Chapter 7
Fractional Integro-Differential Equations

7.1 Introduction

Fractional calculus is a generalization of the classical differentiation and integration of non-integer order. Fractional calculus is as old as differential calculus. Fractional differential and integral equations have applications in many fields including engineering, science, finance, applied mathematics, bio-engineering, radiative transfer, neutron transport, and the kinetic theory of gases, see, e.g., [16, 33, 35, 36, 71, 74]. Noteworthy progress upon the study of ordinary and partial fractional differential equations have recently been made, see, e.g., Abbas et al. [6], Baleanu et al. [19], Diethelm [51], Kilbas et al. [77], Miller and Ross [91], Podlubny [101], and Samko et al. [103]. Further, some recent results upon the existence and attractivity of solutions to various integral equations of two variables have been obtained by many people including Abbas et al. [2, 3, 5].

In this chapter, we study the existence, uniqueness, estimates, and global asymptotic stability for some classes of fractional integro-differential equations with finite delay. In order to achieve our goal, we make extensive use of some fixed-point theorems as well as the so-called Pachpatte techniques.

Recently, Pachpatte [98] obtained some existence and uniqueness results as well as some other properties of solutions to certain Volterra integral and integro-differential equations in two variables. The main tools utilized in his analysis are based upon the applications of the Banach fixed point theorem coupled with the so-called Bielecki type norm and certain integral inequalities with explicit estimates. Using integral inequalities and a fixed-point approach, we improve some of the above-mentioned results and study the global attractivity of solutions for the system of partial fractional integro-differential equations in the form,

$$\mathbb{D}_{\theta}^{r}u(t,x) = f(t,x,u_{(t,x)},(Gu)(t,x)), \quad \text{for } (t,x) \in J := \mathbb{R}_+ \times [0,b], \tag{7.1}$$

$$u(t,x) = \phi(t,x), \text{ if } (t,x) \in \tilde{J} := [-\alpha,\infty) \times [-\beta,b] \backslash (0,\infty) \times (0,b], \tag{7.2}$$

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$$\begin{cases} u(t,0) = \varphi(t); \ t \in \mathbb{R}_+, \\ u(0,x) = \psi(x); \ x \in [0,b], \end{cases} \tag{7.3}$$

where

$$(Gu)(t,x) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} g(t,x,s,y,u_{(s,y)})dyds, \tag{7.4}$$

$\alpha, \beta, b > 0$, $\theta = (0,0)$, $r = (r_1, r_2) \in (0,1] \times (0,1]$, $\mathbb{R}_+ = [0,\infty)$, I^r_θ is the left-sided mixed Riemann–Liouville integral of order r, $\mathbb{D}^r_\theta$ is the standard Caputo fractional derivative of order r, $f : J \times \mathscr{C} \to \mathbb{R}$, $g : J_1 \times \mathscr{C} \to \mathbb{R}$ are given continuous functions, $J_1 := \{(t,x,s,y) : 0 \le s \le t < \infty, \ 0 \le y \le x \le b]\}$, $\varphi : \mathbb{R}_+ \to \mathbb{R}$, $\psi : [0,b] \to \mathbb{R}$ are absolutely continuous functions with $\lim_{t\to\infty} \varphi(t) = 0$, and $\psi(x) = \varphi(0)$ for each $x \in [0,b]$, $\Phi : \tilde{J} \to \mathbb{R}$ is continuous with $\varphi(t) = \Phi(t,0)$ for each $t \in \mathbb{R}_+$, and $\psi(x) = \Phi(0,x)$ for each $x \in [0,b]$, $\Gamma(.)$ is the Gamma function defined by

$$\Gamma(\xi) = \int_0^\infty t^{\xi-1}e^{-t}dt; \ \xi > 0,$$

and $\mathscr{C} := C([-\alpha,0] \times [-\beta,0])$ is the space of continuous functions on $[-\alpha,0] \times [-\beta,0]$ with the standard norm

$$\|u\|_{\mathscr{C}} = \sup_{(t,x)\in[-\alpha,0]\times[-\beta,0]} |u(t,x)|.$$

If $u \in C := C([-\alpha,\infty) \times [-\beta,b])$, then for any $(t,x) \in J$ define $u_{(t,x)}$ by

$$u_{(t,x)}(\tau,\xi) = u(t+\tau, x+\xi); \ \text{for } (\tau,\xi) \in [-\alpha,0] \times [-\beta,0].$$

