduction in school. Negative numbers may be difficult for many to learn because they seem to contradict prior learning about whole numbers, but also because significantly less time is devoted to them compared to whole numbers (Bofferding, 2014; Fuson, 1988, Ulrich, Tillema, Hackenberg, & Norton, 2014). In fact, in the United States and internationally, standards for learning negative numbers are pushed to middle school (e.g., Ginsburg, Leinwand, & Decker, 2009; National Governor's Association for Best Practices & Council of Chief State School Officers, 2010; van den Heuvel-Panhuizen & Wijers, 2005). However, children frequently have positive reactions to learning about negative numbers, demonstrating a willingness to play around with and think about these cryptic numbers (e.g., Bofferding, Aqazade, & Farmer, 2018; Featherstone, 2000; Wessman-Enzinger, 2018). This chapter focuses on students' endeavors to understand negative numbers, highlighting the complexity involved in this process as well as the

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insightfulness that students demonstrate as they grapple with the formal (culturally accepted) meaning of negative numbers.

12.1 From Whole Numbers to Negative Numbers[1]

The body of literature on children's understanding of whole numbers is quite broad; in comparison, research on children's understanding of negative numbers is in its infancy. Early accounts of negative number learning primarily come from teachers and professors detailing their methods for teaching negative number operations (e.g., Ashlock & West, 1967; Cotter, 1969; Snell, 1970) or highlighting students' initial encounters with negative numbers and integer arithmetic (e.g., Cochran, 1966; Malpas & Matthews, 1975). However, few of these accounts describe students' understanding of negative numbers themselves, as a prerequisite to operations. Fuson (1992) identified a need to extend current whole number research to negative numbers. Just as students' conceptions of arithmetic operations change based on their understanding of whole numbers, students' conceptions of arithmetic operations change based on their understanding of negative numbers. Several concepts comprise students' understanding of whole numbers: their understanding of the number sequence, of units, of numbers' values or cardinality, and of numerals or notation (Case, 1996; Fuson, 1988; Steffe, von Glasersfeld, Richards, & Cobb, 1983). Similar concepts comprise students' understanding of negative numbers, with some caveats. Negative numbers do not exist as tangible entities in the same way that positive numbers do; their values are more abstract and directed. Further, in addition to numerals, negative numbers also involve a sign to designate that they are negative. I discuss each of these concepts in turn as well as how students put these concepts together through the lens of conceptual change and mental models.

12.2 Framework Theory of Conceptual Change and Mental Models

As posited by Fuson (1988) and demonstrated by several of the examples included later in this chapter, students' difficulties with negative numbers arise from the numbers "being too closely connected to the cardinal numbers, that is, from inadequate differentiation from the cardinal numbers" (p. 406). This interference is described nicely by the framework theory approach to conceptual change.

Based on the framework theory of conceptual change, children's conceptions align with one of a set of mental models that change over time in response to their everyday experiences and learning opportunities (Vosniadou, 1994, 2007;

[1] For the sake of this chapter, the use of *number* refers to integers unless otherwise specified.

Vosniadou & Brewer, 1992). Accordingly, children develop a framework theory for number that is based on their experiences with the world and that they use to make sense of new numbers (Vosniadou, Vamvakoussi, & Skopeliti, 2008); namely, they construct meaning for whole numbers and base experiences with new numbers, such as negative numbers, on their whole number and operations understanding (Bofferding, 2014). Children's *initial mental models* for negative numbers reflect an attempt to interpret negatives by using their framework theory. Therefore, they interpret negatives as similar to positives, ignoring information that does not fit their theory.

As they learn about negative numbers or encounter them in new ways, children try to assimilate the new information within their current conceptual structure, which causes some changes to their conceptual structure and usually results in *synthetic mental models* for negative numbers (a hybrid of their framework theory and formal understanding). The synthetic mental models are unstable (Vosniadou, 2007), and when children have enough experiences so that they can successfully reorganize their mental structures to accommodate the new information in a way that reflects their culture's accepted use of a concept, they are classified as having a *formal mental model* for negative numbers. The conceptual change process is messy, so at any particular moment, a student may be transitioning from exhibiting an initial mental model to a synthetic mental model (*transition I mental model*) or from exhibiting a synthetic mental model to a formal mental model (*transition II mental model*) (Bofferding, 2014).

In previous work, I outlined how children's integer (including negative integers) mental models for order and value align with conceptual change theory and involve a combination of children's understanding of the number sequence including negatives, the meanings of the minus sign, and integer values (Bofferding, 2014). As if that were not complicated enough, when doing addition and subtraction with integers, children also need to reorganize their conceptual structures for whole number addition and subtraction, to accommodate negatives, as addition no longer only means "getting larger" and subtraction does not necessarily mean "getting smaller" when negatives are involved (Bruno & Martinón, 1999).

12.3 From a Whole Number Sequence to a Negative Number Sequence

An important conceptual structure for whole number understanding is the number sequence conceptual structure. Students' number sequence conceptual structures go through several changes. As detailed by Fuson (1988), at the first level, a student's number sequence exists as a *string* where the student recites the entire sequence to produce number words, and some of the words may be grouped together and considered as one. There are no reported accounts of children using the negative number sequence in this manner, perhaps because children typically learn the negative number sequence after moving beyond the string level for the positive number sequence, making the starting point of the number sequence ambiguous.

At the second level, a student's whole number sequence exists as an *unbreakable list* where the number words are distinguishable and the student can use the sequence to count perceptual unit items (Steffe et al., 1983) but must start counting at one (Fuson, 1988). At the third level, the *breakable chain* level, the student can begin their count at any word and count on or back using perceptual unit items (Fuson, 1988). From there, at the *numeral chain* level, the student can use the numbers in the sequence as sequence unit items (Fuson, 1988) and do not need to use other objects for counting, and at the final *bidirectional chain* level, the process of counting forward or backward is done with ease starting at any number.

The list of counting words for positive numbers has four main features identified by Fuson (1988):

1. The list is comprised of number words. [number words feature]
2. The list is a list (the words are said in one standard order that is stable over repeated productions of the list). [stable-word feature]
3. Each word in the list is unique (it appears once in the list). [unique-word feature]
4. The list has a decade structure between 20 and 100. [decade structure feature] (p. 389).

The same features hold with the inclusion of negative numbers, although a word designating negativity is included before each number word (number words feature), and the starting point for integers is ambiguous[2] (stable-word feature). There are several accounts of children making up a non-standard but stable sequence in place of standard negative numbers, exhibiting nearly formal (transition II) mental models for the integer sequence. These include using children's names to represent positions to the left of zero on a number line (Aze, 1989), which violates the number words feature; using the prefix "zero cousin minus" before each numeral (Wilcox, 2008); or adding a prefix of "zero" or "something" before each numeral (Bishop, Lamb, Philipp, Schappelle, & Whitcare, 2011).

Students will often continue counting beyond the part of the whole number list they know (Fuson, 1988), either violating the stable-order feature or unique-word feature of the list. Determining a violation of the unique-word feature is tricky because it "might reflect a child's awareness of the repeated pattern but a failure to notice or to have learned the additional differentiating syllables" (Fuson, 1988, p. 391) of other numbers in the sequence that contain names of other numbers (e.g., 1 versus 21). These violations also occur when children are asked to continue the number sequence into the negatives. Bofferding et al. (2018) asked second and fifth graders to fill in missing numbers on a number path. The number path had the numbers 5, 4, and 3 filled-in starting from the right end and empty boxes on the left, so that students could write in negative numbers to −11, if they knew about them. After

[2] If we only consider the negative integer sequence, the starting point is −1, just as 1 is the starting point of the positive integer sequence.

1	2	3	4	5	0	1	2	3	4	5	0	1	2	**3**	**4**	**5**

Fig. 12.1 Number path with a descending number pattern

16	15	14	13	12	11	10	9	8	7	6	0	1	2	**3**	**4**	**5**

Fig. 12.2 Number path with a continued number sequence to the left of zero

11	10	9	8	7	6	5	4	3	2	1	0	1	2	**3**	**4**	**5**

Fig. 12.3 Number path with positive numbers on both sides of zero

filling in 2, 1, and 0, several students asked what they should put in the other empty boxes. Although some students just left the boxes blank, others made descending number patterns, reflecting initial mental models (see Fig. 12.1; Bofferding et al., 2018).

Others exhibited initial mental models by continuing the forward number sequence with "6" to the left of zero (see Fig. 12.2; Bofferding et al., 2018).

Many also wrote in positive numbers symmetric around 0 (see Fig. 12.3; Bofferding et al., 2018).

Bofferding (2014) found similar results when asking first graders to fill in missing numbers on a number line. While the number pattern example clearly violates the unique-word feature, a similar conclusion cannot be made without additional evidence about those who wrote in symmetric positive numbers; students who write or say, "One" instead of "Negative one" might just not know the differentiating symbol "-" or term "negative." In many cases, Bofferding et al. (2018) had additional evidence from interviews or students' solutions to arithmetic problems that they exhibited initial mental models and considered the numbers as positive. These results indicate that students' willingness to violate the unique-word feature for natural numbers can occur much later than age 5 as found by Fuson (1988). Students who understand the unique-word feature but do not know or remember how to write negatives will often make up their own notation, such as "1`" (Bofferding et al., 2018, p. 12), "m1" (Liebeck, 1990, p. 226), "S1" (Bishop et al., 2011, p. 353), or "N1" (Bofferding, 2014; Bofferding et al., 2018) to represent −1.

In terms of learning the standard negative number list (starting with −1), even young students can make significant progress. For example, 23 first graders were asked to count backward from 10 before and after an intervention in which they said the numbers they passed through while moving on a board game (in the form of a horizontal number path) with squares labeled −10 to 10 (see Bofferding & Hoffman, 2014 for more details). On the pretest, all but one student correctly said the forward sequence to 10, and all but two of the students could say the backward sequence to 1 or 0; one of them continued to −10. On the posttest, all correctly said the forward number sequence to 10, and all but one said the backward sequence to 1 or 0; they made significant gains in reciting the backward number sequence into the negatives (Bofferding & Hoffman, 2014), with 10 of them doing so. These students had played the counting game for three sessions of 15 min each, and most were able to say the negative number sequence from 0 to −10 as they moved without correction before the end of the first session. Around a dozen needed a prompt in subsequent sessions but quickly corrected themselves after the prompt. Unsurprisingly, additional students in a control group who did not practice the sequence did not make significant gains in counting backwards into the negatives (Bofferding & Hoffman, 2014).

On the other hand, in a similar study with 23 kindergarteners, all but three correctly recited the forward number sequence, and only 12 of them could say the backward number sequence to 1 or 0 on the pretest with none of them continuing into the negatives after prompting. On the posttest, all but four correctly said the forward sequence and only 11 could say the backward number sequence to 1 or 0; the kindergarteners did not make significant gains in reciting the negative number sequence after playing the game, as only two of them recited it.

Given the gains made by the first graders as opposed to the kindergarteners, the evidence suggests that students' whole number sequence conceptual structure needs to be at the *breakable chain* level (where students can produce the backward number sequence) in order for the board game experiences with the negative number sequence to make an impact; even then, producing the new sequence is difficult for many. Of the eight kindergarteners who received the board game intervention and counted back from 10 to 0 on the posttest, only two counted into the negatives (25%) compared to 0% of the 15 who did not count back from 10 to 0; of the 23 first graders who received the board game intervention and counted back from 10 to 0 on the posttest, ten also counted into the negatives (43%). The extent to which the difficulty of saying the negative number sequence is due to interference from children first learning the positive number sequence (or not considering the negative numbers as part of the regular counting down sequence) is not known, although clearly a factor.

One way the coordination between whole numbers and negative numbers manifests is through the use of multiple zeroes, reflecting a transition I mental model. When counting backward, some students will say or write, "3, 2, 1, 0, 0, 0…" (Bofferding, 2014; Bofferding et al., 2018; Wilcox, 2008). These students often explain that zero is nothing, which just continues (e.g., Wilcox, 2008). The assertion that zero is the end of the sequence hints at the strength of whole number knowledge, but the willingness of students to violate the unique-word feature suggests that some students may realize that numbers continue indefinitely in both directions. The extension of the number sequence in both directions from zero is a fifth feature of the extended integer list.

Perhaps the easiest way to remember the negative number sequence is to use an analogy with positive numbers and just add the word "negative" before each numeral name; however, students usually base the analogy on the whole numbers, which leads them to include negative zero in the sequence (or substitute negative zero for zero), (Bofferding, 2014; Bofferding et al., 2018). In fact, when saying the negative number sequence, "Negative three, negative two, negative one," my 3-year-old con-

Table 12.1 Descriptions of the mental model levels in terms of the integer sequence (integer order)

Mental models	Description of the integer sequence (order) mental models
Initial	Students only say or write positive or whole numbers. (e.g., "5, 4, 3, 2, 1, 0")
Transition I	Students include repeated zeroes (e.g., "5, 4, 3, 2, 1, 0, 0, 0…")
Synthetic	Students include negatives but they are reversed (e.g., "5, 4, 3, 2, 1, 0, −10, −9, −8"), or they write negatives connected to positives in non-standard ways (e.g., 0, 1, 2, 3, 4, 5, −1, −2, −3, −4, −5)
Transition II	Students include −0 or use their own consistent labels for negatives (e.g., "5, 4, 3, 2, 1, 0, −0, −1, −2, −3… or write n5, n4, n3, n2, n1, 0, 1, 2, 3, 4, 5)
Formal	Students say or write the integer sequence (e.g., "5, 4, 3, 2, 1, 0, −1, −2…")

tinued, "Negative zero," exhibiting this transition II mental model for the integer sequence. Others also seem to pay attention to the symmetry of the numbers but leave out zero (−3, −2, −1, 1, 2, 3 or − 3, −2, 1, 2, 3). Finally, some students will make an analogy with the positive or whole numbers and begin their negative number sequence with a large negative (e.g., −10 or − 100), reversing the negatives and patterning the sequence after the positive numerals. Therefore, their backward counting sequences will be "3, 2, 1, -10, -9..." or "3, 2, 1, -100,[3] -99…" (Bofferding, 2014; Bofferding et al., 2018). Widjaja, Stacey, and Steinle (2011) found similar results for a preservice teacher's placement of negative decimals (i.e., the preservice teacher labeled −1.2 closer to 0 than −0.5), so it would be worthwhile to explore students' negative numerical sequences for numbers beyond integers. For a description of students' integer sequence or integer order responses in terms of the conceptual change mental model levels, see Table 12.1 (see also Bofferding, 2014).

Just as practice with the whole number sequence can increase young students' performance on counting tasks (Griffin, Case, & Capodilupo, 1995), practicing the negative number sequence can increase students' performance with the sequence if they have sufficient understanding of the whole number sequence (Bofferding & Hoffman, 2014). Further, on a number placement task with a scale from −1000 to 0, sixth and seventh graders' performance paralleled findings for a placement task with positive numbers, although the accuracy and linearity of their placements was less than typically found for positive numbers. Yet, accuracy and linearity of placing the negative numbers was higher for the older students who likely had more experience with negatives (Young & Booth, 2015).

In another study, second graders analyzed worked examples of integer addition and subtraction problems illustrated with a gingerbread man moving up and down a number path labeled from −10 to 10 that was situated on a hill (see Fig. 12.4). For example, when comparing 5 − 3 and −5 − 3, they pointed out that the gingerbread man started at different numbers (one had a negative sign and one did not), moved the same number of spaces, moved in the same direction, and ended at different numbers. They also practiced solving similar problems. After this exposure to the

[3] Students will pick a large negative number to start their negative sequence.

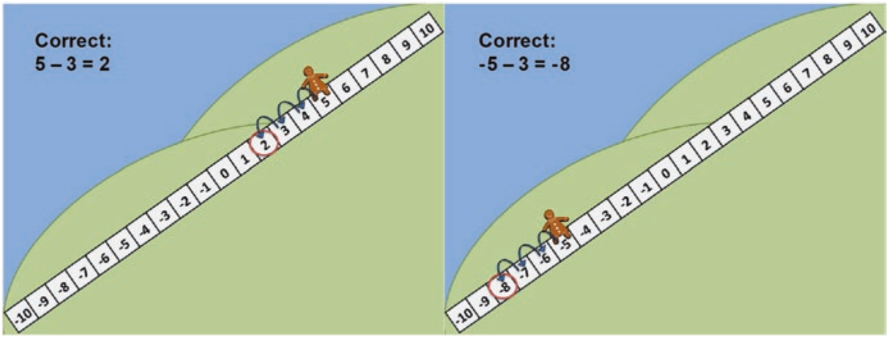

Fig. 12.4 Example of contrasting worked examples that students compared and contrasted

written, integer counting sequence on a number path, there was an increase in second-grade students' ability to correctly extend the whole number sequence into the negatives (Aqazade, Bofferding, & Farmer, 2016).

12.4 Application of the Negative Number Sequence in Addition and Subtraction

Students can use the extended integer number sequence to effectively solve integer addition and subtraction problems via counting-based strategies, especially if they interpret adding a positive number as going up in the sequence and subtracting a positive number as going down in the sequence (Bishop, Lamb, Philipp, Whitacre, & Schappelle, 2014). As with whole number problems, students might use fingers to help them keep track of their counts. For example, Violet, a second grader, sequentially raised five fingers as she counted from −8 to −4 in order to solve −9 + 5 (Bishop, Lamb, Philipp, Whitacre, & Schappelle, 2014), and Timmy, another second grader, used a similar strategy to solve −9 + 2 (Aqazade, Bofferding, & Farmer, 2017). In fact, students across the ages use counting sequence strategies, sometimes with the aid of fingers, number lists or paths, or number lines (Bishop, Lamb, Philipp, Whitacre, & Schappelle, 2014, Bishop, Lamb, Philipp, Whitacre, Schappelle, & Lewis, 2014; Bofferding, 2010; Wessman-Enzinger, 2017). In some cases, providing young students with a number path, which is a count model (Neagoy, 2012)—as opposed to a number line, which is a measurement model—can help them leverage the counting sequence and solve problems that they would not be able to solve without the support. For example, when asked to solve −3 + 2, one first grader answered "5." However, when given a number path to solve −3 + 1, the student correctly answered, "-2."

Students can also use the integer sequence to solve more difficult integer problems, such as missing addend problems (e.g., −5 + __ = 7 or __ + −5 = 7). If the missing number is the first addend, students might use trial and error, starting at different numbers until they find the right combination (Bishop, Lamb, Philipp, Whitacre, & Schappelle, 2014). As with whole number chunking strategies, some students solve problems with negative numbers by chunking to zero and calculating beyond zero. For example, when solving $3 - 5$, one second grader first solved $3 - 3 = 0$ and then knew the two left over would result in negative two (Bishop, Lamb, Philipp, Whitacre, & Schappelle, 2014; see also Schwarz, Kohn, & Resnick, 1993–1994).

12.5 From Unsigned Numerals to Signed Numerals

As discussed in the previous section, students may continue their number sequence into the negatives without knowing the formal notation for negatives, so they may use invented notations or leave out any indication that the numbers are negative. Once again, students' knowledge of whole number operations may contribute to this phenomenon because up until this point, children associated the minus sign with the operation of subtraction. However, subtraction is just one of the three meanings of the minus sign. Teachers have discussed the three meanings of the minus sign in mathematics education journals as early as the 1920s (e.g., Barber, 1926). These three meanings involve interpreting the minus sign as a subtraction sign, which corresponds to the subtraction operation or taking the difference (a binary function); interpreting the minus sign as a negative sign, which corresponds to negative numbers or answers to equations (a unary function); and interpreting the minus sign as indicating multiplication by negative one, which corresponds to taking the opposite (a symmetric function; Gallardo & Rojano, 1994; Vlassis, 2004, 2008).

Students' interpretations of the minus sign as a subtraction, negative, or opposite sign are based on their knowledge (or lack thereof) of negative numbers as well as the placement of the symbol in relation to the numerals around it. In one study, eighth graders demonstrated a set of rules for interpreting minus signs in polynomials:

> Placed at the beginning of the expression, the minus is considered as attached to the number. Placed between two like terms, students explain that the minus is used for subtracting, and between two unlike terms, that it is used for splitting, operating, or making the following term negative. (Fagnant, Vlassis, & Crahay, 2005, p. 89)

Before providing students with negative number instruction, I interviewed 61 first graders on a series of integer problems (see Bofferding, 2014). One set asked students to look at two equations ($-5 - 3 + -1 = -10$ and $3 - -2 + 4 = 6 + 3$) and circle the minus signs that tell them to subtract. Overall, 43% of the students (26/61) only circled the subtraction signs and not the negative signs. When asked how they knew which minuses meant to subtract, some students referenced spatial features and indicated that the negative signs were smaller or higher. Similar to the eighth graders described previously, one first grader indicated that the subtraction signs are "in the middle of two numbers." Such spatial cues are important; in equations without negative signs, moving the subtraction sign closer to the numeral following it (a right-shift) may prompt adults to interpret the sign and numeral together as a negative number (Jiang, Cooper, & Alibali, 2014).

Further work should investigate the role spatial cues can play in helping students distinguish negatives as well as learn about the relation between subtraction and negatives. For example, as suggested by Jiang et al. (2014), teachers could capitalize on students' potential of seeing a right-shifted subtraction sign as a negative sign to help them make sense of the equivalence between adding a negative and subtracting a positive. Booth and Davenport (2013) found that end-of-year sixth, seventh, and eighth graders' conceptual knowledge of minus signs (e.g., $3 - 4x$ is equivalent to $3 + -4x$ but not $4x - 3$) was the greatest predictor of equation-solving success, but students had a difficult time encoding the minus signs (i.e., subtraction and negative signs) when asked to recreate equations that they viewed; "They deleted them, changed them into an addition sign (or other operation), and included additional negative or subtraction signs" (J. Booth, personal communication, March 22, 2018).

12.6 Use of the Three Meanings of the Minus Sign in Addition and Subtraction

Students' interpretations of the minus sign contribute to their interpretations of integer addition and subtraction problems and their subsequent solution strategies. In fact, in an investigation of students' performance in Algebra I classes, minus sign errors were common across all topics and were the most persistent (Booth, Barbieri, Eyer, & Paré-Blagoev, 2014); these errors included "Moving, deleting, or adding a negative sign, including subtracting when addition is indicated or addition when subtraction is indicated…Moving a term without changing its sign" (p. 14). To alleviate these errors, there should be more focus on helping students make sense of the multiple meanings of the minus sign (symmetric, opposite sign; binary, subtraction sign; and unary, negative sign). In this section, I describe the meanings of the minus sign more fully and provide examples of how each interpretation could manifest in students' solutions to integer arithmetic problems.

Symmetric meaning. Of the three meanings of the minus sign, the symmetric or opposite meaning is often unfamiliar to students even when they have learned about negative numbers; middle school students who struggle with this meaning may not know that "–x" could be positive or negative depending on the value of "x" (Fagnant et al., 2005; Lamb, Bishop, Philipp, Schappelle, & Whitacre, 2012). At the same time, students, even before learning about negative numbers, utilize the symmetric meaning naturally when they make analogies between negative number problems and whole number problems (e.g., Murray, 1985; Schwarz et al., 1993–1994; Wessman-Enzinger, 2017; Wessman-Enzinger & Bofferding, 2018). For example, during one of my interviews with second graders, one student who had no prior instruction with negatives solved $-8 - -5$ to get -3 as follows, “I think it’s…yeah, negative three…because it’s just like eight minus five, then you’re making it negative.” This student thought of the problem as a whole number problem and then added the negative sign at the end. The same student made a somewhat different analogy to solve $-1 + -7 = -8$, saying, “Because you’re adding one, and it’s just like normal numbers. If you were to go seven, eight, that’s, so it’s seven, negative seven, negative eight.” Here the student related the counting on process for adding one with positive numbers to adding negative one with the negative numbers.

Binary meaning. Before learning about negatives, many students will ignore the extra minus signs (e.g., treat $5 - -3$ as $5 - 3$) (Bofferding, 2010; Murray, 1985) or signs in positions that do not follow a traditional subtraction problem structure, such as when there is a negative at the beginning of a number sentence as in $-5 + 3$ (Bofferding, Aqazade, & Farmer, 2017). Other students will treat negative signs as subtraction signs regardless of their placement. For example, they might solve $5 + -3$ by starting with five, adding three, and then subtracting three (Murray, 1985) or $9 - -1$ by subtracting one twice (Bofferding, 2010; Murray, 1985). An interesting case arises for students who interpret the negative sign in problems such as $-5 + 3$ as a subtraction sign. Because there is no initial number, some students will interpret -5 as a number taken away from itself, solving $5 - 5 + 3$ (Bofferding, 2010; Hughes, 1986). One first grader whom I interviewed (see Bofferding, 2014) presented an interesting version of this idea for $-1 + 8$, “The person who wrote it [the problem] just thinks they really have to show you there used to be a one (points to -1), but it’s gone…so I’m just going to put an eight.”

