Chapter 10
Second-Order Semilinear Evolution Equations

10.1 Introduction

This chapter is aimed at studying the existence of almost periodic and asymptotically almost periodic solutions to some classes of second-order semilinear evolution equations. In order to establish these existence results, we make extensive use of various tools including the Banach fixed point theorem, the Leray–Schauder alternative, the Sadovsky fixed theorem, etc.

Thermoelastic plate systems play an important role in many applications. For this reason, they have been, in recent years, of a great interest to many researchers. Among other things, the study of the controllability and stability of those thermoelastic plate systems has been considered by many researchers including [14, 18, 24, 44, 65, 79], and [92]. In Sect. 10.2, we study the existence of almost periodic mild solutions to some thermoelastic plate systems with almost periodic forcing terms using mathematical tools such as evolution families and real interpolation spaces.

The main goal of Sect. 10.3 consists of studying the existence of asymptotically almost periodic solutions to some classes of second-order partial functional-differential equations with unbounded delay. The abstract results will, subsequently, be utilized to study the existence of asymptotically almost periodic solutions to some integro-differential equations, which arise in the theory of heat conduction within fading memory materials.

T. Diagana, *Semilinear Evolution Equations and Their Applications*,
https://doi.org/10.1007/978-3-030-00449-1_10

10.2 Almost Periodic Solutions to Some Thermoelastic Plate Systems

10.2.1 Introduction

In this section, we study the existence of almost periodic solutions to thermoelastic plate systems using tools such as evolution families and real interpolation spaces. For that, our main strategy consists of studying its corresponding abstract version. Next, we subsequently use our obtained abstract results to study the existence of almost periodic solutions to these thermoelastic plate systems with almost periodic coefficients.

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ be a bounded subset, which is sufficiently regular and let $a, b : \mathbb{R} \mapsto \mathbb{R}$ be positive functions. The main concern in this section consists of studying the existence (and uniqueness) of almost periodic mild solutions to thermoelastic plate systems given by

$$\begin{cases} u_{tt} + \Delta^2 u + a(t)\Delta\theta \quad = f_1(t, \nabla u, \nabla\theta), \text{ if } t \in \mathbb{R}, \ x \in \Omega \\ \theta_t - b(t)\Delta\theta - a(t)\Delta u_t = f_2(t, \nabla u, \nabla\theta), \text{ if } x \in \Omega, \\ \theta = u = \Delta u = 0, \quad \text{on } \mathbb{R} \times \partial\Omega \end{cases} \tag{10.1}$$

where u, θ are respectively the vertical deflection and the variation of temperature of the plate, the functions f_1, f_2 are continuous and (globally) Lipschitz, and the symbols ∇ and Δ stand respectively for the first and second differential operators given by, $\nabla u = (u_{x_1}, u_{x_2}, \ldots, u_{x_N})$ and

$$\Delta v = \sum_{j=1}^{N} v_{x_j x_j}.$$

Assuming that the coefficients a, b and the forcing terms f_1, f_2 are almost periodic in the first variable (in $t \in \mathbb{R}$) uniformly in the other ones, it will be shown that Eq. (10.1) has a unique almost periodic mild solution.

Recall that a particular case of Eq. (10.1) was investigated by Leiva et al. [80] in the case when not only the coefficients a, b were constant but also there was no gradient terms in the semilinear terms f_1 and f_2. Consequently, the results of this section can be seen as a natural generalization of the results of Leiva et al.

To study the existence of almost periodic solutions to Eq. (10.1), we first study its corresponding abstract semilinear evolution equation and then use the obtained results to establish our existence results. In order to achieve that, let $\mathbb{H} = L^2(\Omega)$ and let A to be the linear operator defined by

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega) \ \text{ and } \ A\varphi = -\Delta\varphi \ \text{ for each } \ \varphi \in D(A).$$

Setting

$$x := \begin{pmatrix} u \\ u_t \\ \theta \end{pmatrix},$$

then Eq. (10.1) can be easily recast in $\mathbb{X} := D(A) \times \mathbb{H} \times \mathbb{H}$ in the following form

$$x'(t) = A(t)x(t) + f(t, x(t)), \ t \in \mathbb{R}, \tag{10.2}$$

where $A(t)$ are the time-dependent linear operators defined by

$$A(t) = \begin{pmatrix} 0 & I_{\mathbb{X}} & 0 \\ -A^2 & 0 & a(t)A \\ 0 & -a(t)A & -b(t)A \end{pmatrix} \tag{10.3}$$

whose constant domains D are given by

$$D = D(A(t)) = D(A^2) \times D(A) \times D(A), \ \ t \in \mathbb{R}.$$

Moreover, the semilinear term f is defined only on $\mathbb{R} \times \mathbb{X}_\alpha$ for some $\frac{1}{2} < \alpha < 1$ by

$$f(t, u, v, \theta) = \begin{pmatrix} 0 \\ f_1(t, \nabla u, \nabla\theta) \\ f_2(t, \nabla u, \nabla\theta) \end{pmatrix},$$

where $\mathbb{X}_\alpha$ is the real interpolation space between $\mathbb{X}$ and $D(A(t))$ given by $\mathbb{X}_\alpha = \mathbb{H}_{1+\alpha} \times \mathbb{H}_\alpha \times \mathbb{H}_\alpha$, with $\mathbb{H}_\alpha = (L^2(\Omega), D(A))_{\alpha,\infty} = L^2(\Omega)^A_{\alpha,\infty}$, and $\mathbb{H}_{1+\alpha}$ is the domain of the part of A in $\mathbb{H}_\alpha$.

In Sect. 10.2.3, we show that the family of operators $A(t)$ given in Eq. (10.3) satisfies the Aquistapace–Terreni condition. The fact that each operator $A(t)$ is sectorial was shown in [80]; however, for the sake of clarity and completeness, a complete proof will be given, as we have to determine the precise constants in order to comply with assumption (H.820) from Chap. 8 of this book. Finally, by applying the abstract result developed in Sect. 8.2 of Chap. 8, we prove that the thermoelastic plate system Eq. (10.1) has a unique almost periodic solution

$$\begin{pmatrix} u \\ \theta \end{pmatrix}$$

in $\mathbb{H}_{1+\alpha} \times \mathbb{H}_\alpha$.

10.2.2 Assumptions on the Coefficients of the Thermoelastic System

Let $a, b : \mathbb{R} \mapsto \mathbb{R}$ be positive functions and let $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ be a bounded subset whose boundary $\partial\Omega$ is sufficiently regular. The main objective here consists of studying Eq. (10.1) in the case when the positive real-valued functions a, b are undervalued by a_0, b_0, and $a, b \in C_b^{0,\mu}(\mathbb{R}) \cap AP(\mathbb{R})$, where u, θ are the vertical deflection and the temperature of the plate. Further, it will be assumed that

$$\max_{t\in\mathbb{R}} b^2(t) < 3(a_0^2 + 1). \tag{10.4}$$

In addition to the above assumptions, we suppose that the functions $f_1, \ f_2 : \mathbb{R} \times H_0^1(\Omega) \times H_0^1(\Omega) \to L^2(\Omega)$ are defined by

$$f_i(t, u, \theta)(x) = f_i(t, \nabla u(x), \nabla\theta(x)) = \frac{K d_i(t)}{1 + |\nabla u(x)| + |\nabla\theta(x)|}$$

for $x \in \Omega, \ t \in \mathbb{R}, \ i = 1, 2$, where $d_i : \mathbb{R} \mapsto \mathbb{R}$ are almost periodic functions.

It is hard to see that the functions f_i $(i = 1, 2)$ are jointly continuous. Further, f_i $(i = 1, 2)$ are globally Lipschitz, that is, there exists $L > 0$ such that

$$\left\| f_i(t, u, \theta) - f_i(t, v, \eta) \right\|_{L^2(\Omega)} \leq L\Big(\|u - v\|^2_{H_0^1(\Omega)} + \|\theta - \eta\|^2_{H_0^1(\Omega)} \Big)^{\frac{1}{2}}$$

for all $t \in \mathbb{R}, u, v, \eta$ and $\theta \in H_0^1(\Omega)$.

10.2.3 Existence of Almost Periodic Solutions

In order to apply the results of Chap. 8 to this setting, we need to check that some assumptions hold.

Theorem 10.1 ([20, Baroun, Boulite, Diagana, and Maniar]) *Under previous assumptions, the thermoelastic plate system Eq. (10.1) has a unique almost periodic solution*

$$\begin{pmatrix} u \\ \theta \end{pmatrix}$$

in $\mathbb{H}_{1+\alpha} \times \mathbb{H}_\alpha$*, whenever* L *is small enough.*

Proof In order to show that $A(t)$ satisfies the Acquistapace–Terreni conditions, we will proceed in two main steps.

Step 1—Let $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \to \infty$ be the sequence eigenvalues of A with each eigenvalue being of a finite multiplicity γ_n equal to the dimension of the corresponding eigenspace and $\{\phi_{n,k}\}$ is a complete orthonormal set of eigenvectors for A. For all $x \in D(A)$ we have

$$Ax = \sum_{n=1}^{\infty} \lambda_n \sum_{k=1}^{\gamma_n} \langle x, \phi_{n,k} \rangle \phi_{n,k} := \sum_{n=1}^{\infty} \lambda_n E_n x,$$

with $\langle \cdot, \cdot \rangle$ being the inner product in $\mathbb{H}$.

Obviously, E_n is a complete family of orthogonal projections in $\mathbb{H}$ and so each $x \in \mathbb{H}$ can be written as

$$x = \sum_{n=1}^{\infty} \sum_{k=1}^{\gamma_n} \langle x, \phi_{n,k} \rangle \phi_{n,k} = \sum_{n=1}^{\infty} E_n x.$$

Consequently, for $\begin{pmatrix} w \\ v \\ \theta \end{pmatrix} \in D(A(t))$, the linear operators $A(t)$ can be rewritten as follows,

$$\begin{aligned}
A(t)z &= \begin{pmatrix} 0 & I & 0 \\ -A^2 & 0 & a(t)A \\ 0 & -a(t)A & -b(t)A \end{pmatrix} \begin{pmatrix} w \\ v \\ \theta \end{pmatrix} \\
&= \begin{pmatrix} v \\ -A^2 w + a(t)A\theta \\ -a(t)Av - b(t)A\theta \end{pmatrix} = \begin{pmatrix} \displaystyle\sum_{n=1}^{\infty} E_n v \\ \displaystyle -\sum_{n=1}^{\infty} \lambda_n^2 E_n w + a(t) \sum_{n=1}^{\infty} \lambda_n E_n \theta \\ \displaystyle -a(t) \sum_{n=1}^{\infty} \lambda_n E_n v - b(t) \sum_{n=1}^{\infty} \lambda_n E_n \theta \end{pmatrix} \\
&= \sum_{n=1}^{\infty} \begin{pmatrix} 0 & 1 & 0 \\ -\lambda_n^2 & 0 & a(t)\lambda_n \\ 0 & -a(t)\lambda_n & -b(t)\lambda_n \end{pmatrix} \begin{pmatrix} E_n & 0 & 0 \\ 0 & E_n & 0 \\ 0 & 0 & E_n \end{pmatrix} \begin{pmatrix} w \\ v \\ \theta \end{pmatrix} \\
&= \sum_{n=1}^{\infty} A_n(t) P_n z,
\end{aligned}$$

where

$$P_n := \begin{pmatrix} E_n & 0 & 0 \\ 0 & E_n & 0 \\ 0 & 0 & E_n \end{pmatrix}, \ n \geq 1,$$

and

$$A_n(t) := \begin{pmatrix} 0 & 1 & 0 \\ -\lambda_n^2 & 0 & a(t)\lambda_n \\ 0 & -a(t)\lambda_n & -b(t)\lambda_n \end{pmatrix}, \ n \geq 1. \tag{10.5}$$

Obviously, the characteristic equation associated with $A_n(t)$ is given by

$$\lambda^3 + b(t)\lambda_n\lambda^2 + (1 + a(t)^2)\lambda_n^2\lambda + b(t)\lambda_n^3 = 0. \tag{10.6}$$

