Chapter 3

The Leibniz Rule

We will show that the derivative as a map on classical function spaces of analysis is characterized by the Leibniz rule as well as the chain rule. This is a consequence of results in this and the next chapter. We first study the solutions of the Leibniz rule equation as a map on the k-times continuously differentiable functions C^k. There are many examples of derivations in algebra and differential geometry generalizing the Leibniz rule for the derivative of products of functions. However, on C^k there are only few examples of derivations. A priori, we assume neither linearity nor continuity of the derivations which we characterize. However, the continuity of the operator is a consequence of the results. Various solutions are actually non-linear.

3.1 The Leibniz rule in C^k

To formulate the basic result, we use the following notation:

Let $I \subset \mathbb{R}$ be an open set. In particular, $I = (-\infty, a), (a, b), (b, \infty)$ with $a, b \in \mathbb{R}$ or $I = \mathbb{R}$ are natural choices. For $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ let

$$C^k(I) := \big\{ f : I \to \mathbb{R} \mid f \text{ is } k\text{-times continuously differentiable on } I \big\}.$$

We denote the continuous functions also by $C(I) := C^0(I)$ and put $C^\infty(I) = \bigcap_{k \in \mathbb{N}} C^k(I)$. The basic result for the Leibniz rule operator equation is

Theorem 3.1 (Leibniz rule). *Let $k \in \mathbb{N}_0$ and $I \subset \mathbb{R}$ be an open set. Suppose that $T : C^k(I) \to C(I)$ is an operator satisfying the* Leibniz rule equation

$$T(f \cdot g) = Tf \cdot g + f \cdot Tg, \quad f, g \in C^k(I). \tag{3.1}$$

Then there are continuous functions $c, d \in C(I)$ such that, if $k \in \mathbb{N}$,

$$Tf = c\, f \ln|f| + d\, f', \quad f \in C^k(I). \tag{3.2}$$

Conversely, any map T given by (3.2) *satisfies* (3.1). *For $k = 0$, if $T : C(I) \to C(I)$ satisfies* (3.1), *there is $c \in C(I)$ such that $Tf = c\, f \ln|f|$.*

H. König, V. Milman, *Operator Relations Characterizing Derivatives*,
https://doi.org/10.1007/978-3-030-00241-1_3

Since $\lim_{x\to 0}\ x\ln|x| = 0$, $0\ln|0|$ should be read as 0.

Remarks. (a) The formulas (3.1) and (3.2) are meant pointwise, e.g., (3.2):

$$(Tf)(x) = c(x)f(x)\ln|f(x)| + d(x)f'(x), \quad f \in C^k(I),\ x \in I.$$

Thus the solutions of the Leibniz rule are linear combinations of the derivative and the "entropy solution" $f\ln|f|$ which acts as a "derivative" on spaces of continuous functions. Note that neither continuity nor linearity is imposed on the operator T; in fact, $Tf = f\ln|f|$ is a non-linear solution.

(b) For $k \geq 2$, there are not more solutions than for $k = 1$. Hence, $T : C^k(I) \to C(I)$ naturally extends by the same formula to $T : C^1(\mathbb{R}) \to C(\mathbb{R})$. Therefore $C^1(I)$ is the "natural domain" for the Leibniz formula among the $C^k(I)$-spaces.

(c) If T also maps $C^2(I)$ into $C^1(I)$, it has the form $Tf = d\,f'$ with $d \in C^1(I)$, since in general $f\ln|f| \notin C^1(I)$ for $f \in C^2(I)$. "Initial" conditions like $T(\mathrm{Id}) = 1$ and $T(2\,\mathrm{Id}) = 2$ together with (3.1) also imply that $Tf = f'$ is the derivative.

(d) If the image of T does not consist of continuous or at least measurable functions, there are different solutions of the Leibniz rule equation. Let $F(\mathbb{R})$ denote the space of all functions $f : \mathbb{R} \to \mathbb{R}$, and $H : \mathbb{R} \to \mathbb{R}$ be an additive but not linear function, as constructed after Proposition 2.1. Let $c \in F(\mathbb{R})$ and define $T : C(\mathbb{R}) \to F(\mathbb{R})$ by

$$Tf(x) = c(x)f(x)H\big(\ln|f(x)|\big), \quad f \in C(\mathbb{R}),\ x \in \mathbb{R},$$

with $Tf(x) := 0$ if $f(x) = 0$. Then T satisfies the Leibniz rule

$$T(f\cdot g) = Tf\cdot g + f\cdot Tg.$$

(e) For $k \geq 2$, there are more solutions of (3.1) on the *positive* C^k-tfunctions than those given in (3.2), cf. Corollary 3.4.

The proof of Theorem 3.1 consists of two steps. The first is to show *localization*, i.e., that T is defined pointwise in the sense that there is a function $F : I \times \mathbb{R}^{k+1} \to \mathbb{R}$ such that for all $f \in C^k(I)$ and $x \in I$

$$Tf(x) = F(x, f(x), \dots, f^{(k)}(x)).$$

At that point no regularity of F is known. The operator equation (3.1) then is equivalent to a functional equation for the *representing* function F. The second step of the proof is to analyze the structure of F and to prove the continuity of the coefficient functions occurring there, by using the fact that the image of T consists of continuous functions. In the case of Theorem 3.1, we have to show that F does not depend on the variables $\alpha_j = f^{(j)}(x)$ for $j \geq 2$ and that the functions c, d in (3.2) are continuous. To find the solutions of other operator equations in later chapters, we will use the same basic strategy in the proofs, although with very different representing functions.

To prove Theorem 3.1, we first show that T is "localized on intervals".

Lemma 3.2. *Suppose $T : C^k(I) \to C(I)$ satisfies* (3.1). *Then $T(\mathbb{1}) = T(-\mathbb{1}) = 0$. If $J \subset I$ is open and $f_1, f_2 \in C^k(I)$ satisfy $f_1|_J = f_2|_J$, then $Tf_1|_J = Tf_2|_J$.*

Proof. For any $f \in C^k(I)$, $T(f) = T(f \cdot \mathbb{1}) = T(f) \cdot \mathbb{1} + T(\mathbb{1}) \cdot f$, which implies $T(\mathbb{1}) = 0$. Moreover $0 = T(\mathbb{1}) = T((-\mathbb{1})^2) = -2T(-\mathbb{1})$, $T(-\mathbb{1}) = 0$. If $J \subset I$ is open and $f_1|_J = f_2|_J$, let $x \in J$ be arbitrary and choose $g \in C^k(I)$ with $g(x) = 1$ and $\operatorname{supp} g \subset J$. Then $f_1 \cdot g = f_2 \cdot g$ and hence by (3.1)

$$f_1 \cdot Tg + Tf_1 \cdot g = T(f_1 \cdot g) = T(f_2 \cdot g) = f_2 \cdot Tg + Tf_2 \cdot g,$$

which implies $Tf_1(x) = Tf_2(x)$ for any $x \in J$, yielding $Tf_1|_J = Tf_2|_J$. □

Localization on intervals always implies pointwise localization.

Proposition 3.3. *Let $k \in \mathbb{N}_0$ and $I \subset \mathbb{R}$ be an open set. Suppose $T : C^k(I) \to C(I)$ satisfies, for all open intervals $J \subset I$, that*

$$\big[f_1|_J = f_2|_J \Longrightarrow Tf_1|_J = Tf_2|_J, \quad f_1, f_2 \in C^k(I)\big]. \tag{3.3}$$

Then there is a function $F : I \times \mathbb{R}^{k+1} \to \mathbb{R}$ such that

$$Tf(x) = F\big(x, f(x), f'(x), \dots, f^{(k)}(x)\big) \tag{3.4}$$

holds for all $x \in I$ and $f \in C^k(I)$. It suffices to have (3.3) *only for all intervals J of the form $J = (-\infty, x) \cap I$ and $J = (x, \infty) \cap I$ with $x \in I$.*

Proof. Let $x_0 \in I$ be arbitrary but fixed. For any $f \in C^k(I)$, let g be the Taylor polynomial of order k at x_0. Let $J_1 := (-\infty, x_0) \cap I$ and $J_2 := (x_0, \infty) \cap I$ and define

$$h(x) := \begin{cases} f(x), & x \in \overline{J_1}, \\ g(x), & x \in J_2. \end{cases}$$

Then $h \in C^k(I)$ and $f|_{J_1} = h|_{J_1}$, $h|_{J_2} = g|_{J_2}$. By assumption $Tf|_{J_1} = Th|_{J_1}$ and $Th|_{J_2} = Tg|_{J_2}$. Since Tf, Th and Tg are continuous functions and $\{x_0\} = \overline{J_1} \cap \overline{J_2}$, we find $Tf(x_0) = Th(x_0) = Tg(x_0)$. Since g only depends on $(x_0, f(x_0), \dots, f^{(k)}(x_0))$, so does $Tg(x_0)$. Therefore, $Tf(x_0) = Tg(x_0)$ only depends on these values, i.e., there is a function $F : I \times \mathbb{R}^{k+1} \to \mathbb{R}$ such that

$$Tf(x_0) = F\big(x_0, f(x_0), \dots, f^{(k)}(x_0)\big),$$

for any $f \in C^k(I)$, $x_0 \in I$. □

Proof of Theorem 3.1. (i) We will first show that for any $f > 0$, $\frac{Tf}{f}$ depends linearly on $\ln f$ and its derivatives, and then that no derivatives of order ≥ 2 show up in the formula for T. By Lemma 3.2 and Proposition 3.3 there is a function $F : I \times \mathbb{R}^{k+1} \to \mathbb{R}$ such that, for any $f \in C^k(I)$ and $x \in I$,

$$Tf(x) = F\big(x, f(x), f'(x), \ldots, f^{(k)}(x)\big).$$

Define a map $S : C^k(I) \to C(I)$ by

$$Sg(x) := T(\exp(g))(x)/\exp(g)(x), \quad g \in C^k(I),\ x \in I.$$

Then $Sg(x) = F(x, \exp(g)(x), \ldots, \exp(g)^{(k)}(x))/\exp(g)(x)$ depends only on $x, g(x)$ and all derivatives of g up to $g^{(k)}(x)$. Hence, there is a function $G : I \times \mathbb{R}^{k+1} \to \mathbb{R}$ such that

$$Sg(x) = G\big(x, g(x), \ldots, g^{(k)}(x)\big), \quad g \in C^k(I),\ x \in I.$$

For any $g_1, g_2 \in C^k(I)$, by the Leibniz rule equation on $C^k(I)$,

$$S(g_1 + g_2) = T(e^{g_1} \cdot e^{g_2})/(e^{g_1} \cdot e^{g_2}) = T(e^{g_1})/e^{g_1} + T(e^{g_2})/e^{g_2} = Sg_1 + Sg_2.$$

Since for any $\alpha = (\alpha_j)_{j=0}^k$, $\beta = (\beta_j)_{j=0}^k \in \mathbb{R}^{k+1}$ and $x \in I$, there are $g_1, g_2 \in C^k(I)$ with $g_1^{(j)}(x) = \alpha_j$, $g_2^{(j)}(x) = \beta_j$ for all $j \in \{0, \ldots, k\}$, we have

$$G(x, \alpha + \beta) = G(x, \alpha) + G(x, \beta), \quad x \in I,\ \alpha, \beta \in \mathbb{R}^{k+1}.$$

Since $Sg = T(e^g)/e^g$ is a continuous function on I, we also know that

$$G(x, g(x), \ldots, g^{(k)}(x))$$

is a continuous function of $x \in I$ for all $g \in C^k(I)$. By Theorem 2.6, there is a continuous function $c : I \to \mathbb{R}^{k+1}$ so that $G(x, \alpha) = \langle c(x), \alpha \rangle = \sum_{j=0}^k c_j(x)\alpha_j$, writing $c = (c_j)_{j=0}^k$, with continuous coefficient functions $c_j \in C(I)$.