Other students who treat an initial minus sign as a subtraction sign will insert a number, such as zero, at the beginning so that they can subtract (Bofferding, 2010). Inserting 0 at the beginning (e.g., $0 - 5 + 3$) would lead to the correct answer and could help students make a connection between subtracting a positive and adding a negative; however, many students refuse to subtract a larger number from a smaller one. For example, one student (4.A07) when solving $-1 + -7$ read the problem as “zero minus one plus minus seven” (i.e., $0 - 1 + -7$). This student reversed numbers in order to subtract smaller numbers from larger ones and calculated $1 - 0 = 1$, ignored the plus sign, and then did $7 - 1 = 6$ to solve this problem.

Unary meaning. Students who productively use the unary meaning of the minus sign interpret negatives in a variety of ways. Table 12.2 displays some of the conceptual models (Wessman-Enzinger & Mooney, 2014) and contexts used to give meaning to negative numbers.

Just as the minus sign has multiple meanings, negative numbers also have multiple meanings, as illustrated in Table 12.2. They can represent deficits (e.g., money owed, points in the hole), a number in a sequence (e.g., points or locations before or below zero on a number line or path), and movements in a negative direction (e.g., going down). Cable (1971) argued that it is confusing to talk about negatives as both

Table 12.2 Popular models and contexts for negative numbers

Conceptual models	Contexts	Instructional models	Examples
Bookkeeping	Gaining and losing money or points	Number line	Mukhophadyay, Resnick, and Schauble (Mukhopadhyay, Resnick, & Schauble, 1990) Stephan and Akyuz (2012) Whitacre, Bishop, Philipp, Lamb, & Schappelle (2014)
Counterbalance	Protons/electrons Net worth Net score Floats and anchors	Colored chips Double Abacus	Cotter (1969) Stephan and Akyuz (2012) Liebeck (1990) Wessman-Enzinger and Bofferding (2014) Williams, Linchevski, and Kutscher (2008) Ulrich (2012) Pettis and Glancy (2015)
Relativity	Stones in a bag	Number line	Coles (2016)
Translation	Changes in elevation Changes in temperature Elevator movements Traveling	(Empty) Number line Number path	Swanson (2010) Bell (1993) Bofferding (2018) Thompson and Dreyfus (1988) Aqazade, Bofferding, and Farmer (2016)

points and displacements on a number line and advocated using negatives to represent points and R2 or L2 to indicate a displacement of 2 in a particular direction (right or left) on the number line. However, even young students routinely use positive numbers as both points in the positive number sequence and as displacements when they count using the number sequence or an empty number line (e.g., showing 2 + 2 as starting at 2, a hop of 2, and landing at 4 using an empty number line), so using negatives in both situations may not be problematic. At an advanced level, Ulrich (2012) advocates strongly for interpreting negative numbers (and signed quantities in general) as changes in quantity or as one-dimensional vectors, i.e., directed differences.

Multiple interpretations. Students need to be able to think about positive numbers flexibly; likewise, they need to understand the many uses of negative numbers, the unary meaning of the minus sign. The net-worth instructional unit utilized by Stephan and Akyuz (2012) involved seventh graders interpreting negatives in multiple ways. The students talked about negatives as debts, treating them as quantities with values opposite that of positive integers. However, in order to operate with them, they also treated negatives as ordered points on a number line, as jumps on the number line, or as changes in net worth. Use of the number line facilitated the class in making arguments about the relative values of the negative numbers.

Physical interpretations. One difficulty attached to the unary meaning of the minus sign (i.e., interpreting negatives as debts or other types of quantities) is in representing negative values. Students, even in the fifth grade, struggle with problems such as 3 − 5, answering zero or reversing the numbers and answering two (Bofferding, 2010; Murray, 1985; Peled, Mukhopadhyay, & Resnick, 1989) because you cannot take away more than you have. Any physical representation of negatives, such as with the colored chip instructional models[4] (see Nurnberger-Haag, 2018), involves imposing a negative label onto the physical objects and treating them as opposite in value to their absolute value counterparts. For example, a first grader who solved 3 − 9 used fingers to represent the initial amount and demonstrate the taking-away process, while counting the nine he was taking away (Bofferding & Wessman-Enzinger, 2017). After putting down the three fingers, he continued to represent his count by raising six fingers, which he interpreted as negative six. This process leveraged the number sequence with the help of fingers. Similarly, Alice, a fifth grader, solved −18 + 12 = −6 by drawing 18 tallies in one color and crossing off 12 using a second color; her process involved interpreting the quantities as opposites (Wessman-Enzinger, 2015). Certainly, these strategies can be effective as long as students do not forget what the physical objects they use represent.

Directed interpretations. The terms *directed magnitudes, directed differences,* and *directed displacements* hint at the importance of attending to the sign of integers. On a number line, a displacement of −3 represents the same amount of movement as a displacement of 3, but they are in opposite directions. One challenge in interpreting integers as *directed displacements* identified by Thompson and Dreyfus (1988) is considering the direction and amount of displacement multiplicatively. In their work with two second graders, Thompson and Dreyfus found that the students initially interpreted a turtle's movement in terms of its displacement, qualifying the movement with a direction (symmetric meaning of the minus sign); eventually, they were able to talk about the movement as a directional displacement (unary meaning of the minus sign).

A benefit of interpreting integers as directed displacements or one-dimensional vectors is that this interpretation can support thinking about relative changes (Thompson & Dreyfus, 1988; Ulrich, 2012; Wessman-Enzinger & Mooney, 2014). In thinking about relative changes in opposite directions (e.g., +59, −88, +29), students might be more likely to reason about changes in relation to zero, making additive inverses to get to zero; Ulrich (2013) suggests this conception involves interpreting addends as "reified composite units" characteristic of someone who has constructed a generalized number sequence (p. 260). Further, interpreting integers as one-dimensional vectors makes the commutative property of addition more obvious because the order of two changes in quantities does not make a difference

[4] In colored chip models, one color of counters (i.e., chips) represents positives and another color of counters represents negatives. A positive counter cancels out a negative counter. For −3 + 4, one would put out three negative chips and four positive chips. Three of the positive and negative chip pairs would cancel out, leaving one positive chip, or an answer of 1.

(Ulrich, 2012). A drawback to the interpretation of integers as directed displacements is that it lends itself well to the addition operation, interpreting combinations of transformations, but not the subtraction operation—especially the take-away meaning of subtraction. Ulrich (2012) argues that students instead need to interpret subtraction problems (e.g., $3 - -5$) as missing addend problems (e.g., $-5 + \_\_ = 3$) or as an addition of the additive inverse of the subtrahend (e.g., $3 + 5$).

An alternative way of interpreting integer addition and subtraction problems is in terms of directed operations (Bofferding, 2014). Typically with whole number operations, students think of addition as "getting more" and subtraction as "getting less." With a directed operation interpretation, students think of addition as getting more positive or more negative and subtracting as getting less positive or less negative, depending on the sign of the integer being added or subtracted. Such an interpretation acknowledges that numbers exist on a continuum where *more positive* is equivalent to *less negative* and where magnitudes can be interpreted from either a positive or negative perspective. In my current research, students (as young as second grade) who can reason in this way can effectively solve problems such as $3 - -1 = 4$ by reasoning that getting less negative means moving in the positive direction (see Table 12.3 for a description of mental models as they relate to conceptions of addition and subtraction).

Multiple perspectives. Some contexts used to teach negative numbers, in particular gaining and losing money, can be solved without using negative numbers at all. When given a question about borrowing money and asked to write an equation representing the situation, 80% of the seventh graders interviewed wrote an equation with only positive numbers, taking the perspective of the borrower and detailing how much money they would owe (e.g., *I owe $13* instead of *I have -$13*) (Whitacre, Bishop, Philipp, Lamb, & Schappelle, 2014). After being pressed, most

Table 12.3 Description of integer addition and subtraction mental models

Mental model	Description
Initial	Students ignore negative signs in operations and subtract larger minus smaller numbers; plus sign means more, so go up toward greater absolute values; subtraction sign means less, so go down toward lesser absolute values
Transition I	Students treat negative signs as subtraction signs (either subtract the number from itself or from another number); they subtract larger minus smaller numbers; they may use multiple signs (e.g., to solve $5 + -3$, they start with five, add three, and then subtract three)
Synthetic	Students treat negatives as positive (use their absolute values) and make the answers negative; plus sign either means go up OR go in the direction of greater magnitude; subtraction sign means either go down OR go in the direction of smaller magnitude
Transition II	Students can subtract smaller minus larger numbers; their responses are mostly consistent with the formal level but may revert to synthetic interpretations
Formal	Students understand the directed nature of the operations; adding a positive means go more positive; adding a negative means go more negative; subtracting a positive means go less positive; subtracting a negative means go less negative

Table 12.4 Descriptions of students' minus sign mental models in the context of integers

Mental model	Description
Initial	The minus sign corresponds to the binary (subtraction) operation, but minus signs are ignored if they are not between two numbers
Transition I	The minus sign corresponds to the binary (subtraction) operation, and minus signs attached to a number indicate that that number is subtracted from itself; a negative sign appearing after a plus or subtraction sign is interpreted as a second subtraction sign
Synthetic	The unary minus sign is interpreted as a symmetric minus sign (i.e., students work with the numerals first and attach the minus sign to the resulting numeral)
Transition II	There is non-consistent use of the unary meaning of the minus sign
Formal	A minus sign attached to a number corresponds to the unary meaning, and a minus sign between two numbers corresponds to the binary meaning

of the students agreed that negative numbers could be used to represent the situation if they took a loss perspective or interpreted the situation from the perspective of the lender (Whitacre et al., 2014). Prather and Alibali (2008) observed a similar effect with undergraduate students. Those who did not have a strong understanding of arithmetic principles (e.g., when adding a positive number and negative number, increasing either operand increases the sum) were less likely to use negative numbers in their equations to represent word problem situations than those with a stronger grasp of principles. See Table 12.4 for alignment between students' interpretations of the minus signs in relation to the mental model levels for integers.

12.7 From Cardinal Quantities to Integer Values

The multiple meanings of negative numbers make interpreting their values, especially compared to whole numbers, challenging. Students may refer to order-based or magnitude-based reasoning in their interpretations of negatives (Whitacre et al., 2017). For example, one second grader explained that −9 is 9 below 0 (Aqazade et al., 2017). During interviews, 10- and 11-year-old students described −5 as "you have to get 5 to get to 0" (order-based), as "5 below 0" (order-based), and as "less than 0" (magnitude-based) (Murray, 1985, p. 148). Similarly, other fourth-grade students (generally 9- and 10-year-olds) described negatives as "below 0" or "to the left of 0" (order-based), "smaller than 0" (magnitude-based), and "0 minus something" or "subtracting a large number from a small number" (magnitude- and operation-based) (Hativa & Cohen, 1995, p. 425). The latter two responses might involve order- or magnitude-based reasoning but could be more appropriately categorized as difference-based reasoning, aligning to the relativity and translation conceptual models. These responses are also close to another interpretation of negatives: as a shift in a negative direction (e.g., Galbraith, 1974).

Interpreting negatives as shifts in a negative direction would not help students make integer comparisons, as a shift of five in a negative direction is equivalent in magnitude to a shift of five in a positive direction. Further, magnitude-based reasoning can lead to students interpreting negatives' values as equivalent to their positive counterparts—essentially using absolute values, an initial mental model conception (Bofferding, 2014)—or as equivalent to zero, a transition I mental model conception (Bofferding, 2014; Hativa & Cohen, 1995; Schwarz et al., 1993–1994). Based on formal mathematics, which preferences order-based reasoning (i.e., linear values), positive numbers have greater value than negative numbers, and negatives with smaller absolute values have greater linear values than negatives with larger absolute values.

Traditionally, integer tasks test students' understanding of these formalities while having students compare two integers (i.e., positive versus positive, positive versus negative, negative versus negative, and comparisons with zero). In one such task with students in second, fourth, seventh, and 11th grades, Whitacre et al. (2017) found clear trends by grade level and problem type. Students with more negative number experience had higher performance. Further, positive-positive comparisons were easiest, followed by positive-negative comparisons, zero-negative comparisons, and negative-negative comparisons. Within the negative-negative comparisons, Whitacre et al. found that near comparisons (−5 versus −6) were harder than far comparisons (−5 versus −100). In general, students in their sample used order-based reasoning the most to justify their comparisons and were most successful when they used this type of reasoning.

Whitacre et al.'s (2017) study focused on students' identification of the larger integer (in terms of linear value). However, changing the language of the comparison question can alter students' performance. For example, Bofferding and Farmer (2018) asked 88 second- and 70 fourth graders to select the *hottest* of three integer temperatures. In both grades, students had the highest performance when all three integers were positive and the lowest performance when all three were negative, similar to Whitacre et al.'s (2017) findings. However, when asked to identify the *coldest* of three integer temperatures, students had the hardest time with the positive-negative comparisons (Bofferding & Farmer, 2018). These mixed comparisons were more prone to interference from students' magnitude reasoning. When all three integers were negative, the students also did better when determining which integer was *coldest* as opposed to *hottest* (Bofferding & Farmer, 2018); the perspective of the question also influences students' thinking.

Whitacre et al.'s (2017) results are consistent with reaction-time studies in psychology and the integer order and value mental models described by Bofferding (2014). Students exhibiting mental models at Bofferding's initial and transition I levels would likely only do well on positive-positive comparisons or positive-zero comparisons because they interpret negatives as positives or as worth zero. These nuances are not obvious without interviewing students. However, adding in a "tie" option to the comparisons can help clarify students' conceptions when interviews are not feasible. In some of my later work, we gave 47 first graders who completed all phases of a larger study (see Bofferding & Hoffman, 2014) a series of integer comparisons. Students chose between two integers or a “tie” option. In comparisons with options “4, -4, tie” and “3, -3, tie,” 21 of the 47 first graders chose “tie” for both sets; five others selected “tie” for just one set. These responses are consistent with a conception of treating negatives as equivalent to positives (or ignoring the negative signs). Likewise, five first graders chose “tie” for negative-negative or negative-zero comparisons, responses consistent with interpreting negatives as worth nothing or as equivalent to each other.

At the next set of levels articulated by Bofferding (2014), synthetic and transition II, students understand that negatives are less than positives and zero; therefore, positive-negative comparisons would be the next easiest. These students would still struggle with negative-negative comparisons, arguing that, for example, −6 is higher in the negatives or a larger negative than −2 (Bofferding, 2014). The language used in comparison questions can add to or decrease students’ focus on absolute value. In the same comparison study described above, the first graders completed comparisons where they were asked to determine which of the integers was higher, more, closer to 10, and farther from 10. The *higher* and *more* responses were significantly correlated, suggesting that the students interpreted *higher* in terms of magnitude (Bofferding & Hoffman, 2015)—the use of *higher* to denote magnitude as opposed to order in the number sequence was identified as a source of confusion among users on Pinterest.com who tagged material related to negative numbers (Hertel & Wessman-Enzinger, 2017). Further, students did better on the order-based comparisons *closer to 10* and *farther from 10* (Bofferding & Hoffman, 2015), possibly due to the linear board game some of them played during which they said the integer sequence multiple times (see Bofferding & Hoffman, 2014).

This latter result suggests that order-based comparison language may prime students’ order-based reasoning, which students across grade levels used successfully a higher percentage of the time than magnitude-based reasoning in Whitacre et al.’s (2017) study. Wessman-Enzinger (2018) presented a compelling exchange among three fifth-grade students who were trying to decide who should go first in a game based on who drew the highest card (they drew −8, −7, and − 4). Their exchange illustrates the difficulty some students have in aligning magnitude-based conceptions with order-based ones, as one student argued that −8 was highest because it was the biggest negative while the other two argued that −4 was highest because it was closest to one.

Finally, students at transition II and formal levels demonstrate an understanding that negatives with larger absolute values (or farther from zero in the number sequence) are considered to be smaller than negatives with smaller absolute values (or closer to zero in the number sequence). Reaction-time studies often evaluate how quickly adults can determine which of two integers is larger, with more recent studies also focusing on middle-school students (e.g., Varma & Schwartz, 2011). These studies aim to clarify how people extend their positive integer understanding to include negative integers, often finding evidence that people use their knowledge of positives plus a set of rules to deal with negatives or that they include negative integers on their mental number lines (Varma & Schwartz, 2011).

Table 12.5 Descriptions of mental models for integer values

Mental model	Description
Initial	Students treat negatives as worth positives (ignores the negative signs)
Transition I	Students treat negatives as worth zero (or greater than zero but less than positives)
Synthetic	Students know positives are greater than negatives; they use absolute value to determine which negative is greater: they treat larger negatives as greater than smaller negatives
Transition II	Students alternate between using absolute value and linear value to determine which negative is greater
Formal	Students use linear values to determine which number is greater: positives are greater than negatives and smaller negatives are greater than larger negatives

If people use rules to make the comparisons, they should be faster on comparisons of positive and negatives because they can use the rule that positives are greater than negatives; however, if they consult an extended number line, comparisons of numbers far apart should be easier than those close together (i.e., exhibit a distance effect) (see Whitacre et al., 2017 for a more detailed summary of these two hypotheses). Varma and Schwartz (2011) presented additional evidence suggesting that adults use a combination of an extended mental number line plus rules about the symmetric nature of the number line in order to make integer comparisons; further, based response patterns, they found that middle-school children, who may not have reorganized their mental number lines, appeared to use rules for making the comparisons. See Table 12.5 for a summary of the integer value mental model levels (see also Bofferding, 2014).

A noticeable weakness in the reaction-time studies is they did not use order-based questions (i.e., Which integer is *closer to* [a set number]?) in addition to the magnitude-based questions focused on identifying which integer is *more* or *larger*. People's willingness to interpret large negatives as *large* (far from 0 or 1) may account for some of the discrepancies noted in their results, especially as professionals in areas such as chemistry, physics, and computer science refer to negatives further from 0 as *large negatives*. Further, without the option to select "tie," some participants' performance (especially in the case of younger students) might be overestimated based on their random selection of integers. A more important issue is whether it is even meaningful to ask comparison questions with magnitude-laden vocabulary unless the point of reference is made explicit. The terms "larger" and "more" leave themselves open to people interpreting them as "larger negative" or larger positive" or "more positive" or "more negative." Future comparison studies should be organized around a context with explicit referents (e.g., hottest or coldest temperature to help clarify students' conception; see Bofferding & Farmer, 2018, for further discussion of these issues).

12.8 From Whole Number to Integer Addition and Subtraction

During the conceptual change process, students' conceptions of any of the elements of integer understanding (sequence or order, sign, value, operation) might change, while the others remain stable. This results in a variety of responses on integer arithmetic problems. In the following sections, I illustrate relations among the elements in terms of the mental models and how students might solve integer addition and subtraction problems, providing examples of how students integrate their separate mental models. To do this, I draw on examples from the literature as well as my own interviews with first, second, and fifth graders.

12.8.1 Initial Integer Mental Models

There are two versions of initial mental models for integers. The first, whole number mental model, is characteristic of students who ignore all negative signs and continue to operate with negative numbers as if they were positive (Bofferding, 2014). For example, when given "5, 4, 3" on a number path with several empty squares, one second grader (4.A06) filled in "2, 1, 0" in the empty squares and left the squares for negatives blank. Further, when solving problems, the student ignored negative signs. When solving $-9 + 2$, the student said, "Nine plus two…I know this one. This one's eleven."

The second version of the initial mental model, absolute value mental model, is similar in that students treat negatives as having positive values, although they can properly order them (Bofferding, 2014). For example, when filling in the number path, one second grader (4.B03) correctly filled in "2, 1, 0, -1, -2…-11" in the empty squares. However, when reading $3 + -3$, the student said, "Three plus three" and provided an answer of six.

12.8.2 Transition I Integer Mental Models

Students exhibiting transition I mental models pay attention to the negative sign and either interpret it as a subtraction sign or interpret negative numbers as worth zero (Bofferding, 2014). One fifth grader (3.T01) correctly filled in negative numbers on the number path but sometimes interpreted negative signs as subtraction signs and sometimes treated negatives as worth zero. When solving $4 + -6$, the student answered 2, effectively solving $6 - 4 = 2$. However, when solving $3 - -1$ and $-7 - 3$, the same student answered 3 for both, treating the negatives, -1 and -7, as worth nothing. Others have also noted students' tendency to treat negatives as worth zero (e.g., Aqazade et al., 2016; Schwarz et al., 1993–1994). Another second grader

(3.D02) who only wrote in positive numbers on a number path interpreted negative signs as subtraction signs, solving 9 − −2 to get 5 in the following way, "Nine in my head and took away two, two times," subtracting the two twice.

12.8.3 Synthetic Integer Mental Models

Synthetic mental models involve over-focusing on magnitude when working with negative numbers although these students know that positives are considered greater than negatives (Bofferding, 2014). Some evidence that students exhibit synthetic mental models is that they will solve arithmetic problems as if they are positive and then make the answer negative (Kuchemann, 1980); this sometimes results in correct answers, as in the case of interpreting −4 − −3 as 4 − 3 and then making that answer (1) negative (−1). One second grader (4.A05) who primarily exhibited a synthetic mental model initially interpreted negatives as worth zero saying, "Three plus negative three would be three. Because you're not adding anything." However, in subsequent problems, the student frequently solved the problems as positive and added a negative sign. For −8 − 5, the student explained, "Negative eight minus five equals negative—I think it's—yeah, negative three. Because it's just like eight minus five, then you're making it negative." The student used similar reasoning when solving 3 − −1, answering −2.

One of the difficulties students experience when they solve negative integer problems involves knowing in which direction to count (Bishop, Lamb, Philipp, Whitacre, Schappelle, & Lewis, 2014). Students who exhibit synthetic mental models, often use the same interpretation of addition (or subtraction) when adding (or subtracting) positive numbers as they do for negative numbers. The same second grader as above (4.A05) who solved −8 − 5 as −(8 − 5) also solved −1 + −7 saying, "Negative one plus negative seven—negative eight. Because you're adding one, and it's just like normal numbers. If you were to go seven, eight, that's—so it's seven—negative seven, negative eight." The student used magnitude reasoning to think about addition as getting more in magnitude (addition should result in a number with larger absolute value). Likewise, the student incorrectly answered −16 for −8 + 8. Similarly, Violet from Bishop, Lamb, Philipp, Whitacre, and Schappelle's study (2014) had difficulty solving problems involving adding or subtracting negatives although she could correctly reason about integer problems if she added or subtracted a positive number to a negative one.

Instead of focusing on magnitude, some students focus on a direction that they associate with an operation. People tend to associate subtraction signs with a leftward direction (Pinhas, Shaki, & Fischer, 2014), leading them to count to the left on a number line or count down. Likewise, many students will count in the same direction regardless of whether they are subtracting a positive number or a negative number. For example, one fifth grader (3.N13) provided answers consistent with starting at the initial number and counting down the second number for subtraction problems. On $1 - -6$, the student answered -5, consistent with starting at 1 and counting down 6, and on $-3 - 3$, the student answered -6, consistent with starting at -3 and counting down 3.

12.8.4 Transition II Integer Mental Models

Students exhibiting transition II integer mental models sometimes interpret larger negatives to have larger values and sometimes interpret them to have smaller values than smaller negatives (Bofferding, 2014); likewise, they may be uncertain in which direction to count when adding or subtracting.

12.8.5 Formal Integer Mental Models

Formal integer mental models describe responses consistent with the culture's understanding of negatives (Bofferding, 2014). One second grader (3.H07) correctly filled in a number path with negatives and knew which numbers were considered higher or closer to positive ten. The student correctly solved problems such as $1 - -6$ by explaining, "I did one minus a negative six, so I went up six." The student also clarified, "Because anything taking away a negative is going up." Likewise, the student correctly knew that subtracting a positive meant going down and correctly answered problems such as $4 - 5 = -1$ and $-7 - 3 = -10$. Although this student had a perfect score on integer subtraction problems, his mental model for the addition operation was still largely tied to increases in magnitude, which led him to give inconsistent answers for integer addition problems.

12.9 Wrapping Up

Overall, the process of conceptual change in terms of transitioning from whole number understanding to negative number understanding is complex. As illustrated, although students may display a formal understanding of integer order and values, their solutions to integer addition and subtraction problems may be inconsistent if they have not revised their mental models of addition and subtraction. Further, students' thinking might change depending on the context and language used to elicit their understanding (Bofferding & Farmer, 2018), but starting the process of learning negatives early (i.e., at least by second grade) has benefits (Aqazade et al., 2017). In order to further our understanding of negative number cognition, it is imperative that, as we move forward, we draw on multiple perspectives, including mathematics education, neuroscience, linguistics, and psychology.

In my current work, we draw on multiple perspectives through a series of experimental studies incorporating ideas from cognitive science, mathematics education, and linguistics. We investigate children's understanding of integer values through questions using both order-based language (e.g., which number is closest to positive or regular ten) and magnitude-based language with the perspective indicated (e.g., which temperature is hottest, which number is most positive, which temperature is coldest, which number is least negative). Further, we make use of linguistic theory to help explain why certain terms might be easier for students to interpret and why certain comparisons might be easier (see, e.g., Bofferding & Farmer, 2018). Our interventions use methods that have proven fruitful in cognitive psychology (e.g., contrasting cases, worked examples, preparation for future learning); they involve children comparing, contrasting, and analyzing a series of worked examples of integer addition or subtraction problems (sometimes paired to highlight a contrast between examples) and participating in a lesson on integer addition and subtraction. We draw on conceptual change theory and literature on the multiple meanings of the minus in our data analysis (see, e.g., Aqazade et al., 2016; Bofferding et al., 2017).

