Chapter 8
First-Order Semilinear Evolution Equations

In this chapter we study and establish the existence of classical and (bounded and almost periodic) mild solutions to some semilinear evolutions including nonautonomous ones.

8.1 First-Order Autonomous Evolution Equations

8.1.1 Existence of Mild and Classical Solutions

Let $J \subset \mathbb{R}$ be an interval whose infimum, inf J, is zero.

Consider the first-order evolution equation

$$\begin{cases} u'(t) &= Au(t) + f(t), \quad t > 0 \\ u(0) &= u_0 \end{cases} \tag{8.1}$$

where $A : D(A) \subset \mathbb{X} \mapsto \mathbb{X}$ is a sectorial linear operator whose associated analytic semi-group will be denoted $(T(t))_{t\geq 0}$ and $f : J \mapsto \mathbb{X}$ is a continuous function.

Our main objective in this subsection consists of studying the existence of solutions to Eq. (8.1) when J is either $[0, T]$ or $\mathbb{R}_+ = [0, \infty)$ where $T > 0$ is a constant.

In this chapter, various types of solutions will be discussed. We basically follow and adopt definitions from Lunardi [87, Definition 4.1.1, Pages 123-124].

Definition 8.1 Let $f : [0, T] \mapsto \mathbb{X}$ be a continuous function and let $u_0 \in \mathbb{X}$. A function $u \in C([0, T]; D(A)) \cap C^1([0, T]; \mathbb{X})$ that satisfies,

$$u'(t) = Au(t) + f(t) \text{ for each } t \in [0, T] \text{ and } u(0) = u_0,$$

is called a *strict solution* to Eq. (8.1) on the interval $J = [0, T]$.

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Definition 8.2 Let $f : [0, T] \mapsto \mathbb{X}$ be a continuous function and let $u_0 \in \mathbb{X}$. A function $u \in C([0, T]; \mathbb{X})$ is called a *strong solution* to Eq. (8.1) on the interval $J = [0, T]$ if there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset C([0, T]; D(A)) \cap C^1([0, T]; \mathbb{X})$ such that

$$\sup_{t\in[0,T]} \|u_n(t) - u(t)\| \to 0 \text{ and } \sup_{t\in[0,T]} \|u_n'(t) - Au_n(t) - f(t)\| \to 0$$

as $n \to \infty$.

Definition 8.3 Let $f : (0, T] \mapsto \mathbb{X}$ be a continuous function. Any function $u \in C([0, T]; \mathbb{X}) \cap C((0, T]; D(A)) \cap C^1((0, T]; \mathbb{X})$ that satisfies,

$$u'(t) = Au(t) + f(t) \text{ for each } t \in (0, T] \text{ and } u(0) = u_0,$$

is called a *classical solution* to Eq. (8.1) on the interval $J = [0, T]$.

Definition 8.4 Let $f : [0, \infty) \mapsto \mathbb{X}$ be a continuous function. A function $u : [0, \infty) \mapsto \mathbb{X}$ is said to be a strict (respectively, classical or strong) solution to Eq. (8.1) on the interval $J = [0, \infty)$, if for every $T > 0$, the restriction of the function u to the interval $[0, T]$ is a strict (respectively, classical or strong) solution to Eq. (8.1) on the interval $[0, T]$.

Definition 8.5 Let $f \in L^1((0, T); \mathbb{X})$ and let $u_0 \in \mathbb{X}$. A function u is called a *mild solution* to Eq. (8.1) if it can be written as follows,

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)ds, \quad t \in [0, T]. \tag{8.2}$$

Let us make a few remarks upon the notions which we have just introduced.

