Chapter 2

Regular Solutions of Some Functional Equations

The derivative D is an operator which acts as a map from the continuously differentiable functions $C^1(\mathbb{R})$ on $\mathbb{R}$ to the continuous functions $C(\mathbb{R})$. It satisfies the Leibniz and the chain rule

$$D(f \cdot g) = Df \cdot g + f \cdot Dg,$$
$$D(f \circ g) = (Df) \circ g \cdot Dg, \qquad f, g \in C^1(\mathbb{R}).$$

In this book, we show that operators $T : C^1(\mathbb{R}) \to C(\mathbb{R})$ obeying either the Leibniz or the chain rule operator equation

$$T(f \cdot g) = Tf \cdot g + f \cdot Tg, \tag{2.1}$$
$$T(f \circ g) = (Tf) \circ g \cdot Tg, \qquad f, g \in C^1(\mathbb{R}) \tag{2.2}$$

are close to the standard derivative. Actually, we completely establish the form of the solutions of either equation. We also consider more general operator equations modeling second-order derivatives or the Laplacian. Only very mild conditions are imposed on the map T.

The basic question mentioned already in the introduction is: Are classical operators in analysis like differential operators characterized by very simple properties such as (2.1) or (2.2), and additional initial conditions, e.g., $T(-2\,\mathrm{Id}) = -2$?

Chapters 3 and 4 will be devoted to determine and describe all solutions of either equation (2.1) or (2.2). The first step in solving equations like (2.1) and (2.2) is to show that the operator T is *localized*, i.e., that there is a function $F : \mathbb{R}^3 \to \mathbb{R}$, such that

$$Tf(x) = F\big(x, f(x), f'(x)\big), \quad f \in C^1(\mathbb{R}),\ x \in \mathbb{R}.$$

At this point, the function F and its possible regularity is unknown, but the operator equation for T translates into a functional equation for F, in the above

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cases into either

$$F(x,\alpha_0\beta_0,\alpha_1\beta_0+\alpha_0\beta_1)=F(x,\alpha_0,\alpha_1)\beta_0+F(x,\beta_0,\beta_1)\alpha_0,$$

or

$$F(x,z,\alpha_1\beta_1)=F(y,z,\alpha_1)F(x,y,\beta_1),$$

for all $x,y,z,\alpha_0,\beta_0,\alpha_1,\beta_1\in\mathbb{R}$.

Functional equations, of course, are a classic subject, and there is a vast literature on the topic, cf., e.g., the books of Aczél [A], Aczél, Dhombres [AD], Járai [J], Székelyhidi [Sz] or the recent book by Rassias, Thandapani, Ravi, Senthil Kumar [RTRS]. Much less is known about the operator equations which we will discuss in this book, and the specific functional equations which they generate.

In this chapter, we determine the solutions of a few functional equations which originate by localization and various reduction steps from the operator equations we will study, identifying the representing function F up to some parametric functions. To be self-contained, we provide the proofs of these results, even though most of them are found in, e.g., [A] or [AD] or in more generality in [J] or [Sz]. Some of the proofs are new, and we present them in more detail. In this chapter we do not outline the general theory of functional equations as done, e.g., in [J] or [Sz], but rather only solve those functional equations which will be relevant in later chapters.

To show the regularity of the parameter functions occurring in the representing function F, we prove some new general continuity results under assumptions which are easily verified in the case of the operator equations which we investigate. A general reference when solutions of functional equations are smooth is Járai [J].

2.1 Regularity results for additive and multiplicative equations

We start with the classical question when additive functions are linear.

Proposition 2.1. *Let* $f:\mathbb{R}\to\mathbb{R}$ *be measurable and* additive*, i.e., satisfy the* Cauchy equation

$$f(x+y)=f(x)+f(y),\quad x,y\in\mathbb{R}.$$

Then f *is linear: there is* $c\in\mathbb{R}$ *such that* $f(x)=cx$ *for all* $x\in\mathbb{R}$.

Clearly, additive functions satisfy $f(rx)=rf(x)$ for all $r\in\mathbb{Q}$. Thus, continuous additive functions are linear, $f(x)=cx$ with $c=f(1)$, as already noted by Cauchy.

Proof. Fix $x\neq 0$ and define functions $\varphi,\psi:\mathbb{R}\to\mathbb{R}$ by

$$\varphi(t):=f(t)-\frac{f(x)}{x}t,\quad \psi(t):=\frac{1}{1+|\varphi(t)|},\quad t\in\mathbb{R}.$$

By assumption φ and ψ are measurable with $0 \le \psi \le 1$. Hence, ψ is integrable on finite intervals. Note that $\varphi(x) = 0$, $\varphi(t+x) = \varphi(t) + \varphi(x) = \varphi(t)$, and $\psi(t+x) = \psi(t)$. Thus φ and ψ are periodic with period x. Therefore,

$$\int_0^x \psi(t)dt = \frac{1}{2}\int_0^{2x} \psi(t)dt = \int_0^x \psi(2t)dt,$$
$$0 = \int_0^x \big(\psi(t) - \psi(2t)\big)dt = \int_0^x \frac{|\varphi(t)|}{(1+|\varphi(t)|)(1+2|\varphi(t)|)}dt,$$

using $|\varphi(2t)| = 2|\varphi(t)|$. We conclude that $\varphi = 0$ almost everywhere, i.e., $f(t) = \frac{f(x)}{x}t$ for almost all $t \in \mathbb{R}$. In particular, for $x = 1$, $f(t) = f(1)t$ for almost all $t \in \mathbb{R}$. Hence, for any $x \neq 0$, there is $0 \neq t_0 \in \mathbb{R}$ such that $f(t_0) = \frac{f(x)}{x}t_0$ and $f(t_0) = f(1)t_0$. Hence, $\frac{f(x)}{x} = \frac{f(t_0)}{t_0} = f(1)$, $f(x) = f(1)x$ for all $x \neq 0$. Obviously, this also holds for $x = 0$. □

In general, additive functions are not linear: Let $X \subset \mathbb{R}$ be a Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$ (assuming the axiom of choice) and $g : X \to \mathbb{R}$ be an arbitrary function. Any $x \in \mathbb{R}$ can be written uniquely as $x = \sum_{i\in J} \lambda_i x_i$, $x_i \in X$, $\lambda_i \in \mathbb{Q}$, J a finite index set. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \sum_{i\in J} g(x_i)\lambda_i x_i, \quad x = \sum_{i\in J} \lambda_i x_i.$$

Then f is additive but not linear, unless g is constant. These pathological functions need to be unbounded on any small interval.

Proposition 2.2. *Let $I \in \mathbb{R}$ be a non-empty open interval and $f : \mathbb{R} \to \mathbb{R}$ be additive and bounded on I. Then f is linear, $f(x) = cx$ with $c \in \mathbb{R}$.*

Proof. Let $|I| \ge \delta > 0$ and $M := \sup_{x\in I} |f(x)|$. Then for any $t \in \mathbb{R}$ with $|t| < \delta$ there are $x, y \in I$ with $t = x - y$,

$$|f(t)| = |f(x-y)| = |f(x) - f(y)| \le 2M.$$

Using the additivity again, we find for any $s \in \mathbb{R}$ with $|s| < \delta/n$ that $|f(s)| \le 2M/n$. Let $u \in \mathbb{R}$ be arbitrary. Then, for any $n \in \mathbb{N}$, there is $r \in \mathbb{Q}$ with $|u - r| < \delta/n$. We find

$$\begin{aligned}\big|f(u) - uf(1)\big| &= \big|f(u) - f(r) + rf(1) - uf(1)\big| \\ &\le \big|f(u-r)\big| + |r-u|f(1) \le \big(2M + \delta f(1)\big)/n,\end{aligned}$$

which yields $f(u) = f(1)u$ for all $u \in \mathbb{R}$. □

The multiplicative analogue of Proposition 2.1 is

Proposition 2.3. *Let* $K : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ *be measurable, not identically zero and* multiplicative*, i.e.,*

$$K(uv) = K(u)K(v), \quad u, v \in \mathbb{R}.$$

Then there is $p \in \mathbb{R}$ *such that, for all* $u \in \mathbb{R}$*, either* $K(u) = |u|^p$ *or* $K(u) = |u|^p \operatorname{sgn}(u)$*.*

Proof. Since K is not identically zero, $K(u) \neq 0$ if $u \neq 0$. Therefore, we may define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \ln |K(e^x)|$. Then, for any $x, y \in \mathbb{R}$, $f(x + y) = f(x) + f(y)$. Since f is measurable, too, by Proposition 2.1 there is $p \in \mathbb{R}$ such that $f(x) = px$ for all $x \in \mathbb{R}$. Hence, $|K(u)| = u^p$ for any $u > 0$. Since $K(u) = K(\sqrt{u})^2 > 0$, we get $K(u) = u^p$ for $u > 0$. Further, $K(-1)^2 = K(1)^2 = K(1) = 1$ implies that $K(-1) \in \{+1, -1\}$. Then $K(-u) = K(-1)K(u)$ implies that $K(u) = |u|^p$ or $K(u) = |u|^p \operatorname{sgn}(u)$, depending on whether $K(-1) = 1$ or $K(-1) = -1$. □

For the complex version of this result, we assume continuity. For $z \in \mathbb{C} \setminus \{0\}$, let $\operatorname{sgn} z := \frac{z}{|z|}$. Also put $\operatorname{sgn} 0 := 0$.

