Connected Vertex Cover for $(sP_1 + P_5)$-Free Graphs

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Abstract. The Connected Vertex Cover problem is to decide if a graph G has a vertex cover of size at most k that induces a connected subgraph of G. This is a well-studied problem, known to be NP-complete for restricted graph classes, and, in particular, for H-free graphs if H is not a linear forest. On the other hand, the problem is known to be polynomial-time solvable for sP_2-free graphs for any integer $s \geq 1$. We prove that it is also polynomial-time solvable for $(sP_1 + P_5)$-free graphs for every integer $s \geq 0$.

1 Introduction

A set S of vertices in a graph G forms a *vertex cover* of G if every edge of G is incident with a vertex of S. The set S is an *independent set* if no two vertices in S are adjacent. These definitions lead to two classical graph problems, which are both NP-complete: the Vertex Cover problem is to decide if a given graph G has a vertex cover of size at most k for a given integer k; the Independent Set problem is to decide if a given graph G has an independent set of size at least ℓ for a given integer ℓ. A set S of at least k vertices of a graph G on n vertices is a vertex cover if and only if $V_G \setminus S$ is an independent set (of size at most $n - k$). Hence Vertex Cover and Independent Set are polynomially equivalent. A vertex cover of a graph G is connected if it induces a connected subgraph of G. In our paper, we focus on the corresponding decision problem.

Connected Vertex Cover *Instance:* a graph G and an integer k. *Question:* does G have a connected vertex cover S with $\vert S\vert \leq k$?

In 1977, Garey and Johnson [9] proved that Connected Vertex Cover is NP-complete for planar graphs of maximum degree 4. More recently, Priyadarsini and Hemalatha [18] and Fernau and Manlove [8] strengthened this result to 2-connected planar graphs of maximum degree 4 and planar bipartite graphs of maximum degree 4, respectively. Wanatabe et al. [22] proved that Connected Vertex Cover is NP-complete even for 3-connected graphs. Very recently,

This work was supported by The Leverhulme Trust (Grant RPG-2016-258).

A. Brandstädt et al. (Eds.): WG 2018, LNCS 11159, pp. 279–291, 2018.
https://doi.org/10.1007/978-3-030-00256-5_23

Munaro [16] proved the same for line graphs of planar cubic bipartite graphs and for planar bipartite graphs of arbitrarily large girth, and Li et al. [13] showed NP-completeness for 4-regular graphs.

We now turn to tractable cases. Ueno et al. [21] proved that Connected Vertex Cover is polynomial-time solvable for graphs of maximum degree at most 3. Escoffier et al. [7] proved the same result for chordal graphs. As Vertex Cover is also polynomial-time solvable for chordal graphs [10], the authors of [7] proposed a general study on the complexity of Connected Vertex Cover on graph classes for which Vertex Cover is polynomial-time solvable. This leads us to the research question of our paper:

For which classes of graphs do the complexities of Vertex Cover *and* Connected Vertex Cover *coincide?*

This question was addressed by Chiarelli et al. [6] who considered classes of graphs characterized by a single forbidden induced subgraph H. Such graphs are called H-free. They observed that the results of Munaro [16] imply that Connected Vertex Cover is NP-complete for H-free graphs if H contains a cycle or a claw. Using Poljak's construction [17], Vertex Cover is readily seen to be NP-complete for graphs of arbitrarily large girth and thus for H-free graphs whenever H contains a cycle. When H is the claw, Vertex Cover becomes polynomial-time solvable for H-free graphs [15,20]. Hence, there exist graphs H such that Connected Vertex Cover and Vertex Cover have different complexities when restricted to H-free graphs (assuming $\mathsf{P} \neq \mathsf{NP}$).

So the complexity of Connected Vertex Cover is known for H-free graphs unless H is a linear forest (the disjoint union of one or more paths). Even the case where H is a single path on r vertices (denoted P_r) is settled neither for Vertex Cover nor for Connected Vertex Cover; it is not known if there exists an integer r such that Vertex Cover or Connected Vertex Cover is NP-complete for P_r-free graphs. Lokshtanov et al. [14] proved that Independent Set, and thus Vertex Cover, is polynomial-time solvable for P_5-free graphs. Recently, Grzesik et al. [11] extended this to P_6-free graphs. We also note that if Vertex Cover is polynomial-time solvable on H-free graphs for some graph H, then it is polynomial-time solvable on $(P_1 + H)$-free graphs. This follows from the folklore observation that to solve the complementary problem of Independent Set on a $(P_1 + H)$-free graph one solves the problem on each H-free graph obtained by removing a vertex and all its neighbours.

