Chapter 3
Semi-Group of Linear Operators

3.1 Introduction

In this chapter, we collect some of the classical results on strongly continuous semi-groups and evolution families needed in the sequel. All the materials presented here can be found in most of the classical books on semi-groups and evolution families.

While Sect. 3.2 follows Diagana [47], Bezandry and Diagana [27], Chicone and Latushkin [37], Engel and Nagel [55], and Pazy [100], Sect. 3.3 is based upon the following sources: Chicone and Latushkin [37], Lunardi [87], Diagana [47], as well as some articles such as Acquistapace and Terreni [8], Baroun et al. [20], and Schnaubelt [105]. Most of the proofs will not be provided, and therefore, the reader is referred to the above-mentioned references.

3.2 Semi-Group of Operators

3.2.1 Basic Definitions

Definition 3.1 A family of bounded linear operators $(T(t))_{t\in\mathbb{R}_+} : \mathbb{X} \mapsto \mathbb{X}$ is said to be a strongly continuous semi-group or a C_0-semi-group, if

i) $T(0) = I$;
ii) $T(t+s) = T(t)T(s)$ for all $t, s \in \mathbb{R}_+$; and
iii) $\lim\limits_{t\searrow 0} T(t)x = x$ for each $x \in \mathbb{X}$.

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If $(T(t))_{t\in\mathbb{R}_+}$ is a C_0-semi-group, then it is the so-called infinitesimal generator, is a linear operator $A : D(A) \subset \mathbb{X} \mapsto \mathbb{X}$ defined by

$$D(A) := \left\{ u \in \mathbb{X} : \lim_{t\searrow 0} \frac{T(t)u - u}{t} \text{ exists} \right\}$$

and

$$Au := \lim_{t\searrow 0} \frac{T(t)u - u}{t}$$

for every $u \in D(A)$.

Example 3.2 Let $p \geq 1$ and let $\mathbb{X} = L^p(\mathbb{R}^n)$ be equipped with its corresponding L^p-norm $\|\cdot\|_{L^p}$.

Definition 3.3 A function $f : \mathbb{R}^n \mapsto \mathbb{C}$ is said to be rapidly decreasing if it is infinitely many times differentiable, that is, $f \in C^\infty(\mathbb{R}^n)$, and

$$\lim_{\|x\|\to\infty} \|x^k D^\alpha f(x)\| = 0$$

for all $k \in \mathbb{N}$ and for all the multi-index $\alpha \in \mathbb{N}^n$.

The Schwartz space is defined by

$$\mathscr{S}(\mathbb{R}^n) = \left\{ f \in C^\infty(\mathbb{R}^n) : f \text{ is rapidly decreasing} \right\}.$$

The Schwartz space is endowed with the family of semi-norms defined by

$$\|f\|_{k,\alpha} = \sup_{x\in\mathbb{R}^n} \|x^k D^\alpha f(x)\|$$

for all $f \in \mathscr{S}(\mathbb{R}^n)$, which makes it a Fréchet space that contains $C_c^\infty(\mathbb{R}^n)$ (class of functions of class C^∞ with compact support) as a dense subspace.

Consider the family of operators in $L^p(\mathbb{R}^n)$ defined by

$$T(t)f(s) := (4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{\frac{-\|s-r\|^2}{4t}} f(r)dr$$

for all $t > 0$, $s \in \mathbb{R}^n$, and $f \in L^p(\mathbb{R}^n)$.

Proposition 3.4 ([55]) *The family of linear operators $T(t)$ given above, for $t > 0$ and with $T(0) = I$, is a strongly continuous semi-group on $L^p(\mathbb{R}^n)$ whose infinitesimal generator A coincides with the closure of the Laplace operator*

$$\Delta f(x) = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2} f(x_1, x_2, ..., x_n)$$

defined for every f in the Schwartz space $\mathscr{S}(\mathbb{R}^n)$.