Rescaling as follows, $\lambda/\lambda_n = -\rho$, Eq. (10.6) can be recast as follows,

$$\rho^3 - b(t)\rho^2 + (1 + a(t)^2)\rho - b(t) = 0. \tag{10.7}$$

Using the well-known Routh–Hurwitz theorem we deduce that the real part of the roots $\rho_1(t)$, $\rho_2(t)$, $\rho_3(t)$ of Eq. (10.7) are positive. By a simple calculation one can also verify that Eq. (10.4) does ensure that the roots ρ_1, ρ_2, and ρ_3 are simple and are uniformly separated. In particular, one root is real and the other ones are complex with imaginary part sufficiently far from 0. Consequently, the eigenvalues of $A_n(t)$ are simple and are given by $\sigma_i(t) = -\lambda_n\rho_i(t)$, $i = 1, 2, 3$. Therefore, the matrices $A_n(t)$ are diagonalizable and can be written as follows,

$$A_n(t) = K_n(t)^{-1} J_n(t) K_n(t), n \geq 1,$$

with

$$K_n(t) = \begin{pmatrix} 1 & 1 & 1 \\ \lambda_n\rho_1(t) & \lambda_n\rho_2(t) & \lambda_n\rho_3(t) \\ \dfrac{a(t)\rho_1(t)}{\rho_1(t) - b(t)}\lambda_n & \dfrac{a(t)\rho_2(t)}{\rho_2(t) - b(t)}\lambda_n & \dfrac{a(t)\rho_3(t)}{\rho_3(t) - b(t)}\lambda_n \end{pmatrix},$$

$$J_n(t) = \begin{pmatrix} -\lambda_n\rho_1(t) & 0 & 0 \\ 0 & -\lambda_n\rho_2(t) & 0 \\ 0 & 0 & -\lambda_n\rho_3(t) \end{pmatrix}$$

and

$$K_n(t)^{-1} = \frac{1}{a(a(t), b(t))\lambda_n} \begin{pmatrix} a_{11}(t) & -a_{12}(t) & a_{13}(t) \\ -a_{21}(t) & a_{22}(t) & -a_{23}(t) \\ a_{31}(t) & -a_{32}(t) & a_{33}(t) \end{pmatrix},$$

where

$$a_{11}(t) = \frac{a(t)\rho_3(t)\rho_2(t)(\rho_2(t)-\rho_3(t))}{(\rho_3(t)-b(t))(\rho_2(t)-b(t))}, \quad a_{12}(t) = \frac{a(t)\rho_3(t)\rho_1(t)(\rho_1(t)-\rho_3(t))}{(\rho_3(t)-b(t))(\rho_1(t)-b(t))},$$

$$a_{13}(t) = \frac{a(t)\rho_2(t)\rho_1(t)(\rho_1(t)-\rho_2(t))}{(\rho_2(t)-b(t))(\rho_1(t)-b(t))}, \quad a_{21}(t) = \frac{a(t)b(t)(\rho_2(t)-\rho_3(t))}{(\rho_3(t)-b(t))(\rho_2(t)-b(t))},$$

$$a_{22}(t) = \frac{a(t)b(t)(\rho_1(t)-\rho_3(t))}{(\rho_3(t)-b(t))(\rho_1(t)-b(t))}, \quad a_{23}(t) = \frac{a(t)b(t)(\rho_1(t)-\rho_2(t))}{(\rho_2(t)-b(t))(\rho_1(t)-b(t))},$$

$$a_{31} = (\rho_3(t)-\rho_2(t)), \qquad a_{32} = (\rho_3(t)-\rho_1(t)), \qquad a_{33} = (\rho_2(t)-\rho_1(t)),$$

$$\begin{aligned} a(a(t),b(t)) = {} & \frac{a(t)\rho_3(t)\rho_2(t)}{(\rho_3(t)-b(t))} + \frac{a(t)\rho_1(t)\rho_3(t)}{(\rho_1(t)-b(t))} + \frac{a(t)\rho_2(t)\rho_1(t)}{(\rho_2(t)-b(t))} \\ & - \frac{a(t)\rho_1(t)\rho_2(t)}{(\rho_1(t)-b(t))} - \frac{a(t)\rho_3(t)\rho_1(t)}{(\rho_3(t)-b(t))} - \frac{a(t)\rho_2(t)\rho_3(t)}{(\rho_2(t)-b(t))}. \end{aligned}$$

From the fact that $b(\cdot)$ is not a solution to Eq. (10.7), it can be shown that the matrix operators $K_n(t)$ and $K_n^{-1}(t)$ are well defined and $K_n(t)P_n(t) : Z := \mathbb{H} \times \mathbb{H} \times \mathbb{H} \mapsto \mathbb{X}$, $K_n^{-1}(t)P_n(t) : \mathbb{X} \mapsto Z$.

We claim that the roots $\rho_i(t), i = 1,2,3,$ of Eq. (10.7) are bounded. Indeed, setting $l(t) = \rho(t) - \frac{b(t)}{3}$, then Eq. (10.7) becomes

$$l(t)^3 + p(t)l(t) + q(t) = 0,$$

where $p(t) := (1 + a(t)^2) - \frac{b(t)^2}{3}$, $q(t) := \frac{2}{27}b(t)^3 - (2 - a(t)^2)\frac{b(t)}{3}$.
Since q is bounded and

$$|q(t)| = |l(t)||l(t)^2 + p(t)| \geq |l(t)||l(t)|^2 - |p(t)|,$$

then l is also bounded. Thus the boundedness of b yields the above claim.

Define the sector S_θ as

$$S_\theta = \{\lambda \in \mathbb{C} : |\arg(\lambda)| \leq \theta,\ \lambda \neq 0\},$$

where

$$0 \leq \sup_{t\in\mathbb{R}} |\arg(\rho_i(t))| < \frac{\pi}{2},\ i = 1,2,3$$

and

$$\frac{\pi}{2} < \theta < \pi - \max_{i=1,2,3} \sup_{t\in\mathbb{R}} \{|\arg(\rho_i(t))|\}.$$

For $\lambda \in S_\theta$ and $z \in \mathbb{X}$, one has

$$R(\lambda, A(t))z = \sum_{n=1}^{\infty}(\lambda - A_n(t))^{-1} P_n z$$
$$= \sum_{n=1}^{\infty} K_n(t)(\lambda - J_n(t)P_n)^{-1} K_n^{-1}(t) P_n z.$$

Hence,

$$\|R(\lambda, A(t))z\|^2 \leq \sum_{n=1}^{\infty} \|K_n(t)P_n(\lambda - J_n(t)P_n)^{-1} K_n^{-1}(t)P_n\|^2_{B(\mathbb{X})} \|P_n z\|^2$$
$$\leq \sum_{n=1}^{\infty} \|K_n(t)P_n\|^2_{B(Z,\mathbb{X})} \|(\lambda - J_n(t)P_n)^{-1}\|^2_{B(Z)}$$
$$. \ \|K_n^{-1}(t)P_n\|^2_{B(\mathbb{X},Z)} \|P_n z\|^2.$$

Using Eq. (10.7) and the fact that $b(t) > b_0$ par assumption, it follows that

$$|\rho(t) - b(t)| \geq \frac{a(t)^2 |\rho(t)|}{1 + |\rho(t)|^2}, \qquad \inf_{t\in\mathbb{R}} |\rho(t)| > 0. \tag{10.8}$$

Consequently, from the assumption $a(t) > a_0$ it follows that

$$\inf_{t\in\mathbb{R}} |\rho(t) - b(t)| > 0. \tag{10.9}$$

Moreover, for $z := \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \in Z$, we have

$$\|K_n(t)P_n z\|^2 = \lambda_n^2 \|E_n z_1 + E_n z_2 + E_n z_3\|^2$$
$$+\lambda_n^2 \|\rho_1(t)E_n z_1 + \rho_2(t)E_n z_2 + \rho_3(t)E_n z_3\|^2$$
$$+\lambda_n^2 \left\| \frac{a(t)\rho_1(t)}{\rho_1(t) - b(t)} E_n z_1 + \frac{a(t)\rho_2(t)}{\rho_2(t) - b(t)} E_n z_2 + \frac{a(t)\rho_3(t)}{\rho_3(t) - b(t)} E_n z_3 \right\|^2.$$

Therefore, there is $C_1 > 0$ such that

$$\|K_n(t)P_n z\|_H \leq C_1 \lambda_n \|z\|_Z \quad \text{for all} \quad n \geq 1 \text{ and } t \in \mathbb{R}.$$

Arguing as above, for $z := \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \in \mathbb{X}$, one can show

$$\|K_n^{-1}(t)P_n z\| \leq \frac{C_2}{\lambda_n}\|z\| \quad \text{for all} \quad n \geq 1 \text{ and } t \in \mathbb{R}.$$

Now, for $z \in Z$, we have

$$\begin{aligned}
\|(\lambda - J_n P_n)^{-1} z\|_Z^2 &= \left\| \begin{pmatrix} \frac{1}{\lambda+\lambda_n \rho_1(t)} & 0 & 0 \\ 0 & \frac{1}{\lambda+\lambda_n \rho_2(t)} & 0 \\ 0 & 0 & \frac{1}{\lambda+\lambda_n \rho_3(t)} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \right\|_Z^2 \\
&\leq \frac{1}{(\lambda + \lambda_n \rho_1(t))^2}\|z_1\|^2 + \frac{1}{(\lambda + \lambda_n \rho_2(t))^2}\|z_2\|^2 \\
&\quad + \frac{1}{(\lambda + \lambda_n \rho_3(t))^2}\|z_3\|^2.
\end{aligned}$$

Let $\lambda_0 > 0$. Obviously, the function defined by

$$\eta(\lambda) := \frac{1 + |\lambda|}{|\lambda + \lambda_n \rho_i(t)|}$$

is continuous and bounded on the closed set $\Sigma := \{\lambda \in \mathbb{C}/|\lambda| \leq \lambda_0, \ |\arg \lambda| \leq \theta\}$.

On the other hand, it is clear that η is bounded for $|\lambda| > \lambda_0$. Thus η is bounded on S_θ. If we take

$$N = \sup \left\{ \frac{1 + |\lambda|}{|\lambda + \lambda_n \rho_i(t)|} \ : \ \lambda \in S_\theta, n \geq 1 \, ; \ i = 1, 2, 3, \ t \in \mathbb{R} \right\}.$$

Therefore,

$$\|(\lambda - J_n P_n)^{-1} z\|_Z \leq \frac{N}{1 + |\lambda|}\|z\|_Z, \quad \lambda \in S_\theta.$$

Consequently,

$$\|R(\lambda, A(t))\| \leq \frac{K}{1 + |\lambda|}$$

for all $\lambda \in S_\theta$ and $t \in \mathbb{R}$.

The operators $A(t)$ are invertible and their inverses are given by

$$A(t)^{-1} = \begin{pmatrix} -a(t)^2 b(t)^{-1} A^{-1} & -A^{-2} & -a(t)b(t)^{-1}A^{-2} \\ I & 0 & 0 \\ -a(t)b(t)^{-1} & 0 & -b(t)^{-1}A^{-1} \end{pmatrix}, \quad t \in \mathbb{R}.$$

Hence, for $t, s, r \in \mathbb{R}$, one has

$$\begin{aligned}
&(A(t) - A(s))A(r)^{-1} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ -a(r)b(r)^{-1}(a(t)-a(s))A & 0 & -b(r)^{-1}(a(t)-a(s)) \\ -(a(t)-a(s))A + a(r)b(r)^{-1}(b(t)-b(s))A & 0 & -b(r)^{-1}(b(t)-b(s)) \end{pmatrix},
\end{aligned}$$

and hence

$$\begin{aligned}
\|(A(t) - A(s))A(r)^{-1}z\| \le{}& \sqrt{3}(\|a(r)b(r)^{-1}(a(t)-a(s))Az_1\| \\
&+\|b(r)^{-1}(a(t)-a(s))z_3\| + \|(a(t)-a(s))Az_1\| \\
&+\|a(r)b(r)^{-1}(b(t)-b(s))Az_1\| \\
&+\|b(r)^{-1}(b(t)-b(s))z_3\|) \\
\le{}& \sqrt{3}(|a(r)b(r)^{-1}\|t-s|^{\mu}\|Az_1\| + |b(r)^{-1}|\|t-s|^{\mu}\|z_3\| \\
&+|t-s|^{\mu}\|Az_1\| + \|a(r)b(r)^{-1}||t-s|^{\mu}\|Az_1\| \\
&+|b(r)^{-1}\|t-s|^{\mu}\|z_3\|) \\
\le{}& (2\sqrt{3}|a(r)b(r)^{-1}| + 1)|t-s|^{\mu}\|Az_1\| \\
&+2\sqrt{3}|a(r)b(r)^{-1}|\|t-s|^{\mu}\|z_3\|.
\end{aligned}$$

Consequently,

$$\|(A(t) - A(s))A(r)^{-1}z\| \le C|t-s|^{\mu}\|z\|.$$

Step 2—For every $t \in \mathbb{R}$, $A(t)$ generates an analytic semigroup $(e^{\tau A(t)})_{\tau \ge 0}$ on $\mathbb{X}$. Using similar computations as above, one can show that

$$\sup_{t,s \in \mathbb{R}} \|A(t)A(s)^{-1}\| < \infty$$

and for every $t, s \in \mathbb{R}$ and $0 < \mu \le 1$,

$$\|A(t)A(s)^{-1} - Id\| \le L'k|t-s|^{\mu}$$

with constant $L' \ge 0$ and k is the Lipschitz constant of the functions a and b.

