Chapter 9
Semilinear Fractional Evolution Equations

9.1 Introduction

Let $\alpha \in (0, 1]$. The main objective of this chapter consists of acquainting the reader with the fast-growing theory of fractional evolution equations. More precisely, we study sufficient conditions for the existence of classical (respectively, mild) solutions for the inhomogeneous fractional Cauchy problem

$$\begin{cases} \mathbb{D}_t^\alpha u(t) = Au(t) + f(t), & t > 0 \\ u(0) = u_0 \in \mathbb{X} \end{cases}$$

and its corresponding semilinear evolution equation

$$\begin{cases} \mathbb{D}_t^\alpha u(t) + Au(t) = F(t, u(t)), & t > 0, \\ u(0) = u_0 \in \mathbb{X}, \end{cases}$$

where $\mathbb{D}_t^\alpha$ is the fractional derivative of order α in the sense of Caputo, $A : D(A) \subset \mathbb{X} \mapsto \mathbb{X}$ is a closed linear operator on a complex Banach space $\mathbb{X}$ (respectively, $A \in \Sigma_\omega^\gamma(\mathbb{X})$ where $\gamma \in (-1, 0)$ and $0 < \omega < \frac{\pi}{2}$), and $f : [0, \infty) \mapsto \mathbb{X}$ and $F : [0, \infty) \times \mathbb{X} \mapsto \mathbb{X}$ are continuous functions satisfying some additional conditions. Under some appropriate assumptions, various existence results are discussed.

The main tools utilized to establish the existence of classical (respectively, mild) solutions to the above-mentioned fractional evolutions are the so-called $(\alpha, \alpha)^\beta$-resolvent families $S_\alpha^\beta(\cdot)$ and almost sectorial operators. Additional details on these classes of operators can be found in Keyantuo et al. [75] and Wang et al. [110]. For additional readings upon the topics discussed in this chapter, we refer to [6, 21–23, 37, 42], etc.

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9.2 Fractional Calculus

Let $(\mathbb{X}, \|\cdot\|)$ be a complex Banach space. If $J \subset \mathbb{R}$ is an interval and if $(V, \|\cdot\|) \subset \mathbb{X}$ is a (normed) subspace, then $C(J; V)$ (respectively, $C^{(k)}(I; V)$ for $k \in \mathbb{N}$) will denote the collection of all continuous functions from J into V (respectively, the collection of all functions of class C^k which go from J into V).

Definition 9.1 If $f : \mathbb{R}_+ \mapsto \mathbb{R}$ and $g : \mathbb{R}_+ \mapsto \mathbb{X}$ are functions, we define their convolution, if it exists, as follows:

$$(f * g)(t) := \int_0^t f(t-s)g(s)ds, \quad t \geq 0.$$

Definition 9.2 If $u : \mathbb{R}_+ \mapsto \mathbb{X}$ is a function, then its Riemann–Liouville fractional derivative of order β is defined by

$$D_t^\beta u(t) := \frac{d^n}{dt^n}\Big[\int_0^t g_{n-\beta}(t-s)u(s)ds\Big], \qquad t > 0$$

where $n := \lceil \beta \rceil$ is the smallest integer greatest than or equal to β, and

$$g_\beta(t) := \frac{t^{\beta-1}}{\Gamma(\beta)}, \qquad t > 0, \quad \beta > 0,$$

with $g_0 = \delta_0$ (the Dirac measure concentrated at 0).

Note that $g_{\alpha+\beta} = g_\alpha * g_\beta$ for all $\alpha, \beta \geq 0$.

Definition 9.3 The Caputo fractional derivative of order $\beta > 0$ of a function $u : \mathbb{R}_+ \mapsto \mathbb{X}$ is defined by

$$\mathbb{D}_t^\beta u(t) := D_t^{n-\beta} u^{(n)}(t) = \int_0^t g_{n-\beta}(t-s)u^{(n)}(s)ds,$$

where $n := \lceil \beta \rceil$.

We have the following additional relationship between Riemann–Liouville and Caputo fractional derivatives:

$$\mathbb{D}_t^\beta f(t) = D_t^\beta \Big(f(t) - \sum_{k=0}^{n-1} f^{(k)}(0) g_{k+1}(t) \Big), \quad t > 0,$$

where $n := \lceil \beta \rceil$.

Definition 9.4 If $f : \mathbb{R}_+ \mapsto \mathbb{X}$ is integrable, then its Laplace transform is defined by

$$(\mathbb{L}f)(z) = \widehat{f}(z) := \int_0^\infty e^{-zt} f(t)dt$$

provided this integral converges absolutely for some $z \in \mathbb{C}$.