Remark 8.6

i) If u is a strict solution to Eq. (8.1), then the function u satisfies,

$$u'(t) = Au(t) + f(t) \text{ for each } t \in [0, T] \text{ and } u(0) = u_0.$$

Consequently,

$$u'(0) = Au(0) + f(0) = Au_0 + f(0)$$

which yields two things. First, the previous equation makes sense only if u_0 belongs to $D(A)$. Second, $u'(0) = Au_0 + f(0)$ must belong to $\overline{D(A)}$.

ii) If u is a classical solution, then one must have $u_0 \in \overline{D(A)}$. In this event, if in addition, $f \in L^1((0, T); \mathbb{X}) \cap C((0, T]; \mathbb{X})$, then

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)ds, \quad t \in [0, T].$$

iii) If u is a mild solution to Eq. (8.1), then using the fact that A is a sectorial linear operator [see ii) of Proposition 3.12], it follows that there exists a constant $C > 0$ such that u satisfies the following estimate,

$$\|u\| \leq C\Big(\|u_0\| + \int_0^t \|f(s)\|ds\Big) \text{ for all } t \in [0, T].$$

Theorem 8.7 ([88, Lemma 4.2.5, Pages 45 and 46]) *Let $f : (0, T] \mapsto \mathbb{X}$ be a bounded continuous function and let $u_0 \in \overline{D(A)}$. If u is a mild solution to Eq. (8.1), then the following statements are equivalent.*

i) $u \in C((0, T]; D(A))$;
ii) $u \in C^1((0, T]; \mathbb{X})$;
iii) u is a classical solution to Eq. (8.1).

If $f \in C([0, T]; \mathbb{X})$, then the following statements are equivalent:

i) $u \in C([0, T]; D(A))$;
ii) $u \in C^1([0, T]; \mathbb{X})$;
iii) u is a strict solution to Eq. (8.1).

8.1.2 Existence of Almost Periodic Solutions

The main objective here consists of studying the existence of bounded and almost periodic solutions to first-order evolution equations in the case when the analytic semi-group $(T(t))_{t\geq 0}$ associated with our sectorial operator $A : D(A) \subset \mathbb{X} \mapsto \mathbb{X}$ is hyperbolic, that is,

$$\sigma(A) \cap i\mathbb{R} = \{\emptyset\}. \tag{8.3}$$

From Proposition 3.12 it follows that there exist constants $M_0, M_1 > 0$ such that

$$\|T(t)\| \leq M_0 e^{\omega t}, \quad t > 0, \tag{8.4}$$

$$\|t(A - \omega)T(t)\| \leq M_1 e^{\omega t}, \quad t > 0. \tag{8.5}$$

Since the semi-group $(T(t))_{t\geq 0}$ is assumed to be hyperbolic, then there exists a projection P and constants $M, \delta > 0$ such that $T(t)$ commutes with P, $N(P)$ is invariant with respect to $T(t)$, $T(t) : R(Q) \mapsto R(Q)$ is invertible, and the following hold

$$\|T(t)Px\| \leq Me^{-\delta t}\|x\| \qquad \text{for } t \geq 0, \tag{8.6}$$

$$\|T(t)Qx\| \leq Me^{\delta t}\|x\| \qquad \text{for } t \leq 0, \tag{8.7}$$

where $Q := I - P$ and, for $t \leq 0$, $T(t) := (T(-t))^{-1}$.

Definition 8.8 Let $\alpha \in (0,1)$. A Banach space $(\mathbb{X}_\alpha, \|\cdot\|_\alpha)$ is said to be an intermediate space between $D(A)$ and $\mathbb{X}$, or a space of class $\mathscr{J}_\alpha$, if $D(A) \subset \mathbb{X}_\alpha \subset \mathbb{X}$ and there is a constant $c > 0$ such that

$$\|x\|_\alpha \leq c\|x\|^{1-\alpha}\|x\|_A^\alpha, \qquad x \in D(A), \tag{8.8}$$

where $\|\cdot\|_A$ is the graph norm of A.

Precise examples of the intermediate space $\mathbb{X}_\alpha$ include $D((-A^\alpha))$ for $\alpha \in (0,1)$, the domains of the fractional powers of A, the real interpolation spaces $D_A(\alpha,\infty)$, $\alpha \in (0,1)$, defined as the space of all $x \in \mathbb{X}$ such

$$[x]_\alpha = \sup_{0<t\leq 1} \|t^{1-\alpha}AT(t)x\| < \infty.$$

with the norm

$$\|x\|_\alpha = \|x\| + [x]_\alpha,$$

the abstract Hölder spaces $D_A(\alpha) := \overline{D(A)}^{\|.\|_\alpha}$ as well as complex interpolation spaces $[\mathbb{X}, D(A)]_\alpha$.

