Chapter 5
Nonautonomous Difference Equations

5.1 Introduction

An autonomous difference equation is an equation of the form

$$x(t+1) = f_0(x(t)), \quad t \in \mathbb{Z}.$$

Although these equations play an important role when it comes to studying some models arising in population dynamics, they do not take into account some important parameters such as environmental fluctuations or seasonal changes. Nonautonomous difference equations, that is, equations of the form

$$x(t+1) = f_1(t, x(t)), \quad t \in \mathbb{Z},$$

seem to be more suitable to capture environmental fluctuations and seasonal changes, see, e.g., Elaydi [54].

The main objective of this chapter is two-fold. We first extend the theory of almost periodic sequences built in $\mathbb{Z}_+$ by Diagana et al. [49] to $\mathbb{Z}$. Next, we make extensive use of dichotomy techniques to find sufficient conditions for the existence of almost periodic solutions for the class of semilinear systems of difference equations given by

$$x(t+1) = A(t)x(t) + h(t, x(t)), \quad t \in \mathbb{Z} \tag{5.1}$$

where $A(t)$ is a $k \times k$ almost periodic matrix function defined on $\mathbb{Z}$, and the function $h : \mathbb{Z} \times \mathbb{R}^k \to \mathbb{R}^k$ is almost periodic in the first variable uniformly in the second one. As in the case of $\mathbb{Z}_+$, our existence results are, subsequently, applied to discretely reproducing populations with overlapping generations.

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Recall once again that $\ell^\infty(\mathbb{Z})$, the Banach space of all bounded $\mathbb{R}^k$-valued sequences, is equipped with the *sup norm* defined for each $x = \{x(t)\}_{t\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z})$, by

$$\|x\|_\infty = \sup_{t\in\mathbb{Z}} \|x(t)\|.$$

In order to deal with the existence of almost periodic solutions to the above-mentioned nonautonomous difference equations, we need to introduce the concepts of bi-almost periodicity and positively bi-almost periodicity for sequences.

Definition 5.1 A sequence $L : \mathbb{Z}\times\mathbb{Z} \mapsto \mathbb{R}^k$ is called bi-almost periodic if for every $\varepsilon > 0$, there exists a positive integer $N_0(\varepsilon)$ such that any set consisting of $N_0(\varepsilon)$ consecutive integers contains at least one integer σ for which

$$\|L(t+\sigma, s+\sigma) - L(t,s)\| < \varepsilon$$

for all $t, s \in \mathbb{Z}$. The collection of such sequences is denoted $bAP(\mathbb{Z}\times\mathbb{Z}, \mathbb{R}^k)$.

Let $\widetilde{\mathbb{T}}$ be the set defined by

$$\widetilde{\mathbb{T}} := \Big\{(t,s) \in \mathbb{Z}\times\mathbb{Z} : t \geq s\Big\}.$$

Definition 5.2 A sequence $L : \widetilde{\mathbb{T}} \mapsto \mathbb{R}^k$ is called positively bi-almost periodic if for every $\varepsilon > 0$, there exists a positive integer $N_0(\varepsilon)$ such that any set consisting of $N_0(\varepsilon)$ consecutive integers contains at least one integer σ for which

$$\|L(t+\sigma, s+\sigma) - L(t,s)\| < \varepsilon$$

for all $(t,s) \in \widetilde{\mathbb{T}}$. The collection of such sequences will be denoted $bAP(\widetilde{\mathbb{T}}, \mathbb{X})$.

Obviously, every bi-almost periodic sequence is positively bi-almost periodic with the converse being untrue.

Example 5.3 Classical examples of bi-almost periodic sequences L include those which are of the form $L(t,s) = h(t-s)$ for all $(t,s) \in \mathbb{Z}\times\mathbb{Z}$, where $h = (h(t)_{t\in\mathbb{Z}}$ is periodic, that is, there exists $0 \neq \omega \in \mathbb{Z}$ such that $h(t+\omega) = h(t)$ for all $t \in \mathbb{Z}$.