Does instruction in integer order, value, and symbols lead to changes in students' approaches to integer arithmetic? Yes! Because of the number of factors involved in our studies, the data is messy, but reflecting on the data as a whole, one result is clear: as children begin to notice negative signs and wonder about their values and order, they interpret the integer addition and subtraction problems differently and thus solve them differently. For example, one second grader (2.W05) started out by ignoring all negative signs in addition problems with negatives and determined which of two integers was larger based on their absolute values (again ignoring the negatives). During the intervention, the student began to notice the negative signs, and on the posttest, although 2.W05 still chose which integer was larger based on absolute value, the student solved all addition problems by adding the absolute values of the numbers and making the answer negative (Aqazade et al., 2016).

We are still in the process of sorting through all of the data based on the students' pretest interpretations and the instructional interventions they experienced. However, we see some general shifts emerging in students' integer addition and subtraction strategies as they learn more about negatives: shifting from ignoring negative signs in arithmetic problems to making all answers negative, shifting from ignoring negative signs to treating them as subtraction signs, shifting from considering large negatives to be larger than small negatives and operating based on absolute value to interpreting negative values based on their linear order and operating based on moving or counting in a particular direction.

Does instruction in interpretations of integer operations, particularly addition and subtraction, lead to changes in students' conceptions of integer order, value, and symbols? This is less clear. In particular, we need a more robust understanding of how early instruction in the difference meaning of subtraction (especially in terms of directed distances) could impact students' learning and understanding of integers as signed or directed quantities (see Kilhamn, 2018, for an interesting discussion of some of these ideas). Likewise, we need a deeper understanding of the impacts of early language use around operations (e.g., treating adding as meaning *getting* *more*) versus directed language use (e.g., *getting more positive*). It is through targeting the collective knowledge and practices across the fields of linguistics, psychology, mathematics education, and others (and acknowledging the immense potential that young children have) that we will be able to answer these questions and craft instruction and interventions to help students more fully understand the wonderful world of negative numbers.

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Chapter 13
Integers as Directed Quantities

Nicole M. Wessman-Enzinger

Abstract Mathematics education researchers have long pursued—and many still pursue—an ideal instructional model for operations on integers. In this chapter, I argue that such a pursuit may be futile. Additionally, I highlight that ideas of relativity have been overlooked; and, I contend that current uses of translation within current integer instructional models do not align with learners' inventions. Yet, conceptions of relativity and translation are essential for making sense of integers as directed quantities. I advocate for drawing on learners' unique conceptions and actions about directed number in developing instructional models. Providing evidence of student work from my research, I illustrate the powerful constructions of relativity and translation as students engage with directed quantities.

Keywords Conceptual models · Integers · Integer addition and subtraction · Integer instructional models · Integer operations · Number line

13.1 Introduction: Pursuit of the Ideal Instructional Model for Integers

The perfect model for teaching and learning operations on integers is the holy grail of integer research in mathematics education. After taking over 1500 years to formally account for integers (e.g., Henley, 1999), mathematicians and educators have sought the perfect model for integer operations through various contexts, including the number line (e.g., Heeffer, 2011; Schubring, 2005; Wessman-Enzinger, 2018a). Yet, the use of the much-vaunted number line broke down for nineteenth century mathematicians for the operations of multiplication and division (Heeffer, 2011). Centuries later, even our social media is proliferated with math teacher chats, groups, and tweets posting and discussing their wonderings about instructional models for integers. For example, a recent post in a large Facebook group of

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mathematics education teachers and researcher quickly welcomed 33 different comments (not including replies) on the following question (Mathematics Education Researchers, 2017):

> Does anybody know a good model of negative integer (number) operations? I am working with middle school teacher and students. It's hard to find a visual model that illustrates the meaning of negative number division/multiplication such as 10/−2. Is there a good explanation that makes sense of the operation with a good connection with the division topic?

This holy grail—the ideal model for teaching and learning of integer operations—will always remain mythical. We will never find the perfect instructional model for all integer operations, despite our commitment, because all models for integers break down at some point (Galbraith, 1974). Although all models have affordances and limitations (Vig, Murray, & Star, 2014), integer instructional models have particular limitations (Peled & Carraher, 2008) because of the constraints on the physical embodiment of negative numbers (Martínez, 2006).

For a few instructional models, some of these affordances and limitations are highlighted in Table 13.1. In an article about mythical creatures in *The New Yorker*, Schultz (2017) commented, "One of the strangest things about the human mind is that it can reason about unreasonable things" (para. 2). Despite the affordances and limitations of instructional models for integers, we can still reason about them. Perhaps this is why an ideal instructional model for integers has been pursued so vigorously.

Table 13.1 highlights only a diminutive portion of the different instructional models that have been proposed across decades of integer research (e.g., Bruno & Martinon, 1996; Janvier, 1985; Liebeck, 1990; Linchevski & Williams, 1999; Schwarz, Kohn, & Resnick, 1993; Thompson & Dreyfus, 1988). There will always be a hunt for the ideal integers instructional model. This pursuit of the unattainable is not so uncommon: "The relative plausibility of impossible beings tells you a lot how the mind works" (Schultz, 2017). It seems that an ideal instructional model for integer operations might exist, for we know that robust models exist for whole number operations. Furthermore, studies with instructional models for integers make the existence of such a model seem plausible because these investigations provide interesting results and insights into students' thinking (e.g., Bofferding, 2014; Tsang, Blair, Bofferding, & Schwartz, 2015). Consequently, math educators and psychologists will continue to pursue better instructional models for integer operations (e.g., Moreno & Mayer, 1999; Pettis & Glancy, 2015; Stephan and Akyuz, 2012; Tsang et al., 2015).

Yet, we can do better than pursuing an ideal integer instructional model. In this chapter, rather than presenting more top-down integer instructional models, I instead point to how conceptualizing integers as directed quantities is a powerful conceptual tool. We should focus on the constructions of learners and the integer models they create prior to their use of integer instructional models made by adults. It is notable that many of our instructional models (those formed by adults) incorporate ideas of movement and measurement metaphors (Chiu, 2001; Lakoff & Núñez, 2000), which align to larger mathematical ideas. This is likely because our students naturally employ ideas of movement and measurement; yet, we need to understand what learners' integer constructions around movement and measurement for inte-

Table 13.1 Affordances and limitations of some instructional models for integer operations

Instructional model(s)	Sample reference(s)	Affordances	Limitations
Two-colored chips	Liebeck (1990), Murray (2018), Vig et al. (2014)	The use of two-colored chips builds on children's experiences with discrete, physical objects. These types of models work well with integer addition (e.g., $-2 + 3 = \square$). These types of models also work well for integer multiplication and division, where one factor is a negative integer and one factor is a positive integers (e.g., $-2 \times 3 = \square$, $6 \div -2 = \square$).	These types of models are not intuitive for integer subtraction because "zero pairs" must unnaturally be added into set of two-colored chips (e.g., $-2 - 3 = \square$). These types of models do not work for the multiplication of negative integers or the division of two negative integers, unless extra chips are available to imagine taking away chips. The use of two-colored chips may be used differently and does not inherently dictate a particular instructional model.
Traditional debt and credit contexts	Wessman-Enzinger and Mooney (2014), Whitacre et al. (2015)	Debts and credits exist in the world, and students might connect the integers to related contexts. Learners may think about debts and credits in relation to integer addition and subtraction in ways that do not involve traditional notions of money (e.g., owing candy bars, lost pencils).	Although we apply negative integers to debts, we do not have to. Even secondary students do not necessarily connect traditional debt and credit contexts naturally to the negative integers.
"Net worth" context paired with empty number line	Stephan and Akyuz (2012)	The "net worth" is used in ways that emulate counterbalancing of quantities, which is natural for learners. "Net worth" modifies traditionally used ideas of debts and credits, which are prevalent in standards and curricula. "Net worth" is an intuitive space for learners to make sense of integer addition and subtraction.	The particular instructional model that Stephan and Akyuz (2012) use is a blended model, paring "net worth" with empty number lines. "Net worth" by itself does not naturally dictate use of an empty number line (as no linear movement is inherently a part of it). When "net worth" is paired with the use of the empty number line, learners use the change/displacements only on the number line, which is slightly different than "net worth" where all of the quantities remain present. This instructional model works well for addition, subtraction, and multiplication. It does not work as well for division (Stephan, personal communication, April 10, 2018).

(continued)

Table 13.1 (continued)

Instructional model(s)	Sample reference(s)	Affordances	Limitations
Movements on a number line	Nurnberger-Haag (2007)	This model incorporates use of the number line and critiques of other number lines that are built on sets of rules and procedures. This instructional model is designed for all four integer operations. This model support physical embodiments of integer operations and students will physically move as they make sense of integer addition and subtraction.	Ultimately, this instructional model (despite critiques of other models) also builds on its own set of rules and procedures that are not necessarily intuitive or invented by learners. “Adding and Subtracting Integers: (Remember to add or subtract only two numbers at a time.) 1. Start on a number line at the first number of the problem 2. Always start with a positive attitude! (Face the positive direction.) 3. Turn the ______ direction for every ‘–’ sign after the first number. Whatever direction you end up facing is the direction you will walk 4. Walk the number of steps indicated by the absolute value of the second number.” (p. 119) Like the two-colored chip model above, there are different interpretations of “walking” on a number line.
Folding number line	Tsang, Blair, Bofferding, and Schwartz (2015)	This model capitalizes on evidence in cognitive research that humans are drawn to symmetry and supports work in embodied cognition that we think about things we physically experience. This model supports the conceptual development of symmetry and works quite well with the addition of integers.	Extending this instructional model beyond integer addition is complicated, if not impossible.

gers look like. Conceptualizing integers as directed quantities, with movement and measurement, requires mathematical ideas of translation and relativity.

13.2 Definitions of Relativity and Translation

The conceptualization of integers as directed quantities requires using integers as a relative number (Gallardo, 2002; Thompson & Dreyfus, 1988). The starting point and directions that are attributed as positive and negative numbers are arbitrary, even if intentionally determined making integers inherently relative. *Relativity* entails using the integers as comparative numbers or relative numbers (Wessman-Enzinger, 2015). The integers describe relative positions. Zero represents the point of reference, which may be intentionally or arbitrarily selected. Distinctively, the zero does not represent a quantity of nothing, but is treated as a referent for comparison, as one reasons about integers with relativity.

The conceptualization of integers as directed quantities includes both movement and measurement as operations with integers are performed (e.g., Bofferding, 2014; Chiu, 2001; Lakoff & Núñez, 2000; Thompson & Dreyfus, 1988). These ideas of linear movement point to conceptualizations of translations. Translation entails using integers as vectors (Wessman-Enzinger, 2015). Integers are often treated as vectors moving right or left or up and down a linear model, coordinate plane, or three-dimensional space. Zero may be conceptualized as a vector or a translation of no movement. Similar to conceptualizations of relativity, the zero can also represent any arbitrary point with the addition and subtraction of positive and negative numbers representing the translation in one direction or another from the relative zero (Thompson & Dreyfus, 1988).

When conceptualizing integers with translation, distance may be used without direction specified, called *absolute value* (Wessman-Enzinger & Bofferding, 2018); for example, the distance between -2 and -3 is 1 (going from -2 to -3, or -3 to -2). Although it is possible to conceptualize distance without direction, it is still considered to be drawing upon translation because all distance must be conceptualized with direction at some point. When the direction of the distance is explicit, allowing for negative distances, this is called *directed value* (Wessman-Enzinger & Bofferding, 2018); e.g., the distance from -2 to -3 is -1 and from -3 to -2 is 1. Moving in "more" and "least" negative (or positive) directions support use of directed value (Bofferding, 2014; Bofferding & Farmer, 2018). Translation may also be employed with the use of counting strategies because counting fundamentally draws on movement and order (Bofferding & Wessman-Enzinger, 2018; Wessman-Enzinger, 2015).

These definitions of translation and relativity describe two broad types of conceptualizations that learners construct as they engage with integer operations. Learners' constructions of relativity and translation are powerful conceptual tools for making sense of integers as directed number. We should focus more on the conceptual tools learners construct within instruction rather than top-down integer instructional models.

13.3 Directed Numbers as a Powerful Conceptual Tool

> *You have the negatives like a thought thing. It's kind of mental. And, you can like literally take away so many apples or slices of pie from someone and you can still have it. And the other person would still end up having some. Whereas, negatives, if you have something and you take something away from them and they don't have any, you can still keep taking more. But, you don't really have anything. You still won't.* (Drake, Grade 8)

In this excerpt, we see Drake, a student with 3 years of experience operating with negative integers, struggling with the abstract nature of the physical embodiment of these numbers. Negative integers, as Drake points out, cannot be physically modeled with discrete objects in our world and are abstracted mathematical objects. Although all numbers are abstract, learning about the negative integers demands a different realm of abstraction (Fischbein, 1987).

Learners use manipulatives or hands-on activities as they learn whole numbers and fractions (e.g., Martin & Schwartz, 2005; Moyer, 2001; Siegler & Ramani, 2009). For these reasons, mathematics educators might think that one affordance of using physical objects with the teaching and learning of integers is that learners draw upon something familiar (e.g., Bolyard & Moyer-Packenham, 2006). Embodied cognitive scientists and psychologists also recognize that our experiences and actions impact our thoughts (e.g., Barsalou, 2008; Goldin-Meadow, Cook, & Mitchell, 2009; Lakoff & Núñez, 2000; Tsang et al., 2015). Yet, there are obstacles when extending previous experiences with whole number and physical objects to negative integers; negative integers are not naturally extended in the physical realm and have limitations in physical embodiment (e.g., Peled & Carraher, 2008; Martínez, 2006). Negative integers, for instance, have to be mapped to the physical objects representing them. For example, the use of two-colored chips, or a cancellation model, is one way that integers are represented with physical objects, where the negative integers are represented by red chips and positive integers by black chips (e.g., Liebeck, 1990). A negative integer, -*n*, is modeled with *n* objects that need to be physically present and countable. Then, -*n* is represented, by extension, with each countable object representing -1. A consequence of this type of modeling with physical objects is that some problems, such as 2 − -1, may not be intuitive and modeling them with physical objects can be challenging (Bofferding & Wessman-Enzinger, 2017; Vig et al., 2014).

Consequently, inaugural learning experiences with integers need to overcome traditional notions of the physical embodiment of number. Specifically, these learning experiences need to support the transition from discrete and static ways of thinking about number to thinking about number as continuous directed quantities. One way to so is to provide learners with opportunities to *create* their own models, rather

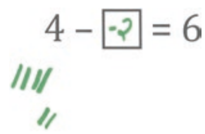


Fig. 13.1 Alice's drawing of discrete objects that supports transitioning from discrete to continuous objects

than giving them instructional models. Learners may create models that bridge discrete and continuous representations of integers (see, e.g., Fig. 13.1).

Figure 13.1 illustrates work from a Grade 5 student, Alice, who drew two sets of discrete objects, 4 tallies to the left, and 2 tallies to the right when solving 4 − □ = 6.

Alice: [Draws four tally marks. Thinks for a bit and draws two more tally marks lower and to the right. Then writes -2 in the box.] I did four minus negative two and I got six because … I did four right here (points to upper tallies) and two (points to lower tallies). And, then this is six.

Teacher-researcher: How did you know it was -2?

Alice: Well, because I did two… I did it backwards (moves pen across 4 − -2 = 6). If I did two plus four I got six. So, then I thought it would be negative two.

Teacher-researcher: What do you mean by backwards?

Alice: If like six (points at 6) minus two would give you four [6 − 2 = 4]. So, I thought four minus negative two would give you six [4 − -2 = 6].

Alice used additive inverses, changing 4 − □ = 6 to 6 + □ = 4. She used 6 − 2 (instead of stating 6 + -2) when she solved this. Building on her discrete representations, she made analogies to whole number addition and subtraction (e.g., "working backwards," comparing to 6 − 2). Her representation of discrete objects, paired with addition and subtraction, points to potential for developing notions of directed number. Instructional experiences could connect Alice's invented reasoning to her drawing. A teacher could ask, "In what ways is Alice's drawing related to her strategy?" Then, her drawing could leverage ideas of movement; that 4 − □ = 6 and 6 + □ = 4 can represent equivalent situations. Or, her representation could be built upon and turned into a continuous model (e.g., her tallies can be related to spaces on a number line).

As learners transition from thinking about whole number operations to integer operations, a wealth of significant conceptual changes need to occur (Bofferding, 2014). As Drake's excerpt above illustrates, learners need to transition from physically operating with number to "thought things." Some of the potential challenges of transitioning from thinking about whole numbers to integers are highlighted below:

- Whole numbers can be physically embodied naturally with counting objects (e.g., Smith, Sera, & Gattuso, 1988); integers have limitations with physical embodiment, especially with counting physical objects (Martínez, 2006; Lakoff & Núñez, 2000).
- Whole number units are positive (Steffe, 1983); integer units are positive units or negative units.
- Whole number direction is one-directional; integer direction is two-directional (Bofferding, 2014).
- Whole numbers have similar order and magnitude, $2 < 5$ and $|2| < |5|$; integers have different order and magnitude, $-2 > -5$ and $|-2| < |-5|$ (Bofferding, 2010, 2014; Wessman-Enzinger, 2018a, c).
- Integers are relative numbers (Gallardo, 2002) in ways that only positive numbers are not.

Engaging with directed number as an inventive, playful "thought thing," outside of pre-determined instructional models, may help learners make these transitions

(Bofferding, Aqazade, & Farmer, 2018; Wessman-Enzinger, 2018b, c). Directed quantities—an inherent part of making sense of integer operations (Poirier & Bednarz, 1991; Ulrich, 2013; Thompson & Dreyfus, 1988)—is a rich place to enter discussion about what thinking about integer operations entails: relativity and translation. Although thinking and learning about integers as directed quantities may have challenges, I argue that conceptualizing integers as directed quantity offers more than any singular instructional model. The following sections delineate some of the ways children construct directed quantities through the lens of the mathematical ideas of relativity and translation. Learners make sense of directed number in powerful ways (e.g., Bofferding, 2014; Bishop, Lamb, Philipp, Whitacre, Schappelle, & Lewis, 2014; Bishop, Lamb, Philipp, Whitacre, & Schappelle, 2016), negating the need to find the mythical, perfect instructional model for integer addition and subtraction. Examples of student work later in this chapter highlight that if we build on learners' created models, rather than giving top-down models, there is no longer a need for a singular, ideal integer instructional model. The tenants of directed number, like translation and relativity, are often overlooked in descriptions of children's thinking; examining these specific ways of thinking can provide insight into the most robust types of models. Yet, if we build on learners' constructions of number as the "instructional model" instead, then we need to describe their constructions of translation and relativity in more depth.

In the following sections, I describe translation and relativity as components of understanding ways learners construct directed quantities. Specifically, I address the following points:

1. There are rich historical backgrounds that support the conceptualizations of translation and relativity of integers; as a society we grappled with ideas of translation and relativity for centuries.
2. Existing research highlights the capabilities and thinking of learners as they engage with integer addition and subtraction; yet, how learners construct ideas of relativity is underrepresented.
3. Many different contextual situations and problem types support different ways of thinking about translation and relativity of integers; one instructional model alone cannot fulfill these needs.
4. Children create powerful ways of thinking with translation and relativity that are significantly different than traditional integer instructional models. Children's unique constructions will point us in better directions for thinking and learning in instructional spaces.

13.4 Coordinating Relativity and Integers as Directed Quantities

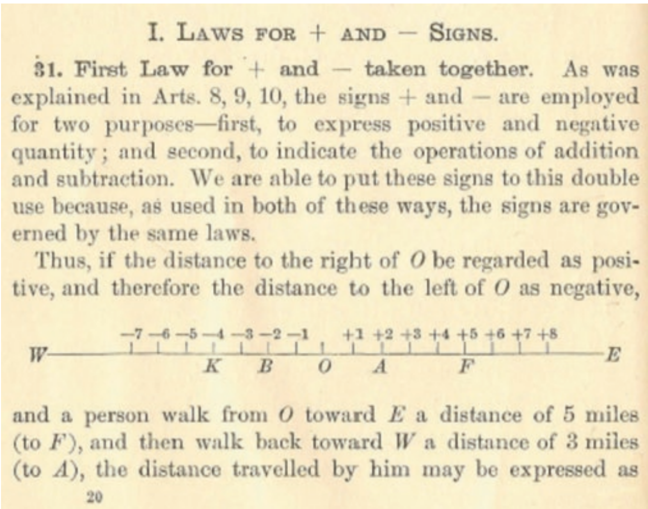
The idea of relativity is a mathematical concept that extends itself beyond integer operations (e.g., choosing to use a Cartesian coordinate plane or polar coordinate plane is an example of relativity). In this section, I discuss the idea of coordinating relativity with integer addition and subtraction as a conceptual tool to make sense of integers as directed number.

13.4.1 A Historical Perspective of Relativity

Nineteenth century North American arithmetic and algebra texts first present integers as relative numbers (Wessman-Enzinger, 2018a). Some of the first illustrations of the number line in North American school mathematics arithmetic and algebra texts included a relative number line. It is worth noting that the relative number line in Fig. 13.2 does not highlight zero, but rather a point “O,” where numbers to the right are positive numbers and numbers to the left are negative numbers.

The mathematical concept of relativity, which is foundational for using integers as directed number, evolved over time (Wessman-Enzinger, 2018a). Our modern definition of integers, with integers as a subset of the real numbers and rational numbers, prioritizes the integers as objects and overlooks relativity. Current curricula and standards do not support extensive time to build the integers conceptually (e.g., integers are often not in the elementary curricula). Our modern curricula and standards, in fact, omit ideas of relativity with integers. Yet, without relativity traditional instructional cancellation models do not work well with subtraction. Consider, for example, illustrating 2 – -3 with a two-colored chip model. One needs to use relativity to represent 2 in multiple ways in order to remove -3 chips.

Our modern curricula and standards even treat the integers as though they are fixed objects on a number line (e.g., negatives must always go on the left side of the number line). These types of ideas, such as negative integers being placed anywhere



I. Laws for + and – Signs.

31. First Law for + and – taken together. As was explained in Arts. 8, 9, 10, the signs + and – are employed for two purposes—first, to express positive and negative quantity; and second, to indicate the operations of addition and subtraction. We are able to put these signs to this double use because, as used in both of these ways, the signs are governed by the same laws.

Thus, if the distance to the right of *O* be regarded as positive, and therefore the distance to the left of *O* as negative,

and a person walk from *O* toward *E* a distance of 5 miles (to *F*), and then walk back toward *W* a distance of 3 miles (to *A*), the distance travelled by him may be expressed as

20

Fig. 13.2 Illustration of a relative number line in Durrell and Robbins (1897, p. 20) on the number line (e.g., negatives on the right side instead of the left side), are absent from curricula and standards. Our standard documents (e.g., National Council of Teachers of Mathematics, 2000; National Governors Association Center for Best Practices [NGA] & Council of Chief State School Officers [CCSSO], 2010) do not highlight the relativity of integers. The implication is that learners must implicitly think about and use relativity; yet, relativity is essential for constructing directed number. Learners need time to conceptualize and build their meanings of relativity, which they might do naturally if allowed to construct models for themselves.

Despite the lack of emphasis on developing relativity in modern standards and curricula, researchers have reflected on it. Gallardo (2002) points to different understandings of integers, making explicit that one of those includes recognizing integers as relative numbers. Carraher, Schliemann, and Brizuela (2001) reflected on an N-number line, where the ordering is centered on N (e.g., $N - 3$, $N - 2$, $N - 1$, N, $N + 1$, $N + 2$, $N + 3$). A distinguishing element of this N-number line is that N is unknown and could be represented by any number, thus incorporating the idea of relativity. The N-number line presented by Carraher et al. captures the essence of "relative numbers" and "relative number lines" found in early arithmetic and algebra texts in the nineteenth century (see, e.g., Durell and Robbins, 1897; Loomis, 1857). For example, Loomis (1857) began his introduction of the negative integers by describing the order of the negative integers through the context of the thermometer. After discussing the thermometer and ordering, Loomis commented on relativity in reference to contexts beyond temperature:

> It has already been remarked, in Art. 5, that algebra differs from arithmetic in the use of negative quantities, and it is important that the beginner should obtain clear ideas of their nature. In many cases, the terms positive and negative are merely relative. They indicated some sort of *opposition* between two classes of quantities, such that if one class should be added, the other ought to be subtracted. Thus, if a ship sails alternately northward and southward, and the motion in one direction is called positive, and the motion in the opposite direction should be considered negative. (pp. 18–19)

In this description, the integers are described as a relative number, where two directions are provided in "opposition" from an arbitrary referent.