On the other hand, we have

$$e^{\tau A(t)}z = \sum_{n=0}^{\infty} K_n(t)^{-1} P_n e^{\tau J_n} P_n K_n(t) P_n z, \ z \in \mathbb{X}.$$

Then,

$$\|e^{\tau A(t)}z\| = \sum_{n=0}^{\infty} \|K_n(t)^{-1} P_n\|_{B(\mathbb{X},Z)} \|e^{\tau J_n} P_n\|_{B(Z)} \|K_n(t) P_n\|_{B(Z,\mathbb{X})} \|P_n z\|,$$

with for each $z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \in Z$

$$\begin{aligned}
\|e^{\tau J_n} P_n z\|_Z^2 &= \left\| \begin{pmatrix} e^{-\lambda_n \rho_1(t)\tau} E_n & 0 & 0 \\ 0 & e^{-\lambda_n \rho_2(t)\tau} E_n & 0 \\ 0 & 0 & e^{-\lambda_n \rho_3(t)\tau} E_n \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \right\|_Z^2 \\
&\leq \|e^{-\lambda_n \rho_1(t)\tau} E_n z_1\|^2 + \|e^{-\lambda_n \rho_2(t)\tau} E_n z_2\|^2 + \|e^{-\lambda_n \rho_3(t)\tau} E_n z_3\|^2 \\
&\leq e^{-2\delta\tau} \|z\|_Z^2,
\end{aligned}$$

where $\delta = \lambda_1 \inf_{t \in \mathbb{R}} \{Re(\rho_1(t)), Re(\rho_2(t)), Re(\rho_3(t))\}$.

Therefore

$$\|e^{\tau A(t)}\| \leq C e^{-\delta\tau}, \quad \tau \geq 0. \tag{10.10}$$

Using the continuity of the functions a, b and the spectral identity

$$R(\lambda, A(t)) - R(\lambda, A(s)) = R(\lambda, A(t)) \, (A(t) - A(s)) \, R(\lambda, A(s))$$

it follows that the mapping $J \ni t \mapsto R(\lambda, A(t))$ is strongly continuous for $\lambda \in S_\theta$ where $J \subset \mathbb{R}$ is an arbitrary compact interval. Therefore, $A(t)$ satisfies the assumptions of [104, Corollary 2.3] and thus, the evolution family $U(t, s)$ is exponentially stable. The step 2 is complete.

To complete the proof, we have to show $(A(\cdot))^{-1} \in AP(\mathbb{R}, B(\mathbb{X}))$ (See assumption (H.823) of [Sect. 8.3, Chap. 8]). Let $\epsilon > 0$, and $\tau = \tau_\epsilon \in \mathscr{P}(\epsilon, a, b)$. We have

$$A(t)^{-1} - A(t+\tau)^{-1} = A(t+\tau)^{-1}(A(t+\tau) - A(t))A(t)^{-1}, \tag{10.11}$$

and,

$$A(t+\tau) - A(t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & (a(t+\tau)-a(t))A \\ 0 & -(a(t+\tau)-a(t))A & -(b(t+\tau)-b(t))A \end{pmatrix}.$$

Therefore, for $z := \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \in D$, one has

$$\begin{aligned} \|(A(t+\tau)-A(t))z\| &\leq \|(a(t+\tau)-a(t))Az_3\| + \|(a(t+\tau)-a(t))Az_2\| \\ &\quad + \|(b(t+\tau)-b(t))Az_3\| \\ &\leq \varepsilon\|Az_2\| + \varepsilon\|Az_3\| \\ &\leq \varepsilon\|z\|_D, \end{aligned}$$

and using Eq. (10.11) ($\|\cdot\|_D$ being the graph norm with respect to the domain $D = D(A(t))$), we obtain

$$\begin{aligned} \|A(t+\tau)^{-1}y - A(t)^{-1}y\| &\leq \|A(t+\tau)^{-1}(A(t+\tau)-A(t))A(t)^{-1}y\| \\ &\leq \|A(t+\tau)^{-1}\|_{B(\mathbb{X})} \\ &\quad \times \|(A(t+\tau)-A(t))\|_{B(D,\mathbb{X})}\|A(t)^{-1}y\|_D, \quad y \in \mathbb{X}. \end{aligned}$$

Since $\|A(t)^{-1}y\|_D \leq c\|y\|$, then

$$\|A(t+\tau)^{-1}y - A(t)^{-1}y\| \leq c'\varepsilon\|y\|.$$

Consequently, $A(t)^{-1}$ is almost periodic.

Finally, for L sufficiently small, all assumptions of Theorem 8.21 are satisfied and thus the thermoelastic system Eq. (10.1) has a unique almost periodic mild solution

$$\begin{pmatrix} u \\ \theta \end{pmatrix}$$

with values in the interpolation space $\mathbb{H}_{1+\alpha} \times \mathbb{H}_\alpha$.

10.3 Existence Results for Some Second-Order Partial Functional Differential Equations

The main focus in this section consists of studying the existence of asymptotically almost periodic solutions to some classes of second-order partial functional differential equations with unbounded delay. The abstract results will, subsequently, be utilized to studying the existence of asymptotically almost periodic solutions to some integro-differential equations which arise in the theory of heat conduction within fading memory materials.

10.3.1 Introduction

Our main concern in this section consists of studying the existence of asymptotically almost periodic solutions to the class of second-order abstract partial functional differential equations of the form

$$\frac{d}{dt}\left[x'(t) - g(t, x_t)\right] = Ax(t) + f(t, x_t), \qquad t \in I, \tag{10.12}$$

$$x_0 = \varphi \in \mathscr{B}, \tag{10.13}$$

$$x'(0) = \xi \in \mathbb{X}, \tag{10.14}$$

where A is the infinitesimal generator of a strongly continuous cosine family $(C(t))_{t\in\mathbb{R}}$ of bounded linear operators on $\mathbb{X}$, the history $x_t : (-\infty, 0] \to \mathbb{X}$, $x_t(\theta) := x(t+\theta)$, belongs to an abstract phase space $\mathscr{B}$ defined axiomatically, and f, g are some appropriate functions.

Recall that the abstract Cauchy systems of the form, Eqs. (10.12)–(10.14) arise, for instance, in the theory of heat conduction in materials with fading memories, see, e.g., Gurtin–Pipkin [62] and Nunziato [96]. In the classical theory of heat conduction, it is assumed that the internal energy and the heat flux depend linearly upon the temperature u as well as its gradient ∇u. Under these conditions, the classical heat equation describes sufficiently well the evolution of the temperature in different types of materials. However, this description is not satisfactory for materials with fading memories. In the theory developed in [62, 96], the internal energy and the heat flux are described as functionals of u and u_x. Upon some physical conditions, they established that the temperature $u(t, \xi)$ satisfies the integro-differential equation

$$c\frac{\partial^2 u(t,\xi)}{\partial t^2} = \beta(0)\frac{\partial u(t,\xi)}{\partial t} + \int_0^\infty \beta'(s)\frac{\partial u(t-s,\xi)}{\partial t}ds + \alpha(0)\triangle u(t,\xi) + \int_0^\infty \alpha'(s)\triangle u(t-s,\xi)ds, \tag{10.15}$$

where $\beta(\cdot)$ is the energy relaxation function, $\alpha(\cdot)$ is the stress relaxation function and c is the density. Assuming that $\beta(\cdot)$ is smooth enough and that $\nabla u(t,\xi)$ is approximately constant at t, we can rewrite the previous equation in the form

$$\frac{\partial^2 u(t,\xi)}{\partial t^2} = \frac{\partial}{\partial t}\left[\frac{\beta(0)}{c}u(t,\xi) + \frac{1}{c}\int_0^\infty \beta'(s)u(t-s,\xi)ds\right] + d\triangle u(t,\xi).$$

By making the function $\beta(\cdot)$ explicitly dependent on the time t, we can consider the situation in which the material is submitted to an aging process so that the hereditary properties are lost as the time goes to infinity. In this case, the previous equation takes the form

$$\begin{aligned}\frac{\partial^2 u(t,\xi)}{\partial t^2} = &\frac{\partial}{\partial t}\left[\frac{\beta(t,0)}{c}u(t,\xi) + \frac{1}{c}\int_0^\infty \frac{\partial\beta(t,s)}{\partial s}u(t-s,\xi)ds\right] \\ &+ d\triangle u(t,\xi),\end{aligned} \tag{10.16}$$

which can be transformed into the abstract systems of the form Eqs. (10.12)–(10.14) assuming that the solution $u(\cdot)$ is known on $[0,\infty)$.

10.3.2 Preliminaries and Notations

In the rest of this section, if W is an arbitrary metric space, then the notation $B_r(x,W)$ stands for the closed ball in W, centered at x with radius r. The linear operator $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ considered here will be assumed to be the infinitesimal generator of a strongly continuous cosine family $(C(t))_{t\in\mathbb{R}}$ of bounded linear operators on $\mathbb{X}$ and $(S(t))_{t\in\mathbb{R}}$ denote the associated sine function, which is defined by

$$S(t)x = \int_0^t C(s)xds, \mathscr{H}x \in \mathbb{X}, t \in \mathbb{R}.$$

For further details upon cosine function theory and their applications to the second-order abstract Cauchy problem, we refer the reader to Fattorini [57] and Travis and Webb [106, 107].

Recall that Travis and Webb [106] studied the existence of solutions to the second-order abstract Cauchy problem,

$$x''(t) = Ax(t) + h(t), \quad t \in [0,b], \tag{10.17}$$

$$x(0) = w, \quad x'(0) = z, \tag{10.18}$$

where $h \in L^1([0,b];\mathbb{X})$.

The corresponding semilinear case was also done by Travis and Webb [107].

Recall that a mild solution for the system Eqs. (10.17)–(10.18) is any function x that is given by

$$x(t) = C(t)w + S(t)z + \int_0^t S(t-s)h(s)ds, \quad t \in [0,b]. \tag{10.19}$$

In this section the definition of the phase space $\mathscr{B}$ will be as in [69]. Namely, $\mathscr{B}$ will be a vector space of functions mapping $(-\infty, 0]$ into $\mathbb{X}$ endowed with a semi-norm $\|\cdot\|_{\mathscr{B}}$. Moreover, we will assume that the following axioms hold,

(A) If $x : (-\infty, \sigma + b] \to \mathbb{X}$, $b > 0$, is such that $x_\sigma \in \mathscr{B}$ and $x|_{[\sigma,\sigma+b]} \in C([\sigma, \sigma + b]; \mathbb{X})$, then for every $t \in [\sigma, \sigma + b)$ the following conditions hold:

(i) x_t is in $\mathscr{B}$,
(ii) $\| x(t) \| \le H \| x_t \|_{\mathscr{B}}$,
(iii) $\| x_t \|_{\mathscr{B}} \le K(t-\sigma) \sup\{\| x(s) \| : \sigma \le s \le t\} + M(t-\sigma) \| x_\sigma \|_{\mathscr{B}}$,

where $H > 0$ is a constant, $K, M : [0,\infty) \to [1,\infty)$ with K being continuous, M is locally bounded, and H, K, M are independent of $x(\cdot)$.

(A1) For the function $x(\cdot)$ in (**A**), the function $t \to x_t$ is continuous from $[\sigma, \sigma + a)$ into $\mathscr{B}$.

(B) The space $\mathscr{B}$ is complete.

(**C2**) If $(\varphi^n)_{n\in\mathbb{N}}$ is a uniformly bounded sequence in $C((-\infty, 0]; \mathbb{X})$ formed by functions with compact support and $\varphi^n \to \varphi$ in the compact-open topology, then $\varphi \in \mathscr{B}$ and

$$\|\varphi^n - \varphi\|_{\mathscr{B}} \to 0$$

as $n \to \infty$.

Example 10.2 (The Phase Space $\mathbf{C_r} \times \mathbf{L^p}(\rho; \mathbb{X})$*)* Let $r \ge 0$, $1 \le p < \infty$ and let $\rho : (-\infty, -r] \to \mathbb{R}$ be a nonnegative measurable function which satisfies the conditions (g-5), (g-6) in the terminology of [69]. Briefly, this means that ρ is a locally integrable function and that there exists a nonnegative locally bounded function γ on $(-\infty, 0]$ such that

$$\rho(\xi + \theta) \le \gamma(\xi)\rho(\theta),$$

for all $\xi \le 0$ and $\theta \in (-\infty, -r) \setminus N_\xi$, where $N_\xi \subseteq (-\infty, -r)$ is a set whose Lebesgue measure is zero.

