Duality on Value Semigroups

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Abstract We consider value semigroup ideals of fractional ideals on certain curve singularities. These satisfy natural axioms defining the class of good semigroup ideals. On this class we develop a purely combinatorial counterpart of the duality on Cohen–Macaulay rings. This is joint work with Philipp Korell and Mathias Schulze.

1 Introduction and Motivation

Let R be a complex algebroid curve with s branches and normalization $R \to \overline{R} \cong \mathbb{C}[[t_1]] \times \cdots \times \mathbb{C}[[t_s]]$. Then, there is a multivaluation map

$$\nu = (\nu_1, \dots, \nu_s)\colon \overline{R} \to (\mathbb{Z} \cup \{\infty\})^s, \quad x \mapsto (\mathrm{ord}_{t_1}(x), \dots, \mathrm{ord}_{t_s}(x))$$

which associates to R its *value semigroup* $\Gamma_R = \nu(\{x \in R \mid x \text{ non zero-divisor}\})$

The value semigroup of a curve singularity is an important combinatorial invariant with a long history. It determines the topological type of plane curves. In case R is an irreducible curve Kunz [1] showed that R is Gorenstein if and only if its value semigroup Γ_R is symmetric.

Example 1 Consider the plane algebroid curve $R = \mathbb{C}[[x, y]]/\langle x^7 - y^4 \rangle \cong \mathbb{C}[[t^4, t^7]]$. Then R is Gorenstein and $\Gamma_R = \langle 4, 7\rangle$ is symmetric.

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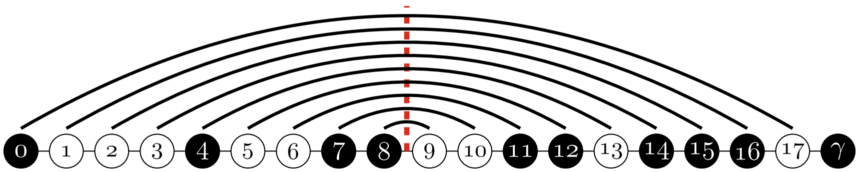

Later Delgado [2] introduced a notion of symmetry in the reducible case, and extended Kunz's result. D'Anna used Delgado's symmetry to define a *canonical semigroup ideal*. Based on this definition, he characterized canonical ideals of R in terms of their value semigroup ideals. More recently, Pol [3] proved a formula for the value semigroup of the dual of a fractional ideal. Our aim is to generalize both the duality results by D'Anna and by Pol.

2 Good Value Semigroups

Including complex algebroid curves as a special case we consider *admissible* rings in the following sense: let R be a one-dimensional semilocal Cohen–Macaulay ring that is analytically reduced, residually rational and has large residue fields (i.e. $|R/\mathfrak{m}| \geq |\{\text{branches of} \widehat{R_{\mathfrak{m}}}\}|$ for any $\mathfrak{m}$ maximal ideal of R). Value semigroup (ideals) are then defined as follows.

Definition 2 Let R be an admissible ring, and let $\mathfrak{V}_R$ be the set of (discrete) valuation rings of Q_R over R with corresponding valuations $\nu = (\nu_V)_{V \in \mathfrak{V}_R} \colon Q_R \to (\mathbb{Z} \cup \{\infty\})^{\mathfrak{V}_R}$. To each regular fractional ideal $\mathcal{E}$ of R we associate its *value semigroup ideal* $\Gamma_{\mathcal{E}} := \nu(\mathcal{E}^{\text{reg}}) \subset \mathbb{Z}^{\mathfrak{V}_R}$. If $\mathcal{E} = R$, then the monoid Γ_R is called the *value semigroup* of R.

If $\mathcal{E}$ is a regular fractional ideal of R, then $\Gamma_{\mathcal{E}}$ is a semigroup satisfying particular properties, that we consider for any subset $E \subset \mathbb{Z}^s$:

(E0) there exists an $\alpha \in \mathbb{Z}^s$ such that $\alpha + \mathbb{N}^s \subset E$;
(E1) for any $\alpha, \beta \in E$, their component-wise minimum $\min\{\alpha, \beta\} \in E$;
(E2) for any $\alpha, \beta \in E$ with $\alpha_j = \beta_j$ for some j there exists an $\epsilon \in E$ such that $\epsilon_j > \alpha_j = \beta_j$ and $\epsilon_i \geq \min\{\alpha_i, \beta_i\}$ with equality if $\alpha_i \neq \beta_i$.