For $f \in C^k(I)$, $f > 0$, let $g := \ln f$. Then $f = \exp g$ and

$$Tf(x) = f(x)S\big(\ln f\big)(x) = f(x)\sum_{j=0}^{k} c_j(x)(\ln f)^{(j)}(x). \tag{3.5}$$

Conversely, this formula defines a map on the *strictly positive* functions into the continuous functions satisfying the Leibniz rule since

$$\big(\ln(fg)\big)^{(j)} = (\ln f)^{(j)} + (\ln g)^{(j)}, \quad f, g \in C^k(I).$$

(ii) Let us now consider the Leibniz rule for $T : C^k(I) \to C(I)$ when the functions are negative. Suppose $f \in C^k(I)$ and $x \in I$ are given with $f(x) < 0$. Then there is an open interval $J \in I$, $x \in J$ with $f|_J < 0$. Choose $g \in C^k(I)$ with

$g < 0$ on I and $f|_J = g|_J$. Then $Tf(x) = Tg(x)$. To determine $Tf(x)$, we may therefore assume that $f < 0$ on I. Then $f = -|f|$ and by the Leibniz rule and Lemma 3.2

$$T(f) = T(-|f|) = -T(|f|) + |f|T(-\mathbb{1}) = -T(|f|).$$

Using (3.5), we find

$$\begin{aligned} Tf = -T(|f|) &= -|f| \sum_{j=0}^{k} c_j (\ln |f|)^{(j)} \\ &= f \sum_{j=0}^{k} c_j (\ln |f|)^{(j)}, \qquad f \in C^k(I). \end{aligned}$$

To be defined on $C^k(I)$, Tf needs to be continuous also for f and x with $f(x) = 0$. However, for $j \geq 2$, $f(\ln |f|)^{(j)}$ is of order $O(|f|^{-(j-1)})$ as $|f| \searrow 0$, if $f' \neq 0$. Therefore, using localization, in the above formula $c_2 = \cdots = c_k = 0$ is required for $T : C^k(I) \to C(I)$ to be well defined.

To be more specific, let $k \geq 2$, $x_0 \in I$ and choose $\epsilon_0 > 0$ with $(x_0 - 2\epsilon_0, x_0 + 2\epsilon_0) \subset I$ and consider $f(x) := x - x_0$. Let $0 < \epsilon < \epsilon_0$ and h be a strictly positive function with $h|_{(x_0+\epsilon,\infty)\cap I} = f|_{(x_0+\epsilon,\infty)\cap I}$, i.e., h has to bend upwards for $x < x_0 + \epsilon$ in a smooth way. Applying the above formula for h, we get for $Tf(x_0 + \epsilon) = Th(x_0 + \epsilon)$

$$Tf(x_0 + \epsilon) = Th(x_0 + \epsilon) = c_0(x_0 + \epsilon)\epsilon \ln \epsilon + \sum_{j=1}^{k} c_j(x_0 + \epsilon)(-1)^{j-1}(j-1)!\ \epsilon^{1-j}.$$

Since Tf and $c_0, \ldots, c_k$ are continuous functions, this implies for $\epsilon \to 0$ that $c_k(x_0) = \cdots = c_2(x_0) = 0$. This means that

$$Tf = c_0 f \ln |f| + c_1 f'.$$

This also holds when f has isolated zeros x, $f(x) = 0$, since $\lim_{y \to 0}\ y \ln |y| = 0$. Note that $Tf(x) = 0$ in this case since we have continuous functions on both sides. This is true by continuity of Tf, too, if x is a limit of isolated zeros of f. If $f|_J$ is zero on a non-trivial interval $J \subset I$, $Tf|_J = 0$. □

Corollary 3.4. *Let $k \in \mathbb{N}$ and $I \subset \mathbb{R}$ be an open set. Suppose that $T : C^k(I) \to C(I)$ satisfies the Leibniz rule equation* (3.1). *Then there are continuous functions $c_0, \ldots, c_k \in C(I)$ such that for every strictly positive function $f \in C^k(I)$, $f > 0$ and all $x \in I$*

$$Tf(x) = f(x) \sum_{j=0}^{k} c_j(x)\ (\ln f)^{(j)}(x).$$

Conversely, T defined this way satisfies equation (3.1) *for all positive functions $f \in C^k(I)$.*

This is a corollary to the proof of Theorem 3.1, which yielded (3.5) for positive functions $f > 0$. Note, however, that we need T to be defined and to satisfy (3.1) for *all functions* $f \in C^k(I)$, and not only for the strictly positive ones, since in the proof of Lemma 3.2 the operator T is applied to functions $f_1 g = f_2 g$ which are zero on a large part of the set I. For $k \geq 2$, there are more solution operators T on the positive functions than on all functions. For $k = 1$, we just recover (3.2).

3.2 The Leibniz rule on $\mathbb{R}^n$

Theorem 3.1 gives the solutions of the Leibniz rule on $I \subset \mathbb{R}$. It has an analogue for functions on n-dimensional domains $I \subset \mathbb{R}^n$. For $n \in \mathbb{N}$, $k \in \mathbb{N}_0$, open sets $I \subset \mathbb{R}^n$ and finite-dimensional real Banach spaces X let

$$C^k(I, X) := \{f : I \to X \mid f \text{ is } k\text{-times continuously differentiable on I}\},$$

with $C(I, X) := C^0(I, X)$ denoting the continuous functions. In this section, we include the image space X of functions in the notation $C^k(I, X)$ to indicate whether X is, e.g., $\mathbb{R}$ or $\mathbb{R}^n$. Let $L(\mathbb{R}^n, \mathbb{R}^n)$ denote the continuous linear maps for $\mathbb{R}^n$ into itself. The derivative $T = D$ maps $C^1(I, \mathbb{R})$ into $C(I, \mathbb{R}^n)$. The following theorem extends Theorem 3.1 to this n-dimensional setting. We did not directly state the result in the more general form, since its proof is a bit more elaborate and requires further notations.

Theorem 3.5. *Let $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ and $I \subset \mathbb{R}^n$ be an open set. Suppose that $T : C^k(I, \mathbb{R}) \to C(I, \mathbb{R}^n)$ satisfies the Leibniz rule*

$$T(f \cdot g) = Tf \cdot g + f \cdot Tg, \quad f, g \in C^k(I, \mathbb{R}).$$

Then there are continuous functions $c \in C(I, \mathbb{R}^n)$ and $d \in C(I, L(\mathbb{R}^n, \mathbb{R}^n))$ such that for all $f \in C^k(I, \mathbb{R})$ and all $x \in I$

$$Tf(x) = c(x) f(x) \ln |f(x)| + d(x)(f'(x)).$$

For $k = 0$, d should be zero. Conversely, any such map T satisfies the Leibniz rule.

Note that on the right-hand side of the Leibniz formula we have pointwise multiplications of scalar and $\mathbb{R}^n$-valued functions. In the result, $d(x)$ is a matrix operating on the vector $f'(x)$, and $c(x)$ is a vector multiplying the scalar entropy expression $f(x) \ln |f(x)|$ for any $x \in I$.

For $k \geq 2$ there are no more solutions than for $k = 1$. Therefore T extends by the same formula to $C^1(I, \mathbb{R})$, so that $C^1(I, \mathbb{R})$ is the "natural" domain of T. If $d = 0$, T even extends to $C(I, \mathbb{R})$.

The Leibniz rule immediately implies $T\mathbb{1} = 0$ for the function $\mathbb{1}$ on $I \subset \mathbb{R}^n$. If $J \subset I$ is open and $f_1, f_2 \in C^k(I, \mathbb{R})$ satisfy $f_1|_J = f_2|_J$, we claim that $Tf_1|_J = Tf_2|_J$: Let $x \in J$ be arbitrary and choose $g \in C^k(I, \mathbb{R})$ with $g(x) = 1$ and support of g in J. Then $f_1 \cdot g = f_2 \cdot g$ and hence by the Leibniz rule $(f_1 - f_2) \cdot Tg = (Tf_1 - Tf_2) \cdot g$, so $Tf_1(x) = Tf_2(x)$, $Tf_1|_J = Tf_2|_J$. Therefore we have localization on (small) open sets. We now show that this implies pointwise localization, as in the 1-dimensional case.

For $0 \le l \le k$, the l-th derivative $f^{(l)}(x)$ of $f \in C^k(I, \mathbb{R})$, $I \subset \mathbb{R}^n$ open, at $x \in I$ is an l-multilinear form $f^{(l)}(x) : \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{l} \to \mathbb{R}$ which we may identify with the vector of all l-th order partial derivatives of f at x, a vector in $\mathbb{R}^{n^l}$. By Schwarz' theorem, the iterated partial derivatives do not depend on the order of taking them, so that we have only $M(n,l) := \binom{n+l-1}{n-1}$ different l-th order partial derivatives, indexed by $(\frac{\partial^l f(x)}{\partial x_{i_1} \cdots \partial x_{i_l}})_{1 \le i_1 \le \cdots \le i_l \le n}$. As in Theorem 2.6, we will identify $f^{(l)}(x)$ with this vector in $\mathbb{R}^{M(n,l)}$ to allow for independent choices of the values of these derivatives. Together the function and all derivatives of order $\le k$ constitute

$$N(n,k) := \sum_{l=0}^{k} M(n,l) = \binom{n+k}{n}$$

independent variables. In this setup, we have:

Proposition 3.6. *Let $m, n \in \mathbb{N}$, $k \in \mathbb{N}_0$, $I \subset \mathbb{R}^n$ be open and $T : C^k(I, \mathbb{R}) \to C(I, \mathbb{R}^m)$ be an operator. Suppose that for all open subsets $J \subset I$ and all $f_1, f_2 \in C^k(I, \mathbb{R})$ with $f_1|_J = f_2|_J$ we have that $Tf_1|_J = Tf_2|_J$. Then there is a function $F : I \times \mathbb{R}^{N(n,k)} \to \mathbb{R}^m$ such that*

$$Tf(x) = F\big(x, f(x), f'(x), \dots, f^{(k)}(x)\big)$$

for all $f \in C^k(I, \mathbb{R})$ and $x \in I$.