7.2 Preliminaries and Notations

Let $a, b > 0$ and $L^1([0,a] \times [0,b])$ be the space of Lebesgue-integrable functions $u : [0,a] \times [0,b] \to \mathbb{R}$ equipped with the norm,

$$\|u\|_1 = \int_0^a \int_0^b |u(t,x)|dxdt.$$

By $C := C(J)$ we denote the space of all continuous functions from J into $\mathbb{R}$. Similarly, by $BC := BC([-\alpha,\infty) \times [-\beta,b])$ we denote the Banach space of all bounded and continuous functions from $[-\alpha,\infty) \times [-\beta,b]$ into $\mathbb{R}$ equipped with the standard sup norm which we denoted by

$$\|u\|_{BC} = \sup_{(t,x)\in[-\alpha,\infty)\times[-\beta,b]} |u(t,x)|.$$

Definition 7.1 For $u_0 \in BC$ and $\eta \in (0, \infty)$, we denote by $B(u_0, \eta)$, the closed ball in BC centered at u_0 with radius η.

Definition 7.2 ([108]) Let $r = (r_1, r_2) \in (0, \infty) \times (0, \infty)$, $\theta = (0, 0)$ and $u \in L^1([0, a] \times [0, b])$. The left-sided mixed Riemann–Liouville integral of order r of u is defined by

$$(I_\theta^r u)(t, x) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} u(s, y)dyds.$$

In particular,

$$(I_\theta^\theta u)(t, x) = u(t, x),\ (I_\theta^\sigma u)(t, x) = \int_0^t \int_0^x u(s, y)dyds;$$
$$\text{for almost all } (t, x) \in [0, a] \times [0, b],$$

where $\sigma = (1, 1)$.
For instance, $I_\theta^r u$ exists for all $r_1, r_2 > 0$, when $u \in L^1([0, a] \times [0, b])$. Moreover

$$(I_\theta^r u)(t, 0) = (I_\theta^r u)(0, x) = 0;\ t \in [0, a],\ x \in [0, b].$$

Example 7.3 Let $\lambda, \omega \in (-1, 0) \cup (0, \infty)$ and $r = (r_1, r_2) \in (0, \infty) \times (0, \infty)$, then

$$I_\theta^r t^\lambda x^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda+r_1)\Gamma(1+\omega+r_2)} t^{\lambda+r_1} x^{\omega+r_2},$$
$$\text{for almost all } (t, x) \in [0, a] \times [0, b].$$

By $1 - r$ we mean $(1 - r_1, 1 - r_2) \in [0, 1) \times [0, 1)$. Denote by $D_{tx}^2 := \frac{\partial^2}{\partial t \partial x}$, the mixed second order partial derivative.

Definition 7.4 ([108]) Let $r \in (0, 1] \times (0, 1]$ and $u \in L^1([0, a] \times [0, b])$. Recall that the Caputo fractional derivative of order r of u is defined by the expression

$$\begin{aligned}\mathbb{D}_\theta^r u(t, x) &= (I_\theta^{1-r} D_{tx}^2 u)(t, x) \\ &= \frac{1}{\Gamma(1-r_1)\Gamma(1-r_2)} \int_0^t \int_0^x \frac{(D_{sy}^2 u)(s, y)}{(t-s)^{r_1}(x-y)^{r_2}} dyds.\end{aligned}$$

The case when $\sigma = (1, 1)$ is included and we have

$$(\mathbb{D}_\theta^\sigma u)(t, x) = (D_{xy}^2 u)(t, x),\ \text{for almost all } (t, x) \in [0, a] \times [0, b].$$

Example 7.5 Let $\lambda, \omega \in (-1, 0) \cup (0, \infty)$ and $r = (r_1, r_2) \in (0, 1] \times (0, 1]$, then

$$\mathbb{D}_\theta^r t^\lambda x^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda-r_1)\Gamma(1+\omega-r_2)} t^{\lambda-r_1} x^{\omega-r_2},$$

for almost all $(t, x) \in [0, a] \times [0, b]$.