Proposition 2.4. *Let* $f : \mathbb{C} \to \mathbb{C}$ *be continuous, not identically zero and multiplicative,*

$$f(zw) = f(z)f(w), \quad z, w \in \mathbb{C}.$$

Then there are $p \in \mathbb{C}$ *with* $\operatorname{Re}(p) \geq 0$ *and* $m \in \mathbb{Z}$ *such that*

$$f(z) = |z|^p (\operatorname{sgn} z)^m, \quad z \in \mathbb{C}.$$

We prove Proposition 2.4 by applying the following proposition which we need later not only for functions defined on $\mathbb{C}$ but on $\mathbb{C}^n$. For $z = (z_j)_{j=1}^n$, $d = (d_j)_{j=1}^n \in \mathbb{C}^n$, we denote by $\langle \cdot, \cdot \rangle$ the linear form – not the scalar product – on $\mathbb{C}^n$, $\langle d, z \rangle = \sum_{j=1}^n d_j z_j$. Moreover we put $\bar{z} = (\bar{z}_j)_{j=1}^n$.

Proposition 2.5. *Let* $n \in \mathbb{N}$ *and suppose that* $F : \mathbb{C}^n \to \mathbb{C} \setminus \{0\}$ *is continuous and satisfies*

$$F(z + w) = F(z) \cdot F(w), \quad z, w \in \mathbb{C}^n.$$

Then there are $c, d \in \mathbb{C}^n$ *such that*

$$F(z) = \exp(\langle c, z \rangle + \langle d, \bar{z} \rangle), \quad z \in \mathbb{C}^n.$$

Proof of Proposition 2.5. Write $z \in \mathbb{C}^n$ as $z = x + iy$, $x, y \in \mathbb{R}^n$ and F in polar decomposition form,

$$F(z) = G(x + iy) \exp\big(iH(x + iy)\big).$$

where $G : \mathbb{C}^n \to \mathbb{R}_{>0}$ is continuous and $H : \mathbb{C}^n \to \mathbb{R}$ may be chosen to be continuous, too, since it may be constructed from continuous branches. Note that H is defined on $\mathbb{C}^n$ and not on n-fold products of strips, so that it does not yield an injective representation of F. (E.g., for $n = 1$ and $F(z) = \exp(2z)$, we would have $H(x+iy) = 2y$ and we would not identify $2y = +\pi$ and $-\pi$ for $y = +\frac{\pi}{2}$ and $y = -\frac{\pi}{2}$.) Then, for all $x, y, u, v \in \mathbb{R}^n$,

$$G\big((x+u)+i(y+v)\big) = G(x+iy)G(u+iv),$$
$$H\big((x+u)+i(y+v)\big) = H(x+iy) + H(u+iv) + 2\pi k,$$

for some $k \in \mathbb{Z}$ which is independent of x, y, u, v since H is continuous. Define $\Phi : \mathbb{R}^{2n} \to \mathbb{R}$ by either $\Phi(x,y) := \ln G(x+iy)$ or $\Phi(x,y) := H(x+iy)+2\pi k$. Then Φ is continuous and additive,

$$\Phi(x+u, y+v) = \Phi(x,y) + \Phi(u,v).$$

Selecting $u = y = 0$ and renaming v as y, we get $\Phi(x,y) = \Phi(x,0) + \Phi(0,y)$ and similarly $\Phi(x+u,0) = \Phi(x,0) + \Phi(u,0)$. If $x = (x_j)_{j=1}^n = \sum_{j=1}^n x_j e_j$, where (e_j) denotes the canonical unit vector basis in $\mathbb{R}^n$, we have by additivity $\Phi(x,0) = \sum_{j=1}^n \Phi(x_j e_j, 0)$. Proposition 2.1 yields that there are $\alpha_j, \beta_j \in \mathbb{R}$ such that $\Phi(x_j e_j, 0) = \alpha_j x_j$ and $\Phi(0, y_j e_j) = \beta_j y_j$. Hence with $\alpha = (\alpha_j)_{j=1}^n$, $\beta = (\beta_j)_{j=1}^n$, $a := \frac{1}{2}(\alpha - i\beta)$ and $b := \frac{1}{2}(\alpha + i\beta) \in \mathbb{C}^n$,

$$\Phi(x,y) = \langle \alpha, x\rangle + \langle \beta, y\rangle = \langle a, z\rangle + \langle b, \bar{z}\rangle.$$

This means that $G(z) = \exp(\Phi(x,y)) = \exp(\langle a, z\rangle + \langle b, \bar{z}\rangle)$, and with different vectors $\tilde{a}, \tilde{b} \in \mathbb{C}^n$, $H(z) = \langle \tilde{a}, z\rangle + \langle \tilde{b}, \bar{z}\rangle - 2\pi k$, so that

$$F(z) = \exp(\langle c, z\rangle + \langle d, \bar{z}\rangle), \quad c := a + i\tilde{a},\ d := b + i\tilde{b} \in \mathbb{C}^n. \qquad \square$$

Proof of Proposition 2.4. We have $f(w) \neq 0$ for $w \neq 0$ since $f \not\equiv 0$. Define $F : \mathbb{C} \to \mathbb{C} \setminus \{0\}$ by $F(z) := f(\exp z)$. Then F is continuous and

$$F(z+w) = F(z)F(w), \quad z, w \in \mathbb{C}.$$

By Proposition 2.5 with $n = 1$, $F(z) = \exp(cz + d\bar{z})$, hence $f(w) = w^c \bar{w}^d$, $w \in \mathbb{C}$. For $w \neq 0$, let $\operatorname{sgn}(w) := \frac{w}{|w|}$. Then $f(w) = |w|^p \operatorname{sgn}(w)^q$ with $p = c + d \in \mathbb{C}$ and $q = c - d \in \mathbb{C}$. Since f is continuous, q has to be an integer, $q = m \in \mathbb{Z}$. Since f is bounded near zero, $\operatorname{Re}(p) \geq 0$ is required. $\square$

In later applications of Proposition 2.1, the measurable additive function f will actually depend on parameters or independent variables, so the linearity factor c will depend on these parameters. To prove the continuous dependence of c on the variables, we use the following result. Before formulating it, we introduce some notations. Let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For $n \in \mathbb{N}, k \in \mathbb{N}_0$, $I \subset \mathbb{R}^n$ open, let

$$C^k(I, \mathbb{R}) := \{f : I \to \mathbb{R} \mid f \text{ is } k\text{-times continuously differentiable}\}$$

and $C^\infty(I,\mathbb{R}) := \bigcap_{k \in \mathbb{N}} C^k(I,\mathbb{R})$, $C(I,\mathbb{R}) := C^0(I,\mathbb{R})$. Let $l \in \mathbb{N}$, $f \in C^l(I,\mathbb{R})$. By Schwarz' theorem, the l-th derivative $f^{(l)}(x)$ of f at $x \in I$ can be represented by the $M(n,l) := \binom{n+l-1}{n-1}$ independent l-th order partial derivatives

$\left(\frac{\partial^l f(x)}{\partial x_{i_1} \cdots \partial x_{i_l}}\right)_{1 \le i_1 \le \cdots \le i_l \le n}$. For $k \in \mathbb{N}$, let $N(n,k) := \sum_{l=0}^{k-1} M(n,l) = \binom{n+k-1}{n}$. Then, using this representation of derivatives, we put

$$J_k(x,f) := \big(f(x), \ldots, f^{(k-1)}(x)\big) \in \mathbb{R}^{N(n,k)}, \quad f \in C^{k-1}(I,\mathbb{R}),\ x \in I.$$

Theorem 2.6. *Let $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ and $I \subset \mathbb{R}^n$ be an open set, possibly unbounded. Let $B : I \times \mathbb{R}^{N(n,k)} \to \mathbb{R}$ be a function satisfying*

(a) $B(x, v_1 + v_2) = B(x, v_1) + B(x, v_2), \quad x \in I\ ,\ \ v_i \in \mathbb{R}^{N(n,k)}$.

