Chapter 1
Banach and Hilbert Spaces

1.1 Introduction

In this chapter we present the basic material on metric, Banach, and Hilbert spaces needed in the sequel. By design, every Hilbert space is a Banach space—with the converse being untrue. Banach and Hilbert spaces play a central role in many areas and subareas of mathematical analysis as most of the spaces encountered and utilized in practical problems turn out to be either Hilbert spaces or Banach spaces.

Recall that Banach spaces were named after the Polish mathematician Stefan Banach who introduced them in the mathematical literature around 1920–1922. Basically, a Banach space is a normed vector space that is complete, that is, every Cauchy sequence in it must converge in it. For instance $\mathbb{Q}$ the field of rational numbers equipped with the standard absolute value is not complete; its completion is in fact $\mathbb{R}$, the field of real numbers. Standard examples of Banach spaces include, but are not limited to, $\mathbb{R}^d$, $\mathbb{C}^d$, $\ell^p(\mathbb{N})$, L^p (Lebesgue spaces), $W^{k,p}$ (Sobolev spaces), $C^{k,\alpha}$ (Hölder spaces), $B^s_{p,q}$ (Besov spaces), $H^p(\mathbb{S}^1)$ (Hardy spaces), and BMO (functions of bounded mean oscillation)—when they are equipped with their respective standard norms.

In this chapter, we first study some of the basic properties of metric spaces and then use these to deduce those of Banach and Hilbert spaces.

One should stress upon the fact that the introductory material presented in this chapter can be found in any good book in (nonlinear) functional analysis. Consequently, some proofs will be omitted.

T. Diagana, *Semilinear Evolution Equations and Their Applications*,
https://doi.org/10.1007/978-3-030-00449-1_1

1.2 Metric Spaces

This section is essentially devoted to metric spaces and their basic properties. Among other things, the following notions will be introduced and studied in the context of metric spaces: convergence, completeness, continuity, compact metric spaces, and the Banach fixed-point principle.

1.2.1 Basic Definitions and Examples

Definition 1.1 A pair (M, d) consisting of a nonempty set M and a mapping (metric or distance) $d : M \times M \mapsto [0, \infty)$ is called a metric space, if the mapping d fulfills the following properties,

i) $d(x, y) = 0$ if and only if $x = y$;
ii) $d(x, y) = d(y, x)$; and
iii) $d(x, z) \leq d(x, y) + d(y, z)$

for all $x, y, z \in M$.

The inequality iii) appearing in Definition 1.1 is commonly known as the *triangle inequality*. Further, elements of the set M are called points and the quantity $d(x, y)$ is referred to as the distance between the points $x, y \in M$.

If (M, d) is a metric space, then using the triangle inequality, it can be easily shown that the following property holds,

$$\Big| d(x, y) - d(y, z) \Big| \leq d(x, z)$$

for all $x, y, z \in M$.

Example 1.2

(1) Let d_0 and d_1 be two metrics upon a nonempty set M. Consider the mapping d defined by, $d(x, y) = \alpha d_0(x, y) + \beta d_1(x, y)$ for all $x, y \in M$, where $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$. It is not hard to see that the pair (M, d) is a metric space.
(2) Let M be an arbitrary nonempty set which we endow with the so-called discrete metric d_s defined by

$$d_s(x, y) := \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

It can be easily shown that (M, d_s) is a metric space.

(3) Fix $p \geq 1$. Then $(\mathbb{R}^n, d_p)$ is a metric space, where

$$d_p(\alpha, \beta) := \left(\sum_{k=1}^{n} |\alpha_k - \beta_k|^p \right)^{1/p}$$

for all $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n), \quad \beta = (\beta_1, \beta_2, ..., \beta_n) \in \mathbb{R}^n$.

(4) Consider the unit circle, $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ and let $\mathscr{A}(\mathbb{S}^1)$ stand for the collection of all functions $f : \mathbb{S}^1 \mapsto \mathbb{C}$ for which

$$\int_0^{2\pi} |f(e^{i\theta})|^2 d\theta < \infty.$$

It can be easily seen that $(\mathscr{A}(\mathbb{S}^1), \rho)$ is a metric space, where the metric ρ is defined by,

$$\rho(f, g) := \left(\int_0^{2\pi} |f(e^{i\theta}) - g(e^{i\theta})|^2 d\theta \right)^{\frac{1}{2}}$$

for all $f, g \in \mathscr{A}(\mathbb{S}^1)$.

As usual, if (M, d) is a metric space, then the metric d enables us to define the notions of balls and spheres in M. Indeed, the (open) ball centered at $x \in M$ with radius $r > 0$ is defined by $B(x, r) = \{y \in M : d(x, y) < r\}$. Similarly, the (closed) ball centered at $x \in M$ with radius $r \geq 0$ is defined by $\overline{B}(x, r) = \{y \in M : d(x, y) \leq r\}$. By the sphere $S(x, r)$, centered at $x \in M$ with radius $r \geq 0$, we mean the set of all points defined by $S(x, r) = \{y \in M : d(x, y) = r\}$.

Definition 1.3 Let (M, d) be a metric space and let $\mathscr{O} \subset M$ be a subset. The set $\mathscr{O}$ is said to be an open set if for all $x \in M$, there exists $r > 0$ such that $B(x, r) \subset \mathscr{O}$.

Classical examples of open sets in a metric space (M, d) include M itself and the empty set, $\emptyset$. Recall that arbitrary unions of open sets of M are also open sets of M. Further, finite intersections of open sets of M are also open sets of M.

Definition 1.4 Let (M, d) be a metric space and let $\mathscr{O} \subset M$ be a subset. A point $x \in M$ is said to be an interior point of $\mathscr{O}$ if and only if there exists $r > 0$ such that $B(x, r) \subset \mathscr{O}$.The collection of all interior points of $\mathscr{O}$ is denoted $\text{Int}(\mathscr{O})$.

It can be shown that a subset $\mathscr{O}$ of M is open if and only if it contains all of its interior points, that is, $\mathscr{O} = \text{Int}(\mathscr{O})$.

Definition 1.5 Let (M, d) be a metric space and let $\mathscr{O} \subset M$ be a subset. The set $\mathscr{O}$ is said to be a closed set if its complement $\mathscr{O}^C = M \setminus \mathscr{O}$ is an open set.

Classical examples of closed sets of a metric space (M, d) include singletons $\{x\}$, M, and $\emptyset$.

Definition 1.6 Let (M, d) be a metric space and let $\mathcal{O} \subset M$ be a subset. A point $x \in M$ is said to be an adherent point of $\mathcal{O}$ if and only if for all $r > 0$, the following holds, $B(x, r) \cap \mathcal{O} \neq \{\emptyset\}$. The collection of all adherent points of $\mathcal{O}$ is called the closure of $\mathcal{O}$ and is denoted $\overline{\mathcal{O}}$.

It can be shown that a subset $\mathcal{O}$ of M is closed if and only if it contains all of its adherent points, that is, $\mathcal{O} = \overline{\mathcal{O}}$. In other words, $\mathcal{O}$ is closed if and only if for any sequence $(x_n)_{n\in\mathbb{N}} \subset \mathcal{O}$ such that $d(x_n, x) \to 0$ for some $x \in M$ as $n \to \infty$, then one must have $x \in \mathcal{O}$.

Definition 1.7 Let (M, d) be a metric space and let $\mathcal{O} \subset M$ be a subset. The subset $\mathcal{O}$ is said to be bounded it is included in some ball $B(x, r)$. Otherwise, the set $\mathcal{O}$ is said to be unbounded.

It is easy to see that $\mathcal{O} \subset M$ is bounded if and only if its diameter, $\operatorname{diam}(\mathcal{O}) := \sup\limits_{x,y\in\mathcal{O}} d(x, y)$, is finite, that is, $\operatorname{diam}(\mathcal{O}) < \infty$.

If (M, d) is a metric space, then the metric d enables us to define the notion of convergence in M.

Definition 1.8 Let (M, d) be a metric space. A sequence $(x_n)_{n\in\mathbb{N}} \subset M$ is said to converge to some $x \in M$ with respect to the metric d, if $d(x_n, x) \to 0$ as $n \to \infty$. Equivalently, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for all $n \geq N$.

If a sequence $(x_n)_{n\in\mathbb{N}} \subset M$ converges to some $x \in M$ with respect to the metric d, then we write $\lim\limits_{n\to\infty} x_n = x$.

Proposition 1.9 *Let (M, d) be a metric space. If a sequence $(x_n)_{n\in\mathbb{N}} \subset M$ converges, then its limit is unique.*

Proof Suppose $(x_n)_{n\in\mathbb{N}} \subset M$ converges to two limits $x, y \in M$. Then, using the triangle inequality it follows that $0 \leq d(x, y) \leq d(x, x_n) + d(x_n, y)$. Letting $n \to \infty$ in the previous inequality, it follows that $d(x, y) = 0$ which yields $x = y$.

Definition 1.10 Let (M, d) be a metric space. A sequence $(x_n)_{n\in\mathbb{N}} \subset M$ is called a Cauchy sequence, if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m \geq N$.