In the sequel, we need the following lemma

Lemma 7.6 ([1]) *Let $f \in L^1([0, a] \times [0, b])$. A function $u \in AC([0, a] \times [0, b])$ is a solution to the problem*

$$\begin{cases} (\mathbb{D}_\theta^r u)(t, x) = f(t, x); \ (t, x) \in [0, a] \times [0, b], \\ u(t, 0) = \varphi(t); \ t \in [0, a], \ u(0, x) = \psi(x); \ x \in [0, b], \\ \varphi(0) = \psi(0), \end{cases}$$

if and only if u satisfies

$$u(t, x) = \mu(t, x) + (I_\theta^r f)(t, x); \ (t, x) \in [0, a] \times [0, b],$$

where

$$\mu(t, x) = \varphi(t) + \psi(x) - \varphi(0).$$

Denote by $D_1 := \frac{\partial}{\partial t}$, the partial derivative of a function defined on J (or J_1) with respect to the first variable, $D_2 := \frac{\partial}{\partial x}$, $D_2 D_1 := \frac{\partial^2}{\partial t \partial x}$. In the sequel we will make use of the following Lemma due to Pachpatte.

Lemma 7.7 ([98]) *Let $u, e, p \in C(J)$, $k, D_1 k, D_2 k, D_2 D_1 k \in C(J_1)$ be positive functions. If $e(t, x)$ is nondecreasing in each variable $(t, x) \in J$ and*

$$u(t, x) \leq e(t, x) + \int_0^t \int_0^x p(s, y) \times \left[u(s, y) + \int_0^s \int_0^y k(s, y, \tau, \xi) u(\tau, \xi) d\xi d\tau \right] dy ds; \ (t, x) \in J, \quad (7.5)$$

then,

$$u(t, x) \leq e(t, x) \left[1 + \int_0^t \int_0^x p(s, y) \exp\left(\int_0^s \int_0^y [p(\tau, \xi) + A(\tau, \xi)] d\xi d\tau \right) dy ds \right] \quad (7.6)$$

for all $(t, x) \in J$, *where*

$$A(t,x) = k(t,x,s,y) + \int_0^t D_1 k(t,x,s,y)ds + \int_0^x D_2 k(t,x,s,y)dy + \int_0^t \int_0^x D_2 D_1 k(t,x,s,y)dyds; \ (t,x) \in J. \tag{7.7}$$

Let G be an operator from $\emptyset \neq \Omega \subset BC$ into itself and consider the solutions of equation

$$(Gu)(t,x) = u(t,x). \tag{7.8}$$

Now we review the concept of attractivity of solutions to Eq. (7.8). For $u_0 \in BC$ and $\eta \in (0, \infty)$, we denote by $B(u_0, \eta)$, the closed ball in BC centered at u_0 with radius η.

Definition 7.8 ([5]) Solutions to Eq. (7.8) are locally attractive if there exists a ball $B(u_0, \eta)$ in the space BC such that for any arbitrary solutions $v = v(t,x)$ and $w = w(t,x)$ to Eq. (7.8) belonging to $B(u_0, \eta) \cap \Omega$, we have that, for each $x \in [0, b]$,

$$\lim_{t\to\infty} (v(t,x) - w(t,x)) = 0. \tag{7.9}$$

Definition 7.9 When the limit to Eq. (7.9) is uniform with respect to $B(u_0, \eta)$, solutions to Eq. (7.8) are said to be locally attractive (or equivalently that solutions to Eq. (7.8) are asymptotically stable).

Definition 7.10 ([5]) The solution $v = v(t,x)$ of Eq. (7.8) is said to be globally attractive if Eq. (7.9) holds for each solution $w = w(t,x)$ of Eq. (7.8). If condition Eq. (7.9) is satisfied uniformly with respect to the set Ω, solutions of Eq. (7.8) are said to be globally asymptotically stable (or uniformly globally attractive).

7.3 Main Results

Prior to getting into technical considerations and estimates, let us define what we mean by a solution to the system Eqs. (7.1)–(7.3).

Definition 7.11 A function $u \in BC$ whose mixed derivative D^2_{tx} exists and is integrable, is said to be a solution to the system Eqs. (7.1)–(7.3), if u satisfies Eqs. (7.1) and (7.3) on J and that Eq. (7.2) on $\tilde{J}$ holds.

7.3.1 Existence and Uniqueness

Our first result is devoted to the existence and uniqueness of a solution to Eqs. (7.1)–(7.3).