13.4.2 A Contextual Perspective of Relativity

> *Say you are down five runs in the first inning of a baseball game. And you end up losing by fifteen runs. You would have to have ten runs in the other innings to be down by fifteen runs.* (Joseph, Grade 8, -5 + □ = -15)

Joseph, posing a story for -5 + □ = -15, makes use of integers as relative numbers with an unknown referent. When Joseph posed this story for the first time, I remember initially thinking this was quite a novel context—and then, I reflected on the mathematics he employed. What is the score of the game? Although the score of the game is unknown, the zero in Joseph's context represents a "tied game." Joseph drew on the relativity of the negative integers, illustrating runs below the tied score (i.e., the unknown referent).

Many contexts implicitly use integers as relative number: up and down runs in a baseball game without a known score (Wessman-Enzinger & Mooney, 2014); increases and decreases in money in a piggy bank with an unknown amount of money (Ulrich, 2012); getting on and off a train with an unknown number of riders (Bishop, Lamb, Philipp, Schappelle, & Whitacre, 2010). As learners coordinate their understanding of relativity with the integers in contexts, they must first determine a relative position, which points out the need for learners coordinate what a relative zero is.

Herbst (1997) reflected on how translation is related to relativity. He wrote, "The statement of an addition of the number line involves the juxtaposition of two arrows, a relative position" (p. 38). Herbst's reference to this relative position is similar to referencing a relative zero, or a starting location that uses 0. Similarly, Marthe (1982) used a river problem for investigating the thinking and learning of integer addition and subtraction. In this problem, the positive integers represented moving upstream and the negative integers represented moving downstream. This upstream and downstream movement is relative to the initial starting point on the river. Wherever one starts at on the river, represents the zero. Exactly where one starts at this river is unknown; yet, everything is measured from this point. This is a relative zero. Because part of conceptualizing relativity requires using zero as an unknown reference (with an infinite number of possibilities), this may be challenging for learners.

Ulrich (2012) referred to this use of zero as an unspecified reference point. Similar to Joseph's story where we do not know the score of the tied game, Ulrich defined an unspecified reference point as being able to conceptualize changes without an actual quantity known. Ulrich highlighted that this ability to think about relativity and use an "unknown" reference point, like Joseph did in his story, impacts students later in mathematics. For example, unknown reference points are important when working with vectors and matrices in linear algebra. Although we use conceptions of relativity beyond making sense of directed number, we lack explicit explorations of how these types of conceptions develop early on with directed number. The next section provides an example how a Grade 5 student constructed use of relativity with directed number.

13.4.3 Illustration of Relativity and Directed Quantities: The Case of Jace

Figure 13.3 illustrates the work of a Grade 5 student, Jace, for the same number sentence type (i.e., $-a + \square = b$, where $|a| < |b|$; Murray, 1985) at two different points in a 12-week teaching experiment (Steffe & Thompson, 2000) focused on integer addition and subtraction. Jace produced the drawing for $-4 + \square = 10$ during his first individual open number sentence session with me and produced the drawing for $-6 + \square = 15$ during his second individual open number sentence session. In the first session, Jace created an empty number line with -4 on the left and 10 on the right. He then used three sets of distances, with varying direction: -4 to 0 (a distance of 4,

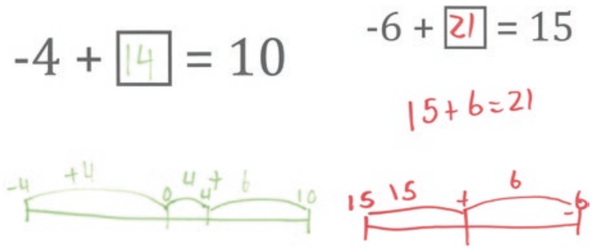


Fig. 13.3 Two number line drawings produced by Jace, a Grade 5 student, illustrating use of relativity with directed number

left to right), 0 to 4 (a distance of 4, left to right), and 10 to 4 (a distance 6, right to left). After summing his three distances, he concluded that the solution to -4 + □ = 10 is 14. In contrast, when solving -6 + □ = 15 2 weeks later, Jace placed -6 on the right side and 15 on the left of the empty number line. This differed from the first session in that the negative numbers are represented on the right side of the number line, rather than the left side. Also, he found two distances, instead of three: first examining 15 to 0 (a distance of 15, left to right) and then going from -6 to 0 (a distance of 6, right to left).

First, Jace's drawings of empty number lines highlight that the use of integers as a directed number ultimately requires using integers as relative number. Whether the integers are on the left or the right side of the number line, both of these representations are correct. Although our culture, curricula, representations of mental models, models in mathematics education, and research place negative numbers on the left side of a horizontal number line or on the bottom of a vertical number line, children do not necessarily attend to these conventions.

Second, Jace started to develop conceptions of zero as a referent. In the first drawing, he produced a number line with zero as a number on the number line. By the second drawing, we see that he, in fact, omitted an explicit 0. Yet, in both cases, he drew on 0 as a flexible referent to find the distances.

Third, Jace's strategy of finding the distance implicitly used directed number relatively for determining the absolute value or distance. Jace physically illustrated directed number from one relative number to another with motions and drawings, using these motions flexibly (sometimes right to left, other times left to right). When solving -6 + □ = 15, for example, Jace first moved his marker from left to right (i.e., 15 to 0) and then right to left (i.e., -6 to 0). Jace accounts for incremented distances verbally and flexibly, writes only these distances above his number line, and uses these distances to determine the solution of the directed distance (i.e., he sums of the two absolute values required to translate -6 to 15).

13.4.4 Connecting Themes of Relativity Across Mathematics Education and Developmental Psychology

Although the use of integers as relative numbers seems underemphasized in both mathematics education (i.e., not mentioned in curricula or standards documents) and developmental psychology (i.e., relativity has not been directly investigated), relativity is addressed both explicitly and implicitly in mathematics education and developmental psychology research. As illustrated above, the relativity of integers is an integral part of mathematics and thinking about directed number. What entails the starting point of the physical movement (i.e., our unknown referent)? Are the positive integers on the right or left side of a horizontal number line? Is the movement associated with adding or subtracting one-dimensional, or is that relative as well? These notions of relativity have implications for embodied cognition (e.g., Lakoff & Núñez, 2000), mental models (Bofferding, 2014), and integer instructional models (e.g., Saxe, Diakow, & Gearhart, 2013). Not only are integers relative numbers (Gallardo, 2002; Schwarz et al., 1993), conceptualizing relativity is implicitly imbedded in our work. Now, we need to learn more about how learners construct conceptualizations about relativity as it pertains to directed number.

13.5 Coordinating Translation and Integers as Directed Quantities

In contrast to the relativity of integers, coordinating translation and the integers as directed quantities represents a prominent theme in both mathematics education and developmental psychology from describing thinking to describing integer instructional models. In terms of integer instructional models, Herbst (1997) discussed the use of the number line metaphor as a way to make sense of integer addition and subtraction. Lakoff and Núñez's (2000) identification of order as a foundational component of mathematical cognition supports and informs the use of integers with translation. Whether talking about integer instructional models or ways of thinking about directed number, negative numbers may be constructed as point locations within this motion metaphor. Using a motion metaphor draws on the idea of symmetry on the number line (Herbst, 1997; Lakoff & Núñez, 2000). Ubiquitous pedagogical approaches support thinking about the addition or subtraction of integers as translations (e.g., Nurnberger-Haag, 2007; Tillema, 2012).

Although number lines (e.g., Saxe et al., 2013) and movement on linear scales (e.g., Nurnberger-Haag, 2007) are prevalent pedagogical tools, children do not necessarily construct movement or use number lines like top-down integer instructional models dictate (Wessman-Enzinger, 2018b). Rather, children create unique uses of movement and number line. These learner-generated constructions provide a conceptual tool for making sense of integers as directed number.

13.5.1 A Historical Perspective of Translation

The concept of a number line is foundational, not only for informing thinking and learning about translation, but also for informing current research in mathematics education on student thinking about number, and specifically negative integers (e.g., Bofferding, 2014; Saxe et al., 2013). Although historical developments of a concept, such as number line, may not parallel educational and psychological developments, a deep understanding of the past can offer researchers and educators perspectives on the present and help them make decisions for the future. As Sfard (2008) pointed out, "one becomes … bewildered when one notices the strange similarity between children's misconceptions and the early historical versions of the concepts" (p. 17).

Historically, we know that although some mathematicians had conceived of the number line in the seventeenth and eighteenth centuries (e.g., Wallis, 1685), most mathematicians and educators did not refer to number lines when attempting to make sense of operations with negative integers (Heeffer, 2011). Rather, mathematicians during the seventeenth and eighteenth centuries often made sense of negative integers by using contexts, such as debts of money, or incorporated geometrical approaches within explanations of rules of operations with negative integers (Wessman-Enzinger, 2018a). Heeffer (2011) presented historical evidence that mathematicians struggled in the past using number lines with operations, such as division, in their efforts to make sense of negative numbers and their operations. Indeed, the number line as a pedagogical tool evolved over several centuries to be incorporated into school mathematics (Wessman-Enzinger, 2018a)—with illustrations of the number line itself delayed for centuries after verbal descriptions of it. And, texts that included references to number line often paired it with contextual situations.

The historical struggle of mathematicians connecting operations with integers to the number line points to conceptual struggles of using the integers and number line; however, these are not necessarily places where the number line actually breaks down as contemporary learners engage with integers. Reflecting on potential breaking points for integers and number line, Liebeck (1990) stated, "The number line, then, emphasizes ordinality at the expense of cardinality" (p. 237). Liebeck hinted at the idea that the number line is not an infallible tool and certain integer instructional models, like number lines, offer different affordances. The number line is an important pedagogical tool, but specific tools may support some ways of reasoning more than others. Liebeck points to a conceptual leap that a child may have to undertake to begin to use the number line with integer operations—ordinality over cardinality.

In terms of using a number line, integer operations are often paired contrived rules. For example, Nicodemus (1993) described a "Linesman" where a human is standing on a number line facing right, negative number represents facing the opposite direction or walking backwards, and addition and subtraction represent moving forwards or backwards. Herbst (1997) also found these types of rules in a textbook analysis. For example, when considering the number sentence $2 - -3$, it is suggested that one conceptualizes starting at 2 on the number line, turning around, and walking backwards three spaces on the number line, ending at 5.

These types of rules may not be intuitive to children, yet metaphors of movement are (e.g., Lakoff & Núñez, 2000). However, even these intuitions of movement do not guarantee that children will construct our cultural convention of a number line and translations on that number line. How will children use number line and integer operations, without us imposing our conventions and models on them?

We know that a major challenge that children may have, for example, with the number line is that the distance unit between the tick marks is to be used, not the tick marks themselves (Carr & Katterns, 1984; Ulrich, 2012). When learners count the tick marks, rather than the distances between the tick marks, they will end up with one more (or one less) than anticipated (Barrett et al., 2012). A major assumption with the number line as an integer instructional model is that learners will be able to extend their previous knowledge about whole numbers and the number line to operations with the integers and the number line. Ernest (1985) stated that the "number line model does not have any compelling inner logic. Instead it assumes familiarity with underlying representational conventions, which are to some extent arbitrary" (p. 418). Major assumptions of using the number line as an integer instructional model are that it is used in similar ways and that it supports learners' ways of thinking. Yet, we know that the same integer instructional model is not used in the same way by teachers (e.g., Murray, 2018). We also know that children create unique and sophisticated ways of working with integers (e.g., Bofferding, 2014; Bishop et al. 2016) that often surprise us.

Although the number line can certainly be tool for extending whole number reasoning with integers, we have to re-evaluate ways that it is developed and used. It took centuries for mathematicians to develop and use the number line; our students need time to develop use of the number line, particularly with integers. Learners may extend their use of a number line with whole number by using a number path (Bofferding & Farmer, 2018) incorporating negative integers. They may or may not use the number line as mathematically or culturally expected (e.g., Wessman-Enzinger & Bofferding, 2014; Wessman-Enzinger, 2018b). We cannot expect learners to create number lines that necessarily align with our cultural conventions.

13.5.2 A Contextual Perspective of Translation

Most research literature that discusses *transformations* of integers is specifically focused on translations (Marthe, 1979; Thompson & Dreyfus, 1988; Vergnaud, 1982) in contextualized situations for addition and subtraction only. While some researchers have pointed to using translation as a way to think about integer addition and subtraction (e.g., Wheeler, Nesher, Bell, and Gattegno, 1981), other researchers, like Marthe (1979) and Vergnaud (1982), have provided problem types that support translation as well. Bell (1982), Marthe (1979), and Vergnaud (1982) presented integer addition and subtraction as beginning with a relative number or initial starting point, using a translation, and then ending at a relative number or final ending point. Supporting this work, Bishop, Lamb, Philipp, Whitacre, and Schappelle (2014) shared that the children in their study solved integer problems with translation: "Starting point + Change = Ending Point." Directed number can be conceptualized as more than just "Starting point + Change = Ending Point," but also can be used with distances or difference (Bofferding & Wessman-Enzinger, 2017; Selter, Prediger, Nührenbörger, & Hußmann, 2012; Whitacre, Schoen, Champagne, &

Goddard, 2016). Thus, a variety of both contexts and problem types provide different opportunities for conceptualizing the integers (Wessman-Enzinger & Mooney, 2014; Wessman-Enzinger & Tobias, 2015).

Although many of the contexts used for whole numbers include discrete objects without movement (Carpenter, Fennema, Franke, Levi, & Empson, 2015), learners also engage in contexts with linear movement that supports translation as it begins to work with negative integers. We also know that different problem types support different ways of reasoning for whole number (Carpenter et al., 2015); this is likely the case for integers and translation as well.

For translation problem types, Marthe (1979) classified different problem types for additive structures for integers. The first category was S_iTS_f, where the initial state (S_i) is translated (T) to the final state (S_f). Marthe then described that any of S_i, T, or S_f could be the unknowns in a given problem. A second category Marthe described was $T_1T_2T_3$. He described T_1, T_2, and T_3 as "transformations" although they can also be described as linear translations. From this problem type, Marthe described that there are three subsequent problems that can be posed, where T_1, T_2, or T_3 are unknowns, and T_1, T_2, or T_3 have differing magnitudes and signs. Marthe provided contextual examples of each of these problems. For example, for the problem type $T_1T_2T_3$ with T_2 unknown, T_1 and T_3 with opposite signs, and $|T_1| < |T_3|$, Marthe provided the example, "A car makes an initial journey of 20 km upstream. Then it makes a second journey. If it had made only one journey from its starting point to its destination, it would have made a journey of 25 km downstream. Describe the second journey" (p. 156). Marthe stated that this problem type is more challenging than STS.

Temperature is an example of a context for connecting integer operations to directed number, with both translation and relativity (Altiparmak & Özdoğan, 2010; Beatty, 2010; Bofferding & Farmer, 2018). Using the context of temperature, we modified the Marthe (1979) problem types to include a distinction between directed distance and undirected distance, with state-state-translation (SST) and state-state-distance (SSD), respectively (Wessman-Enzinger & Tobias, 2015). When a problem is posed with two given relative numbers and the translation is unknown, this is classified as an SST problem. Whereas, when a problem with two numbers and a distance, without a clearly distinguished direction, this is considered to be an SSD problem (see Table 13.2 for the distinction between SST and SSD). Consider the SST problem type posed by a prospective teacher: "It was 12° outside Wednesday. It was 17 below zero degrees Thursday. How much had the temperature dropped since Wednesday?" Compare this to the SSD problem type posed by a prospective teacher: "One day in New York it is -14 degrees out. In Maine the same day it was -20°. What is the difference between the two states' temperatures?" The distinguishing feature of the SSD problem type from the SST is that no direction is provided in the problem. The problem types modified from Marthe (1979) are summarized in Table 13.2 below.

Similarly, in terms of the STS problem type, Vergnaud (1982) pointed out that the minus sign can illustrate a translation, or the minus sign can represent the inversion of a directed translation, which is more challenging. The "minus sign" is used

Table 13.2 Relativity and translation problem types from Wessman-Enzinger and Tobias (2015)

Problem type	Description
STS	A problem posed with a relative number and a translation, with the second relative number as the unknown
TTT	A problem posed with two given translations and the third translation is unknown
SST	A problem posed with two given relative numbers and the translation is unknown
SSD	A problem posed with two relative numbers and a distance, without specified direction

for finding differences; yet, the plus sign can also mean a difference between two directed numbers of different signs. Vergnaud, for example, provided "$x + (+4) = -3$, $x = (-3) - (+4) = -(3 + 4)$" (p. 73) and stated, "My view is that equalities and equations do not fit equally well all situations met and handled by learners, but only a few of them" (p. 74). In terms of using translation, Vergnaud made an important distinction that thinking about moving backwards two units from 1 may be represented by both the expression $1 - 2$ or $1 + -2$; however, these expressions may not *conceptually* represent this situation equally for the student.

As Vergnaud highlights, I have similarly found in my own work that children's thinking about contextual problems with integers, and the number sentences they write, do not always match the context (Wessman-Enzinger, in press). Three Grade 5 children (Alice, Jace, Kim) solved the following problem:

> The warmest recorded temperature of the North Pole is about 5° Celsius. The warmest recorded temperature of the South Pole is about -9° Celsius. Which place has the warmest recorded temperature? And, how much warmer is it?

Alice, Jace, and Kim each wrote different number sentences: $5 - -9 = 14$, $-9 + 5 = 4$, and $5 + 9 = 14$, respectively (Fig. 13.4).

How learners conceptualize this problem does not necessarily coordinate with the problem type, but it might. Here we see that both Alice and Kim obtained the solution of 14, yet Kim did not even use subtraction (i.e., $9 + 5 = 14$). For Kim, $5 - -9 = 14$ did not conceptually match this context; she stated that she did not agree that subtraction should be involved when one is adding distances. Similarly, Alice did not agree with $9 + 5 = 14$ initially because she stated that +9 is not in the context of the problem; -9 is. Alice's conceptualization matched the problem type (SST); but, Kim's conceptualization of the problem did not. In this vein, although various problem types for integers may provide insight into how learners solve problems, they do not necessarily solve the problems with translations as we expect.

Thompson and Dreyfus (1988) provided a rich instructional context in a microworld, called INTEGERS, for two Grade 6 students in order to investigate conceptions about integers. Within the microworld, the Grade 6 students solved contextual problems that were often of the problem type TTT, even illustrating directed numbers as linear vectors on a horizontal number line. For example, they constructed two different translations of a turtle and determined the net translation of the turtle with the vectors. Thompson and Dreyfus conducted the teaching experiment using

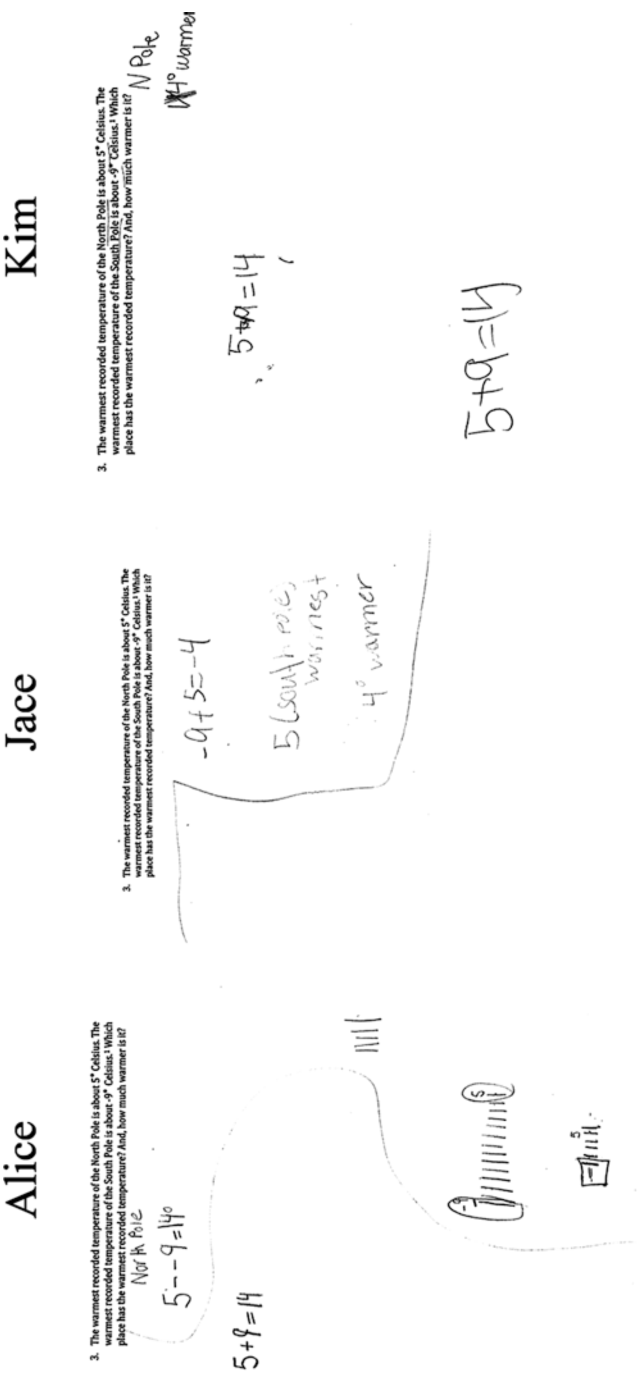


Fig. 13.4 Alice, Jace, and Kim's written work for the North Pole Problem

these net translations for 6 weeks. Similar to Thompson and Dreyfus (1988), but using a person instead of turtle, Liebeck (1990) included a number line with a person that moved along this number line. Liebeck's activity differed from Thompson and Dreyfus as it did not incorporate visualized directed vectors. Liebeck's activity also supported a different problem type—STS. Students in Liebeck's study started at different relative points, such as 2 or -5, translated the person from that point, and then found the ending point. Addition and subtraction of integers was described as "when we add, we move forwards" and "when we take away, we move backwards," respectively (p. 233). Liebeck provided a table for students to record the starting place, moving forwards or backwards, the ending place, and then the "answer" or the number sentence. This use of the person moving on the number line related to the STS problem. Both the contexts from Thompson and Dreyfus (1988) and from Liebeck (1990) used a conventional number line (e.g., partitioned, negatives on the left) and interpreted addition and subtraction unidirectionally (i.e., subtraction moves left on a conventional horizontal number line).

The contexts of Thompson and Dreyfus (1988) and Liebeck (1990) facilitated students' thinking about integers and translation, and there are many other contexts that may also support thinking about integers and translation. Some of these contexts include: a timeline with BC and AD dates (Gallardo, 2003); temperature increasing and decreasing (Wessman-Enzinger & Tobias, 2015); traveling up and down a river (Marthe, 1979); riding in an elevator (Iannone & Cockburn, 2006; Larsen & Saldanha, 2006); and balloons moving up and down (Janvier, 1985; Reeves & Webb, 2004). Despite all of these contexts supporting linear movement and directed number, some of the contexts support different types of conceptualizations of translation. The context used by Thompson and Dreyfus (1988) supports net translations (i.e., TTT); the context used by Liebeck (1990) and grounding metaphors with movement support identifying a relative number and translating to another relative number (i.e., STS); and, other contexts, like temperature (e.g., Wessman-Enzinger, in press), support using directed and undirected distance (e.g., SST, SSD). While there is often a quest for a "perfect" instructional model or a meaningful context for integers, these examples illustrate how working within a variety of contexts and problem types provides different opportunities to think about and work with integers as directed number, all of which are crucial for understanding integers.

Selter et al. (2012) differentiated between the *take-away* and *difference* models of subtraction. These models are related to both the problem types discussed above and to conceptualizations of translation. SST and SSD problem types are directly related to the difference model of subtraction, with one representing a directed distance and the other an undirected distance, and STS problem types seem related to the take-away model, with the change or "take-away" as a directed movement. Although STS, SST, and SSD are presented above as problem types in contexts, these problem types also point to ways that learners may conceptualize translations with integer addition and subtraction. Interpreting integer subtraction often requires a transition from take-away models of subtraction to distance models of subtraction (Bofferding & Wessman-Enzinger, 2017; Whitacre, Schoen, Champagne, & Goddard, 2016). Yet, our top-down instructional models for integer addition and subtraction do not explicitly support these transitions. Learners are capable of inventing their own constructions, using their own conceptions of translation to create ways to deal with integer subtraction and transition. Learners' ways of reasoning can be supported, in alignment to structures we understand (e.g., distance models, SST, SSD), without top-down integer instructional models (e.g., walk backwards, turn around, on a number line).

13.5.3 Illustration of Translation and Directed Quantities: The Case of Kim

The following example illustrates a student construction that differs substantially from conventional integer instructional models that support translation and directed quantities. Figure 13.5 illustrates a drawing of an empty number line from a Grade 5 student, Kim, at the end of a 12-week teaching experiment. Kim's number line highlights how she used directed quantity in unconventional ways. It is important to note that in this teaching experiment I did not provide any instructional models (e.g., chips models, number line models) to the students, and provided students only open number sentences or contexts without illustrations. The following excerpt of transcript is from when Kim solved -1 − □ = 8 shown in Fig. 13.5.