Among other things, if $\lceil\alpha\rceil = n \geq 1$, then

$$\widehat{\mathbb{D}_t^\alpha f}(z) = z^\alpha \widehat{f}(z) - \sum_{k=0}^{n-1} z^{\alpha-k-1} f^{(k)}(0)$$

and

$$\widehat{D_t^\alpha f}(z) = z^\alpha \widehat{f}(z) - \sum_{k=0}^{n-1} (g_{n-\alpha} * f)^{(k)}(0) z^{n-1-k}$$

where z^α is uniquely defined as $z^\alpha = |z|^\alpha e^{i \arg z}$ with $-\pi < \arg z < \pi$.

Let $k \in \mathbb{N}$. If $u \in C^{k-1}(\mathbb{R}_+; \mathbb{X})$ and $v \in C^k(\mathbb{R}_+; \mathbb{X})$, then for every $t \geq 0$,

$$\begin{aligned} \frac{d^k}{dt^k}[(u * v)(t)] &= \sum_{j=0}^{k-1} u^{(k-1-j)}(t) v^{(j)}(0) + (u * v^{(k)})(t) \\ &= \sum_{j=0}^{k-1} \frac{d^{k-1}}{dt^{k-1}} \left[(g_j * u)(t) v^{(j)}(0) \right] + (u * v^{(k)})(t). \end{aligned} \tag{9.1}$$

Define the generalized Mittag–Leffler special function $E_{\alpha,\beta}$ by

$$\begin{aligned} E_{\alpha,\beta}(z) &= \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + \beta)} \\ &= \frac{1}{2\pi i} \int_\Gamma \frac{\lambda^{\alpha-\beta} e^\lambda}{\lambda^\alpha - z} d\lambda, \quad \alpha,\ \beta > 0, z \in \mathbb{C}, \end{aligned}$$

where Γ is a contour which starts and ends at $-\infty$ and encircles the disc

$$|\lambda| \leq |z|^{1/\alpha}$$

counter-clockwise.

In what follows, we set

$$E_\alpha(z) := E_{\alpha,1}(z),$$

and

$$e_\alpha(z) := E_{\alpha,\alpha}(z).$$

9.3 Inhomogeneous Fractional Differential Equations

9.3.1 Introduction

Let $\alpha \in (0, 1]$. In this section, we study the existence of classical (respectively, mild) solutions for the inhomogeneous fractional Cauchy problem

$$\begin{cases} \mathbb{D}_t^{\alpha} u(t) = Au(t) + f(t) \\ u(0) = u_0 \in \mathbb{X} \end{cases} \tag{9.2}$$

where $\mathbb{D}_t^{\alpha}$ is the fractional derivative of order α in the sense of Caputo, $A : D(A) \subset \mathbb{X} \mapsto \mathbb{X}$ is a closed linear operator on a complex Banach space $\mathbb{X}$, and $f : \mathbb{R}_+ \mapsto \mathbb{X}$ is a function satisfying some additional conditions.

As it was pointed out in Keyantuo et al. [75], Caputo fractional derivative is more appropriate for equations of the form Eq. (9.2) than the Riemann–Liouville fractional derivative. Indeed, Caputo fractional derivative requires that the solution u of the above Cauchy problem be known at $t = 0$ while that of Riemann–Liouville requires that it be known in a right neighborhood of $t = 0$.

Recall that if $\alpha = 1$, then there are two situations that can be considered. If A is the infinitesimal generator of a strongly continuous semi-group, then semi-group techniques can be used to establish the existence of solutions to Eq. (9.2). Now, if A is not the infinitesimal generator of a strongly continuous semi-group, the concept of exponentially bounded β-times integrated semi-groups can be utilized to deal with existence of solutions to the above Cauchy problem. Similarly, if $\alpha \in (0, 1)$, a family of strongly continuous linear operators $S_\alpha : \mathbb{R}_+ \mapsto B(\mathbb{X})$ can be used to establish the existence of solutions to the above Cauchy problem. Unfortunately, the previous concept is inappropriate for some important practical problems, see details in Keyantuo et al. [75]. This in fact is one of the main reasons that led Keyantuo et al. to introduce the concept of $(\alpha, \alpha)^{\beta}$-resolvent families (respectively, $(\alpha, 1)^{\beta}$-resolvent families), which generalizes naturally all the above-mentioned cases. Such a new concept will play a central role in this section.

9.3.2 Basic Definitions

Definition 9.5 ([75]) Let $A : D(A) \subset \mathbb{X} \mapsto \mathbb{X}$ be a closed linear operator and let $\alpha \in (0, 1]$ and $\beta \geq 0$. The operator A is called an $(\alpha, \alpha)^{\beta}$-resolvent family if there exist $\omega \geq 0$, $M \geq 0$, and a family of strongly continuous functions $T_\alpha^\beta : [0, \infty) \mapsto B(\mathbb{X})$ (respectively, $T_\alpha^\beta : (0, \infty) \mapsto B(\mathbb{X})$ in the case when $\alpha(1 + \beta) < 1$) such that,

i) $\left\| (g_1 * T_\alpha^\beta)(t) \right\| \leq M e^{\omega t}$ for all $t > 0$;

ii) $\left\{\lambda^{\alpha} : \Re e\,\lambda > \omega\right\} \subset \rho(A)$; and

$$(\lambda^{\alpha} I - A)^{-1} u = \lambda^{\alpha\beta} \int_0^{\infty} e^{-\lambda t} T_{\alpha}^{\beta}(t) u\, dt, \quad \Re e\,\lambda > \omega, \quad u \in \mathbb{X}.$$