For a given hyperbolic analytic semi-group $(T(t))_{t\geq 0}$, it can be checked that similar estimations as both Eqs. (8.6) and (8.7) still hold with the α-norms $\|\cdot\|_\alpha$. In fact, as the part of A in $R(Q)$ is bounded, it follows from Eq. (8.7) that

$$\|AT(t)Qx\| \leq C'e^{\delta t}\|x\| \text{ for } t \leq 0.$$

Thus from Eq. (8.8) there exists a constant $c(\alpha) > 0$ such that

$$\|T(t)Qx\|_\alpha \leq c(\alpha)e^{\delta t}\|x\| \quad \text{for } t \leq 0. \tag{8.9}$$

In addition to the above, the following holds

$$\|T(t)Px\|_\alpha \leq \|T(1)\|_{B(\mathbb{X},\mathbb{X}_\alpha)}\|T(t-1)Px\|, \quad t \geq 1,$$

and hence from Eq. (8.6), one obtains

$$\|T(t)Px\|_\alpha \leq M'e^{-\delta t}\|x\|, \qquad t \geq 1,$$

where M' depends on α. For $t \in (0,1]$, by Eqs. (8.5) and (8.8),

$$\|T(t)Px\|_\alpha \leq M''t^{-\alpha}\|x\|.$$

Hence, there exist constants $M(\alpha) > 0$ and $\gamma > 0$ such that

$$\|T(t)Px\|_\alpha \leq M(\alpha)t^{-\alpha}e^{-\gamma t}\|x\| \qquad \text{for } t > 0. \tag{8.10}$$

Consider the differential equation

$$u'(t) = Au(t) + f(t), \quad t \in \mathbb{R} \tag{8.11}$$

where $A : D(A) \subset \mathbb{X} \mapsto \mathbb{X}$ is a sectorial linear operator for which Eq. (8.3) holds and $f : \mathbb{R} \mapsto \mathbb{X}$ is a bounded continuous function.

Definition 8.9 A function $u \in BC(\mathbb{R}, \mathbb{X})$ is called a *mild solution* to Eq. (8.11) on $\mathbb{R}$ if for all $\tau \in \mathbb{R}$,

$$u(t) = T(t-\tau)u(\tau) + \int_\tau^t T(t-s)f(s)ds, \quad t \geq \tau. \tag{8.12}$$

Proposition 8.10 *If $f \in BC(\mathbb{R}, \mathbb{X})$, then Eq. (8.11) has a unique mild solution $u \in BC(\mathbb{R}, \mathbb{X})$ given by*

$$u(t) = \int_{-\infty}^t T(t-s)Pf(s)ds - \int_t^\infty T(t-s)(I-P)f(s)ds, \quad t \in \mathbb{R}. \tag{8.13}$$

Moreover, if $f \in C^{0,\alpha}(\mathbb{R}, \mathbb{X})$ for some $\alpha \in (0, 1)$, then u given above is a strict solution to Eq. (8.11) that belongs to $C^{0,\alpha}(\mathbb{R}, D(A))$.

Proof Clearly, the function given in Eq. (8.13), that is,

$$u(t) = \int_{-\infty}^t T(t-s)Pf(s)ds - \int_t^\infty T(t-s)(I-P)f(s)ds, \quad t \in \mathbb{R}$$

is well defined and satisfies

$$u(t) = T(t-s)u(s) + \int_s^t T(t-s)f(s)ds, \quad \text{for all } t, s \in \mathbb{R}, \quad t \geq s. \tag{8.14}$$

Consequently, u is a mild solution to Eq. (8.11).