Theorem 1 **([11]).** *For every $s \geq 0$,* Vertex Cover *can be solved in polynomial time for $(sP_1 + P_6)$-free graphs.*

By using the concept of the price of connectivity [3,5,12], Chiarelli et al. [6] proved that Connected Vertex Cover is polynomial-time solvable for sP_2-free graphs for any integer $s \geq 1$. For Vertex Cover this follows by combining two classical results [2,19] (as is well-known). No other complexity results are known for Connected Vertex Cover for H-free graphs if H is a linear forest.

Our Contribution. We continue the study of [6,7] and prove the following result, which includes polynomial-time solvability for P_5-free graphs.

Theorem 2. *For every $s \geq 0$,* Connected Vertex Cover *can be solved in polynomial time for $(sP_1 + P_5)$-free graphs.*

Our Method. It is easy to construct graphs with a minimum connected vertex cover that do not contain a minimum vertex cover; see the graph G_1 in Fig. 1. We also note that the difference between a minimum vertex cover and a minimum connected vertex cover in an $(sP_1 + P_5)$- free graph is at most 3 if $s = 0$ and at most $3s+10$ if $s \geq 1$ [12]. We cannot exploit this property directly as that would require an algorithm to enumerate all minimum vertex covers in polynomial time. Moreover, the graph G_2 in Fig. 1 shows that even if this were possible, it is not immediately obvious how to proceed; one cannot necessarily hope to find a minimum connected vertex cover by extending a minimum vertex cover. As an extra complication, for Connected Vertex Cover one cannot extend results on H-free graphs to results on $(sP_1 + H)$-free graphs in a straightforward way (certainly one cannot use the technique for Vertex Cover described before Theorem 1).

Our method is based on an analysis of the structure of dominating sets in $(sP_1 + P_5)$-free graphs using a characterization of P_5-free graphs due to Bacsó and Tuza [1]. We translate the problem into a problem in which we try to extend a partial vertex cover into a full connected vertex cover. We solve this extension variant of Connected Vertex Cover by using Theorem 1 (applied to the smaller class of $(sP_1 + P_5)$-free graphs). We show how to do this in Sect. 3 and then show how to use this result to prove Theorem 2 in Sect. 4. An important ingredient of our proof is to reduce the size of the input graph by contracting an edge between two vertices u and v whenever we detect that u and v will belong to the connected vertex cover. This idea stems from the observation that a connected graph G on n vertices has a connected vertex cover of size k if and only if G contains the star $K_{1,n-k}$ on $n-k+1$ vertices as a contraction.

2 Preliminaries

Let $G = (V, E)$ be a graph. For a set $S \subseteq V$, the graph $G[S]$ denotes the subgraph of G induced by S, and we say that S is *connected* if $G[S]$ is connected. We write $G - S = G[V \setminus S]$, and if $S = \{u\}$ we may simply write $G - u$. For a vertex $u \in V$, we write $N_G(u) = \{v \mid uv \in E\}$ to denote the neighbourhood of u. For a set $S \subseteq V$, we write $N_G(S) = (\bigcup_{u \in S} N_G(u)) \setminus S$. A subset $D \subseteq V$ is a *dominating* set of G if every vertex of $V \setminus D$ is adjacent to at least one vertex of D. An edge uv of a graph $G = (V, E)$ is *dominating* if $\{u, v\}$ is dominating. The *contraction* of an edge $uv \in E$ is the operation that replaces u and v by a new vertex adjacent to precisely those vertices of $V \setminus \{u, v\}$ adjacent to u or v in G. Recall that for a graph H, we say that another graph G is *H-free* if it does not contain an induced subgraph isomorphic to H. The *disjoint union* $G + H$ of two

