Chapter 6
Singular Difference Equations

6.1 Introduction

The mathematical problem which consists of studying the existence of solutions to singular difference equations with almost periodic coefficients is an important one as almost periodicity, according to Henson et al. [66], is more likely to accurately describe many phenomena occurring in population dynamics than periodicity. In the previous chapter, the existence of almost periodic solutions to some classes of nonautonomous non-singular difference equations was obtained. These results were utilized to study the effect of almost periodicity upon the Beverton-Holt model.

In this chapter, we study and establish the existence of Bohr (respectively, Besicovitch) almost periodic solutions to the following class of singular systems of difference equations,

$$Ax(t+1) + Bx(t) = f(t, x(t)) \tag{6.1}$$

where $f : \mathbb{Z} \times \mathbb{R}^N \to \mathbb{R}^N$ is Bohr (respectively, Besicovitch) almost periodic in $t \in \mathbb{Z}$ uniformly in the second variable, and A, B are $N \times N$ square matrices satisfying $\det A = \det B = 0$.

Recall that singular difference equations of the form Eq. (6.12) arise in many applications including optimal control, population dynamics, economics, and numerical analysis [52]. The main result discussed in this chapter can be summarized as follows: if $\lambda A + B$ is invertible for all $\lambda \in \mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ and if f is Bohr (respectively, Besicovitch) almost periodic in $t \in \mathbb{Z}$ uniformly in the second variable and under some additional conditions, then Eq. (6.12) has a unique Bohr (respectively, Besicovitch) almost periodic solution.

The chapter is organized as follows: Sect. 6.1 serves as an introduction but also provides preliminary tools needed in the sequel. In Sect. 6.2, some preliminary

T. Diagana, *Semilinear Evolution Equations and Their Applications*,
https://doi.org/10.1007/978-3-030-00449-1_6

results corresponding to the case $f(t, x(t)) = C(t)$ are obtained. Section 6.3 is devoted to the main results of this chapter. In Sect. 6.4, we make use of the main results in Sect. 6.3 to study the existence of Bohr (respectively, Besicovitch) almost periodic solutions for some second-order (and higher-order) singular systems of difference equations.

Let $x = (x(t))_{t\in\mathbb{Z}}$ be a sequence. Define $P(x)$ as follows

$$P(x) := \sup_{k\in\mathbb{Z}} \limsup_{N\to\infty} \left[\frac{1}{N}\sum_{j=k+1}^{k+N} \|x(j)\|^2\right]^{\frac{1}{2}}.$$

Set

$$\widetilde{B} = \Big\{x = (x(t))_{t\in\mathbb{Z}} : P(x) < \infty\Big\}.$$

It is not hard to see that P is a semi-norm on $\widetilde{B}$. Consider the following equivalence relation on $\widetilde{B}$: $x, y \in \widetilde{B}$, $x \sim y$ if and only if, $P(x - y) = 0$. The quotient space

$$B := \widetilde{B}/\sim$$

endowed with $P(\cdot)$ is a normed vector space.

Definition 6.1 A sequence $x = (x(t))_{t\in\mathbb{Z}}$ is called Besicovitch almost periodic if it belongs to the closure of trigonometric polynomials under the semi-norm P. The collection of all Besicovitch almost periodic sequences will be denoted $B^2(\mathbb{Z}, \mathbb{R}^N)$.

Definition 6.2 A sequence $F : \mathbb{Z} \times \mathbb{R}^N \mapsto \mathbb{R}^N$, $(t, u) \mapsto F(t, u)$ is called Besicovitch almost periodic in $t \in \mathbb{Z}$ if $t \mapsto F(t, u)$ belongs to $B^2(\mathbb{Z}, \mathbb{R}^N)$ uniformly in $u \in \mathbb{R}^N$.

6.2 The Case of a Linear Equation

In this section, we consider the case when the forcing term f does not depend on x, that is, $f(t, x(t)) = C(t)$ where $(C(t))_{t\in\mathbb{Z}}$ is almost periodic. Namely, we study the existence of almost periodic solutions for the singular difference equation

$$Ax(t+1) + Bx(t) = C(t), \quad t \in \mathbb{Z} \tag{6.2}$$

where $C : \mathbb{Z} \mapsto \mathbb{R}^N$ is Bohr (respectively, Besicovitch) almost periodic.