In this chapter, we are aimed at finding sufficient conditions for the existence of almost periodic solutions to the class of semilinear systems of difference equations given by

$$x(t+1) = A(t)x(t) + f(t, x(t)), \quad t \in \mathbb{Z} \tag{5.2}$$

where $A(t)$ is a $k\times k$ almost periodic matrix function defined on $\mathbb{Z}$, and the function $f : \mathbb{Z}\times\mathbb{R}^k \to \mathbb{R}^k$ is almost periodic in the first variable uniformly in the second one.

To study the existence of solutions to Eq. (5.2), we make extensive use of the fundamental solutions to the system

$$x(t+1) = A(t)x(t), \quad t \in \mathbb{Z} \tag{5.3}$$

to examine almost periodic solutions of the system of difference equations

$$x(t+1) = A(t)x(t) + g(t), \quad t \in \mathbb{Z} \tag{5.4}$$

where $g : \mathbb{Z} \mapsto \mathbb{R}^k$ is almost periodic.

5.2 Discrete Exponential Dichotomy

Define the state transition matrix associated with $A(t)$ as follows

$$X(t,s) = \prod_{r=s}^{t-1} A(r), \quad X(t,t) = I,$$

for $t > s$.

Definition 5.4 ([65, Definition 7.6.4, p. 229]) Equation (5.3) is said to have a discrete exponential dichotomy if there exist $k \times k$ projection matrices $P(t)$ with $t \in \mathbb{Z}$ and positive constants M and $\beta \in (0,1)$ such that,

(i) $A(t)P(t) = P(t+1)A(t)$;
(ii) The matrix $A(t)\Big(R(P(t))\Big)$ is an isomorphism from $R(P(t))$ onto $R(P(t+1))$;
(iii) $\|X(t,r)P(r)x\| \leq M\beta^{r-t}\|x\|$, for $t < r$, $x \in \mathbb{R}^k$;
(iv) $\|X(t,r)(I - P(r))x\| \leq M\beta^{t-r}\|x\|$, for $r \leq t$, $x \in \mathbb{R}^k$.

By repeated application of [(i), Definition 5.4], we obtain

$$P(t)X(t,s) = X(t,s)P(s). \tag{5.5}$$

If Eq. (5.3) has a discrete dichotomy, then we define its associated Green function G by setting

$$G(t,s) = \begin{cases} -X(t,s)P(s) & \text{if } t < s, \\ X(t,s)(I - P(s)) & \text{if } t \geq s. \end{cases}$$

In view of the above, we have

$$\|G(t,s)\| \leq \begin{cases} M\beta^{s-t} \text{ if } t < s, \\ M\beta^{t-s} \text{ if } t \geq s. \end{cases}$$

Remark 5.5 It should be mentioned that if $t \mapsto A(t)$ is almost periodic and if Eq. (5.3) has discrete dichotomy, then the Green operator function $G(t, s)Y \in bAP(\mathbb{T}, \mathbb{R}^k)$ uniformly for all Y in any bounded subset of $\mathbb{R}^k$.

We have the following characterization for the discrete exponential dichotomies:

Theorem 5.6 ([65, Theorem 7.6.5, p. 230]) *The following statements are equivalent,*

i) Equation (5.3) has a discrete exponential dichotomy;
ii) For every bounded $\mathbb{R}^k$-valued sequence g, Eq. (5.4) has a unique bounded solution.

If Eq. (5.3) has a discrete exponential dichotomy, then Theorem 5.6 ensures the existence and uniqueness of a bounded solution to Eq. (5.4) whenever $g : \mathbb{Z} \mapsto \mathbb{R}^k$ is a bounded sequence. Moreover, it can be shown that such a solution is given by

$$\begin{aligned}\overline{x}(t) &= \sum_{r=-\infty}^{\infty} G(t, r+1)g(r) \\ &= \sum_{r=-\infty}^{t-1} X(t, r+1)(I - P(r+1))g(r) - \sum_{r=t}^{\infty} X(t, r+1)P(r+1)g(r)\end{aligned}$$

for all $t \in \mathbb{Z}$.