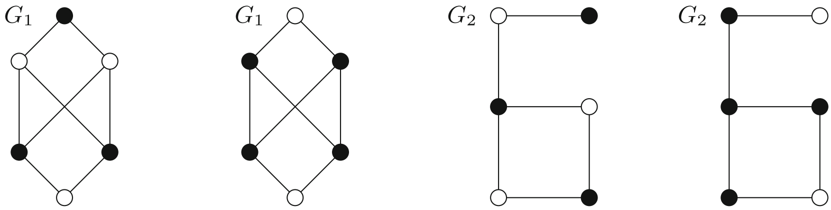

Fig. 1. An example of a P_5-free graph G_1 with a minimum connected vertex cover (coloured black in the right-hand drawing) that contains no minimum vertex cover (there are exactly two, indicated by the sets of black and white vertices in the left-hand drawing). The graph G_2 is an example of a $(P_1 + P_5)$-free graph with a minimum vertex cover (coloured black in the left hand drawing) that is not contained in any minimum connected vertex cover; clearly any connected vertex cover that contains it has at least five vertices and an example of a minimum connected vertex cover on four vertices is indicated by the vertices coloured black in the right-hand drawing.

vertex-disjoint graphs G and H is the graph $(V_G \cup V_H, E_G \cup E_H)$. The disjoint union of r copies of a graph G is denoted by rG. A *linear forest* is the disjoint union of one or more paths. The following, straightforward lemma holds for any linear forest.

Lemma 1. *Let G be a connected (sP_1+P_5)-free graph for some $s \geq 0$. The graph obtained from G after contracting an edge is also connected and $(sP_1 + P_5)$-free.*

We will use the following result of Bacsó and Tuza [1] as a lemma.

Lemma 2. **([1]).** *Every connected P_5-free graph G has a dominating set D, computable in $O(n^3)$ time, that induces either a P_3 or a complete graph.*

Note that it is not difficult to compute the set D in polynomial time; this also follows from a more general result of Camby and Schaudt [4] for P_r-free graphs $(r \geq 1)$.

Proofs of some lemmas are omitted due to space restrictions.

3 An Auxiliary Problem

In this section we prove that a variant of Connected Vertex Cover can be solved in polynomial time for $(sP_1 + P_5)$-free graphs for every integer $s \geq 0$.

To prove Theorem 2 we will solve a polynomial number of instances of this variant, which we show can be solved in polynomial time for $(sP_1 + P_5)$-free graphs for every $s \geq 0$. We introduce the variant by first describing its input. Let G be a connected graph, let $J \subseteq V_G$ be a subset of the vertex set of G and let y be a vertex of J. We call the triple (G, J, y) *cover-complete* if it has the following properties (see also Fig. 2):

(A) J is an independent set;
(B) y is adjacent to every vertex of $G - J$;
(C) the neighbours of each vertex in $J \setminus \{y\}$ form an independent set in $G - J$.

We now describe the problem.

Connected Vertex Cover Completion
Instance: a cover-complete triple (G, J, y).
Goal: find a smallest connected vertex cover S of G such that $J \subseteq S$.

We will show how to solve this problem in polynomial time for $(sP_1 + P_5)$-free graphs for any $s \geq 0$.

Let (G, J, y) be a cover-complete triple, where G is a connected $(sP_1 + P_5)$-free graph. For a vertex $w \in N_G(J \setminus \{y\})$, we write $J_w = N_G(w) \cap J$. Note that, by (B), $y \in J_w$. Let G' be the graph obtained from G by contracting every edge of $G[J_w \cup \{w\}]$. As $G[J_w \cup \{w\}]$ is connected, contracting its edges reduces it to a single vertex which we denote y_w. We say that we have *set-contracted* G into G' via w and that we *contracted* $J_w \cup \{w\}$ into y_w.