Theorem 7.12 *Assume that the following assumptions hold,*

(H.61) *The function φ is continuous and bounded with*

$$\varphi^* = \sup_{(t,x)\in\mathbb{R}_+\times[0,b]} |\varphi(t,x)|;$$

(H.62) *There exist positive functions $p_1, p_2 \in BC(J)$ such that*

$$|f(t,x,u_1,u_2) - f(t,x,v_1,v_2)| \le p_1(t,x)\|u_1 - v_1\|_{\mathscr{C}} + p_2(t,x)|u_2 - v_2|,$$

for each $(t,x) \in J$, $u_1, v_1 \in \mathscr{C}$ and $u_2, v_2 \in \mathbb{R}$. Moreover, assume that the function

$$t \to \int_0^t \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} f(s,y,0,(G0)(s,y))dyds$$

is bounded on J with

$$f^* = \sup_{(t,x)\in J} \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1}|f(s,y,0,(G0)(s,y))|dyds;$$

(H.63) *There exists a positive function $q \in BC(J_1)$ such that*

$$|g(t,x,s,y,u) - g(t,x,s,y,v)| \le q(t,x,s,y)|u-v|,$$

for each $(t,x,s,y) \in J_1$ and $u, v \in \mathbb{R}$.

If

$$p_1^* + p_2^* q^* < 1, \tag{7.10}$$

where

$$p_i^* = \sup_{(t,x)\in J} \Big[\frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} p_i(s,y)dyds\Big];\ i = 1,2,$$

and

$$q^* = \sup_{(t,x)\in J} \Big[\frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} q(t,x,s,y)dyds\Big],$$

then the system (7.1)–(7.3) has a unique solution on $[-\alpha,\infty)\times[-\beta,b]$.

Proof Define the nonlinear operator $N : BC \to BC$ by

$$(Nu)(t,x) = \begin{cases} \Phi(t,x), & (t,x) \in \tilde{J}, \\ \varphi(t) + \dfrac{1}{\Gamma(r_1)\Gamma(r_2)} \displaystyle\int_0^t \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} & \\ \times f(s,y,u_{(s,y)},(Gu)(s,y))dyds, & (t,x) \in J. \end{cases} \tag{7.11}$$

Clearly, the function $(t,x) \mapsto (Nu)(t,x)$ is continuous on $[-\alpha,\infty) \times [-\beta,b]$. The next step consists of showing that $N(u) \in BC$ for each $u \in BC$. Indeed, for each $(t,x) \in \tilde{J}$, we have

$$|\Phi(t,x)| \leq \sup_{(t,x)\in\tilde{J}} |\Phi(t,x)| := \Phi^*,$$

and so $\Phi \in BC$.

From (H.62), and for arbitrarily fixed $(t,x) \in J$, we have

$$\begin{aligned} |(Nu)(t,x)| &= \Big|\varphi(t) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} \\ &\quad \times f(s,y,u_{(s,y)},(Gu)(s,y))dyds\Big| \\ &\leq |\varphi(t)| + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} \\ &\quad |f(s,y,u_{(s,y)},(Gu)(s,y)) - f(s,y,0,(G0)(s,y))|dyds \\ &\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1}|f(s,y,0,(G0)(s,y))|dyds \\ &\leq |\varphi(t)| + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} \\ &\quad \times \left(p_1(s,y)|u_{(s,y)}| + p_2(s,y)|(Gu)(s,y)|\right) dyds \\ &\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1}|f(s,y,0,(G0)(s,y))|dyds \\ &\leq \varphi^* + f^* + p_1^*\|u\|_{BC} + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} \\ &\quad \times p_2(s,y)|(Gu)(s,y) - (G0)(s,y)|dyds. \end{aligned} \tag{7.12}$$

Now (H.63) yields

$$\begin{aligned}
&|(Gu)(t,x)-(G0)(t,x)| \\
&\quad\le \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} \\
&\qquad\times |g(t,x,s,y,u(s,y))-g(t,x,s,y,0)|dyds \\
&\quad\le \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1}q(t,x,s,y)|u(s,y)|dyds \\
&\quad\le q^*\|u\|_{BC}.
\end{aligned}$$

From (7.12) we get

$$\begin{aligned}
|(Nu)(t,x)| &\le \varphi^*+f^*+p_1^*\|u\|_{BC} \\
&\quad+\frac{q^*\|u\|_{BC}}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1}p_2(s,y)dyds \\
&\le \varphi^*+f^*+p_1^*\|u\|_{BC}+p_2^*q^*\|u\|_{BC} \\
&\le \varphi^*+f^*+(p_1^*+p_2^*q^*)\|u\|_{BC}.
\end{aligned}$$

thus $N(u)\in BC$.