(b) $B(\,\cdot\,, J_k(\,\cdot\,; f))$ *is a continuous function from I to $\mathbb{R}$ for all $f \in C^\infty(I,\mathbb{R})$.*

Then there is a continuous function $c : I \to \mathbb{R}^{N(n,k)}$ so that

$$B(x,v) = \big\langle c(x), v \big\rangle, \quad x \in I,\ v \in \mathbb{R}^{N(n,k)}.$$

By $\langle\,\cdot\,,\,\cdot\,\rangle$ we denote the standard scalar product on the appropriate $\mathbb{R}^N$-space, here $N = N(n,k)$. Then

$$B\big(x, J_k(x,f)\big) = \sum_{l=0}^{k-1} \langle c_l(x), f^{(l)}(x) \rangle, \quad x \in I,\ f \in C^{k-1}(I,\mathbb{R}),$$

with continuous functions $c_l : I \to \mathbb{R}^{M(n,l)}$.

For $k = 0$, the variable v and $J_k(\,\cdot\,; f)$ are not present in (a) and (b).

Proof. To keep the notation simple, we give the proof only in dimension $n = 1$, although the arguments in higher dimensions follow the same basic idea. For $n = 1$, we may assume that I is an open interval. We proceed by induction on $k = N(1,k)$. For $k = 0$ there is nothing to prove. Assume $k \in \mathbb{N}$ and that the result holds for $k - 1$.

(i) Define $A := \{x \in I \mid B(x, \cdot, 0, \ldots, 0) : \mathbb{R} \to \mathbb{R} \text{ is discontinuous}\}$. We claim that A has no accumulation points in I. Assume to the contrary that $x_m \in A \to x_\infty \in I$. We may assume that (x_m) is strictly monotone, say decreasing, so that $x_m > x_{m+1} > x_\infty$. Fix a smooth, non-negative cut-off function $\psi \in C^\infty(\mathbb{R})$ with $\psi|_{\mathbb{R} \setminus [-1,1]} = 0$, $\max \psi = \psi(0) = 1$ and $\psi^{(l)}(0) = 0$ for all $l \in \mathbb{N}$. Denote $c_l := \max |D^l \psi|$. For $m \in \mathbb{N}$, let

$$\delta_m := \min\left(\tfrac{1}{2} \min\big\{|x_m - x_j| : 1 \le j \le \infty,\ m \ne j\big\},\ \tfrac{1}{2^m}\right).$$

By assumption (a), $B(x_m, \cdot, 0 \ldots, 0) : \mathbb{R} \to \mathbb{R}$ is an additive function which is discontinuous for each $m \in \mathbb{N}$. By Proposition 2.2 it must be unbounded on $(0, \epsilon)$ for any $\epsilon > 0$. Therefore, we may choose $0 < y_m < \exp\big(-\frac{1}{\delta_m}\big)$ with $|B(x_m, y_m, 0, \ldots, 0)| > 1$. Define

$$g_m(x) := y_m \psi\left(\frac{x - x_m}{\delta_m}\right), \quad x \in I.$$

Then $g_m \in C^\infty(I)$ with $g_m(x_m) = y_m$, $g_m^{(l)}(x_m) = 0$ for all $l \in \mathbb{N}$ and $g_m(x) = 0$ for all $x \in I$ with $|x - x_m| > \delta_m$. Moreover, $|D^l g_m| \le c_l y_m \delta_m^{-l}$. Define $g := \sum_{m\in\mathbb{N}} g_m$. We find, for any $l \in \mathbb{N}_0$,

$$\sum_{m\in\mathbb{N}} |D^l g_m| \le c_l \sum_{m\in\mathbb{N}} y_m \delta_m^{-l} \le c_l \sum_{m\in\mathbb{N}} \delta_m^{-l} \exp\left(-\frac{1}{\delta_m}\right) < \infty,$$

so that $g \in C^\infty(I)$. Note that $g(x_m) = y_m$ since we have by definition of δ_j for any $m \neq j$ that $|x_m - x_j| \ge 2\delta_j$ so that $g_j(x_m) = 0$. Since $x_m \to x_\infty$ and $y_m \to 0$, we have by continuity that $g(x_\infty) = 0$. Also $g^{(l)}(x_m) = 0$ for all $l \in \mathbb{N}$, and again by continuity $g^{(l)}(x_\infty) = 0$ for all $l \in \mathbb{N}$. Since $B(\,\cdot\,, J_k(\,\cdot\,, g))$ is a continuous function by assumption (b),

$$B\big(x_m, J_k(x_m, g)\big) \longrightarrow B\big(x_\infty, J_k(x_\infty, g)\big) = B(x_\infty, 0, \ldots, 0) = 0.$$

However, $|B(x_m, J_k(x_m, g))| = |B(x_m, y_m, 0, \ldots, 0)| > 1$, which is a contradiction. Therefore, A has no accumulation points in I and its complement in I is dense in I.

(ii) We next claim that A is empty. Take any $x_0 \in I$. By (i) there is a sequence (x_m) with $x_m \notin A$, $x_m \to x_0$. For all $y_0 \in \mathbb{R}$, $B(\,\cdot\,, y_0, 0, \ldots, 0)$ is continuous on $\mathbb{R}$, applying (b) to the constant function $f(x) = y_0$, and therefore, $B(x_m, y_0, 0, \ldots, 0) \to B(x_0, y_0, 0, \ldots, 0)$. Hence, $B(x_m, \cdot, 0, \ldots, 0) \to B(x_0, \cdot, 0, \ldots, 0)$ pointwise. This implies that $B(x_0, \cdot, 0, \ldots, 0)$ is a measurable function, being the pointwise limit of continuous functions. By (a), $B(x_0, \cdot, 0, \ldots, 0)$ is additive so that Proposition 2.1 yields that $B(x_0, \cdot, 0, \ldots, 0)$ is linear and hence continuous so that $x_0 \notin A$. Hence, $A = \emptyset$.

We conclude that $B(x, y, 0, \ldots, 0) = c_0(x)y$ for some function $c_0 : I \to \mathbb{R}$. Since $c_0(x) = B(x, 1, 0, \ldots, 0)$, c_0 is continuous by assumption (b). Finally write

$$\begin{aligned} B(x, y_0, \ldots, y_{k-1}) &= B(x, y_0, 0, \ldots, 0) + B(x, 0, y_1, \ldots, y_{k-1}) \\ &= c_0(x)y_0 + B(x, 0, y_1, \ldots, y_{k-1}). \end{aligned}$$

Note that conditions (a), (b) also hold for $B(x, 0, y_1, \ldots, y_{k-1})$ as a function from $I \times \mathbb{R}^{k-1}$ to $\mathbb{R}$. Thus, by induction assumption, $B(x, 0, y_1, \ldots, y_{k-1}) = \sum_{j=1}^{k-1} c_j(x)y_j$, $c_j \in C(I)$. Hence,

$$B(x, y_0, \ldots, y_{k-1}) = \sum_{j=0}^{k-1} c_j(x)y_j = \big\langle c(x), y \big\rangle$$

with $c(x) = (c_j(x))_{j=0}^{k-1}$, $y = (y_j)_{j=0}^{k-1}$. □

Theorem 2.6 will be used in the next chapter to analyze the solutions of the Leibniz rule operator equation. We will also study perturbations of the Leibniz rule equation. To show that the solutions of the perturbed equations are perturbations of the solutions of the unperturbed Leibniz rule equation, we need a more technical variant of Theorem 2.6 in dimension $n = 1$ which we will apply in Chapter 5.

Proposition 2.7. *Let $k \in \mathbb{N}$, $I \subset \mathbb{R}$ be an open set and $B, \widetilde{B}, \Psi : I \times \mathbb{R}^k \to \mathbb{R}$ be functions, Ψ measurable, and $M : I \to \mathbb{R}_+$ be a locally bounded function such that*

(i) $\widetilde{B}(x,v) = B(x,v) + \Psi(x,v)$, $\quad x \in I$, $v \in \mathbb{R}^k$.

(ii) $\widetilde{B}(x, v_1 + v_2) = \widetilde{B}(x,v_1) + \widetilde{B}(x,v_2)$, $\quad x \in I$, $v_1, v_2 \in \mathbb{R}^k$.

(c) $B(\cdot, J_k(\cdot, f))$ *is a continuous function from I to $\mathbb{R}$ for all $f \in C^\infty(\mathbb{R})$.*

(d) $\sup\{|\Psi(x,v)| \mid v \in \mathbb{R}^k\} \le M(x) < \infty$, $\quad x \in I$.

Then $\widetilde{B}(x,\cdot)$ is linear for all $x \in I$, i.e., there is $c(x) \in \mathbb{R}^k$ such that $\widetilde{B}(x,v) = \langle c(x), v\rangle$ for all $v \in \mathbb{R}^k$.