Proposition 3.5 ([55]) *If $(T(t))_{t\in\mathbb{R}_+}$ is a C_0-semi-group, then there exists constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that*

$$\|T(t)\| \leq M\, e^{\omega t} \ \text{ for all } \ t \in \mathbb{R}_+.$$

Theorem 3.6 ([100]) *Let $(T(t))_{t\in\mathbb{R}_+}$ be a C_0-semi-group, then the following hold,*

i) For each $x \in \mathbb{X}$,

$$\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} T(s)x ds = T(t)x.$$

ii) For each $x \in \mathbb{X}$, $\int_0^t T(s)x ds \in D(A)$ and

$$A\Big(\int_0^t T(s)x ds\Big) = T(t)x - x.$$

iii) For all $x \in D(A)$,

$$T(t)x - T(s)x = \int_s^t T(r)Ax dr = \int_s^t AT(r)x dr.$$

Proposition 3.7 ([47]) *If $(T(t))_{t\in\mathbb{R}_+} : \mathbb{X} \to \mathbb{X}$ is a C_0-semi-group, then the following hold,*

i) the infinitesimal generator A of $T(t)$ is a closed densely defined operator;
ii) the following differential equation holds

$$\frac{d}{dt} T(t)x = AT(t)x = T(t)Ax,$$

holds for every $x \in D(A)$;
iii) for every $x \in \mathbb{X}$, we have $T(t)x = \lim_{s\searrow 0} (\exp(tA_s))x$, with

$$A_s x := \frac{T(s)x - x}{s},$$

where the above convergence is uniform on compact subsets of $\mathbb{R}_+$; and
iv) if $\lambda \in \mathbb{C}$ such that $\Re e\, \lambda > \omega$, then the integral

$$R(\lambda, A)x := (\lambda I - A)^{-1} x = \int_0^\infty e^{-\lambda t}\, T(t)x\, dt,$$

gives rise to a bounded linear operator $R(\lambda, A)$ *on* $\mathbb{X}$ *whose range is* $D(A)$ *and satisfies the following identity,*

$$(\lambda I - A)\ R(\lambda, A) = R(\lambda, A)(\lambda I - A) = I.$$

If $(T(t))_{t\in\mathbb{R}_+} : \mathbb{X} \to \mathbb{X}$ is a C_0-semi-group, then we know that there exist constants $\omega \in \mathbb{R}$ and $M \geq 1$ such that

$$\|T(t)\| \leq Me^{t\omega}$$

for all $t \in \mathbb{R}_+$.

Now, if $\omega = 0$, then $(T(t))_{t\in\mathbb{R}_+}$ is a C_0-semi-group that is uniformly bounded. If in addition, $M = 1$, then $(T(t))_{t\in\mathbb{R}_+}$ is said to be a C_0-semi-group of contraction. In what follows, we study necessary and sufficient conditions so that A is the infinitesimal generator of a C_0-semi-group of contraction.

Theorem 3.8 (Hille–Yosida) *A linear operator* $A : D(A) \to \mathbb{X}$ *is the infinitesimal generator of a* C_0*-semi-group* $(T(t))_{t\in\mathbb{R}_+}$ *of contraction if and only if,*

i) A is a densely defined closed operator; and
ii) the resolvent $\rho(A)$ *of* A *contains* $[0, \infty)$ *and for all* $\lambda > 0$*,*

$$\left\|(\lambda I - A)^{-1}\right\| \leq \frac{M}{\lambda}. \tag{3.1}$$

For the proof of the Hille-Yosida's theorem, we refer the reader to Pazy [100, Pages 8-9].

3.2.2 *Analytic Semi-Groups*

Definition 3.9 ([87, Page 34]) A family of bounded linear operators $T(t) : \mathbb{X} \mapsto \mathbb{X}$ satisfying the following conditions (semi-group),

i) $T(0) = I$;
ii) $T(t + s) = T(t)T(s)$ for all $t, s \geq 0$

is called an analytic semi-group, if $(0, \infty) \mapsto B(\mathbb{X})$, $t \mapsto T(t)$ is analytic.

Definition 3.10 An analytic semi-group $(T(t))_{t\geq 0}$ is said strongly continuous, if the function $[0, \infty) \mapsto \mathbb{X}$, $t \mapsto T(t)x$ is continuous for all $x \in \mathbb{X}$.