The space $C_r \times L^p(\rho; \mathbb{X})$ consists of all classes of functions $\varphi : (-\infty, 0] \to \mathbb{X}$ such that φ is continuous on $[-r, 0]$, Lebesgue-measurable, and $\rho\|\varphi\|^p$ is Lebesgue integrable on $(-\infty, -r)$. The semi-norm on $C_r \times L^p(\rho; \mathbb{X})$ is defined by

$$\|\varphi\|_{\mathscr{B}} := \sup\left\{\|\varphi(\theta)\| : -r \le \theta \le 0\right\} + \left(\int_{-\infty}^{-r} \rho(\theta)\|\varphi(\theta)\|^p d\theta\right)^{1/p}.$$

The space $\mathscr{B} = C_r \times L^p(\rho; \mathbb{X})$ satisfies axioms **(A)-(A1)-(B)**. Moreover, when $r = 0$ and $p = 2$, we can take

$$H = 1, \quad M(t) = \gamma(-t)^{1/2}, \quad \text{and} \quad K(t) = 1 + \left(\int_{-t}^{0} \rho(\theta)\, d\theta \right)^{1/2}$$

for $t \geq 0$, see [69, Theorem 1.3.8] for details.

Some of our existence results require some additional assumptions upon the phase space $\mathscr{B}$.

Definition 10.3 Let $\mathscr{B}_0 = \{\psi \in \mathscr{B} : \psi(0) = 0\}$. The phase space $\mathscr{B}$ is called a fading memory space if $\| S(t)\psi \|_{\mathscr{B}} \to 0$ as $t \to \infty$ for every $\psi \in \mathscr{B}_0$. We say that $\mathscr{B}$ is a uniform fading memory space if $\|S(t)\|_{\mathscr{L}(\mathscr{B}_0)} \to 0$ as $t \to \infty$.

For further details upon phase spaces, we refer the reader to for instance [69].

Let $I \subset \mathbb{R}$ be an interval. Recall that the spaces $BC(I; \mathscr{V}) = C_b(I; \mathscr{V})$ and $C_0([0, \infty); \mathscr{V})$ are defined respectively by

$$\begin{aligned} BC(I, \mathscr{V}) &= C_b(I; \mathscr{V}) \\ &= \left\{ x : I \to \mathscr{V},\ x \text{ is continuous and } \|x\| = \sup_{t \in I} \| x(t) \| < \infty \right\}, \end{aligned}$$

$$C_0([0, \infty); \mathscr{V}) = \left\{ x \in C_b([0, \infty); \mathscr{V}) : \lim_{t \to \infty} \|x(t)\| = 0 \right\},$$

and both spaces are endowed with their corresponding sup-norms.

10.3.3 Existence of Local and Global Mild Solutions

We establish the existence of mild solutions to Eqs. (10.12)–(10.14) in the particular cases when $I = [0, a]$ and $I = [0, \infty)$. Suppose, $I = [0, a]$ or $I = [0, \infty)$ and let $N, \widetilde{N}$ be positive constants such that

$$\|C(t)\| \leq N$$

and

$$\|S(t)\| \leq \widetilde{N}$$

for every $t \in I$.

Our existence results require the following general assumption,

($\mathbf{H_1}$) The functions $f, g : I \times \mathscr{B} \to \mathbb{X}$ satisfy the following conditions:

(i) The functions $f(t, \cdot),\ g(t, \cdot) : \mathscr{B} \to \mathbb{X}$ are continuous a.e. $t \in I$.

(ii) For each $\psi \in \mathscr{B}$, the functions $f(\cdot,\psi), g(\cdot,\psi) : I \to \mathbb{X}$ are strongly measurable.

(iii) There exist integrable functions $m_f, m_g : I \to [0,\infty)$ and continuous nondecreasing functions $W_f, W_g : [0,\infty) \to (0,\infty)$ such that

$$\| f(t,\psi) \| \le m_f(t) W_f(\| \psi \|_{\mathscr{B}}), \quad (t,\psi) \in I \times \mathscr{B},$$
$$\| g(t,\psi) \| \le m_g(t) W_g(\| \psi \|_{\mathscr{B}}), \quad (t,\psi) \in I \times \mathscr{B}.$$

Motivated by the concept of mild solution given in Eq. (10.19), we adopt the following concept of mild solution for Eqs. (10.12)–(10.14).

Definition 10.4 A function $x : (-\infty,0] \cup I \to \mathbb{X}$ is called a mild solution of the abstract Cauchy problem Eqs. (10.12)–(10.14) on I, if $x_0 = \varphi$ and

$$x(t) = C(t)\varphi(0) + S(t)[\xi - g(0,\varphi)] + \int_0^t C(t-s)g(s,x_s)ds + \int_0^t S(t-s)f(s,x_s)ds, \qquad t \in I.$$

In the rest of this section, we set $W := \max\{W_f, W_g\}$.

10.3.4 Existence of Solutions in Bounded Intervals

Recall that the existence of mild solutions to Eqs. (10.12)–(10.14) in the case when $I = [0,a]$ can be obtained through the results in [67]. However, for the sake of clarity and completeness, we provide the reader with the proof of the next theorem, as some of the ideas in this proof are also needed in the sequel.

Theorem 10.5 *Suppose that assumption* ($\mathbf{H_1}$) *holds and that for every* $0 < t \le a$ *and* $r > 0$, *the sets* $U(t,r) = \{S(t)f(s,\psi) : s \in [0,t], \| \psi \|_{\mathscr{B}} \le r\}$ *and* $g(I \times B_r(0,\mathscr{B}))$ *are relatively compact in* $\mathbb{X}$. *If*

$$K_a \int_0^a (N m_g(s) + \widetilde{N} m_f(s))\, ds < \int_c^\infty \frac{ds}{W(s)}, \tag{10.20}$$

where

$$c = (K_a N H + M_a) \| \varphi \|_{\mathscr{B}} + K_a \widetilde{N}(\|\xi\| + \|g(0,\varphi)\|),$$

$$K_a = \sup_{s\in[0,a]} K(s),$$

and

$$M_a = \sup_{s\in[0,a]} M(s),$$

then the system Eqs. (10.12)–(10.14) has a mild solution.

Proof Let the vector space

$$\mathscr{B}C = \left\{x : (-\infty, a] \to \mathbb{X} : x|_{(-\infty,0]} \in \mathscr{B},\ x|_{[0,a]} \in C([0, a]; \mathbb{X})\right\}$$

be endowed with the norm defined by

$$\|x\|_{\mathscr{B}C} = \|x|_{(-\infty,0]}\|_{\mathscr{B}} + \|x|_{[0,a]}\|_a$$

for all $x \in \mathscr{B}C$.

On this space, we define the map $\Gamma : \mathscr{B}C \to \mathscr{B}C$ by $(\Gamma x)_0 = \varphi$ and

$$\Gamma x(t) = C(t)\varphi(0) + S(t)[\xi - g(0, \varphi)] + \int_0^t C(t - s)g(s, x_s)ds + \int_0^t S(t - s) f(s, x_s)ds, \quad t \in I.$$

It is then easy to see that Γx is well defined and that $\Gamma x \in \mathscr{B}C$. Moreover, by using the phase space axioms and the Lebesgue Dominated Convergence Theorem, one can prove that Γ is a continuous function from $\mathscr{B}C$ into $\mathscr{B}C$.

In order to apply Theorem 1.81, we establish an *a priori* estimate for the solution of the integral equation $x = \lambda\Gamma x$, $\lambda \in (0, 1)$. Let $x^\lambda \in \mathscr{B}C$ be a solution of $x = \lambda\Gamma x$, $\lambda \in (0, 1)$. For $t \in I$, we get

$$\| x^\lambda(t) \| \le NH \| \varphi \|_{\mathscr{B}} + \widetilde{N}(\| \xi \| + \| g(0, \varphi) \|) + \int_0^t (Nm_g(s) + \widetilde{N}m_f(s))W(\| x_s^\lambda \|_{\mathscr{B}})ds$$

which yields

$$\| x_t^\lambda \|_{\mathscr{B}} \le (K_a NH + M_a) \| \varphi \|_{\mathscr{B}} + K_a\widetilde{N}(\| \xi \| + \| g(0, \varphi) \|) + K_a \int_0^t (Nm_g(s) + \widetilde{N}m_f(s))W(\| x_s^\lambda \|_{\mathscr{B}})ds.$$

Denoting by $\beta_\lambda(t)$ the right-hand side of the last inequality, we find that

$$\beta_\lambda'(t) \le K_a(Nm_g(t) + \widetilde{N}m_f(t))W(\beta_\lambda(t)),$$

and hence,

$$\int_{\beta_\lambda(0)=c}^{\beta_\lambda(t)} \frac{ds}{W(s)} \le K_a \int_0^t (Nm_g(s) + \widetilde{N}m_f(s))\,ds < \int_c^\infty \frac{ds}{W(s)},$$

which enables to conclude that the set of functions $\{\beta_\lambda : \lambda \in (0,1)\}$ is bounded. As a consequence of the previous fact, $\{x^\lambda : \lambda \in (0,1)\}$ is bounded in $C(I, X)$ as

$$\| x^\lambda(t) \| \le H \| x_t^\lambda \| \le \beta_\lambda(t)$$

for every $t \in I$.

On the other hand, from [68, Lemma 3.1] we deduce that Γ is completely continuous on $\mathscr{B}C$. The existence of a mild solution for Eqs. (10.12)–(10.14) is now a consequence of Theorem 1.81.

In many situations of practical interest, the sine function $S(t)$ is compact. This is the motivation for the next result.

Corollary 10.6 *Suppose that assumption* $(\mathbf{H_1})$ *holds and that* $S(t)$ *is compact for all* $t \ge 0$ *and the set* $g(I \times B_r(0, \mathscr{B}))$ *is relatively compact in* $\mathbb{X}$ *for every* $r > 0$*. If Eq. (10.20) holds, then the system Eqs. (10.12)–(10.14) has a mild solution.*

Remark 10.7 Recall that except when the space $\mathbb{X}$ is a finite dimensional space, the cosine function is not compact, and that for this reason, the compactness assumption on the function g cannot be removed. For more on this and related issues, we refer the reader to for instance the work of Travis and Webb [106, pp. 557].

Using similar ideas as in the proof of Theorem 10.5, we can prove the following local existence result.

Theorem 10.8 *Suppose that assumption* $(\mathbf{H_1})$ *holds and that for every* $0 < t \le a$ *and* $r > 0$*, the sets*

$$U(t,r) = \left\{ S(t)f(s,\psi) : s \in [0,t],\ \|\psi\|_{\mathscr{B}} \le r \right\}$$

and

$$g(I \times B_r(0, \mathscr{B}))$$

are relatively compact in $\mathbb{X}$*.*

Then there exists a mild solution to Eqs. (10.12)–(10.14) on $[0, b]$ *for some* $0 < b \le a$*.*

10.3.5 Existence of Global Solutions

In this subsection, our discussions will be upon the existence of mild solutions defined on the interval $I = [0, \infty)$. For that, we suppose that M, K are positive constants such that $M(t) \leq M$ and $K(t) \leq K$ for every $t \geq 0$ and that the functions m_f, m_g are locally integrable.

We need the following notations

$$W = \max\{W_f, W_g\}, \quad m = \max\{m_f, m_g\}, \quad \gamma(s) = Nm_g(s) + \widetilde{N}m_f(s).$$

Remark 10.9 Recall that if $\mathscr{B}$ is a fading memory space, then the functions $M(\cdot)$, $K(\cdot)$ are bounded on $[0, \infty)$. For further details on this and related issues, we refer the reader to [69, Proposition 7.1.5].

Let $h : [0, \infty) \to (0, \infty)$ be a continuous nondecreasing function with $h(0) = 1$ and such that $h(t) \to \infty$ as $t \to \infty$.