Proposition 1.11 *Let (M, d) be a metric space. Every convergent sequence is a Cauchy sequence. Further, every Cauchy sequence is bounded.*

Proof

i) Let $(x_n)_{n\in\mathbb{N}}$ be a convergent sequence in the metric space (M, d). This means that there exists $x \in M$ such that $d(x_n, x) \to 0$ as $n \to \infty$. Equivalently, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x) < \frac{\varepsilon}{2}$ for all $n \geq N$. Now, using the triangle inequality it follows that $d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ for all $n, m \geq N$. Consequently, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

ii) Suppose that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. For $\varepsilon = 1$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < 1$ for $n, m \geq N$. In particular, $d(x_n, x_N) < 1$ for all $n \geq N$. Setting $r = 1 + \max\{1, d(x_1, x_N), d(x_2, x_N), ..., d(x_{N-1}, x_N)\}$, one can easily see that $(x_n)_{n\in\mathbb{N}} \subset B(x_N, r)$ which yields the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded.

1.2.2 Complete Metric Spaces

Definition 1.12 A metric space (M, d) is said to be complete, if every Cauchy sequence in it converges in it.

Classical examples of complete metric spaces include $\mathbb{R}^n$ equipped with its corresponding Euclidean metric defined by

$$d(x, y) = \left(\sum_{k=1}^{n} |x_k - y_k|^2\right)^{\frac{1}{2}}$$

for all $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2..., y_n) \in \mathbb{R}^n$, and $BC(\mathbb{R}, M)$ the collection of all bounded continuous functions which go from $\mathbb{R}$ into a complete metric space (M, d), when it is equipped with the sup norm metric, $d_\infty(f, g) = \sup_{t\in\mathbb{R}} d(f(t), g(t))$ for all $f, g \in BC(\mathbb{R}, M)$, etc.

A classical example of a metric space which is not complete is $\mathbb{Q}$; the field of rational numbers; when it is equipped with the standard absolute value defined by $d_0(x, y) = |x-y|$ for all $x, y \in \mathbb{Q}$. There are obviously various ways of constructing a Cauchy sequence in $\mathbb{Q}$ which diverges. Let us exhibit one here. Indeed, consider the recurrent sequence $(x_n)_{n\in\mathbb{N}}$ given by,

$$\begin{cases} x_1 = q \in \mathbb{N}, \\ \\ x_{n+1} = \dfrac{2}{3}x_n + \dfrac{1}{x_n}, \quad \text{for all } \; n \in \mathbb{N}. \end{cases}$$

It is not hard to show that not only $(x_n)_{n\in\mathbb{N}}$ is a sequence of rational numbers but also that $\lim\limits_{n\to\infty} x_n = \sqrt{3} \in \mathbb{R} \setminus \mathbb{Q}$. This shows that $(\mathbb{Q}, d_0)$ is not complete.

The larger question regarding the completion of a given arbitrary metric space (M, d) is the following: if (M, d) is not complete, are there ways of making it complete? The answer is a "yes." Indeed, suppose that (M, d) is not complete and let

$$CS(M, d) := \Big\{\text{all Cauchy sequences in } \; (M, d)\Big\}.$$

Define an equivalence relation upon elements of $CS(M, d)$ as follows: two sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ in $CS(M, d)$ are said to be equivalent which we denote $(x_n)_{n\in\mathbb{N}} \sim (y_n)_{n\in\mathbb{N}}$ if and only if, for $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$d(x_n, y_m) < \varepsilon \text{ for all } n, m \geq N.$$

Consider

$$\widetilde{CS(M, d)} = CS(M, d)/\sim$$

and define the mapping $\widetilde{d} : \widetilde{CS(M, d)} \times \widetilde{CS(M, d)} \mapsto [0, \infty)$ as follows: if $x, y \in \widetilde{CS(M, d)}$, that is, $x = [(x_n)_n]$ and $y = [(y_n)_{n\in\mathbb{N}}]$ (equivalence classes of $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$), then

$$\widetilde{d}(x, y) = \lim_{n\to\infty} d(x_n, y_n).$$

It is easy to check that $\widetilde{d}$ is a metric upon $\widetilde{CS(M, d)}$. Moreover, the metric space $(\widetilde{CS(M, d)}, \widetilde{d})$, by construction, is complete.

1.2.3 Continuous Functions

In the sequel, the pairs (M, d), (M', ρ), (M_1, d_1), (M_2, d_2), and (M_3, d_3) stand for metric spaces.

Definition 1.13 A function $f : (M, d) \mapsto (M', \rho)$ is said to be continuous at $x_0 \in M$, if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in M$, $d(x_0, x) < \delta$ yields $\rho(f(x_0), f(x)) < \varepsilon$. The function f is said to be continuous on M, if it is continuous at each point of M.

Recall that the continuity of a function $f : (M, d) \mapsto (M', \rho)$ at $x_0 \in M$ is equivalent to its sequential continuity at $x_0 \in M$, that is, for any arbitrary sequence $(x_n)_{n\in\mathbb{N}} \subset M$ that converges to some x_0, we have that $f(x_n)$ converges to $f(x_0)$. In general, it is easier to prove the continuity of a function using the sequential continuity than the general definition of continuity given in Definition 1.13.

We have the following composition result for continuous functions on metric spaces whose proof is left to the reader as an exercise.

Proposition 1.14 *Let $f : M_1 \mapsto M_2$ and $g : M_2 \mapsto M_3$ be given functions. Let $x_0 \in M_1$ be such that f is continuous at x_0 and that g is continuous at $f(x_0)$. Then, gof, the composition of f with g, is also continuous at x_0.*

Definition 1.15 A function $f : (M, d) \mapsto (M', \rho)$ is said to be uniformly continuous if, for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in M$, $d(x, y) < \delta$ yields $\rho(f(x), f(y)) < \varepsilon$.

Obviously, every uniformly continuous function is continuous. However, the converse, except in the case when M is a compact metric space, is in general not true (see for instance Theorem 1.20).

1.2.4 Compact Metric Spaces

Definition 1.16 An open covering of (M, d) is a collection of open sets whose union is M.

Definition 1.17 A metric space (M, d) is said to be compact if every open covering of it has a finite subcovering.

Definition 1.18 A subset $\mathcal{O}$ of a metric space (M, d) is called relatively compact if its closure $\overline{\mathcal{O}}$ is a compact subset of M.

Theorem 1.19 *A metric space (M, d) is compact if and only if every sequence of elements of M has a subsequence which converges.*

Proof The proof is left to the reader as an exercise.

Theorem 1.20 *If (M, d) is a compact metric space and if $f : M \mapsto M'$ is a continuous function, then f is uniformly continuous.*

Proof Let $\varepsilon > 0$ and $x \in M$ be given. Using the fact that f is continuous, we deduce that there exists $\delta_x > 0$ such that $d(x, y) < \delta_x$ yields $\rho(f(x), f(y)) < \frac{\varepsilon}{2}$. Since M is compact, it follows that there exists a finite number of points $x_1, x_1, ..., x_d$ such that the following holds,

$$M \subset \bigcup_{j=1}^{d} B\left(x_j, \frac{\delta_{x_j}}{2}\right).$$

Now let $\delta = \frac{1}{2}\min\{\delta_{x_j} : j = 1, 2, ..., d\}$. Clearly, $\delta > 0$. Further, for all $(x, y) \in M \times M$ satisfying $d(x, y) < \delta$, one can find $j_0 \in \{1, 2, ..., d\}$ such that x and y belong to $B(x_{j_0}, \delta_{j_0})$, which yields $\rho(f(x), f(x_{j_0})) < \frac{\varepsilon}{2}$ and $\rho(f(y), f(x_{j_0})) < \frac{\varepsilon}{2}$. Using the triangle inequality it follows that,

$$\rho(f(x), f(y)) \leq \rho(f(x), f(x_{j_0})) + \rho(f(x_{j_0}), f(y)) < \varepsilon,$$

which shows that the function f is uniformly continuous.

Let $C(M, M')$ denote the collection of all continuous functions from M into M'. If $M' = \mathbb{C}$, then $C(M, \mathbb{C})$ will be denoted by $C(M)$. If M is compact, then $C(M)$ is a metric space when it is equipped with the metric, $d_\infty(u, v) = \max_{x \in M} |u(x) - v(x)|$ for all $u, v \in C(M)$.

Definition 1.21 A sub-collection $\Gamma \subset C(M)$ is called uniformly bounded if there exists $C > 0$ such that $|u(x)| \leq C$ for every $x \in M$ and $u \in \Gamma$.

Definition 1.22 A sub-collection $\Gamma \subset C(M, M')$ is said to be equi-continuous if for all $x_0 \in M$ and for all $\varepsilon > 0$ there exists $\delta = \delta(x_0, \varepsilon) > 0$ such that $d(x, x_0) < \delta$ yields $\rho(f(x), f(x_0)) < \varepsilon$ for all $f \in \Gamma$.

Example 1.23 Let $\Gamma \subset C(M)$ be defined as the family of all K-Lipschitz functions on M where $K \geq 0$. That is, all functions $u : M \mapsto \mathbb{C}$ such that $|u(x) - u(y)| \leq Kd(x, y)$ for all $x, y \in M$. It can be easily shown that the family Γ is equi-continuous.

Theorem 1.24 (Arzelà-Ascoli Theorem) *A sub-collection $\Gamma \subset C(M)$ is relatively compact if and only if,*

(a) Γ is equi-continuous; and
(b) Γ is uniformly bounded.

Proof The proof is left to the reader as an exercise.

Corollary 1.25 *A sub-collection $\Gamma \subset C(M)$ is compact if and only if it is closed, uniformly bounded, and equi-continuous.*

Proof The proof is left to the reader as an exercise.