It is well known (see for instance Lunardi [87, Page 33]) that if $A : D(A) \subset \mathbb{X} \mapsto \mathbb{X}$ is a sectorial linear operator, then $T(t)$ defined by

$$T(t) = \frac{1}{2i\pi}\int_{w+\gamma_{r,\eta}} e^{\lambda t}(\lambda I - A)^{-1}d\lambda, \quad t > 0 \tag{3.2}$$

is analytic, where $r > 0$, $\eta \in (\frac{\pi}{2}, \pi)$ and $\gamma_{r,\eta}$ is the curve in the complex plane defined by

$$\{\lambda \in \mathbb{C} : \ |\arg \lambda| = \eta, \ \ |\lambda| \geq r\} \cup \{\lambda \in \mathbb{C} : |\arg \lambda| \leq \eta, \ \ |\lambda| = r\},$$

which we assume to be oriented counterclockwise.

Definition 3.11 ([87, Page 34]) If $A : D(A) \subset \mathbb{X} \mapsto \mathbb{X}$ is a sectorial linear operator, then the family of linear operators $\{T(t) : \ \ t \geq 0\}$ defined in Eq. (3.2) is called the analytic semi-group associated with the operator A.

Proposition 3.12 ([87, Proposition 2.1.1]) *Let A be a sectorial operator and let $T(t)$ be the analytic semi-group associated with it. Then the following hold:*

i) $T(t)u \in D(A^n)$ *for all* $t > 0$, $u \in \mathbb{X}$, $n \in \mathbb{N}$. *If* $u \in D(A^n)$, *then*

$$A^n T(t)u = T(t)A^n u, \ \ t \geq 0;$$

ii) there exist constants $M_0, M_1, \ldots$ *such that*

$$\left\| T(t) \right\| \leq M_0 e^{\omega t}, \ \ t > 0, \ \ \textit{and}$$

$$\left\| t^n (A - \omega I)^n T(t) \right\| \leq M_n e^{\omega t}, \ \ t > 0; \ \ \textit{and}$$

iii) the mapping $t \to T(t)$ *belongs to* $C^{\infty}((0, \infty), B(\mathbb{X}))$ *and*

$$\frac{d^n}{dt^n} T(t) = A^n T(t), \ \ t > 0, \ \forall n \in \mathbb{N}.$$

Proposition 3.13 ([87, Proposition 2.1.9]) *Let* $(T(t))_{t>0}$ *be a family of bounded linear operators on* $\mathbb{X}$ *such that* $t \mapsto T(t)$ *is differentiable, and*

i) $T(t + s) = T(t)T(s)$ *for all* $t, s > 0$;
ii) there exist $\omega \in \mathbb{R}$, $M_0, M_1 > 0$ *such that*

$$\left\| T(t) \right\| \leq M_0 e^{\omega t}, \ \ \ \left\| tT'(t) \right\| \leq M_1 e^{\omega t}, \ \ \ \forall t > 0;$$

iii) either: there exists $t > 0$ *such that* $T(t)$ *is one-to-one, or: for every* $x \in \mathbb{X}$, $\lim_{t \to 0} T(t)x = x$.

Then $t \mapsto T(t)$ *is analytic in* $(0, \infty)$ *with values in* $B(\mathbb{X})$, *and there exists a unique sectorial operator* $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ *such that* $(T(t))_{t \geq 0}$ *is the semi-group associated with* A.

3.2.3 Hyperbolic Semi-Groups

Definition 3.14 ([55]) A strongly continuous semi-group $(T(t))_{\in\mathbb{R}_+}$ is called

i) Uniformly exponentially stable, if there exists $\varepsilon > 0$ such that

$$\lim_{t\to\infty} e^{\varepsilon t}\|T(t)\| = 0.$$

ii) Uniformly stable, if

$$\lim_{t\to\infty} \|T(t)\| = 0.$$

iii) Strongly stable, if

$$\lim_{t\to\infty} \|T(t)x\| = 0 \text{ for all } x \in \mathbb{X}.$$

Definition 3.15 A semi-group $(T(t))_{t\in\mathbb{R}_+} : \mathbb{X} \mapsto \mathbb{X}$ is said to be hyperbolic if the Banach space $\mathbb{X}$ can be decomposed as a direct sum