Let $C_{0,h}(\mathbb{X})$ denote the space defined by

$$C_{0,h}(\mathbb{X}) = \left\{ x \in C([0, \infty); \mathbb{X}) : \lim_{t\to\infty} \frac{\| x(t) \|}{h(t)} = 0 \right\},$$

which we equip with the norm

$$\|x\|_h = \sup_{t\geq 0} \frac{\| x(t) \|}{h(t)}.$$

Recall the following well-known compactness criterion:

Lemma 10.10 *A set $B \subset C_0([0, \infty); \mathbb{X})$ is relatively compact in $C_0([0, \infty); \mathbb{X})$ if and only if,*

(a) *B is equi-continuous;*
(b) $\lim_{t\to\infty} \| x(t) \| = 0$*, uniformly for $x \in B$;*
(c) *The set $B(t) = \{x(t) : x \in B\}$ is relatively compact in $\mathbb{X}$ for every $t \geq 0$.*

Proof The proof is left to the reader as an exercise.

The main existence result of this subsection can now be formulated as follows:

Theorem 10.11 *Under assumption* $(\mathbf{H_1})$*, if the following conditions hold:*

(a) *for every $t \in I$ and each $r \geq 0$ the sets*

$$\left\{ S(t)f(s, \psi) : (s, \psi) \in [0, t] \times B_r(0, \mathscr{B}) \right\}$$

and

$$g([0,t] \times B_r(0,\mathscr{B}))$$

are relatively compact in $\mathbb{X}$*;*

(b) *for every* $L \geq 0$*,*

$$\frac{1}{h(t)} \int_0^t m(s) W(Lh(s))\, ds \to 0$$

as $t \to \infty$ *and*

$$\limsup_{r\to\infty} \frac{1}{r} \int_0^\infty \gamma(s) \frac{W((K+M)rh(s))}{h(s)} ds < 1.$$

Then the system Eqs. (10.12)–(10.14) has a mild solution on $[0,\infty)$.

Proof On the space

$$\mathscr{B}C_{0,h}(\mathbb{X}) = \{x : \mathbb{R} \to \mathbb{X} : x_0 \in \mathscr{B},\ x|_I \in C_{0,h}(\mathbb{X})\}$$

endowed with the norm defined by

$$\|x\|_{\mathscr{B}C_{0,h}} = \|x_0\|_{\mathscr{B}} + \|x|_I\|_h,$$

we define the map $\Gamma : \mathscr{B}C_{0,h}(\mathbb{X}) \to \mathscr{B}C_{0,h}(\mathbb{X})$ by $(\Gamma x)_0 = \varphi$ and

$$\Gamma x(t) = C(t)\varphi(0) + S(t)[\xi - g(0,\varphi)] + \int_0^t C(t-s)g(s,x_s)ds$$
$$+ \int_0^t S(t-s)f(s,x_s)ds, \quad t \geq 0.$$

It is easy to prove that the expression $\Gamma x(\cdot)$ is well defined for each $x \in \mathscr{B}C_{0,h}(\mathbb{X})$. On the other hand, using the fact that $\|x_s\|_{\mathscr{B}} \leq (K+M) \parallel x \parallel_{\mathscr{B}C_{0,h}} h(s)$ for $s \in I$, we find that

$$\frac{\|\Gamma x(t)\|}{h(t)} \leq \frac{NH\|\varphi\|_{\mathscr{B}} + (\|\xi\| + \|g(0,\varphi)\|)}{h(t)} \tag{10.21}$$
$$+\frac{1}{h(t)} \int_0^t [Nm_g(s) + \widetilde{N}m_f(s)]W((K+M) \parallel x \parallel_{\mathscr{B}C_{0,h}} h(s))ds,$$

which implies, from condition (**c**), that $\frac{\|\Gamma x(t)\|}{h(t)}$ converges to zero as $t \to \infty$. This shows that Γ is a well-defined map from $\mathscr{B}C_{0,h}(\mathbb{X})$ into $\mathscr{B}C_{0,h}(\mathbb{X})$. Note that the inequality (10.21) shows also that $\frac{\|\Gamma x(t)\|}{h(t)} \to 0$, as $t \to \infty$, uniformly for x in bounded sets of $\mathscr{B}C_{0,h}(\mathbb{X})$.

In the sequel we prove that Γ verifies the hypotheses of Theorem 1.81. We begin by proving that Γ is continuous. Let $(u^n)_n$ be a sequence in $\mathscr{B}C_{0,h}(\mathbb{X})$ and $u \in \mathscr{B}C_{0,h}(\mathbb{X})$ such that $u^n \to u$ as $n \to \infty$. Clearly, $g(s, u^n_s) \to g(s, u_s)$, $f(s, u^n_s) \to f(s, u_s)$ a.e. $s \in I$ as $n \to \infty$, and

$$\| f(s, u^n_s)\| \leq m_f(s) W_f(\beta h(s)), \quad s \geq 0,$$

$$\| g(s, u^n_s)\| \leq m_g(s) W_g(\beta h(s)), \quad s \geq 0,$$

where $\beta = (K + M)L$ and $L > 0$ is such that

$$\sup\{\| u \|_{\mathscr{B}C_{0,h}(\mathbb{X})}, \|u^n\|_{\mathscr{B}C_{0,h}(\mathbb{X})} : n \in \mathbb{N}\} \leq L.$$

Since the functions on the right-hand side of the above inequalities (involving f and g) are integrable on $[0, t]$, we conclude that

$$\| \Gamma u^n(t) - \Gamma u(t) \| \to 0 \ \text{ as } \ n \to \infty$$

uniformly for t in bounded intervals. Moreover, using the argument that the set of functions $\{u^n : n \in \mathbb{N}\}$ is bounded in $\mathscr{B}C_{0,h}(\mathbb{X})$, for each $\epsilon > 0$ there exists $T_\epsilon > 0$ such that $\frac{\|\Gamma u^n(t) - \Gamma u(t)\|}{h(t)} \leq \epsilon$, for all $n \in \mathbb{N}$ and every $t \geq T_\epsilon$. Combining these properties we obtain that $\Gamma u^n \to \Gamma u$ in $\mathscr{B}C_{0,h}(\mathbb{X})$. Thus, Γ is continuous.

On the other hand, if $x^\lambda \in \mathscr{B}C_{0,h}(\mathbb{X})$ is a solution of the integral equation $\lambda \Gamma z = z$, $0 < \lambda < 1$, for $t \geq 0$, we obtain that

$$\frac{\|x^\lambda(t)\|}{h(t)} \leq \frac{NH \| \varphi \|_{\mathscr{B}} + \widetilde{N}(\| \xi \| + \|g(0, \varphi) \|)}{h(t)} + \frac{1}{h(t)} \int_0^t \gamma(s) W((K + M) \| x^\lambda \|_{\mathscr{B}C_{0,h}(\mathbb{X})} h(s)) ds,$$

and hence

$$\| x^\lambda \|_{\mathscr{B}C_{0,h}(\mathbb{X})} \leq (1 + NH) \| \varphi \|_{\mathscr{B}} + \widetilde{N}(\| \xi \| + \|g(0, \varphi) \|) + \int_0^\infty \gamma(s) \frac{W((K + M) \| x^\lambda \|_{\mathscr{B}C_{0,h}(\mathbb{X})} h(s))}{h(s)} ds.$$

From the previous estimates, if the set $\{\| x^\lambda \|_{\mathscr{B}C_{0,h}(\mathbb{X})} : 0 < \lambda < 1\}$ is unbounded, we deduce the existence of a sequence $(r^n)_{n\in\mathbb{N}}$ with $r^n \to \infty$ such that

$$1 \leq \liminf_{n\to\infty} \frac{1}{r_n} \int_0^\infty \gamma(s) \frac{W((K + M) r_n h(s))}{h(s)} ds$$

which is absurd, therefore the set $\{\| x^\lambda \|_{\mathscr{B}C_{0,h}(\mathbb{X})} : 0 < \lambda < 1, \}$ is bounded.

Arguing as in the proof of Theorem 10.5, we can prove that

$$\left\{ \Gamma x(t) : x \in B_r(0, \mathscr{B}C_{0,h}(\mathbb{X})) \right\}$$

is relatively compact in $\mathbb{X}$ for every $t \geq 0$ and that

$$\left\{ \frac{\Gamma x}{h} : x \in B_r(0, \mathscr{B}C_{0,h}(\mathbb{X})) \right\}$$

is equi-continuous on $[0, \infty)$. Moreover, from our previous remarks we know that $\frac{\Gamma x(t)}{h(t)} \to 0$ as $t \to \infty$, uniformly for $x \in B_r(0, \mathscr{B}C_{0,h}(\mathbb{X}))$. Consequently, we have shown that the set

$$\left\{ \frac{\Gamma x}{h} : x \in B_r(0, \mathscr{B}C_{0,h}(\mathbb{X})) \right\}$$

fulfills the conditions of Lemma 10.10, which yields it is relatively compact in $C_0(\mathbb{X})$. Therefore, $\Gamma B_r(0, \mathscr{B}C_{0,h}(\mathbb{X})))$ is relatively compact in $\mathscr{B}C_{0,h}(\mathbb{X})$.

The existence of a mild solution for the system Eqs. (10.12)–(10.14) on $[0, \infty)$ follows from Theorem 1.81.

10.3.6 Existence of Asymptotically Almost Periodic Solutions

In this subsection we study the existence of asymptotically almost periodic solutions for the abstract system Eqs. (10.12)–(10.14). For that, suppose that there exist two positive constants N and $\widetilde{N}$ such that

$$\|C(t)\| \leq N$$

and

$$\|S(t)\| \leq \widetilde{N}$$

for every $t \geq 0$.

Let us recall the following definitions which are needed in the sequel:

Definition 10.12 An operator function $F : [0, \infty) \to B(\mathscr{V}, \mathscr{W})$ is said to be:

(a) strongly continuous if for every each $x \in \mathscr{V}$, the function $F(\cdot)x : [0, \infty) \to \mathscr{W}$ is continuous;

(b) pointwise almost periodic (respectively, pointwise asymptotically almost periodic) if $F(\cdot)x \in AP(\mathscr{W})$ for every $x \in \mathscr{V}$ (respectively, $F(\cdot)x \in AAP(\mathscr{W})$ for every $x \in \mathscr{V}$);

(c) almost periodic (respectively, asymptotically almost periodic) if $F(\cdot) \in AP(B(\mathscr{V}, \mathscr{W}))$ (respectively, $F(\cdot) \in AAP(B(\mathscr{V}, \mathscr{W}))$).

Remark 10.13 Note that if the sine function $S(\cdot)$ is uniformly bounded and pointwise almost periodic, then the cosine function $C(\cdot)$ is also pointwise almost periodic, see, e.g., [64, Lemma 3.1] and [64, Theorem 3.2] for details.

Lemma 10.14 ([113, Chapter 6]) *Let $V \subseteq AP(\mathbb{X})$ be a set with the following properties:*

(a) *V is uniformly equi-continuous on $\mathbb{R}$;*
(b) *for each $t \in \mathbb{R}$, the set $V(t) = \{x(t) : x \in V\}$ is relatively compact in $\mathbb{X}$;*
(c) *V is equi-almost periodic, that is, for every $\varepsilon > 0$ there is a relatively dense set $\mathscr{H}(\varepsilon, V, \mathbb{X}) \subset \mathbb{R}$ such that*

$$\|x(t+\tau) - x(t)\| \leq \varepsilon, \quad x \in V, \quad \tau \in \mathscr{H}(\varepsilon, V, \mathbb{X}),\ t \in \mathbb{R}.$$

Then, V is relatively compact in $AP(\mathbb{X})$.

Remark 10.15 As an immediate consequence of this characterization, one can assert that if $F : \mathbb{R} \to B(\mathbb{X}, \mathbb{Y})$ is almost periodic and U is a relatively compact subset of $\mathbb{X}$, then $V = \{F(\cdot)x : x \in U\}$ is relatively compact in $AP(\mathbb{Y})$. For the sine function, we can strengthen this property.

Proposition 10.16 *Assume that the sine function $S(\cdot)$ is almost periodic and that $U \subseteq \mathbb{X}$. If the set $\{S(t)x : x \in U,\ t \geq 0\}$ is relatively compact in $\mathbb{X}$, then $V = \{S(\cdot)x : x \in U\}$ is relatively compact in $AP(\mathbb{X})$.*

Proof Let us fix $\delta > 0$. Since $S(\delta)U$ is relatively compact in $\mathbb{X}$, by using Remark 10.15, we can claim that $V_\delta = \{S(\cdot)S(\delta)x : x \in U\}$ is relatively compact in $AP(\mathbb{X})$. On the other hand, for each $\varepsilon > 0$ there is $\delta > 0$ such that

$$\|(I - C(s))S(t)x\| \leq \varepsilon$$

for all $0 \leq s \leq \delta$, every $x \in U$ and all $t \geq 0$.