Proof. Fix $x_0 = (x_{0i})_{i=1}^n \in I$. By assumption, $Tf_1(x_0) = Tf_2(x_0)$ for every two functions $f_1, f_2 \in C^k(I, \mathbb{R})$ which coincide on a small open neighborhood of x_0 in I. To prove that $Tf(x_0)$ depends only on $(x_0, f(x_0), \dots, f^{(k)}(x_0))$, we may therefore assume that I is a (possibly small) open cube or ball centered at x_0. Let $f \in C^k(I, \mathbb{R})$. Define, for $x = (x_i)_{i=1}^n \in I$ and $i \in \{1, \dots, n\}$ the i-th partial k-th order Taylor approximation to f at x_0 by

$$h_i(x) := \sum_{l=0}^{k} \frac{1}{l!} f^{(l)}(x_{01}, \dots, x_{0i}, x_{i+1}, \dots, x_n)((x - x_0)_{[i]}, \dots, (x - x_0)_{[i]}),$$

where $(x - x_0)_{[i]} := (x_1 - x_{01}, \dots, x_i - x_{0i}, 0, \dots, 0) \in \mathbb{R}^n$. Here we consider $f^{(l)}$ as an l-multilinear form from $\mathbb{R}^n \times \cdots \mathbb{R}^n$ to $\mathbb{R}$. Note that $h := h_n$ is the k-th order Taylor approximation to f at x_0. Let $h_0 := f$. Then the functions h_0 and h_1 join C^k-smoothly at the intersection of the hyperplane $x_1 = x_{01}$ with I, since by definition of $(x - x_0)_{[1]}$ only the iterated derivatives with respect to x_1 occur non-trivially in h_1. Similarly h_{i-1} and h_i join C^k-smoothly at the intersection of the hyperplane $x_i = x_{0i}$ with I, for all $i \in \{2, \dots, n\}$. Therefore, putting

$$g_i(x) := \begin{cases} h_{i-1}(x), & x \in I,\ x_i < x_{0i}, \\ h_i(x), & x \in I,\ x_i \geq x_{0i} \end{cases}$$

for $i \in \{1, \dots, n\}$, we have that $g_i \in C^k(I, \mathbb{R})$. On $J_i^- := \{x \in I \mid x_i < x_{0i}\}$ and $J_i^+ := \{x \in I \mid x_i > x_{0i}\}$, we have $h_{i-1}|_{J_i^-} = g_i|_{J_i^-}$, $g_i|_{J_i^+} = h_i|_{J_i^+}$. Hence, also using that the image of T consists of continuous functions,

$$(Th_{i-1})(x_0) = (Tg_i)(x_0) = (Th_i)(x_0),$$

since $x_0 \in \overline{J_i^-} \cap \overline{J_i^+}$. We conclude

$$(Tf)(x_0) = (Th_1)(x_0) = \cdots = (Th_n)(x_0) = (Th)(x_0).$$

However, h only depends on $(x_0, f(x_0), f'(x_0), \cdots, f^{(k)}(x_0))$. Therefore, there exists a function of these parameters which determines $Tf(x_0)$. Identifying $f^{(l)}(x_0)$ with vectors of iterated partial derivatives in $\mathbb{R}^{M(n,l)}$ as described before, this means that there is a function $F : I \times \mathbb{R}^{N(n,k)} \to \mathbb{R}^m$ such that

$$Tf(x_0) = F\big(x_0, f(x_0), f'(x_0), \cdots, f^{(k)}(x_0)\big)$$

for all $x_0 \in I$, $f \in C^k(I, \mathbb{R})$, with $N(n,k) := \sum_{l=0}^k M(n,l)$. $\square$

Proof of Theorem 3.5. We adapt the proof of Theorem 3.1 to the multidimensional setting. By Proposition 3.6 for $m = n$ and the localization on (small) open sets which we proved before formulating Proposition 3.6, there is a function $F : \mathbb{R}^{N(n,k)} \to \mathbb{R}^n$ such that for all $f \in C^k(I, \mathbb{R})$, $x \in I$

$$Tf(x) = F(x, f(x), f'(x), \dots, f^{(k)}(x)).$$

Define $S : C^k(I, \mathbb{R}) \to C(I, \mathbb{R}^n)$ by

$$Sg(x) := T(\exp(g))(x)/\exp(g)(x), \quad g \in C^k(I, \mathbb{R}),\ x \in I.$$

Then $Sg(x) = F(x, \exp(g)(x), \dots, \exp(g)^{(k)}(x))/\exp(g)(x)$ depends only on $x, g(x)$ and all derivatives of g up to $g^{(k)}(x)$. Therefore there is a function $G : I \times \mathbb{R}^{N(n,k)} \to \mathbb{R}^n$ such that

$$Sg(x) = G(x, g(x), \dots, g^{(k)}(x)), \quad g \in C^k(I, \mathbb{R}),\ x \in I.$$

For any $g_1, g_2 \in C^k(I, \mathbb{R})$ by the Leibniz rule

$$\begin{aligned} S(g_1 + g_2) &= T(\exp(g_1) \cdot \exp(g_2))/(\exp(g_1) \cdot \exp(g_2)) \\ &= T(\exp(g_1))/\exp(g_1) + T(\exp(g_2))/\exp(g_2) = Sg_1 + Sg_2, \end{aligned}$$

i.e., S is additive in the function and derivative variables. We split any $\alpha \in \mathbb{R}^{N(n,k)}$ as $\alpha = (\alpha_l)_{l=0}^k$ where $\alpha_l \in \mathbb{R}^{M(n,l)}$. Then for any $x \in I$ and any $\alpha = (\alpha_l)_{l=0}^k$ and $\beta = (\beta_l)_{l=0}^k \in \mathbb{R}^{N(n,k)}$ there are functions $g_1, g_2 \in C^k(I, \mathbb{R})$ such that $g_1^{(l)}(x) = \alpha_l$ and $g_2^{(l)}(x) = \beta_l$ for all $l \in \{0, \dots, k\}$. Recall that all iterated partial derivatives with indices $1 \le i_1 \le \cdots \le i_l \le n$ can be chosen independently. Therefore the additivity of S is equivalent to the additivity of G in the sense that

$$G(x, \alpha + \beta) = G(x, \alpha) + G(x, \beta), \quad x \in I,\ \alpha, \beta \in \mathbb{R}^{N(n,k)}.$$

Since $Sg = T(\exp(g))/\exp(g)$ is a continuous function, we have that $G(x, g(x), \cdots, g^{(k)}(x))$ is a continuous function of x for all $g \in C^k(I, \mathbb{R})$. By Theorem 2.6, applied with k instead of $k-1$ to any coordinate function $G_i : I \to \mathbb{R}$ of $G = (G_i)_{i=1}^n$ (with respect to the canonical unit vector basis of $\mathbb{R}^n$) separately, there is a continuous function $c : I \to L(\mathbb{R}^{N(n,k)}, \mathbb{R}^n)$ so that

$$G(x, \alpha) = c(x)(\alpha) = \sum_{l=0}^{k} c_l(x)(\alpha_l), \quad x \in I,\ \alpha = (\alpha_l)_{l=0}^k \in \mathbb{R}^{N(n,k)},$$

with direct sum splitting $c(x) = \sum_{l=0}^k c_l(x)$, $c_l \in L(\mathbb{R}^{M(n,l)}, \mathbb{R}^n)$. The direct sum splitting of c is a result of the coordinatewise application of Theorem 2.6.

For $f \in C^k(I, \mathbb{R})$ with $f > 0$, let $g := \ln f$. Then $f = \exp(g)$ and

$$Tf(x) = f(x)\ S(\ln f)(x) = f(x) \sum_{l=0}^{k} c_l(x)((\ln f)^{(l)}(x)). \tag{3.6}$$

Here the l-th derivative of $\ln f \in C^k(I, \mathbb{R})$ at x is identified with a vector in $\mathbb{R}^{M(n,l)}$. For $l \ge 2$, in the regular derivative sense

$$(\ln f)^{(l)}(x) = (\frac{f'}{f})^{(l-1)}(x) = (-1)^{l-1}(l-1)!(\frac{f'(x)}{f(x)})^l + P_l(f(x), \dots, f^{(l)}(x)),$$

where $f'(x)^l$ is the (tensor product) l-multilinear form

$$f'(x)^l(y_1, \dots, y_l) = \prod_{j=1}^{l} \langle f'(x), y_j \rangle, \quad y_1, \dots, y_l \in \mathbb{R}^n,$$

and P_l is a sum of quotients of terms containing powers of $f(x)$ of order $\le l-1$ in the denominator and tensor product terms of derivatives in the numerator. Therefore for $f(x) \searrow 0$, the order of singularity of $f(x)\ (\ln f)^{(l)}(x)$ is $\frac{f'(x)^l}{f(x)^{l-1}}$, if $f'(x) \ne 0$, up to terms of smaller growth. Since Tf is continuous and hence bounded on compact sets of I also for functions having zeros in I, in (3.6) we need $c_k(x) = \cdots = c_2(x) = 0$, $x \in I$. To be more precise, suppose that $k \ge 2$, that $x = 0 \in I$ for simplicity of notation and that the cube of side-length $\epsilon_0 > 0$ centered at 0 is contained in I. Choose any $b = (b_i)_{i=1}^n \in (\mathbb{R}_{>0})^n$ and consider $f(x) := \langle b, x\rangle$ and $I_\epsilon := \{x = (x_i)_{i=1}^n \in I \mid x_i > \frac{\epsilon}{2}, i \in \{1, \dots, n\}\}$ for any $0 < \epsilon < \epsilon_0$. Let $\mathbb{1} := (1)_{i=1}^n \in \mathbb{R}^n$. Then $f|_{I_\epsilon} \geq \frac{\epsilon}{2}\langle b, \mathbb{1}\rangle > 0$ and

$$\frac{\partial^l}{\partial x_{i_1} \cdots \partial x_{i_l}}(\ln f)(x) = (-1)^{l-1}(l-1)!\, \frac{\prod_{j=1}^l b_{i_j}}{\langle b, x\rangle^l}$$

for $x \in I_\epsilon$, $l \in \mathbb{N}$. Put $\psi_l(b) := (-1)^{l-1}(l-1)!\ (\prod_{j=1}^l\ b_{i_j})_{1 \leq i_1 \leq \cdots \leq i_l \leq n}$. Let $h \in C^k(I, \mathbb{R})$ be a smooth strictly positive extension of $f|_{I_\epsilon}$ to I. By localization, $Tf(\epsilon\mathbb{1}) = Th(\epsilon\mathbb{1})$ since $\epsilon\mathbb{1} \in I_\epsilon$. Applying (3.6) to h yields at the point $\epsilon\mathbb{1}$ with $h|_{I_\epsilon} = f|_{I\epsilon}$

$$\begin{aligned} Tf(\epsilon\mathbb{1}) = Th(\epsilon\mathbb{1}) &= c_0(\epsilon\mathbb{1})\ \langle b, \epsilon\mathbb{1}\rangle\ \ln(\langle b, \epsilon\mathbb{1}\rangle) \\ &\qquad + \sum_{l=1}^k\ c_l(\epsilon\mathbb{1})(\psi_l(b))\ \langle b, \epsilon\mathbb{1}\rangle^{-(l-1)}. \end{aligned}$$