Definition 3 A submonoid S of $\mathbb{N}^s$ with group of differences $D_S = \mathbb{Z}^s$ is called a *good semigroup* if properties (E0), (E1), and (E2) hold for $E = S$.

A *semigroup ideal* of S is subset $E \subset \mathbb{Z}^s$ such that $E + S \subset E$ and $\alpha + E \subset S$ for some $\alpha \in \mathbb{Z}^s$. It is called a *good semigroup ideal* of the good semigroup S if it satisfies (E1) and (E2).

Proposition 4 *Let R be an admissible ring. Then,*

(i) the value semigroup Γ_R is a good semigroup;
(ii) for any regular fractional ideal $\mathcal{E}$ of R, $\Gamma_{\mathcal{E}}$ is a good semigroup ideal of Γ_R. □

On value semigroup ideals there is a distance function that mirrors the relative length of fractional ideals.

Definition 5 Let S be a good semigroup, and let $E \subset D_S$ be a subset. Then $\alpha, \beta \in E$ with $\alpha < \beta$ are called *consecutive* in E if $\alpha < \delta < \beta$ implies $\delta \notin E$ for any $\delta \in D_S$. For $\alpha, \beta \in E$, a chain of points $\alpha^{(i)} \in E$,

$$\alpha = \alpha^{(0)} < \cdots < \alpha^{(n)} = \beta, \tag{1}$$

is said to be *saturated of length n* if $\alpha^{(i)}$ and $\alpha^{(i+1)}$ are consecutive in E for all $i = 0, \dots, n-1$. If E satisfies

(E4) for fixed $\alpha, \beta \in E$, any two saturated chains (1) in E have the same length n;

then we call $d_E(\alpha, \beta) := n$ the *distance* of α and β in E.

D'Anna [4, Prop. 2.3] proved that any good semigroup ideal E satisfies property (E4).

Definition 6 For a good semigroup ideal E, the *conductor of E* is defined as $\gamma^E := \min\{\alpha \in E \mid \alpha + \mathbb{N}^s \subset E\}$. We denote $\gamma := \gamma^S$ and $\tau := \gamma - \mathbf{1}$.

Definition 7 Let S be a good semigroup, and let $E \subset F$ be two semigroup ideals of S satisfying property (E4). Then we call

$$d(F \backslash E) := d_F(\mu^F, \gamma^E) - d_E(\mu^E, \gamma^E)$$

the *distance* between E and F.

In the following, we collect the main properties of the distance function $d(-\backslash-)$. It follows from the definition that it is additive, as proven by D'Anna in [4, Prop. 2.7]:

Lemma 8 *Let $E \subset F \subset G$ be semigroup ideals of a good semigroup S satisfying properties (E1) and (E4). Then $d(G \backslash E) = d(G \backslash F) + d(F \backslash E)$.* □

Moreover, the distance function detects equality as formulated in [4, Prop. 2.8] and proved in [5, Prop. 4.2.6].

Proposition 9 *Let S be a good semigroup, and let E, F be good semigroup ideals of S with $E \subset F$. Then $E = F$ if and only if $d(F \backslash E) = 0$.* □

The length of a quotient of fractional ideals corresponds to the distance between the corresponding good semigroup ideals; see [4, Prop. 2.2] and [5, Prop. 4.2.7].

Proposition 10 *Let R be an admissible ring. If $\mathcal{E}, \mathcal{F}$ are two regular fractional ideals of R such that $\mathcal{E} \subset \mathcal{F}$ then, $\ell_R(\mathcal{F}/\mathcal{E}) = d(\Gamma_{\mathcal{F}} \backslash \Gamma_{\mathcal{E}})$.* □

As a corollary, one can check equality of fractional ideals through their value semigroups:

Corollary 11 *Let R be an admissible ring, and let $\mathcal{E}, \mathcal{F}$ be two regular fractional ideals of R such that $\mathcal{E} \subset \mathcal{F}$. Then $\mathcal{E} = \mathcal{F}$ if and only if $\Gamma_{\mathcal{E}} = \Gamma_{\mathcal{F}}$.* □

3 Canonical Ideals and Main Results

The following is the canonical semigroup ideal as defined by D'Anna in [4].