Theorem 5.7 *Suppose $t \mapsto A(t)$ is almost periodic and that Eq. (5.3) has a discrete exponential dichotomy. If $g \in AP(\mathbb{Z})$, then Eq. (5.4) has a unique almost periodic solution which can be expressed as*

$$\overline{x}(t) = \sum_{r=-\infty}^{t-1} X(t, r+1)(I-P(r+1))g(r) - \sum_{r=t}^{\infty} X(t, r+1)P(r+1)g(r). \tag{5.6}$$

Proof Since every almost periodic sequence is bounded, it follows from Theorem 5.6 that Eq. (5.4) has a unique bounded solution given by Eq. (5.6). Moreover,

$$\begin{aligned}\|\overline{x}(t)\| &\leq \left\{ \sum_{r=-\infty}^{t-1} \|X(t, r+1)(I - P(r+1))\| \right. \\ &\qquad \left. + \sum_{r=t}^{\infty} \|X(t, r+1)P(r+1)\| \right\} \|g\|_\infty \\ &\leq \left\{ \frac{M}{1-\beta} + \frac{M\beta}{1-\beta} \right\} \|g\|_\infty \\ &= \frac{M(1+\beta)}{1-\beta} \|g\|_\infty\end{aligned}$$

which yields

$$\|\overline{x}\|_\infty \leq \frac{M(1+\beta)}{1-\beta}\,\|g\|_\infty\,.$$

To complete the proof, one has to show that $\overline{x} \in AP(\mathbb{Z})$. For that, write $\overline{x} = M(g) - N(g)$ where

$$M(g)(t) := \sum_{r=-\infty}^{t-1} X(t,r+1)(I-P(r+1))g(r)$$

and

$$N(g)(t) = \sum_{r=t}^{\infty} X(t,r+1)P(r+1)g(r).$$

Let us show that $t \mapsto Mg(t)$ is almost periodic. Indeed, since g is almost periodic, for every $\varepsilon > 0$ there exists a positive integer $N_0(\varepsilon)$ such that any set consisting of $N_0(\varepsilon)$ consecutive integers contains at least one integer τ for which

$$\|g(t+\tau)-g(t)\| < \varepsilon$$

for all $t \in \mathbb{Z}$.

Setting $Q(t) = I - P(t)$, we obtain,

$$\begin{aligned}
&M(g)(t+\tau) - M(g)(t)\\
&\quad= \sum_{r=-\infty}^{t+\tau-1} X(t+\tau,r+1)Q(r+1)g(r) - \sum_{r=-\infty}^{t-1} X(t,r+1)Q(r+1)g(r)\\
&\quad= \sum_{r=-\infty}^{t-1} X(t+\tau,r+1+\tau)Q(r+\tau+1)g(r+\tau)\\
&\qquad - \sum_{r=-\infty}^{t-1} X(t,r+1)Q(r+1)g(r)\\
&\quad= \sum_{r=-\infty}^{t-1} X(t+\tau,r+1+\tau)Q(r+1+\tau)\big[g(r+\tau)-g(r)\big]\\
&\qquad + \sum_{r=-\infty}^{t-1} \big[X(t+\tau,r+1+\tau)Q(r+1+\tau) - X(t,r+1)Q(r+1)\big]g(r).
\end{aligned}$$

Clearly,

$$\| \sum_{r=-\infty}^{t-1} X(t+\tau, r+1+\tau) Q(r+1+\tau) \Big[g(r+\tau) - g(r) \Big] \| < c_1(\beta, M)\varepsilon$$

From Remark 5.5 it follows that

$$\| \sum_{r=-\infty}^{t-1} \big[X(t+\tau, r+1+\tau) Q(r+1+\tau) \\ - X(t, r+1) Q(r+1) \big] g(r) \| < c_2(\beta, M)\varepsilon,$$

and hence

$$\| M(g)(t+\tau) - M(g)(t) \| < c_3(\beta, M)\varepsilon$$

for each $t \in \mathbb{Z}$.

Using similar ideas as the previous ones, one can easily see that $N(g) \in AP(\mathbb{Z})$. This completes the proof.

Suppose that there exists $L > 0$ such that

$$\| f(t, x) - f(t, y) \| \leq L \| x - y \|$$

for all $t \in \mathbb{R}$ and $x, y \in \mathbb{R}^k$.