Proof. (i) We adapt the previous proof and first claim that

$$A := \bigl\{x \in I \bigm| \widetilde{B}(x, \cdot, 0, \ldots, 0) : \mathbb{R} \to \mathbb{R} \text{ is discontinuous}\bigr\}$$

has no accumulation point in I. If this would be false, there would be a sequence of pairwise disjoint, say strictly decreasing points $x_m \in A$ with $x_m \to x_\infty \in I$. Since M is locally bounded,

$$K := \max\bigl(M(x_\infty), \sup\{M(x_m) \mid m \in \mathbb{N}\}\bigr) < \infty.$$

Since $\widetilde{B}(x_m, \cdot, 0, \ldots, 0)$ is discontinuous and additive, by Proposition 2.2, it attains arbitrarily large values in any neighborhood of zero. Again, choosing δ_m and $0 < y_m < \exp(-1/\delta_m)$ as in the previous proof, such that $|\widetilde{B}(x_m, y_m, 0, \ldots, 0)| > 3K + 1$, we define $g \in C^\infty(I)$ as before with

$$g(x_m) = y_m, \quad g(x_\infty) = 0, \quad g^{(l)}(x_m) = g^{(l)}(x_\infty) = 0,$$

for all $m, l \in \mathbb{N}$. By assumption (c)

$$\begin{aligned} B(x_m, y_m, 0, \ldots, 0) &= B\bigl(x_m, J_k(x_m, g)\bigr) \\ \longrightarrow B\bigl(x_\infty, J_k(x_\infty, g)\bigr) &= B(x_\infty, 0, \ldots, 0). \end{aligned}$$

But $\widetilde{B}(x_\infty, \cdot)$ is additive, hence $\widetilde{B}(x_\infty, 0) = 0$. Since $B = \widetilde{B} - \Psi$ and $|\Psi(x_m, \cdot)| \le K$, we arrive at the contradiction

$$2K < \lim_{m\to\infty} \bigl|B(x_m, y_m, 0, \ldots, 0)\bigr| = \bigl|B(x_\infty, 0, \ldots, 0)\bigr| \le K.$$

(ii) Fix an arbitrary point $x_0 \in I$. By (i) there are $x_m \notin A$ with $x_m \to x_0$. Therefore, $\widetilde{B}(x_m, \cdot, 0, \ldots, 0)$ is continuous for all $m \in \mathbb{N}$ and, by assumption (c), $B(\cdot, y_0, 0, \ldots, 0)$ is continuous for any $y_0 \in \mathbb{R}$. Thus

$$B(x_m, y_0, 0, \ldots, 0) \longrightarrow B(x_0, y_0, 0, \ldots, 0).$$

Hence $B(x_0, \cdot, 0, \ldots, 0)$ is the pointwise limit of the functions

$$B(x_m, \cdot, 0, \ldots, 0) = \widetilde{B}(x_m, \cdot, 0, \ldots, 0) - \Psi(x_m, \cdot, 0, \ldots, 0),$$

therefore measurable, so that

$$\big|\widetilde{B}(x_0,\,\cdot\,,0,\ldots,0)\big| \le K + \big|B(x_0,\,\cdot\,,0,\ldots,0)\big|,$$

i.e., $\widetilde{B}(x_0,\,\cdot\,,0,\ldots,0)$ is additive and bounded by a measurable function. By a result of Kestelman [Ke] – similar to Proposition 2.2 but slightly more general – $\widetilde{B}(x_0,\,\cdot\,,0,\ldots,0)$ is linear, i.e.,

$$\widetilde{B}(x_0,y_0,0,\ldots,0) = c_0(x_0)y_0.$$

Induction on k using

$$\widetilde{B}(x_0,y_0,\ldots,y_{k-1}) = \widetilde{B}(x_0,y_0,0,\ldots,0) + \widetilde{B}(x_0,0,y_1,\ldots,y_{k-1})$$

ends the proof. □

In the case of the chain rule operator equation studied in chapter 4, we will need different regularity results, yielding the regularity of a function from the property that certain differences of the function are regular.

Proposition 2.8. (a) *Let $L:\mathbb{R}\to\mathbb{R}$ be a function such that for any $b\in\mathbb{R}$*

$$\varphi(x) := L(x) - L(bx), \quad x\in\mathbb{R}$$

defines a continuous function $\varphi\in C(\mathbb{R})$. Then L is the pointwise limit of continuous functions and hence measurable.

(b) *Let $0<a\le 1$ and $L\in C(\mathbb{R})$ be a continuous function such that*

$$\psi(x) := L(x) - aL\left(\tfrac{x}{2}\right), \quad x\in\mathbb{R}$$

defines a C^1-function $\psi\in C^1(\mathbb{R})$. Then L is a C^1-function, $L\in C^1(\mathbb{R})$.

Proof. (i) For $b=1/2$, $\varphi(x) = L(x) - L(x/2)$ is continuous and for $n\in\mathbb{N}$

$$\sum_{j=0}^{n-1}\left(\varphi\left(\frac{x}{2^j}\right) - \varphi\left(\frac{1}{2^j}\right)\right) = \big(L(x) - L(1)\big) + \left(L\left(\frac{1}{2^n}\right) - L\left(\frac{x}{2^n}\right)\right).$$

For $b=x$, $\widetilde{\varphi}(y) = L(y) - L(xy)$ is continuous in $y=0$, hence,

$$\lim_{n\to\infty}\left(L\left(\frac{1}{2^n}\right) - L\left(\frac{x}{2^n}\right)\right) = \widetilde{\varphi}(0) = 0.$$

Therefore, the limit exists for $n\to\infty$ in the above equation and

$$L(x) = L(1) + \sum_{j=0}^{\infty}\left(\varphi\left(\frac{x}{2^j}\right) - \varphi\left(\frac{1}{2^j}\right)\right).$$

Hence L is the pointwise limit of continuous functions.

(ii) Fix $M > 0$ and let $x, x_1 \in [-M, M]$. For any $n \in \mathbb{N}$

$$\sum_{j=0}^{n-1} a^j \left(\psi\left(\frac{x}{2^j}\right) - \psi\left(\frac{x_1}{2^j}\right)\right) = \left(L(x) - L(x_1)\right) - a^n \left(L\left(\frac{x}{2^n}\right) - L\left(\frac{x_1}{2^n}\right)\right).$$

Since L is continuous, the last term on the right-hand side tends to 0 for $n \to \infty$. Since $\psi \in C^1(\mathbb{R})$, ψ' is uniformly continuous in $[-M, M]$ and bounded in modulus, say by N. Let $\epsilon > 0$. Then there is $\delta > 0$ such that for all $y, z \in [-M, M]$ with $|y - z| < \delta$, we have $|\psi'(y) - \psi'(z)| < \epsilon/2$. Assume $|x - x_1| < \delta$. Then, by the mean-value theorem,

$$\psi\left(\frac{x}{2^j}\right) - \psi\left(\frac{x_1}{2^j}\right) = \psi'\left(\frac{x(j)}{2^j}\right)\frac{x - x_1}{2^j},$$

for some $x(j)$ between x and x_1. Since $\left|\frac{x(j)}{2^j} - \frac{x_1}{2^j}\right| \leq |x - x_1| < \delta$, we find

$$\begin{aligned}\left|\sum_{j=1}^{n-1} a^j \cdot \frac{\psi\left(\frac{x}{2^j}\right) - \psi\left(\frac{x_1}{2^j}\right)}{x - x_1} - \sum_{j=0}^{n-1}\left(\frac{a}{2}\right)^j \psi'\left(\frac{x_1}{2^j}\right)\right| \\ = \left|\sum_{j=0}^{n-1}\left(\frac{a}{2}\right)^j \left(\psi'\left(\frac{x(j)}{2^j}\right) - \psi'\left(\frac{x_1}{2^j}\right)\right)\right| \leq \frac{\epsilon}{2}\sum_{j=0}^{n-1}\left(\frac{a}{2}\right)^j \leq \epsilon.\end{aligned}$$

Moreover,

$$\begin{aligned}\left|a^n \frac{L\left(\frac{x}{2^n}\right) - L\left(\frac{x_1}{2^n}\right)}{x - x_1}\right| = \left|\sum_{j=n}^{\infty} a^j \cdot \frac{\psi\left(\frac{x}{2^j}\right) - \psi\left(\frac{x_1}{2^j}\right)}{x - x_1}\right| \\ = \left|\sum_{j=n}^{\infty}\left(\frac{a}{2}\right)^j \psi'\left(\frac{x(j)}{2^j}\right)\right| \leq N\sum_{j=n}^{\infty}\left(\frac{a}{2}\right)^j \longrightarrow 0,\end{aligned}$$

uniformly in $x, x_1 \in [-M, M]$ for $n \to \infty$. Therefore,

$$L'(x_1) = \lim_{x \to x_1} \frac{L(x) - L(x_1)}{x - x_1} = \sum_{j=0}^{\infty}\left(\frac{a}{2}\right)^j \psi'\left(\frac{x_1}{2^j}\right)$$

exists and $\psi' \in C(\mathbb{R})$ implies $L' \in C(\mathbb{R})$, $L \in C^1(\mathbb{R})$. □

2.2 Functional equations with two unknown functions

In this section we discuss the solutions of some functional equations which involve two unknown functions. It is an interesting subject by itself which was studied intensively, cf., e.g., the books by Aczél [A], Aczél, Dhombres [AD] and Székelyhidi [Sz]. We will use these results in Chapters 7, 8 and 9 to study operator equations which are inspired by the Leibniz rule or by the chain rule of the second order. Several theorems in this section are special cases of results in [Sz]. Our intention here is to give direct proofs.