Since Tf, $c_0, \dots, c_k$ are continuous at 0, we get for $\epsilon \to 0$ that $c_k(0)(\psi_k(b)) = 0$ for any $b \in (\mathbb{R}_{>0})^n$. This implies $c_k(0) = 0$. Recall that $c_k \in L(\mathbb{R}^{M(n,k)}, \mathbb{R}^n)$. If $k \geq 3$, we find successively in the same way $c_{k-1}(0) = 0, \dots, c_2(0) = 0$. Therefore $c_2 = 0, \dots, c_k = 0$ on I and hence

$$Tf(x) = c_0(x) f(x) \ln f(x) + c_1(x)(f'(x))$$

for positive C^k-functions f. Note here that $c_0(x)$ can be identified with a vector in $\mathbb{R}^n$ and $c_1(x) \in L(\mathbb{R}^n, \mathbb{R}^n)$. For general $f \in C^k(I, \mathbb{R})$, which may be also negative or zero, it has to be modified to

$$Tf(x) = c_0(x) f(x) \ln|f(x)| + c_1(x)(f'(x)).$$

This is shown similarly as in part (ii) of the proof of Theorem 3.1 by proving that T is odd, $T(-f) = -T(f)$. $\square$

3.3 An extended Leibniz rule

We study in this section some families of operator equations to which the Leibniz rule belongs. These families turn out to be very rigid, in the sense that they admit only very few "isolated" solutions, in our view a manifestation of the exceptional role which the derivative plays in analysis.

We return to functions of one variable. Looking at derivations from a more general point of view, we keep the operator $T : C^k(I) \to C(I)$, $k \in \mathbb{N}$, but replace the identity operation on the right-hand side of the Leibniz rule by some more general operators $A_1, A_2 : C^k(I) \to C(I)$ and study the solutions of the extended Leibniz rule operator equation

$$T(f \cdot g) = Tf \cdot A_1 g + A_2 f \cdot Tg, \quad f, g \in C^k(I).$$

Thus $A_1 = A_2 = \mathrm{Id}$ is the classical case of the Leibniz rule. Choosing $A_1 f = A_2 f = 1$ for all $f \in C^k(I)$ would result in the equation $T(f \cdot g) = Tf + Tg$ mapping products to sums, as the logarithm does on the positive reals. However, choosing $g = 0$, we conclude immediately that this equation only admits the trivial solution $T = 0$. Therefore, adding operators A_1, A_2 to the formula plays a "tuning" role, helping to create reasonable operators T which in some sense map products to sums on classical function spaces.

The maps A_1, A_2 should be rather different from T since, for $A_1 = A_2 = \frac{1}{2}T$, we would have the multiplicative equation $T(f \cdot g) = Tf \cdot Tg$, where bijective solutions $T : C^k(I) \to C^k(I)$ have a very different form, e.g., for $k = 0$, $Tf(x) = |f(u^{-1}(x))|^{p(x)} \{\mathrm{sgn}\, f(u^{-1}(x))\}$ where $u : I \to I$ is a homeomorphism, cf. Milgram [M], or for $k \in \mathbb{N}$, $Tf(x) = f(u^{-1}(x))$, where $u : I \to I$ is a diffeomorphism, cf. Mrčun, Šemrl [MS] or Artstein-Avidan, Faifman, Milman [AFM].

Though, for $A_1 = A_2 =: A$, the operators T and A are closely intertwined by the equation $T(f \cdot g) = Tf \cdot Ag + Af \cdot Tg$, there is more variability when solving an operator equation for two unknown operators. Typically we have to impose a weak assumption of "non-degeneration", to guarantee that the operators are localized and avoid examples like the above proportional one or the following:

Example. Define $T : C^k(\mathbb{R}) \to C(\mathbb{R})$ and $A : C^k(\mathbb{R}) \to C(\mathbb{R})$ by

$$Tf(x) := f(x) - f(x+1)\,, \quad Af(x) := \frac{1}{2}(f(x) + f(x+1)).$$

Then for all $f, g \in C^k(\mathbb{R})$, $T(f \cdot g) = Tf \cdot Ag + Af \cdot Tg$ since the mixed terms cancel. This means that both operators are not localized. Here for functions with small support $\mathrm{supp}\, f \subset (-\frac{1}{2}, \frac{1}{2})$, we have $Tf(x) = 2Af(x) = f(x)$ for all $x \in (-\frac{1}{2}, \frac{1}{2})$. To be able to prove localization, we have to avoid that T and A are "locally homothetic", i.e., homothetic on functions with small support. To exclude this type of "resonance" situation between T and A, we introduce the following condition for the pair (T, A).

Definition. Let $k \in \mathbb{N}$, $I \subset \mathbb{R}$ be an open set and $T, A : C^k(I) \to C(I)$ be operators. The pair (T, A) is C^k*-non-degenerate* if, for every open interval $J \subset I$ and $x \in J$, there are functions $g_1, g_2 \in C^k(I)$ with support in J such that $z_i := (Tg_i(x), Ag_i(x)) \in \mathbb{R}^2$ are linearly independent in $\mathbb{R}^2$ for $i = 1, 2$. We also assume that, for every $x \in \mathbb{R}$, there is $g \in C^k(\mathbb{R})$ with $Tg(x) = 0$ and $Ag(x) \neq 1$.

The first condition here is weaker than asking that T and A are not proportional.

We will assume a weak continuity assumption to simplify the proof of the main theorem.

Definition. For $k \in \mathbb{N}$, a map $A : C^k(I) \to C(I)$ is *pointwise continuous* provided that, for any sequence $(f_n)_{n\in\mathbb{N}}$ of $C^k(I)$-functions and $f \in C^k(I)$ such that $f_n^{(j)} \to f^{(j)}$ converge uniformly on all compact subsets of I for all $j \in \{0, \dots, k\}$, we have pointwise convergence $\lim_{n\to\infty} Af_n(x) = Af(x)$ for every $x \in I$.

We now state the main result for the extended Leibniz rule equation.

Theorem 3.7 (Extended Leibniz rule)**.** *Let $k \in \mathbb{N}_0$. Assume that $I \subset \mathbb{R}$ is an open interval and that $T, A_1, A_2 : C^k(I) \to C(I)$ are operators satisfying*

$$T(f \cdot g) = Tf \cdot A_1 g + A_2 f \cdot Tg, \quad f, g \in C^k(I). \tag{3.7}$$

Suppose that (T, A_1) are C^k-non-degenerate and that T, A_1 and A_2 are pointwise continuous. Then T, A_1 and A_2 are localized.

There are three possible families of solutions for T and A_1, A_2, given by the formulas below. They might be defined on disjoint subsets I_1, I_2 and I_3 of the interval I, being combined to yield a globally non-degenerate solution so that T and A_1, A_2 have ranges in the continuous functions on I.

More precisely, there are three pairwise disjoint subsets I_1, I_2, I_3 of I, one or two of them possibly empty, with I_2, I_3 open, such that $I = I_1 \cup I_2 \cup I_3$, and there are functions $a, d_0, \dots, d_k, p : I \to \mathbb{R}$ with $p > 0$ which are continuous on $I \setminus N$ where $N := \partial I_2 \cup \partial I_3$, and functions $\gamma \in C(I)$ and $q \in C(I_3)$ with $q > 0$ such that $A_1 - A_2 = 2\gamma T$ on $C^k(I)$, and putting $A := \frac{1}{2}(A_1 + A_2)$, we have for all $f \in C^k(I)$ and $x \in I_1$,

$$\begin{aligned} Tf(x) &= a(x)\Big(\sum_{l=0}^{k} d_l(x)\ (\ln|f|)^{(l)}(x)\Big)|f(x)|^{p(x)}\{\operatorname{sgn} f(x)\}, \\ Af(x) &= |f(x)|^{p(x)}\{\operatorname{sgn} f(x)\}, \end{aligned} \tag{3.8}$$

and for $x \in I_2$,

$$\begin{aligned} Tf(x) &= a(x)\ \sin\Big(\sum_{l=0}^{k} d_l(x)\ (\ln|f|)^{(l)}(x)\Big)\ |f(x)|^{p(x)}\{\operatorname{sgn} f(x)\}, \\ Af(x) &= \cos\Big(\sum_{l=0}^{k} d_l(x)\ (\ln|f|)^{(l)}(x)\Big)|f(x)|^{p(x)}\{\operatorname{sgn} f(x)\}, \end{aligned} \tag{3.9}$$

and for $x \in I_3$,

$$\begin{aligned} Tf(x) &= \frac{1}{2}a(x)\Big(|f(x)|^{p(x)}\{\operatorname{sgn} f(x)\} - |f(x)|^{q(x)}[\operatorname{sgn} f(x)]\Big), \\ Af(x) &= \frac{1}{2}\Big(|f(x)|^{p(x)}\{\operatorname{sgn} f(x)\} + |f(x)|^{q(x)}[\operatorname{sgn}(x)]\Big). \end{aligned} \tag{3.10}$$

The terms $\{\operatorname{sgn} f(x)\}$ *and* $[\operatorname{sgn} f(x)]$ *may be present in both formulas for* T *and* A *or not at all, yielding different solutions.*

The solution (3.8) *requires that* $p(x) \geq \max\{l \leq k \mid d_l(x) \neq 0\}$ *to guarantee that the range of* T *consists of continuous functions.*

In (3.10), $p(x) = 0$ *or* $q(x) = 0$ *are allowed, too, if the corresponding sign-terms do not occur.*

Conversely, let $A_1 := A + \gamma T$, $A_2 := A - \gamma T$ *where* T *and* A *are given by the above formulas. Then* (T, A_1, A_2) *satisfy* (3.7).