1.2.5 Banach Fixed-Point Principle

Definition 1.26 A mapping $T : (M, d) \mapsto (M', \rho)$ is said to be Lipschitz if there exists $K \geq 0$ (Lipschitz constant) such that

$$\rho(T(x), T(y)) \leq Kd(x, y)$$

for all $x, y \in M$. In the case when $0 \leq K < 1$, then the map T is said to be a strict contraction.

Example 1.27 Let $\mathscr{O} \subset M$ be a subset. If $x \in M$, ones defines the distance between the point x and the set $\mathscr{O}$ as follows,

$$d(x, \mathscr{O}) := \inf_{y \in \mathscr{O}} d(x, y).$$

Proposition 1.28 *The map* $D : M \mapsto \mathbb{R}_+$, $x \mapsto D(x) := d(x, \mathscr{O})$ *is Lipschitz with* 1 *as its Lipschitz constant. That is,*

$$|d(x, \mathscr{O}) - d(y, \mathscr{O})| \leq d(x, y)$$

for all $x, y \in M$.

Proof Using the triangle inequality and a property of the infimum it follows that $D(x) \leq d(x, e) \leq d(x, y) + d(y, e)$ for all $x, y \in M$ and $e \in \mathscr{O}$, which yields $D(x) - d(x, y) \leq d(y, e)$ for all $x, y \in M$ and $e \in \mathscr{O}$. Using the fact that the infimum is the greatest element that is less than or equal to all elements, we deduce that $D(x) - d(x, y) \leq D(y)$ which in turn yields $D(x) - D(y) \leq d(x, y)$ for all $x, y \in M$. Since x and y are arbitrary elements of M, replacing x with y in the previous inequality, one gets, $D(y) - D(x) \leq d(y, x) = d(x, y)$ for all $x, y \in M$. Combing the last two inequalities, we deduce that $|D(x) - D(y)| \leq d(x, y)$ for all $x, y \in M$.

If $T : (M, d) \mapsto (M, d)$ is a mapping, then one defines its fixed-points by $F_T = \{x \in M : T(x) = x\}$.

Remark 1.29 Note that if T is a strict contraction, then F_T cannot contain more than one element. Indeed, suppose $x = Tx$ and $y = Ty$, then $d(x, y) = d(T(x), T(y)) \leq Kd(x, y) < d(x, y)$, this impossible. Consequently, F_T cannot contain more than one element.

Theorem 1.30 (Banach Fixed-Point Theorem [47]) *If* (M, d) *is a complete metric space and if* $T : M \mapsto M$ *is a strict contraction, then it has a unique fixed-point.*

Proof Let $x_0 \in M$. Define the sequence $x_n = T^n x_0$ which yields $x_{n+1} = Tx_n$ and $x_n = Tx_{n-1}$. Consequently, $d(x_{n+1}, x_n) \leq K^n d(x_0, x_1)$ where $0 \leq K < 1$ is the Lipschitz constant. Similarly, for all $n, m \in \mathbb{N}$ with $n > m$, we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_{n+1}, x_n) + d(x_n, x_{n-1}) + ... + d(x_m, x_{m-1}) \\ &\leq \Big(K^n + K^{n-1} + ... + K^{m-1}\Big) d(x_1, x_0) \\ &\leq \frac{K^n}{1-K} d(x_1, x_0). \end{aligned}$$

This yields the sequence $(x_n)_{n\in\mathbb{N}} \subset M$ which is a Cauchy sequence and since (M, d) is complete, it follows that there exists $x \in M$ such that $\lim_{n\to\infty} d(x_n, x) = 0$. Since T is continuous, it follows that $Tx = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = x$ which yields $x \in F_T$. Using Remark 1.29, we deduce that $F_T = \{x\}$.

1.2.6 Equilibrium Points for the Discrete Logistic Equation

This subsection is an application to the Banach fixed-point theorem and is based upon [72, Chapter 3]. Indeed, we make use of the Banach fixed-point theorem to determine equilibrium points of the (discrete) logistic equation. One of the most celebrated discrete dynamical systems arising in dynamic of population is that of the logistic equation, which is given by the following nonlinear difference equation,

$$x(t+1) = 4\sigma x(t)(1 - x(t)),$$

where $\sigma \in [0, 1]$ and $x(t) \in [0, 1]$ for all $t \in \mathbb{N}$.

In many concrete applications, $x(t)$ stands for the size of the population being studied at the generation t of a reproducing population. Recall that the linear part of the logistic equation, that is, $x(t+1) = 4\sigma x(t)$, describes, depending upon σ, the exponential growth (if $\sigma > \frac{1}{4}$) or decay (if $\sigma < \frac{1}{4}$) of a population subject to constant birth or death rate.

Clearly, the discrete logistic equation is of the form, $x(t+1) = Sx(t)$, where the function S (logistic map) is defined by

$$S : [0, 1] \mapsto [0, 1], \quad x \mapsto 4\sigma x(1 - x).$$

Obviously, equilibrium points of the logistic equation correspond to the fixed-points of the logistic map S. It can be easily shown that for $\sigma \in [0, \frac{1}{4}]$, the point $x_0 = 0$ is the only fixed-point (equilibrium point of the discrete logistic equation) of S. For $\sigma \in (\frac{1}{4}, 1]$, fixed-points of the logistic map S are given by the points of the form, $x(\sigma) := 1 - (4\sigma)^{-1}$. Consequently, the logistic map S has infinity many fixed-points (or equilibrium points for the logistic equation) given by

$$F_S = \Big\{0\Big\} \cup \Big\{x_\sigma := 1 - (4\sigma)^{-1} : \sigma \in (\frac{1}{4}, 1]\Big\}.$$

1.3 Banach Spaces

In the rest of this book, unless otherwise stated, $\mathbb{F}$ stands for the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$.

1.3.1 Basic Definitions

Let $\mathbb{X}$ be a vector space over the field $\mathbb{F}$. A norm on $\mathbb{X}$ is a mapping $\|\cdot\| : \mathbb{X} \mapsto \mathbb{R}_+$ satisfying the following properties,

i) $\|x\| = 0$ if and only if $x = 0$;
ii) $\|\lambda x\| = |\lambda|\ \|x\|$; and
iii) $\|x + y\| \leq \|x\| + \|y\|$

for all $x, y \in \mathbb{X}$ and $\lambda \in \mathbb{F}$.

The pair $(\mathbb{X}, \|\cdot\|)$ is then called a normed vector space. From a given normed vector space $(\mathbb{X}, \|\cdot\|)$, one can construct a metric space $(\mathbb{X}, d)$, where the metric d is defined by $d(x, y) := \|x - y\|$ for all $x, y \in \mathbb{X}$.

In what follows, we introduce two important notions which play a crucial role in many fields: separable and uniformly convex normed vector spaces. For that, we need to introduce a few notions.

Definition 1.31 Let $(\mathbb{X}, \|\cdot\|)$ be a normed vector space and let $\mathscr{D} \subset \mathbb{X}$ be a subset. Then, $\mathscr{D}$ is said to be *dense*, if $\mathbb{X} = \overline{\mathscr{D}}$. Equivalently, for each $x \in \mathbb{X}$, there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset \mathscr{D}$ such that $d(x_n, x) = \|x_n - x\| \to 0$ as $n \to \infty$.

Definition 1.32 A set $\mathscr{D}$ is called *countable* if it is finite or has the same cardinality as $\mathbb{N}$ (i.e., there exists a bijection between $\mathscr{D}$ and $\mathbb{N}$). A set $\mathscr{D}$ is called *uncountable* if it is infinite and not countable.

Definition 1.33 A normed vector space $(\mathbb{X}, \|\cdot\|)$ is said to be separable if it contains a countable dense subset $\mathscr{D}$.

Classical examples of separable normed vector spaces include, but are not limited to, $(\mathbb{R}, |\cdot|)$, $(\mathbb{C}, |\cdot|)$, and $(\ell^p(\mathbb{N}), \|\cdot\|_p)$ for $1 \leq p < \infty$, where $\ell^p(\mathbb{N})$ is vector space consisting of all sequences $x = (x_n)_{n\in\mathbb{N}}$ with $x_n \in \mathbb{C}$ for all $n \in \mathbb{N}$ and

$$\|x\|_p := \Big(\sum_{k=1}^{\infty} |x_k|^p\Big)^{1/p} < \infty.$$

It is also well known that $\ell^\infty(\mathbb{N})$, the vector space of all bounded sequences (the vector space consisting of all sequences $x = (x_n)_{n\in\mathbb{N}}$ with $x_n \in \mathbb{C}$ for all $n \in \mathbb{N}$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$ with $M \geq 0$ being a constant), is not separable, when it is equipped with its natural sup-norm $\|\cdot\|_\infty$ defined, for each $x = (x_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N})$, by

$$\|x\|_\infty = \sup_{n\in\mathbb{N}} |x_n|.$$

Definition 1.34 A normed vector space $(\mathbb{X}, \|\cdot\|)$ is said to be uniformly convex if for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$x, y \in \mathbb{X},\ \|x\| \leq 1,\ \|y\| \leq 1 \text{ and } \|x - y\| \geq \varepsilon \text{ yields } \left\|\frac{x + y}{2}\right\| < 1 - \delta.$$

While the normed vector spaces $(\mathbb{R}, |\cdot|)$, $(\mathbb{C}, |\cdot|)$, and $(\ell^p(\mathbb{N}), \|\cdot\|_p)$ for $1 < p < \infty$ are uniformly convex, $\ell^\infty(\mathbb{N})$ is not.