> Negative one minus something would equal eight. So if I did nine, it would be negative ten. So I turned it into a negative nine and so, it's sort of like this (starts drawing a number line). Here's a negative one (marks negative one on the number line), here's zero (marks zero on the number line). That's really far. Then right here is eight (marks eights on the number line). Then, nine, that's, they're both negative so you're going to subtract regularly. So like five minus three, you are going to subtract regularly with positive numbers, but it's negative numbers this time. When you do subtract it nine is a lot greater than the starting off number. So, it's going to hit zero when it's lost one (mark number line). And, then there's eight remaining over and then you can just like go into the positive though (waves hand to the right). You know, keep going with your remaining eight and get eight.

Kim constructed empty number lines like the one in Fig. 13.5, where she used STS as a strategy for integer subtraction: she started at a relative number and translated right to a second relative number. Integer subtraction like this conventionally may be thought of as the distance between two states, where -1 − □ = 8 would be conceptualized as the distance between -1 and -9. Instead, Kim uniquely used motion and a directed number starting at -1 and translating "-9" units to 8. Kim stated, "it's going to hit zero when it's lost one," decomposing the -9 units to -1 and

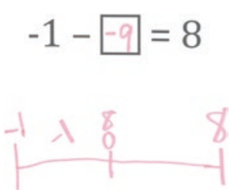


Fig. 13.5 Kim's unique use of translation and directed quantity with integer subtraction

-8 and using negative distance or directed number. Kim created a strategy where subtraction *moved right* (see Fig. 13.5), when traditionally subtraction involves moving left on this type of conventional number line (i.e., negative numbers on the left and positive numbers on the right). Comparing this to whole number reasoning, where addition moves right, marks this type of reasoning a powerful construction. Furthermore, she conceptualized distance as negative (see the “-1” written above the empty number line). Comparing the use of negative distance to whole number reasoning, where distance is always positive and not directional, marks another area of a distinct invention.

Kim’s empty number line drawing and use of directed quantity highlight the uniqueness of her constructions, relative to typical instructional models for integer addition and subtraction. Kim’s example provides evidence of a sophisticated and unique mathematical construction from a Grade 5 student. Her construction does not align well with current integer instructional models, yet does draw on the ideas of motion.

Using movement on the empty number line, Kim used her translations with addition and subtraction flexibly. That is, addition moved right on her empty number line (with positive directed distance) and subtraction also moved right on her empty number line (with negative directed distance). Comparing this to reasoning with whole number, where all directed distance is positive—addition moves right and subtraction moves left on a number line like hers—is novel. Furthermore, many instructional models for integer addition and subtraction maintain this type of whole number reasoning with the integer models (i.e., where addition moves right only and subtraction moves left only, but uses integer operations). Thus, Kim’s construction and flexibility of using both addition and subtraction for moving right on her number line is powerful. Kim’s construction offers a perspective on integer addition and subtraction where distance is relative (distance can be positive or negative) and movement is relative (subtraction can move right or left). Kim’s invention highlights a way of thinking about integer subtraction absent from current integer instructional models and even subtraction models of take-away and distance.

13.5.4 Connecting Themes of Translation Across Mathematics Education and Developmental Psychology

Although the use of integers with translation is emphasized in both mathematics education and developmental psychology (e.g., use of movement on number line), the use of translation that is represented, both explicitly and implicitly, may be different from learners’ constructions. As illustrated above, Kim used translation in an unconventional way. Her interpretation of subtraction with movement to the right marks a unique construction. She uniquely “lost” negative distance. What are other ways that children may create and construct translation? How are these unique constructions related to conceptions of relativity? For instance, if distance is interpreted as positive, then subtraction may be interpreted as a translation to the left (on a conventional number line). And, if distance is interpreted negative, subtraction may be interpreted as movement to the right (or left). Learning more about the depth of learners' constructions of translation and how this is related to conceptions of relativity has implications for embodied cognition (e.g, Lakoff & Núñez, 2000), mental models (Bofferding, 2014), and integer instructional models (e.g., Saxe et al., 2013), as it impacts the ways we leverage learning.

13.6 Concluding Remarks

The examples from students discussed here are intended to highlight and extend key themes in the literature: children are capable of creating robust and sophisticated constructions of translation and relativity in relation to integers as directed quantities, but we need to explore these constructions more in depth. Additionally, the examples are intended to challenge typical notions of what instructional models for integers entail. We must abandon the search for the holy grail of integer research—the illusive, infallible integer instructional model. Instead, let us take up pursuit of learners' robust and sophisticated constructions of integer operations.

Rather than using integer instructional models from top-down perspectives (instructional models created by teachers and researchers), we can draw on learners' constructions as the instructional models. As we look more towards learners' constructions, we should focus on overlooked ideas of relativity, paired with translation, for insight into directed quantity. Children have produced mathematical ideas (such as relativity) that have been overlooked in our own integer work. Yet, the ideas that the children have constructed are essential to directed quantity. As we learn more about conceptualizations of translation and relativity in relation to directed quantity, we can investigate how to leverage these student-constructed ideas to other advanced mathematics.

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Chapter 14
Cognitive Science Foundations of Integer Understanding and Instruction

Sashank Varma, Kristen P. Blair, and Daniel L. Schwartz

Abstract This chapter considers psychological and neuroscience research on how people understand the integers, and how educators can foster this understanding. The core proposal is that new, abstract mathematical concepts are built upon known, concrete mathematical concepts. For the integers, the relevant foundation is the natural numbers, which are understood by reference to a mental number line (MNL). The integers go beyond the natural numbers in obeying the additive inverse law: for any integer x, there is an integer $-x$ such that $x + (-x) = 0$. We propose that practicing applying this law, such as when students learn that the same quantity can be added or subtracted from both sides of an equation, transforms the MNL. In particular, perceptual mechanisms for processing visual symmetry are recruited to represent the numerical symmetry between the integers x and $-x$. This chapter reviews psychological and neuroscience evidence for the proposed learning progression. It also reviews instructional studies showing that the hypothesized transformation can be accelerated by novel activities that engage symmetry processing compared to conventional activities around number lines and cancellation. Ultimately, these instructional insights can guide future psychological and neuroscience studies of how people understand the integers in arithmetic and algebraic contexts.

Keywords Integers · Distance effect · Intraparietal sulcus · Mental number line · Additive inverse law · Symmetry processing · Analog-x model · Bundling hypothesis

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14.1 Introduction

When psychology and neuroscience ask how people understand mathematical concepts, they search for fundamental mechanisms of mind and brain. Studies from these fields have demonstrated that adults possess magnitude representations on which natural number concepts are constructed (Moyer & Landauer, 1967); have tracked the increasing precision of these representations over development (e.g., Sekuler & Mierkiewicz, 1977; Xu & Spelke, 2000); and have identified neural correlates of these representations (e.g., Pinel, Dehaene, Rivière, & Le Bihan, 2001). Central to cognitive science is the question of how these basic cognitive capacities are organized to understand culturally constructed number systems.

Education asks a different question. What experiences best support the learning of new, evermore abstract mathematical concepts? Research, for example, has investigated the ideal sequencing of concepts and procedures in mathematics instruction (Rittle-Johnson, Schneider, & Star, 2015; Rohrer & Taylor, 2007). It has also examined how to use concrete manipulatives to teach more abstract concepts (e.g., Martin & Schwartz, 2005). Ideally, the work of cognitive science can inform the educational enterprise of improving learning.

In this chapter, we develop the cognitive science foundations of how people understand integers and how these foundational insights contribute to instruction. The integers consist of a perceptually available number class, the natural numbers {0, 1, 2, …}, coupled with the much less perceptually obvious negative integers {−1, −2, …}. When walking in the woods, people can count the number of squirrels on their fingers, but they will not have an easy way to count the number of negative squirrels.

The integers are a relatively new human construction. The concept of negative numbers as debts arose as early as 250 BCE in China and seventh century India, but for much of history the idea of negative numbers was absurd, and the modern system of negative numbers did not arise until the nineteenth century (Gallardo, 2002; Hefendehl-Hebeker, 1991). The integers provide an excellent point of contact for psychology, neuroscience, and education because they are an important abstract concept that students need to learn. They also represent a quantitative system that is culturally constructed. Unlike the perceptual sense of magnitude, which helps understand that 5 is bigger than 4, negative numbers do not exhibit an obvious mapping to basic perceptual abilities. Thus, they represent a test-bed for researchers from all three disciplines to study how an abstract mathematical concept can be nurtured from fundamental cognitive and perceptual-motor capacities.

14.2 A Learning Progression for Integer Understanding

How might one understand numerical expressions such as "−4", questions about magnitude such as "which is greater −4 or 3?", and questions about arithmetic expressions such as "−4 + 3"? One intuition might be that people do so by reference to a mental number line (MNL), organized and oriented in the mind's eye in the same way as physical number lines are organized and oriented in the world. Zero would be in the middle, negative integers on the left side, and positive integers on the right. We call this model *analog+* because it extends the well-established MNL for natural numbers (Moyer & Landauer, 1967; Sekuler & Mierkiewicz, 1977).

An alternative intuition might be that negative integers are too abstract to represent directly, and that people reason about them using positive numbers and rules for manipulating the negative and positive signs. For example, to decide if −4 is less than −7, one might reason that 7 is greater than 4, but with negative numbers, one reverses the decision, so −4 is greater than −7. To decide if −4 is greater than 3, one might apply a rule that negative numbers are always less than positive numbers. We call this model *symbol+* under the assumption that mapping is via symbolic rules, and negative magnitudes are not accessed directly.

Recent research indicates both analog+ and symbol+ have merit. People obviously can reason about integers in these ways, as demonstrated by the fact that one can understand the verbal descriptions of each model in the preceding paragraphs. Surprisingly, however, adults appear to rely on yet a third model that lends more sophistication to their abilities to reason about integers. In the following, we describe this model and offer hypotheses for how it develops and how instruction can support it.

In doing so, we build on our earlier proposals (Blair, Tsang, & Schwartz, 2014; Schwartz, Blair, & Tsang, 2012) to develop a learning progression for how people come to understand abstract mathematical concepts such as the integers. This proposal is depicted in Fig. 14.1. New mathematical concepts are built upon known mathematical concepts, but they can also incorporate additional perceptual primitives that provide structure not found in the original mathematical concepts.

For the integers, the relevant foundational concepts come from knowledge of the natural numbers. As previewed above, psychological and neuroscience evidence suggests that natural numbers are understood by reference to magnitude representations organized as an MNL. These representations support judgments such as deciding which of 1 and 9 is greater.

When people first learn about the integers, they reason about them using the rules of the governing symbol system, i.e., according to symbol+. This is not surprising: Conventional classroom instruction introduces procedures for handling this new, abstract number class by reference to the procedures for handling natural numbers—the more concrete number class that children have already mastered.

Children learn the integers, but the standard instruction does not capture the key law that creates the class of integers. This is the *additive inverse* law, which states that any integer plus its “inverse” equals zero: $x + -x = 0$. Our proposal is that as children learn algebra, they practice applying the additive inverse law in its colloquial form: the same quantity can be added or subtracted from both sides of an equation. This practice transforms their understanding of integers, extending the MNL for natural numbers “to the left” of zero, to also include the negative integers. Critically, this new MNL is not a simple extension of the positive number line as suggested by analog+, but rather a transformation that incorporates the *symmetry* between pairs of additive inverses x and $-x$ in a novel way. In doing so, it combines the mind’s capacity for representing magnitudes with its capacity for processing symmetry. We call this transformed mental representation *analog-x*.

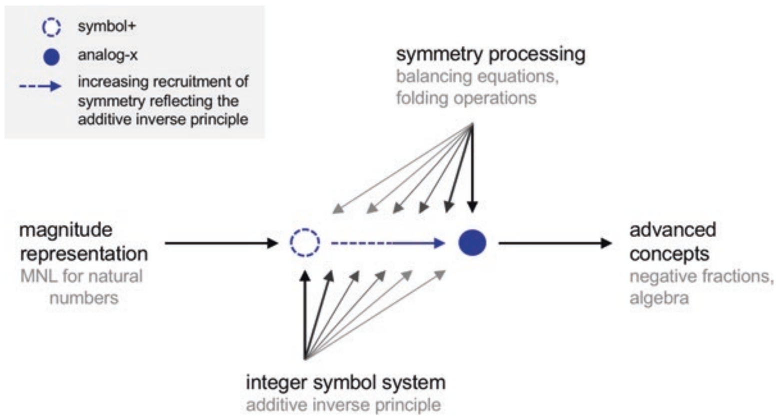

Fig. 14.1 Learning progression for integer understanding. People understand the natural numbers using magnitude representations. Initially, they reason about integers directly, by using the rules of the defining symbol system, most notably the additive inverse principle (symbol+). Through experiences with balancing equations, they recruit symmetry processing, transforming their mental representation of integers to directly reflect the additive inverse principle (analog-x). The transformed representation supports learning of advanced mathematical concepts

In the remainder of this chapter, we develop the case for the learning progression depicted in Fig. 14.1. We begin with a review of psychological and neuroscience studies of how adults mentally represent the integers, and how this representation shifts over development. This research has primarily evaluated the analog+ and symbol+ models and found both wanting. We next introduce the analog-x model, which accounts for many of these challenging findings, and consider additional evidence for its proposals. We then selectively review classroom-based research, focusing on examples of symmetry-based instruction, which then feeds back to further inform our understanding of mental representations.

14.3 Cognitive and Developmental Science Studies of Integer Understanding

Our review of the literature begins with a consideration of some cognitive and developmental studies of how adults and children understand integers. Psychologists and neuroscientists utilize a set of standard laboratory paradigms for investigating the mental representation and processing of symbolic numbers, particularly with respect to how they relate to perceptual-motor primitives for comparing physical magnitudes. In this section, we selectively review studies that have used some of these paradigms to reveal how people understand negative integers. Our focus is on findings that distinguish the analog+ and symbol+ models, and that motivate the analog-x model developed in the next section.

14.3.1 Distance Effect

The comparison paradigm dominates studies of numerical cognition. In this paradigm, people are presented with two numbers and make a judgment about which one is greater (or lesser) as quickly as possible while maintaining high accuracy. Response times are then used to make inferences about underlying mental representations and processes. Moyer and Landauer (1967) found that when adults compare pairs of one-digit natural numbers, the farther apart the numbers, the faster the judgment (e.g., 1 vs. 9 is judged faster than 1 vs. 3). From this *distance effect*, they inferred that people understand one-digit natural numbers using magnitude representations. More precisely, people possess an MNL for natural numbers, organized and oriented in space with smaller numbers "on the left" and larger numbers "on the right."[1] When comparing which of two numbers is greater, they map them to points on the MNL and discriminate which point is "to the right." The farther apart the points, the easier the discrimination, and thus the faster the judgment. The distance effect has been extended to infants and children (Sekuler & Mierkiewicz, 1977; Xu & Spelke, 2000) and to multi-digit natural numbers, rational numbers, and irrational numbers (Dehaene, Dupoux, & Mehler, 1990; Patel & Varma, 2018; Schneider & Siegler, 2010; Varma & Karl, 2013).

Recently, psychological researchers have used the comparison paradigm to investigate the mental representation of integers. Two kinds of comparisons have received the bulk of attention. For negative comparisons, where both numbers are negative integers, adults and children show a distance effect. For example, they compare −1 vs. −9 faster than they compare −1 vs. −3 (Tzelgov, Ganor-Stern, & Maymon-Schreiber, 2009; Varma & Schwartz, 2011). This finding is consistent with analog+, which proposes that negative integers are represented as points on the extended MNL to the left of zero. The greater the distance between two points, the easier it is to discriminate which point is farther "to the right," just as it is when comparing natural numbers; hence the distance effect. This finding is also consistent with symbol+, which proposes that comparisons of negative integers are first mapped to comparisons of positive integers (e.g., which is greater, −1 vs. −9? → which is lesser, 1 vs. 9?); the positive integers are compared using the MNL for natural numbers (e.g., $1 < 9$); and these judgments are mapped back to the negative integer domain (e.g., $1 < 9 \rightarrow -1 > -9$). It is the middle step, where the mapped positive integers are compared using the MNL, that produces the distance effect. Thus, negative comparisons cannot differentiate analog+ and symbol+ because both models predict a distance effect.

[1] Cultural differences may influence the left-right orientation of the number line, based on whether numbers are read from left-to-right or right-to-left in one's native language. However, there is also evidence that a left-to-right orientation may be innate. (See Zohar-Shai, Tzelgov, Karni, and Rubinsten (2017) for a review of this literature.) This chapter does not address this culture difference. Instead, it assumes a left-right orientation, consistent with the data from English-speaking countries.

What can differentiate the two models are mixed comparisons, where one integer is negative and the other positive (e.g., −1 vs. 2 and −1 vs. 7). Analog+ proposes that the two integers are mapped to points on the extended MNL and discriminated, and therefore predicts a distance effect. By contrast, symbol+ proposes that the rule "positives are greater than negatives" is applied. Because this rule does not rely on magnitude representations, there should be no effect of distance.[2] These conflicting predictions mean that, in principle, the data can be used to choose between the two models. However, in practice, this has proven difficult. One difficulty is that relatively few psychological studies have looked for distance effects (or a lack thereof) for mixed comparisons. Another difficulty is that those that have done so have found inconsistent results. Nevertheless, some inferences are possible.

No study has found a conventional distance effect for mixed comparisons, which would be consistent with analog+. Some studies have found no effect of distance, consistent with symbol+. For example, Tzelgov et al. (2009) found no effect of distance for mixed comparisons of the form $-x$ vs. y, where the integers have different absolute values (e.g., −2 vs. 4); see also Ganor-Stern, Pinhas, Kallai, and Tzelgov (2010). Remarkably, other studies have found an *inverse* distance effect! Tzelgov et al. (2009) found an inverse distance effect for mixed comparisons of the form $-x$ vs. x, where the integers have the same absolute value (e.g., −1 vs. 1 is judged faster than −4 vs. 4). Varma and Schwartz (2011) also found an inverse distance effect for mixed comparisons of the form x vs. y (e.g., −1 vs. 2 is judged faster than −1 vs. 7); see also Krajcsi and Igács (2010). These mixed findings limit the strength of the inferences that can be drawn about the mental representation and processing of negative integers. With this caveat in mind, the remainder of this chapter assumes that the inverse distance effect is "real" (although we note several other inconsistencies in the literature below and give reasons for them in Sect. 14.6).

14.3.2 SNARC Effect

Further evidence for people's mental representation of number comes from the Spatial-Numerical Response Codes (SNARC) effect. This is the finding that smaller numbers are associated with the left side of space and larger numbers with the right side of space, reflecting their respective locations on the MNL as conventionally oriented. This effect was first documented in a study where adults judged the parity of one-digit natural numbers (Dehaene, Bossini, & Giraux, 1993). Adults were faster to judge the parity of small numbers (e.g., 2) when the response (e.g., "even") was made on the left vs. right side of space, and faster to judge the parity of large numbers (e.g., 9) when the response (e.g., "odd") was made on the right vs. left side of space. The SNARC effect for one-digit natural numbers has been replicated many times

[2] In addition to mixed comparisons, zero comparisons can also differentiate the analog+ and symbol+ models. These are comparisons where one of the two numbers is zero (e.g., −2 vs. 0). See Varma and Schwartz (2011) for further discussion.

(Gevers & Lammertyn, 2005). However, this effect extends inconsistently to other number classes such as multi-digit natural numbers and rational numbers (Bonato, Fabbri, Umiltà, & Zorzi, 2007; Toomarian & Hubbard, 2018; Varma & Karl, 2013).

Analog+ and symbol+ agree in predicting that the SNARC effect extends to integers. However, they make different predictions regarding the form of this extension. Analog+ predicts a continuous SNARC effect, with negative integers responded to faster on the left vs. right side of space and positive integers showing the opposite pattern. This is because it proposes that negative integers correspond to points "to the left" of zero on the MNL (and positive integers to points "to the right" of zero). By contrast, symbol+ predicts a piecewise SNARC effect, with negative integers showing an *inverse* SNARC effect (and positive integers a conventional SNARC effect). The inverse SNARC effect results because negative integers are mapped to positive integers before processing them (i.e., $-x \rightarrow |-x| \rightarrow x$). Thus, large negative integers are processed as small positive integers (e.g., $-1 \rightarrow 1$) and small negative integers as large positive integers (e.g., $-9 \rightarrow 9$), yielding an inverted SNARC effect.

That analog+ and symbol+ predict different SNARC effects means that, in principle, the data can be used to choose between them. Unfortunately, the literature is full of mixed results. Some studies have found the continuous SNARC effect predicted by analog+ (Fischer, 2003) whereas others have found the piecewise SNARC effect predicted by symbol+ (Fischer & Rottman, 2005). Shaki and Petrusic (2005) showed that these different findings are due in part to differences in methodology. They had adults make positive comparisons (e.g., 1 vs. 2) and negative comparisons (e.g., −1 vs. −2), holding the distance between each pair of numbers constant. When positive comparisons and negative comparisons were intermixed in the same block of trials, participants showed a continuous SNARC effect consistent with analog+. However, when these different comparison types were segregated in different blocks, participants showed the piecewise SNARC effect predicted by symbol+. This study suggests that adults possess multiple integer representations and choose among them based on task demands. We return to this flexibility in Sect. 14.6.

14.3.3 Number Line Estimation Task

The number line estimation (NLE) paradigm has also been used to investigate the mental representation of integers. In this paradigm, participants are presented with a number and a number line with only the endpoints labeled and have to mark the position of the number on the number line with a pencil or computer pointer. This task was originally used with children and with natural numbers in the ranges 1–100 to 1−1000. Not surprisingly, the error in children's estimates decreases over development. The more interesting finding is that the pattern of errors also changes over development. The pattern for older children is veridical, with linearly spaced numbers. By contrast, the pattern for younger children is logarithmic, with exaggerated spaces between smaller numbers and compressed spaces between larger numbers (Siegler & Opfer, 2003). These developmental trends have been extended to rational numbers, whether expressed as fractions or decimal proportions. In both cases, children as young as 10 years old already make linear estimates, with error decreasing with further development into adulthood (Iuculano & Butterworth, 2011; Siegler, Thompson, & Schneider, 2011). Finally, for irrational numbers, adults make linear and accurate estimates of radical expressions such as $\sqrt{2}$ and $\sqrt{90}$ (Patel & Varma, 2018).

Analog+ and symbol+ do not make strong predictions about performing the NLE task on integers, and how this performance changes over development. For this reason, we simply present some of the core findings. First, there appears to be a logarithmic-to-linear shift with development in the estimation of negative integers, one that parallels that for natural numbers. Brez, Miller, and Ramirez (2015) found evidence that second graders rely on logarithmically scaled representations when estimating numbers in the range −1000 to 0, just as they do when estimating numbers in the range 0–1000. This representation shifts over elementary school, and by fourth (and especially sixth) grade, children exhibit linear representations for both ranges. By middle school, children's estimates are linear in the much larger range −10,000 to 0 and also in the combined range −1000 to 1000 (Young & Booth, 2015).

14.4 Neuroimaging Studies

Additional insight into the mental representation and processing of integers can be gained from neuroscience studies. We focus here on functional Magnetic Resonance Imaging (fMRI) studies that have utilized the comparison paradigm, as these are of greatest relevance to competitively evaluating the analog+ and symbol+ models.

Chassy and Grodd (2012) identified areas that show greater activation when adults make negative comparisons (e.g.. −3 vs. −2) vs. positive comparisons (e.g., 5 vs. 4). One such area was the superior parietal lobule (SPL). This area is adjacent to the intraparietal sulcus (IPS), which prior studies have identified as a neural correlate of the MNL for natural numbers. Specifically, the IPS shows a neural distance effect when comparing natural numbers, with greater activation for harder near-distance comparisons (e.g., 1 vs. 2) than for easier far-distance comparisons (e.g., 1 vs. 9) (Pinel et al., 2001). The researchers interpreted activation of the SPL similarly, as evidence that negative integers also have magnitude representations.[3] This interpretation is consistent with the extended MNL representation proposed by analog+ but not with the mapping rules of symbol+.

Stronger evidence would be provided by an experiment that looked for distance effects and that included mixed comparisons. Blair, Rosenberg-Lee, Tsang, Schwartz, and Menon (2012) provided such evidence in an fMRI study of adults

[3] SPL and IPS are also associated with visuospatial reasoning (e.g., Zacks, 2008). Thus, it is possible that they are recruited here not to process the magnitudes of positive integers and negative integers, but rather to process their symmetric relationship about zero. We consider the role of symmetry processing in integer understanding below, when describing the analog-x model.

who made positive, negative, and mixed comparisons of pairs of integers in which the distance varied systematically. Their results concerning positive vs. negative comparisons largely replicated those of Chassy and Grodd (2012). A finding of interest involved a representational similarity analysis. In this kind of analysis, the neural response patterns elicited by different stimuli are compared. The idea is that the more dissimilar the patterns for two stimuli, the more distinct the representations. The researchers focused on the IPS and the neural patterns elicited when people make near- vs. far-distance comparisons. They found that near- vs. far-distance comparisons elicited more distinct neural patterns for positive integers than for negative integers. This implies that positive integers may have a “sharper tuning” in IPS than negative integers. This finding is consistent with analog+, suggesting that negative integer magnitudes are less well differentiated than positive integer magnitudes. With regard to mixed comparisons, this study did not find a behavioral effect of distance, but the neuroimaging data told a more complex story. No areas were more active for mixed comparisons than for negative comparisons or positive comparisons. This null result is inconsistent with symbol+, which predicts recruitment of areas in prefrontal cortex associated with rule application (i.e., “positives are greater than negatives”).