Definition 9.6 ([75]) Let $A : D(A) \subset \mathbb{X} \mapsto \mathbb{X}$ be a closed linear operator and let $\alpha \in (0, 1]$ and $\beta \geq 0$. The operator A is called an $(\alpha, 1)^{\beta}$-resolvent family generator if there exist $\omega \geq 0$, $M \geq 0$, and a family of strongly continuous functions $S_{\alpha}^{\beta} : \mathbb{R}_{+} \mapsto B(\mathbb{X})$ such that,

i) $\left\|(g_1 * S_{\alpha}^{\beta})(t)\right\| \leq M e^{\omega t}$ for $t \geq 0$;
ii) $\left\{\lambda^{\alpha} : \Re e \lambda > \omega\right\} \subset \rho(A)$; and

$$\lambda^{\alpha - 1} (\lambda^{\alpha} I - A)^{-1} u = \lambda^{\alpha\beta} \int_0^{\infty} e^{-\lambda t} S_{\alpha}^{\beta}(t) u\, dt, \quad \Re e\,\lambda > \omega, \quad u \in \mathbb{X}.$$

Remark 9.7 A family of strongly continuous functions $T_{\alpha}^{\beta}(t)$ that satisfies items i)–ii) of Definition 9.5 is called the $(\alpha, \alpha)^{\beta}$-resolvent family generated by the linear operator A. And there is uniqueness of the $(\alpha, \alpha)^{\beta}$-resolvent family (respectively, $(\alpha, 1)^{\beta}$-resolvent family) associated with a given operator A.

In fact, there is a relationship between these two new notions. It is not hard to show that if A generates an $(\alpha, \alpha)^{\beta}$-resolvent family T_{α}^{β}, then it generates an $(\alpha, 1)^{\beta}$-resolvent family S_{α}^{β} (see [75] for details) and that both T_{α}^{β} and S_{α}^{β} are linked through the following identity,

$$S_{\alpha}^{\beta}(t)x = (g_{1-\alpha} * T_{\alpha}^{\beta})(t)x, \quad t \geq 0, \quad x \in \mathbb{X}.$$

Let us now collect a few additional properties of both $(\alpha, 1)^{\beta}$- and $(\alpha, \alpha)^{\beta}$-resolvent families.

Proposition 9.8 ([75]) *Let $A : D(A) \subset \mathbb{X} \mapsto \mathbb{X}$ be a closed linear operator and let $\alpha \in (0, 1]$ and $\beta \geq 0$. If A generates an $(\alpha, 1)^{\beta}$-resolvent family S_{α}^{β}, then the following hold,*

i) $S_{\alpha}^{\beta}(t)(D(A)) \subset D(A)$ *and*

$$A S_{\alpha}^{\beta}(t)x = S_{\alpha}^{\beta}(t) A x$$

for all $x \in D(A)$ and $t \geq 0$.
ii) For all $x \in D(A)$,

$$S_{\alpha}^{\beta}(t)x = g_{\alpha\beta+1}(t)x + \int_0^t g_{\alpha}(t-s) A S_{\alpha}^{\beta}(s)x ds, \quad t \geq 0.$$

iii) For all $x \in \mathbb{X}$, $(g_\alpha * S_\alpha^\beta)(t)x \in D(A)$,

$$S_\alpha^\beta(t)x = g_{\alpha\beta+1}(t)x + A\int_0^t g_\alpha(t-s)S_\alpha^\beta(s)xds, \quad t \geq 0.$$

iv) $S_\alpha^\beta(0) = g_{\alpha\beta+1}(0)$*;* $S_\alpha^\beta(0) = I$ *if* $\beta = 0$ *and* $S_\alpha^\beta(0) = 0$ *if* $\beta > 0$.

Proposition 9.9 ([75]) *Let* $A : D(A) \subset \mathbb{X} \mapsto \mathbb{X}$ *be a closed linear operator and let* $\alpha \in (0, 1]$ *and* $\beta \geq 0$*. If A generates an* $(\alpha,\alpha)^\beta$*-resolvent family* T_α^β*, then the following hold,*

i) $T_\alpha^\beta(t)(D(A)) \subset D(A)$ *and*

$$AT_\alpha^\beta(t)x = T_\alpha^\beta(t)Ax$$

for all $x \in D(A)$ *and* $t > 0$.