We deduce from above that

$$\| \ S(t)x - \frac{1}{\delta}S(t)S(\delta)x \ \| = \| \ \frac{1}{\delta}\int_0^\delta (I - C(s))S(t)x ds \ \| \leq \varepsilon,$$

for every $t \geq 0$.

This property and the decomposition

$$S(\cdot)x = \frac{1}{\delta}S(\cdot)S(\delta)x + S(\cdot)x - \frac{1}{\delta}S(\cdot)S(\delta)x,$$

imply that $V \subseteq \frac{1}{\delta}V_\delta + B_\varepsilon(0, C_b(\mathbb{X}))$, which in turn proves that V is relatively compact in $AP(\mathbb{X})$.

Remark 10.17 Recall that the assumption on the compactness of the set $\{S(t)x : x \in U,\ t \geq 0\}$ in Proposition 10.16 is verified, for instance, in the case when the sine function is almost periodic.

Lemma 10.18 *Assume that $S(\cdot)$ is pointwise almost periodic and that U is a bounded subset of $\mathbb{X}$. If one of the following conditions holds:*

(i) *U is relatively compact.*
(ii) *$S(\cdot)$ is almost periodic and $S(t)$ is compact for every $t \in \mathbb{R}$.*

Then $\{S(t)x : x \in U,\ t \geq 0\}$ is relatively compact in $\mathbb{X}$.

Proof The proof is left to the reader as an exercise.

Recall that the case (ii) includes the case of periodic sine functions.

For asymptotically almost periodic functions, we have a similar characterization of compactness given in the next lemma.

Lemma 10.19 *Let $V \subseteq AAP(\mathbb{X})$ be a set with the following properties:*

(a) *V is uniformly equi-continuous on $[0, \infty)$;*
(b) *for each $t \geq 0$, the set $V(t) = \{x(t) : x \in V\}$ is relatively compact in $\mathbb{X}$;*
(c) *V is equi-asymptotically almost periodic, that is, for every $\varepsilon > 0$ there are $L(\varepsilon, V, \mathbb{X}) \geq 0$ and a relatively dense set $\mathscr{H}(\varepsilon, V, \mathbb{X}) \subseteq [0, \infty)$ such that*

$$\|x(t+\tau) - x(t)\| \leq \varepsilon, \quad x \in V,\ t \geq L(\varepsilon, V, \mathbb{X}),\ \tau \in \mathscr{H}(\varepsilon, V, \mathbb{X}).$$

Then, V is relatively compact in $AAP(\mathbb{X})$.

Proof The proof is left to the reader as an exercise.

Remark 10.20 If $f \in AAP(\mathbb{X})$, then it can be decomposed in a unique fashion as $f = f_1 + f_2$, where $f_1 \in AP(\mathbb{X})$ and $f_2 \in C_0(\mathbb{X})$. Let $V \subseteq AAP(\mathbb{X})$ and $V_i = \{f_i : f \in V\},\ i = 1, 2$. It follows from the above-mentioned results that V is relatively compact in $AAP(\mathbb{X})$ if, and only if, V_1 is relatively compact in $AP(\mathbb{X})$ and V_2 is relatively compact in $C_0(\mathbb{X})$.

We will be using the next proposition.

Proposition 10.21 *Let $(\mathscr{V}_i, \|\cdot\|_{\mathscr{V}_i})$, $i = 1, 2$, be Banach spaces and $V \subseteq L^1([0, \infty), \mathscr{V}_1)$. If $F_1 : [0, \infty) \to B(\mathscr{V}_1, \mathscr{V}_2)$ and $F_2 : [0, \infty) \to B(\mathscr{V}_2)$ are strongly continuous functions of bounded linear operators which satisfy*

(a) *$\displaystyle\int_L^\infty F_1(s)x(s)ds \to 0$ in $\mathscr{V}_2$ when $L \to \infty$, uniformly for $x \in V$;*
(b) *For each $t \geq 0$, the set $\{x(s) : x \in V,\ 0 \leq s \leq t\}$ is relatively compact in $\mathscr{V}_1$,*

then the sets

$$W(t) = \left\{ \int_0^t F_1(s)x(s)ds : x \in V \right\},\quad t \geq 0,$$

and

$$W = \bigcup_{0 \le t \le \infty} W(t)$$

are relatively compact in $\mathscr{V}_2$. *Moreover, if* F_2 *is uniformly bounded on* $[0, \infty)$ *and*

$$\int_t^{t+h} F_1(s)x(s)ds \to 0,$$

as $h \to 0$, *uniformly for* $x \in V$, *then the set* $U = \{z_x : x \in V\}$, *where*

$$z_x(t) = F_2(t) \int_t^\infty F_1(s)x(s)ds,$$

is relatively compact in $C_0(\mathscr{V}_2)$.

Proof Let $(K_t)_{t \ge 0}$ be a family of compacts sets such that $\{x(s) : x \in V, s \in [0, t]\} \subseteq K_t$ for every $t \ge 0$. Since F_1 is strongly continuous, then the set

$$F_1 K_t = \{F_1(s)y : y \in K_t, 0 \le s \le t\}$$

is relatively compact in $\mathscr{V}_2$.

Let $(\widetilde{K}_t)_{t \ge 0}$ be a nondecreasing family of compact and absolutely convex sets such that $F_1 K_t \subset \widetilde{K}_t$ for every $t \ge 0$.

From the mean value theorem for the Bochner integral (see [90, Lemma 2.1.3]), we infer that $W(t) \subseteq t\widetilde{K}_t$ for all $t > 0$. On the other hand, for each $\varepsilon > 0$ there is a constant $L \ge 0$ such that

$$\left\| \int_L^\infty F_1(s)x(s)ds \right\|_{\mathscr{V}_2} \le \varepsilon$$

for all $x \in V$.

Using the sets $\widetilde{K}_t$ it follows that $W \subseteq L\widetilde{K}_L + B_\varepsilon(0, \mathscr{V}_2)$, which yields W is relatively compact in $\mathscr{V}_2$. Thus, the sets $W(t)$, $t \ge 0$, and W are relatively compact in $\mathscr{V}_2$.

To establish the last assertion, we make use of Lemma 10.10. The hypothesis (**b**) of Lemma 10.10 can be easily obtained as an immediate consequence of (**a**) and the fact that F_2 is uniformly bounded. Moreover, for every $t \ge 0$ and $x \in V$, we have that

$$\int_t^\infty F_1(s)x(s)ds \in \overline{W - W(t)} \subset \overline{W - W} = W_1,$$

which proves that the set

$$U(t) = \left\{ F_2(t) \int_t^\infty F_1(s)x(s)ds : x \in V \right\}$$

is relatively compact in $\mathscr{V}_2$ for every $t \geq 0$.

Finally, we prove that U is equi-continuous. To this end, we fix $t \geq 0$. Since elements

$$\int_t^\infty F_1(\xi)x(\xi)\,d\xi, \quad x \in V,$$

are in the compact set W_1 (which is independent of t), and the family $(F_2(t))_{t\geq 0}$ is strongly continuous in $\mathscr{V}_2$, for $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\| F_2(t+s)x - F_2(t)x \| \leq \varepsilon, \quad x \in W_1,$$

$$\| \int_t^{t+s} F_1(\xi)x(\xi)d\xi \| \leq \varepsilon, \quad x \in V,$$

for every $0 <| s |< \delta$ with $t + s \geq 0$. Consequently, for $x \in V$ and $0 <| s |< \delta$ such that $t + s \geq 0$, we get,

$$\begin{aligned}
&\| F_2(t+s) \int_{t+s}^\infty F_1(\xi)x(\xi)\,d\xi - F_2(t) \int_t^\infty F_1(\xi)x(\xi)\,d\xi \| \\
&\leq \| (F_2(t+s) - F_2(t)) \int_{t+s}^\infty F_1(\xi)x(\xi)d\xi \| \\
&\quad + \| F_2(t) \| \| \int_{t\wedge(t+s)}^{t\vee(t+s)} F_1(\xi)x(\xi)\,d\xi \| \\
&\leq \sup\{\| (F_2(t+s)y - F_2(t)y \|: y \in W_1\} + \| F_2(t) \| \varepsilon, \\
&\leq (1 + \sup_{\theta\geq 0} \| F_2(\theta) \|)\varepsilon,
\end{aligned}$$

which implies that U is equi-continuous at t.

In the next results, for a locally integrable function $x : [0, \infty) \to X$, we denote by $z_x,\ y_x : [0, \infty) \to \mathbb{X}$ the functions given by

$$z_x(t) = \int_0^t C(t-s)x(s)ds$$

and

$$y_x(t) = \int_0^t S(t-s)x(s)ds.$$

Proposition 10.22 *Assume that $S(\cdot)$ is pointwise almost periodic and that $V \subseteq L^1([0,\infty),\mathbb{X})$ is a set with the following properties:*

(a) $\displaystyle\int_L^\infty \|x(s)\| ds \to 0$ *when $L \to \infty$, uniformly for $x \in V$;*

(b) $\displaystyle\int_t^{t+s} \|x(\xi)\|\, d\xi \to 0$, *when $s \to 0$, uniformly for $x \in V$ and $t \geq 0$;*

(c) *for each $t \geq 0$ the set $\{x(s) : 0 \leq s \leq t,\ x \in V\}$ is relatively compact.*

Then the sets $\{y_x : x \in V\}$ and $\{z_x : x \in V\}$ are relatively compact in $AAP(\mathbb{X})$.

Proof We first establish that each function y_x is asymptotically almost periodic. For $x \in V$, we can write

$$\begin{aligned} y_x(t) &= S(t)\int_0^t C(s)x(s)\,ds \;-\; C(t)\int_0^t S(s)x(s)\,ds \\ &= S(t)\int_0^\infty C(s)x(s)\,ds \;-\; S(t)\int_t^\infty C(s)x(s)\,ds \\ &\quad - C(t)\int_0^\infty S(s)x(s)\,ds \;+\; C(t)\int_t^\infty S(s)x(s)\,ds. \end{aligned}$$

Since the sine function $S(\cdot)$ is pointwise almost periodic, it follows from [64, Lemma 3.1] and [64, Theorem 3.2] that $C(\cdot)$ is also pointwise almost periodic. Therefore, the first and third terms on the right-hand side define almost periodic functions while the second and fourth terms are functions that vanish at ∞. Thus, $y_x \in AAP(\mathbb{X})$.

From Proposition 10.21, we know that the integrals

$$\int_0^\infty C(s)x(s)ds$$

and

$$\int_0^\infty S(s)x(s)ds,$$

for $x \in V$, are included in a compact subset of $\mathbb{X}$, which implies that the set formed by the functions

$$S(\cdot)\int_0^\infty C(s)x(s)ds - C(\cdot)\int_0^\infty S(s)x(s)ds,\ \ x \in V,$$

is relatively compact in $AP(\mathbb{X})$. The same Proposition enables us to infer that the set

$$\{t \to C(t)\int_t^\infty S(s)x(s)ds - S(t)\int_t^\infty C(s)x(s)ds : x \in V\}$$

is relatively compact in $C_0(\mathbb{X})$. This shows that $\{y_x : x \in V\}$ is relatively compact in $AAP(\mathbb{X})$.

We now prove that the set $\{z_x : x \in V\}$ is relatively compact in $AAP(\mathbb{X})$. For that we first show that the functions z_x, $x \in V$, are uniformly continuous. First of all, fix $L > 0$. Since $C(\cdot)$ is pointwise almost periodic, from (**c**) we have that

$$\|(C(t+s) - C(t))x(\xi)\| \to 0,$$

as $s \to 0$, uniformly for $t \geq 0$, $0 \leq \xi \leq L$ and $x \in V$. Therefore,

$$\begin{aligned}
&\|z_x(t+s) - z_x(t)\| \\
&\quad \leq \int_0^{t\wedge(t+s)} \|C(t+s-\xi)x(\xi) - C(t-\xi)x(\xi)\|d\xi \\
&\qquad + \|\int_{t\wedge(t+s)}^{t\vee(t+s)} C(t+s-\xi)x(\xi)d\xi\| \\
&\quad \leq \int_0^L \sup_{t\geq 0, x\in V} \|(C(t+s-\xi) - C(t-\xi))x(\xi)\|d\xi \\
&\qquad + 2N\int_L^\infty \|x(\xi)\|d\xi + N\int_{t\wedge(t+s)}^{t\vee(t+s)} \|x(\xi)\|d\xi.
\end{aligned}$$

Using conditions (**a**) and (**b**) we can appropriately choose L to show that the right-hand side of the above inequality converges to 0 as $s \to 0$, uniformly in $t \geq 0$ and $x \in V$, which proves that each function z_x is uniformly continuous on $[0, \infty)$. Moreover, from the above, it is clear that the set $\{z_x : x \in V\}$ is uniformly equi-continuous on $[0, \infty)$. Since z_x is the derivative of y_x, it follows from [113, Theorem 5.2] that $\{z_x : x \in V\}$ is a uniformly equi-continuous subset of $AAP(\mathbb{X})$. Moreover, from Proposition 10.21 we obtain that $\{z_x(t) : x \in V\}$ is relatively compact, for all $t \geq 0$.