6.2.1 Existence of a Bohr Almost Periodic Solution

Define the resolvent $\rho(A, B)$ by

$$\rho(A, B) := \Big\{\lambda \in \mathbb{C} : \lambda A + B \quad \text{is invertible}\Big\}.$$

Theorem 6.3 *If $\mathbb{S}^1 \subseteq \rho(A, B)$, then Eq. (6.2) has a unique almost periodic solution.*

Proof The strategy here consists of adapting our setting to that of Campbell [34]. Indeed, setting $\hat{A} = (A + B)^{-1}A$, $\hat{B} = (A + B)^{-1}B$, and $\hat{C(t)} = (A + B)^{-1}C(t)$, one can easily see that Eq. (6.2) is equivalent to,

$$\hat{A}x(t + 1) + \hat{B}x(t) = \hat{C(t)}, \quad t \in \mathbb{Z}. \tag{6.3}$$

Using the identity, $\hat{A} + \hat{B} = I_N$, it follows that $\hat{A}\hat{B} = \hat{B}\hat{A}$. Consequently, one can find a common basis of trigonalization for $\hat{A}$ and $\hat{B}$. That is, there exists an invertible matrix T such that

$$\hat{A} = T^{-1}\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} T, \quad \hat{B} = T^{-1}\begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} T,$$

where A_1, B_2 are invertible and A_2, B_1 are nilpotent.

Recall that here, $A_i + B_i = I_N$ for $i = 1, 2$. Consequently, writing

$$Tx(t) = \begin{pmatrix} w(t) \\ v(t) \end{pmatrix}$$

and

$$T\hat{C(t)} = \begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix},$$

where $(\alpha(t))_t$ and $(\beta(t))_t$ are almost periodic, it follows that Eq. (6.3) can be rewritten as

$$\begin{cases} A_1 w(t + 1) + B_1 w(t) = \alpha(t) \\ A_2 v(t + 1) + B_2 v(t) = \beta(t). \end{cases} \tag{6.4}$$

Using the fact that both A_1 and B_2 are invertible, one can see that Eq. (6.4) is equivalent to,

$$\begin{cases} w(t + 1) + A_1^{-1}B_1 w(t) = A_1^{-1}\alpha(t) \\ B_2^{-1}A_2 v(t + 1) + v(t) = B_2^{-1}\beta(t). \end{cases} \tag{6.5}$$

Let us now put our main focus upon the first equation appearing in Eq. (6.5), that is, the equation given by

$$w(t+1) - (-A_1^{-1}B_1)w(t) = A_1^{-1}\alpha(t), \quad t \in \mathbb{Z}. \tag{6.6}$$

Obviously, $t \mapsto (A_1^{-1}\alpha(t))_{t\in\mathbb{Z}}$ is almost periodic. We shall now prove that $-A_1^{-1}B_1$ has no eigenvalue that belongs to $\mathbb{S}^1$. From that, we will deduce that Eq. (6.6) has a unique almost periodic solution. For that, consider a nonzero eigenvalue λ of $-A_1^{-1}B_1$. Let $x_1 \neq 0$ be an eigenvector for $-A_1^{-1}B_1$, that is, $-A_1^{-1}B_1x_1 = \lambda x_1$. Consequently,

$$(\lambda A_1 + B_1)x_1 = 0,$$

from which we deduce that

$$(\lambda\hat{A} + \hat{B})T^{-1}\begin{pmatrix} x_1 \\ 0 \end{pmatrix} = 0.$$

Using the fact that

$$T^{-1}\begin{pmatrix} x_1 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

we deduce that $\lambda\hat{A} + \hat{B}$ is not invertible, thus this is the case for $\lambda A + B$ too. With the assumption made, this proves that $|\lambda| \neq 1$. Consequently, there exists a unique almost periodic solution $(w(t))_{t\in\mathbb{Z}}$ to

$$w(t+1) - (-A_1^{-1}B_1)w(t) = A_1^{-1}\alpha(t).$$

For the second equation appearing in Eq. (6.5), setting $V(t) = v(-t)$, it becomes, by changing t in $-t$,

$$V(t) + B_2^{-1}A_2V(t-1) = B_2^{-1}\beta(-t). \tag{6.7}$$

Using similar arguments as above, one can see that Eq. (6.7) has a unique almost periodic solution $(V(t))_{t\in\mathbb{Z}}$, so the second equation appearing in Eq. (6.5) has a unique almost periodic solution $(v(t))_{t\in\mathbb{Z}}$. Since Eqs. (6.5) and (6.2) are equivalent, we obtain existence and uniqueness of an almost periodic solution to Eq. (6.2). The proof is complete.