Definition 1.35 A normed vector space $(\mathbb{X}, \|\cdot\|)$ is said to be a *Banach space* if the metric space $(\mathbb{X}, d)$, where $d(x, y) := \|x - y\|$ for all $x, y \in \mathbb{X}$, is complete.

Classical examples of Banach spaces include finite-dimensional normed vector spaces. Obviously, there are plenty of normed vector spaces which are not Banach spaces (not complete). Indeed, for $a < b$, consider $C[a, b]$, the collection of all continuous functions $f : [a, b] \mapsto \mathbb{R}$. It is clear that $C[a, b]$ is a vector space over $\mathbb{R}$ which we equip with the norm given, for $p \in [1, \infty)$, by

$$\|f\|_p := \Big(\int_a^b |f(t)|^p dt \Big)^{\frac{1}{p}} \text{ for all } f \in C[a, b].$$

It is not hard to see that the normed vector space $(C[a, b], \|\cdot\|_p)$ is not complete.

Obviously, Banach spaces are more interesting for applications than incomplete normed vector spaces. Consequently, in what follows, our main focus will be on Banach spaces and their basic properties.

The proof of the next theorem presents no difficulty and hence is left to the reader as an exercise.

Theorem 1.36 *If* $(\mathbb{X}, \|\cdot\|)$ *is a Banach space and if* $\mathbb{Y} \subset \mathbb{X}$ *is a subspace, then* $(\mathbb{Y}, \|\cdot\|)$ *is a Banach space if and only if* $\mathbb{Y}$ *is closed.*

1.3.2 The Quotient Space

Definition 1.37 Let $\mathbb{L}$ be a subspace of the vector space $\mathbb{X}$. The cosets of $\mathbb{L}$ are defined by the sets, $[x] = x + \mathbb{L} = \{x + \ell : \ell \in \mathbb{L}\}$.

Define the quotient $\mathbb{X} \setminus \mathbb{L}$ as follows $\mathbb{X} \setminus \mathbb{L} = \{[x] : x \in \mathbb{X}\}$. The canonical projection of $\mathbb{X}$ onto $\mathbb{X} \setminus \mathbb{L}$ is defined by $\pi : \mathbb{X} \mapsto \mathbb{X} \setminus \mathbb{L}$, $x \mapsto [x]$. It is not hard to see that π is surjective and that $Ker(\pi) = \{x \in \mathbb{X} : \pi(x) = 0\} = \mathbb{L}$. Further, $\mathbb{X} \setminus \mathbb{L}$ is a vector space over $\mathbb{F}$ ($[x + y] = [x] + [y]$ and $[\lambda x] = \lambda[x]$ for all $x, y \in \mathbb{X}$ and $\lambda \in \mathbb{F}$) called the quotient space. Furthermore, the mapping $\|\cdot\| : \mathbb{X} \setminus \mathbb{L} \mapsto [0, \infty)$ defined by

$$\|[x]\| = \|x + \mathbb{L}\| = d(x, \mathbb{L}) = \inf_{y \in \mathbb{L}} \|x - y\|$$

for each $[x] \in \mathbb{X} \setminus \mathbb{L}$, is a norm on the quotient vector space $\mathbb{X} \setminus \mathbb{L}$. Indeed, for all $x, y \in \mathbb{X}$ and $\lambda \in \mathbb{F}$, using the fact that $\mathbb{L}$ is closed, we have $d(x, \mathbb{L}) = 0$ if and only if $x \in \mathbb{L}$. Thus $\|[x]\| = \|x + \mathbb{L}\| = 0$ if and only if $[x] = x + \mathbb{L} = 0 + \mathbb{L}$.

Suppose $\lambda \neq 0$. Then, we have

$$\|\lambda[x]\| = \|\lambda(x + \mathbb{L})\| = d(\lambda x, \mathbb{L}) = d(\lambda x, \lambda\mathbb{L}) = |\lambda| d(x, \mathbb{L}) = |\lambda| \|[x]\|.$$

Now, if $\lambda = 0$, it easily follows that

$$\|0(x + \mathbb{L})\| = \|0 + \mathbb{L}\| = 0 = |0|\|x + \mathbb{L}\|.$$

Now let $x_1, y_1 \in \mathbb{L}$, then

$$\begin{aligned}\|(x + \mathbb{L}) + (y + \mathbb{L})\| &= \|(x + y) + \mathbb{L}\| \\ &\leq \|x + y + x_1 + y_1\| \\ &\leq \|x + x_1\| + \|y + y_1\|,\end{aligned}$$

which yields

$$\|(x + \mathbb{L}) + (y + \mathbb{L})\| \leq \|x + \mathbb{L}\| + \|y + \mathbb{L}\|$$

That is,

$$\|[x] + [y]\| \leq \|[x]\| + \|[y]\|.$$

Recall that if $\mathbb{L}$ a closed subspace of the normed vector space $(\mathbb{X}, \|\cdot\|)$, then the canonical projection π is linear and continuous as $\|\pi(x)\| \leq \|x\|$ for all $x \in \mathbb{X}$.

Theorem 1.38 *Let* $(\mathbb{X}, \|\cdot\|)$ *be Banach space and let* $\mathbb{L} \subset \mathbb{X}$ *is a closed subspace. Then the quotient normed vector space* $(\mathbb{X} \setminus \mathbb{L}, \|\cdot\|)$ *is a Banach space.*

1.3.3 $L^p(\Omega, \mu)$ Spaces

Definition 1.39 Let Ω be a set and let $\mathscr{F}$ be a σ-algebra of measurable sets, that is, $\mathscr{F} \subset \mathscr{P}(\Omega)$ is a subset and satisfies the following conditions:

i) $\emptyset \in \mathscr{F}$;
ii) if $\Gamma \subset \mathscr{F}$, then its complement Γ^C belongs to $\mathscr{F}$; and
iii) $\bigcup_{j=1}^{\infty} \Gamma_j \in \mathscr{F}$ whenever $\Gamma_j \in \mathscr{F}$ for all j.

Elements of $\mathscr{F}$ are then called measurable sets.

Definition 1.40 The mapping $\mu : \Omega \mapsto [0, \infty]$ is called a measure, if it satisfies the following conditions:

i) $\mu(\emptyset) = 0$; and
ii) $\mu(\bigcup_{j=1}^{\infty} \Gamma_j) = \sum_{j=1}^{\infty} \mu(\Gamma_j)$ for any disjoint countable family $(\Gamma_j)_{j\in\mathbb{N}}$ of elements of $\mathscr{F}$.

The σ-algebra $\mathscr{F}$ is said to be σ-finite if there exists a disjoint countable family $(\Gamma_j)_{j\in\mathbb{N}}$ of elements of $\mathscr{F}$ such that $\Omega = \bigcup_{j=1}^{\infty} \Gamma_j$ with $\mu(\Omega_j) < \infty$ for all j.

The space $(\Omega, \mathscr{F}, \mu)$ is called a measure space if Ω is a set, $\mathscr{F}$ is a σ-algebra on Ω, and μ is a measure on Ω.

Let $p \in [1, \infty)$ and let $(\Omega, \mathscr{F}, \mu)$ be a measure space. The space $L^p(\Omega, \mu)$ (also denoted $L^p(\Omega)$) is defined as the collection of all measurable functions $f : \Omega \mapsto \mathbb{C}$ such that

$$\|f\|_{L^p} = \|f\|_p := \left[\int_{\Omega} |f(x)|^p d\mu\right]^{\frac{1}{p}} < \infty.$$

Recall that in $L^p(\Omega, \mu)$, two functions f and g are equal, if they are equal almost everywhere on Ω. Obviously, $(L^p(\Omega, \mu), \|\cdot\|_p)$ is a normed vector space.

Similarly, one defines $L^\infty(\Omega, \mu)$ (also denoted $L^\infty(\Omega)$) as the set of all measurable functions $f : \Omega \mapsto \mathbb{C}$ such that there exists a constant $M \geq 0$ such that $|f(x)| \leq M$ a.e. $x \in \Omega$. Now define, for all $f \in L^\infty(\Omega, \mu)$,

$$\|f\|_\infty := \inf\left\{M : |f(x)| \leq M \text{ a.e. } x \in \Omega\right\}.$$

Obviously, $(L^\infty(\Omega, \mu), \|\cdot\|_\infty)$ is a normed vector space.

In view of the above, $(L^p(\Omega), \|\cdot\|_p)$ is a normed vector space for all $p \in [1, \infty]$.

Example 1.41 Take $\Omega = \mathbb{R}$ and $d\mu = dx$ (Lebesgue measure). Consider the function defined by $f(x) = e^{-|x|}$ for all $x \in \mathbb{R}$. One can easily see that $f \in L^p(\mathbb{R}, dx)$ for all $p \in [1, \infty]$.

Proposition 1.42 *Let (Ω, μ) be a measure space. If $1 \leq p \leq q < \infty$ and if $0 < \mu(\Omega) < \infty$, then*

$$\|f\|_p \leq [\mu(\Omega)]^r \|f\|_q$$

for any measurable function f, where $r = p^{-1} - q^{-1}$.

Proof The proof is left to the reader as an exercise.

Corollary 1.43 *Let (Ω, μ) be a measure space. If $1 \leq p \leq q < \infty$ and if $0 < \mu(\Omega) < \infty$, then the injection*

$$L^q(\Omega, \mu) \hookrightarrow L^p(\Omega, \mu)$$

is continuous.

Proof The proof is left to the reader as an exercise.