The second derivative D^2 satisfies the Leibniz and the chain rule type formulas

$$\begin{aligned} D^2(f\cdot g) &= D^2 f\cdot g + f\cdot D^2 g + 2Df\cdot Dg, \\ D^2(f\circ g) &= (D^2 f\circ g)\cdot (Dg)^2 + (Df)\circ g\cdot D^2 g, \qquad f,g\in C^2(\mathbb{R}). \end{aligned}$$

To understand the structure of these equations, we will later consider a more general setting: We will study operators $T : C^2(\mathbb{R}) \to C(\mathbb{R})$ and $A, A_1, A_2 : C^1(\mathbb{R}) \to C(\mathbb{R})$ satisfying one of the following equations

$$\begin{aligned} T(f\cdot g) &= Tf\cdot g + f\cdot Tg + Af\cdot Ag, \\ T(f\circ g) &= Tf\circ g\cdot A_1 g + (A_2 f)\circ g\cdot Tg, \qquad f,g\in C^2(\mathbb{R}). \end{aligned}$$

Under mild assumptions, it will turn out that there are not too many choices of operators (T, A) or (T, A_1, A_2) satisfying any one of these operator equations. To solve them, after localization, we have to find the solutions of some specific functional equations which involve two unknown functions.

We now discuss the solutions of these functional equations. The results of this section will only be used later in Chapters 7, 8 and 9.

Proposition 2.9. *Let $m \in \mathbb{N}$ and assume that $F, B : \mathbb{R}^m \to \mathbb{R}$ are functions such that for any $\alpha, \beta \in \mathbb{R}^m$,*

$$F(\alpha+\beta) = F(\alpha) + F(\beta) + B(\alpha)B(\beta). \tag{2.3}$$

Then there are additive functions $c, d : \mathbb{R}^m \to \mathbb{R}$ and $\gamma \in \mathbb{R}$ such that F and B have one of the following three forms:
Either

(a) $F(\alpha) = -\gamma^2 + d(\alpha)$, $B(x) = \gamma$,

or

(b) $F(\alpha) = \frac{1}{2}c(\alpha)^2 + d(\alpha)$, $B(\alpha) = c(\alpha)$,

or

(c) $F(\alpha) = \gamma^2(\exp(c(\alpha)) - 1) + d(\alpha)$, $B(\alpha) = \gamma(\exp(c(\alpha)) - 1)$,

for any $\alpha \in \mathbb{R}^m$.

Conversely, these functions satisfy equation (2.3).

Proof. (i) If $B = 0$, then F is additive and we are in case (a) with $\gamma = 0$. Therefore, we may assume that $B \neq 0$. Choose $a \in \mathbb{R}^m$ with $B(a) \neq 0$. For $\alpha \in \mathbb{R}^m$, define functions $f, b : \mathbb{R}^m \to \mathbb{R}$ by

$$f(\alpha) := F(\alpha + a) - F(\alpha) - F(a), \quad b(\alpha) := B(\alpha + a) - B(\alpha).$$

Then by (2.3)

$$f(\alpha + \beta) = f(\alpha) + b(\alpha)B(\beta), \quad \alpha, \beta \in \mathbb{R}^m. \tag{2.4}$$

For $\alpha = 0$, $f(\beta) = f(0) + b(0)B(\beta)$. Inserting this back into (2.3), we find

$$b(0)\big(B(\alpha + \beta) - B(\alpha)\big) = b(\alpha)B(\beta). \tag{2.5}$$

Suppose first $b(0) = 0$. Since $B(a) \neq 0$, (2.5) implies that $b \equiv 0$ identically and that $f = f(0)$ is a constant function. Since $f(\alpha) = B(a)B(\alpha)$ by (2.3), also B is constant, $B = f(0)/B(a) =: \gamma$. Let $d(\alpha) := F(\alpha) + \gamma^2$. Then by (2.3)

$$d(\alpha + \beta) = F(\alpha + \beta) + \gamma^2 = \big(F(\alpha) + F(\beta) + \gamma^2\big) + \gamma^2 = d(\alpha) + d(\beta),$$

i.e., d is additive on $\mathbb{R}^m$ and F and B have the form given in (a).

(ii) Assume now $b(0) \neq 0$. Putting $\alpha = 0$ in (2.5), we find that $B(0) = 0$. Moreover,

$$B(\alpha + \beta) = B(\alpha) + \frac{b(\alpha)}{b(0)}B(\beta). \tag{2.6}$$

Suppose first that b is a constant function. Then $c(\alpha) := B(\alpha)$ is additive and $d(a) := F(\alpha) - \frac{1}{2}c(\alpha)^2$ satisfies

$$\begin{aligned} d(\alpha + \beta) &= F(\alpha + \beta) - \tfrac{1}{2}\big(c(\alpha) + c(\beta)\big)^2 \\ &= \big(F(\alpha) + F(\beta) + B(\alpha)B(\beta)\big) - \tfrac{1}{2}c(\alpha)^2 - \tfrac{1}{2}c(\beta)^2 - c(\alpha)c(\beta) \\ &= d(\alpha) + d(\beta). \end{aligned}$$

Hence, d is additive and F and B have the form given in (b).

(iii) Now assume $b(0) \neq 0$ and that b is not constant. Choose $\alpha_0 \in \mathbb{R}^m$ with $b(\alpha_0) \neq b(0)$. Since the left-hand side of (2.6) is symmetric in α and β, we have

$$B(\alpha) + \frac{b(\alpha)}{b(0)}B(\beta) = B(\beta) + \frac{b(\beta)}{b(0)}B(\alpha).$$

For $\beta = \alpha_0$, $B(\alpha) = \frac{B(\alpha_0)}{b(\alpha_0) - b(0)}(b(\alpha) - b(0))$, and by (2.4),

$$f(\alpha) - f(0) = b(0)B(\alpha) = \gamma\big(b(\alpha) - b(0)\big), \tag{2.7}$$

with $\gamma := b(0)B(\alpha_0)/(b(\alpha_0) - b(0))$. For $\gamma = 0$, $B = 0$, and we are again in case (a).

So assume $\gamma \neq 0$. Then, by (2.4) and (2.7),

$$\begin{aligned}\gamma\big(b(\alpha+\beta)-b(0)\big) &= f(\alpha+\beta)-f(0)\\ &= f(\alpha)-f(0)+b(\alpha)B(\beta)\\ &= \gamma\big(b(\alpha)-b(0)\big)+\frac{b(\alpha)}{b(0)}\gamma\big(b(\beta)-b(0)\big)\\ &= \gamma\left(\frac{b(\alpha)b(\beta)}{b(0)}-b(0)\right).\end{aligned}$$

Hence, $\widetilde{b}(\alpha) := b(\alpha)/b(0)$ satisfies $\widetilde{b}(\alpha+\beta) = \widetilde{b}(\alpha)\widetilde{b}(\beta)$, $\widetilde{b}(\alpha) = \widetilde{b}\big(\frac{\alpha}{2}\big)^2 > 0$. Note that $\widetilde{b}(\alpha) \neq 0$, since otherwise $\widetilde{b}(0) = \widetilde{b}(\alpha)\widetilde{b}(-\alpha) = 0$, but $\widetilde{b}(0) = 1$. Therefore, $c(\alpha) := \ln \widetilde{b}(\alpha)$ is additive and $b(\alpha) = b(0)\exp(c(\alpha))$. This yields $B(\alpha) = \gamma(\exp(c(\alpha))-1)$. Put similarly as above $d(\alpha) := F(\alpha)-\gamma^2(\exp(c(\alpha))-1)$. Then (2.3) and the additivity of c yield

$$\begin{aligned}d(\alpha+\beta) &= \big(F(\alpha)+F(\beta)+B(\alpha)B(\beta)\big)-\gamma^2\big(\exp(c(\alpha))\exp(c(\beta))-1\big)\\ &= d(\alpha)+d(\beta),\end{aligned}$$

i.e., d is additive. Therefore, we have the solution given in (c),

$$F(\alpha) = \gamma^2\big(\exp(c(\alpha))-1\big)+d(\alpha). \qquad \square$$

In the case $m = 1$, we need a multiplicative analogue of Proposition 2.9.