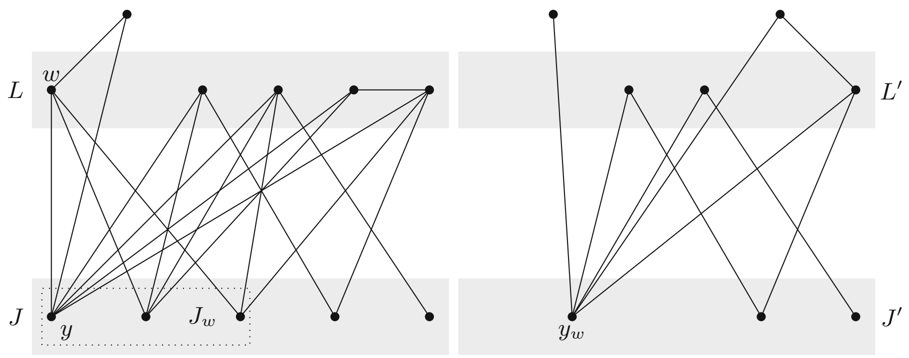

Fig. 2. An example of a cover-complete triple (G, J, y) and the cover-complete triple (G', J', y_w) obtained from set-contracting G via vertex w. The sets $J' = (J \setminus J_w) \cup \{y_w\}$, $L = N_G(J \setminus \{y\})$ and $L' = N_{G'}(J' \setminus \{y_w\})$ are also displayed (the latter two sets will be formally introduced later).

Lemma 3. *Let (G, J, y) be a cover-complete triple, where G is a connected $(sP_1 + P_5)$-free graph for some $s \geq 0$. Let $w \in N_G(J \setminus \{y\})$, and let G' be the graph obtained from G after set-contracting via w. Let $J' = (J \setminus J_w)) \cup \{y_w\}$ and $y' = y_w$. Then the following hold:*

1. *G' is a connected $(sP_1 + P_5)$-free graph;*
2. *(G', J', y') is a cover-complete triple;*
3. *A set $S \subseteq V_G$ is a (smallest) connected vertex cover of G that contains $J \cup \{w\}$ if and only if $(S \setminus (J \cup \{w\}) \cup J'$ is a (smallest) connected vertex cover of G' that contains J'.*

Let (G, J, y) be a cover-complete triple. We define $L_J = N_G(J \setminus \{y\})$. If there is no ambiguity, we will just write $L = L_J$. Note that, by (C), L is the union of a number of independent sets, but L itself might not be independent. However we can deduce the following lemma, which follows immediately from property (C).

Lemma 4. *Let (G, J, y) be a cover-complete triple. If w_1 and w_2 are two adjacent vertices in L, then no vertex of $J \setminus \{y\}$ is adjacent to both w_1 and w_2.*

We introduce two key definitions. Two vertices $w_1, w_2 \in L$ form a *pseudo-dominating pair* if w_1 and w_2 are non-adjacent; w_1 has a neighbour $x_1 \in J$ not adjacent to w_2; and w_2 has a neighbour $x_2 \in J$ not adjacent to w_1. Three vertices $w_1, w_2, w_3 \in L$ form a *pseudo-dominating triple* if w_1 is adjacent to neither w_2 nor w_3; w_2 and w_3 are adjacent; J contains two distinct vertices x_1 and x_2 such that $x_1 \in N_G(w_1) \setminus N_G(\{w_2, w_3\})$ and $x_2 \in (N_G(w_1) \cap N_G(w_2)) \setminus N_G(w_3)$. See the illustrations in Fig. 3, from which we also observe that no pseudo-dominating pair or pseudo-dominating triple can be found in a P_5-free graph.

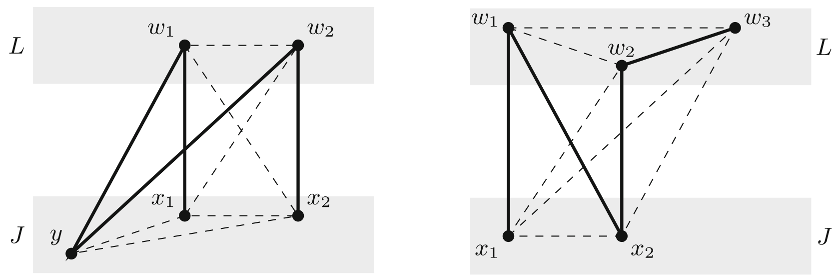

Fig. 3. Examples, on the left, of a pseudo-dominating pair (w_1, w_2), and, on the right, of a pseudo-dominating triple (w_1, w_2, w_3). As easily seen, the presence of either implies the existence of at least one induced P_5.

Let S be a connected vertex cover of G that contains J. Recall that J is an independent set. A subset $L^* \subseteq L \cap S$ is a *connector* of S if $J \cup L^*$ is connected.