Finally, we establish that $\{z_x : x \in V\}$ is equi-asymptotically almost periodic. For a given $\varepsilon > 0$, there exists $L_\varepsilon > 0$ such that

$$\int_{L_\varepsilon}^\infty \|x(s)\|ds \leq \varepsilon/6N$$

for all $x \in V$.

In addition, since the set $\{C(\cdot)x(s) : 0 \leq s \leq L_\varepsilon\}$ is equi-almost periodic, there is a relatively dense set $P_\varepsilon \subseteq [0,\infty)$ such that

$$\|C(\xi+\tau)x(s) - C(\xi)x(s)\| \leq \frac{\varepsilon}{3L_\varepsilon},$$

for all $\xi \geq 0,\ 0 \leq s \leq L_\varepsilon$ and every $\tau \in P_\varepsilon$. Hence, for $t \geq L_\varepsilon$ and $\tau \in P_\varepsilon$, we obtain

$$\begin{aligned}\|z_x(t+\tau) - z_x(t)\| &\leq \int_0^t \|C(t+\tau-s)x(s) - C(t-s)x(s)\|ds \\ &\quad + \int_t^{t+\tau} \|C(t+\tau-s)x(s)\|\, ds \\ &\leq \int_0^{L_\varepsilon} \|C(t+\tau-s)x(s) - C(t-s)x(s)\|ds \\ &\quad +3N\int_{L_\varepsilon}^{\infty} \|x(s)\|\, ds \\ &\leq \varepsilon\end{aligned}$$

which shows the assertion.

One completes the proof by applying Lemma 10.19 to the set $\{z_x : x \in V\}$.

Using this result and proceeding as in the proof of Proposition 10.16 we obtain the compactness of $\{y_x : x \in V\}$ with some weaker conditions.

Proposition 10.23 *Assume that $S(\cdot)$ is almost periodic and that $V \subseteq L^1([0,\infty),\mathbb{X})$ is uniformly bounded and satisfies the following properties:*

(a) $\int_L^{\infty} \| x(s) \|\, ds \to 0$, *when* $L \to \infty$, *uniformly for* $x \in V$;

(b) $\int_t^{t+s} \|x(\xi)\|\, d\xi \to 0$, *as* $s \to 0$, *uniformly for* $t \geq 0$ *and* $x \in V$;

(c) *for each* $t,\ \delta \geq 0$, *the set* $\{S(\delta)x(s) : 0 \leq s \leq t,\ x \in V\}$ *is relatively compact in* $\mathbb{X}$.

Then $\{y_x : x \in V\}$ *is relatively compact in* $AAP(\mathbb{X})$.

Proof Define for all $x \in V$, the function

$$\widetilde{y}_x(t) = \int_0^t S(s)x(s)ds.$$

Let $0 < \varepsilon < t \leq a$. Since the function $s \to S(s)$ is Lipschitz continuous, we can choose points $0 = t_1 < t_2 \ldots < t_n = t$ such that $\| S(s) - S(s') \| \leq \varepsilon$ for $s, s' \in [t_i, t_{i+1}]$ and $i = 1, 2, \ldots .n-1$. For $x \in V$, then from the Mean Value Theorem for the Bochner integral (see [90, Lemma 2.1.3]), we find that

$$\begin{aligned}\widetilde{y}_x(t) &= \sum_{i=1}^{n-1}\int_{t_i}^{t_{i+1}}(S(s)-S(t_i))x(s)ds + \sum_{i=1}^{n-1}\int_{t_i}^{t_{i+1}} S(t_i)x(s)ds \\ &\in \mathscr{C}_\varepsilon + \sum_{i=1}^{n-1}(t_{i+1}-t_i)\,\overline{co(\{S(t_i)z(s) : s \in [0,t_i], z \in V\})} \\ &\subset \mathscr{C}_\varepsilon + \mathscr{K}_\varepsilon,\end{aligned}$$

where $\mathscr{K}_\varepsilon$ is compact and $diam(\mathscr{C}_\varepsilon) \to 0$ as $\varepsilon \to 0$. This proves that $W_1(t) = \{\widetilde{y}_x(t);\ x \in V\}$ is relatively compact in $\mathbb{X}$. Moreover, proceeding as in the proof of Proposition 10.21, we infer that

$$W = \bigcup_{0\le t\le\infty} W(t)$$

and

$$U = \left\{\int_t^\infty S(s)x(s)ds : x \in V,\ t \ge 0\right\}$$

are also relatively compact in $\mathbb{X}$.

To complete the proof, we consider one more time the decomposition

$$\begin{aligned}y_x(t) = S(t)\int_0^\infty C(s)x(s)ds - S(t)\int_t^\infty C(s)x(s)ds \\ -C(t)\int_0^\infty S(s)x(s)ds + C(t)\int_t^\infty S(s)x(s)ds.\end{aligned}$$

Since the cosine function is pointwise almost periodic (see Remark 10.13), we infer from Remark 10.15 and Lemma 10.10 that the set of functions

$$\left\{t \to -C(t)\int_0^\infty S(s)x(s)\,ds + C(t)\int_t^\infty S(s)x(s)\right\}$$

is relatively compact in $AAP(\mathbb{X})$. Moreover, using the fact that $S(\cdot)$ is almost periodic and that $S(t)$ is a compact operator for every $t \ge 0$, we can prove from Remark 10.15 and Lemma 10.10 that the set of functions

$$\left\{t \to S(t)\int_0^\infty C(s)x(s)ds - S(t)\int_t^\infty C(s)x(s)ds : x \in V\right\}$$

is also completely continuous in $AAP(\mathbb{X})$.

The main result of this subsection can be formulated as follows:

Theorem 10.24 *Assume that $S(\cdot)$ is almost periodic and that condition* ($\mathbf{H_1}$) *holds with $m_f(\cdot)$ and $m_g(\cdot)$ in $L^1([0,\infty))$. Suppose, in addition, that for every $t \geq 0$ and each $r \geq 0$ the sets $\{S(t)f(s,\psi) : (s,\psi) \in [0,t] \times B_r(0,\mathscr{B})\}$ and $g([0,t] \times B_r(0,\mathscr{B}))$ are relatively compact in $\mathbb{X}$. If*

$$K \int_0^\infty (Nm_g(s) + \widetilde{N}m_f(s))ds < \int_c^\infty \frac{ds}{W(s)}, \tag{10.22}$$

where $c = (KNH + M) \parallel \varphi \parallel_{\mathscr{B}} + K\widetilde{N}(\parallel \xi \parallel + \parallel g(0,\varphi) \parallel)$, then there exists a mild solution $u(\cdot) \in AAP(\mathscr{B},\mathbb{X})$ to the system Eqs. (10.12)–(10.14).

Proof Let $\mathscr{B}AAP = \{x : \mathbb{R} \to \mathbb{X} : x_0 \in \mathscr{B},\ x\,|_{[0,\infty)} \in AAP(\mathbb{X})\}$ endowed with the semi-norm $\|x\|_{\mathscr{B}AAP} := \|x_0\|_{\mathscr{B}} + \sup_{t\geq 0}\|x(t)\|$ and $\Gamma : \mathscr{B}AAP \to \mathscr{B}AAP$ be the operator defined by

$$\begin{aligned}\Gamma x(t) &= C(t)\varphi(0) + S(t)[\xi - g(0,\varphi)] \\ &\quad + \int_0^t C(t-s)g(s,x_s)ds + \int_0^t S(t-s)f(s,x_s)ds,\end{aligned}$$

for $t \geq 0$, and $(\Gamma x)_0 = \varphi$.

By the integrability of the functions $m_f(\cdot)$ and $m_g(\cdot)$ and proceeding as in the proof of Proposition 10.22 for the functions $f(s,x_s)$ and $g(s,x_s)$, we infer that $\Gamma(x) \in AAP(\mathscr{B},\mathbb{X})$. Furthermore, if we take a sequence $(x^n)_n$ that converges to x in the space $AAP(\mathscr{B},\mathbb{X})$, then $S(t-s)f(s,x^n_s) \to S(t-s)f(s,x_s)$ and $C(t-s)g(s,x^n_s) \to C(t-s)g(s,x_s)$, as $n \to \infty$, a.e. for $t,s \in [0,\infty]$. Let $L = \sup\{\|x\|_{\mathscr{B}C}, \|x^n\|_{\mathscr{B}C} : n \in \mathbb{N}\}$ and $\beta = (K+M)L$. From the inequalities

$$\begin{aligned}\|C(t-s)g(s,x^n_s) - C(t-s)g(s,x_s)\| &\leq 2Nm_g(s)W_g(\beta),\\ \|S(t-s)f(s,x^n_s) - S(t-s)f(s,x_s)\| &\leq 2\widetilde{N}m_f(s)W_f(\beta),\end{aligned}$$

and using the integrability of $m_f(\cdot)$ and $m_g(\cdot)$, we conclude that $\parallel \Gamma x^n - \Gamma x \parallel_{\mathscr{B}AAP} \to 0$ when $n \to \infty$. Thus, Γ is a continuous map from $AAP(\mathscr{B},\mathbb{X})$ into $AAP(\mathscr{B},\mathbb{X})$.

On the other hand, proceeding as in the proof of Theorem 10.5, we conclude that the set of functions $\{x^\lambda \in AAP(\mathscr{B},\mathbb{X}) : \lambda\Gamma(x^\lambda) = x^\lambda,\ 0 < \lambda < 1\}$ is uniformly bounded on $[0,\infty)$.

Finally, we show that Γ is completely continuous. In order to establish this assertion, we take a bounded set $V \subseteq AAP(\mathscr{B},\mathbb{X})$. Since the sets of functions $\Lambda_1 = \{s \to g(s,x_s) : x \in V\}$ and $\Lambda_2 = \{s \to f(s,x_s) : x \in V\}$ satisfy the hypotheses of Propositions 10.22 and 10.23, respectively, we deduce that $\Gamma(V)$ is relatively compact in $AAP(\mathbb{X})$. The assertion is now a consequence of Theorem 1.81.

In Theorem 10.26 below, we prove the existence of an asymptotically almost periodic mild solution to Eqs. (10.12)–(10.14) assuming that $g(\cdot)$ satisfies an appropriate Lipschitz condition. For that, we need the following lemma.

Lemma 10.25 *If $\mathscr{B}$ is a fading memory space and $z \in BC(\mathbb{R}; \mathbb{X})$ is a function such that $z_0 \in \mathscr{B}$ and $z \in AAP(\mathbb{X})$, then $t \to z_t \in AAP(\mathscr{B})$.*

Theorem 10.26 *Assume that the sine function $S(\cdot)$ is almost periodic and that $\mathscr{B}$ is a fading memory space. Suppose, in addition, that the following conditions hold:*

(a) *For every $t \geq 0$ and each $r \geq 0$, the set*

$$\Big\{S(t)f(s,\psi) : (s,\psi) \in [0,t] \times B_r(0,\mathscr{B})\Big\}$$

is relatively compact in $\mathbb{X}$.

(b) *There exists a function $L_g \in L^1([0,\infty))$ such that*

$$\| g(t,\psi_1) - g(t,\psi_2) \| \leq L_g(t) \| \psi_1 - \psi_2 \|_{\mathscr{B}}, \quad (t,\psi_j) \in [0,\infty) \times \mathscr{B}.$$

(c) *The condition $(\mathbf{H_1})$ is valid with m_g, m_f in $L^1([0,\infty))$ and*

$$(K+M) \liminf_{\xi \to \infty} \frac{W(\xi)}{\xi} \int_0^\infty (N m_g(s) + \widetilde{N} m_f(s)) ds < 1. \qquad (10.23)$$

Then there exists a mild solution $u(\cdot) \in AAP(\mathbb{X})$ of (10.12)–(10.14).

