Chapter 7
Wave Dynamics: Propagation and Resonance in Inhomogeneous Plasma

Plasma is a dispersive medium and wave propagation can be described by the Eikonal equation derived by Landau. Wave energy is also defined. If there is no dissipation in the plasma, Lagrange–Hamilton formulation is applicable and the conservation law is obtained. The zero approximation as the dispersive medium is the dissipationless cold plasma approximation ignoring the thermal effect. Cutoff and resonance occurs in this approximation.

The wave propagation in non-uniform plasma is important in confined plasma and energy absorption occurs in the resonance layer. The drift wave appears universally in the confined plasma and is unstable above the critical temperature gradient, producing turbulence through wave–wave interactions.

7.1 Eikonal Equation: Dynamics of Wave Propagation

Wave propagation in a dispersive media with a local dispersion relation $\omega = \Omega(\boldsymbol{k}, \boldsymbol{x}, t)$ cannot be described by a simple plane–wave approximation $e^{i(k \cdot x - \Omega t)}$ but is described by the Eikonal form $\boldsymbol{x} = \boldsymbol{A} e^{i\zeta(x,t)} + cc$ with its phase (or eikonal) ζ and amplitude $\boldsymbol{A}$ [1]. The angular frequency ω and wave number $\boldsymbol{k}$ are related to eikonal ζ by the following equation,

$$\omega = -\frac{\partial \zeta}{\partial t}\ , \tag{7.1}$$

$$\boldsymbol{k} = \frac{\partial \zeta}{\partial \boldsymbol{x}}\ . \tag{7.2}$$

From the analytical condition of the second-order derivative $\partial^2 \zeta / \partial t \partial \boldsymbol{x}$, ω and $\boldsymbol{k}$ should satisfy following equation:

$$\frac{\partial \boldsymbol{k}}{\partial t} = -\frac{\partial \omega}{\partial \boldsymbol{x}}\ . \tag{7.3}$$

From the expression of local dispersion relation $\omega = \Omega(\boldsymbol{k}, \boldsymbol{x}, t)$, we obtain

$$\frac{\partial \omega}{\partial \boldsymbol{x}} = \boldsymbol{v}_g \cdot \frac{\partial \boldsymbol{k}}{\partial \boldsymbol{x}} + \frac{\partial \Omega}{\partial \boldsymbol{x}} |_{\boldsymbol{k}} \quad . \tag{7.4}$$

Here the group velocity of the wave packet $\boldsymbol{v}_g = \partial\omega/\partial\boldsymbol{k} = \partial\Omega/\partial\boldsymbol{k}|_x$ is used. From Equations 7.3 and 7.4, we can see that the velocity of the wave packet center $\mathrm{d}\boldsymbol{x}/\mathrm{d}t$ and the time variation of the wave number in the moving frame with the group velocity $\mathrm{d}\boldsymbol{k}/\mathrm{d}t = \partial\boldsymbol{k}/\partial t + \boldsymbol{v}_g \partial\boldsymbol{k}/\partial\boldsymbol{x}$ satisfy the following Hamilton equation:

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \left(\frac{\partial \Omega}{\partial \boldsymbol{k}}\right)_{\boldsymbol{x}} \quad , \tag{7.5}$$

$$\frac{\mathrm{d}\boldsymbol{k}}{\mathrm{d}t} = -\left(\frac{\partial \Omega}{\partial \boldsymbol{x}}\right)_{\boldsymbol{k}} \quad . \tag{7.6}$$