$$\mathbb{X} = \mathbb{X}_s \oplus \mathbb{X}_u$$

where $\mathbb{X}_s$ (stable) and $\mathbb{X}_u$ (unstable) are two $T(t)$-invariant closed subspaces such that $T_s(t)$ the restriction of $T(t)$ to $\mathbb{X}_s$ and $T_u(t)$ the restriction of $T(t)$ to $\mathbb{X}_u$ satisfy the following properties,

i) $(T_s(t))_{t\in\mathbb{R}_+}$ is a semi-group that is uniformly exponentially stable on $\mathbb{X}_s$; and
ii) $(T_u(t))_{t\in\mathbb{R}_+}$ is a semi-group that is invertible on $\mathbb{X}_u$ and its inverse $((T_u)^{-1}(t))_{t\in\mathbb{R}_+}$ is uniformly exponentially stable on $\mathbb{X}_u$.

Remark 3.16 It can be shown that a C_0-semi-group $(T(t))_{t\in\mathbb{R}_+}$ is hyperbolic if and only if there exist a projection $P : \mathbb{X} \mapsto \mathbb{X}$ and some constants $M, \delta > 0$ such that $T(t)P = PT(t)$ for all $t \in \mathbb{R}_+$, $T(t)(N(P)) = N(P)$, and

i) $\|T(t)x\| \le Me^{-\delta t}\|x\|$ for all $t \in \mathbb{R}_+$ and $x \in R(P)$; and
ii) $\|T(t)x\| \ge \frac{1}{M}e^{\delta t}\|x\|$ for all $t \in \mathbb{R}_+$ and $x \in N(P)$.

Proposition 3.17 ([55, Proposition 3.1.3]) *For a C_0-semi-group $(T(t))_{t\in\mathbb{R}_+}$, the following statements are equivalent,*

i) $(T(t))_{t\in\mathbb{R}_+}$ is hyperbolic.
ii) $\sigma(T(t)) \cap \mathbb{S}^1 = \emptyset$ for one (for all) $t > 0$, where $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$.

Definition 3.18 A C_0-semi-group $(T(t))_{t\in\mathbb{R}_+}$ is said to have the circular spectral mapping theorem, if

$$\mathbb{S}^1 . T(t) \setminus \{0\} = \mathbb{S}^1 e^{t\sigma(A)} \text{ for one (for all) } t > 0,$$

where A is the infinitesimal generator of $(T(t))_{t\in\mathbb{R}_+}$.

Proposition 3.19 ([55, Theorem 3.1.5]) *Let $(T(t))_{t\in\mathbb{R}_+}$ be a C_0-semi-group that has the circular spectral mapping theorem. Then, the following statements are equivalent,*

i) $(T(t))_{t\in\mathbb{R}_+}$ *is hyperbolic.*
ii) $\sigma(T(t)) \cap \mathbb{S}^1 = \emptyset$ *for one (for all)* $t > 0$.
iii) $\sigma(A) \cap i\mathbb{R} = \emptyset$.

3.3 Evolution Families

This section introduces evolution families which play a central role when it comes to dealing with nonautonomous differential equations on Banach spaces. For more on evolution equations and related issues, we refer to the following books: Chicone and Latushkin [78] and Lunardi [87].

3.3.1 Basic Definitions

Let $J \subset \mathbb{R}$ be an interval (possibly unbounded).

Definition 3.20 A family of bounded linear operators $\mathscr{U} = \{U(t,s) : t, s \in J, \ \ t \geq s\}$ on $\mathbb{X}$ is called an evolution family (also called "evolution systems," "evolution operators," "evolution processes," "propagators," or "fundamental solutions") if the following hold,

i) $U(t,s)U(s,r) = U(t,r)$ for $t, s, r \in J$ such that $t \geq s \geq r$;
ii) $U(t,t) = I$ for all $t \in J$.

The evolution family $\mathscr{U}$ is called strongly continuous if, for each $x \in \mathbb{X}$, the function, $J \times J \mapsto \mathbb{X}$, $(t,s) \mapsto U(t,s)x$, is continuous for all $s, t \in J$ with $t \geq s$.