For the uniqueness, let v be another mild solution to Eq. (8.11). Thus using the projections P and $Q = I - P$, one obtains

$$Pv(t) = T(t-s)Pv(s) + \int_s^t T(t-s)Pf(s)ds, \quad \text{for all } t, s \in \mathbb{R}, \quad t \geq s, \tag{8.15}$$

and

$$Qv(t) = T(t-s)Qv(s) + \int_s^t T(t-s)Qf(s)ds, \quad \text{for all } t, s \in \mathbb{R}, \quad t \geq s. \tag{8.16}$$

Using the fact that v is bounded and Eqs. (8.9)–(8.10), letting $s \to -\infty$ in Eq. (8.15) (respectively, letting $s \to \infty$ in Eq. (8.16)), we obtain

$$Pv(t) = \int_{-\infty}^t T(t-s)Pf(s)ds, \quad \text{for all } t \in \mathbb{R},$$

and

$$Qv(t) = -\int_t^\infty T(t-s)Qf(s)ds, \quad \text{for all } \ t \in \mathbb{R},$$

which yields

$$v(t) = Pv(t) + Qv(t) = \int_{-\infty}^t T(t-s)Pf(s)ds$$
$$- \int_t^\infty T(t-s)Qf(s)ds = u(t), \quad \forall t \in \mathbb{R}.$$

Therefore, $u = v$.

Using [88, Lemma 3.3.1 and Lemma 3.3.3], it can be shown that if $f \in C^{0,\alpha}(\mathbb{R}, \mathbb{X})$ for some $\alpha \in (0, 1)$, then u belongs to $C^{0,\alpha}(\mathbb{R}, D(A))$.

We have

Corollary 8.11 *If $f \in AP(\mathbb{X})$, then Eq. (8.11) has a unique mild solution $u \in AP(\mathbb{X})$ given by*

$$u(t) = \int_{-\infty}^t T(t-s)Pf(s)ds - \int_t^\infty T(t-s)(I-P)f(s)ds, \quad t \in \mathbb{R}. \tag{8.17}$$

In particular, if f is continuous and T-periodic, then the mild solution u is also T-periodic.

Proof Using the fact that $AP(\mathbb{X}) \subset BC(\mathbb{R}, \mathbb{X})$, it follows that Eq. (8.11) has a unique mild solution $u \in BC(\mathbb{R}, \mathbb{X})$ given by

$$u(t) = \int_{-\infty}^t T(t-s)Pf(s)ds - \int_t^\infty T(t-s)(I-P)f(s)ds, \quad t \in \mathbb{R}.$$

To complete the proof, we have to show that $u \in AP(\mathbb{X})$. Since $f \in AP(\mathbb{X})$, for all $\varepsilon > 0$, there exists $\ell(\varepsilon) > 0$ such that for all $a \in \mathbb{R}$, the interval $(a, a + \ell(\varepsilon))$ contains a τ such that

$$\|f(t+\tau) - f(t)\| < \varepsilon \tag{8.18}$$

for all $t \in \mathbb{R}$.

Now

$$u(t+\tau) - u(t) \ = \int_{-\infty}^0 T(-s)P\Big(f(s+t+\tau) - f(s+t)\Big)ds$$
$$+ \int_0^\infty T(-s)Q\Big((s+t+\tau) - f(s+t)\Big)ds$$

which, for $\alpha \in (0, 1)$, yields

$$\begin{aligned}
\|u(t+\tau)-u(t)\|_\alpha &\leq \int_{-\infty}^{0} \|T(-s)P\Big(f(s+t+\tau)-f(s+t)\Big)\|_\alpha ds \\
&\quad + \int_{0}^{\infty} \|T(-s)Q\Big((s+t+\tau)-f(s+t)\Big)\|_\alpha ds \\
&\leq c(\alpha)\int_{-\infty}^{0} e^{\delta s}\|f(s-t-\tau)-f(s-t)\|ds \\
&\quad + M(\alpha)\int_{0}^{\infty} e^{-\gamma s}s^{-\alpha}\|f(s-t-\tau)-f(s-t)\|ds
\end{aligned}$$

by using Eqs. (8.9)–(8.10).