ii) For all $x \in D(A)$,

$$T_\alpha^\beta(t)x = g_{\alpha(\beta+1)}(t)x + \int_0^t g_\alpha(t-s)AT_\alpha^\beta(s)xds, \quad t \geq 0.$$

iii) For all $x \in \mathbb{X}$, $(g_\alpha * T_\alpha^\beta)(t)x \in D(A)$,

$$T_\alpha^\beta(t)x = g_{\alpha(\beta+1)}(t)x + A\int_0^t g_\alpha(t-s)T_\alpha^\beta(s)xds, \quad t > 0.$$

iv) If $\beta > 0$*, then for every* $x \in \overline{D(A)}$,

$$\frac{1}{\Gamma(\alpha(1+\beta))}\lim_{t\to 0} t^{1-\alpha(1+\beta)}T_\alpha^\beta(t)x = x$$

if $\alpha(1+\beta) < 1$*;* $T_\alpha^\beta(0)x = x$ *if* $\alpha(1+\beta) = 1$*; and* $T_\alpha^\beta(0)x = 0$ *if* $\alpha(1+\beta) > 1$.

v) If $\alpha(1+\beta) > 1$*, then all the above equalities occur for* $t \geq 0$. .

A strongly continuous function $h : [0,\infty) \mapsto \mathbb{X}$ is called exponentially bounded if there exist constants $M, \omega \geq 0$ such that

$$\|h(t)\| \leq Me^{\omega t}$$

for all $t > 0$. In particular, an $(\alpha,\alpha)^\beta$-resolvent family T_α^β (respectively, $(\alpha, 1)^\beta$-resolvent family S_α^β) is exponentially bounded, there exist constants $M, \omega \geq 0$ such that

$$\|T_\alpha^\beta(t)\| \leq Me^{\omega t}$$

for all $t > 0$ (respectively, there exist constants $M', \omega' \geq 0$ such that

$$\|S_\alpha^\beta(t)\| \leq M' e^{\omega' t}$$

for all $t \geq 0$).

For more on $(\alpha, \alpha)^\beta$-resolvent families (respectively, $(\alpha, 1)^\beta$-resolvent families), we refer the reader to [75].

9.3.3 Existence of Classical and Mild Solutions

Definition 9.10 ([75]) A continuous function A function $u : [0, \infty) \mapsto D(A)$ is said to be a classical solution to Eq. (9.2) if $g_{1-\alpha} * (u - u(0)) : [0, \infty) \mapsto \mathbb{X}$ is a continuous function and Eq. (9.2) holds.

Definition 9.11 ([75]) A continuous function $u : [0, \infty) \mapsto \mathbb{X}$ is said to be a mild solution to Eq. (9.2) if $(g_\alpha * u)(t) \in D(A)$ for all $t \geq 0$ and

$$u(t) = u_0 + A\int_0^t g_\alpha(t-s)u(s)ds + \int_0^t g_\alpha(t-s)f(s)ds, \quad t \geq 0. \tag{9.3}$$

Theorem 9.12 ([75]) *Let $\alpha \in (0, 1]$ and $\beta \geq 0$ and set $n = \lceil\beta\rceil$ and $k = \lceil\alpha\beta\rceil$. Suppose that A is the generator of an $(\alpha, 1)^\beta$-resolvent family S_α^β. Then the following hold,*

i) For every $f \in C^{(k+1)}(\mathbb{R}_+; \mathbb{X})$, $f^{(l)}(0) \in D(A^{n+1-l})$ for $l = 0, 1, \ldots, k$, $\mathbb{D}_t^{\alpha\beta} f$ is exponentially bounded and $u_0 \in D(A^{n+1})$, Eq. (9.2) has a unique classical solution given by

$$u(t) = D_t^{\alpha\beta} S_\alpha^\beta(t)u_0 + D_t^{\alpha\beta}\mathbb{D}_t^{1-\alpha}(S_\alpha^\beta * f)(t), \quad t \geq 0. \tag{9.4}$$

ii) For every $f \in C^{(k)}(\mathbb{R}_+; \mathbb{X})$, $f^{(l)}(0) \in D(A^{n-l})$ for $l = 0, 1, \ldots, k-1$, $\mathbb{D}_t^{\alpha\beta} f$ is exponentially bounded and $u_0 \in D(A^n)$, Eq. (9.2) has a unique mild solution given by Eq. (9.4).