Gullick, Wolford, and Temple (2012) conducted a study similar to Blair et al. (2012). The results were comparable overall, but one finding is worth highlighting. There was an inverse distance effect for mixed comparisons of the form $-x$ vs. y where both $-x < y$ and $|x| < |y|$ (e.g., −3 vs. 5). This was true behaviorally, with far-distance comparisons slower than near-distance comparisons, and this was also true neurally, with far-distance comparisons eliciting greater activation in IPS and SPL than near-distance comparisons. These inverse behavioral and neural distance effects are inconsistent with both analog+, which predicts conventional distance effects, and with symbol+, which predicts no effects of distance.

To summarize, these neuroimaging studies of adults provide limited insight into the representation and processing of negative integers. Negative comparisons elicit greater activation than positive comparisons in IPS and SPL, areas associated with the MNL and visuospatial processing (Blair et al., 2012; Chassy & Grodd, 2012; Gullick et al., 2012). In addition, negative comparisons do not elicit greater activation than positive comparisons in prefrontal areas associated with rule processing (Gullick et al., 2012). These findings can be interpreted as evidence for analog+ and against symbol+, respectively. However, neither of these models can explain the inverse distance effect that Gullick et al. (2012) found for (a subset of) mixed comparisons, both behaviorally and in the activations of IPS and SPL.

By contrast, the findings are clearer from the lone neuroimaging study that has investigated how children understand negative integers. Gullick and Wolford (2013) had fifth and seventh graders make negative comparisons and positive comparisons. The important finding was that for the fifth graders, negative comparisons elicited greater activation than positive comparisons in prefrontal areas associated with rule processing. For seventh graders, however, there was no such difference. This suggests that younger children reason according to symbol+. This also suggests that older children might have shifted to a new model of integer understanding, whether because of development, experience, or instruction. We consider a candidate model next.

14.5 Analog-x

The psychological and neuroscience literatures on integer understanding are small and in some cases inconsistent. Nevertheless, they support tentative inferences about the nature of the underlying mental representations and processes.

We begin with mixed comparisons, where people judge whether a positive integer or negative integer is greater, because this case provides the most leverage for choosing between possible models. Analog+ proposes that integers are understood with respect to an extended MNL, where negative integers are located "to the left" of zero. It predicts a standard distance effect for mixed comparisons, with far-distance pairs (e.g., -1 vs. 7) judged faster than near-distance pairs (e.g., -1 vs. 2). Because no study in the literature has found support for this prediction, analog+ can be ruled out. Symbol+ proposes that negative integers are not understood directly, by reference to magnitude representations, but rather indirectly, by applying rules. In particular, mixed comparisons are made by applying the rule "positive integers are greater than negative integers." Because this rule makes no reference to the magnitudes of the integers, symbol+ predicts no effect of distance. Varma and Schwartz (2011) found support for this prediction among sixth graders who had just learned about negative numbers. This makes sense if conventional instruction builds new procedures for working with integers on top of known procedures for working with natural numbers, which students have already mastered. Some studies of adults have also found support for this prediction (Ganor-Stern et al., 2010; Tzelgov et al., 2009).

However, our assessment is that adults likely reason according to a different model. This follows from numerous other studies of adults that have instead found an inverse distance effect for mixed comparisons (Gullick et al., 2012 for comparisons of the form $-x$ vs. y where both $-x < y$ and $|x| < |y|$; Krajcsi & Igács, 2010; Varma & Schwartz, 2011; Tzelgov et al., 2009, for mixed comparisons of the form $-x$ vs. x). In these studies, adults are faster to judge near-distance pairs (e.g., -1 vs. 2) than far-distance pairs (e.g., -1 vs. 7). The inverse distance effect is inconsistent with the predictions of the analog+ and symbol+ models and raises the question of how adults understand the integers? We address it here by describing a third model and reviewing evidence for its key proposal that adults understand integers by combining magnitude representations with symmetry processing.

14.5.1 Integer Understanding = Magnitude Representations + Symmetry Processing

The natural numbers coupled with the addition operation form a system that obeys the commutative law $x + y = y + x$, the associative law $(x + y) + z = x + (y + z)$, and the identity law $x + 0 = x$, with 0 the additive identity. Critically, the integers bring additional structure: they also obey the inverse law, which states that for every x, there is a corresponding $-x$ such that their sum is the identity $x + -x = 0$.

Extending one's understanding of number from the natural numbers to the integers requires understanding the additional structure brought by the inverse law. Initially, this understanding is explicit. When children first learn about the integers, they apply the governing laws in a deliberate and controlled manner to work with integers in arithmetic expressions and algebraic equations. This is one sense in which they reason according to symbol+. With development and experience, however, children's integer understanding shifts. They come to an implicit understanding of the integers, such that they no longer recruit rule-based processing as heavily. Rather, they gain an intuitive understanding of how integers can and cannot be manipulated in arithmetic and algebraic contexts. This raises the question of what it means to have an intuitive understanding of the integers, in particular to understand that additive inverse law that enriches them beyond the natural numbers.

Analog-x provides an answer to this question. It proposes that adults understand negative integers as they understand natural numbers, with reference to magnitude representations. That is, there is an MNL for integers. Critically, it is not the MNL proposed by analog+: it does not *extend* the MNL for natural numbers "to the left." Rather, it *reflects* the MNL for natural numbers to directly represent the inverse relationship between the pairs $-x$ and x. In this way, analog-x combines the mind's capacity for representing magnitudes with its capacity for processing symmetry.

Figure 14.2 depicts the combination of magnitude and symmetry mechanisms proposed by analog-x. At the center is a reference axis that helps locate the natural number MNL and the negative integer MNL. The natural number MNL is shown above the reference axis. Its nonlinear form captures the psychophysical scaling of magnitude representations. The magnitude of a natural number is given by the height of the corresponding point above the reference axis. Natural numbers are compared in the usual way, by discriminating their magnitudes (i.e., heights). As the examples in Fig. 14.2 show, the model predicts a distance effect for positive comparisons (i.e., 1 vs. 8 is more discriminable than 1 vs. 3).

A new proposal is that the MNL for negative integers is a reflection of the MNL for natural numbers about the reference axis. This reflective organization has two important consequences. First, it directly models the inverse relationship between $-x$ and x, in the vertical alignment of the corresponding points. In this way, analog-x captures people's intuitive understanding of the additional structure that the integers bring over the natural numbers. Negative integers are compared in the same way as natural numbers, by discriminating the corresponding magnitudes (i.e., heights). The model predicts a distance effect for negative comparisons, as the examples in Fig. 14.2 show (i.e., −1 vs. −8 is more discriminable than −1 vs. −3).

The second consequence of the reflective relationship between the natural number and negative integer MNLs concerns mixed comparisons. Specifically, this reflective relationship predicts the inverse distance effect observed by some researchers (Gullick et al., 2012 for comparisons of the form $-x$ vs. y where both $-x < y$ and $|x| < |y|$; Krajcsi & Igács, 2010; Varma & Schwartz, 2011; Tzelgov et al., 2009, for mixed comparisons of the form $-x$ vs. x). Positive and negative integers that are close together on the standard number line (e.g., −2 vs. 1), and thus hard to discriminate, correspond to magnitudes (i.e., heights) that are quite different in the

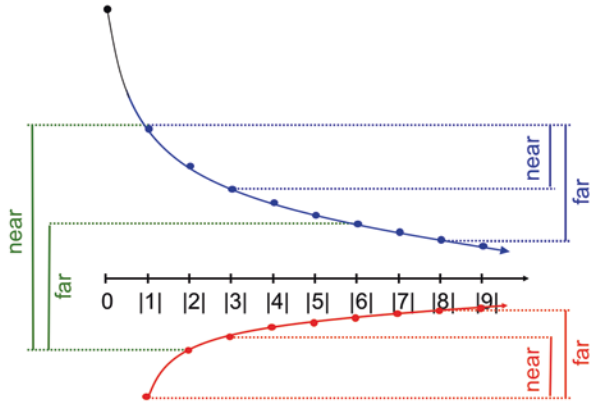

Fig. 14.2 The analog-x model. The reference axis at the center helps locate the natural number MNL (above) and the negative integer MNL (below), which are reflections of each other. It predicts conventional distance effects for positive comparisons and for negative comparisons, as shown by the projections on the right. Critically, it predicts an inverse distance effect for mixed comparisons across the two MNLs, as shown by the projection on the left

analog-x representation, and thus easy to discriminate. The reverse is true for positive and negative integers that are far apart on the standard number line (e.g., −2 vs. 6): the corresponding heights in the analog-x representation are quite similar, and thus difficult to discriminate.[4]

14.5.2 Studies of Symmetry and Integer Processing

A novel proposal of analog-x is that the integer MNL encodes the additive inverse law using symmetry processing. Tsang and Schwartz (2009) tested this proposal in a behavioral study of adults. They developed an integer bisection paradigm where participants are presented with pairs of integers and have to name the midpoints as quickly as possible. They predicted that performance would be best for two cases where the symmetry of integers about 0 could be exploited. The first is for symmetric pairs of the form $(-x, x)$, where the midpoint is 0. Computing the midpoint should be particularly easy because in analog-x, the corresponding points are vertically aligned to capture the additive inverse relationship between x and $-x$. The second case is for pairs of the form $(-x, 0)$ and $(0, x)$, where 0—the point of symmetry—can be used to anchor midpoint estimation. They further predicted that symmetric processing would confer some advantage for pairs close to these two cases, e.g., (−6, 8) because it is almost symmetric, and (−1, 13) because it is almost

[4]The analog-x model shown in Fig. 14.2 can be formalized and quantitatively fit to the data. See Varma and Schwartz (2011) for the details.

anchored. Their results supported these predictions. Response times were fastest for bisections that were symmetric, anchored, or nearly so; see Fig. 14.3a.

Tsang, Rosenberg-Lee, Blair, Schwartz, and Menon (2010) followed up this behavioral study with an fMRI study investigating the neural correlates of integer bisection. They predicted that bisection of symmetric pairs (e.g., (−7, 7)) or nearly symmetric pairs (e.g., (−6, 8)) would produce greater activation in posterior areas associated with processing visual symmetry. This prediction was supported. In particular, the more symmetric the pair of integers being numerically bisected, the greater the activation in left lateral occipital cortex, an area associated with processing visual symmetry in dot patterns (Sasaki, Vanduffel, Knutsen, Tyler, & Tootell, 2005); see Fig. 14.3b.

14.6 Instructional Studies and the Symmetry of Positive and Negative Integers

Analog-x is a model of how adults understand the integers. Its key proposal is that symmetry processing is recruited to represent that additive inverse law, resulting in a transformed MNL as shown in Fig. 14.2. It is this symmetry that allows analog-x to predict the inverse distance effect for mixed comparisons and privileged performance on the integer bisection task for symmetrical and anchored pairs. In contrast, symbol+ provides a better characterization of children's understanding of integers. Behaviorally, children show no effect of distance (Varma & Schwartz, 2011), and neurally, they show increased recruitment of prefrontal areas associated with deliberate rule processing (Gullick & Wolford, 2013). This raises the question of the factors that drive the progression on how integers are understood, from applying symbolic rules to referencing a transformed MNL.

One hypothesized factor is learning algebra. This requires practicing applying the additive inverse law in its colloquial form: The same quantity can be added or subtracted from both sides of an equation. This practice could transform children's understanding of integers, restructuring their MNL to directly incorporate the symmetry between pairs of additive inverses x and $-x$; see Fig. 14.1. Evidence for this developmental claim could come from a longitudinal study tracking changes in the integer representation over schooling. Unfortunately, no such study has been run to date.

Another perspective on how the integer representation changes over development comes from instructional studies of how best to teach the integers to children. Some of these interventions have emphasized the use of standard number lines and can be understood as consistent with analog+ (Hativa & Cohen, 1995; Moreno & Mayer, 1999; Schwarz, Kohn, & Resnick, 1993; Thompson & Dreyfus, 1988). Others have focused on teaching rules for manipulating negative numbers (e.g., the SR condition of Moreno & Mayer, 1999), consistent with the symbol+ model. Still others have focused directly on the additive inverse principle, using different colored chips or other discrete entities to represent positive and negative quantities, which cancel each other out (Bolyard, 2006; Liebeck, 1990; Linchevski & Williams,

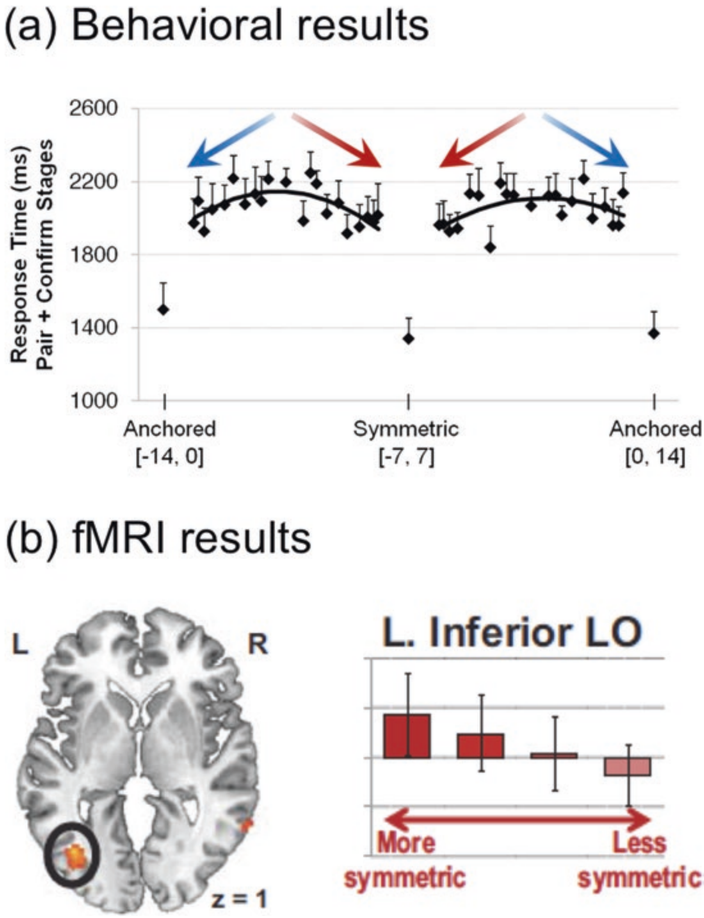

Fig. 14.3 Integer bisection paradigm. (**a**) Bisection of integer pairs is privileged when the numbers are either more symmetric around 0 or more anchored to 0. (**b**) The greater the numerical symmetry of the pair, the greater the activation in left lateral occipital cortex, an area associated with processing of visual symmetry

1999; Streefland, 1996). A potential downside of these discrete cancellation-based approaches is that they do not emphasize order, and thus are isolated from linear magnitude representations of number (Bofferding, 2014).

Three recent studies have moved beyond instructional approaches aligned with analog+ and symbol+ or focused on the additive inverse principle in isolation. These studies have developed new approaches to instruction that focus on the symmetry of the positive and negative integers about zero, and as a result they are better aligned with analog-x.

14.6.1 Instructional Approaches Incorporating Symmetry

Two recent studies that have started from a conceptual analysis of the elements necessary to understand integers have derived instructional approaches that incorporate a focus on symmetry. Saxe, Earnest, Sitabkhan, Haldar, Lewis, and Zheng (2010) designed one part of their instruction around the task of marking the position of an integer on a standard number line, where other numbers might already be marked. They identified five principles necessary for successful performance. The fifth principle was understanding symmetry and absolute value: "For every positive number, there is a negative number that is the same distance from 0" (p. 440). Their instructional materials included problems that required reasoning about this symmetry, such as locating −150 on a number line where 0 and 150 were already marked. Their learning assessments included items measuring understanding of this symmetry, such as judging as correct or incorrect a number line where −1000 and 1000 were marked but were not equidistant from 0 (which was also marked) and providing a justification for why.

Bofferding (2014) developed and evaluated new instructional approaches for teaching the integers to first graders.[5] These approaches derived in part from a conceptual analysis of what it means to understand the integers, which revealed three meanings of the "−" sign. The first and second meanings are familiar: as a mark distinguishing negative integers from positive integers (e.g., −7 vs. 7) and as the name of the subtraction function (e.g., 9 − 3). The third meaning had been previously overlooked in the education literature: as the name of the "symmetric function" for "taking the opposite" (e.g., −(7) = −7). This study also revealed the roles symmetry plays in the mental models children have for the integers. Only the most sophisticated of these models represents that positive integers and negative integers are symmetric about zero. In addition, only these models correctly distinguish the values vs. magnitudes of negative integers (e.g., $8 < 9$ but $-8 > -9$), which is critical for making "more" vs. "less" (and "high" vs. "low") judgments of negative integers.

In these studies, symmetry is thoughtfully incorporated into the instruction and models of student learning. An important limitation is that the value of symmetry for learning is not tested directly.

14.6.2 An Instructional Study Directly Comparing Symmetry to other Approaches

We see convergence in the psychological, neuroscience, and mathematics education literatures that symmetry plays a critical role in what it means to understand the integers. A study that builds on this convergence is Tsang, Blair, Bofferding, and

[5] This study is notable in testing children much younger than those in prior psychological and educational studies.

Schwartz (2015), which directly compared an instructional approach that incorporated symmetry to more traditional number line and cancellation approaches. The instructional approaches were built around three manipulatives embodying different underlying models. The "jumping" approach modeled arithmetic operations as movements along an extended number line (Fig. 14.4a); it corresponds to analog+. The "stacking" approach modeled arithmetic operations on the cancellation of discrete items (Fig. 14.4b). The "folding" approach combined elements of jumping (i.e., directed magnitudes) and stacking (i.e., cancellation) (Fig. 14.4c). What is novel about this approach is that adding or subtracting integers requires bringing the two operands into alignment using symmetric processing.

Post-test measures found substantial evidence for the efficacy of the folding approach, and thus for the use of symmetry. When estimating the position of a negative integer on a number line where the corresponding positive integer was already marked, the folding group was most likely to use a symmetry strategy, which was associated with more accurate performance. More importantly, the folding group performed best on far-transfer problems such as estimating the position of negative fractions on number lines and solving missing operand problems (e.g., $1 + -4 = [\ ] + -2$), which had been not covered in class. These far-transfer findings are evidence for the analog-x proposal that symmetry is particularly important when students learn pre-algebra and must apply the additive inverse law to manipulate equations.

The results of Tsang et al. (2015) suggest that including symmetry in integer instruction allows learners to generalize to solve new types of negative number problems that they had not directly been taught, including those that focus on the additive inverse property. These findings bring useful questions back to the study of mental representations of number. For example, there are relatively few neuroimaging studies of integer processing in general, and even fewer where the participants are children. How do different instructional approaches affect children's neural representations of integers as they become more fluent? Does an instructional approach that focuses on symmetry and the additive inverse property increase the recruitment of brain areas associated with visual symmetry, even when learners are reasoning about symbolic numbers?

14.7 Conclusion

This chapter has considered how adults understand an abstract mathematical concept, the integers, and how educators can foster this understanding in children. It has built a corridor of explanation from neuroimaging data to response times to hands-on activities in the classroom. The result is a clearer picture of how magnitude representations and symmetry processing support integer understanding, and how these capacities are coordinated and integrated through learning.

Our first proposal is that acquiring a new, abstract mathematical concept requires mastering the governing symbol system. More novel is our second proposal: mas-

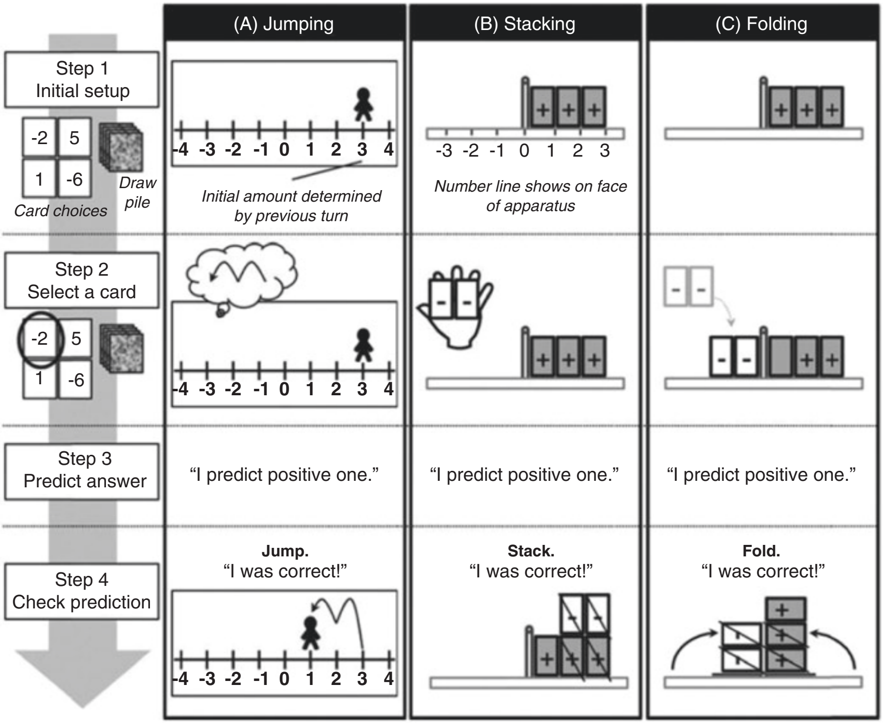

Fig. 14.4 Actions taken by students when evaluating the equation 3 + −2 = 1 in the (**a**) jumping, (**b**) stacking, and (**c**) folding instructional conditions. These correspond to analog+, symbol+, and analog-x, respectively. (Note. From "Learning to 'see' less than nothing: Putting perceptual skills to work for learning numerical structure," by J. M. Tsang, K. P. Blair, L. Bofferding, and D. L. Schwartz, 2015, *Cognition and Instruction*, *33*, p. 167. Copyright 2015 by Taylor & Francis. https://www.tandfonline.com/toc/hcgi20/current. Reprinted with permission)

tery enriches the mental representation of known concepts to reflect the unique properties of the new concept, and it does so by recruiting additional perceptuo-motor capacities. In this way, people can build intuition for ideas quite far from perceptual-motor experience (Blair et al., 2014; Schwartz et al., 2012). Specifically, analog-x makes the surprising claim that the MNL for natural numbers is transformed through symmetry processing to directly encode that $-x$ and x are additive inverses. We speculate that this transformation is accelerated when students learn algebra, and practice applying the additive inverse law in its colloquial form ("the same quantity can be added to or subtracted from both sides of the equal sign") to manipulate equations. It is an open question whether this transformation can be accelerated further, for example, by developing instructional activities where younger children coordinate magnitude representations and symmetry processing of integers. The folding condition of Tsang et al. (2015) offers initial evidence that this might be possible.

Our review began with neuroscience and psychological studies and progressed towards educational studies. We end by considering a path less often trodden: how education can inform psychology and neuroscience. Educational research can guide future lab studies of how analog-x (and analog+ and symbol+) scale to arithmetic and algebraic contexts. For example, there are few psychological studies of how people understand arithmetic operations on integers (e.g., Prather & Alibali, 2008), and the neural correlates of this understanding (e.g., Gullick & Wolford, 2014). By contrast, there is an extensive mathematics education literature on different approaches for teaching integer arithmetic (Hativa & Cohen, 1995; Liebeck, 1990; Linchevski & Williams, 1999; Moreno & Mayer, 1999; Saxe et al., 2010; Schwarz et al., 1993; Streefland, 1996; Thompson & Dreyfus, 1988). This asymmetry represents an opportunity for psychological and neuroscience research, as many of the phenomena that have been documented in the classroom merit further study in the lab. One example is Bofferding's (2014) proposal that to understand the integers is to understand three meanings of the "–" sign, including its easily overlooked meaning as a "symmetric function" for reversing the sign of an integer expression. Another example is the Tsang et al. (2015) finding that understanding the symmetric organization of positive integers and negative integers about zero is associated with better performance on pre-algebra problems demanding sensitivity to the meaning of the "=" sign (i.e., missing operand problems). What mental and neural mechanisms undergird understanding the "–" sign as a "symmetric function" and pre-algebraic reasoning about integers?