Lemma 5. *Let (G, J, y) be a cover-complete triple, where G is an $(sP_1 + P_5)$-free graph for some $s \geq 0$. Let S be a connected vertex cover of G that contains J. If S contains both vertices of a pseudo-dominating pair w_1, w_2, then S has a connector of size at most $s + 1$ that contains both w_1 and w_2.*

Lemma 6. *Let (G, J, y) be a cover-complete triple, where G is an $(sP_1 + P_5)$-free graph for some $s \geq 0$. Let S be a connected vertex cover of G that contains J. If S contains all three vertices of a pseudo-dominating triple w_1, w_2, w_3, then S has a connector of size at most $s + 2$ that contains $\{w_1, w_2, w_3\}$.*

Let (G, J, y) be a cover-complete triple. Let S be a connected vertex cover of G that contains J. If S contains both vertices of some pseudo-dominating pair of

G or all three vertices of some pseudo-dominating triple of G, then S is of *type 1*. Otherwise S must contain at most one vertex of any pseudo-dominating pair and at most two vertices of any pseudo-dominating triple of G. In that case we say that S is of *type 2*. We observe that G might have connected vertex covers of only one type.

We will now see, in Lemma 8, how to find a smallest type 1 connected vertex cover of a graph G of a cover-complete triple (G, J, y) in polynomial time (if it exists). After that we shall prove how to find a smallest type 2 connected vertex cover of G in polynomial time (if it exists). To compute these sets we need the following lemma, which uses Theorem 1 in its proof.

Lemma 7. *Let $(G, \{y\}, y)$ be a cover-complete triple, where G is an $(sP_1 + P_5)$-free graph for some $s \geq 0$. Then it is possible to compute a smallest connected vertex cover of G that contains y in polynomial time.*

Using Lemmas 5–7, we can now prove the following lemma.

Lemma 8. *Let (G, J, y) be a cover-complete triple. Then it is possible to find in polynomial time a smallest type 1 connected vertex cover of G.*

Let (G, J, y) be a cover-complete triple. Using Lemma 8 we can find a smallest type 1 connected vertex cover of G. However, it might be possible that G has a smaller connected vertex cover of type 2. To investigate this, we introduce two reduction rules that will transform a cover-complete triple (G, J, y) into a triple (G', J', y') with $|J'| < |J|$. We say that such a rule is *safe* if the following holds:

1. If G is (sP_1+P_5)-free and connected, then G' is (sP_1+P_5)-free and connected.
2. (G', J', y') is cover-complete.
3. Given a smallest connected vertex cover S' of G' that contains J', it is possible, in polynomial time, to find a smallest connected vertex cover S of G that contains J.

Rule 1. Set-contract via x whenever x is a vertex in $L \cap N_G(w_1) \cap N_G(w_2)$ for some pseudo-dominating pair (w_1, w_2).

Rule 2. For any vertex $w_5 \in L$ that is not adjacent to any vertex of a clique of four vertices w_1, w_2, w_3, w_4 in L, delete w_5 and set-contract via u for every $u \in L \cap N_G(w_5)$.

Lemma 9. *Rules 1 and 2 are safe.*

We call a cover-complete triple (G, J, y) *free* if G has no pseudo-dominating pair with a common neighbour in L, and moreover, $G[L]$ is $(P_1 + K_4)$-free. By exhaustively applying Rules 1 and 2 in arbitrary order, which we may safely do due to Lemma 9, we have the following lemma.

Lemma 10. *A cover-complete triple (G, J, y) can be modified, in polynomial time, into a free cover-complete triple (G', J', y) with the following properties:*

1. *If* G *is* (sP_1+P_5)*-free and connected, then* G' *is* (sP_1+P_5)*-free and connected.*
2. *Given a smallest connected vertex cover* S' *of* G' *that contains* J'*, it is possible to find in polynomial time a smallest connected vertex cover* S *of* G *that contains* J*.*

Let (G, J, y) be a free cover-complete triple. A connector of a connected vertex cover S of G is *minimal* if it does not properly contain a smaller connector of S.