Proposition 2.10. *Assume that $F, B : \mathbb{R} \to \mathbb{R}$ are functions such that, for any $\alpha, \beta \in \mathbb{R}$,*

$$F(\alpha\beta) = F(\alpha)\beta + F(\beta)\alpha + B(\alpha)\ B(\beta). \tag{2.8}$$

Then there are additive functions $c, d : \mathbb{R} \to \mathbb{R}$, and there is $\gamma \in \mathbb{R}$ such that F and B have one of the following three forms:

(a) $F(\alpha) = \alpha\ (c(\ln|\alpha|)-\gamma^2)$, $B(\alpha) = \gamma\alpha$;

(b) $F(\alpha) = \alpha\big(\frac{1}{2}c(\ln|\alpha|)^2+d(\ln|\alpha|)\big)$, $B(\alpha) = \alpha c(\ln|\alpha|)$;

(c) $F(\alpha) = \alpha\big(\gamma^2[\{\operatorname{sgn}\alpha\}\exp(c(\ln|\alpha|))-1]+d(\ln|\alpha|)\big)$,
$B(\alpha) = \alpha\gamma\big[\{\operatorname{sgn}\alpha\}\exp(c(\ln|\alpha|))-1\big]$.

In (c)*, there are two possibilities, with* $\operatorname{sgn}\alpha$ *present in both F and B and the other one with* $\operatorname{sgn}\alpha$ *replaced by* 1.

Conversely, these functions satisfy equation (2.8).

Proof. (i) For $a \in \mathbb{R}$, define $f(a) := F(\exp(a))/\exp(a)$, $g(a) := B(\exp(a))/\exp(a)$. Then (2.8) implies

$$f(a+b) = f(a)+f(b)+g(a)g(b), \quad a, b \in \mathbb{R}.$$

The solutions of this equation ($m = 1$) were given in Proposition 2.9, e.g., in case (b) with additive functions $c, d : \mathbb{R} \to \mathbb{R}$,

$$f(a) = \tfrac{1}{2}c(a)^2 + d(a), \quad g(a) = c(a).$$

Then for $\alpha > 0$, $a := \ln \alpha$, so that $\alpha = \exp(a)$,

$$F(\alpha) = \alpha\left(\tfrac{1}{2}c(\ln\alpha)^2 + d(\ln\alpha)\right), \quad B(\alpha) = \alpha c(\ln\alpha).$$

The cases (a) and (c) are similar, which yields Proposition 2.10 if $\alpha > 0$.

(ii) We will now determine $F(\alpha)$ and $B(\alpha)$ for negative α. In all cases except one, F and B turn out to be odd functions. The exceptional one is the case of the third solution when the $\operatorname{sgn}\alpha$-term appears. Unfortunately, this requires distinguishing several cases in the basic equation (2.9) below. Choosing $\beta = -1$ in (2.8) and exchanging α and $-\alpha$, we find

$$F(\alpha) + F(-\alpha) = F(-1)\alpha + B(-1)B(\alpha) = -F(-1)\alpha + B(-1)B(-\alpha),$$
$$\text{i.e., } B(-1)B(-\alpha) = B(-1)B(\alpha) + 2F(-1)\alpha.$$

For $\alpha = 1$, $B(-1)^2 = B(-1)B(1) + 2F(-1)$. Hence,

$$\begin{aligned} B(-1)B(-\alpha) &= B(-1)\big[B(\alpha) + (B(-1) - B(1))\alpha\big], \\ F(\alpha) + F(-\alpha) &= B(-1)\left[B(\alpha) + \tfrac{1}{2}(B(-1) - B(1))\alpha\right]. \end{aligned} \tag{2.9}$$

If $B(-1) = 0$, also $F(-1) = 0$ and (2.9) implies that $F(-\alpha) = -F(\alpha)$ and, using (2.8),

$$\begin{aligned} B(-\alpha)B(\beta) &= F(-\alpha\beta) - F(-\alpha)\beta + F(\beta)\alpha \\ &= -F(\alpha\beta) + F(\alpha)\beta + F(\beta)\alpha = -B(\alpha)B(\beta), \end{aligned}$$

i.e., F and B are odd functions, which means that in cases (a), (b) and (c) $\ln\alpha$ has to be replaced by $\ln|\alpha|$ for $\alpha < 0$.

(iii) Now assume $B(-1) \neq 0$. In cases (b), (c), we know $B(1) = F(1) = 0$. Then by (2.9)

$$B(-\alpha) = B(\alpha) + B(-1)\alpha, \quad F(\alpha) + F(-\alpha) = B(-1)\left[B(\alpha) + \tfrac{1}{2}B(-1)\alpha\right]. \tag{2.10}$$

Using (2.10) for $\alpha\beta$ instead of α and (2.8), we find

$$\begin{aligned} B(-1)\left[B(\alpha\beta) + \tfrac{1}{2}B(-1)\alpha\beta\right] &= F(\alpha\beta) + F(-\alpha\beta) \\ &= \big(F(\alpha) + F(-\alpha)\big)\beta + \big(B(\alpha) + B(-\alpha)\big)B(\beta) \\ &= B(-1)\left[B(\alpha)\beta + \tfrac{1}{2}B(-1)\alpha\beta\right] + 2B(\alpha)B(\beta) + B(-1)B(\beta)\alpha \\ &= 2\left(B(\alpha) + \tfrac{1}{2}B(-1)\alpha\right)\left(B(\beta) + \tfrac{1}{2}B(-1)\beta\right). \end{aligned}$$

Therefore, $\varphi(\alpha) := \frac{2}{B(-1)} B(\alpha) + \alpha$ is multiplicative, $\varphi(\alpha\beta) = \varphi(\alpha)\varphi(\beta)$. For positive $\alpha > 0$, this occurs only in case (c) when

$$B(\alpha) = \alpha\gamma\big[\exp(c(\ln\alpha)) - 1\big].$$

This identifies $\gamma = \frac{1}{2}B(-1)$, and for $\alpha < 0$ we have

$$B(\alpha) = \alpha\gamma\big[\operatorname{sgn}\alpha \cdot \exp(c(\ln|\alpha|)) - 1\big],$$

from the multiplicity of φ, where the term $\operatorname{sgn}\alpha$ has to be present since otherwise $B(-1) = 0$. For $\alpha < 0$, we get from (2.10) and the known form of $F(-\alpha)$ for $-\alpha = |\alpha| > 0$

$$\begin{aligned} F(\alpha) &= -F(-\alpha) + 2\gamma\big(B(\alpha) + \gamma\alpha\big) \\ &= \alpha\big[\gamma^2(\exp(c(\ln|\alpha|)) - 1) + d(\ln|\alpha|)\big] \\ &\qquad + 2\gamma\big[-\gamma\alpha(\exp(c(\ln|\alpha|)) + 1) + \gamma\alpha\big] \\ &= \alpha\big[\gamma^2(\operatorname{sgn}\alpha\exp(c(\ln|\alpha|)) - 1) + d(\ln|\alpha|)\big]. \end{aligned}$$

In this case B and F are not odd, in the other cases of (b) and (c) they are odd.

(iv) It remains to consider case (a) for $\alpha < 0$, when $B(-1) \neq 0$. Then $B(1) = \gamma$ and (2.9) yields for $\alpha > 0$ that $B(-\alpha) = B(-1)\alpha$ and

$$F(\alpha) + F(-\alpha) = \tfrac{1}{2}B(-1)\big(\gamma + B(-1)\big)\alpha.$$

Using this for $\alpha\beta$ instead of α and (2.8) we find

$$\begin{aligned} \tfrac{1}{2}B(-1)\big(\gamma + B(-1)\big)\alpha\beta &= F(\alpha\beta) + F(-\alpha\beta) \\ &= \big(F(\alpha) + F(-\alpha)\big)\beta + \big(B(\alpha) + B(-\alpha)\big)B(\beta) \\ &= \tfrac{1}{2}B(-1)\big(\gamma + B(-1)\big)\alpha\beta + \big(\gamma + B(-1)\big)\alpha B(\beta), \end{aligned}$$

hence, $B(-1) = -\gamma$, $B(-\alpha) = -\gamma\alpha = -B(\alpha)$, $F(-\alpha) = -F(\alpha)$, so that B and F are odd functions, which means, in the formula of (a), that $\ln\alpha$ has to replaced by $\ln|\alpha|$ for $\alpha < 0$. □

In Chapter 3 we will need the solution of a functional equation which resembles the addition formula for the sin function. We first consider the complex case.