To conclude, one makes use of both Eq. (8.18) and the Lebesgue dominated convergence theorem.

8.2 Semilinear First-Order Evolution Equations

8.2.1 Existence of Mild and Classical Solutions

Consider the first-order semilinear evolution equation

$$\begin{cases} u'(t) &= Au(t) + F(t, u(t)), \quad t > 0 \\ u(0) &= u_0 \end{cases} \tag{8.19}$$

where $A : D(A) \subset \mathbb{X} \mapsto \mathbb{X}$ is a sectorial linear operator whose corresponding analytic semi-group is $(T(t))_{t\geq 0}$ and $F : [0, T] \times \mathbb{X} \mapsto \mathbb{X}$ is a jointly continuous function and locally Lipschitz in the second variable, that is, there exist $R > 0$ and $L > 0$ such that

$$\|F(t, u) - F(t, v)\| \leq L\, \|u - v\| \tag{8.20}$$

for all $t \in [0, T]$ and $u, v \in B(0, R)$.

Let $J = [0, T_0)$ or $[0, T_0]$ where $T_0 \leq T$. As in the linear case, we have the following definitions for classical, strict, and mild solutions.

Definition 8.12 ([88]) A function $u : J \mapsto \mathbb{X}$ is said to be a strict solution to Eq. (8.19) in J, if u is continuous with values in $D(A)$ and differentiable with values in $\mathbb{X}$ in the interval J, and satisfies Eq. (8.19).

Definition 8.13 ([88]) A function $u : J \mapsto \mathbb{X}$ is said to be a classical solution to Eq. (8.19) in J, if u is continuous with values in $D(A)$ and differentiable with values in $\mathbb{X}$ in the interval $J \setminus \{0\}$, continuous in the interval J with values in $D(A)$, and satisfies Eq. (8.19).

Definition 8.14 ([88]) A function $u : J \mapsto \mathbb{X}$ is called a *mild solution* to Eq. (8.19) if it is continuous with values in $D(A)$ in the interval $J \setminus \{0\}$, and satisfies

$$u(t) = T(t)u_0 + \int_0^t T(t-s)F(s, u(s))ds, \quad t \in J. \tag{8.21}$$

Under some suitable conditions (see [87]) it can be shown that every mild solution to Eq. (8.19) is a classical (or strict) solution.

Theorem 8.15 ([88]) *Suppose $F : [0, T] \times \mathbb{X} \mapsto \mathbb{X}$ is jointly continuous and satisfies Eq. (8.20). Then for every $v \in \mathbb{X}$ there exist constants $r, \delta > 0$, $K > 0$ such that for $\|u_0 - v\| \le r$, then Eq. (8.19) has a unique mild solution $u = u(\cdot, u_0) \in BC((0, \delta]; \mathbb{X})$. The mild solution u belongs to $C([0, \delta]; \mathbb{X})$ if and only if $u_0 \in \overline{D(A)}$. Further, for $u_0, u_1 \in B(v, r)$, the following holds,*

$$\|u(t, u_0) - u(t, u_1)\| \le K\|u_0 - u_1\|, \quad t \in [0, \delta].$$

8.2.2 *Existence Results on the Real Number Line*

Consider the differential equation

$$u'(t) = Au(t) + F(t, u(t)), \quad t \in \mathbb{R} \tag{8.22}$$

where $A : D(A) \subset \mathbb{X} \mapsto \mathbb{X}$ is a sectorial linear operator for which Eq. (8.3) holds and $F : \mathbb{R} \times \mathbb{X}_\alpha \mapsto \mathbb{X}$ for some $\alpha \in (0, 1)$ is a jointly continuous function and globally Lipschitz in the second variable, that is, there exists a constant $L > 0$ such that

$$\|F(t, u) - F(t, v)\| \le L\, \|u - v\|_\alpha \tag{8.23}$$

for all $t \in \mathbb{R}$ and $u, v \in \mathbb{X}_\alpha$.