Lemma 11. *Let* (G, J, y) *be a free cover-complete triple that has a pseudo-dominating pair* (w_1, w_2)*. Then every minimal connector* L^* *of every type 2 connected vertex cover* S *of* G *has size at most* 5*.*

Lemma 12. *Let* (G, J, y) *be a free cover-complete triple that has no pseudo-dominating pair. It is possible to find in polynomial time a clique* $K \subseteq L$ *with* $N_G(K) \cap J = J$*.*

We are now ready to prove the following theorem.

Theorem 3. *For every* $s \geq 0$*,* Connected Vertex Cover Completion *can be solved in polynomial time for* $(sP_1 + P_5)$*-free graphs.*

Proof. Let $s \geq 0$ and let (G, J, y) be a cover-complete triple, where G is an $(sP_1 + P_5)$-free graph. We first apply Lemma 10 to obtain a free cover-complete triple (G', J', y') in polynomial time. By the same lemma, G' is $(sP_1 + P_5)$-free. Our aim is to find a smallest connected vertex cover of G' that contains J' in polynomial time, so that we can apply statement 2 of Lemma 10. We first compute in polynomial time a smallest type 1 connected vertex cover S^* of G' using Lemma 8. We now need to compute a smallest type 2 connected vertex cover S' of G' and compare $|S'|$ with $|S^*|$.

First suppose that G' contains a pseudo-dominating pair. We guess a minimal connector of size at most 5 and apply Lemma 3 on its vertices. (By guess, we mean choose a set of up to 5 vertices and test to see if they form a minimal connector. We eventually look at all such sets.) If we obtain an instance of the form $(G'', \{y''\}, y'')$, then we apply Lemma 7. Then we uncontract all contracted edges to get a connected vertex cover of G' of type 2. By Lemma 11, doing this for every guessed minimal connector of size at most 5 gives us a smallest type 2 connected vertex cover S' of G'. As we process each guess in polynomial time and there are at most $O(n^5)$ guesses, we find S' in polynomial time. We compare S' and S^* and choose the smaller of the two.

Now suppose that G' has no pseudo-dominating pair. Let $L' = N_{G'}(J' \setminus \{y'\})$. By Lemma 12, we can obtain in polynomial time a clique $K \subseteq L'$ with $N_{G'}(K) \cap J' = J'$. Let $K = \{w_1, \ldots, w_r\}$ for some $r \geq 1$. As K is a clique, every vertex cover contains at least $r - 1$ vertices of K. We will do as follows: first we will find in polynomial time a smallest connected vertex cover of G' that contains $J' \cup K$, and then we will find in polynomial time, for $i = 1, \ldots, r$, a smallest connected vertex cover of G' that contains $J' \cup (K \setminus \{w_i\})$ and that does not contain w_i. As there are $O(n)$ cases, the total time is polynomial.

We start by computing a smallest connected vertex cover of G' that contains $J' \cup K$ by set-contracting via each vertex of K. By Lemma 3, this yields a cover-complete triple $(G'', \{y''\}, y'')$ to which we apply Lemma 7. Then we uncontract all contracted edges in polynomial time. By Lemma 3, this yields a smallest connected vertex cover S_K of G' that contains $J' \cup K$.

We now show how to compute, in polynomial time, a smallest connected vertex cover of G' that contains $J' \cup (K \setminus \{w_1\})$ and that does not contain w_1. The case $i \geq 2$ is done in the same way.

Let $A = L' \setminus N_{G'}(w_1)$ consist of all non-neighbours of w_1 in L'. As $G'[L']$ is $(K_4 + P_1)$-free by definition, we find that $G'[A]$ is K_4-free. As w_1 is not in the connected vertex cover we are looking for we remove w_1, and we set-contract via each neighbour of w_1 in L. By Lemma 3, we may now consider the resulting cover-complete triple (G'', J'', y'') where G'' is connected and $(sP_1 + P_5)$-free. As G' had no pseudo-dominating pairs, we have that G'' has no pseudo-dominating pairs. We write $L'' = N_{G''}(J'' \setminus \{y''\})$. As $L'' \subseteq A$, we find that $G''[L'']$ is K_4-free.

Claim. Every minimal connector L^* of every connected vertex cover of G'' that contains J'' has size at most 3.