Remark 6.4 Notice that the continuous operator

$$\mathscr{T} : (x(t))_{t\in\mathbb{Z}} \to (Ax(t+1) + Bx(t))_{t\in\mathbb{Z}}$$

is invertible and maps the Banach space of (Bohr) almost periodic sequences into itself. It follows from the *Bounded Inverse Theorem* that $\mathcal{T}^{-1}$ is also continuous. Consequently, there exists a constant $M > 0$ such that for all $(C(t))_{t\in\mathbb{Z}}$,

$$\|(x(t))_t\|_\infty \le M\|(C(t))_t\|_\infty.$$

An immediate consequence of Theorem 6.3 is the following:

Corollary 6.5 *Let $A = (a_{ij})$ and $B = (b_{ij})$ be $N \times N$ square matrices and suppose that*

$$\forall i,\ \ ||a_{ii}| - |b_{ii}|| > \sum_{j\neq i} \left(|a_{ij}| + |b_{ij}|\right).$$

Then Eq. (6.2) has a unique almost periodic solution.

Proof Indeed, let $\lambda \in \mathbb{S}^1$ and let $c_{ij} = a_{ij}\lambda + b_{ij}$, so that $\lambda A + B = (c_{ij})$.
Now

$$|c_{ii}| = |\lambda a_{ii} + b_{ii}| \ge ||\lambda a_{ii}| - |b_{ii}|| = ||a_{ii}| - |b_{ii}||$$

and

$$\sum_{j\neq i} |c_{ij}| = \sum_{j\neq i} |a_{ij}\lambda + b_{ij}| \le \sum_{j\neq i} \left(|a_{ij}| + |b_{ij}|\right)$$

so thus for all i,

$$|c_{ii}| > \sum_{j\neq i} |c_{ij}|,$$

which yields $\mathbb{S}^1 \subseteq \rho(A, B)$.

In view of the above, using Theorem 6.3 it follows that Eq. (6.2) has a unique almost periodic solution.

Remark 6.6 Let us mention that there exist infinitely many pairs of matrices (A, B) satisfying the assumption of Corollary 6.5.

6.2.2 *Existence of Besicovitch Almost Periodic Solution*

In this subsection, we suppose $(C(t))_{t\in\mathbb{Z}}$ is Besicovitch almost periodic and study the existence of Besicovitch almost periodic solutions to Eq. (6.2). Here, the proof is more straightforward, using tools from Fourier analysis.

Indeed, write $C(t) \sim \sum_{\alpha\in[0,2\pi)} c_\alpha \hat{e}_\alpha(t)$, where $\hat{e}_\alpha(t) = e^{i\alpha t}$ and $(c_\alpha)_\alpha \in \ell^2([0,2\pi),\mathbb{R}^N)$. We look for a solution in the following form,

$$x(t) \sim \sum_{\alpha\in[0,2\pi)} a_\alpha \hat{e}_\alpha(t), \quad (a_\alpha)_\alpha \in \ell^2([0,2\pi),\mathbb{R}^N).$$

Now

$$Ax(t+1) + Bx(t) \sim \sum_{\alpha\in[0,2\pi)} (\hat{e}_\alpha(1)Aa_\alpha + Ba_\alpha)\hat{e}_\alpha(t).$$

By the uniqueness of the Fourier-Bohr expansion, Eq. (6.2) is equivalent to,

$$\forall\alpha \in [0,2\pi), \quad (\hat{e}_\alpha(1)A + B)a_\alpha = c_\alpha.$$

Since $\hat{e}_\alpha(1) \in \mathbb{S}^1$, given that $\hat{e}_\alpha(1)A + B$ is invertible, so we obtain a candidate

$$\forall\alpha \in [0,2\pi), \quad a_\alpha = (\hat{e}_\alpha(1)A + B)^{-1}c_\alpha.$$