Theorem 5.8 *Suppose that $t \mapsto A(t)$ is almost periodic and that Eq. (5.3) has a discrete exponential dichotomy. Further, we assume that $(t, w) \mapsto f(t, w)$ is almost periodic in $t \in \mathbb{Z}$ uniformly in $w \in B$ where $B \subset \mathbb{R}^k$ is an arbitrary bounded subset. Then Eq. (5.2) has a unique almost periodic solution given by*

$$z(t) = \sum_{r=-\infty}^{t-1} X(t, r+1) Q(r+1) f(r, z(r)) - \sum_{r=t}^{\infty} X(t, r+1) P(r+1) f(r, z(r)), \tag{5.7}$$

whenever L is small enough.

Proof Using the composition of almost periodic sequences (Theorem 4.40) it follows that $r \mapsto g(r) := f(r, z(r))$ belongs to $AP(\mathbb{Z})$ whenever $z \in AP(\mathbb{Z})$.

Let Δ be the nonlinear operator defined by

$$(\Delta z)(t) := \sum_{r=-\infty}^{\infty} G(t, r+1) g(r) \quad \text{for all} \quad t \in \mathbb{Z}.$$

Using the proof of Theorem 5.7, one can easily see that Δ is well defined as it maps $AP(\mathbb{Z})$ into itself.

Now for all $u, v \in AP(\mathbb{Z})$,

$$\|(\Delta u)(t) - (\Delta v)(t)\| \leq \frac{M(1+\beta)}{1-\beta} \|f(t, u(t)) - f(t, u(t))\|,$$

and hence

$$\|\Delta u - \Delta v\|_\infty \leq \frac{ML(1+\beta)}{1-\beta} \|u - v\|_\infty.$$

Thus the nonlinear operator Δ is a strict contraction whenever L is small enough, that is,

$$\frac{ML(1+\beta)}{1-\beta} < 1.$$

To conclude, we make use of the classical Banach fixed point principle.

5.3 The Beverton-Holt Model with Overlapping Generations

To illustrate the results of the previous section, we consider the following theoretical discrete-time population model,

$$x(t+1) = f(t, x(t)) + \gamma x(t), \quad t \in \mathbb{Z} \tag{5.8}$$

where $x(t)$ is the total population size in generation t, $\gamma \in (0, 1)$ is the constant "probability" of surviving per generation, and $f : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ models the birth or recruitment process.

In order to induce almost periodic effects on the population model, we consider the general model in the form,

$$x(t+1) = f(t, x(t)) + \gamma_t x(t), \quad t \in \mathbb{Z}. \tag{5.9}$$

where both $\{\gamma_t\}_{t\in\mathbb{Z}}$ and $f(t, x(t))$ belong to $AP(\mathbb{Z})$ and $\gamma_t \in (0, 1)$ for all $t \in \mathbb{Z}$.

Recall that Eq. (5.9) was studied by Franke and Yakubu [59] in $\mathbb{Z}_+$ and when recruitment function is of the form:

$$f(t, x(t)) = K_t(1 - \gamma_t), \tag{5.10}$$

and (with the periodic Beverton-Holt recruitment function)

$$f(t, x(t)) = \frac{(1-\gamma_t)\mu K_t x(t)}{(1-\gamma_t)K_t + (\mu - 1 + \gamma_t)x(t)}, \tag{5.11}$$

where the carrying capacity K_t is p-periodic, that is, $K_{t+p} = K_t$ for all $t \in \mathbb{Z}_+$ and $\mu > 1$ [43, 59].

Among other things, they have shown that the periodically forced recruitment functions Eqs. (5.10) and (5.11) generate globally attracting cycles in Eq. (5.9) (see details in [59]).

In this section, we extend these results to the almost periodic case in $\mathbb{Z}$. For that, we make use of Theorem 5.8 to show that if both $\{K_t\}_{t\in\mathbb{Z}}$ and $\{\gamma_t\}_{t\in\mathbb{Z}}$ are almost periodic, then Eq. (5.9) has a unique almost periodic solution.