Theorem 1.44 (Riesz-Fisher) *The space $(L^p(\Omega, \mu), \|\cdot\|_p)$ is a Banach space for any $1 \leq p \leq \infty$.*

Proof The proof is left to the reader as an exercise.

Theorem 1.45 *Let $p \in [1, \infty]$. If $u \in L^1(\mathbb{R})$ and let $v \in L^p(\mathbb{R}^n)$, then for almost every $x \in \mathbb{R}^n$, the function $y \mapsto u(x-y)v(y)$ is integrable on $\mathbb{R}^n$. And the convolution of u and v defined by*

$$(u * v)(x) := \int_{\mathbb{R}^n} u(x-y)v(y)dy$$

is well defined. Further, $u * v \in L^p(\mathbb{R}^n)$ and

$$\|u * v\|_{L^p} \leq \|u\|_{L^1} \|v\|_{L^p}.$$

Proof See the book by Brézis [31].

More generally,

Theorem 1.46 (Young) *Let $p, q \in [1, \infty]$ such that $r^{-1} = p^{-1} + q^{-1} - 1 \geq 0$. If $u \in L^p(\mathbb{R}^n)$ and let $v \in L^q(\mathbb{R}^n)$, then $u * v \in L^r(\mathbb{R}^n)$ and*

$$\|u * v\|_{L^r} \leq \|u\|_{L^p} \|v\|_{L^q}.$$

Proof See the book by Brézis [31].

Let $\Omega \subset \mathbb{R}^n$ be a subset and let $d\mu = dx$ (Lebesgue measure). Let $L^p_{loc}(\Omega)$ for $1 \leq p < \infty$ stand for the collection of all measurable functions $f : \Omega \mapsto \mathbb{C}$ such that

$$\left(\int_{\Omega'} |f(x)|^p dx \right)^{\frac{1}{p}} < \infty \tag{1.1}$$

for any $\Omega' \subset \Omega$ bounded closed subset.

Clearly, $L^p_{loc}(\Omega)$ is a vector space. Further, $L^p(\Omega)$ is a subspace of $L^p_{loc}(\Omega)$. The natural topology of $L^p_{loc}(\Omega)$ is given as follows: a sequence $(f_n)_{n \in \mathbb{N}} \in L^p_{loc}(\Omega)$ is said to converge to some $f \in L^p_{loc}(\Omega)$ if $\|f_n - f\|_p \to 0$ as $n \to \infty$ in $L^p(\Omega')$ for any $\Omega' \subset \Omega$ bounded closed subset. Although $L^p_{loc}(\Omega)$ equipped with such a type of convergence is a topological vector space, it is not a Banach space.

1.3.4 Sobolev Spaces

If $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ with $\alpha_i \in \mathbb{Z}_+$ for $i = 1, ..., n$, one defines the length $|\alpha|$ of α as follows: $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$. In this event, the differential operator D^α is defined by

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} ... \partial x_n^{\alpha_n}}.$$

Definition 1.47 Let $1 \leq p \leq \infty$ and let $k \in \mathbb{N}$. Suppose $\Omega \subset \mathbb{R}^n$ is an open subset. The Sobolev spaces $W^{k,p}(\Omega)$ is the collection of all functions $u : \Omega \mapsto \mathbb{F}$ belonging to the set

$$W^{k,p}(\Omega) := \Big\{ u \in L^p(\Omega) : \ D^\alpha u \in L^p(\Omega) \ \text{ for } \ |\alpha| \leq k \Big\}. \tag{1.2}$$

If $p = 2$, the Sobolev space $W^{k,2}(\Omega)$ is denoted by $H^k(\Omega)$. The space $W^{k,p}(\Omega)$ is equipped with the norm defined by

$$\|u\|_{k,p} = \Big(\sum_{|\alpha| \leq k} \|D^\alpha u\|_p^p \Big)^{\frac{1}{p}}, \quad \text{if} \quad 1 \leq p < \infty, \tag{1.3}$$

and

$$\|u\|_{k,\infty} = \max_{|\alpha| \leq k} |D^\alpha u|_\infty \quad \text{if} \quad p = \infty. \tag{1.4}$$

Definition 1.48 If $f : \Omega \mapsto \mathbb{F}$ is a function, then its support denoted $supp(f)$ is defined by $Supp(f) := \overline{\{x \in \Omega : f(x) \neq 0\}}$.

Example 1.49 If $E \subset \mathbb{R}$ is a subset, then the support of the characteristic function χ_E of the set E defined by $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ if $x \notin E$, is $\overline{E}$ (closure of E).

Definition 1.50 The notation $C_0^\infty(\Omega)$ stands for the collection of all functions $u : \Omega \mapsto \mathbb{R}$ (or $\mathbb{C}$) of class C^∞ with compact support in Ω.

Definition 1.51 The Sobolev space $W_0^{k,p}(\Omega)$ is defined to be the closure of $C_0^\infty(\Omega)$ in the space $W^{k,p}(\Omega)$, that is,

$$W_0^{k,p}(\Omega) = \overline{C_0^\infty(\Omega)}^{W^{k,p}(\Omega)}.$$

If $p = 2$, then the Sobolev space $W_0^{k,2}(\Omega)$ is denoted by $H_0^k(\Omega)$. Further, if $\Omega = \mathbb{R}^n$, then $W_0^{k,p}(\mathbb{R}^n) = W^{k,p}(\mathbb{R}^n)$.

Theorem 1.52 ([47]) *The Sobolev space $W^{k,p}(\Omega)$ is a Banach space.*

Definition 1.53 Let $1 \leq p < \infty$ and let $s = k + \sigma$ where $k \in \mathbb{N}$ and $\sigma \in (0, 1)$. The Sobolev space $W^{s,p}(\Omega)$ is defined by

$$W^{s,p}(\Omega) := \Big\{ u \in W^{k,p}(\Omega) : \frac{|D^\alpha u(x) - D^\alpha u(y)|}{\|x - y\|^{\sigma + \frac{n}{p}}} \in L^p(\Omega \times \Omega), \ \forall \alpha, \ \ |\alpha| = k \Big\},$$

whose norm is given by

$$\|u\|_{s,p} = \left(\|u\|_{k,p}^p + \sum_{|\alpha|=k} \int_{\Omega\times\Omega} \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{\|x-y\|^{p\sigma+n}} dxdy \right)^{1/p}.$$

If $p = 2$, then $W^{s,2}(\Omega)$ is denoted $H^s(\Omega)$. If $s \geq 0$, then we define $W_0^{s,p}(\Omega)$ to be the closure of the space $C_0^\infty(\Omega)$ in the Sobolev space $W^{s,p}(\Omega)$. In particular, $W_0^{s,2}(\Omega)$ is denoted by $H_0^s(\Omega)$.

1.3.5 Embedding Theorems for Sobolev Spaces

In this subsection, we collect some important and useful embedding theorems on Sobolev spaces including the Sobolev–Gagliardo–Nirenberg's Theorem and the Poincaré's Theorem. Although most of the proofs of these theorems are omitted, we will be referring the reader to the appropriate references.

Theorem 1.54 (Sobolev–Gagliardo–Nirenberg) *Let $p \in [1, n)$. Then the following embedding holds,*

$$W^{1,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)$$

where $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$. Moreover, there exists a constant $C(p, n)$ such that

$$\|u\|_{L^q} \leq C\|\nabla u\|_{L^p}$$

for all $u \in W^{1,p}(\mathbb{R}^n)$, where the gradient ∇u of u is defined by the vector

$$\nabla u = grad\, u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial u}{\partial x_n}\right).$$

Proof See Brézis [32].

Theorem 1.55 *Let $k \in \mathbb{N}$ and let $p \in [1, \infty)$. Then the following continuous embeddings hold,*

$$W^{k,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n) \text{ where } \frac{1}{q} = \frac{1}{p} - \frac{k}{n}, \text{ if } \frac{1}{p} - \frac{k}{n} > 0,$$

$$W^{k,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n) \text{ for all } q \in [p, \infty), \text{ if } \frac{1}{p} - \frac{k}{n} = 0,$$

$$W^{k,p}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n) \text{ if } \frac{1}{p} - \frac{k}{n} < 0.$$

Proof See Brézis [32].

Theorem 1.56 *Let $\Omega \subset \mathbb{R}^n$ be an open subset of class C^1 with bounded boundary $\partial\Omega$. If $p \in [1, \infty]$, we have the following continuous embeddings,*

$$W^{1,p}(\Omega) \subset L^q(\Omega) \text{ where } \frac{1}{q} = \frac{1}{p} - \frac{k}{n}, \text{ if } p < n,$$

$$W^{1,p}(\Omega) \subset L^q(\Omega) \text{ for all } q \in [p, \infty), \text{ if } p = n,$$

$$W^{1,p}(\Omega) \subset L^\infty(\Omega) \text{ if } p > n.$$

Proof See Brézis [32].

We also have

Theorem 1.57 (Rellich–Kondrachov) *Let $\Omega \subset \mathbb{R}^n$ be a bounded subset and of class C^1. If $p \in [1, \infty]$, we have the following compact embeddings,*

$$W^{1,p}(\Omega) \subset L^q(\Omega) \text{ for all } q \in [1, r) \ \frac{1}{r} = \frac{1}{p} - \frac{1}{n}, \text{ if } p < n,$$

$$W^{1,p}(\Omega) \subset L^q(\Omega) \text{ for all } q \in [p, \infty), \text{ if } p = n,$$

$$W^{1,p}(\Omega) \subset C(\overline{\Omega}) \text{ if } p > n.$$

Proof See Brézis [32].