We need now to prove that $(a_\alpha)_\alpha \in \ell^2([0,2\pi),\mathbb{R}^N)$. Since $\mathbb{S}^1$ is compact, then the function $\mathbb{S}^1 \mapsto (0,\infty)$, $\lambda \to \|(\lambda A + B)^{-1}\|$ is bounded and so let $M > 0$ be such that

$$\forall\lambda \in \mathbb{S}^1, \quad \|(\lambda A + B)^{-1}\| \le M.$$

Clearly,

$$|a_\alpha|^2 \le M^2 |c_\alpha|^2.$$

This yields $(a_\alpha)_\alpha \in \ell^2([0,2\pi),\mathbb{R}^N)$. Further,

$$\|(x(t))_{t\in\mathbb{Z}}\|_2 \le M\|(C(t))_{t\in\mathbb{Z}}\|_2.$$

Notice here that we have a formula for M which is given by

$$M = \sup_{\lambda\in\mathbb{S}^1} \|(\lambda A + B)^{-1}\|.$$

Remark 6.7 In the case of assumptions of Theorem 6.5, one can actually compute explicitly a bound for M. Indeed, let us consider

$$\theta := \min_{i=1,2,\ldots,N} \{||a_{ii}| - |b_{ii}|| - \sum_{j\neq i}(|a_{ij}| + |b_{ij}|)\}.$$

Then

$$M \leq \frac{\sqrt{n}}{\theta}.$$

Set $c_{ij} = a_{ij}\lambda + b_{ij}$. Given $\lambda \in \mathbb{S}^1$, let us consider the system $Y = (\lambda A + B)X$ and fix i_0 such that $|X_{i_0}| = \max_i |X_i|$.

Now

$$|Y|_2 \geq |Y_{i_0}| = |\sum_j (\lambda a_{i_0 j} + b_{i_0 j} X_j)| \geq$$

$$|a_{i_0 i_0}|.|X_{i_0}| - \sum_{j \neq i_0} |\lambda a_{i_0 j} + b_{i_0 j}||X_j| \geq \theta |X|_\infty,$$

thus

$$|X|_2 \leq \sqrt{n}|X|_\infty \leq \frac{\sqrt{n}}{\theta}|Y|_2.$$

We apply this with $Y = c_\alpha$ and $X = a_\alpha$.

6.3 The Semilinear Equation

First of all, note that from Sect. 6.2, we deduce that the linear operator

$$\mathscr{T} : (x(t))_{t \in \mathbb{Z}} \to (Ax(t+1) + Bx(t))_{t \in \mathbb{Z}}$$

is bijective and bi-continuous from $AP(\mathbb{Z}, \mathbb{R}^N)$ (respectively, from $B^2(\mathbb{Z}, \mathbb{R}^N)$ into itself) into itself.

Using similar arguments as above and the composition of almost periodic sequences, we can obtain the existence of Bohr (respectively, Besicovich) almost periodic solutions to Eq. (6.12).

Theorem 6.8 *Suppose $\mathbb{S}^1 \subseteq \rho(A, B)$ and that $f \in AP(\mathbb{Z}, \mathbb{R}^N)$. Further, suppose that $x \mapsto f(t, x)$ is K-Lipschitzian. Then for sufficiently small K, Eq. (6.12) has a unique Bohr almost periodic solution.*

Theorem 6.9 *Suppose $\mathbb{S}^1 \subseteq \rho(A, B)$ and that $f : b\mathbb{Z} \times \mathbb{R}^N \to \mathbb{R}^N$ is Caratheodory, $f(., 0) \in \ell^2(b\mathbb{Z}, \mathbb{R}^N)$. Further, we suppose that $x \mapsto f(t, x)$ is K-Lipschitzian. Then for sufficiently small K, Eq. (6.12) has a unique Besicovitch almost periodic solution.*

Let X be either $AP(\mathbb{Z}, \mathbb{R}^N)$ or $B^2(\mathbb{Z}, \mathbb{R}^N)$. From the assumptions upon f, the Nemytskii operator for f is given by

$$\mathscr{N}_f : ((x(t))_{t\in\mathbb{Z}}) \mapsto (f(t, x(t)))_{t\in\mathbb{Z}},$$

which maps X into itself. Moreover, $\mathscr{T}$ is bi-continuous from X to itself.