We prove the claim by showing that L^* is a clique, which implies that L^* has size at most 3, as $G''[L'']$ is K_4-free. Suppose instead that L^* is not a clique. Then L^* contains two non-adjacent vertices w_1 and w_2. As L^* is a minimal connector, w_1 has a neighbour in J'' not adjacent to w_2, and vice versa. But then (w_1, w_2) is a pseudo-dominating pair of G'': this is not possible, as G'' has no pseudo-dominating pairs. This contradiction proves the claim.

We now guess a minimal connector by considering all subsets in L'' that have size at most 3. For each guess we apply Lemma 3 on its vertices. If we obtain an instance $(G''', \{y'''\}, y''')$, then we apply Lemma 7. Then we uncontract all contracted edges to obtain in polynomial time a connected vertex cover of G'' that contains J''. We take the smallest one of these connected vertex covers of G''. For this connected vertex cover of G'', we uncontract all contracted edges again to obtain in polynomial time a smallest connected vertex cover S_{w_1} of G' that contains $J' \cup (K \setminus \{w_1\})$ and that does not contain w_1.

As mentioned, we pick the smallest one out of the connected vertex covers S_K and S_{w_i}, $1 \leq i \leq r$, to obtain a smallest type 2 connected vertex cover of G', the size of which we compare with the size of S^*. We pick the smallest one.

Thus we obtain in polynomial time a smallest connected vertex cover of G' that contains J' (both in the case where G' has a pseudo-dominating pair and in the case where G' has no pseudo-dominating pair). As stated, it remains to apply statement 2 of Lemma 10 to find in polynomial time a smallest connected vertex cover of G that contains J. The correctness of our algorithm follows immediately from the above case analysis and the description of the cases. □

4 Our Main Result

In this section we prove Theorem 2. We need two more lemmas (we use Lemma 2 to prove the first one).

Lemma 13. *Let $s \geq 0$ and let G be a connected $(sP_1 + P_5)$-free graph. Then G has a connected dominating set D that is either a clique or has size at most $2s^2 + s + 3$. Moreover, D can be found in $O(n^{2s^2+s+3})$ time.*

Lemma 14. *Let J be an independent set in a connected graph G such that J has a vertex y that is adjacent to every vertex of $G - J$. Let J' consist of those vertices of $J \setminus \{y\}$ that have two adjacent neighbours in $G - J$ (or equivalently, in G). Then a subset S is a connected vertex cover of G that contains J if and only if $S \setminus J'$ is a connected vertex cover of $G - J'$ that contains $J \setminus J'$.*

We are now ready to prove our main result.

Theorem 2 (Restated). *For every $s \geq 0$,* Connected Vertex Cover *can be solved in polynomial time for $(sP_1 + P_5)$-free graphs.*

Proof. Let G be an (sP_1+P_5)-free graph for some $s \geq 0$. We may assume without loss of generality that G is connected. By Lemma 13 we can first compute in $O(n^{2s^2+s+3})$ time a connected dominating set D that either has size at most $2s^2 + s + 3$ or is a clique. We note that, if D is a clique, any vertex cover of G contains all but at most one vertex of D. This leads to a case analysis where we guess the subset $D^* \subseteq D$ of vertices not in a minimum connected vertex cover of G. Because $|D^*| \leq 2s^2 + s + 3$, the number of guesses is polynomial. For each guess of D^*, we compute a smallest connected vertex cover S_{D^*} that contains all vertices of $D \setminus D^*$ and no vertex of D^*. Then, in the end, we return one that has minimum size overall.

Let D^* be a guess. We first show the following claim (proof omitted).

Claim 1. We may assume without loss of generality that $D \setminus D^*$ is connected.

Case 1. $D^* = \emptyset$.
We compute a minimum vertex cover S' of $G - D$ in polynomial time by Theorem 1. Clearly $S' \cup D$ is a vertex cover of G. As D is a connected dominating set, $S' \cup D$ is a connected vertex cover of G. Let $S_\emptyset = S' \cup D$. As S' is a minimum vertex cover of $G - D$, $S_\emptyset$ is a smallest connected vertex cover of G that contains all vertices of D. We remember $S_\emptyset$, which we found in polynomial time.