Here, Ω plays the role of the Hamiltonian and $\boldsymbol{k}$ the role of canonical momentum. They are also called Eikonal equations. The variational principle to give this eikonal equation is given by $\delta \int \mathcal{L}\,\mathrm{d}t = 0$ with $\mathcal{L} = \boldsymbol{k} \cdot \boldsymbol{x} - \Omega$. Substituting the eikonal expression into the Maxwell equation, $\boldsymbol{E}, \boldsymbol{B} \sim \hat{\boldsymbol{E}}, \hat{\boldsymbol{B}} \mathrm{e}^{\mathrm{i}\xi(x,t)} + cc$, we obtain

$$\boldsymbol{k} \times \hat{\boldsymbol{B}} = -i\mu_0 \hat{\boldsymbol{J}} - \frac{\omega}{c^2} \hat{\boldsymbol{E}} \ , \tag{7.7}$$

$$\boldsymbol{k} \times \hat{\boldsymbol{E}} = \omega \hat{\boldsymbol{B}} \ . \tag{7.8}$$

We assume that the following linear relation holds using the electrical conductivity tensor σ (see Section 7.3 for details):

$$\hat{\boldsymbol{J}} = \boldsymbol{\sigma} \hat{\boldsymbol{E}} \ . \tag{7.9}$$

This is Ohm's law and Equation 7.7 leads to

$$\boldsymbol{k} \times \hat{\boldsymbol{B}} = -\frac{\omega}{c^2} \boldsymbol{K}(\omega, \boldsymbol{k}) \cdot \hat{\boldsymbol{E}} \ , \tag{7.10}$$

$$\boldsymbol{K}(\omega, \boldsymbol{k}) = \boldsymbol{I} + \frac{\mathrm{i}\boldsymbol{\sigma}}{\varepsilon_0 \omega} \ . \tag{7.11}$$

where $\boldsymbol{K}$ is the dielectric tensor. From Equations 7.8 and 7.10, vectors $\boldsymbol{k}$, $\boldsymbol{K}(\omega, \boldsymbol{k}) \cdot \hat{\boldsymbol{E}}$, and $\boldsymbol{B}$ are orthogonal within the Eikonal approximation. Eliminating $\boldsymbol{B}$ from these equations, we obtain

$$\boldsymbol{M} \cdot \hat{\boldsymbol{E}} = 0 \ , \tag{7.12}$$

$$\boldsymbol{M} = (\boldsymbol{k}\boldsymbol{k} - k^2 \boldsymbol{I})/k_0^2 + \boldsymbol{K} \tag{7.13}$$

where $k_0 = \omega/c$. As is well known in electromagnetism, the following Poynting theorem holds among the electromagnetic energy $\boldsymbol{B}^2/2\mu_0 + \varepsilon_0 \boldsymbol{E}^2/2$, Joule losses $\boldsymbol{J} \cdot \boldsymbol{E}$, and Poynting vector $\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{B}/\mu_0$:

$$\frac{\partial}{\partial t}\left(\frac{\boldsymbol{B}^2}{2\mu_0} + \frac{\varepsilon_0 \boldsymbol{E}^2}{2}\right) = -\boldsymbol{J} \cdot \boldsymbol{E} - \nabla \cdot \boldsymbol{S} \ . \tag{7.14}$$

This equation shows that the Poynting vector $\boldsymbol{S}$ is the energy flux density. If we take time and space derivatives of the amplitude in slowly changing medium up to the first order $(\omega + \mathrm{i}\partial/\partial t, \mathrm{i}\boldsymbol{k} + \nabla)$, $\boldsymbol{J}$ and $\boldsymbol{B}$ are given as follows,

$$\hat{\boldsymbol{J}} + \delta\hat{\boldsymbol{J}} = \boldsymbol{\sigma}\left(\omega + \mathrm{i}\frac{\partial}{\partial t}\right)(\hat{\boldsymbol{E}} + \delta\hat{\boldsymbol{E}}) = \boldsymbol{\sigma}(\omega)\hat{\boldsymbol{E}} + \boldsymbol{\sigma}(\omega)\delta\hat{\boldsymbol{E}} + \frac{\partial\boldsymbol{\sigma}}{\partial\omega}\mathrm{i}\frac{\partial\hat{\boldsymbol{E}}}{\partial t}\ , \tag{7.15}$$

$$\mathrm{i}\omega\delta\hat{\boldsymbol{B}} = \mathrm{i}\boldsymbol{k} \times \delta\hat{\boldsymbol{E}} + \frac{\partial\hat{\boldsymbol{B}}}{\partial t} + \nabla \times \hat{\boldsymbol{E}}\ . \tag{7.16}$$