Definition 12 We call the semigroup ideal

$$K_S^0 := \left\{\alpha \in \mathbb{Z}^s \mid \Delta^S(\tau - \alpha) = \emptyset\right\}.$$

the *normalized canonical semigroup ideal of* S, where

$$\Delta^S(\delta) := \Delta(\delta) \cap S = (\cup_{i \in I}\{\beta \in \mathbb{Z}^s \mid \delta i = \beta_i,\ \delta_j < \beta_j \ \forall\ j \neq i\}) \cap S$$

Definition 13 Let S be a good semigroup. Then S is called *symmetric* if $S = K_S^0$.

As mentioned in the introduction, Delgado proved that $S = \Gamma_R$ is symmetric if and only if R is Gorenstein. D'Anna [4] generalized this result: a regular fractional ideal $\mathcal{K}$ with $R \subset \mathcal{K} \subset \overline{R}$ is canonical if and only if $\Gamma_{\mathcal{K}} = K_S^0$. Recall that by definition a fractional ideal $\mathcal{K}$ is *canonical* if $\mathcal{K} : (\mathcal{K} : \mathcal{E}) = \mathcal{E}$ for any regular fractional ideal $\mathcal{E}$.

Definition 14 Let K be a good semigroup ideal of a good semigroup S. We call K a *canonical semigroup ideal* of S if $K \subset E$ implies $K = E$ for any good semigroup ideal E with $\gamma^K = \gamma^E$.

In analogy with this definition, we give a characterization of canonical semigroup ideals; see [5, Thm 5.2.7].

Theorem 15 *For a good semigroup ideal K of a good semigroup S the following are equivalent:*

(a) K is a canonical semigroup ideal;
(b) there exists an α such that $\alpha + K = K_S^0$;
(c) for all good semigroup ideals E one has $K - (K - E) = E$.

Moreover, if K satisfies these equivalent conditions, then $K - E$ is a good semigroup ideal for any good semigroup ideal E. □

Given this characterization, it is natural to ask if taking the dual commutes with taking the semigroup. In the Gorenstein case, Pol [3] gave a positive answer.

Theorem 16 *If R is a Gorenstein admissible ring then,*

$$\Gamma_{R:\mathcal{E}} = \left\{\alpha \in \mathbb{Z}^s \mid \Delta^E(\tau - \alpha) = \emptyset\right\} = \Gamma_R - \Gamma_{\mathcal{E}}$$

for any regular fractional ideal $\mathcal{E}$ of R.

Our main result extends Pols result beyond the Gorenstein case.

Theorem 17 *Let $\mathcal{K}$ be a canonical ideal of R and let $K := \Gamma_{\mathcal{K}}$. Then, the following diagram commutes:*

$$\begin{array}{ccc}
\left\{\begin{array}{r}\text{regular fractional}\\ \text{ideals of } R\end{array}\right\} & \xrightarrow{\mathcal{E}\mapsto\mathcal{K}:\mathcal{E}} & \left\{\begin{array}{r}\text{regular fractional}\\ \text{ideals of } R\end{array}\right\} \\
{\scriptstyle \mathcal{E}\mapsto\Gamma_{\mathcal{E}}}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{E}\mapsto\Gamma_{\mathcal{E}}} \\
\left\{\begin{array}{r}\text{good semigroup}\\ \text{ideals of } \Gamma_R\end{array}\right\} & \xrightarrow{E\mapsto K-E} & \left\{\begin{array}{r}\text{good semigroup}\\ \text{ideals of } \Gamma_R\end{array}\right\}
\end{array}$$

Proof It is not restrictive to assume R local and $R \subset \mathcal{K} \subset \overline{R}$. Hence $K := \Gamma_{\mathcal{K}} = K_S^0$ by D'Anna [4].