Corollary 9.13 ([75]) *Let $\alpha \in (0, 1]$ and $\beta \geq 0$ and set $n = \lceil\beta\rceil$ and $k = \lceil\alpha\beta\rceil$. Suppose that A generates an $(\alpha, \alpha)^\beta$-resolvent family T_α^β. And let S_α^β be the $(\alpha, 1)^\beta$-resolvent family generated by A. Then the following hold:*

(a) For every $f \in C^{(k+1)}(\mathbb{R}_+; \mathbb{X})$, $f^{(j)}(0) \in D(A^{n+1-j})$ for $j = 0, 1, \ldots, k$, $\mathbb{D}_t^{\alpha\beta} f$ is exponentially bounded, and for every $u_0 \in D(A^{n+1})$, the unique classical solution to Eq. (9.2) is given by

$$u(t) = D_t^{\alpha\beta}\Big[S_\alpha^\beta(t)u_0 + \int_0^t T_\alpha^\beta(t-s)f(s)ds\Big], \quad t \geq 0. \tag{9.5}$$

(b) *For every* $f \in C^{(k)}(\mathbb{R}_+; \mathbb{X})$, $f^{(j)}(0) \in D(A^{n-j})$ *for* $j = 0, 1, \ldots, k-1$, $\mathbb{D}_t^{\alpha\beta} f$ *is exponentially bounded and for every* $u_0 \in D(A^n)$, *the unique mild solution to Eq. (9.2) is given by Eq. (9.5).*

9.4 Semilinear Fractional Differential Equations

9.4.1 Preliminaries and Notations

Let $\gamma \in (-1, 0)$ and let S_μ^0 (with $0 < \mu < \pi$) be the open sector defined by

$$\{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \mu\}$$

and let S_μ be its closure, that is,

$$S_\mu := \{z \in \mathbb{C} \setminus \{0\} : |\arg z| \leq \mu\} \cup \{0\}.$$

Definition 9.14 ([110]) Let $\gamma \in (-1, 0)$ and let $0 < \omega < \pi/2$. The set $\Sigma_\omega^\gamma(\mathbb{X})$ stands for the collection of all closed linear operators $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ satisfying

i) $\sigma(A) \subset S_\omega$; and
ii) for every $\omega < \mu < \pi$ there exists a constant C_μ such that

$$\|(zI - A)^{-1}\| \leq C_\mu |z|^\gamma \tag{9.6}$$

for all $z \in \mathbb{C} \setminus S_\mu$.

Definition 9.15 A linear operator $A : D(A) \subset \mathbb{X} \mapsto \mathbb{X}$ that belongs to $\Sigma_\omega^\gamma(\mathbb{X})$ will be called an almost sectorial operator on $\mathbb{X}$.

Among other things, recall that if $A \in \Sigma_\omega^\gamma(\mathbb{X})$, then $0 \in \rho(A)$. Further, there exist almost sectorial operators which are not sectorial, see, e.g., [109]. There are many examples of almost sectorial operators in the literature, see, e.g., Wang et al. [110].

Let $\alpha \in (0, 1)$. Our main objective in this section consists of studying the existence of solutions to the following semilinear fractional differential equations

$$\begin{cases} \mathbb{D}_t^\alpha u(t) + Au(t) = f(t, u(t)), & t > 0, \\ u(0) = u_0 \in \mathbb{X}, \end{cases} \tag{9.7}$$

where $\mathbb{D}_t^\alpha$ is the Caputo fractional derivative of order α, $A \in \Sigma_\omega^\gamma(\mathbb{X})$ with $0 < \omega < \frac{\pi}{2}$, and $f : [0, \infty) \times \mathbb{X} \mapsto \mathbb{X}$ is a jointly continuous function.

Suppose that $A \in \Sigma_\omega^\gamma(\mathbb{X})$ such $-1 < \gamma < 0$ and $0 < \omega < \pi/2$. Define operator families $\{\mathscr{S}_\alpha(t)\}|_{t\in S^0_{\frac{\pi}{2}-\omega}}$, $\{\mathscr{T}_\alpha(t)\}|_{t\in S^0_{\frac{\pi}{2}-\omega}}$ by

$$\mathscr{S}_\alpha(t) := E_\alpha(-zt^\alpha)(A) = \frac{1}{2\pi i}\int_{\Gamma_\theta} E_\alpha(-zt^\alpha)(zI-A)^{-1}dz,$$
$$\mathscr{T}_\alpha(t) := e_\alpha(-zt^\alpha)(A) = \frac{1}{2\pi i}\int_{\Gamma_\theta} e_\alpha(-zt^\alpha)(zI-A)^{-1}dz,$$

where the integral contour $\Gamma_\theta := \{\mathbb{R}_+e^{i\theta}\}\cup\{\mathbb{R}_+e^{-i\theta}\}$ is oriented counter-clockwise and $\omega < \theta < \mu < \frac{\pi}{2} - |\arg t|$.

We have

Theorem 9.16 ([110]) *For each fixed $t \in S^0_{\frac{\pi}{2}-\omega}$, $\mathscr{S}_\alpha(t)$ and $\mathscr{T}_\alpha(t)$ are linear and bounded operators on $\mathbb{X}$. Moreover, there exist constants $C_s = C(\alpha,\gamma) > 0$, $C_p = C(\alpha,\gamma) > 0$ such that for all $t > 0$,*

$$\|\mathscr{S}_\alpha(t)\| \le C_s t^{-\alpha(1+\gamma)}, \quad \|\mathscr{T}_\alpha(t)\| \le C_p t^{-\alpha(1+\gamma)}. \tag{9.8}$$

9.4.2 Existence Results

Definition 9.17 ([110]) A continuous function $u : (0,T] \mapsto \mathbb{X}$ is called a mild solution to Eq. (9.7) if it satisfies,

$$u(t) = \mathscr{S}_\alpha(t)u_0 + \int_0^t (t-s)^{\alpha-1}\mathscr{T}_\alpha(t-s)f(s,u(s))ds$$

for all $t \in (0,T]$.