In addition, mathematics education research can potentially reframe how we understand the inconsistent results of some of the psychological studies reviewed above. This was true for the distance effect and the SNARC effect, with different studies finding evidence consistent with the three different models of integer understanding (i.e., analog+, symbol+, and analog-x). These inconsistencies are deeply problematic for psychologists and neuroscientists because they make it impossible to choose between competing models, and ultimately to make scientific progress. The conventional explanation for mixed findings is noise in the signal: the samples are too small, the methods are too varied, and so on. Mathematics education research offers a different perspective on this heterogeneity. The participants in these studies learned about the integers in classrooms spread across the United States and indeed the world. We have seen that different instructional approaches are aligned with the three different models of integer understanding. Thus, it is possible that some of the inconsistencies observed in psychological studies are not the product of noise in the data or even individual differences in basic cognitive abilities. Rather, they may be the product of instructional differences. Understanding this systematic variation is a goal for future research.

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Chapter 15
Commentary on Negative Numbers: Aspects of Epistemology, Cognition, and Instruction

Guershon Harel

Abstract This commentary reviews each of the three content chapters in the integers section and offers questions to promote further discussion. In addition to the themes raised in the three chapters, I introduce the role of formal mathematical structure in generalizing systems of number, from natural numbers to integers, and analogously, from real numbers to complex numbers. Integers, in particular, are structured by algebraic relations, which imply, for example, that $(-1) \cdot (-1) = 1$. Historical observations and anecdotal evidence of children's reasoning pertaining to this role raise important cognitive, pedagogical, and metaphysical questions.

Keywords Additive inverse · Algebraic invariance · Complex numbers · Epistemological obstacles · Negative numbers · Structural reasoning · Symbols

Understanding students' understanding of negative numbers is a central, shared goal of the three chapters comprising this portion of the book. The objective is not merely academic, of only theoretical interest; rather, it is geared toward the improvement of the learning and teaching of negative numbers. Collectively, the authors of these chapters make a significant contribution to our understanding of the complexity inherent in the cognition of negative numbers, of the instructional implications of this cognition, and of the existing gaps in our understanding of the developmental interdependency among the cognitive, neurological, linguistic, and educational facets of negative numbers. In this brief commentary, I did not attempt—nor could I, if endeavored—to capture the richness of the findings, instructional implications, and open research questions discussed in the three chapters. Instead, I choose to reflect on a central theme of each chapter, while focusing on a common concern that engendered questions not addressed in the chapters. At the heart of this concern is the structure of negative numbers and their symbolic representation.

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15.1 Contributions Through a Common Three-Tier Thesis

Although not organized in this manner, each of the chapters offers a three-tier thesis: a *diagnostic* tier framing the problem; a *clinical* tier offering ways to study the problem; and a *prognostic* tier proposing pedagogical paths to solve the problem.

Nicole Wessman-Enzinger.
Based on an extensive review of current research, *Wessman-Enzinger* concludes that the search for an ideal instructional model for integers—one that can help children conceptualize the domain of integers as a "natural" extension of the domain of whole numbers—is futile. Such a model, she maintains, does not exist, for the simple reason that the emergence of negative numbers marks a breakdown in humans' early schemes of number and quantity. Hence, no model can possibly overcome this conceptual discontinuity. She supports this claim by examining the limitations and affordances of a list of instructional models for integer operations, stressing that the list is "only a diminutive portion of the different instructional models that have been proposed across decades of integer research."

Wessman-Enzinger offers an alternative strategy for studying the problem: redirect the energy that, for decades, has been invested in the search for an ideal model, toward trying to understand children's constructions of constituent elements of negative numbers. She suggests using the insights gained from studying children's constructions to develop instructional activities that help children build bottom-up their own model, where negative numbers are conceptualized as a set of collinear vectors stemming from an arbitrary point on a line (i.e., the *origin*), whereby forming a *directed number line*. Underlying this conceptualization are two constructs: *relativity* and *translation*. *Relativity* refers to the arbitrariness of two elements of a number line: the location of the origin on the number line and the assignment of the vector direction as positive or negative. *Translation* refers to the movement of vectors along the number line together with their magnitude.

Sashank Varma, Kristen Blair, and Daniel Schwarz.
Approaching development of knowledge from a cognitive science perspective, Varma, Blair, and Schwarz consider the problem of how children and adults understand integers, and how educators can foster adults' understanding in children. They discussed three cognitive models that account for, at least partially, learners' understandings of negative numbers. The models build upon the foundation of natural numbers, whose conceptualization is analogous to the structure of the positive number line, referred to as the mental number line (MNL). The first of these three models is the *analog+* model—an extension of the MNL for natural numbers. It is "organized and oriented in the mind's eye in the same way as physical number lines are organized and oriented in the world. Zero would be in the middle, negative integers on the left side, and positive integers on the right." The second model is called *symbol+*. It is a model where judgments, such as those involving order relations between negative numbers, are made on the basis of knowledge of order relations between natural numbers and symbolic rules for manipulating negative and positive

signs (e.g., $a\langle b \leftrightarrow -a\rangle - b$). The third model, called *analog*−*x*, is Varma et al.'s novel contribution to the field. According to *analog*−*x*, the MNL for natural numbers is reflected symmetrically, rather than extended, creating in the process the crucial concept of additive inverse: for any number x, there exists a number $-x$, such that $x + (-x) = 0$; and with it, the entire domain of integers. Thus, *analog*−*x* combines in it (a) magnitude representation, modeled by the MNL for natural numbers; (b) the integer symbol system, modeled by *symbol*+ through the application of the additive inverse property; and (c) the symmetry processing developed and enhanced through equation-solving activities.

Varma et al. (this volume) draw two pedagogically important implications relevant to the focus of this commentary. The first implication is that a necessary cognitive prerequisite for the acquisition of new, abstract mathematical concepts is a mastery of the governing symbol system. The second implication is that such mastery "enriches the mental representation of known concepts to reflect the unique properties of the new concept, and it does so by recruiting additional perceptuo-motor capacities. In this way, people can build intuition for ideas quite far from perceptual-motor experience."

Laura Bofferding.
What conceptual changes take place with children as their understandings of negative numbers evolve toward the institutionalized, taken-to-be-shared meaning of these numbers? Approached from the perspective of the theory of conceptual change, Bofferding addresses this question by examining (a) the interpretations that children generate for constituent dimensions of negative numbers: *notation*, *order relation*, *numerical values*, and *addition and subtraction operations*; and (b) children's use of these interpretations as they attempt to solve addition and subtraction problems involving negative numbers. She demonstrates the complexity of this conceptual change and the role that children's prior knowledge—acquired through schooling or everyday experience—plays as they transition from the domain of whole numbers to the domain of negative numbers. As Bofferding points out, it's of particular importance that the transition is not unidirectional, but dialectical. Students' mental structures of whole numbers affect their understanding of negative numbers and, conversely, as they attempt to make sense of negative numbers, they reorganize their conceptual structures for whole numbers.

Children's struggle to make sense of the notation of negative numbers is an example of this complex dual transition. Bofferding discusses the three meanings attached to the minus sign: (a) as a subtraction operation; (b) as a symbol denoting a negative number; and (c) as a multiplication by negative one. In school mathematics, spatial cues are often used to differentiate among these meanings. Bofferding encouragingly finds that instruction "focuse[d] on integer order, value, and symbols lead to changes in students' approaches to integer arithmetic." For example, she reports a shift "from considering large negatives to be larger than small negatives and operating based on absolute value to interpreting negative values based on their linear order and operating based on moving or counting in a particular direction."

15.1.1 Negative Numbers: Symbolism and Structure

Structural reasoning is a fundamental, ubiquitous, practice in mathematics. "Look for and make use of structure" is one of the eight standards for mathematical practices called for by the Common Core State Standards in Mathematics (CCSSM, 2010). The meaning and use of structural reasoning in the mathematics education literature is diverse, characterized by a wide range of abilities (see Harel, 2013b; Harel & Soto, 2016). Relevant to our discussion here is the question of how to plant the seeds for structural reasoning as children struggle to make sense of negative numbers. The concerns and ideas discussed in the three chapters inspired attention to the following manifestations of structural reasoning.

1. *Presumed Meaning*. The ability to carry out *purposeful* actions on meaningless symbols using structural rules, with the a priori assumption that meaning for the symbols will be constructed subsequently.
2. *Formation of conceptual entities*. The ability to conceive of a string of symbols representing, simultaneously, an operation and an outcome (e.g., $2 \cdot x,\ 2 + x, \sqrt{x}$) as a conceptual entity, as an input for further operations.
3. *Multiple Interpretations*. The ability to interpret mathematical symbols in multiple ways.

The meaning of these characteristics of structural reasoning and their relevance to the three chapters under discussion will become clearer as the rest of the commentary unfolds. We begin with a brief historical account.

15.2 A Snapshot into Leibniz and Euler's Views of Negative Numbers

At the heart of the difficulties children encounter with negative numbers is the transition from whole numbers to negative numbers. This, of course, is not unique to negative numbers. Any extension of one domain of numbers into a larger domain of numbers involves what Brousseau (1997) calls epistemological obstacles—obstacles that are unavoidable due to the meaning inherent in the concept; the well-documented phenomenon concerning obstacles learners encounter as they transition from whole-number operators (multipliers or divisors) to fraction operators is an example.

Epistemological obstacles observed with individuals usually have traces in the history of mathematics. Those involved in the conceptualization of negative numbers are a case in point. Consider the following argument by Antoine Arnauld (1612–1694), a theologian and mathematician. Applying the conceptualization of a number as a representation of a quantity, measure, or capacity, Arnauld questionedthe validity of negative numbers by pointing to the equality, $\frac{+1}{-1} = \frac{-1}{+1}$. Since $+1 > -1$, how could, he asked, a greater to a smaller equal a smaller to a greater?

(Kline, 1972). Not less fascinating is Leibniz' response to Arnauld's claim. According to Kline, Leibniz agreed with Arnauld's objection but "argued that one can calculate with such proportions *because their form is correct, just as one calculates with imaginary quantities*" (emphasis added; Kline, 1972, p. 252).

What might Leibniz have meant by his analogy to complex numbers? The history of the development of complex numbers began with the sixteenth-century discovery of a solution formula to cubic equations, primarily by Cardano (1501–1576). As the mathematicians of the time looked into this new result, they encountered baffling behaviors (Trignol, 1980). One of the most perplexing was that in certain cases the cubic formula yielded (at the time) meaningless solutions involving the square roots of negative numbers when real roots are known. However, when these solutions were treated as if they were meaningful expressions, the manipulation yielded the expected meaningful solutions. (e.g., for the cubic equation, $x^3 = 15x + 4$,the cubic formula yields the solution, $x = \sqrt[3]{2+\sqrt{-121}} + \sqrt[3]{2+\sqrt{-121}}$. But when this solution is manipulated by algebraic rules, *pretending as if the expressions involving the square roots of negative numbers are meaningful*, the manipulation results in the solution 4, as expected. For an extended discussion of this history and its instructional implications, see Harel (2013b). In his response, it is safe to assume, Leibniz was referring to this very process. Hence, he argued, the meaningfulness ofthe equality $\frac{+1}{-1} = \frac{-1}{+1}$ is structural; it is derived from the symbolic rules of algebra.

Euler's explanation to the rule "negative multiplied by negative is positive" is equally as intriguing. Euler first justifies this rule, "negative multiplied by positive is negative," through the interpretation of negative numbers as *debt.* Then he writes:

> It remains to resolve the case in which − is multiplied by −; or, for example, $-a$ by $-b$. It is evident, at first sight, with regard to the latter that the product will be ab; but is doubtful whether the sign +, or the sign −, is to be placed before it; all we know is, that it must be one or the other of these signs. Now, I say that it cannot be the sign −; for $-a$ by $+b$ gives $-ab$, and $-a$ by $-b$ cannot produce the same result as $-a$ by $+b$; but must produce a contrary result, that is to say, $+ab$; consequently, we have the following rule: − multiplied by + produces +, that is the same as + multiplied by +.

Both Leibniz and Euler's justifications resorted to the *structure of symbols* rather than the quantitative referents represented by the symbols involved. Leibniz did so by analogizing symbolic manipulations of negative numbers to those applied successfully earlier in the development of complex numbers. While Euler remained in the context of negative numbers, he also rested his case on symbolic structure considerations—that two different symbolic rules cannot yield the same outcome.

15.3 Presumed Meaning

In discussing ways children's constructions could be leveraged to ideas of translation, or movement, Wessman-Enzinger brings up an excerpt of a response by Alice, a fifth grader who was asked to solve the equation $4 - \square = 6$. The following dialogue ensued.

Alice: Well, because I did two… I did it backwards (moves pen across $4 - -2 = 6$). If I did two plus four, I got six. So, then I thought it would be negative two.

W-E: What do you mean by backwards?

Alice: If like six (points at 6) minus two would give you four. So, I thought four minus negative two would give you six.

A key question here is this: How did Alice conceptualize the equation? If "*subtraction makes bigger*" was not part of her scheme, her response points to potential for reasoning along the following lines:

Assuming it is meaningful to take away a number from 4 *to get* 6, *then* 6 *plus that number must be* 4. *What number could that be? It must be a number that, when added to* 6, *the result would be* 4. −2 *is that number*.

The theorem in action (Vergnaud, 1994) underlying this reasoning is, "if the difference between a and b is c, then the sum of c and b is a.", as suggested by Wessman-Enzinger. Reasoning where one assumes existence of a number satisfying an equation inconsistent with her or his prior knowledge and acts upon that number in accordance with a structural rule, corresponds to the reasoning applied in the development of the complex numbers and argued by Leibnitz in the case of negative numbers.

Fascinated by Alice's response, I posed the same question to Lei, a mathematically advanced 5-year-old child, who was taught negative numbers through the traditional models of "debt," "temperature," etc., but was not exposed to equations of the form $4 - \square = 6$.

Lei: [the answer is] −2.

I: How did you get −2?

Lei: Has to be a negative number, because when you take away a number from 4, you never get 6, because 6 is bigger than 4. Then I understood it has to be a negative number.

The subsequent dialogue with Lei was less clear, but he mentioned the fact that $6 + (-2) = 4$.

Lei's response, too, is reminiscent of Leibniz' defense of the use of structure-based arguments, but it also resembles Euler's justification-by-elimination. Lei's response might be interpreted as follows: The only additive connecter between 4 and 6 is 2, again perhaps using his MNL for natural numbers. But, the solution to the equation cannot be (positive) 2, since it is not possible to subtract 2 from 4 and get 6. The remaining option is, therefore, negative 2.

This brings me to the enduring question of how to teach the rule, "negative times negative is positive." Anecdotally, I observed that elementary school teachers typically respond positively to the following structural justification. As the reader can see, the justification includes elements of Euler's considerations, but it goes beyond them in that it uses the concept of additive inverse and its uniqueness, as well as other basic properties of integers (e.g., associativity and distributivity).

- Assume the product $(-1) \cdot (-1)$ is meaningful, resulting in an integer. Then it must have a *unique* additive inverse.
- We suspect that its additive inverse is 1 or -1 (along the line of reasoning expressed by Euler). Claim: -1 is the additive inverse of $(-1) \cdot (-1)$. Here is why:
- $[(-1) \cdot (-1)] + (-1) = [(-1) \cdot (-1)] + (-1) \cdot 1 = (-1) \cdot [(-1) + 1] = (-1) \cdot 0 = 0$
- By the uniqueness property of additive inverse, $(-1) \cdot (-1) = 1$

I hasten to add that if we have learned any lesson from the New Math era, this explanation in its current form is entirely inadequate for elementary school students. However, if indeed fifth-grade children are capable of reasoning structurally, as the three authors seem to suggest, and following Wessman-Enzinger's call to use children's own constructions as guide for negative number instruction, the following questions are of cognitive and pedagogical interest.

1. *Are young children capable of reasoning in terms of presumed meaning?*
2. *Should we encourage the development of curricula that utilize this ability to help students overcome the transition from whole numbers to negative numbers, especially in dealing with problems involving multiplicative operations?*
3. *What are the characteristics of learning trajectories* (Simon, 1995) *that lead up to the development of this ability?*
4. *How can we avoid the risk that the presumed meaning kind of reasoning would diminish attention to quantitative reasoning?*

As the above discussion demonstrates, the ideas of additive inverse and uniqueness of additive inverse are key in the conceptualization of negative numbers. Varma et al.'s central claim is that the *analog*$-x$ model provides an answer to "the question of what it means to have an intuitive understanding of the integers, in particular to understand that additive inverse law that enriches them beyond the natural numbers." *Analog*$-x$ is proposed as a model that accounts for adult's understanding, including, I assume, the institutionalized understanding of the integers. It is significant to note that the model implies—mathematically, due to its symmetry processing—an essential condition about the additive inverse property: *uniqueness*—that for any integer x, there exists a *unique* integer $-x$, such that $x + (-x) = 0$. This property, implicit in the model, together with the commutative property, the associative property, and the closure of the binary operation, $+(\cdot, \quad \cdot)$, makes the integers a *group*. Without this property, the structure of the integers as we know it crumbles; for example, the equality we have just discussed, $(-1) \cdot (-1) = 1$, is no longer valid. This raises several questions:

5. *What foundations, perceptual or conceptual, provide impetus for the development of the uniqueness property?*
6. *What formal and informal experiences engender or enhance this development?*
7. *Where in the learning progression of the integers proposed by the authors, particularly in the analog*$-x$ *model does the uniqueness property fall?*

15.4 Formation of Conceptual Entities

Students' difficulties with negative numbers are often attributed in part to the "abstract nature" of the concept. APOS theory (Dubinsky, 1991) pours cognitive meaning into the term "abstract" by specifying the types of abstractions learners must go through in order to cope with mathematical concepts and ideas. *Abstraction* in APOS theory involves four levels of conceptualizations: *action, process, object, and schema* (hence, the acronym). Relevant to our discussion here are the first three levels. The essential feature of understanding the symbol $-x$ at the level of process conception is that the learner can imagine taking *any* natural number x and through the processing symmetry claimed by *analog*$-x$, transform it into a new integer, $-x$. The key here is the ability to reason in terms of a universal quantifier, expressed by the pronoun, *any.* Without this ability, the learner might be able to deal with $-x$ at the level of action conception; that is, given a specific number (e.g., 4) the learner can produce its negative counterpart. But he or she would be handicapped in dealing with algebraic expressions involving the minus sign, as was pointed out by Bofferding.

Entailed from APOS theory is that there is a reflexive developmental relationship between the formation of negative numbers as objects and the use of their symbolic representation as an input of a relation (e.g., an equation or a function). For example, in dealing with the equation, $\sqrt{-x} = 7$, $-x$ is an input for the function $t \rightarrow \sqrt{t}$. Reasoning conceptually, not procedurally, along the line, "The square of what number of the form $-x$ is 7?" is an indication of conceiving $-x$ as an object, for it is taken as an argument of a function. The point of this discussion is that negative numbers are perhaps the earliest opportunity to engage students in reasoning about mathematical symbols at the level of object conception. Clearly, a critical ingredient in this reasoning is the concept of additive inverse.

Varma et al. speculate that the acquisition of this concept is accelerated when students learn to solve equations, a practice that involves applying the additive inverse property to isolate the equation's unknown—definitely a structure-based practice. They raise the question whether this capacity can be accelerated further through certain instructional activities. Equations and inequalities involving absolute values (e.g., $|1 - x| - |x| > 2$) are likely to be instrumental in both strengthening students' practice of additive inverse and reasoning at the level of object conception.

15.5 Multiple Interpretations

Given the critical role negative numbers play in algebra and the difficulties algebra students have in encoding their symbolic designations, Bofferding is right to call for further investigation into the role of spatial cues in helping students differentiate among the meanings assigned to the minus sign. Yet a faulty interpretation associated with negative numbers common among algebra students is that the very appearance of the negative sign in front of an expression implies that the expression is negative. This misconception manifests itself, for example, in the difficulty students have with the algebraic definition of absolute value. On the one hand, students learn that the absolute value of an integer is always positive, since it represents the distance of a point on the number line from the origin. On the other hand, the algebraic definition states that $|x| = -x$ (when $x < 0$), a negative value by the appearance of the latter equality.

Mathematically, there is a unified meaning for the negative sign. The symbol "$-x$" denotes the additive inverse of an integer x. The binary operation, $a - b$, is "a plus the additive inverse of b", that is, $a - b = a + (-b)$. And $(-1) \cdot x = -x$ is a theorem (not a predetermined property, an axiom, such as $1 \cdot x = x$) asserting that the product of -1 and an integer x is the additive inverse of x (which, as stated, is denoted by $-x$). Clearly, this unified meaning of the minus sign is not suitable for young children, but questions for researchers and curriculum developers are:

8. *Should this meaning be promoted in school mathematics?*
9. *If so, how to intellectually necessitate it* (Harel, 2013a) *for the students*? And
10. *In what stage of their conceptual development?*

The intellectual value of constructing a unified meaning cannot be overestimated, as pointed out by Jacques Hadamard: "The creation of a word or a notation for a class of ideas may be, and often is, a scientific fact of great importance, because it means connecting these ideas together in our subsequent thought" (Edwards, 1979, p. 89; quoted by Moreno-Armella, 2014).

The suggestion to consider the merit of teaching this unified meaning is not to avoid the multiple meanings of the minus sign. Rather, as Bofferding indicates, "students need to be able to think about positive numbers flexibly; likewise, they need to understand the many uses of negative numbers, the unary meaning of the minus sign." The unavoidable difficulties associated with the multiple interpretations of negative numbers detailed by Bofferding should be welcomed, not circumvented. It is through these difficulties that learners are likely to construct one of the most valuable—indeed essential—ways of thinking in mathematics: that symbols can have multiple interpretations and that it is advantageous to attribute different interpretations to symbols in the process of solving problems.

A quote from Otte (2006) by Moreno-Armella (2014) captures this sentiment: "a mathematical object, such as a 'number' or 'function' does not exist independently of the totality of its possible representations, [and the object] must not be confused with any particular representation, either" (p. 17). Moreno-Armella continues: "Each representation provides a door of access, but what one 'finds' through it must not be confused with the object under consideration." (p. 628)

15.6 Concluding Remarks

A piece of literature—a journal article, a poem, an essay—is judged not only by the factual observations it makes or positions it expresses, but also by the questions it engenders. Each of the chapters discussed in this commentary raises questions entailed from a literature review or a research investigation. As I read and reflected on these chapters, more questions arose. In this narrative I focused on those pertaining to the role of symbolism and structure in the development of negative numbers, both in history and with individual learners. There remains, however, one question—metaphysical in nature and belonging to the philosophy of meaning—about the ability to carry out purposeful actions on meaningless symbols using structural rules, with the a priori assumption that meaning for the symbols will be constructed subsequently.

11. Are purposeful actions on meaningless symbols using structural rules meaningful? If so, what meaning do they have?

As discussed, these actions had enormous success in the development of the complex numbers and negative numbers. The success of these actions accelerated later in the nineteenth century, turning into part of the mathematical practice of the time. They were known as *operational method* (Friedman, 1991). The mathematical area known as *functional analysis* initially emerged to put the method on solid logical grounds. We would be hard pressed to ignore this historical development, especially if children invent such actions spontaneously in dealing with negative numbers. The final question, then, is:

12. Should the presumed meaning approach be considered by curriculum developers and teachers? If yes, how?

Elsewhere (Harel, 2008) I defined two ways of thinking associated with the use of symbols: the *non-referential symbolic* and *algebraic invariance*. The *non-referential symbolic* is a way of thinking by which one operates on symbols as if they possess a life of their own, not as representations of entities in a coherent reality. *Algebraic invariance*, on the other hand, is a way of thinking by which one recognizes that algebraic expressions are not manipulated haphazardly, but rather with the purpose of arriving at a desired form and maintaining certain properties of the expression invariant. The difference between the two ways of thinking is critical for mathematics educators. While the latter includes the ability to pause at will to probe into a referential meaning for the symbols involved, the latter does not. The goal of instruction is to help children develop the *presumed meaning* practice in the context of negative numbers in such a manner that it becomes an instantiation of the algebraic invariance way of thinking, not the non-referential way of thinking.

A significant part of this commentary addressed what Varma et al. call *adults' understanding* of integers, with reference to the institutionalized knowledge of this domain at various points in the historical development of mathematics, including the modern era. I did so because the ultimate goal of mathematics education is to help students gradually refine and modify their understandings toward those that have been institutionalized—those the mathematics community at large accepts as correct and useful in solving mathematical and scientific problems. As a mathematics educator, not a cognitive scientist, the approach I have taken here dovetails with Varma et al.'s succinct distinction between the research focus of a cognitive scientist and that of a mathematics educator: "Central to cognitive science is the question of how … basic cognitive capacities are organized to understand culturally constructed number systems. Education asks a different question. What experiences best support the learning of new, evermore abstract mathematical concepts?"

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Chapter 16
Synergizing Research on Constructing Number: Themes and Prospects

Martha W. Alibali and Anderson Norton

Abstract The overarching theme of this book can be simply stated: Building on a foundation of biologically based abilities, children construct number via sensorimotor and mental activity. In this chapter, we return to this theme, and we connect it to three additional themes that emerge across chapters: comparing competing models for conceptual change; consideration of multiple concepts for natural numbers, fractions, and integers; and understanding interrelations of conceptual and procedural knowledge in the construction of number. We close by suggesting ways that psychologists and mathematics educators might move forward with interdisciplinary research that addresses important questions about the construction of number. Indeed, the chapters in this volume chart many possible paths.