Remarks. (i) Theorem 3.7 shows that basically only three different types of combinations of operators (T, A_1, A_2) satisfying the extended Leibniz rule (3.7) are possible. For $k > 1$, the first one is similar to the one for positive functions in Corollary 3.4. Note that $(\ln|f|)^{(k)}|f|^p = a_k|f|^{p-k}(f')^k + Q_{k,p}$ where, for $p \geq k$, $Q_{k,p}$ is a polynomial in the function f and its derivatives, so that $Tf(x)$ is well defined by (3.8) for $p \geq p(x)$ (in the limit) also for functions f having zeros in x, and equation (3.8) provides the solution in this situation, too. In (3.8), Tf depends linearly on the highest derivative $f^{(k)}$, although with a factor which is a power of f, e.g., for $k = 2$, $Tf = ff'' - (f')^2, Af = f^2$.

(ii) For $k = 1$, the first solution is similar to the one of the Leibniz rule in Theorem 3.1, namely $Tf = c_0 f \ln|f| + c_1 f'$. Since (3.7) reminds of the addition formula for the sin-function when logarithmic arguments occur, the second solution is not surprising, cf. Proposition 2.13.

(iii) Note that only very few tuning operators A yield possible solutions of (3.7), and that they then determine the main operator T to a large extent. E. g. choosing A to be given by $Af = |f|^p\{\operatorname{sgn} f\}$, we get that Tf is a linear combination of terms $(\ln|f|)^{(l)}\ |f|^p\ \{\operatorname{sgn} f\}$.

(iv) The following example shows that the three solutions in Theorem 3.7 may be combined on different subintervals of I to form a non-degenerate solution.

Example. Let $I := (-1, 1)$ and $f \in C(I)$. Define maps T, A on $C(I)$ by

$$Tf(x) := \begin{cases} \frac{1}{x}\sin(x\ln|f(x)|)\ f(x), & x \in (-1,0), \\ \ln|f(x)|\ f(x), & x = 0, \\ \frac{1}{x}(|f(x)|^x - 1)\ f(x), & x \in (0,1), \end{cases}$$

$$Af(x) := \begin{cases} \cos(x\ln|f(x)|)\ f(x), & x \in (-1,0), \\ f(x), & x = 0, \\ \frac{1}{2}(|f(x)|^x + 1)\ f(x), & x \in (0,1). \end{cases}$$

On $I_1 := \{0\}$, the pair (T, A) has the form of the first solution (3.8), on $I_2 := (-1, 0)$ the form of the second solution (3.9) and on $I_3 := (0, 1)$ the form of the third solution (3.10). Note, however, that for $x \to 0$, $d(x) = x \to 0$, $p_3(x) - q(x) =$

$x \to 0$ and that $c_2(x) = c_3(x) = \frac{1}{x}$ have a singularity at 0. Nevertheless, Tf and Af define continuous functions on I since $\lim_{y\to 0} \frac{\sin(y)}{y} = 1$ and

$$\lim_{x\to 0} \frac{1}{x}(|f(x)|^x - 1) = \ln |f(x)| \text{ for } f(x) \neq 0.$$

For $f(x) = 0$, there is nothing to prove. Therefore T and A map $C(I)$ into $C(I)$ and satisfy (3.7). The solution is non-degenerate at zero: Just choose functions g_1, g_2 with small support and $g_1(0) = 3, g_2(0) = 2$. Then $(g_i(0) \ln g_i(0), g_i(0)) \in \mathbb{R}^2$ are linearly independent for $i = 1, 2$.

(v) It is also possible to combine the two solutions involving derivative terms, as the following example shows.

Example. Let $I := (-1, 1)$, $p > 1$ and $f \in C^1(I)$. Define maps T, A on $C^1(I)$ by

$$Tf(x) := \begin{cases} \frac{1}{x} \sin(x\frac{f'(x)}{f(x)}) \ |f(x)|^p, & x \in (-1, 0), \\ \frac{f'(x)}{f(x)} \ |f(x)|^p, & x \in [0, 1), \end{cases}$$

$$Af(x) := \begin{cases} \cos(x\frac{f'(x)}{f(x)}) \ |f(x)|^p, & x \in (-1, 0), \\ |f(x)|^p, & x \in [0, 1). \end{cases}$$

On $[0, 1)$, the solution is of the first type (3.8), with $(\ln |f|)' = \frac{f'}{f}$; it could be defined on $\mathbb{R}$ as well. But $p \geq 1$ is required here. On $(-1, 0)$, the solution is of the second type (3.9) and requires only $p > 0$ to yield continuous functions. For $x \to 0$, $d_1(x) = x$ tends to zero and $a(x) = 1/x$ has a singularity. This behavior is needed to join the other solution in a continuous way. We note that there is a delicate point about the continuity at zero. Both solutions are well defined for $p = 1$. However, choosing $p = 1$ does not yield a solution T with range in the continuous functions. Simply take $f(x) = x$. Then for $p = 1$, $Tf(x) = 1$ for $x \geq 0$ while $Tf(x) = \sin(1)$ for $x < 0$; Tf is not continuous at 0. However, for any $p > 1$, the range of T consists of continuous functions, since

$$|\frac{f'(x)}{f(x)}|f(x)|^p - \frac{1}{x} \sin(x\frac{f'(x)}{f(x)})|f(x)|^p \ | \leq 2|f(x)|^{p-1}|f'(x)|$$

as easily seen using $|\sin(t)| \leq |t|$, and this tends to zero as f tends to zero.

(vi) Let $S : C^k(I) \to C(I)$ satisfy the Leibniz rule and $M : C^k(I) \to C(I)$ be multiplicative. Then the pointwise product $T := S \cdot M : C^k(I) \to C(I)$ satisfies equation (3.7) with A being given by $A(f) := f \cdot M(f)$, $f \in C^k(I)$. The solution (3.8) is of this form.

Additional conditions will guarantee in the case $k = 1$ that the solutions have a simple form:

Corollary 3.8. *Assume that $T, A_1, A_2 : C^1(I) \to C(I)$ satisfy (3.7), with $k = 1$, $T \not\equiv 0$, and that (T, A_1) are C^1-non-degenerate and pointwise continuous. Let $A := \frac{1}{2}(A_1 + A_2)$. Suppose further that T maps $C^\infty(I)$ into $C^\infty(I)$.*
Then there are $n, m \in \mathbb{N}_0$ and a function $c \in C^\infty(I)$ such that the solution of (3.7) has one of the following two forms: either

$$Tf = c\ f'\ f^n, \quad Af = f^{n+1},$$

or

$$Tf = c\ (f^n - f^m), \quad Af = \frac{1}{2}(f^n + f^m),$$

for any $f \in C^1(I)$. If additionally $0 \in I$, $T2 = 0$ and $T(2\,\mathrm{Id}) = 2$, we have

$$Tf = f', \quad Af = f.$$

Corollary 3.9. *Assume that $T, A_1, A_2 : C^1(I) \to C(I)$ satisfy (3.7), with $k = 1$, $T \not\equiv 0$, and that (T, A_1) are C^1-non-degenerate and pointwise continuous. Let $A := \frac{1}{2}(A_1 + A_2)$. Suppose further that T maps linear functions into polynomials. Then there are $n, m \in \mathbb{N}_0$ and a polynomial function c such that the solution of (3.7) has one of the following two forms:*
either

$$Tf = c\ f'\ f^n, \quad Af = f^{n+1},$$

or

$$Tf = c\ (f^n - f^m), \quad Af = \frac{1}{2}(f^n + f^m),$$

for any $f \in C^1(I)$. If additionally $T2 = 0$ and $T(2\,\mathrm{Id}) = 2$, we have

$$Tf = f', \quad Af = f.$$

In both corollaries, there is $\gamma \in C(I)$ such that $A_1 = A + \gamma T$ and $A_2 = A - \gamma T$.
Note that the second solution in both corollaries may be extended to any $f \in C(I)$.

We now turn to the proof of Theorem 3.7. We again start by showing that T is localized.

Lemma 3.10. *Under the assumptions of Theorem 3.7, we have*

(i) *$T(0) = T(\mathbb{1}) = 0$ and $A_1(\mathbb{1}) = A_2(\mathbb{1}) = \mathbb{1}$.*

(ii) *If $J \subset I$ is open and $f_1, f_2 \in C^k(I)$ are such that $f_1|_J = f_2|_J$, then $(Tf_1)|_J = (Tf_2)|_J$ and $(A_i f_1)|_J = (A_i f_2)|_J$ for $i = 1, 2$.*

Proof. (i) Choosing $f = 0$ in (3.7), we find for any $x \in I$ and $g \in C^k(I)$

$$T(0)(x)\bigl(1 - A_1 g(x)\bigr) = A_2(0)(x) Tg(x).$$

By the C^k-non-degeneracy assumption, there is $g \in C^k(I)$ with $A_1 g(x) \neq 1$ and $Tg(x) = 0$. Hence, $T(0)(x) = 0$, $T(0) = 0$. Therefore, $0 = A_2(0)(x)Tg(x)$ for all $g \in C^k(I)$ which also yields $A_2(0) = 0$. Taking $g = 0$ in (3.7), we get

$$Tf(x) A_1(0)(x) = T(0)(x)\bigl(1 - A_2 f(x)\bigr) = 0,$$

for all $x \in I$, $f \in C^k(I)$. Hence also $A_1(0)$ =0.

Next, choose $f = 1$ in (3.7) to find

$$Tg(x)(1 - A_2(\mathbb{1})(x)) = T(\mathbb{1})(x) A_1 g(x), \quad x \in I,\ g \in C^k(I).$$

By C^k-non-degeneracy, there are functions $g_1, g_2 \in C^k(I)$ such that $(Tg_i(x), A_1 g_i(x)) \in \mathbb{R}^2$ are linearly independent for $i = 1, 2$. Therefore the previous equation with $g = g_1$ and $g = g_2$ implies $A_2(\mathbb{1}) = \mathbb{1}$, $T(\mathbb{1}) = 0$. Taking $g = \mathbb{1}$ in (3.7), we find similarly

$$Tf(x)(1 - A_1(\mathbb{1})(x)) = T(\mathbb{1})(x) A_2 f(x) = 0,$$

for all $f \in C^k(I)$. This yields $A_1(\mathbb{1}) = \mathbb{1}$.