Theorem 5.9 *Let*

$$f(t, x(t)) = \frac{(1-\gamma_t)\mu K_t x(t)}{(1-\gamma_t)K_t + (\mu - 1 + \gamma_t)x(t)},$$

where both $\{K_t\}_{t\in\mathbb{Z}}$ *and* $\{\gamma_t\}_{t\in\mathbb{Z}}$ *are almost periodic, each* $\gamma_t \in (0, 1)$, $K_t > 0$ *and* $\mu > 1$. *Then Eq. (5.9) has a unique almost periodic solution whenever*

$$\sup\{\gamma_t \mid_{t\in\mathbb{Z}}\} < \frac{1}{\mu+1}.$$

Proof First of all, note that Eq. (5.9) is in the form of Eq. (5.2), where $A(t)$ and f can be taken respectively as follows

$$A(t) = \gamma_t,$$

and

$$f(t, x(t)) = \frac{(1-\gamma_t)\mu K_t x(t)}{(1-\gamma_t)K_t + (\mu - 1 + \gamma_t)x(t)}.$$

Now

$$\begin{aligned}
&|f(t,x) - f(t,y)| \\
&\le \frac{(1-\gamma_t)^2 \mu K_t^2 |x-y|}{(1-\gamma_t)^2K_t^2 + (\mu-1+\gamma_t)(1-\gamma_t)K_t(x+y) + (\mu-1+\gamma_t)^2xy} \\
&\le \mu |x-y|.
\end{aligned}$$

Consequently, f is Lipschitz with the Lipschitz constant $L = \mu$. Similarly, take $M < \mu^{-1}$ and $\beta = \sup\{\gamma_t \mid_{t\in\mathbb{Z}}\}$. Clearly, Eq. (5.9) has a unique almost periodic solution whenever

$$\sup\{\gamma_t \mid_{t\in\mathbb{Z}}\} < \frac{1-\mu M}{1+\mu M}.$$

Similarly, if $f(t, x(t)) = K_t(1-\gamma_t)$, then $f(t,x) - f(t,y) = 0$ which yields Eq. (5.9) has a unique almost periodic solution.

Corollary 5.10 *Let the recruitment function be* $f(t, x(t)) = K_t(1 - \gamma_t)$, *where both* $\{K_t\}_{t\in\mathbb{Z}}$ *and* $\{\gamma_t\}_{t\in\mathbb{Z}}$ *are almost periodic, each* $\gamma_t \in (0, 1)$ *and* $K_t > 0$. *Then Eq. (5.9) has a unique globally asymptotically stable almost periodic solution whenever*

$$\sup\{\gamma_t \mid_{t\in\mathbb{Z}}\} < 1.$$

5.4 Exercises

1. Prove Theorem 5.6.
2. Use dichotomy techniques to study the existence of almost periodic solutions to the semilinear difference equation with delay given by

$$u(t+1) = A(t)u(t) + f(t, u(t), u(t-1)), \quad t \in \mathbb{Z}$$

   where $t \mapsto A(t)$ is a $d \times d$ almost periodic matrix and $f : \mathbb{Z} \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is almost periodic in $t \in \mathbb{Z}$ uniformly in the second and the third variables.
3. Use dichotomy techniques to study the existence of almost periodic solutions to the functional difference equation given by

$$u(t+1) = A(t)u(t) + f(t, u(h_1(t)), u(h_2(t)), u(h_3(t))), \quad t \in \mathbb{Z}$$

   where $t \mapsto A(t)$ is a $d \times d$ almost periodic matrix, the sequence $h_j : \mathbb{Z} \mapsto \mathbb{Z}$ with $h_j(\mathbb{Z}) = \mathbb{Z}$ for $j = 1, 2, 3$, and $f : \mathbb{Z} \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is almost periodic in $t \in \mathbb{Z}$ uniformly in the other variables.

5.5 Comments

The main references for this chapter include Diagana [47], Diagana et al. [49] and Henry [65]. Some parts of this chapter are based upon the following references: Diagana [46] and Araya et al. [17]. For additional readings on this topic, we refer to Diagana [47], Lizama and Mesquita [84], and Henry [65].