Theorem 9.18 ([110]) *Let $A \in \Sigma_\omega^\gamma(\mathbb{X})$ such that $-1 < \gamma < -\frac{1}{2}$ and $0 < \omega < \frac{\pi}{2}$. Suppose that $f : (0,T]\times\mathbb{X} \to \mathbb{X}$ is continuous with respect to t and that there exist constants $M, N > 0$ such that*

$$\|f(t,x) - f(t,y)\| \le M(1 + \|x\|^{\nu-1} + \|y\|^{\nu-1})\|x-y\|,$$
$$\|f(t,x)\| \le N(1+\|x\|^\nu),$$

for all $t \in (0,T]$ and for each $x, y \in \mathbb{X}$, where ν is a constant in $[1, -\frac{\gamma}{1+\gamma})$. Then, for every $u_0 \in \mathbb{X}$, there exists a $T_0 > 0$ such that Eq. (9.7) has a unique mild solution defined on $(0, T_0]$.

Proof Fix $r > 0$ and consider the metric space $(F_r(T, u_0), \rho_T)$ where

$$F_r(T, u_0) = \Big\{u \in C((0, T]; \mathbb{X}) : \quad \rho_T(u, \mathscr{S}_\alpha(t)u_0) \le r\Big\},$$

$$\rho_T(u_1, u_2) = \sup_{t\in(0,T]} \|u_1(t) - u_2(t)\|.$$

It can be shown that it is not difficult to see that the metric space $(F_r(T, u_0), \rho_T)$ is complete.

Now for all $u \in F_r(T, u_0)$,

$$\|s^{\alpha(1+\gamma)}u(s)\| \le s^{\alpha(1+\gamma)}\|u - \mathscr{S}_\alpha(t)u_0\| + s^{\alpha(1+\gamma)}\|\mathscr{S}_\alpha(t)u_0\| \le L.$$

where $L := T^{\alpha(1+\gamma)}r + C_s\|u_0\|$.

Let $T_0 \in (0, T]$ such that

$$C_pN\frac{T_0^{-\alpha\gamma}}{-\alpha\gamma} + C_pNL^\nu T_0^{-\alpha(\nu(1+\gamma)+\gamma)}\beta(-\gamma\alpha, 1 - \nu\alpha(1 + \gamma)) \le r, \tag{9.9}$$

$$MC_p\frac{T_0^{-\alpha\gamma}}{-\alpha\gamma} + 2L^{\rho-1}T_0^{-\alpha(\gamma+(1+\gamma)(\nu-1))}\beta(-\alpha\gamma, 1 - \alpha(1 + \gamma)(\nu - 1)) \le \frac{1}{2}, \tag{9.10}$$

where $\beta(\eta_1, \eta_2)$ with $\eta_i > 0, i = 1, 2$ stands for the usual Beta function.

Suppose $u_0 \in \mathbb{X}$ and consider the mapping Γ^α given by

$$(\Gamma^\alpha u)(t) = \mathscr{S}_\alpha(t)u_0 + \int_0^t (t - s)^{\alpha-1}\mathscr{P}_\alpha(t - s)f(s, u(s))ds, \quad u \in F_r(T_0, u_0).$$

From the assumptions upon f, Theorem 9.16, and [110, Theorem 3.2], we deduce that $(\Gamma^\alpha u)(t) \in C((0, T]; \mathbb{X})$ and

$$\begin{aligned}
&\|(\Gamma^\alpha u)(t) - \mathscr{S}_\alpha(t)u_0\| \\
&\le C_pN\int_0^t (t - s)^{-\alpha\gamma-1}(1 + \|u(s)\|^\nu)ds \\
&\le C_pN\frac{T_0^{-\alpha\gamma}}{-\alpha\gamma} + \int_0^t C_pNL^\nu(t - s)^{-\alpha\gamma-1}s^{-\nu\alpha(1+\gamma)}ds \\
&\le C_pN\frac{T_0^{-\alpha\gamma}}{-\alpha\gamma} + C_pNL^\nu T_0^{-\alpha(\nu(1+\gamma)+\gamma)}\beta(-\gamma\alpha, 1 - \nu\alpha(1 + \gamma)) \\
&\le r,
\end{aligned}$$

by using Eq. (9.9).

In view of the above, one can see that Γ^α maps $F_r(T_0, u_0)$ into itself.