Let $\mathcal{E} \subset \mathcal{F}$ be regular fractional ideals of R. Proposition 10 then yields

$$d(\Gamma_{\mathcal{K}:\mathcal{E}} \backslash \Gamma_{\mathcal{K}:\mathcal{F}}) = \ell_R((\mathcal{K}:\mathcal{E})/(\mathcal{K}:\mathcal{F})) = \ell_R(\mathcal{F}/\mathcal{E}) = d(\Gamma_{\mathcal{F}} \backslash \Gamma_{\mathcal{E}}) =: n.$$

Notice that $\ell_R((\mathcal{K}:\mathcal{E})/(\mathcal{K}:\mathcal{F})) = \ell_R(\mathcal{F}/\mathcal{E})$ as K is canonical. There is a composition series of regular fractional ideals

$$\mathcal{C}_{\mathcal{E}} = \mathcal{E}_0 \subsetneq \mathcal{E}_1 \subsetneq \cdots \subsetneq \mathcal{E}_l = \mathcal{E} \subsetneq \mathcal{E}_{l+1} \subsetneq \cdots \subsetneq \mathcal{E}_{l+n} = \mathcal{F},$$

where $\mathcal{C}_{\mathcal{E}}$ is the conductor of $\mathcal{E}$. By Corollary 11, applying Γ yields a chain of good semigroup ideals of Γ_R

$$C_{\Gamma_{\mathcal{E}}} = \Gamma_{\mathcal{E}_0} \subsetneq \Gamma_{\mathcal{E}_1} \subsetneq \cdots \subsetneq \Gamma_{\mathcal{E}_l} = \Gamma_{\mathcal{E}} \subsetneq \Gamma_{\mathcal{E}_{l+1}} \subsetneq \cdots \subsetneq \Gamma_{\mathcal{E}_{l+n}} = \Gamma_{\mathcal{F}}.$$

By Corollary 11 and Theorem 15(c), dualizing with K yields a chain of good semigroup ideals of Γ_R

$$\begin{aligned}\Gamma_{\mathcal{K}:\mathcal{C}_{\mathcal{E}}} = \Gamma_{\mathcal{K}} - \Gamma_{\mathcal{C}_{\mathcal{E}}} = K - C_{\Gamma_{\mathcal{E}}} = K - C_{\Gamma_{\mathcal{E}_0}} \supsetneq \cdots \supsetneq K - \Gamma_{\mathcal{E}_l} = K - \Gamma_{\mathcal{E}} \\ \supsetneq K - \Gamma_{\mathcal{E}_{l+1}} \supsetneq \cdots \supsetneq K - \Gamma_{\mathcal{E}_{l+n}} = K - \Gamma_{\mathcal{F}} \supset \Gamma_{\mathcal{K}:\mathcal{F}}. \quad (2)\end{aligned}$$

By Theorem 15, $K - \Gamma_{\mathcal{E}_i}$ is a good semigroup ideal of S for all $i = 0, \ldots, l + n$. Hence, using Proposition 9, we obtain $d(K - \Gamma_{\mathcal{E}_i} \backslash K - \Gamma_{\mathcal{E}_{i+1}}) \geq 1$ for all $i = 0, \ldots, l + n - 1$. On the other hand, by Proposition 10,

$$d(\Gamma_{\mathcal{K}:\mathcal{C}_{\mathcal{E}}} \setminus \Gamma_{\mathcal{K}:\mathcal{F}}) = \ell_R(\mathcal{K} : \mathcal{C}_{\mathcal{E}}/\mathcal{K} : \mathcal{F}) = \ell_R(\mathcal{F}/\mathcal{C}_{\mathcal{E}}) = l + n.$$

By Lemma 8 and (2), it follows that $d(K - \Gamma_{\mathcal{E}_i} \setminus K - \Gamma_{\mathcal{E}_{i+1}}) = 1$ for all $i = 0, \ldots, l + n - 1$ and $d(K - \Gamma_{\mathcal{F}} \setminus \Gamma_{\mathcal{K}:\mathcal{F}}) = 0$. By Proposition 9 the latter is equivalent to the second claim.

In particular, this implies the following

Corollary 18 *Let $\mathcal{K}$ be a fractional ideal of an admissible ring R. Then $\mathcal{K}$ is canonical if and only if $K := \Gamma_{\mathcal{K}}$ canonical.*

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