Let $u,\ v\in BC$. Using our assumptions, for each $(t,x)\in J$, we obtain,

$$\begin{aligned}
&|(Nu)(t,x)-(Nv)(t,x)| \le \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} \\
&\quad\times |f(s,y,u_{(s,y)},(Gu)(s,y))-f(s,y,v_{(s,y)},(Gv)(s,y))|dyds \\
&\quad\le \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} \\
&\quad\times (p_1(s,y)\|u_{(s,y)}-v_{(s,y)}\|_{\mathcal{C}}+p_2(s,y)|(Gu)(s,y)-(Gv)(s,y)|)dyds \\
&\quad\le \frac{\|u-v\|_{BC}}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1}p_1(s,y)dyds \\
&\quad+\frac{\|u-v\|_{BC}}{\Gamma^2(r_1)\Gamma^2(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} \\
&\quad\times p_2(s,y)\left(\int_0^s\int_0^y (s-\tau)^{r_1-1}(y-\xi)^{r_2-1}q(s,t,\tau,\xi)d\xi d\tau\right)dyds \\
&\quad\le (p_1^*+p_2^*q^*)\|u-v\|_{BC}.
\end{aligned}$$

From Eq. (7.10), it follows from the Banach fixed-point principle (Theorem 1.30) that N has a unique fixed point in BC which is the solution to Eqs. (7.1)–(7.3).

7.3.2 Estimates for Solutions

Theorem 7.13 *Set*

$$d = \varphi^* + f^*. \tag{7.13}$$

Suppose that assumptions (H.61)–(H.63) hold and that we have,

(H.64) $p_1 = p_2$ *and there exists a positive function* $p \in BC(J)$ *such that,*

$$p_1(s, y) \leq \Gamma(r_1)\Gamma(r_2)(t-s)^{1-r_1}(x-y)^{1-r_2}p(s, y), \text{ for each } (t, x, s, y) \in J_1,$$

(H.65) $k, D_1k, D_2k, D_2D_1k \in BC(J_1)$, *where*

$$k(t, x, s, y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)}(t-s)^{r_1-1}(x-y)^{r_2-1}q(t, x, s, y).$$

For any solution u to Eqs. (7.1)–(7.3) on $[-\alpha, \infty)\times[-\beta, b]$, *then for each* $(t, x) \in J$,

$$|u(t, x)| \leq d\left[1 + \int_0^t \int_0^x p(s, y)\exp\left(\int_0^s \int_0^y [p(\tau, \xi) + A(\tau, \xi)]d\xi d\tau\right)dyds\right], \tag{7.14}$$

where $A(t, x)$ is defined by Eq. (7.7).

Proof Using the fact that u is a solution to Eqs. (7.1)–(7.3) and from our assumptions, we have, for each $(t, x) \in J$,

$$\begin{aligned}
|u(t, x)| &\leq |\varphi(t)| \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1}|f(t, x, 0, (G0)(t, x))|dyds \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} \\
&\quad \times |f(s, y, u_{(s,y)}, (Gu)(s, y)) - f(s, y, 0, (G0)(s, y))|dyds \\
&\leq \varphi^* + f^* + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1}p_1(s, y)\Bigg[\|u_{(s,y)}\|_{\mathscr{C}} \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^s \int_0^y (s-\tau)^{r_1-1}(y-\xi)^{r_2-1}q(s, y, \tau, \xi)|u(\tau, \xi)|d\xi d\tau\Bigg]dyds
\end{aligned}$$

$$
\begin{aligned}
&\le d + \int_0^t \int_0^x p(s,y)\Bigg[\|u_{(s,y)}\|_{\mathscr{C}} \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^s \int_0^y q(s,y,\tau,\xi)|u(\tau,\xi)|d\xi d\tau\Bigg] dyds \\
&\le d + \int_0^t \int_0^x p(s,y)\left[\|u_{(s,y)}\|_{\mathscr{C}} + \int_0^s \int_0^y k(s,y,\tau,\xi)|u(\tau,\xi)|d\xi d\tau\right] dyds.
\end{aligned}
$$

Consider the function w defined by

$$
w(t,x) = \sup\{\|u(s,y)\| : -\alpha \le s \le t,\ \ -\beta \le y \le x\},\ 0 \le t < \infty,\ 0 \le x \le b.
$$

Let $(t^*, x^*) \in [-\alpha, t] \times [-\beta, x]$ be such that $w(t,x) = |u(t^*, x^*)|$. If $(t^*, x^*) \in \tilde{J}$, then $w(t,x) = \|\Phi\|_{\mathscr{C}}$ and the previous inequality holds. If $(t^*, x^*) \in J$, then by the previous inequality, we have for $(t,x) \in J$,

$$
w(t,x) \le d + \int_0^t \int_0^x p(s,y)\left[w(s,y) + \int_0^s \int_0^y k(s,y,\tau,\xi) w(\tau,\xi) d\xi d\tau\right] dyds.
$$

From Lemma 7.7, we get

$$
w(t,x) \le d\left[1 + \int_0^t \int_0^x p(s,y) \exp\left(\int_0^s \int_0^y [p(\tau,\xi) + A(\tau,\xi)] d\xi d\tau\right) dyds\right];
$$

$$
(t,x) \in J. \tag{7.15}
$$

But, for every $(t,x) \in J$, $\|u_{(t,x)}\|_{\mathscr{C}} \le w(t,x)$. Hence, Eq. (7.15) yields Eq. (7.14).