(ii) Let $J \subset I$ be given and $f_1, f_2 \in C^k(I)$ with $f_1|_J = f_2|_J$. Let $g \in C^k(I)$ with $\operatorname{supp} g \subset J$. Then $f_1 \cdot g = f_2 \cdot g$. By (3.7)

$$\begin{aligned} Tf_1 \cdot A_1 g + A_2 f_1 \cdot Tg &= T(f_1 \cdot g) = T(f_2 \cdot g) \\ &= Tf_2 \cdot A_1 g + A_2 f_2 \cdot Tg, \\ \bigl(Tf_1(x) - Tf_2(x)\bigr) \cdot A_1 g(x) &= \bigl(A_2 f_2(x) - A_2 f_1(x)\bigr) \cdot Tg(x), \quad x \in I. \end{aligned}$$

For a given $x \in J$, choose $g_1, g_2 \in C^k(I)$ with support in J such that $(Tg_i(x), A_1 g_i(x)) \in \mathbb{R}^2$ are linearly independent for $i \in 1, 2$. The previous equation then yields for $g = g_1$ and $g = g_2$ that $Tf_1(x) = Tf_2(x)$, $A_2 f_1(x) = A_2 f_2(x)$, i.e., $Tf_1|_J = Tf_2|_J$, $A_2 f_1|_J = A_2 f_2|_J$. The argument for $A_1 f_1|_J = A_1 f_2|_J$ is similar. □

Proof of Theorem 3.7. (i) Assume that (T, A_1, A_2) satisfy the extended Leibniz rule (3.7). Then for all $f, g \in C^k(I)$ and $x \in I$, using the symmetry in f and g,

$$T(f \cdot g)(x) = Tf(x) A_1 g(x) + A_2 f(x) Tg(x) = Tg(x) A_1 f(x) + A_2 g(x) Tf(x),$$

hence $Tf(x)(A_1 g(x) - A_2 g(x)) = Tg(x)(A_1 f(x) - A_2 f(x))$. If $A_1 \not\equiv A_2$, there is $g \in C^k(I)$ and $x \in I$ such that $A_1 g(x) \neq A_2 g(x)$. Then $Tg(x) \neq 0$ since otherwise $Tf(x) = 0$ for all $f \in C^k(I)$ which would contradict the assumption of non-degeneration of (T, A_1), and therefore $A_1 f(x) - A_2 f(x) = 2\gamma(x) Tf(x)$ holds for all $f \in C^k(I)$, where $\gamma(x) := \frac{A_1 g(x) - A_2 g(x)}{2Tg(x)}$. Since $Tf, A_1 f, A_2 f$ are continuous functions, so is γ. Thus $A_1 - A_2 = 2\gamma T$. Clearly, $A_1 f(x) = A_2 f(x)$ is possible for some x or all $x \in I$, with $\gamma(x) = 0$. Put $A := \frac{1}{2}(A_1 + A_2)$. Then $A_1 = A + \gamma T$, $A_2 = T - \gamma T$. Equation (3.7) holds for (T, A) if A_1 and A_2 are replaced by the one operator A.

In the following, we write equation (3.7) with T and A and analyze the structure of these operators.

(ii) By Lemma 3.10 and Proposition 3.3 there are functions $\widetilde{F}, \widetilde{B} : I \times \mathbb{R}^{k+1} \to \mathbb{R}$ such that for all $f \in C^k(I)$ and $x \in I$

$$Tf(x) = \widetilde{F}(x, f(x), \ldots, f^{(k)}(x)), \quad Af(x) = \widetilde{B}(x, f(x), \ldots, f^{(k)}(x)).$$

We introduce operators $S, R : C^k(I) \to C(I)$ by $Sh := T(\exp h)$, $Rh := A(\exp h)$ for all $h \in C^k(I)$. Since the derivatives of $\exp h$ of order l can be written as a function of h and its derivatives of order $\leq l$, the operators S and R are localized as well, i.e., there exist functions $F, B : I \times \mathbb{R}^{k+1} \to \mathbb{R}$ such that for all $h \in C^k(I)$ and $x \in I$

$$Sh(x) = F(x, h(x), \ldots, h^{(k)}(x)), \quad Rh(x) = B(x, h(x), \ldots, h^{(k)}(x)).$$

Equation (3.7) yields for $h_1, h_2 \in C^k(I)$

$$\begin{aligned} S(h_1 + h_2) = T(\exp h_1 \exp h_2) &= T(\exp h_1)A(\exp h_2) + A(\exp h_1)T(\exp h_2) \\ &= S(h_1)R(h_2) + R(h_1)S(h_2). \end{aligned} \tag{3.11}$$

Let $\alpha = (\alpha_j)_{j=0}^k$, $\beta = (\beta_j)_{j=0}^k \in \mathbb{R}^{k+1}$ and $x \in I$ be arbitrary. Choose $h_1, h_2 \in C^k(I)$ with $h_1^{(j)}(x) = \alpha_j$ and $h_2^{(j)}(x) = \beta_j$ for all $j \in \{0, \ldots, k\}$. Then the operator equation (3.11) is equivalent to the functional equation for F and B

$$F(x, \alpha + \beta) = F(x, \alpha)B(x, \beta) + F(x, \beta)B(x, \alpha) \tag{3.12}$$

for all $\alpha, \beta \in \mathbb{R}^{k+1}$, $x \in I$.

We claim that for any fixed $x \in I$, $B(x, \cdot)$ and $F(x, \cdot)$ are continuous functions on $\mathbb{R}^{k+1}$. To verify this, take a sequence $\alpha_n = (\alpha_{n,j})_{j=0}^k \in \mathbb{R}^{k+1}$ and $\alpha \in \mathbb{R}^{k+1}$ such that $\alpha_n \to \alpha$ in $\mathbb{R}^{k+1}$. Consider the functions $h_n(t) := \sum_{j=0}^k \frac{\alpha_{n,j}}{j!}(t - x)^j$, $h(t) := \sum_{j=0}^k \frac{\alpha_j}{j!}(t - x)^j$. Then $h_n^{(l)} \to h^{(l)}$ and $f_n := \exp(h_n)^{(l)} \to f := \exp(h)^{(l)}$ converge uniformly on all compact subsets of I for any $l \in \{0, \ldots, k\}$. By the assumption of pointwise continuity , we have $Af_n(x) \to Af(x)$ and $Tf_n(x) \to Tf(x)$ for all $x \in I$. This means

$$\begin{aligned} B(x, \alpha_{n,0}, \ldots, \alpha_{n,k}) &= Af_n(x) \to Af(x) = B(x, \alpha_0, \ldots, \alpha_k), \\ F(x, \alpha_{n,0}, \ldots, \alpha_{n,k}) &= Tf_n(x) \to Tf(x) = F(x, \alpha_0, \ldots, \alpha_k). \end{aligned}$$

Therefore for all $x \in I$, $B(x, \cdot)$ and $F(x, \cdot)$ are continuous functions which satisfy (3.12). The solutions of (3.12) were studied in Chapter 2, Corollary 2.12.

(iii) We now determine the form of Tf and Af for strictly positive functions $f > 0$. By Corollary 2.12 for $n = k+1$ there are vectors $b(x), c(x), d(x) \in \mathbb{R}^{k+1}$ and $a(x) \in \mathbb{R}$ such that $F(x, \cdot)$ and $B(x, \cdot)$ have one of the following forms

(a) $F(x,\alpha) = \langle b(x), \alpha\rangle \exp(\langle c(x), \alpha\rangle)$, $B(x,\alpha) = \exp(\langle c(x), \alpha\rangle)$;

(b) $F(x,\alpha) = a(x)\ \exp(\langle c(x), \alpha\rangle)\ \sin(\langle d(x), \alpha\rangle)$,
$B(x,\alpha) = \exp(\langle c(x), \alpha\rangle)\ \cos(\langle d(x), \alpha\rangle)$;

(c) $F(x,\alpha) = a(x)\ \exp(\langle c(x), \alpha\rangle)\ \sinh(\langle d(x), \alpha\rangle)$,
$B(x,\alpha) = \exp(\langle c(x), \alpha\rangle)\ \cosh(\langle d(x), \alpha\rangle)$;

(d) $F(x,\alpha) = a(x)\ \exp(\langle c(x), \alpha\rangle)$, $B(x,\alpha) = \frac{1}{2}\exp(\langle c(x), \alpha\rangle)$, $\alpha \in \mathbb{R}^n$.

Since $A(\mathbb{1}) = \mathbb{1}$ by Lemma 3.10, $1 = A(\mathbb{1})(x) = R(0)(x) = B(x, 0)$. Therefore the last case (d) is impossible here since in that case $B(x, 0) = \frac{1}{2}$.

For positive functions $f \in C^k(I)$, $f > 0$, let $h := \ln f$, $f = \exp h$, so that in case (a) with $b = (b_l)_{l=0}^k$, $c = (c_l)_{l=0}^k$

$$\begin{aligned} Af(x) &= R(\ln f)(x) = B(x, (\ln f)(x), \dots, (\ln f)^{(k)}(x)) \\ &= \exp\Big(\sum_{l=0}^{k} c_l(x)(\ln f)^{(l)}(x)\Big), \\ Tf(x) &= S(\ln f)(x) = F(x, (\ln f)(x), \dots, (\ln f)^{(k)}(x)) \\ &= \Big(\sum_{l=0}^{k} b_l(x)(\ln f)^{(l)}(x)\Big)\ \exp\Big(\sum_{l=0}^{k} c_l(x)(\ln f)^{(l)}(x)\Big). \end{aligned} \tag{3.13}$$

Depending on $x \in I$, one of the formulas (a), (b) or (c) might apply. Let I_1, I_2 and I_3, respectively, denote the subsets of I where $Tf(x)$, $Af(x)$ is determined by (a), (b) and (c), respectively. For (a) and $f > 0$, we just wrote down the formulas in (3.13). However, the sets are restricted by the requirement that Tf and Af have to be continuous functions for all $f \in C^k(I)$. Suppose that the interior of the domain I_1 where (3.13) gives the solution – for $f > 0$ – is not empty. Let us show that the functions $c_0, \dots, c_k$ and $b_0, \dots, b_k$ have to be continuous in the interior of I_1. Indeed, starting with constant functions f, the continuity of Af and Tf yields that c_0 and b_0 are continuous. Then choosing linear functions, it follows that c_1 and b_1 are continuous. Repeat the argument with polynomials of successively higher degree.

Since T and A are localized and have to be well defined also for functions having zeros in the interior of I_1, the formula for Af should never become singular, i.e., unbounded when $f \searrow 0$. The argument for this is exactly the same as in the proof of Theorem 3.1. However, $(\ln f)^{(l)}$ is of order $(\frac{f'}{f})^l$, if $f' \neq 0$ and $l \in \mathbb{N}$, up to terms of smaller order. Therefore we must have $c_1 = \cdots = c_k = 0$ in (3.13) on I_1. Put $p(x) := c_0(x)$. Then for $f > 0$, $x \in I_1$,

$$Af(x) = f(x)^{p(x)}, \quad Tf(x) = \Big(\sum_{l=0}^{k} b_l(x)(\ln f)^{(l)}(x)\Big) \; f(x)^{p(x)}. \tag{3.14}$$

The continuity of Tf for all f requires that $p(x) \geq \max\{l \leq k \mid b_l(x) \neq 0\} =: P(x)$. If $P(x) = 0$, we need $p(x) > 0$. In this case, (3.14) provides a solution of (3.7) for positive f.