We also have the so-called Poincaré's inequality.

Theorem 1.58 (Poincaré's Inequality) *Let $p \in [1, \infty)$ and let $\Omega \subset \mathbb{R}^n$ be an open bounded subset. Then there exists a constant $C = C(\Omega, p) > 0$ such that*

$$\|u\|_{L^p} \leq C\|\nabla u\|_{L^p}$$

for all $u \in W_0^{1,p}(\Omega)$.

Proof See Brézis [32].

Theorem 1.59 *Let $\Omega \subset \mathbb{R}^n$ be an open bounded subset with a smooth boundary $\partial\Omega$. Suppose $0 \leq k \leq m - 1$. Then, we have the following embeddings,*

$$W^{m,p}(\Omega) \subset W^{k,q}(\Omega) \text{ if } \frac{1}{q} \geq \frac{1}{p} - \frac{m-k}{n}$$

$$W^{m,p}(\Omega) \subset W^{k,q}(\Omega) \text{ if } q < \infty, \text{ and } \frac{1}{p} = \frac{m-k}{n}.$$

While the second embedding is compact, the first one is compact only if

$$\frac{1}{q} > \frac{1}{p} - \frac{m-k}{n}.$$

Proof See Adams [10].

1.3.6 Bounded Continuous Functions

Let $J \subset \mathbb{R}$ be an interval (possibly unbounded) and let $(\mathbb{X}, \|\cdot\|)$ be a Banach space.

Definition 1.60 Let $BC(J;\mathbb{X})$ denote the space of all bounded continuous functions $f : J \mapsto \mathbb{X}$. The space $BC(J;\mathbb{X})$ will be equipped with the sup-norm defined by

$$\|f\|_\infty := \sup_{t\in J} \|f(t)\|$$

for all $f \in BC(J;\mathbb{X})$.

Theorem 1.61 *The normed vector space* $(BC(J;\mathbb{X}), \|\cdot\|_\infty)$ *is a Banach space.*

Proof The proof is left to the reader as an exercise.

Proposition 1.62 *If* $(f_n)_{n\in\mathbb{N}} \subset BC(J;\mathbb{X})$ *such that* f_n *converges to some* f *with respect to the sup-norm, then* $f \in BC(J;\mathbb{X})$.

Proof The proof is left to the reader as an exercise.

1.3.7 Hölder Spaces $C^{k,\alpha}(\overline{\Omega})$

Fix once and for all $\alpha \in (0,1)$. Let $J \subset \mathbb{R}$ be an interval (possibly unbounded) and let $(\mathbb{X}, \|\cdot\|)$ be a Banach space.

The space $C^m(J;\mathbb{X})$ ($m \in \mathbb{N}$) stands for the collection of all m-times continuously differentiable functions from J into $\mathbb{X}$ and let $BC^m(J;\mathbb{X})$ stand for the space,

$$BC^m(J;\mathbb{X}) = \{f \in C^m(J;\mathbb{X}) : f^{(k)} \in BC(J;\mathbb{X}), \;\; k = 0, 1, ..., m\}$$

equipped with the norm

$$\|f\|_{BC^m(J;\mathbb{X})} := \sum_{k=0}^{m} \|f^{(k)}\|_\infty \;\text{ for all }\; f \in BC^m(J;\mathbb{X}).$$

Definition 1.63 Hölder spaces of continuous functions $C^{0,\alpha}(J;\mathbb{X})$ and $C^{k,\alpha}(J;\mathbb{X})$ for $\alpha \in (0,1)$ and $k \in \mathbb{N}$ are respectively defined by

$$C^{0,\alpha}(J;\mathbb{X}) = \left\{ f \in BC(J;\mathbb{X}) : [f]_{C^{0,\alpha}(J;\mathbb{X})} = \sup_{t,s\in J, s<t} \frac{\|f(t)-f(s)\|}{(t-s)^{\alpha}} < \infty \right\}$$

equipped with the norm

$$\|f\|_{C^{0,\alpha}(J;\mathbb{X})} = \|f\|_{\infty} + [f]_{C^{0,\alpha}(J;\mathbb{X})}, \quad \text{and}$$

$$C^{k,\alpha}(J;\mathbb{X}) = \left\{ f \in BC^{k}(J;\mathbb{X}) : f^{(k)} \in C^{0,\alpha}(J;\mathbb{X}) \right\}$$

equipped with the norm

$$\|f\|_{C^{k,\alpha}(J;\mathbb{X})} = \|f\|_{BC^{k}(J;\mathbb{X})} + [f^{(k)}]_{C^{0,\alpha}(J;\mathbb{X})}.$$

Proposition 1.64 *The Hölder spaces $C^{0,\alpha}(J;\mathbb{X})$ and $C^{k,\alpha}(J;\mathbb{X})$ for $\alpha \in (0,1)$ and $k \in \mathbb{N}$ equipped with their corresponding norms are respectively Banach spaces.*

Proof The proof is left to the reader as an exercise.

Definition 1.65 The Lipschitz space $Lip(J;\mathbb{X})$ is defined by

$$\text{Lip}(J;\mathbb{X}) = \left\{ f \in BC(J;\mathbb{X}) : [f]_{Lip(J;\mathbb{X})} = \sup_{t,s\in J, s<t} \frac{\|f(t)-f(s)\|}{(t-s)} < \infty \right\},$$

and is equipped with the norm defined by

$$\|f\|_{\tilde{Lip}(J;\mathbb{X})} = \|f\|_{\infty} + [f]_{Lip(J;\mathbb{X})}.$$

Proposition 1.66 *The Lipschitz space $(Lip(J;\mathbb{X}), \|\cdot\|_{\tilde{Lip}(J;\mathbb{X})})$ is a Banach space.*

Proof The proof is left to the reader as an exercise.

Definition 1.67 Let $\Omega \subset \mathbb{R}^N$ be an open subset. The Hölder space $C_b^{0,\alpha}(\overline{\Omega})$ consists of all bounded continuous functions $f : \overline{\Omega} \mapsto \mathbb{C}$ such that

$$[f]_{C_b^{\alpha}(\overline{\Omega})} := \sup_{x\neq y\in\overline{\Omega}} \frac{|f(x)-f(y)|}{\|x-y\|^{\alpha}} < \infty.$$

Theorem 1.68 *The space* $(C_b^{0,\alpha}(\overline{\Omega}), \|\cdot\|_{C_b^{0,\alpha}(\overline{\Omega})})$ *is a Banach space, where the norm* $\|\cdot\|_{C_b^{0,\alpha}(\overline{\Omega})}$ *is defined by*

$$\|f\|_{C_b^{0,\alpha}(\overline{\Omega})} = \|f\|_\infty + [f]_{C_b^{0,\alpha}(\overline{\Omega})}$$

for all $f \in C_b^{0,\alpha}(\overline{\Omega})$.

Proof The proof is left to the reader as an exercise.

Definition 1.69 Let $k \in \mathbb{N}$. The Hölder space $C_b^{k,\alpha}(\overline{\Omega})$ consists of all functions $f : \overline{\Omega} \mapsto \mathbb{C}$ which are k-times continuously differentiable functions with bounded partial derivatives such that $D^\beta f \in C_b^{0,\alpha}(\overline{\Omega})$ for any multi-index β with $|\beta| = k$.

Theorem 1.70 *The space Hölder space* $(C_b^{k,\alpha}(\overline{\Omega}), \|\cdot\|_{C_b^{k,\alpha}(\overline{\Omega})})$ *is a Banach space, where the norm* $\|\cdot\|_{C_b^{k,\alpha}(\overline{\Omega})}$ *is defined by*

$$\|u\|_{C_b^{k,\alpha}(\overline{\Omega})} = \sum_{|\beta|\le k} \|D^\beta u\|_\infty + \sum_{|\beta|=k} [D^\beta]_{C_b^{0,\alpha}(\overline{\Omega})}.$$

Proof The proof is left to the reader as an exercise.

Remark 1.71 Note that the subscript "b" in $C_b^{k,\alpha}(\overline{\Omega})$ should be dropped in the case when the domain Ω is bounded. In other words, if Ω is bounded, then $C_b^{k,\alpha}(\overline{\Omega})$ will be denoted $C^{k,\alpha}(\overline{\Omega})$.

1.3.8 Embedding Theorems for Hölder Spaces

Theorem 1.72 *Let* $\Omega \subset \mathbb{R}^n$ *be an open bounded subset with a smooth boundary* $\partial\Omega$. *Suppose* $0 \le k \le m - 1$. *Then, we have the following compact embedding,*

$$W^{m,p}(\Omega) \subset C^{k,\alpha}(\overline{\Omega}) \ \ if \ \ \frac{n}{p} < m - (k + \alpha) \ \ with \ \ 0 < \alpha < 1.$$

Proof See Adams [10].

Theorem 1.73 *Let* $\Omega \subset \mathbb{R}^n$ *be an open subset with a smooth boundary* $\partial\Omega$. *Suppose* $m \in \mathbb{Z}_+$ *and* α, β *are given such that* $0 < \alpha < \beta \le 1$. *Then, we have the following embeddings,*

$$C_b^{m,\alpha}(\overline{\Omega}) \subset C^m(\overline{\Omega}),$$

$$C_b^{m,\beta}(\overline{\Omega}) \subset C_b^{m,\alpha}(\overline{\Omega}).$$

If Ω is bounded, then embeddings,

$$C^{m,\alpha}(\overline{\Omega}) \subset C^m(\overline{\Omega}),$$

$$C^{m,\beta}(\overline{\Omega}) \subset C^{m,\alpha}(\overline{\Omega}).$$

are compact.