Now for all $u, v \in F_r(T_0, u_0)$, using the assumptions upon f and Theorem 9.16 we deduce that

$$\begin{aligned}
&\|(\Gamma^\alpha u)(t) - (\Gamma^\alpha v)(t)\| \\
&\quad \le C_p M \int_0^t (t-s)^{-\alpha\gamma-1}(1 + \|u(s)\|^{\rho-1} + \|v(s)\|^{\rho-1})\|u(s) - v(s)\| ds \\
&\quad \le C_p M \rho_t(u, v) \int_0^t (t-s)^{-\alpha\gamma-1}(1 + 2L^{\nu-1}s^{-\alpha(\nu-1)(1+\gamma)}) ds \\
&\quad \le 2L^{\rho-1}T_0^{-\alpha(\gamma+(1+\gamma)(\nu-1))}\beta(-\alpha\gamma, 1-\alpha(1+\gamma)(\nu-1))\rho_{T_0}(u, v) \\
&\qquad + MC_p \frac{T_0^{-\alpha\gamma}}{-\alpha\gamma}\rho_{T_0}(u, v).
\end{aligned}$$

Using (9.10), one can easily see that Γ^α is a strict contraction on $F_r(T_0, u_0)$ and so Γ_α has a unique fixed point $u \in F_r(T_0, u_0)$ which, by the Banach Fixed Point Theorem, is the only mild solution to Eq. (9.7) on $(0, T_0]$.

It can be shown that $\mathbb{X}^1 = D(A)$ equipped with the norm defined by $\|x\|_{\mathbb{X}^1} = \|Ax\|$ for all $x \in \mathbb{X}^1$, is a Banach space.

Theorem 9.19 ([110]) *Let $A \in \Theta_\omega^\gamma(\mathbb{X})$ with $-1 < \gamma < -\frac{1}{2}$, $0 < \omega < \frac{\pi}{2}$ and $u_0 \in \mathbb{X}^1$. Suppose there exists a continuous function $M_f(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ and a constant $N_f > 0$ such that the mapping $f : (0, T] \times \mathbb{X}^1 \to \mathbb{X}^1$ satisfies*

$$\begin{aligned}
&\|f(t, x) - f(t, y)\|_{\mathbb{X}^1} \le M_f(r)\|x - y\|_{\mathbb{X}^1}, \\
&\|f(t, \mathscr{S}_\alpha(t)u_0)\|_{\mathbb{X}^1} \le N_f(1 + t^{-\alpha(1+\gamma)}\|u_0\|_{\mathbb{X}^1}),
\end{aligned}$$

for all $0 < t \le T$ and for all $x, y \in \mathbb{X}^1$ satisfying

$$\sup_{t\in(0,T]} \|x(t) - \mathscr{S}_\alpha(t)u_0\|_{\mathbb{X}^1} \le r, \quad \sup_{t\in(0,T]} \|y(t) - \mathscr{S}_\alpha(t)u_0\|_{\mathbb{X}^1} \le r.$$

Then there exists a $T_0 > 0$ such that Eq. (9.7) has a unique mild solution defined on $(0, T_0]$.

Proof Fix $u_0 \in \mathbb{X}^1$ and $r > 0$ and consider

$$F_r''(T, u_0) = \{u \in C((0, T]; \mathbb{X}^1);\ \sup_{t\in(0,T]} \|u - \mathscr{S}_\alpha(t)u_0\|_{\mathbb{X}^1} \le r\}.$$

For any $u \in F_r''(T, u_0)$, using the assumptions upon f and Theorem 9.16, we obtain

$$
\begin{aligned}
&\|(\Gamma^\alpha u)(t) - \mathscr{S}_\alpha(t)u_0\|_{\mathbb{X}^1} \\
&\le \int_0^t (t-s)^{\alpha-1}\|\mathscr{P}_\alpha(t-s)\|||f(s,u(s)) - f(s,\mathscr{S}_\alpha(t)u_0)||_{\mathbb{X}^1} ds \\
&\quad + \int_0^t (t-s)^{\alpha-1}\|\mathscr{P}_\alpha(t-s)\|||f(s,\mathscr{S}_\alpha(t)u_0))||_{\mathbb{X}^1} ds \\
&\le C_p \int_0^t (t-s)^{-\alpha\gamma-1}(M_f(r)r + N_f + N_f s^{-\alpha(1+\gamma)}\|u_0\|)ds \\
&\le C_p(M_f(r)r + N_f)\frac{T^{-\alpha\gamma}}{-\alpha\gamma} + C_p N_f T^{-\alpha(1+2\gamma)}\beta(-\gamma\alpha, 1-\alpha(1+\gamma))\|u_0\|.
\end{aligned}
$$

In view of the above and ideas from the proof of Theorem 9.18, we obtain the desired result.