Theorem 7.14 *Set*

$$
\overline{d} := f^* + \varphi^* p^* (1 + q^*). \tag{7.16}
$$

Suppose that assumptions (H.61)–(H.65) hold. For any solution u to Eq. (7.2) on $[-\alpha, \infty) \times [-\beta, b]$, *we have the following estimates,*

$$
\begin{aligned}
&|u(t,x) - \varphi(t)| \\
&\quad \le \overline{d}\left[1 + \int_0^t \int_0^x p(s,y) \exp\left(\int_0^s \int_0^y [p(\tau,\xi) + A(\tau,\xi)] d\xi d\tau\right) dyds\right]
\end{aligned} \tag{7.17}
$$

for all $(t,x) \in J$, where A is given by Eq. (7.7).

Proof Let $h(t,x) = |u(t,x) - \varphi(t)|$. Using the fact that u is a solution to Eqs. (7.1)–(7.3) combined with our assumptions, it follows that, for each $(t,x) \in J$,

$$h(t,x) \le \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1}$$
$$\times |f(s,y,u_{(s,y)},(Gu)(s,y)) - f(s,y,\varphi(s),(G\varphi)(s))|dyds$$
$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1}|f(s,y,\varphi(s),(G\varphi)(s))|dyds$$
$$\le \overline{d} + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1}$$
$$\times |f(s,y,u_{(s,y)},(Gu)(s,y)) - f(s,y,\varphi(s),(G\varphi)(s))|dyds$$
$$\le \overline{d} + \int_0^t\int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1}p(s,y)$$
$$\times \left[h(s,y) + \int_0^s\int_0^y k(s,y,\tau,\xi)h(\tau,\xi)d\xi d\tau\right]dyds. \tag{7.18}$$

Using Lemma 7.7 and Eq. (7.18), one obtains Eq. (7.17).

7.3.3 Global Asymptotic Stability of Solutions

Our main objective here is to study the global asymptotic stability of solution. For that, we show that under more suitable conditions on the functions involved in Eqs. (7.1)–(7.3) that the solutions go zero exponentially as $t \to \infty$.

Theorem 7.15 *Suppose that assumptions (H.64)–(H.65) hold and that*

(H.66) *There exist constants* $\lambda > 0$ *and* $M \ge 0$ *such that*

$$|\varphi(t)| \le Me^{-\lambda t}; \tag{7.19}$$

$$|f(t,x,u_1,u_2) - f(t,x,v_1,v_2)| \le p_1(t,x)e^{-\lambda t}(\|u_1 - v_1\|_{\mathscr{C}} + |u_2 - v_2|), \tag{7.20}$$

for each $(t,x) \in J,\ u_1, v_1 \in \mathscr{C},\ u_2, v_2 \in \mathbb{R}$,

$$|g(t,x,s,y,u) - g(t,x,s,y,v)| \le q(t,x,s,y)|u - v|; \tag{7.21}$$

for each $(t,x,s,y) \in J_1,\ u,v \in \mathbb{R}$, *and* $f(t,x,0,(G0)(t,x)) = 0$; *for each* $(t,x) \in J$ *and the functions* $p,\ q$ *be as in Theorem 7.13; and*

(H.67) $\int_0^\infty\int_0^x [p(s,y) + A(s,y)]dyds < \infty$, *where A is given by Eq. (7.7).*

If u is any solution of Eqs. (7.1)–(7.3) on $[-\alpha,\infty)\times[-\beta,b]$, *then all solutions to Eqs. (7.1)–(7.3) are uniformly globally attractive on J.*