Proof See Adams and Fournier [11].

1.3.9 The Dual Space

Let $(\mathbb{X}, \|\cdot\|)$ be a normed vector space over the field $\mathbb{F}$. A functional $\xi : \mathbb{X} \mapsto \mathbb{F}$, $\xi \mapsto \xi(x)$ ($\xi(x)$ is also denoted $\langle \xi, x \rangle$), is said to be linear if it satisfies the following identity, $\xi(\lambda x + \mu y) = \lambda\xi(x) + \mu\xi(y)$ for all $\lambda, \mu \in \mathbb{F}$ and for all $x, y \in \mathbb{X}$.

A functional $\xi : \mathbb{X} \mapsto \mathbb{F}$ is called continuous, if there exists a constant $K \geq 0$ such that

$$|\xi(x)| \leq K\|x\| \tag{1.5}$$

for all $x \in \mathbb{X}$.

The collection of all linear continuous functionals $\xi : \mathbb{X} \mapsto \mathbb{F}$, which we denote by $\mathbb{X}^*$, is called the (topological) dual of $\mathbb{X}$. Clearly, $\mathbb{X}^*$ is a vector space over $\mathbb{F}$ as we can add elements of $\mathbb{X}^*$ up and multiply them by scalars and still get continuous linear functionals. One can endow the dual $\mathbb{X}^*$ of $\mathbb{X}$ with a norm which we denote by $\|\cdot\|_*$ and which is defined as follows: the norm $\|\xi\|_*$ of $\xi \in \mathbb{X}^*$, is the smallest constant K satisfying Eq. (1.5). Consequently,

$$\|\xi\|_* = \sup_{0 \neq x \in \mathbb{X}} \frac{|\langle \xi, x \rangle|}{\|x\|}.$$

Using the definition of the norm $\|\cdot\|_*$, it easily follows that $|\langle \xi, x \rangle| \leq \|\xi\|_* \|x\|$ for all $\xi \in \mathbb{X}^*$ and $x \in \mathbb{X}$. Furthermore,

$$\|\xi\|_* = \sup_{\|x\| \leq 1} |\langle \xi, x \rangle| = \sup_{\|x\| = 1} |\langle \xi, x \rangle|.$$

Theorem 1.74 *The normed vector space* $(\mathbb{X}^*, \|\cdot\|_*)$ *is a Banach space.*

Proof The proof is left to the reader as an exercise.

1.3.10 The Schauder Fixed-Point Theorem

Definition 1.75 A nonempty set S is said to be convex if for all $x, y \in S$ and $\lambda \in [0, 1]$, then $\lambda x + (1 - \lambda)y \in S$.

Theorem 1.76 (The Brouwer Fixed-Point Theorem [76]) *Let $S \subset \mathbb{F}^n$ be a nonempty bounded closed convex subset. If the mapping $T : S \mapsto S$ is continuous, then T has at least one fixed-point, that is, $F_T \neq \emptyset$.*

Theorem 1.77 (The Schauder Fixed-Point Theorem [76]) *Let $\mathbb{X}$ be a Banach space and let $S \subset \mathbb{X}$ be a nonempty compact convex subset. If the mapping T : $S \mapsto S$ is continuous, then T has at least one fixed-point, that is, $F_T \neq \emptyset$.*

The following concept which measures the "non-compactness" is due to Kuratowski [78].

Definition 1.78 If $D \subset \mathbb{X}$ is a bounded subset, one defines the measure $\alpha(D)$ of non-compactness of D as follows:

$$\alpha(D) := \inf\Big\{d > 0 : \ D \text{ has a finite covering of diameter less than } d\Big\}.$$

Definition 1.79 Let $D \subset \mathbb{X}$ be a subset. Suppose that the map $P : D \mapsto \mathbb{X}$ is continuous. The map P is called condensing if for any bounded subset D' of D, $\alpha(D') > 0$ yields

$$\alpha(P(D')) < \alpha(D').$$

We have the following generalization of the Schauder's fixed-point due to Sadovsky.

Theorem 1.80 (The Sadovsky Fixed-Point Theorem [76]) *Let D be a nonempty convex, bounded, and closed subset of a Banach space $\mathbb{X}$ and $F : D \to D$ be a condensing map. Then F has a fixed point in D.*

Proof The proof makes use of the Schauder's fixed point theorem (Theorem 1.77). Indeed, fix $x \in D$ and let Γ be the set of all closed convex subsets C of D such that $x \in C$ and F maps C into itself.

Set

$$\Omega = \bigcap_{C \in \Gamma} C \quad \text{and} \quad K = \overline{Conv}\{F(\Omega) \cup \{x\}\},$$

where $Conv$ denotes the convex envelop and $\overline{Conv}$ its closure.

Using the fact that $x \in \Omega$ and that F maps Ω into itself yields one must have $K \subseteq \Omega$, which, in turn, yields $F(K) \subseteq F(\Omega) \subseteq \Omega$. Now from $x \in K$ it follows that $K \in \Gamma$. Consequently, $\Omega \subseteq K$ which yields $\Omega = K$, which, in turn, yields $F(K) = F(\Omega) \subseteq K$ and, therefore,

$$\alpha(K) = \alpha(\overline{Conv}\{F(\Omega) \cup \{x\}\}) = \alpha(\{F(\Omega) \cup \{x\}\}) = \alpha(F(\Omega)) = \alpha(F(K)).$$

Using the fact that F is condensing it follows that $\alpha(K) = 0$, which yields Ω is compact. Consequently, F is a continuous function which maps a convex compact set K into itself and so using Schauder theorem it follows that F has a fixed point.

1.3.11 Leray-Schauder Alternative

We will need the following fixed-point theorem in the sequel.

Theorem 1.81 (Leray-Schauder Alternative [61, Theorem 6.5.4]) *Let D be a closed convex subset of a Banach space $\mathbb{X}$ with $0 \in D$. Let $G : D \to D$ be a completely continuous map. Then, either G has a fixed point in D or the set*

$$\Big\{x \in D : x = \lambda G(x),\ 0 < \lambda < 1\Big\}$$

is unbounded.

1.4 Hilbert Spaces

Hilbert spaces play an important role in many areas including mathematical analysis, physics, quantum mechanics, Fourier analysis, partial differential equations, etc. These spaces, which generalize in a natural fashion the Euclidean space, are named after the German mathematician David Hilbert who introduced them in the mathematical literature. Obviously, a Hilbert space is, by design, a Banach space. Some of their basic properties will be discussed in this section and throughout the entire book. For the uncovered material on Hilbert spaces, we refer the interested reader to some of the classical books in functional analysis, i.e., Brézis [31, 32], Conway [38], Eidelman et al. [53], Naylor and Sell [93], etc.

1.4.1 Basic Definitions

In this section, $\mathscr{H}$ stands for a vector space over the field $\mathbb{F}$ where $\mathbb{F} = (\mathbb{R}, |\cdot|)$ or $(\mathbb{C}, |\cdot|)$.

Definition 1.82 A mapping $a : \mathscr{H} \times \mathscr{H} \mapsto \mathbb{F}$ is said to be a sesquilinear form, if

i) the mapping $x \mapsto a(x, y)$ is linear for all $y \in \mathscr{H}$, that is,

$$a(\lambda x + \mu x', y) = \lambda a(x, y) + \mu a(x', y)$$

for all $x, x', y \in \mathscr{H}$; and

ii) the mapping $y \mapsto a(x, y)$ is anti-linear for all $x \in \mathscr{H}$, that is,

$$a(x, \lambda y + \mu y') = \overline{\lambda} a(x, y) + \overline{\mu} a(x, y')$$

for all $x, y, y' \in \mathscr{H}$.

Definition 1.83 An inner product or scalar product $\langle \cdot, \cdot \rangle$ on $\mathscr{H}$ is a sesquilinear form which goes from $\mathscr{H} \times \mathscr{H}$ into $\mathbb{F}$ and satisfies:

i) $\langle x, x \rangle \geq 0$ for all $x \in \mathscr{H}$;
ii) $\langle x, x \rangle = 0$ if and only if $x = 0$; and
iii) $\langle y, x \rangle = \overline{\langle x, y \rangle}$ for all $x, y \in \mathscr{H}$.

Recall that if $\langle \cdot, \cdot \rangle$ is an inner product on $\mathscr{H}$, then the mapping $\| \cdot \| : \mathscr{H} \mapsto \mathbb{R}_+$ defined by $\|x\| := [\langle x, x \rangle]^{\frac{1}{2}}$ for all $x \in \mathscr{H}$ is a norm on $\mathscr{H}$; called the norm deduced from the inner product $\langle \cdot, \cdot \rangle$. Recall also that the norm $\| \cdot \|$ satisfies various properties including the so-called Cauchy-Schwarz inequality and the parallelogram identity given respectively by,

$$\left| \langle x, y \rangle \right| \leq \|x\| \,.\, \|y\|, \tag{1.6}$$

and

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 \tag{1.7}$$

for all $x, y \in \mathscr{H}$.

While the proof of the Cauchy-Schwarz inequality requires more efforts, that of the parallelogram identity is easy and is based upon the following identities:

$$\|x + y\|^2 = \|x\|^2 + 2\Re e\langle x, y \rangle + \|y\|^2$$

and

$$\|x - y\|^2 = \|x\|^2 - 2\Re e\langle x, y \rangle + \|y\|^2$$

for all $x, y \in \mathscr{H}$.