9.5 Exercises

1. Show that if A generates an $(\alpha,\alpha)^\beta$-resolvent family T_α^β, then it generates an $(\alpha,1)^\beta$-resolvent family S_α^β. Further,

$$S_\alpha^\beta(t)x = (g_{1-\alpha} * T_\alpha^\beta)(t)x, \quad t \ge 0, \quad x \in \mathbb{X}.$$

2. Let $A : D(A) \subset \mathbb{X} \mapsto \mathbb{X}$ be a closed linear operator and let $\alpha \in (0,1]$ and $\beta \ge 0$. Show that if A generates an $(\alpha,1)^\beta$-resolvent family S_α^β, then the following hold,
   a. $S_\alpha^\beta(t)(D(A)) \subset D(A)$ and

   $$AS_\alpha^\beta(t)x = S_\alpha^\beta(t)Ax$$

   for all $x \in D(A)$ and $t \ge 0$.
   b. For all $x \in D(A)$,

   $$S_\alpha^\beta(t)x = g_{\alpha\beta+1}(t)x + \int_0^t g_\alpha(t-s)AS_\alpha^\beta(s)xds, \quad t \ge 0.$$

   c. For all $x \in \mathbb{X}$, $(g_\alpha * S_\alpha^\beta)(t)x \in D(A)$,

   $$S_\alpha^\beta(t)x = g_{\alpha\beta+1}(t)x + A\int_0^t g_\alpha(t-s)S_\alpha^\beta(s)xds, \quad t \ge 0.$$

   d. $S_\alpha^\beta(0) = g_{\alpha\beta+1}(0)$; $S_\alpha^\beta(0) = I$ if $\beta = 0$ and $S_\alpha^\beta(0) = 0$ if $\beta > 0$.

3. Let $A : D(A) \subset \mathbb{X} \mapsto \mathbb{X}$ be a closed linear operator and let $\alpha \in (0, 1]$ and $\beta \geq 0$. Show that if A generates an $(\alpha, \alpha)^\beta$-resolvent family T_α^β, then the following hold,

   a. $T_\alpha^\beta(t)(D(A)) \subset D(A)$ and
   $$AT_\alpha^\beta(t)x = T_\alpha^\beta(t)Ax$$
   for all $x \in D(A)$ and $t > 0$.

   b. For all $x \in D(A)$,
   $$T_\alpha^\beta(t)x = g_{\alpha(\beta+1)}(t)x + \int_0^t g_\alpha(t-s)AT_\alpha^\beta(s)xds, \quad t \geq 0.$$

   c. For all $x \in \mathbb{X}$, $(g_\alpha * T_\alpha^\beta)(t)x \in D(A)$,
   $$T_\alpha^\beta(t)x = g_{\alpha(\beta+1)}(t)x + A\int_0^t g_\alpha(t-s)T_\alpha^\beta(s)xds, \quad t > 0.$$

   d. If $\beta > 0$, then for every $x \in \overline{D(A)}$,
   $$\frac{1}{\Gamma(\alpha(1+\beta))}\lim_{t\to 0} t^{1-\alpha(1+\beta)}T_\alpha^\beta(t)x = x$$
   if $\alpha(1+\beta) < 1$; $T_\alpha^\beta(0)x = x$ if $\alpha(1+\beta) = 1$; and $T_\alpha^\beta(0)x = 0$ if $\alpha(1+\beta) > 1$.

   e. If $\alpha(1+\beta) > 1$, then all the above equalities occur for $t \geq 0$. .

4. Suppose $p \in [1, \infty)$, $\alpha \in (0, 1)$, and $\lambda \in [0, \pi)$. Let A_p be the linear operator defined by $A_p = e^{i\lambda}\Delta_p$ where Δ_p is a realization of the Laplace differential operator on $L^p(\mathbb{R}^d)$. Show that A_p is the generator of an $(\alpha, 1)^\beta$ on $L^p(\mathbb{R}^d)$ for all $\beta \geq 0$.

9.6 Comments

The material discussed in this chapter is mainly based upon the following two sources: Keyantuo et al. [75] and Wang et al. [110]. One should mention that the semilinear case is not treated in [75]. Consequently, an interesting question consists of using the same tools as in [75] to study the existence of classical (respectively, mild) solutions for the semilinear fractional Cauchy problem

$$\begin{cases} \mathbb{D}_t^\alpha u(t) = Au(t) + F(t, u(t)) \\ u(0) = u_0 \in \mathbb{X} \end{cases} \tag{9.11}$$

where $\mathbb{D}_t^\alpha$ is the fractional derivative of order α in the sense of Caputo, $A : D(A) \subset \mathbb{X} \mapsto \mathbb{X}$ is a closed linear operator on a complex Banach space $\mathbb{X}$, and $f : \mathbb{R}_+ \times \mathbb{X} \mapsto \mathbb{X}$ is a jointly continuous function satisfying some additional conditions.

For the proofs of the existence results Theorem 9.12 and Corollary 9.13, we refer the reader to Keyantuo et al. [75].

The proofs of Theorems 9.18 and 9.19 are taken from Wang et al. [110]. For additional readings upon these topics, we refer to [4, 6, 21–23, 37, 42], etc.