Case 2. $1 \leq |D^*| \leq |D|$ (recall that $|D| \leq 2s^2 + s + 3$).
Recall that we are looking for a smallest connected vertex cover of G that contains every vertex of $D \setminus D^*$ but does not contain any vertex of D^*. Hence D^* must be an independent set and $G - D^*$ must be connected (if one of these conditions is false, then we stop considering the guess D^*). Moreover, a vertex cover that contains no vertex of D^* must contain all vertices of $N_G(D^*)$. Hence we can safely contract not only any edge between two vertices of $D \setminus D^*$, but also any edge between two vertices in $N_G(D^*)$ or between a vertex of $D \setminus D^*$ and a vertex in $N_G(D^*)$. We perform edge contractions recursively and as long as possible while remembering all the edges that we contract. Let G^* be the resulting graph.

Note that the set D^* still exists in G^*, as we did not contract any edges with an endpoint in D^*. By Claim 1, the set $D \setminus D^*$ in G corresponds to exactly one vertex of G^*. We denote this vertex by y. We observe the following equivalence.

Claim 2. Every smallest connected vertex cover of G^* that contains y and that does not contain any vertex of D^* corresponds to a smallest connected vertex cover of G that contains $D \setminus D^*$ and that does not contain any vertex of D^*, and vice versa.

As we obtained G^* in polynomial time, and we can uncontract all contracted edges in polynomial time as well, Claim 2 tells us that we may consider G^* instead of G. As G is connected and $(sP_1 + P_5)$-free, G^* is connected and $(sP_1 + P_5)$-free as well by Lemma 1.

We write $J^* = N_{G^*}(D^*)$ and note that y belongs to J^* as D is connected in G. We now consider the graph $G^* - D^*$. As $G - D^*$ is connected, $G^* - D^*$ is connected. By Claim 2, our new goal is to find a smallest connected vertex cover of $G^* - D^*$ that contains J^*. By our procedure, J^* is an independent set of $G^* - D^*$. As D dominates G, we find that $D \setminus D^*$ dominates every vertex of $G - D^*$ that is not adjacent to a vertex of D^*. Hence the vertex y, which corresponds to the set $D \setminus D^*$, is adjacent to every vertex of $(G^* - D^*) - J^*$ in the graph $G^* - D^*$.

Let $J \subseteq J^*$ consist of y and those vertices in J^* whose neighbourhood in $G^* - D^*$ is an independent set. As y is adjacent to every vertex of $(G^* - D^*) - J^*$ in $G^* - D^*$, and we can remember the set $J^* \setminus J$, we can apply Lemma 14 and remove $J^* \setminus J$. That is, it suffices to find a smallest connected vertex cover of the graph $G' = (G^* - D^*) - (J^* \setminus J)$ that contains J.

As J^* is an independent set of $G^* - D^*$, we find that J is an independent set of G'. By definition, $y \in J$. As y is adjacent to every vertex of $(G^* - D^*) - J^*$ in $G^* - D^*$, we find that y is adjacent to every vertex in $G' - J$. By definition, the neighbours of each vertex in $J \setminus \{y\}$ form an independent set in $G' - J$. Hence the triple (G', J, y) is cover-complete. This means that we can apply Theorem 3 to find in polynomial time a smallest connected vertex cover S' of G' that contains J.

We translate S' in polynomial time into a smallest connected vertex cover S^* of $G^* - D^*$ that contains J^* by adding $J^* \setminus J$ to S'. We translate S^* in polynomial time into a smallest connected vertex cover S_{D^*} of G that contains no vertex of D^* by uncontracting any contracted edges.

As mentioned, in the end we pick, in polynomial time, a smallest set of the sets S_{D^*}. This set is then a minimum connected vertex cover of G, which is obtained in polynomial time. We have not sought to optimize the running time of the algorithm so do not provide a detailed analysis, but observe that, for sufficiently large s, it is $n^{O(s^3)}$. The running time is dominated by obtaining a connected $D \setminus D^*$ (in Claim 1). As $D \setminus D^*$ has $O(n^{2s^2+s+3})$ components and the paths required to join them each have $O(s)$ vertices, the time required to find them is $n^{O(s^3)}$. The correctness of our algorithm follows immediately from the above case analysis and the description of the cases. □

5 Future Work

We pose two open problems. First, determine the complexity of Connected Vertex Cover for P_6-free graphs. Second, is there an integer r such that Connected Vertex Cover is NP-complete for P_r-free graphs?

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