Proof From our assumptions, we have, for each $(t, x) \in J$,

$$
\begin{aligned}
|u(t,x)| &\leq |\varphi(t)| + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} \\
&\quad \times |f(s, y, u_{(s,y)}, (Gu)(s,y)) - g(s, y, 0, (G0)(s,y))| dyds \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} |f(s, y, 0, (G0)(s,y))| dyds \\
&\leq Me^{-\lambda t} + \int_0^t \int_0^x p(s,y) e^{-\lambda t} \Big[u_{(s,y)} + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \\
&\quad \times \int_0^s \int_0^y (s-\tau)^{r_1-1}(y-\xi)^{r_2-1} q(s,y,\tau,\xi)|u(\tau,\xi)| d\xi d\tau \Big] dyds. \qquad (7.22)
\end{aligned}
$$

From Eq. (7.22), we get

$$
|u(t,x)|e^{\lambda t} \leq M + \int_0^t \int_0^x p(s,y) \left[u_{(s,y)} + k(s,y,\tau,\xi)|u(\tau,\xi)| d\xi d\tau \right] dyds. \qquad (7.23)
$$

Using Lemma 7.7 to Eq. (7.23) we obtain

$$
|u(t,x)|e^{\lambda t} \leq M \left[1 + \int_0^t \int_0^x p(s,y) \exp\left(\int_0^s \int_0^y [p(\tau,\xi) + A(\tau,\xi)] d\xi d\tau \right) dyds \right]; \quad (t,x) \in J, \qquad (7.24)
$$

Multiplying both sides of Eq. (7.24) by $e^{-\lambda t}$ and in view of (H. 66), we get

$$
|u(t,x)| \leq M \left[e^{-\lambda t} + \int_0^t \int_0^x p(s,y) \exp\Big(-\lambda t + \int_0^s \int_0^y [p(\tau,\xi) + A(\tau,\xi)] d\xi d\tau \Big) dyds \right].
$$

Thus, for each $x \in [0, b]$,

$$
\lim_{t \to \infty} u(t,x) = 0.
$$

Therefore, the solution u goes to zero as $t \to \infty$. Consequently, all solutions to Eqs. (7.1)–(7.3) are uniformly globally attractive on $[-\alpha, \infty) \times [-\beta, b]$.

7.4 Example

To illustrate our previous results, we consider the system of partial fractional integro-differential equations of the form,

$$\mathbb{D}_\theta^r u(t,x) = f(t,x,u_{(t,x)},(Gu)(t,x)); \quad \text{for } (t,x) \in J := \mathbb{R}_+ \times [0,1], \tag{7.25}$$

$$u(t,x) = \frac{1}{1+t^2}; \text{ if } (t,x) \in \tilde{J} := [-1,\infty) \times [-2,1] \backslash (0,\infty) \times (0,1], \tag{7.26}$$

$$\begin{cases} u(t,0) = \dfrac{1}{1+t^2}; \ t \in \mathbb{R}_+, \\ u(0,x) = 1; \ x \in [0,1], \end{cases} \tag{7.27}$$

where

$$(Gu)(t,x) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} g(t,x,s,y,u_{(s,y)})dyds, \tag{7.28}$$

$r_1, r_2 \in (0,1]$,

$$\begin{cases} f(t,x,u,v) = \dfrac{x^2 t^{-r_1} \sin t}{2c(1+t^{-\frac{1}{2}})(1+|u(t+1,x+2)|+|v|)}; \\ \qquad for \ \ (t,x) \in J, \ t \neq 0 \, and \ \ u \in \mathscr{C}, \ v \in \mathbb{R}, \\ f(0,x,u,v) = 0, \end{cases}$$

$$c := \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+r_1)} \left(1 + \frac{\Gamma(\frac{1}{2})e}{\Gamma(\frac{1}{2}+r_1)\Gamma(1+r_2)} \right),$$

$$\begin{cases} g(t,x,s,y,u) = \dfrac{t^{-r_1} s^{-\frac{1}{2}} e^{x-y-\frac{1}{s}-\frac{1}{t}}}{2c(1+t^{-\frac{1}{2}})(1+|u|)}; \ for \ (t,x,s,y) \in J_1, \ st \neq 0 \, and \, u \in \mathbb{R}, \\ g(t,x,0,y,u) = g(0,x,s,y,u) = 0, \end{cases}$$

and

$$J_1 = \{(t,x,s,y) : 0 \le s \le t < \infty, \ 0 \le y \le x \le 1\}.$$

Set

$$\varphi(t) = \frac{1}{1+t^2}; \ t \in \mathbb{R}_+.$$

One can see that (H.61) holds as the function φ is continuous and bounded with $\varphi^* = 1$.