Equation (6.12) is equivalent to,

$$\mathscr{T}((x(t))_{t\in\mathbb{Z}}) = \mathscr{N}_f((x(t))_{t\in\mathbb{Z}}),$$

which is equivalent to finding a fixed point for $\mathscr{T}^{-1} \circ \mathscr{N}_f$. This nonlinear operator is $\|\mathscr{T}^{-1}\|K$-Lipschitzian. Consequently,

$$K < \|\mathscr{T}^{-1}\|^{-1}$$

to obtain the existence of a unique almost periodic solution to Eq. (6.12), we use the Banach fixed-point theorem.

6.4 Second-Order Singular Difference Equations

Of interest is the study of (respectively, Besicovitch) Bohr almost periodic to the following second-order difference equations

$$Ax(t+2) + Bx(t+1) + Cx(t) = f(t, x(t)), \quad t \in \mathbb{Z} \tag{6.8}$$

where A, B, C are $N \times N$-squares matrices with $\det A = \det B = \det C = 0$ and $f : \mathbb{Z} \times \mathbb{R}^N \mapsto \mathbb{R}^N$ is almost periodic in the first variable uniformly in the second one.

In order to study the existence of (respectively, Besicovitch) Bohr almost periodic solutions to Eq. (6.8), one makes extensive use of the results obtained in Sect. 6.3. For that, we rewrite Eq. (6.8) as follows:

$$Lw(t+1) + Mw(t) = F(t, w(t)), \quad t \in \mathbb{Z} \tag{6.9}$$

where

$$L = \begin{pmatrix} B & A \\ I & O \end{pmatrix}, \quad M = \begin{pmatrix} C & O \\ O & -I \end{pmatrix}, \quad F = \begin{pmatrix} f \\ 0 \end{pmatrix}, \quad w(t) = \begin{pmatrix} x(t) \\ x(t+1) \end{pmatrix}$$

with O and I being the $N \times N$ zero and identity matrices.

Lemma 6.10 *$\lambda L + M$ is invertible if and only if $\lambda(A + B) + C$ is invertible.*

Proof The $2N \times 2N$ square matrix $\lambda L + M$ is given by

$$\lambda L + M = \begin{pmatrix} \lambda B + C & \lambda A \\ \lambda I & -I \end{pmatrix}.$$

Consequently, solving the system

$$(\lambda L + M)\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

yields $(\lambda B + C)u + \lambda A v = x$ and $\lambda u - v = y$. If $\lambda^2 A + \lambda B + C$ is invertible, then from $v = \lambda u - y$ it follows that $(\lambda^2 A + \lambda B + C)u = \lambda A y + x$ which yields,

$$u = \Big[\lambda^2 A + \lambda B + C\Big]^{-1}\Big(\lambda A y + x\Big), \quad v = \lambda\Big[\lambda^2 A + \lambda B + C\Big]^{-1}\Big(x + \lambda A y\Big) - y$$

which yields $\lambda L + M$ is invertible.

The proof for the converse can be done using similar arguments as above and hence is omitted.

Set

$$\rho(A, B, C) := \Big\{\lambda \in \mathbb{C} : \lambda^2 A + \lambda B + C \ \text{ is invertible}\Big\}.$$

Using Lemma 6.10, Theorem 6.8, and Theorem 6.9, we obtain the following results:

Theorem 6.11 *Suppose $\mathbb{S}^1 \subseteq \rho(A, B, C)$ and that $f \in AP(\mathbb{Z}, \mathbb{R}^N)$. Further, suppose that $x \mapsto f(t, x)$ is K-Lipschitzian. Then for sufficiently small K, Eq. (6.8) has a unique Bohr almost periodic solution.*

Theorem 6.12 *Suppose $\mathbb{S}^1 \subseteq \rho(A, B, C)$ and that $f : b\mathbb{Z} \times \mathbb{R}^N \to \mathbb{R}^N$ is Caratheodory, $f(., 0) \in \ell^2(b\mathbb{Z}, \mathbb{R}^N)$. Further, we suppose that $x \mapsto f(t, x)$ is K-Lipschitzian. Then for sufficiently small K, Eq. (6.8) has a unique Besicovitch almost periodic solution.*