Definition 8.16 A mild solution to Eq. (8.22) is any function $u : \mathbb{R} \mapsto \mathbb{X}_\alpha$ which satisfies the following variation of constants formula,

$$u(t) = T(t-s)u(s) + \int_s^t T(t-\sigma)F(\sigma, u(\sigma))d\sigma \tag{8.24}$$

for all $t \ge s, \quad t, s \in \mathbb{R}$.

Using Corollary 8.11 in which we let $f(t) := F(t, u(t))$ and under some additional assumptions, we obtain the next theorem.

Theorem 8.17 *Under Eqs. (8.3) and (8.23), if $F \in AP(\mathbb{R} \times \mathbb{X}_\alpha, \mathbb{X})$, then Eq. (8.22) has a unique almost periodic mild solution if L is small enough.*

Remark 8.18 A generalization of Theorem 8.17 to the case when the linear operator A is replaced with $A(t)$ is given by Theorem 8.20.

8.3 Nonautonomous First-Order Evolution Equations

8.3.1 Existence of Almost Periodic Mild Solutions

Consider the nonautonomous differential equation

$$u'(t) = A(t)u(t) + f(t, u(t)) \tag{8.25}$$

where $A(t)$ for $t \in \mathbb{R}$ be a family of linear operators on $\mathbb{X}$ whose domains $D(A(t)) = D$ are constant for all $t \in \mathbb{R}$ and the function $f : \mathbb{R} \times \mathbb{X}_\alpha \mapsto \mathbb{X}$ is continuous and globally lipschitzian, that is, there is $k > 0$ such that

$$\|f(t, x) - f(t, y)\| \leq k \|x - y\|_\alpha \quad \text{for all } t \in \mathbb{R} \text{ and } x, y \in \mathbb{X}_\alpha. \tag{8.26}$$

To study the almost periodicity of the solutions of Eq. (8.25), we assume that the following holds:

(H.820) The family of linear operators $A(t)$ satisfy the Aquistapace–Terreni conditions.
(H.821) The evolution family U generated by $A(\cdot)$ has an exponential dichotomy with constants $N, \delta > 0$ and dichotomy projections $P(t)$ for $t \in \mathbb{R}$.
(H.822) There exists $0 \leq \alpha < \beta < 1$ such that

$$\mathbb{X}_\alpha^t = \mathbb{X}_\alpha \quad \text{and} \quad \mathbb{X}_\beta^t = \mathbb{X}_\beta$$

for all $t \in \mathbb{R}$, with uniform equivalent norms.
(H.823) $R(\omega, A(\cdot)) \in AP(\mathbb{R}, B(\mathbb{X}))$ with pseudo periods $\tau = \tau_\epsilon$ belonging to sets $\mathscr{P}(\epsilon, A)$.

Definition 8.19 By a mild solution Eq. (8.25) we mean every continuous function $x : \mathbb{R} \mapsto \mathbb{X}_\alpha$, which satisfies the following variation of constants formula

$$x(t) = U(t, s)x(s) + \int_s^t U(t, \sigma) f(\sigma, x(\sigma)) d\sigma \quad \text{for all } t \geq s,\ t, s \in \mathbb{R}. \tag{8.27}$$

In order to study the existence of almost periodic mild solution to the semilinear evolution equation Eq. (8.25), we first study the existence of almost periodic mild solution to the inhomogeneous evolution equation

$$x'(t) = A(t)x(t) + g(t), \quad t \in \mathbb{R}. \tag{8.28}$$

We have

Theorem 8.20 *Suppose that assumptions (H.820)–(H.823) hold. If* $g \in BC(\mathbb{R}, \mathbb{X})$*, then*

(i) Equation (8.28) has a unique bounded mild solution $x : \mathbb{R} \mapsto \mathbb{X}_\alpha$ *given by*

$$x(t) = \int_{-\infty}^{t} U(t,s)P(s)g(s)ds - \int_{t}^{+\infty} \widetilde{U}(t,s)Q(s)g(s)ds. \tag{8.29}$$