Example 3.21 If $(T(t))_{t\in\mathbb{R}_+}$ is a C_0-semi-group, then U defined by $U(t,s) = T(t-s)$ for all $t, s \in J = \mathbb{R}_+$ with $t \geq s$, is an evolution family that is strongly continuous.

Definition 3.22 The exponential growth bound $\omega(\mathscr{U})$ of an evolution family $\mathscr{U} = \{U(t,s) : t, s \in J, \ \ t \geq s\}$ on $\mathbb{X}$ is defined by

$$\omega(\mathscr{U}) := \inf\left\{\sigma \in \mathbb{R} : \exists M_\sigma \geq 1, \ \ \|U(t,s)\| \leq M_\sigma e^{\sigma(t-s)} \text{ for all } t, s \in J, \ \ t \geq s\right\}.$$

Definition 3.23 An evolution family $\mathscr{U} = \{U(t,s) : t, s \in \mathbb{R}, \ \ t \geq s\}$ defined on $\mathbb{X}$ is called exponentially bounded if $\omega(\mathscr{U}) < \infty$ and $\mathscr{U}$ is exponentially stable if $\omega(\mathscr{U}) < 0$.

Obviously, the evolution family defined by $U(t,s) = T(t-s)$ where $(T(t))_{t\geq 0}$ is an exponentially stable C_0-semi-group, and is an example of an evolution family which is exponentially stable.

3.3.2 Acquistapace–Terreni Conditions

Definition 3.24 A family of linear operators $(A(t))_{t\in\mathbb{R}}$ (not necessarily densely defined) is said to satisfy the Acquistapace–Terreni conditions whether there exist $\omega \in \mathbb{R}$ and the constants $\theta \in (\frac{\pi}{2}, \pi)$, $L, K \geq 0$, and $\mu, \nu \in (0, 1]$ with $\mu + \nu > 1$ such that

$$\Sigma_\theta \cup \{0\} \subseteq \rho(A(t) - \omega I) \ni \lambda, \quad \|R(\lambda, A(t) - \omega I)\| \leq \frac{K}{1 + |\lambda|} \tag{3.3}$$

and

$$\|(A(t) - \omega I)R(\omega, A(t) - \omega I)\ [R(\omega, A(t)) - R(\omega, A(s))]\ \| \leq K\,|t-s|^{\mu}\,|\lambda|^{-\nu} \tag{3.4}$$

for $t, s \in \mathbb{R}$, $\lambda \in \Sigma_\theta := \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg\lambda| \leq \theta\}$.

Recall that Acquistapace–Terreni conditions were introduced by Acquistapace and Terreni in [8, 9] for $\omega = 0$. Among other things, Eqs. (3.3) and (3.4) yield the existence of an evolution family

$$\mathscr{U} = \{U(t,s) : t, s \in \mathbb{R},\ \ t \geq s\}$$

associated with $A(t)$ such that for all $t, s \in \mathbb{R}$ with $t > s$, then $(t,s) \mapsto U(t,s)$, $\mathbb{R} \times \mathbb{R} \mapsto B(\mathbb{X})$ is strongly continuous and continuously differentiable in $t \in \mathbb{R}$, $U(t,s)\mathbb{X} \subseteq D(A(t))$ for all $t, s \in \mathbb{R}$,

$$\partial_t U(t,s) = A(t)U(t,s),$$

(a)

$$\|A(t)^k U(t,s)\| \leq C\ (t-s)^{-k} \tag{3.5}$$

for $0 < t - s \leq 1$, $k = 0, 1$; and

(b) $\partial_s^+ U(t,s)x = -U(t,s)A(s)x$ for $t > s$ and $x \in D(A(s))$ with $A(s)x \in \overline{D(A(s))}$.