The case (b) for T and A on I_2 yields the formula

$$Af(x) = \exp\Big(\sum_{l=0}^{k} c_l(x)(\ln f)^{(l)}(x)\Big) \; \cos\Big(\sum_{l=0}^{k} d_l(x)(\ln f)^{(l)}(x)\Big),$$

with continuous coefficient functions c_l, d_l on I_2. Continuity for functions with zeros requires that $c_1 = \cdots = c_k = 0$. Then with $p(x) := c_0(x)$, for $f > 0$, $x \in I_2$,

$$\begin{aligned} Af(x) &= \cos\Big(\sum_{l=0}^{k} d_l(x)(\ln f)^{(l)}(x)\Big) \; f(x)^{p(x)}, \\ Tf(x) &= a(x)\sin\Big(\sum_{l=0}^{k} d_l(x)(\ln f)^{(l)}(x)\Big) \; f(x)^{p(x)}, \end{aligned} \tag{3.15}$$

where $p(x) > 0$ is required and a is continuous in I_2. In the last case (c)

$$Af(x) = \exp\Big(\sum_{l=0}^{k} c_l(x)(\ln f)^{(l)}(x)\Big) \; \cosh\Big(\sum_{l=0}^{k} d_l(x)(\ln f)^{(l)}(x)\Big),$$

and here necessarily $c_1 = \cdots = c_k = 0$ and $d_1 = \cdots = d_k = 0$. Then with $p(x) := c_0(x) + d_0(x)$ and $q(x) := c_0(x) - d_0(x)$, $Af(x) = \frac{1}{2}(f(x)^{p(x)} + f(x)^{q(x)})$, $p(x) \geq 0$, $q(x) \geq 0$, yielding for $f > 0$, $x \in I_3$

$$Af(x) = \frac{1}{2}\Big(f(x)^{p(x)} + f(x)^{q(x)}\Big), \quad Tf(x) = a(x)\Big(f(x)^{p(x)} - f(x)^{q(x)}\Big). \tag{3.16}$$

To be non-degenerate, the solution on I_2 given by (3.15) requires that some of the continuous functions d_l are non-zero at any $x \in I_2$, and the one on I_3 given (3.16) requires that $p(x) \neq q(x)$ for any $x \in I_3$. They can be joined to another one of the three solutions only when the d_l or $p - q$ tend to zero and at the same time $|a|$ becomes unbounded. Hence, by continuity of the parameter functions, the subsets I_2 and I_3 are open. Of course, any of the sets I_1, I_2 or I_3 could be empty; the solution may be given on all of I by just one of the formulas, this being the most natural case. However, I_1 is not necessarily open. In the first example in Remark (ii) after Theorem 3.7 we had $I_1 = \{0\}$.

(iv) It remains to determine the formulas for $Tf(x)$ and $Af(x)$ when $f \in C^k(I)$ is negative or zero. Since Af and Tf are continuous and the coefficient functions are continuous on their domains, the localized formulas (3.14), (3.15),(3.16) extend by continuity to $Tf(x)$ and $Af(x)$ when $f(x) = 0$ and x is an isolated zero of f or a limit of isolated zeros. If $x \in J \subset I$, J open and $f|_J = 0$, we know by Lemma 3.10 that $Tf(x) = 0$.

Suppose now that $f \in C^k(I)$ and $x \in I$ are such that $f(x) < 0$. We may assume that $f < 0$ on the full set I, since $Tf(x)$ and $Af(x)$ are determined locally near x with $f(x) < 0$. For constant functions $f(x) = \alpha_0$, $g(x) = \beta_0$, we have

$$Tf(x) = \widetilde{F}(x, \alpha_0, 0, \ldots, 0), \quad Af(x) = \widetilde{B}(x, \alpha_0, 0, \ldots, 0).$$

Therefore the extended Leibniz rule (3.7) yields

$$\begin{aligned}\widetilde{F}(x, \alpha_0\beta_0, 0, \ldots, 0) = \widetilde{F}(x, \alpha_0, 0, \ldots, 0)&\widetilde{B}(x, \beta_0, 0, \ldots, 0) \\ &+ \widetilde{B}(x, \alpha_0, 0, \ldots, 0)\widetilde{F}(x, \beta_0, 0, \ldots, 0).\end{aligned}$$

Proposition 2.13 gives the possible solutions of this functional equation. They imply for constant functions f having negative values, too, that one of the following three cases can occur:

$$\begin{aligned}&Tf = b(\ln|f|)|f|^p\{\operatorname{sgn} f\}, \quad Af = |f|^p\{\operatorname{sgn} f\}, \\ &Tf = b\sin(d\ln|f|)|f|^p\{\operatorname{sgn} f\}, \quad Af = \cos(d\ln|f|)|f|^p\{\operatorname{sgn} f\}, \\ &Tf = b(|f|^p\{\operatorname{sgn} f\} - |f|^q[\operatorname{sgn} f]), \quad Af = \frac{1}{2}(|f|^p\{\operatorname{sgn} f\} + |f|^q[\operatorname{sgn} f]),\end{aligned}$$

leaving out the variable x. The fourth solution in Proposition 2.13 is not applicable since there $B(\mathbb{1}) = \frac{1}{2} \neq 1$.

In the first two cases and in the last case when both $\operatorname{sgn} f$-terms are present or both are absent, we have $T(-\mathbb{1}) = 0$ and $A(-\mathbb{1}) \in \{\mathbb{1}, -\mathbb{1}\}$. Then by (3.7), $T(-f) = Tf\ A(-\mathbb{1}) + Af\ T(-\mathbb{1}) = Tf\ A(-\mathbb{1})$. Hence T is even or odd, depending on whether $A(-\mathbb{1}) = 1$ or $A(-\mathbb{1}) = -1$. For A, we have similarly $A(-f) = Af\ A(-\mathbb{1})$, by the same arguments as in the proof of Proposition 2.13. In the last case, when the $\operatorname{sgn} f$-terms are different, T and A are neither even nor odd. The determination of $T(-f)$ and $A(-f)$ in this case is similar to the last case in the proof of Proposition 2.13. Using this, formulas (3.14), (3.15) and (3.16) yield formulas (3.8), (3.9) and (3.10) in Theorem 3.7 for general functions $f \in C^k(I)$.

Conversely, the operators T and A defined by these formulas satisfy (3.7). To check this, e.g., in the case of (3.9), use the addition formula for the sin-function and $(\ln|fg|)^{(l)} = (\ln|f|)^{(l)} + (\ln|g|)^{(l)}$. This ends the proof of Theorem 3.7. $\square$

Proof of Corollary 3.8. The operator T defined by (3.9) does not map C^∞-functions to C^∞-functions, since – possibly large order – derivatives of Tf will become singular in points where f has zeros. The operator given by (3.8) for $k = 1$ has the form

$$Tf = (bf\ln|f| + af')\ |f|^q\{\operatorname{sgn} f\},$$

$q = p-1$. Choosing for f constant or linear functions, we conclude that $a, b, q \in C^\infty$ is required. Since $|f|^q\{\operatorname{sgn} f\}$ has to be a C^∞-function for any C^∞-function f, we moreover need that $|f|^q\{\operatorname{sgn} f\} = f^n$ for a suitable $n \in \mathbb{N}_0$. If b would not be zero, a suitable derivative of Tf would have a singularity of order $\ln|f|$ when $|f| \searrow 0$. Hence $Tf = af' f^n$ in the case of (3.8). Similarly, the solution (3.10) maps C^∞-functions into C^∞-functions if and only if $Tf = a(f^n - f^m)$ for suitable $n, m \in \mathbb{N}_0$ and $a \in C^\infty$. Both solutions cannot be combined on disjoint subsets partitioning I since f' cannot be continuously approximated by differences $f^N - f^M$, in general. Therefore we have two solutions defined on the full set I.

If additionally $T2 = 0$, the second solution would require $n = m$ and then $T \equiv 0$. Thus only the first solution is possible, with $2 = T(2\,\mathrm{Id})(x) = a(x)2^{n+1}x^n$, i.e., $a(x) = (2x)^{-n}$. Since $x = 0 \in I$ and $a \in C^\infty(I)$, it follows that $n = 0$ and $a \equiv \mathbb{1}$, i.e., $Tf = f'$ and $Af = f$ for all $f \in C^1(I)$. □

Proof of Corollary 3.9. The operator T defined by (3.9) does not map arbitrary linear functions $f(x) = cx$, $c \in \mathbb{R}$ to polynomials, if $T \not\equiv 0$. In the case of (3.8), T again has the form

$$Tf = (bf \ln|f| + af')\ |f|^q\{\operatorname{sgn} f\}.$$

This will not yield polynomials for all linear functions f unless $b \equiv 0$, $q = n \in \mathbb{N}_0$ and a is a polynomial function, i.e., $Tf = af' f^n$, $Af = f^{n+1}$ for all $f \in C^1(I)$.

Again, (3.10) yields the second solution with $p = n, q = m \in \mathbb{N}_0$.

If additionally $T2 = 0$, the second solution requires $n = m$, i.e., $T \equiv 0$. In the case of the first solution $T(2\,\mathrm{Id}) = 2$ gives $2 = T(2\,\mathrm{Id})(x) = a(x)2^{n+1}x^n$, i.e., $a(x) = (2x)^{-n}$. However, a is only a polynomial if $n = 0$, $a \equiv \mathbb{1}$. Then $Tf = f'$ and $Af = f$ for all $f \in C^1(I)$. □

3.4 Notes and References

The basic result on the Leibniz rule equation, Theorem 3.1, is due to König, Milman [KM1]. The case $k = 0$ was shown before by Goldmann, Šemrl [GS].

Lemma 3.2 and Proposition 3.3 are taken from [KM1]. For $k = 1$, Theorems 3.5 and 3.7 were shown in [KM1], too.

The logarithm $F = \log$ satisfies $F(xy) = F(x) + F(y)$ for positive $x, y > 0$. However, there do not exist a function $F : \mathbb{R} \to \mathbb{R}$ and constants $c, d \in \mathbb{R}$ such that $F(xy) = cF(x) + dF(y)$ holds for all real numbers $x, y \in \mathbb{R}$. A function of this type sending products to sums requires replacing the constants c, d by functions, yielding in the simplest case the Leibniz rule in $\mathbb{R}$. On the real line $\mathbb{R}$ or the complex plane $\mathbb{C}$, there is the following version of the Leibniz rule:

Proposition 3.11. (a) *Let $F : \mathbb{R} \to \mathbb{R}$ be a measurable function satisfying*

$$F(xy) = F(x)y + xF(y), \quad x, y \in \mathbb{R}. \tag{3.17}$$

Then there is $d \in \mathbb{R}$ such that $F(x) = d\,x \ln|x|$, $x \in \mathbb{R}$.

(b) *Let $F : \mathbb{C} \to \mathbb{C}$ be a measurable function satisfying*

$$F(zw) = F(z)w + zF(w), \quad z, w \in \mathbb{C}.$$

Then there is $d \in \mathbb{C}$ such that $F(z) = d\, z \ln|z|$, $z \in \mathbb{C}$.