(ii) If $g \in AP(\mathbb{R}, \mathbb{X})$*, then* $x \in AP(\mathbb{R}, \mathbb{X}_\alpha)$*.*

Proof

(i) Since g is bounded, we know from [37] that the function x given by (8.29) is the unique bounded mild solution to Eq. (8.28). To prove that x is bounded in $\mathbb{X}_\alpha$, we make use of Proposition 3.27 to obtain,

$$\begin{aligned}
\|x(t)\|_\alpha &\le c\,\|x(t)\|_\beta \\
&\le c \int_{-\infty}^{t} \|U(t,s)P(s)g(s)\|_\beta\, ds + c \int_{t}^{+\infty} \left\|\widetilde{U}(t,s)Q(s)g(s)\right\|_\beta ds \\
&\le cc(\beta) \int_{-\infty}^{t} e^{-\frac{\delta}{2}(t-s)}(t-s)^{-\beta}\, \|g(s)\|\, ds \\
&\quad + cm(\beta) \int_{t}^{+\infty} e^{-\delta(s-t)}\, \|g(s)\|\, ds \\
&\le cc(\beta)\, \|g\|_\infty \int_0^{+\infty} e^{-\sigma} \left(\frac{2\sigma}{\delta}\right)^{-\beta} \frac{2d\sigma}{\delta} + cm(\beta)\, \|g\|_\infty \int_0^{+\infty} e^{-\delta\sigma} d\sigma \\
&\le cc(\beta)\delta^\alpha \Gamma(1-\beta)\, \|g\|_\infty + cm(\beta)\delta^{-1}\, \|g\|_\infty\,,
\end{aligned}$$

and hence

$$\|x(t)\|_\alpha \le c\,\|x(t)\|_\beta \le c[c(\beta)\delta^\beta \Gamma(1-\beta) + m(\beta)\delta^{-1}]\, \|g\|_\infty\,. \tag{8.30}$$

(ii) Let $\epsilon > 0$ and $\mathscr{P}(\epsilon, A, f)$ be the set of pseudo periods for the almost periodic function $t \mapsto (f(t), R(\omega, A(t)))$. We know, from [89, Theorem 4.5] that x, as an $\mathbb{X}$-valued function is almost periodic. Hence, there exists a number $\tau \in \mathscr{P}(\left(\frac{\varepsilon}{c'}\right)^{\frac{\beta}{\beta-\alpha}}, A, f)$ such that

$$\|x(t+\tau) - x(t)\| \le \left(\frac{\varepsilon}{c'}\right)^{\frac{\beta}{\beta-\alpha}} \quad \text{for all } t \in \mathbb{R}.$$

For $\theta = \frac{\alpha}{\beta}$, the reiteration theorem implies that $\mathbb{X}_\alpha = (\mathbb{X}, \mathbb{X}_\beta)_{\theta,\infty}$. Using the property of interpolation and Eq. (8.30), we obtain

$$\begin{aligned}\|x(t+\tau)-x(t)\|_\alpha &\leq c(\alpha,\beta)\|x(t+\tau)-x(t)\|^{\frac{\beta-\alpha}{\beta}}\|x(t+\tau)-x(t)\|_\beta^{\frac{\alpha}{\beta}}\\ &\leq c(\alpha,\beta)2^{\frac{\alpha}{\beta}}\left(c[c(\beta)\delta^\beta\Gamma(1-\beta)+m(\beta)\delta^{-1}]\,\|g\|_\infty\right)^{\frac{\alpha}{\beta}}\\ &\quad\ \|x(t+\tau)-x(t)\|^{\frac{\beta-\alpha}{\beta}}\\ &:= c'\|x(t+\tau)-x(t)\|^{\frac{\beta-\alpha}{\beta}},\end{aligned}$$

and hence

$$\|x(t+\tau)-x(t)\|_\alpha \leq \varepsilon$$

for $t \in \mathbb{R}$.