Remark 3.25 Recall that if $D(A(t)) = D$ is constant in $t \in \mathbb{R}$, then Eq. (3.4) (see for instance [100]) can be replaced with the identity: there exist constants L and $0 < \mu \leq 1$ such that

$$\|\,(A(t) - A(s))\,R(\omega, A(r))\| \leq L|t-s|^{\mu},\ \ s, t, r \in \mathbb{R}. \tag{3.6}$$

Definition 3.26 An evolution family $\mathscr{U} = \{U(t,s) : t,s \in \mathbb{R}, \quad t \geq s\} \subset B(\mathbb{X})$ is said to have an *exponential dichotomy* (*or is hyperbolic*) if there are projections $P(t)$ that are uniformly bounded and strongly continuous in t and constants $\delta > 0$ and $N \geq 1$ such that

i) $U(t,s)P(s) = P(t)U(t,s)$ for all $t \geq s$;
ii) $U(t,s) : Q(s)\mathbb{X} \to Q(t)\mathbb{X}$ is invertible with inverse $\widetilde{U}(s,t)$; and
iii) $\|U(t,s)P(s)\| \leq Ne^{-\delta(t-s)}$ and $\|\widetilde{U}(s,t)Q(t)\| \leq Ne^{-\delta(t-s)}$ for $t \geq s$ and $t,s \in \mathbb{R}$, where $Q(t) = I - P(t)$.

3.3.2.1 Estimates for $U(t,s)$

Fix once and for all $\alpha \in (0,1)$ let $A : D(A) \subset X \mapsto \mathbb{X}$ be a sectorial operator. We will be using the following real interpolation space in the sequel:

$$\mathbb{X}_\alpha^A := \left\{ x \in \mathbb{X} : \|x\|_\alpha^A := \sup_{r>0} \|r^\alpha (A-\zeta)R(r, A-\zeta)x\| < \infty \right\}.$$

Clearly, $(\mathbb{X}_\alpha^A, \|\cdot\|_\alpha^A)$ is a Banach space.

We also define

$$\mathbb{X}_0^A := \mathbb{X}, \ \|x\|_0^A := \|x\|, \ \mathbb{X}_1^A := D(A), \ \hat{\mathbb{X}}^A := \overline{D(A)}, \ \text{and} \ \|x\|_1^A := \|(\zeta - A)x\|.$$

Obviously, the following continuous embedding holds,

$$D(A) \hookrightarrow \mathbb{X}_\beta^A \hookrightarrow D((\zeta - A)^\alpha) \hookrightarrow \mathbb{X}_\alpha^A \hookrightarrow \hat{\mathbb{X}}^A \subset \mathbb{X}, \tag{3.7}$$

for all $0 < \alpha < \beta < 1$.

Similarly,

$$\mathbb{X}_\beta^A \hookrightarrow \overline{D(A)}^{\|\cdot\|_\alpha^A} \tag{3.8}$$

for $0 < \alpha < \beta < 1$.

If the family of linear operators $A(t)$ for $t \in \mathbb{R}$ satisfies Acquistapace–Terreni conditions, we let

$$\mathbb{X}_\alpha^t := \mathbb{X}_\alpha^{A(t)}, \quad \hat{\mathbb{X}}^t := \hat{\mathbb{X}}^{A(t)}$$

for $0 \leq \alpha \leq 1$ and $t \in \mathbb{R}$.

Recall that the above interpolation spaces are of class $\mathscr{J}_\alpha$ and hence there is a constant $l(\alpha)$ such that

$$\|y\|_\alpha^t \leq l(\alpha)\|y\|^{1-\alpha}\|A(t)y\|^\alpha, \quad y \in D(A(t)). \tag{3.9}$$

Proposition 3.27 ([20, 47]) *For $x \in \mathbb{X}$, $0 \leq \alpha \leq 1$ and all $t > s$, the following hold,*

(a) There is a constant $c(\alpha)$, such that

$$\|U(t,s)P(s)x\|_\alpha^t \leq c(\alpha)e^{-\frac{\delta}{2}(t-s)}(t-s)^{-\alpha}\|x\|. \tag{3.10}$$

(b) There is a constant $m(\alpha)$, such that

$$\|\widetilde{U}(s,t)Q(t)x\|_\alpha^s \leq m(\alpha)e^{-\delta(t-s)}\|x\|. \tag{3.11}$$

3.4 Exercises

1. Let $T(t)$ be the family of linear operators on $L^2(\mathbb{R}^n)$ defined for all $t > 0$ by

$$T(t)f(s) := (4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{\frac{-\|s-r\|^2}{4t}} f(r)dr$$

for all $s \in \mathbb{R}^n$, and $f \in L^2(\mathbb{R}^n)$, and by setting $T(0) = I$.