Proposition 2.11. *Let $n \in \mathbb{N}$ and $F, B : \mathbb{C}^n \to \mathbb{C}$ be continuous functions satisfying*

$$F(z + w) = F(z) \cdot B(w) + F(w) \cdot B(z), \quad z, w \in \mathbb{C}^n. \tag{2.11}$$

Suppose F is not identically zero. Then there are vectors $c_1, c_2, d_1, d_2 \in \mathbb{C}^n$ and there are $k \in \mathbb{C} \setminus \{0\}$ and $\epsilon_1, \epsilon_2 \in \{0, 1\}$, with ϵ_1, ϵ_2 not both zero, such that F and B have one of the following two forms:

(a) $F(z) = (\langle c_1, z\rangle + \langle c_2, \bar{z}\rangle)\exp(\langle d_1, z\rangle + \langle d_2, \bar{z}\rangle)$,
$B(z) = \exp(\langle d_1, z\rangle + \langle d_2, \bar{z}\rangle)$;

(b) $F(z) = \frac{1}{2k}(\epsilon_1 \exp(\langle c_1, z\rangle + \langle c_2, \bar{z}\rangle) - \epsilon_2 \exp(\langle d_1, z\rangle + \langle d_2, \bar{z}\rangle))$,
$B(z) = \frac{1}{2}\big(\epsilon_1 \exp(\langle c_1, z\rangle + \langle c_2, \bar{z}\rangle) + \epsilon_2 \exp(\langle d_1, z\rangle + \langle d_2, \bar{z}\rangle)\big), \qquad z \in \mathbb{C}^n$.

Conversely, these functions satisfy equation (2.11).

In the real case we get

Corollary 2.12. *Let $F, B : \mathbb{R}^n \to \mathbb{R}$ be continuous functions satisfying*

$$F(\alpha + \beta) = F(\alpha)B(\beta) + F(\beta)B(\alpha), \quad \alpha, \beta \in \mathbb{R}^n.$$

Suppose F is not identically zero. Then there are vectors $b, c, d \in \mathbb{R}^n$ and there is $a \in \mathbb{R}$ such that F and B have one of the following four forms:

(a) $F(\alpha) = \langle b, \alpha\rangle \exp(\langle d, \alpha\rangle)$, $B(\alpha) = \exp(\langle d, \alpha\rangle)$;

(b) $F(\alpha) = a\ \exp(\langle c, \alpha\rangle)\ \sin(\langle d, \alpha\rangle)$, $B(\alpha) = \exp(\langle c, \alpha\rangle)\ \cos(\langle d, \alpha\rangle)$;

(c) $F(\alpha) = a\ \exp(\langle c, \alpha\rangle)\ \sinh(\langle d, \alpha\rangle)$, $B(\alpha) = \exp(\langle c, \alpha\rangle)\ \cosh(\langle d, \alpha\rangle)$;

(d) $F(\alpha) = a\ \exp(\langle d, \alpha\rangle)$, $B(\alpha) = \frac{1}{2}\exp(\langle d, \alpha\rangle)$, $\alpha \in \mathbb{R}^n$.

Conversely, these functions satisfy the above functional equation.

Proof of Proposition 2.11. (i) Fix $t \in \mathbb{C}^n \setminus \{0\}$. We claim that F,B and $B(\cdot + t)$ are linearly dependent functions. For all $x, y \in \mathbb{C}^n$

$$F(x+t)B(y)+B(x+t)F(y) = F(x+y+t) = F(x)B(y+t)+B(x)F(y+t). \quad (2.12)$$

Since F is not identically zero, by (2.11) also B is not identically zero. Hence there is $y_1 \in \mathbb{C}^n$ such that $B(y_1) \neq 0$. Choosing $y = y_1$, equation (2.12) shows that $F(\cdot+t)$ is a linear combination of F, B and $B(\cdot+t)$ with coefficients depending on the values $B(y_1)$, $F(y_1)$, $B(y_1+t)$ and $F(y_1+t)$. Inserting this back into (2.12) yields for all $x, y \in \mathbb{C}^n$

$$F(x)\big(B(y)B(y_1+t) - B(y_1)B(y+t)\big) + B(x)\big(B(y)F(y_1+t) - B(y_1)F(y+t)\big) + B(x+t)\big(B(y_1)F(y) - B(y)F(y_1)\big) = 0. \quad (2.13)$$

Suppose $B(y_1)F(y) - B(y)F(y_1) = 0$ holds for all $y \in \mathbb{C}^n$. Then $F = \frac{F(y_1)}{B(y_1)}B$, and already F and B are linearly dependent. Else there is $y_2 \in \mathbb{C}^n$ such that $B(y_1)F(y_2) - B(y_2)F(y_1) \neq 0$, and equation (2.13) shows that F, B and $B(\cdot + t)$ are linearly dependent.

(ii) Assume that $B = kF$ for some $k \in \mathbb{C}$. Then $F(x + y) = 2kF(x)F(y)$, and $k \neq 0$ since F is not identically zero. Proposition 2.5 implies that there are $c_1, c_2 \in \mathbb{C}^n$ such that $F(z) = \frac{1}{2k}\exp(\langle c_1, z\rangle+\langle c_2, \bar{z}\rangle)$, $B(z) = \frac{1}{2}\exp(\langle c_1, z\rangle+\langle c_2, \bar{z}\rangle)$. This is a solution of type (b) with $\epsilon_2 = 0$.

(iii) We may now assume that B and F are linearly independent. Then by (i) there are functions $c_1, c_2 : \mathbb{C}^n \to \mathbb{C}$ such that

$$B(x+t) = c_1(t)F(x) + c_2(t)B(x), \quad x, t \in \mathbb{C}^n. \tag{2.14}$$

The left-hand side is symmetric in x and t. Applying it to $x+y+t$, we get an equation similar to (2.12). The arguments in (i) then show that c_2, B and F are linearly dependent: there are $b_1, b_2 \in \mathbb{C}$ such that

$$c_2(x) = b_1 B(x) + b_2 F(x).$$

Inserting this back into (2.14) and using the symmetry in (x,t), we find

$$\begin{aligned} c_1(t)F(x) + \big(b_1 B(t) + b_2 F(t)\big)B(x) &= B(x+t) \\ &= c_1(x)F(t) + \big(b_1 B(x) + b_2 F(x)\big)B(t), \\ c_1(x) - b_2 B(x) &= \frac{c_1(t) - b_2 B(t)}{F(t)} F(x) =: b_3 F(x), \end{aligned}$$

for any fixed t with $F(t) \neq 0$. Hence $c_1(x) = b_2 B(x) + b_3 F(x)$, and again by (2.14)

$$B(x+t) = \big(b_2 B(t) + b_3 F(t)\big)F(x) + \big(b_1 B(t) + b_2 F(t)\big)B(x).$$

Insert this and formula (2.11) for $F(x+t)$ into (2.12) to find, after some calculation,

$$\big((1-b_1)B(t) - b_2 F(t)\big)\big(F(x)B(y) - F(y)B(x)\big) = 0,$$

for all $x, y, t \in \mathbb{C}^n$. Since B and F are linearly independent, we first conclude that $(1-b_1)B(t) = b_2 F(t)$ for all t, and then that $b_1 = 1$, $b_2 = 0$. Therefore, $c_1 = b_3 F$, $c_2 = B$, and (2.14) yields

$$B(x+t) = b_3 F(t)F(x) + B(t)B(x), \quad x, t \in \mathbb{C}^n.$$

Take $k \in \mathbb{C}$ with $k^2 = b_3$. Using (2.11) again, we find

$$\big(B(x+y) \pm kF(x+y)\big) = \big(B(x) \pm kF(x)\big)\big(B(y) \pm kF(y)\big),$$

so that $f := B \pm kF$ solves the equation $f(x+y) = f(x)f(y)$. Since $f \not\equiv 0$, by Proposition 2.5, there are $c_1, c_2, d_1, d_2 \in \mathbb{C}^n$ such that

$$\begin{aligned} B(z) + kF(z) &= \exp(\langle c_1, z\rangle + \langle c_2, \bar{z}\rangle), \\ B(z) - kF(z) &= \exp(\langle d_1, z\rangle + \langle d_2, \bar{z}\rangle), \end{aligned}$$

which gives solution (b) with $\epsilon_1 = \epsilon_2 = 1$, if $k \neq 0$.