Keywords Numerical development · Conceptual change · Procedural knowledge · Sensorimotor activity · Interdisciplinary research

16.1 Introduction

How do learners construct number, including natural numbers, rational numbers, and integers? The purpose of this volume was to bring together research from different intellectual traditions, including mathematics education, cognitive psychology, developmental psychology, and neuroscience, to address this broad question. Different chapters take different theoretical stances and use different methodological tools. Taken together, they highlight that the development of number depends both on foundational systems that provide an early basis for number, and on activities that build upon, enrich, restructure, or otherwise modify early knowledge. The overarching theme of the book can be simply stated: Building on a foundation of

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biologically based abilities, children construct number via sensorimotor and mental activity.

Several chapters in the volume address biologically based, foundational abilities and systems that engage with sensory input in mathematical ways. Young children's construction of number builds on these foundations. These foundational abilities and systems include the ability to subitize (the focus of Chap. 2, by Clements, Sarama, and MacDonald), the approximate number system (a focus of Chap. 4, by Ulrich and Norton), the ratio processing system (the focus of Chap. 10, by Matthews and Ziols), and perceptual mechanisms for processing visual symmetry (considered in Chap. 14, by Varma, Blair and Schwartz).

Other chapters in this volume focus on later developments and on activities that engage these systems and build on them in specific ways. In some cases, early knowledge seems to stand in the way of later knowledge, causing difficulties or interfering with performance. For example, the shift from reasoning about natural numbers to reasoning about fractions poses many challenges for learners, as discussed by Obersteiner, Dressler, Bieck, and Moeller (Chap. 7). At the same time, early knowledge also positions children to take in and to construct new, mathematically relevant information, both in informal settings, as highlighted by McMullen, Chan, Mazzocco, and Hannula-Sormunen (Chap. 4), and in more formal, educational settings. Understanding precisely *how* later developments build on foundational abilities and systems is one key challenge for contemporary research at the interface of psychology and mathematics education. Ulrich and Norton (Chap. 5) present an attempt at such integration, casting the approximate number system as yielding a sense of *gross quantity*, which is refined through the child's activity to yield progressively richer understanding of *magnitude*, per se.

Many later developments occur primarily in formal educational settings. Several of the chapters focus on elements of these settings, including the activities that children engage in and their consequences for children's thinking. For example, Clements, Sarama, and MacDonald (Chap. 2) consider instructional activities through which children can build on their subitizing skills to discover key properties of natural numbers, such as cardinality. Obersteiner and colleagues (Chap. 7) provide recommendations about instructional practices in light of current knowledge about how natural number knowledge can interfere with fraction knowledge. Mix, Smith, and Crespo (Chap. 5) consider the cognitive processes afforded and encouraged by existing curricular materials for place value. Varma and colleagues (Chap. 14) highlight a range of conceptual models that form the bases for curricular materials about integers. Several of the chapters consider educational practices that bridge from or build on biologically based abilities.

Other chapters in this volume seek to characterize the nature of children's understanding of different types of number and the range and sequence of conceptions that children may hold. For example, Tzur (Chap. 8) describes schemes for conceptualizing fractions that draw on the operations of iterating units and recursive partitioning. Bofferding (Chap. 12) describes children's shifting concepts of negative numbers as they engage in a process of conceptual change, with a focus on how concepts of negative numbers draw on foundational concepts of numerical order, numerical values, and operations. Wessman-Enzinger (Chap. 13) focuses on a particular model of integers as *directed numbers*, and the foundational concepts that are required for such a model—specifically, relativity, or distance from an arbitrary referent (0), and translation, or moving from one number to another.

Across chapters, one key focus is on mechanisms of change in children's thinking. For example, Mix, Smith, and Crespo (Chap. 5) focus on statistical learning and structure mapping as two foundational learning mechanisms that are implicated in children's learning about place value. Simon (Chap. 9) focuses on the Piagetian construct of reflective abstraction as a mechanism by which learners construct concepts, through coordinating actions that derive from activity. Tzur (Chap. 8) focuses on children's reflection on the *effects* of their activity. Beckmann (Chap. 11) suggests that when children experience the outcomes of actions, they may update their expectations in ways that align with Bayesian models of inference—a perspective that underpins a growing body of contemporary research in cognitive development (e.g., Gopnik & Bonawitz, 2014; Sobel & Kushnir, 2013; Xu & Tenenbaum, 2007). Thus, a central theme in understanding and effecting change in children's understanding of number is the importance of *activity*. Moreover, as the chapters in the current volume make clear, activity is often structured in important ways by educational materials, tasks and settings, and by social partners such as teachers and peers.

In this closing chapter, we reflect on the roles of sensorimotor and mental activity in the construction of number. In doing so, we also consider three additional themes that arise repeatedly throughout the book: (1) processes of conceptual change in the development of number concepts, (2) multiple concepts of natural numbers, fractions, and integers, and (3) interrelations of conceptual and procedural knowledge in the construction of number. We close by considering the implications of the work considered in this volume for bringing together psychology and mathematics education.

16.2 Sensorimotor Activity in the Construction of Number

In the opening chapter of this volume, we argued that sensorimotor activity is foundational for constructing mathematical objects, including number. We further proposed that sensorimotor activity could be a meeting point for psychologists and mathematics educators, in the sense that activity is involved in the mechanisms of development, and that specific forms of activity can be encouraged or promoted in educational practices in both formal and informal settings.

Many chapters in this volume have identified or described activities that support the development of mathematical knowledge. For example, Simon (Chap. 9) describes a student's construction of a concept of multiplication that generalizes from natural numbers to fractions—specifically, multiplication as a change of units. Simon conceptualizes the construction of this general concept as a process of "guided reinvention" that involves both physical activity within a specially designed learning environment (involving fraction bars) and reflective abstraction that is socially supported by a teacher. McMullen, Chan, Mazzocco, and Hannula-Sormunen (Chap. 3) describe activities that encourage young children to focus on number in their everyday environments, by arranging toys and other materials in ways that make numerical features highly salient—creating what they call "bait" for noticing and reflecting on numerical properties. Varma, Blair, and Schwartz (Chap. 14) discuss instructional approaches for integers using manipulatives that embody conceptualizations of integers as "jumping" (movement along a number line), "stacking" (cancellation), or "folding" (symmetry) (Tsang, Blair, Bofferding, & Schwartz, 2015). Some chapters consider specifically how activity can build on biologically based systems, such as the approximate number system (Ulrich & Norton, Chap. 5) and the ratio processing system (Matthews & Ziols, Chap. 10). Indeed, both Ulrich and Norton and DeSmedt (Chap. 6) emphasize that sensorimotor activity can "feed back" to affect the approximate number system, in the sense that mathematically relevant activities can enhance the acuity of the system. In this way, children's activity influences how the approximate number system is engaged in their construction of number.

The chapters in this volume consider a range of different ways in which children establish, extend, and organize their mathematical thinking through activities. In this regard, two distinctions are worth highlighting. First, activities may include both physical activities and mental ones. Children's actions with their bodies and with objects in the world provide them many opportunities for taking in, operating on, and generating mathematical information, as emphasized in many contemporary theories of embodied cognition (see Alibali & Nathan, 2018). Some objects are specially designed to afford particular sorts of actions that connect in particular ways to specific mathematical concepts, for example, base-10 blocks, fraction bars, and two-color counters for representing integers. Some activities involve actions on symbolic representations, such as number lines or number statements, that can be visually perceived but cannot be handled, grasped, or moved. Other activities occur in the mental realm. For example, unitizing and disembedding involve mental activity that may not have a specific outward manifestation. Moreover, reflective abstraction describes a process for mentally coordinating such actions.

Second, activities can be engaged in individually or they can occur in social settings. Children engage in many mathematically relevant activities spontaneously and on their own—including physical activities such as counting, measuring, and aligning, and mental activities such as noticing mathematical properties, reflecting on mathematical ideas, and thinking about connections. Understanding the nature of children's spontaneous mathematical activity is an important focus of current research, as demonstrated in the programmatic work of McMullen and colleagues (Chap. 3). Children also engage in mathematically relevant activities in informal social interactions, including conversations and play interactions with peers and adu lts (Gunderson & Levine, 2011; Levine, Ratliff, Huttenlocher, & Cannon, 2012; Levine, Suriyakham, Rowe, Huttenlocher, & Gunderson, 2010). Finally, a great deal of mathematically relevant activity—and in particular, activity relevant to constructing more advanced mathematics—occurs in formal educational contexts.

The activities that occur in instructional contexts are heavily shaped by cultural practices about who goes to school, how schools and classrooms are organized, how teachers and students are expected to interact, and the kinds of curricular materials and technological tools that are available. Many of the chapters in this book describe instructional activities that use particular sorts of curricular materials or technological tools within particular sorts of physical and social contexts. For example, Mix, Smith, and Crespo (Chap. 5) consider the opportunities for learning offered by activities in three different curricula about place value. Tzur (Chap. 8) and Simon (Chap. 9) consider activities that children can engage in with fraction bars—either using simple paper strips ("French fries") or using a specially designed computer application that allows children to make and measure the bars. Broadly speaking, instruction provides opportunities for activities that can be involved in—and can provoke—processes of knowledge change.

A deeper understanding of how children construct knowledge through activities—including both physical and mental activities, and both individual and socially supported ones—can guide the design of instructional approaches, curricular materials, and technological tools that promote engaging in such mathematically relevant activities. This focus on activity is woven through our consideration of other themes that recur throughout the volume.

16.3 Processes of Conceptual Change in the Development of Number

A central issue in understanding the development of number has to do with the "shape" or nature of change. Once an initial conceptual structure is in place, does change involve *elaborating and enriching* that initial structure? Or does change involve *reorganizing and restructuring* that initial structure? Researchers from different theoretical traditions have offered different answers to this question. This general issue is of particular importance for researchers who study fractions and integers, because knowledge of these number types builds on knowledge of natural numbers in many ways.

It is well established that natural number concepts can sometimes interfere with reasoning about other types of numbers. The tendency to inappropriately draw on natural number knowledge in reasoning about fractions has been termed the "natural number bias" (Ni & Zhou, 2005); some of the research documenting the natural number bias in fraction reasoning is reviewed by Obersteiner, Dressler, Bieck, and Moeller (Chap. 7). A similar pattern of interference from natural number knowledge is also evident in people's reasoning about irrational numbers (Obersteiner & Hofreiter, 2017).

Although interference from natural numbers is well documented, it is not observed in every sample or in every setting. Indeed, some research has suggested that reasoning about natural numbers can also *support* reasoning about other number types. For example, drawing on whole number division concepts can support students' reasoning about dividing fractions (Sidney & Alibali, 2015, 2017). With respect to integers, some researchers have argued that negative numbers are actually represented as natural numbers that are bound together with polarity on an as-needed basis (Ganor-Stern, Pinhas, Kallai, & Tzelgov, 2010; Ganor-Stern & Tzelgov, 2008). According to this perspective, negative numbers are created from natural number components when required by the task at hand.

Indeed, because natural numbers are the earliest learned type of number, they provide a basis for learners' concepts of other types of numbers. At issue is just *how* learning about other number types builds on natural numbers. The answer to this question should yield insights into how knowledge of natural numbers and knowledge of fractions and integers are related.

Is the transition from whole numbers to fractions and integers best characterized as an *extension* of natural number knowledge into new realms? Some scholars have argued that this is the case for fractions (e.g., Siegler & Lortie-Forgues, 2014; Siegler, Thompson, & Schneider, 2011). Or is this transition better characterized as a process of reorganization and restructuring, as other scholars have argued (e.g., Vamvakoussi & Vosniadou, 2004)? Shall we, as Tzur (Chap. 8) argues, "embrace bringing forth natural number reasoning, which learners do have available, as a starting point to a reorganization process"? Further, if the shift is best characterized in terms of restructuring, what is the relation between the initial, pre-restructuring knowledge state and the later, post-restructuring knowledge state? Are those states *incommensurable*, in the sense that new concepts are not definable in terms of previous ones, as Carey (1991) describes? Put another way, when a child's view of the number system changes from one based solely on natural numbers to one that incorporates fractions and/or integers, is the child's view of *number* fundamentally changed?

It seems likely that both types of change—that is, change best characterized as enrichment and more fundamental, structural change—occur during the course of children's mathematical development. Some forms of learning may yield enrichment of existing structures, whereas others may involve more far-reaching, structural reorganization. One challenge for researchers and educators is to determine what sorts of learning activities provoke change of each kind, and to understand what sequence of activities may help students to progress in understanding in ways that are optimal.

In this regard, one important question for future study has to do with the notion of *backward transfer* (Cook, 2003; Hohensee, 2014, 2016) which occurs when new learning reaches back to alter previously learned knowledge. Might children's new learning about fractions and integers provoke changes in the knowledge about natural numbers upon which it built? For example, might learning about numerical density for fractions change how learners view numerical density for whole numbers? Although there is only one *natural* number between 3 and 5, there are infinitely many *numbers*; could learning about numerical density for fractions compel children to reconsider their views about numerical density more broadly? As a second example, could learning about negative numbers compel changes in children's views of natural numbers, by highlighting a measurement view of number, rather than a discrete quantity view?

Further, if such backward transfer does occur, what mechanisms might underlie it? Restructuring provoked by new learning could yield novel insights into structures that were previously unanalyzed. For example, learning about place value in the context of multi-digit addition and subtraction (including "carrying" and "borrowing") might provoke insights into numerical representations such as number names, which might initially be understood in unanalyzed form. A child might begin to think of "fourteen" as one ten and four ones, rather than as fourteen countable entities. Properties of numerical language may also support (or prevent) insights into relevant structure, a point discussed by De Smedt (Chap. 6).

Whether change involves restructuring or elaboration may also depend on the nature of people's initial whole number concepts, which presumably depend on instructional experiences, both formal and informal. For example, a concept of whole numbers as places on number line might readily support extension to integers, whereas a concept of whole numbers as the result of a counting operation may be more difficult to extend. It may be that restructuring occurs only when a shift in conceptualization is required.

Different sorts of activities may also promote different sorts of changes in children's knowledge—either more elaborative, enrichment types of changes or more structural changes. For this reason, richer and more complete theories are needed that can both explain and guide the further design of activities to promote desired sorts of changes. Consider, for example, what sorts of activities might promote children's thinking about equipartitioning, about place value, or about the additive inverse. With greater knowledge about how different kinds of activities promote knowledge change, we will be in a better position to design learning trajectories that promote desired sorts of changes in children's thinking.

16.4 Multiple Concepts of Natural Numbers, Fractions, and Integers

Another theme that arises in many of the chapters in this volume is the idea that there are multiple ways to conceptualize numbers of different types. For example, natural numbers may be conceptualized as the result of a counting operation or as a place on a number line. Fractions may be conceptualized as parts of a whole, as measures, as ratios, or as proportions of a collection. Integers may be conceptualized in terms of cancellation, symmetry, or movement along a number line.

Operations also allow for multiple conceptualizations. For example, subtraction can be conceptualized in terms of "taking away" (e.g., If Jordan had 5 candies and gave 3 to Alyn, how many did Jordan have left?) or in terms of "comparing" (e.g., If Jordan has 5 candies and Alyn has 3 candies, how many more candies does Jordan have than Alyn?). Division can be conceptualized in terms of forming groups of a particular size (quotative division) or in terms of forming a particular number of groups (partitive division).

Given that there are multiple ways to conceptualize numbers and operations on numbers, it seems likely that the construction of number may involve both acquiring new conceptualizations for numbers or operations and linking distinct conceptualizations. Moreover, these learning experiences may also be viewed in terms of enrichment and structural change. With respect to fractions, Kieren (1980) described five subconstructs, including part-whole, measurement, ratio, quotient, and operator. Initial research on these subconstructs treated them as separate components necessary for a robust understanding of fractions (e.g., Behr, Lesh, Post, & Silver, 1983)—aligning with an enrichment model of conceptual change. More recent research has linked the subconstructs by demonstrating, for example, how children can reorganize mental actions that comprise their part-whole schemes to construct measurement schemes (e.g., Wilkins & Norton, 2018)—aligning more closely with a reorganization model of conceptual change.

This example suggests close connections between the three themes we have discussed thus far in this chapter: the role of activity in constructing number, different kinds of knowledge change, and multiple concepts of numbers. If students hold multiple concepts of various kinds of number, linking them might yield reorganizations in students' concept of number in general, and these linkages might also rely on activity. For instance, if students have available concepts of partitive and quotative division of whole numbers, we might ask what activities students can engage in to relate those concepts to one another and to apply them to ratios or fractions. Here, we see some progress in recent work (e.g., Beckmann & Izsák, 2015; Sidney & Alibali, 2017).

For each of the number types we have considered, activity can play a central role in establishing and linking differing concepts of mathematical entities and operations. As demonstrated within several of the chapters in this volume, teachers can promote concepts and linkages by engaging students in tasks that elicit relevant activity. For example, Wessman-Enzinger (Chap. 13) describes a task in which a student was asked to represent number sentences of the form $-a + \square = b$, where $|a| < |b|$, on a number line. The student used the number line to locate positions of the numbers, a and b, relative to 0 and then to sum up their distances from 0 ($a + b$). Thus, the task served as an opportunity for the student to link additive reasoning with whole numbers to a concept of integers as directed numbers, using 0 as a referent.

Beckmann (Chap. 11) characterizes rich, interconnected knowledge of multiple concepts of different number types as a form of expertise. With respect to fractions, she notes, "Experts may also be able to move easily, flexibly, and even subconsciously between viewing fractions as stuff, fractions as operating on stuff to produce stuff, and fractions as how much of 1 unit it takes to make some stuff, selecting a view that is appropriate for the purpose at hand." From this perspective, constructing such interconnected knowledge about different concepts of numbers is a case of the development of expertise. Like the development of expertise, it presumably has a protracted developmental course and requires extensive experience (see, e.g., Ericsson, Krampe, & Tesch-Römer, 1993; Macnamara, Hambrick, & Oswald, 2014).

Mathematical symbols can also have multiple interpretations, and students need to learn these different interpretations as well as the relations among them. As one case in point, Bofferding (Chap. 12) discusses multiple interpretations of the minus sign. The value inherent in having multiple interpretations is highlighted by Harel (Chap. 15). In his view, "The unavoidable difficulties associated with the multiple interpretations …. should be welcomed, not circumvented. It is through these difficulties that learners are likely to construct one of the most valuable—indeed essential—ways of thinking in mathematics: that symbols can have multiple interpretations and that it is advantageous to attribute different interpretations to symbols in the process of solving problems."

At the same time, Harel argues that there is also value in constructing a single, unified meaning. Focusing on the multiple interpretations of the minus sign discussed by Bofferding (Chap. 12), he argues that different interpretations can be subsumed into a unified meaning that centers on the concept of additive inverse. He makes a case for the intellectual value of constructing this unified meaning, and raises the question of how instruction should address this unified meaning.

To date, relatively little research has focused on how students acquire and build connections among different concepts of natural numbers, integers, and fractions, among different concepts of mathematical operations, or among different interpretations of mathematical symbols. Likewise, relatively little research has focused on how learners apply or shift among different concepts or interpretations as they attempt to solve problems. Future research is needed to address these issues, and to develop instructional activities to help students link multiple views of number, operations, and mathematical symbols.

16.5 Conceptual and Procedural Knowledge in the Construction of Number

Robust mathematical competence requires not only knowing *how* to solve problems, but also understanding of principles and relationships (Baroody & Dowker, 2003; Crooks & Alibali, 2014; Rittle-Johnson, Schneider, & Star, 2015). Although the interrelations among conceptual and procedural knowledge were not a primary focus of this volume, many of the chapters touch on this general issue in one way or another. In the realm of number, children must not only learn how to operate on different types of numbers (i.e., operations on natural numbers, fractions, and integers), but they must also learn the principles that underlie those procedures, so that they can flexibly apply and adapt procedures as needed for solving problems. They must learn about different conceptions of number and operations and how they relate to one another, as discussed in the previous section, and they must learn how different number types relate to one another.

Mathematical activities vary in their implications vis-à-vis knowledge of concepts and procedures. Indeed, activities are often designed to target one form of knowledge or the other; however, the distinction between the two forms of knowledge can be blurry, and it is often unclear what kinds of knowledge children actually use or activate for particular tasks. If children engage in similar tasks over and over again, their approach to such tasks can become routinized, and they may fail to activate conceptual knowledge when engaging in those tasks. On the other hand, children may choose to reflect about procedures as they implement them, even for tasks that are highly routine, and so tasks that seem procedural at face value may also activate conceptual knowledge, at least for some children. Context matters, as well; for example, children show differential gains in conceptual knowledge from a lesson about division by a fraction, depending on whether the lesson was preceded by a warm-up task about whole number division or one about fraction addition and subtraction (Sidney & Alibali, 2015)—activities which provide different opportunities for structure mapping. Thus, activities may engage conceptual and procedural aspects of knowledge to different degrees, for different learners, at different points in time and in different contexts.

Many mathematical activities involve solving problems, which can involve applying, adapting, or generating procedures of various sorts. Carrying out procedures creates opportunities for noticing regularities, via mechanisms of statistical learning like those described by Mix, Smith, and Crespo (Chap. 5). Regularities may vary for different number types; for example, when adding natural numbers, the sum is always greater than either of the addends, but this is not the case when adding integers (see Prather & Alibali, 2008, for discussion). Noticing such regularities (and considering their scope of application) may provide the basis for inferring relevant concepts. Further, noticing cases in which regularities no longer apply may provoke reflection or spark reorganization.

More generally, the chapters in this volume highlight the many ways in which activities may engage and influence children's knowledge of concepts and procedures. A deeper understanding of the knowledge that children draw on as they engage in different sorts of activities could shed light on mechanisms of knowledge change. Indeed, the chapters in this volume point toward many potentially fruitful avenues for future research on the role of activity in fostering and connecting conceptual and procedural knowledge.

16.6 Bringing Together Psychology and Mathematics Education

A principal aim of this volume was to bring together research on fundamental number concepts from a range of disciplinary perspectives. In framing this volume and in synthesizing across chapters, we have focused in particular on activity, because we view it as a meeting point for mathematics educators and psychologists. Activity is involved in the mechanisms of development, and indeed, many educational practices call for or implement specific forms of activity. Thus, a focus on activity brings together the theoretical and the practical, the basic and the applied. As discussed earlier in this chapter, a focus on activity also highlights considerations of physical versus mental, and individual versus social.

In this volume, we sought to give voice to scholars who work at or near the intersections of psychology and education, to highlight common foci and to encourage both authors and readers to present their work to a broad, multi-disciplinary audience. We recognize that scholars' disciplinary roots "show" in their choices of questions and research methods, in their views of the nature of mathematics, in the kinds of data that they consider and present, and even in their stylistic practices and authorship conventions (Norton & Nurnberger-Haag, 2018). At the same time, we have been pleased to see genuine engagement with and consideration of diverse points of view among the chapter authors.

Embedded within many of the chapters (in more or less explicit form) are ideas and bits of advice from scholars within one discipline for scholars in other disciplines. For example, Ulrich and Norton (Chap. 4) call for more precise definitions of magnitude and number. McMullen and colleagues (Chap. 3) call for an appreciation of children's spontaneous numerical activity (and individual differences therein). Tzur (Chap. 8) calls for a more charitable view of the natural number "bias" in fraction reasoning, encouraging educators (and scholars) to embrace natural number reasoning as a starting point for reorganization. Matthews and Ziols (Chap. 10) call for recognition of non-symbolic ratio processing as a potential basis for fractional reasoning. De Smedt (Chap. 6) calls for greater attention to individual differences in mathematical thinking, and for a greater appreciation of the bidirectional relations between neurocognitive abilities and learning of mathematical content.

It is our hope that this volume will contribute to breaking down some of the "silos" that separate researchers in different disciplines. For this to happen, however, scholars need to appreciate the value of research that uses methods that differ from the ones most prevalent in their own disciplines. Such "methodological openness" is an important foundation for successful collaboration across fields (Alibali & Knuth, 2018). Even for scholars who do not wish to directly collaborate across disciplines, such openness is critical for gaining a deep knowledge of the relevant literature—because there are multiple informative perspectives on every research issue and on every important real-world problem. In our view, scholars from psychology, mathematics education, and neuroscience (as well as other fields) have different and valuable perspectives on development, learning, and teaching. Productive discussions can be had, even when scholars do not agree on what methods are most suitable or even appropriate for answering research questions.

Our aim in this volume was to bring together research from a range of methodological and disciplinary perspectives on how children construct number. Across chapters, this body of work highlights the importance of the foundational knowledge and systems upon which children build numerical knowledge. It also highlights the importance of children's activity, including both sensorimotor and mental activity, which takes place in contexts with different forms of informal and formal social support. It is our hope that this effort to bring together distinct perspectives will spark future collaborations that yield a fuller account of the similar and diverse ways in which children construct natural numbers, fractions, and integers.

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