Proof Let $\mathscr{B}AAP = \{x : \mathbb{R} \to \mathbb{X} : x_0 \in \mathscr{B},\ x\ |_{[0,\infty)} \in AAP(\mathbb{X})\}$ endowed with the semi-norm defined by $\|x\|_{\mathscr{B}AAP} = \|x_0\|_{\mathscr{B}} + \sup_{t \geq 0} \|x(t)\|$. On this space, we define the operators $\Gamma_i : \mathscr{B}AAP \to \mathscr{B}AAP$, $i = 1, 2$, by

$$\Gamma_1 x(t) = C(t)\varphi(0) + S(t)[\xi - g(0,\varphi)] + \int_0^t C(t-s) g(s,x_s) ds,$$

$$\Gamma_2 x(t) = \int_0^t S(t-s) f(s,x_s) ds,$$

for $t \geq 0$, and $(\Gamma_1 x)_0 = \varphi$ and $(\Gamma_2 x)_0 = 0$.

From the proof of Proposition 10.22, we infer that the functions

$$\zeta(t) = \int_0^t S(t-s) g(s,x_s) ds$$

and $\Gamma_2 x$ are asymptotically almost periodic. It is easy to see that

$$\zeta'(t) = \int_0^t C(t-s) g(s,x_s) ds.$$

Moreover, since the function $s \to g(s, x_s)$ is integrable on $[0, \infty)$ and

$$\|\zeta'(t+h) - \zeta'(t)\| \leq \int_0^h N \parallel g(s, x_s) \parallel ds$$
$$+N \int_0^\infty \parallel g(s+h, x_{s+h}) - g(s, x_s) \parallel ds,$$

converge to zero as $h \to 0$, uniformly for $t \in [0, \infty)$, we can conclude from [113, Theorem 5.2] that $\Gamma_1 x$ is also asymptotically almost periodic. This proof that $\Gamma_1 x, \Gamma_2 x$ are well defined and that Γ_1, Γ_2 are functions defined from $\mathscr{B}AAP$ into $\mathscr{B}AAP$.

Let $y : \mathbb{R} \to \mathbb{X}$ be the extension of φ to $\mathbb{R}$ such that

$$y(t) = C(t)\varphi(0) + S(t)[\xi - g(0, \varphi)]$$

for $t \geq 0$ and $\Gamma : \mathscr{B}AAP \to \mathscr{B}AAP$ be the map $\Gamma = \Gamma_1 + \Gamma_2$. We next prove that there exists $r > 0$ such that $\Gamma(B_r(y, \mathscr{B}AAP)) \subset B_r(y, \mathscr{B}AAP)$. Proceeding by contradiction, we suppose that for each $r > 0$ there exist $u^r \in B_r(y, \mathscr{B}AAP)$ and $t^r \geq 0$ such that $\parallel \Gamma u^r(t^r) - y(t^r) \parallel > r$. Consequently,

$$\begin{aligned} r &\leq \parallel \Gamma u^r(t^r) - y(t^r) \parallel \\ &\leq \int_0^{t^r} (Nm_g(s) + \widetilde{N}m_f(s))W(\parallel u_s^r - y_s \parallel_{\mathscr{B}} + \parallel y_s \parallel_{\mathscr{B}})ds \\ &\leq \int_0^\infty (Nm_g(s) + \widetilde{N}m_f(s))W((K+M)r + \rho)ds, \end{aligned}$$

where $\rho = (M + KNH) \parallel \varphi \parallel_{\mathscr{B}} + K\tilde{N} \parallel \xi - g(0, \varphi) \parallel$, which yields

$$1 \leq (K+M) \liminf_{\xi \to \infty} \frac{W(\xi)}{\xi} \int_0^\infty (Nm_g(s) + \widetilde{N}m_f(s))ds.$$

Since this inequality contradicts Eq. (10.23), we obtain the assertion.

Let $r > 0$ such that $\Gamma(B_r(0, \mathscr{B}AAP)) \subset B_r(0, \mathscr{B}AAP)$. Proceeding as in the proof of Theorem 10.24, we can show that the map Γ_2 is completely continuous. Moreover, from the estimate

$$\parallel \Gamma_1 u(t) - \Gamma_1 v(t) \parallel \leq NK \int_0^t L_g(s)ds \parallel u - v \parallel_{\mathscr{B}AAP},$$

we infer that Γ_1 is a contraction on $\mathscr{B}AAP$, which enables us to conclude that Γ is condensing on $B_r(0, \mathscr{B}AAP)$. Now, the assertion is a consequence of Theorem 1.80.

10.3.7 Asymptotically Almost Periodic Solutions to Some Second-Order Integro-differential Systems

This subsection is devoted an illustrative example to the previous subsection and consists of studying the existence of asymptotically almost periodic mild solutions for the second-order partial differential equations given by

$$\frac{\partial}{\partial t}\left[\frac{\partial u(t,\xi)}{\partial t} + \eta(t)u(t,\xi) + \int_{-\infty}^{t} \alpha_1(t,s)u(s,\xi)ds\right] = \frac{\partial^2 u(t,\xi)}{\partial \xi^2}$$
$$+ \int_{-\infty}^{t} \alpha_2(t,s)u(s,\xi)ds, \tag{10.24}$$

for $t \geq 0$ and $\xi \in J = [0, \pi]$, subject to the initial conditions

$$u(t,0) = u(t,\pi) = 0, \qquad t \geq 0, \tag{10.25}$$

$$u(\theta,\xi) = \varphi(\theta,\xi), \qquad \theta \leq 0, \quad \xi \in J, \tag{10.26}$$

$$\frac{\partial u(0,\xi)}{\partial t} = z(\xi), \qquad \xi \in J, \tag{10.27}$$

where $\eta(\cdot) : \mathbb{R} \to \mathbb{R}$, $\alpha_i : \mathbb{R}^2 \to \mathbb{R}$ $(i = 1, 2)$ are continuous functions and φ, ξ are some appropriate functions.

In order to cast the above system into an abstract version of the previous subsection, we let $\mathbb{X} = (L^2(0,\pi); \|\cdot\|_2)$ and consider the operator $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ defined by

$$D(A) = \left\{u \in H^2(0,\pi) : u(0) = u(\pi) = 0\right\}, \quad Au = \frac{d^2u}{dx^2}, \quad u \in D(A).$$

It is well known that A is the infinitesimal generator of a strongly continuous cosine function, $(C(t))_{t\in\mathbb{R}}$ on $\mathbb{X}$. Furthermore, A has discrete spectrum with eigenvalues $-n^2$, $n \in \mathbb{N}$, with corresponding normalized eigenvectors given by

$$z_n(\xi) = \left(\frac{2}{\pi}\right)^{1/2} \sin(n\xi).$$

Moreover, the following properties are fulfilled:

(a) The set $\{z_n : n \in \mathbb{N}\}$ is an orthonormal basis of $\mathbb{X}$;

(b) For $u \in \mathbb{X}$, $C(t)u = \sum_{n=1}^{\infty} \cos(nt)\langle u, z_n\rangle z_n$. It follows from this expression that

$$S(t)u = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n}\langle u, z_n\rangle z_n.$$

Moreover, the sine function $S(\cdot)$ is periodic with $S(t)$ being a compact operator for all $t \in \mathbb{R}$ such that $\max\{\|C(t)\|, \|S(t)\|\} \leq 1$, for every $t \in \mathbb{R}$.

As a phase space we choose the space $\mathscr{B} = C_r \times L^p(\rho; \mathbb{X})$, $r \geq 0$, $1 \leq p < \infty$ (see Example 10.2) and assume that the conditions (g-5)–(g-7) in the terminology of [69] are valid. Note that under these conditions, the space $\mathscr{B}$ is a fading memory space and that there exists $\mathfrak{K} > 0$ such that $\max\{K(t), M(t)\} \leq \mathfrak{K}$ for all $t \geq 0$, see [69, Example 7.1.8] and [69, Proposition 7.1.5] for details.

By assuming that

$$L_g(t) = |\eta(t)| + \left(\int_{-\infty}^{0}\left[\frac{\alpha_1(t, t+\theta)}{\rho(\theta)}\right]^2 d\theta\right)^{1/2},$$

$$m_f(t) = \left(\int_{-\infty}^{0}\left[\frac{\alpha_2(t, t+\theta)}{\rho(\theta)}\right]^2 d\theta\right)^{1/2},$$

are finite, for every $t \geq 0$, we can define the operators $g, f : \mathbb{R}_+ \times \mathscr{B} \to \mathbb{X}$ by the mean of the expressions

$$g(t, \psi)(\xi) = \eta(t)\psi(0, \xi) + \int_{-\infty}^{0} \alpha_1(t, t+s)\psi(s, \xi)ds,$$

$$f(t, \psi)(\xi) = \int_{-\infty}^{0} \alpha_2(t, t+s)\psi(s, \xi)ds.$$

It is easy to see that $g(t, \cdot)$ and $f(t, \cdot)$ are bounded linear operators, as $\| g(t, \cdot) \|_{B(\mathscr{B},\mathbb{X})} \leq L_g(t)$ and $\| f(t, \cdot) \|_{B(\mathscr{B},\mathbb{X})} \leq m_f(t)$ for every $t \geq 0$. The next results are a direct consequence of Theorem 10.26. Thus the details of the proof will be omitted.

Proposition 10.27 *Assume $\varphi \in \mathscr{B}$, $\eta \in \mathbb{X}$ and that $L_g(\cdot), m_f(\cdot)$ are functions in $L^1([0, \infty))$. If*

$$2\mathfrak{K}\int_{0}^{\infty} (L_g(s) + m_f(s))ds < 1, \tag{10.28}$$

then there exists an asymptotically almost periodic mild solution to (10.24)–(10.27).

To complete this subsection, we study the existence of asymptotically almost periodic solutions for the system (10.16). To simplify the description and for sake of brevity, we consider the case when $d = 1$. Assume that the functions $\beta(\cdot)$ and $\frac{\partial \beta(\cdot)}{\partial s}$ are continuous and that the expression

$$L_g(t) = \left| \frac{\beta(t, 0)}{c} \right| + \frac{1}{|c|} \left(\int_{-\infty}^{0} \left[\frac{\partial \beta(t, s)}{\partial s} \rho^{-1}(s) \right]^2 ds \right)^{1/2}$$

defines a function in $L^1([0, \infty))$. By assuming that the solution $u(\cdot)$ of (10.16) is known on $[0, \infty)$, and defining the function $g : \mathbb{R} \times \mathscr{B} \to \mathbb{X}$ by

$$g(t, \psi)(\xi) = \frac{\beta(t, 0)}{c} \psi(0, \xi) + \frac{1}{c} \int_0^{\infty} \frac{\partial \beta(t, s)}{\partial s} \psi(-s, \xi) ds,$$

we can transform system (10.16) into the abstract system (10.12)–(10.14).

Corollary 10.28 *For every $\varphi \in \mathscr{B}$ and $\xi \in \mathbb{X}$, there exists an asymptotically almost periodic mild solution of Eq. (10.16) with $u_0 = \varphi$.*

Proof This result is a particular case of Proposition 10.27. We only observe that the inequality (10.23) is automatically satisfied, as $m_f \equiv 0$.

10.4 Exercises

1. Consider the functions $f_1, \; f_2 : \mathbb{R} \times H_0^1(\Omega) \times H_0^1(\Omega) \to L^2(\Omega)$ are defined by

$$f_i(t, u, \theta)(x) = f_i(t, \nabla u(x), \nabla \theta(x)) = \frac{K d_i(t)}{1 + |\nabla u(x)| + |\nabla \theta(x)|}$$

   for $x \in \Omega, \; t \in \mathbb{R}, \; i = 1, 2$, where $d_i : \mathbb{R} \mapsto \mathbb{R}$ are almost periodic functions.

   a. Show that the functions f_i $(i = 1, 2)$ are jointly continuous.
   b. Show that f_i $(i = 1, 2)$ are globally Lipschitz functions, that is, there exists $L > 0$ such that

$$\| f_i(t, u, \theta) - f_i(t, v, \eta) \|_{L^2(\Omega)} \leq L \left(\|u - v\|^2_{H_0^1(\Omega)} + \|\theta - \eta\|^2_{H_0^1(\Omega)} \right)^{\frac{1}{2}}$$

   for all $t \in \mathbb{R}, u, v, \eta$ and $\theta \in H_0^1(\Omega)$.

2. Prove Corollary 10.6.
3. Prove Theorem 10.8.
4. Prove Lemma 10.10.

5. Prove Lemma 10.18.
6. Prove Lemma 10.19.
7. Prove Lemma 10.25.
8. Prove Proposition 10.27.

10.5 Comments

The existence results of Sect. 10.2 are based upon Baroun et al. [20] and Baroun [29]. For additional reading on thermoelastic systems, we refer the reader to [14, 18, 24, 44, 65, 79], and [92].

The existence results of Sect. 10.2 are based upon Diagana et al. [50]. For additional reading on the topics discussed in this section, we refer the reader to Fattorini [57] and Travis and Webb [106, 107].