Let $p \geq 2$ be an integer. One should mention that the previous techniques can be easily used to study the existence of almost periodic solutions to higher order singular systems of difference equations of the form,

$$A_p x(t+p) + A_{p-1} x(t+p-1) + \ldots + A_1 x(t+1) + A_0 x(t) = f(t, x(t)), \tag{6.10}$$

for all $t \in \mathbb{Z}$, where A_k for $k = 0, 1, 2, \ldots, p$, are $N \times N$-squares matrices with $\det A_k = 0$ for $k = 0, 1, 2, .., p$, and $f : \mathbb{Z} \times \mathbb{R}^N \mapsto \mathbb{R}^N$ is almost periodic in the first variable uniformly in the second one.

Setting

$$\rho(A_p, A_{p-1}, \ldots, A_0) := \Big\{\lambda \in \mathbb{C} : \lambda^p A_p + \lambda^{p-1} A_{p-1} + \ldots + \lambda A_1 + A_0 \text{ is invertible}\Big\},$$

the existence results can be formulated as follows:

Theorem 6.13 *Suppose* $\mathbb{S}^1 \subseteq \rho(A_p, A_{p-1}, \ldots, A_0)$ *and that* $f \in AP(\mathbb{Z}, \mathbb{R}^N)$. *Further, suppose that* $x \mapsto f(t, x)$ *is* K*-Lipschitzian. Then for sufficiently small* K, *Eq. (6.10) has a unique Bohr almost periodic solution.*

Theorem 6.14 *Suppose* $\mathbb{S}^1 \subseteq \rho(A_p, A_{p-1}, \ldots, A_0)$ *and that* $f : b\mathbb{Z} \times \mathbb{R}^N \to \mathbb{R}^N$ *is Caratheodory,* $f(.,0) \in \ell^2(b\mathbb{Z}, \mathbb{R}^N)$. *Further, we suppose that* $x \mapsto f(t, x)$ *is* K*-Lipschitzian. Then for sufficiently small* K, *Eq. (6.10) has a unique Besicovitch almost periodic solution.*

6.5 Exercises

1. Give an example of square matrices A and B that satisfy the assumption of Corollary 6.5.
2. Prove Theorem 6.8.
3. Prove Theorem 6.9.
4. Prove Theorem 6.11.
5. Prove Theorem 6.12.
6. Prove Theorem 6.13.
7. Prove Theorem 6.14.
8. Study the existence of Bohr (respectively, Besicovitch) almost periodic solutions to the following class of nonautonomous singular difference equations,

$$A(t)x(t+1) + B(t)x(t) = f(t, x(t)) \tag{6.11}$$

   where $f : \mathbb{Z} \times \mathbb{R}^N \to \mathbb{R}^N$ is Bohr (respectively, Besicovitch) almost periodic in $t \in \mathbb{Z}$ uniformly in the second variable, and $A(t)$, $B(t)$ are $N \times N$ square matrices satisfying $\det A(t) = \det B(t) = 0$ for all $t \in \mathbb{Z}$.
9. Study the existence of Bohr (respectively, Besicovitch) almost periodic solutions to the following class of nonautonomous singular difference equations,

$$A(t)x(t+2) + B(t)x(t+1) + C(t)x(t) = f(t, x(t)) \tag{6.12}$$

   where $f : \mathbb{Z} \times \mathbb{R}^N \to \mathbb{R}^N$ is Bohr (respectively, Besicovitch) almost periodic in $t \in \mathbb{Z}$ uniformly in the second variable, and $A(t)$, $B(t)$, $C(t)$ are $N \times N$ square matrices satisfying $\det A(t) = \det B(t) = \det C(t) = 0$ for all $t \in \mathbb{Z}$.

6.6 Comments

This chapter is mainly based upon the work by Diagana and Pennequin [48]. Other sources for this chapter include the work of Campbell [34]. For additional readings on the topic discussed in this chapter, we refer the reader for instance to Anh et al. [15] and Du et al. [52].