(iv) If $k = 0$, again by Proposition 2.5, $B(z) = \exp(\langle d_1, z\rangle + \langle d_2, \bar{z}\rangle)$ for suitable $d_1, d_2 \in \mathbb{C}$. Define $G(z) := \frac{F(z)}{B(z)}$. Since $B(z+w) = B(z)B(w)$, equation (2.11) yields

$$G(z+w) = G(z) + G(w), \quad z, w \in \mathbb{C}^n.$$

Hence G is additive and continuous. As in the proof of Proposition 2.5 there are $c_1, c_2 \in \mathbb{C}^n$ such that $G(z) = \langle c_1, z\rangle + \langle c_2, \bar{z}\rangle$, which yields with $F(z) = G(z)B(z)$ the form of F and B given in part (a). □

Proof of Corollary 2.12. Extend $F, B : \mathbb{R}^n \to \mathbb{R}$ to $\widetilde{F}, \widetilde{B} : \mathbb{C}^n \to \mathbb{C}$ by $\widetilde{F}(z) := F(\Re z)$, $\widetilde{B}(z) := B(\Re z)$ with $\Re z = (\Re z_j)_{j=1}^n$ if $z = (z_j)_{j=1}^n$. Here $\Re$ denotes the real part, and below $\Im$ will stand for the imaginary part. Then $\widetilde{F}, \widetilde{B}$ satisfy (2.11) and are real valued. The functions B and F in part (a) of Proposition 2.11 are real valued if and only if $c_1 = \bar{c}_2$ and $d_1 = \bar{d}_2$ yielding the solution in (a), when restricted to $\mathbb{R}^n$, with $b = 2\Re c_1$ and $d = 2\Re d_1$.

The formula for B in part (b) of Proposition 2.11 with $\epsilon_1 = \epsilon_2 = 1$ is real valued if and only if either $c_1 = \bar{c}_2$ and $d_1 = \bar{d}_2$ or $c_1 = \bar{d}_2$ and $c_2 = \bar{d}_1$. In the first case one gets a solution of type (c) with vectors $c = \Re(c_1 + d_1)$, $d = \Re(c_1 - d_1)$, in the second case a solution of type (b) with $c = \Re(c_1 + c_2)$ and $d = \Im(c_1 + c_2)$. In the first case k needs to be real, in the second case purely imaginary. The last solution (d) originates from (b) in Proposition 2.11 for $\epsilon_1 = 1$, $\epsilon_2 = 0$ (or $\epsilon_1 = 0$, $\epsilon_2 = 1$). □

In Chapter 9 we need a multiplicative one-dimensional analogue of Corollary 2.12 which is the following result.

Proposition 2.13. *Let $F, B : \mathbb{R} \to \mathbb{R}$ be continuous functions satisfying*

$$F(xy) = F(x)B(y) + F(y)B(x), \quad x, y \in \mathbb{R}. \tag{2.15}$$

Suppose F is not identically zero. Then there are constants $a, b, c, d \in \mathbb{R}$, $c, d > 0$, so that F and B have one of the following four forms:

(a) $F(x) = b(\ln|x|)|x|^d\{\operatorname{sgn} x\}$, $B(x) = |x|^d\{\operatorname{sgn} x\}$;

(b) $F(x) = b|x|^d \sin(a \ln|x|)\{\operatorname{sgn} x\}$, $B(x) = |x|^d \cos(d \ln|x|)\{\operatorname{sgn} x\}$;

(c) $F(x) = \frac{b}{2}(|x|^c[\operatorname{sgn} x] - |x|^d\{\operatorname{sgn} x\})$, $B(x) = \frac{1}{2}(|x|^c[\operatorname{sgn} x] + |x|^d\{\operatorname{sgn} x\})$;

(d) $F(x) = b|x|^d\{\operatorname{sgn} x\}$, $B(x) = \frac{1}{2}|x|^d\{\operatorname{sgn} x\}$, $x \in \mathbb{R}$.

Here the terms $\{\operatorname{sgn} x\}$ and $[\operatorname{sgn} x]$ may be present or not, simultaneously in F and B. If a sgn*-factor is not present, the corresponding value of c or d could be* 0*, too, Conversely, these functions satisfy the above functional equation.*

Proof. (i) Let $\widetilde{F}(\alpha) := F(\exp\alpha)$, $\widetilde{B}(\alpha) := B(\exp\alpha)$. Then $\widetilde{F}(\alpha+\beta) = \widetilde{F}(\alpha)\widetilde{B}(\beta) + \widetilde{B}(\alpha)\widetilde{F}(\beta)$. Hence $(\widetilde{F}, \widetilde{B})$ have one of the four forms given in Corollary 2.12. Then for $x > 0$, substituting $\alpha = \ln x = \ln|x|$, (F, B) have the form given in Proposition 2.13 with $\operatorname{sgn} x = 1$.

(ii) It remains to determine $F(x)$ and $B(x)$ for $x \le 0$. In the first three cases $F(1) = 0$. Then $0 = F(1) = F((-1)^2) = 2F(-1)B(-1)$. Assume first that $F(-1) = 0$. Then $F(x) = F(-x)B(-1) = F(x)B(-1)^2$, hence $B(-1)^2 = 1$, $B(-1) \in \{1, -1\}$. Thus F is even or odd, depending on whether $B(-1) = 1$ or $B(-1) = -1$. Using $F(x) = F(-x)B(-1)$, the functional equation implies for any $x, y \in \mathbb{R}$

$$\begin{aligned} F(x)B(-y) + B(-1)F(y)B(x) &= F(x)B(-y) + F(-y)B(x) = F(-xy) \\ &= B(-1)F(xy) = B(-1)[F(x)B(y) + F(y)B(x)]. \end{aligned}$$

Therefore $F(x)B(-y) = F(x)B(-1)B(y)$ which yields $B(-y) = B(-1)B(y)$. Hence F and B are both even or both odd. This implies the formulas for F and B for negative x in the first three cases. Since F and B and the right-hand sides are continuous, the values at zero are obtained by taking the limit for $x \to 0$ on both sides.

In the last case $F(1) =: b \neq 0$. Equation (2.15) yields for $y = 1$ that $F(x) = F(x)B(1) + bB(x)$. Since $B \not\equiv 0$, we conclude that $B(1) \neq 1$ and $F(x) = \lambda B(x)$ with $\lambda := \frac{b}{1-B(1)} \neq 0$. Inserting this into (2.15), we get $B(xy) = 2B(x)B(y)$, so that $2B$ is multiplicative on $\mathbb{R}$. By Proposition 2.3, $B(x) = \frac{1}{2}|x|^d\{\operatorname{sgn} x\}$, $F(x) = \frac{\lambda}{2}|x|^d\{\operatorname{sgn} x\}$, so that $b = \frac{\lambda}{2}$. □

2.3 Notes and References

The classical result for measurable additive functions, Proposition 2.1, is due to Fréchet [Fr]. The paper [Fr] is written in Esperanto. Alternative proofs were given by Banach [B] and Sierpinski [S]. The proofs in [Fr] and [B] use the axiom of choice, the one in [S] does not require it. The simple proof presented here is due to Alexiewicz and Orlicz [AO].

The proof of Proposition 2.2 follows Kestelman [Ke], where the linearity of additive functions is shown under the even weaker assumption that f is bounded from above by a measurable function on a set of positive Lebesgue measure. This stronger result is used in the proof of Proposition 2.7.

Proposition 2.3 on measurable multiplicative functions is found, e.g., in Aczél [A], Section 2.1.2.

Proposition 2.5 is shown by Aczél [A] in Section 5.1.1, Theorem 3, in the case of $n = 1$. The generalization to $n > 1$ is straightforward. The result also holds if F is assumed to be only measurable instead of being continuous, cf. Aczél, Dhombres [AD], Theorem 5 of Section 5.1 ($n = 1$). The proof is slightly more elaborate than in the continuous case.

Since Proposition 2.4 follows directly from Proposition 2.5, it is also true if the non-zero function f is only assumed to be multiplicative and measurable.

Theorem 2.6 is due to Faifman, see the Appendix of [KM1].

Proposition 2.8 is a slight extension of Lemma 19 in [AKM].

Proposition 2.9 is a special case of Theorem 10.4. in Székelyhidi [Sz], which is illustrated by the functional equation (10.6b) in this book. Theorem 10.4. also covers solutions of functional equations with more than two unknown functions. In the case $m = 1$, Proposition 2.9 is related to some functional equations in Section 3.1.3 of Aczél [A] and in Chapter 15, Theorem 1 of Aczél, Dhombres [AD] to which this result could be reduced. Our direct proof uses ideas of Section 3.1.3 of Aczél [A].

Proposition 2.11 can be found in Székelyhidi [Sz], Theorem 12.2., as an application of his general theory of functional equations on topological abelian groups, cf. also Theorem 10.4. in [Sz]. We gave a direct proof which was inspired by the book of Aczél [A], where the case $n = 1$ is considered in Section 4.2.5, Theorem 2 and its Corollary. For Corollary 2.12 in the case $n = 1$ cf. Aczél [A], p. 180.