To show the existence of almost periodic solutions to Eq. (8.25), let $y \in AP(\mathbb{R}, \mathbb{X}_\alpha)$ and $f \in AP(\mathbb{R}\times\mathbb{X}_\alpha, \mathbb{X})$. Using the theorem of composition of almost periodic functions (Theorem 4.18) we deduce that the function $g(\cdot) := f(\cdot, y(\cdot)) \in AP(\mathbb{R}, \mathbb{X})$, and from Theorem 8.20, the semilinear equation (Eq. (8.25)) has a unique mild solution $x \in AP\,(\mathbb{R}, \mathbb{X}_\alpha)$ given by

$$x(t) = \int_{-\infty}^t U(t,s)P(s)f(s,y(s))ds - \int_t^{+\infty}\widetilde{U}(t,s)Q(s)f(s,y(s))ds, \quad t\in\mathbb{R}.$$

Define the nonlinear operator $F : AP(\mathbb{R}, \mathbb{X}_\alpha) \mapsto AP(\mathbb{R}, \mathbb{X}_\alpha)$ by

$$\begin{aligned}(Fy)(t) := &\int_{-\infty}^t U(t,s)P(s)f(s,y(s))ds\\ &-\int_t^{+\infty}\widetilde{U}(t,s)Q(s)f(s,y(s))ds, \quad t\in\mathbb{R}.\end{aligned}$$

Now for any $x,\ y \in AP\,(\mathbb{R}, \mathbb{X}_\alpha)$,

$$\begin{aligned}\|Fx(t)-Fy(t)\|_\alpha &\leq c(\alpha)\int_{-\infty}^t e^{-\delta(t-s)}(t-s)^{-\alpha}\,\|f(s,y(s))-f(s,x(s))\|\,ds\\ &\quad + c(\alpha)\int_t^{+\infty} e^{-\delta(t-s)}\,\|f(s,y(s))-f(s,x(s))\|\,ds.\\ &\leq k[c(\alpha)\delta^{-\alpha}\Gamma(1-\alpha)+m(\alpha)\delta^{-1}]\,\|x-y\|_\infty \quad \text{for all } t\in\mathbb{R}.\end{aligned}$$

By taking k small enough, that is, $k < (c(\alpha)\delta^{\alpha}\Gamma(1-\alpha) + m(\alpha)\delta^{-1})^{-1}$, the operator F becomes a contraction on $AP(\mathbb{R}, \mathbb{X}_{\alpha})$ and hence has a unique fixed point in $AP(\mathbb{R}, \mathbb{X}_{\alpha})$, which obviously is the unique $\mathbb{X}_{\alpha}$-valued almost periodic solution to Eq. (8.25).

The previous discussion can be formulated as follows:

Theorem 8.21 *Let* $\alpha \in (0, 1)$. *Suppose that assumptions (H.820)–(H.821)–(H.822)–(H.823) hold and that* $f \in AP(\mathbb{R} \times \mathbb{X}_{\alpha}, \mathbb{X})$ *with* $k < (c(\alpha)\delta^{-\alpha}\Gamma(1-\alpha) + m(\alpha)\delta^{-1})^{-1}$. *Then Eq.* (8.25) *has a unique mild solution* x *in* $AP(\mathbb{R}, \mathbb{X}_{\alpha})$.

8.4 Exercises

1. Prove Theorem 8.7
2. Prove Theorem 8.15.
3. Prove Theorem 8.17

8.5 Comments

The preliminary results of this chapter are taken from Lunardi [87, 88]. Let us point out that the setting of Sect. 8.1.2 follows that of Boulite et al. [30]. The proofs of Theorem 8.20 and Theorem 8.21 discussed follow Baroun et al. [20]. The existence of mild solutions for similar evolution equations can be obtained in the cases when the operator A is not necessarily sectorial. For these cases, we refer the interested readers to Pazy [100] and Engel and Nagel [55]. The existence results obtained when the forcing term is almost periodic can be extended to more general classes of functions including almost automorphic and pseudo-almost periodic or pseudo-almost automorphic functions, see, e.g., Diagana [47].