Show that $(T(t))_{t\geq 0}$ defined above is a C_0-semi-group whose generator is the Laplace operator Δ.

2. Let $BUC(\mathbb{R})$ denote the collection of all real-valued uniformly continuous bounded functions equipped with the sup norm defined by $\|f\|_\infty = \sup_{t\in\mathbb{R}} |f(t)|$. Let $T(t)$ be the family of linear operators on $BUC(\mathbb{R})$ defined by

$$T(t)f(s) = f(t+s)$$

for all $t, s \in \mathbb{R}$, and $f \in BUC(\mathbb{R})$.

Show that $(T(t))_{t\geq 0}$ defined above is a C_0-semi-group whose generator is the linear operator defined by

$$D(A) = \{f \in BUC(\mathbb{R}) : f' \in BUC(\mathbb{R})\}, \text{ and } Af = f'$$

for all $f \in D(A)$.

3. Prove Proposition 3.5.
4. Prove Proposition 3.7.
5. Let $(T(t))_{t\in\mathbb{R}_+}$ be a C_0-semi-group $(T(t))_{t\in\mathbb{R}_+}$. Show that

(a) For each $x \in \mathbb{X}$,

$$\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} T(s)xds = T(t)x.$$

(b) For each $x \in \mathbb{X}$, $\int_0^t T(s)xds \in D(A)$ and

$$A\Big(\int_0^t T(s)xds\Big) = T(t)x - x.$$

(c) For all $x \in D(A)$,

$$T(t)x - T(s)x = \int_s^t T(r)Axdr = \int_s^t AT(r)xdr.$$

6. Prove Theorem 3.8.
7. Prove Proposition 3.17.
8. Prove Proposition 3.19.
9. Let A be the linear operator defined on $C([0, 1])$ by $Af = af'' + bf$ for all $f \in D(A) = \{g \in C^2([0, 1]) : g(0) = g(1) = 0\}$ where $a > 0$ and $b \neq 0$ are constant real numbers.

   (a) Show that $\sigma(A) = \{b - n^2\pi^2\sqrt{a} : n \in \mathbb{N}\}$.
   (b) Show that A is the infinitesimal generator of an analytic semi-group $(T(t))_{t\geq 0}$.
   (c) Show that $(T(t))_{t\geq 0}$ is not strongly continuous at $t = 0$.
   (d) Find conditions on a and b so that the semi-group $(T(t))_{t\geq 0}$ is hyperbolic, that is, $\sigma(A) \cap i\,\mathbb{R} = \emptyset$.

10. Let $A : D(A) \subset \mathbb{X} \mapsto \mathbb{X}$ be a linear operator on a Banach space $\mathbb{X}$ and let $a : \mathbb{R} \mapsto \mathbb{R}$ be a bounded continuous function satisfying

$$\inf_{t\in\mathbb{R}} a(t) = a_0 > 0.$$

   Suppose that A is the infinitesimal generator of a C_0-semi-group $(T(t))_{t\in\mathbb{R}_+}$.

   (a) Show that $\mathscr{U} = \{U(t, s) : t, s \in \mathbb{R},\ \ t \geq s\}$ defined on $\mathbb{X}$ by

$$U(t, s) = T\left(\int_s^t a(r)dr\right)$$

   is an evolution family.
   (b) Show that if $(T(t))_{t\in\mathbb{R}_+}$ is exponentially stable, then so is $\mathscr{U}$.
   (c) Find the generator associated with the evolution family $\mathscr{U}$.

3.5 Comments

The material on semi-groups is taken from various sources. Among them are Diagana [47], Bezandry and Diagana [27], Pazy [100], Chicone and Latushkin [37], Engel and Nagel [55], and Lunardi [87]. Section 3.3 is mainly based upon the following books: Chicone and Latushkin [37], Lunardi [87], Diagana [47], Bezandry and Diagana [27], as well as some articles including Acquistapace and Terreni [8], Baroun et al. [20], and Schnaubelt [105].

Most of the proofs are omitted, and therefore, the reader is referred to the above-mentioned references for proofs and additional material on the different topics discussed in this chapter.