Proof. (a) $F(1) = F(1^2) = 2F(1)$ implies $F(1) = 0$. Similarly $F(-1) = 0$, which implies $F(-x) = -F(x)$. For $xy \neq 0$,

$$\frac{F(xy)}{xy} = \frac{F(x)}{x} + \frac{F(y)}{y}.$$

Hence, $H(s) := F(e^s)/e^s$ is measurable and additive. By Proposition 2.1 there is $d \in \mathbb{R}$ with $H(s) = ds$. Then

$$F(x) = dx \ln|x|.$$

(b) We show by induction on n that for any $n \in \mathbb{N}$ and $z \in \mathbb{C}$, $F(z^n) = nz^{n-1}F(z)$: For $n = 2$ this is the assumption with $z = w$. Assuming this for n, we have $F(z^{n+1}) = F(z^n)z + z^nF(z) = (n+1)z^nF(z)$. Let $\zeta \in \mathbb{C}$ be an n-th root of unity. Then $0 = F(1) = F(\zeta^n) = n\zeta^{n-1}F(\zeta)$ implies that $F(\zeta) = 0$. Define $G(z) := \frac{F(z)}{z}$ for $z \in \mathbb{C} \setminus \{0\}$. Then $G(zw) = G(z) + G(w)$ for all $z, w \in \mathbb{C} \setminus \{0\}$. Hence $\phi : \mathbb{R} \to \mathbb{C}$ given by $\phi(t) := G(\exp(it))$, $t \in \mathbb{R}$, is additive and measurable. By Proposition 2.1 there is $c \in \mathbb{C}$ such that $\phi(t) = ct$ for all $t \in \mathbb{R}$. Since $F(\zeta) = 0$ for all roots of unity ζ, $c = 0$, i.e., $G|_{S^1} = 0$. The polar decomposition of $z \in \mathbb{C} \setminus \{0\}$, $z = |z|\exp(it)$ yields that $G(z) = G(|z|) + G(\exp(it)) = G(|z|)$ and for $z, w \in \mathbb{C} \setminus \{0\}$, $G(|zw|) = G(|z|) + G(|w|)$. Similarly as in part (a) we find $d \in \mathbb{C}$ such that $G(z) = G(|z|) = d\ln|z|$. Hence $F(z) = dz\ln|z|$ for all $z \in \mathbb{C} \setminus \{0\}$. Clearly $F(0) = 0$. □

Remark. The equation

$$F(xy) = F(x)B(y) + B(x)F(y), \quad x, y \in \mathbb{R} \tag{3.18}$$

for unknown functions $F, B : \mathbb{R} \to \mathbb{R}$ is a relaxation of equation (3.17). Proposition 2.13 gives the four (real) solutions of (3.18). The first of these, $B(x) = |x|^d\{\operatorname{sgn} x\}$, $F(x) = b \cdot \ln|x| \cdot B(x)$ has the property that B has a smaller order of growth as $|x| \to \infty$ than F. Comparing this with the operator functional equation (3.7),

$$T(f \cdot g) = Tf \cdot A_1 g + A_2 f \cdot Tg, \quad f, g \in C^k(I),$$

which has an algebraically similar form, the first solution of (3.7) has the property that $A = A_1 = A_2$ has a smaller order of differentiability than T.

We may also consider the Leibniz rule on real or complex spaces of polynomials or analytic functions. For $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, let $\mathcal{P}(\mathbb{K})$ denote the space of polynomials with coefficients in $\mathbb{K}$ and $\mathcal{E}(\mathbb{K})$ be the space of real-analytic functions ($\mathbb{K} = \mathbb{R}$)

or entire functions ($\mathbb{K} = \mathbb{C}$) and $C(\mathbb{K})$ be the space of continuous functions on $\mathbb{K}$. Moreover, let $\mathcal{P}_n(\mathbb{K})$ be the subset of $\mathcal{P}(\mathbb{K})$ consisting of polynomials of degree $\leq n$.

On these spaces, there are different solutions of the Leibniz rule than those given in Theorem 3.1.

Example 1. Define $T : \mathcal{P}(\mathbb{K}) \to \mathcal{P}(\mathbb{K})$ by $Tf := \deg f \cdot f$, $f \in \mathcal{P}(\mathbb{K})$, where $\deg f$ denotes the degree of the polynomial f. Since $\deg(f \cdot g) = \deg f + \deg g$, T satisfies the Leibniz rule $T(f \cdot g) = Tf \cdot g + f \cdot Tg$ on $\mathcal{P}(\mathbb{K})$.

Example 2. Fix $x_0 \in \mathbb{K}$. For $f \in \mathcal{E}(\mathbb{K})$, let $n(f)$ denote the order of zero of f in x_0 (which may be zero if $f(x_0) \neq 0$). Define $T : \mathcal{E}(\mathbb{K}) \to \mathcal{E}(\mathbb{K})$ by $Tf := n(f) \cdot f$. Since $n(f \cdot g) = n(f) + n(g)$, T satisfies the Leibniz rule $T(f \cdot g) = Tf \cdot g + f \cdot Tg$ on $\mathcal{E}(\mathbb{K})$.

However, in both examples the operator T is not pointwise continuous in the sense that there are functions $f_m, f \in \mathcal{P}(\mathbb{K})$ or $\mathcal{E}(\mathbb{K})$ where $f_m \to f$ converges uniformly on compact sets but where $Tf_m(x)$ does not converge to $Tf(x)$ for some $x \in \mathbb{K}$, since the degree and the order of zero are not pointwise continuous operations. Let us therefore assume that $T : \mathcal{P}(\mathbb{K}) \to C(\mathbb{K})$ is pointwise continuous and satisfies the Leibniz rule. Does this guarantee that we have the same solutions as in Theorem 3.1? Again the answer is negative, as the following example due to Faifman [F3] shows:

Example 3 (Faifman).. If $T : \mathcal{P}(\mathbb{K}) \to C(\mathbb{K})$ satisfies the Leibniz rule $T(f \cdot g) = Tf \cdot g + f \cdot Tg$ for all $f, g \in \mathcal{P}(\mathbb{K})$, then for all $f_1, \ldots, f_n \in \mathcal{P}(\mathbb{K})$

$$T(\prod_{j=1}^{n} f_j) = \sum_{j=1}^{n} (\prod_{i=1, i\neq j}^{n} f_i)\; Tf_j. \tag{3.19}$$

Let us first consider the complex case $\mathbb{K} = \mathbb{C}$. Since any polynomial $f \in \mathcal{P}(\mathbb{C})$ factors as a product of linear terms, $f(z) = a\prod_{j=1}^{n}(z - z_j)$, with zeros $z_j \in \mathbb{C}$ and $a \in \mathbb{C} \setminus \{0\}$, it suffices to define $T(az + b)$, in order to define an operator $T : \mathcal{P}(\mathbb{C}) \to C(\mathbb{C})$ by applying (3.19), and then verify that this map T actually satisfies the Leibniz rule. Let $\phi : \mathbb{C} \to \mathbb{C}$ be given by $\phi(z) := z \ln |z|$, with $\phi(0) = 0$. Define

$$T(az + b) := \phi(a)z + \phi(b). \tag{3.20}$$

This map T satisfies the Leibniz rule on $\mathcal{P}_1(\mathbb{C})$ in the sense that $T(c(az + b)) = T(c)(az + b) + cT(az + b)$, since ϕ satisfies the Leibniz rule on $\mathbb{C}$. In terms of the elementary symmetric polynomials we have for $f \in \mathcal{P}_n(\mathbb{C})$

$$f(z) = a\prod_{j=1}^{n}(z - z_j) = \sum_{k=0}^{n}(-1)^k\; (\sum_{1\leq j_1<\cdots<j_k\leq n} a z_{j_1} \cdots z_{j_k})\; z^{n-k}. \tag{3.21}$$

Using (3.20) and requiring that (3.19) holds, yields the formula for $T : \mathcal{P}(\mathbb{C}) \to C(\mathbb{C})$

$$(Tf)(z) = \sum_{k=0}^{n} (-1)^k \ \Big(\sum_{1 \le j_1 < \cdots < j_k \le n} \phi(a z_{j_1} \cdots z_{j_k}) \Big) \ z^{n-k}, \tag{3.22}$$

as induction on $n \in \mathbb{N}$ shows. Conversely, one checks that the operator T defined by (3.22) satisfies the Leibniz rule, using once more that ϕ satisfies it on $\mathbb{C}$. Moreover, this operator T is pointwise continuous on $\mathcal{P}(\mathbb{C})$, i.e., for any $f_m, f \in \mathcal{P}(\mathbb{C})$ with $f_m \to f$ uniformly on compact sets, we have $Tf_m(z) \to Tf(z)$ for any $z \in \mathbb{C}$, since the zeros depend continuously on the polynomials (in appropriate order) and ϕ is continuous. We remark that the pointwise continuity statement also holds, if $\deg f < \liminf_{m\to\infty} \deg f_m$.

Real polynomials $f \in \mathcal{P}(\mathbb{R})$ may be factored into linear and irreducible quadratic factors, the latter corresponding to two complex conjugate zeros. Applying the Leibniz rule (in $\mathbb{C}$) to such factors yields the real variable requirement for T

$$T(x^2 + px + q) = \frac{1}{2}(p \ln |q|)x + q \ln |q|, \quad p^2 < 4q.$$

Using this together with (3.20) and (3.19) then defines a pointwise continuous operator $T : \mathcal{P}(\mathbb{R}) \to C(\mathbb{R})$ satisfying the Leibniz rule. In both cases $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, the image of T is actually again in $\mathcal{P}(\mathbb{K})$.

The question whether pointwise continuous operators $T : \mathcal{E}(\mathbb{K}) \to C(\mathbb{K})$ on the space of entire functions satisfying the Leibniz rule are of the same form as in Theorem 3.1 is open. The previous example does not extend to the space of entire functions $\mathcal{E}(\mathbb{K})$ since the (polynomial) functions given by $f_m(z) = (1 + \frac{z}{m})^m$ tend to $f(z) = \exp(z)$ uniformly on compact sets, but $Tf_m(z) = -z(1 + \frac{z}{m})^{m-1} \ln m$ for fixed $z \neq 0$ is a divergent sequence.

The extended Leibniz rule which was investigated in Theorem 3.7 in the space $C^k(I)$ may also be studied in the Schwartz space of complex-valued rapidly decreasing functions $\mathcal{S}(\mathbb{R}, \mathbb{C})$. The operator solutions Af are then expressed by integer powers of the functions f and their complex conjugates, and the images Tf are linear combinations of logarithmic derivatives of f and its complex conjugate or a difference of powers of f and its complex conjugate. We refer to König, Milman [KM13], where also criteria are given such that A is the identity and T the derivative. The extended Leibniz rule in $\mathcal{S}(\mathbb{R}, \mathbb{C})$ has applications to joint characterizations of the Fourier transform and the derivative [KM13].