For each $u_1, v_1 \in \mathscr{C},\ u_2, v_2 \in \mathbb{R}$ and $(t, x) \in J$, we have

$$\begin{aligned}
&|f(t,x,u_1,u_2) - f(t,x,s,v_1,v_2)| \\
&\leq \frac{1}{2c(1+t^{-\frac{1}{2}})}\left(x^2 t^{-r_1}|\sin t|\right)(|u_1 - v_1| + |u_2 - v_2|),
\end{aligned}$$

and for each $u, v \in \mathbb{R}$ and $(t, x, s, y) \in J_1$, we have

$$|g(t,x,s,y,u) - g(t,x,s,y,v)| \leq \frac{1}{2c(1+t^{-\frac{1}{2}})}\left(t^{-r_1} s^{-\frac{1}{2}} e^{x-y-t-\frac{1}{s}-\frac{1}{t}}\right)|u - v|.$$

Therefore, (H.62) holds with

$$\begin{cases}
p_1(t,x) = p_2(t,x) = \dfrac{x^2 t^{-r_1}|\sin t|}{2c(1+t^{-\frac{1}{2}})};\ t \neq 0, \\
p_1(0,x) = p_2(0,x) = 0,
\end{cases}$$

and assumption (H.63) holds with

$$\begin{cases}
q(t,x,s,y) = \dfrac{1}{2c(1+t^{-\frac{1}{2}})}\left(t^{-r_1} s^{-\frac{1}{2}} e^{x-y-t-\frac{1}{s}-\frac{1}{t}}\right);\ st \neq 0, \\
q(t,x,0,y) = k(0,x,0,y) = 0.
\end{cases}$$

We shall show that Eq. (7.10) holds with $b = 1$. Indeed,

$$\begin{aligned}
&\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} p_1(s,y)\,dy\,ds \\
&\leq \frac{1}{2c(1+t^{-\frac{1}{2}})\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^1 (t-s)^{r_1-1}(1-y)^{r_2-1} x^2 t^{-r_1}\,dy\,ds \\
&\leq \frac{\Gamma(\frac{1}{2}) e t^{-\frac{1}{2}}}{2c(1+t^{-\frac{1}{2}})\Gamma(\frac{1}{2}+r_1)\Gamma(1+r_2)},
\end{aligned}$$

then

$$p_1^* = p_2^* \leq \frac{\Gamma(\frac{1}{2})}{2c\Gamma(\frac{1}{2}+r_1)}.$$

Now

$$\begin{aligned}
&\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1}q(t,x,s,y)dyds\\
&\quad\le \frac{1}{2c(1+t^{-\frac{1}{2}})\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^1 (t-s)^{r_1-1}(1-y)^{r_2-1}t^{-r_1}s^{-\frac{1}{2}}e^x dyds\\
&\quad\le e^x t^{-r_1}t^{-\frac{1}{2}+r_1}\frac{\Gamma(\frac{1}{2})}{2c(1+t^{-\frac{1}{2}})\Gamma(\frac{1}{2}+r_1)\Gamma(1+r_2)}\\
&\quad\le \frac{\Gamma(\frac{1}{2})et^{-\frac{1}{2}}}{2c(1+t^{-\frac{1}{2}})\Gamma(\frac{1}{2}+r_1)\Gamma(1+r_2)},
\end{aligned}$$

then

$$q^* \le \frac{e\Gamma(\frac{1}{2})}{2c\Gamma(\frac{1}{2}+r_1)\Gamma(1+r_2)}.$$

Thus,

$$p_1^* + p_2^* q^* \le \frac{\Gamma(\frac{1}{2})}{2c\Gamma(\frac{1}{2}+r_1)}\left(1+\frac{\Gamma(\frac{1}{2})e}{\Gamma(\frac{1}{2}+r_1)\Gamma(1+r_2)}\right) = \frac{1}{2} < 1,$$

which holds for each $r_1, r_2 \in (0,\infty)$. Consequently Theorem 7.12 yields Eq. (7.25)—(7.27) has a unique solution defined on $[-1,\infty)\times[-2,1]$.

7.5 Comments

This chapter is mainly based upon the following source: Abbas et al. [7] with some slight modifications. For additional readings on similar topics and related issues, we refer the readers to the following references: [16, 33, 35, 36, 71, 74]. Furthermore, recent progress made upon the study of ordinary and partial fractional differential equations can be found in the following books: Abbas et al. [6], Baleanu et al. [19], Diethelm [51], Kilbas et al. [77], Miller and Ross [91], Podlubny [101], and Samko et al. [103].