Let $\mathscr{H}$ be a vector space over $\mathbb{F}$ equipped with the inner product given by, $\langle \cdot, \cdot \rangle$. Two vectors $x, y \in \mathscr{H}$ are said to be orthogonal if $\langle x, y \rangle = 0$. If $\langle x, y \rangle = 0$, then the Pythagorean theorem holds in $\mathscr{H}$, that is,

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2.$$

More generally, if $(x_n)_{n\in\mathbb{N}} \subset \mathscr{H}$ is a sequence such that $\langle x_n, x_m\rangle = 0$ for all $n, m \in \mathbb{N}$ with $n \neq m$, then the series $\sum_{n=1}^{\infty} x_n$ converges if and only if $\sum_{n=1}^{\infty} \|x_n\|^2$ does. In that event, we have the following generalized Pythagorean theorem,

$$\Big\|\sum_{n=1}^{\infty} x_n\Big\|^2 = \sum_{n=1}^{\infty}\Big\|x_n\Big\|^2.$$

If $M \subset \mathscr{H}$ is a subspace, then its orthogonal $M^\perp$ is defined by

$$M^\perp := \Big\{x \in \mathscr{H} : \langle x, y\rangle = 0 \text{ for all } y \in M\Big\}.$$

Proposition 1.84 *If $M \subset \mathscr{H}$ is a subspace, then its orthogonal $M^\perp$ is a closed subspace of $\mathscr{H}$.*

Proof Let $(x_n)_{n\in\mathbb{N}} \subset M^\perp$ be a sequence such that $\|x_n - x\| \to 0$ as $n \to \infty$ for some $x \in \mathscr{H}$. Now, using the fact that, $\langle x, y\rangle = \langle x - x_n + x_n, y\rangle = \langle x - x_n, y\rangle + \langle x_n, y\rangle = \langle x - x_n, y\rangle$ for all $y \in M$ and $n \in \mathbb{N}$, it follows, using the Cauchy-Schwarz inequality, that $|\langle x, y\rangle| \le \|x - x_n\| . \|y\| \to 0$ as $n \to \infty$, which yields $\langle x, y\rangle = 0$, and hence $x \in M^\perp$.

Note that in addition to Proposition 1.84, we also have $\mathscr{H}^\perp = \{0\}$ and $\{0\}^\perp = \mathscr{H}$. Further, if $M \subset N$ where M, N are subspaces of $\mathscr{H}$, then $N^\perp \subset M^\perp$ and $M \subset (M^\perp)^\perp$ with $(M^\perp)^\perp = M$ if M is closed.

Definition 1.85 The space $\mathscr{H}$ is said to be a Hilbert space if $(\mathscr{H}, \|\cdot\|)$ is complete where $\|\cdot\|$ is the norm deduced from the inner product $\langle\cdot,\cdot\rangle$.

One of the most important properties of Hilbert spaces is that of the projection theorem. It plays an important role in many areas.

Theorem 1.86 *Let $\mathscr{H}$ be a Hilbert space and let $\Sigma \subset \mathscr{H}$ be a nonempty closed convex subset. For each $x \in \mathscr{H}$, there exists a unique point $P_\Sigma(x)$ belonging to Σ and called the orthogonal projection of x onto Σ that satisfies the identity*

$$\Big\|x - P_\Sigma(x)\Big\| = \inf_{y\in\Sigma}\Big\|x - y\Big\|.$$

An immediate consequence of Theorem 1.86 is that if $\mathbb{L} \subset \mathscr{H}$ is a closed subspace, then $\mathscr{H}$ can be written as the direct sum of $\mathbb{L}$ and $\mathbb{L}^\perp$ as follows: $\mathscr{H} = \mathbb{L} \oplus \mathbb{L}^\perp$. This means that each $x \in \mathscr{H}$ can be uniquely written as $x = (x - P_\mathbb{L}(x)) + P_\mathbb{L}(x)$ where $P_\mathbb{L}(x) \in \mathbb{L}$ and $x - P_\mathbb{L}(x) \in \mathbb{L}^\perp$. The mapping $P_\mathbb{L} : \mathscr{H} \mapsto \mathscr{H}, x \mapsto P_\mathbb{L}(x)$, is called the orthogonal projection of $\mathscr{H}$ onto $\mathbb{L}$.

1.4.2 Examples of Hilbert Spaces

Classical examples of Hilbert spaces include, but are not limited to, $\mathbb{R}^n$, $\mathbb{C}^n$, $L^2(\Omega)$, and $\ell^2(\mathbb{N})$, equipped with their natural inner products. Other examples of Hilbert spaces include Sobolev spaces $H^k(\Omega)$. Take $k = 1$, $\Omega = \mathbb{R}$, and consider H^1 defined by

$$H^1(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : f' \in L^2(\mathbb{R})\}.$$

Recall that f' appearing in the definition of $H^1(\mathbb{R})$ is the derivative of f in the sense of distributions. The inner product on $H^1(\mathbb{R})$ and its corresponding norm are given respectively by

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(t)\overline{g(t)}dt + \int_{-\infty}^{\infty} f'(t)\overline{g'(t)}dt$$

and

$$\|f\|_{H^1(\mathbb{R})} = \left[\int_{-\infty}^{\infty} |f(t)|^2 dt + \int_{-\infty}^{\infty} |f'(t)|^2 dt\right]^{\frac{1}{2}}$$

for all $f, g \in H^1(\mathbb{R})$.

1.5 Exercises

1. Let $(\mathbb{X}, d)$ be a metric space. Show that

$$|d(x, y) - d(y, z)| \leq d(x, z)$$

   for all $x, y, z \in \mathbb{X}$.
2. Let d_0 and d_1 be two metrics on a nonempty set $\mathbb{X}$. Show that mapping d defined by, $d = \alpha d_0 + \beta d_1$ ($d(x, y) = \alpha d_0(x, y) + \beta d_1(x, y)$ for all $x, y \in \mathbb{X}$) where $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$, is a metric on $\mathbb{X}$.
3. Prove Proposition 1.14.
4. Prove Theorem 1.19.
5. Let $a, b > 0$ and let f be the function given for all $x \in \mathbb{R}^n$ by

$$f(x) = \frac{1}{(1 + \|x\|^a)(1 + (\ln \|x\|)^b)},$$

   where $\|\cdot\|$ denotes the Euclidean norm of $\mathbb{R}^n$. Find conditions under which $f \in L^p(\mathbb{R}^n)$.

6. Show that the Lebesgue space $L^p(\Omega)$ is vector space for all $p \in [1, \infty]$.
7. Prove Theorem 1.44.
8. Let $C[a, b]$ the collection of all continuous functions $f : [a, b] \mapsto \mathbb{R}$.

   a) Show that $C[a, b]$ is a vector space over $\mathbb{R}$.
   b) Show that $C[a, b]$ equipped with the norm given, for $p \in [1, \infty)$, by

$$\|f\|_p := \Big(\int_a^b |f(t)|^p dt \Big)^{\frac{1}{p}} \text{ for all } f \in C[a, b],$$

   is not a Banach space.

   Hint: Construct a Cauchy sequence in $(C[a, b], \|\cdot\|_p)$ which does not converge.
9. Consider the function $f : \mathbb{R}^n \mapsto \mathbb{R}$ defined by $f(x) = \|x\|^{-\alpha}$ where $\|\cdot\|$ is the Euclidean norm of $\mathbb{R}^n$ and $\alpha \in \mathbb{R}$. Show that $f \in L^1_{loc}(\mathbb{R}^n)$ if and only if $\alpha < n$.
10. Let $1 < p \leq \infty$ and let $1 \leq q < \infty$ be such that $p^{-1} + q^{-1} = 1$. Show that the (topological) dual of $L^p(\Omega)$ is $L^q(\Omega)$.
11. Let $s \geq 0$ and $1 \leq p < \infty$. Suppose q is such that $p^{-1} + q^{-1} = 1$. Show that the (topological) dual of $W_0^{s,p}(\Omega)$ is the Sobolev space $W^{-s,q}(\Omega)$.
12. Prove Theorem 1.64.
13. Prove Theorem 1.66.
14. Prove Theorem 1.68.
15. Prove Theorem 1.70.
16. Prove Theorem 1.74.
17. Prove Theorem 1.86.

1.6 Comments

Some of the basic materials of Sects. 1.3.3, 1.3.4, 1.3.6, and 1.3.10 are taken from the following sources: Adams [10], Adams and Fournier [11], Brézis [31, 32] and Diagana [47]. The material covered in Sects. 1.3.7 and 1.3.9 are partially taken from Bezandry and Diagana [29] and Lunardi [87]. For additional readings upon L^p spaces and Sobolev spaces $W^{k,p}$ we refer the reader to Adams [10], Adams and Fournier [11], and Brézis [32]. For additional readings upon basic functional analysis and real analysis, we refer to Conway [38], Diagana [45], Eidelman et al. [53], Kato [73], Rudin [102], Weidmann [111], and Yosida [112].

The proof of Theorem 1.80 follows Khamsi and Kirk [76, Proof of Theorem 7.12, pages 190–191].

For additional references on metric and normed vector spaces, we refer to the book by Oden and Demkowicz [97]. For additional readings on fixed-point theory, we refer to the book by Khamsi and Kirk [76].