Here $\hat{\boldsymbol{B}} = \boldsymbol{k} \times \hat{\boldsymbol{E}}/\omega$. Substituting these equations into Equation 7.14, we can obtain the following equations:

$$\frac{\partial\mathcal{E}}{\partial t} + \nabla \cdot \hat{\boldsymbol{S}} = Q\ , \tag{7.17}$$

$$\mathcal{E} = \frac{1}{2}\left(\varepsilon_0\hat{\boldsymbol{E}}^* \cdot \frac{\partial(\omega\boldsymbol{K}_\mathrm{h})}{\partial\omega} \cdot \hat{\boldsymbol{E}} + \frac{1}{\mu_0}\hat{\boldsymbol{B}}^* \cdot \hat{\boldsymbol{B}}\right), \tag{7.18}$$

$$\hat{\boldsymbol{S}} = Re(\hat{\boldsymbol{E}}^* \times \hat{\boldsymbol{B}})/\mu_0 \tag{7.19}$$

$$Q = \hat{\boldsymbol{E}}^* \cdot \boldsymbol{\sigma}_\mathrm{h} \cdot \hat{\boldsymbol{E}}\ , \tag{7.20}$$

$$\boldsymbol{K}_\mathrm{h} = (\boldsymbol{K} + \boldsymbol{K}^+)/2\ , \tag{7.21}$$

$$\boldsymbol{\sigma}_\mathrm{h} = (\boldsymbol{\sigma} + \boldsymbol{\sigma}^+)/2\ . \tag{7.22}$$

Here, $\boldsymbol{K}_\mathrm{h}$, $\boldsymbol{\sigma}_\mathrm{h}$ are the Hermitian part in each component, $\hat{\boldsymbol{E}}^*$ and $\hat{\boldsymbol{B}}^*$ are complex conjugate vectors, $\boldsymbol{K}^+$ and $\boldsymbol{\sigma}^+$ are complex conjugate tensors. Using Equations 7.12 and 7.13, wave energy $\mathcal{E}$ is given as follows,

$$\mathcal{E} = \frac{\varepsilon_0}{2}\hat{\boldsymbol{E}}^* \cdot \frac{\partial(\omega\boldsymbol{M}_\mathrm{h})}{\partial\omega} \cdot \hat{\boldsymbol{E}}\ , \tag{7.23}$$

$$\mathcal{E} = \omega\mathcal{J}\ , \tag{7.24}$$

$$\mathcal{J} = \frac{\varepsilon_0}{2}\hat{\boldsymbol{E}}^* \cdot \frac{\partial\boldsymbol{M}_\mathrm{h}}{\partial\omega} \cdot \hat{\boldsymbol{E}}\ . \tag{7.25}$$

Here, J is a "wave action" and is the adiabatic invariant if there is no dissipation. The wave energy form seen in Equation 7.18 was derived by M. von Laue in 1905 for a dispersive medium [2]. In non-thermodynamic equilibrium plasma, this wave energy can take a negative value and is called "negative energy wave" [3]. If we regard $(\boldsymbol{x}, \boldsymbol{k})$ as independent variables, the wave action J is conserved along the trajectory in the phase space $(\boldsymbol{x}, \boldsymbol{k})$, in case there is no dissipation. So, J follows

$$\frac{\partial J}{\partial t} + \dot{\boldsymbol{k}} \cdot \frac{\partial J}{\partial\boldsymbol{k}} + \dot{\boldsymbol{x}} \cdot \frac{\partial J}{\partial\boldsymbol{x}} = 0\ . \tag{7.26}$$

Substituting Equations 7.5 and 7.6 into Equation 7.26, we obtain the following equation:

$$\frac{\partial J}{\partial t} + [J, \Omega] = 0\ , \tag{7.27}$$

where $[J, \Omega]$ is the Poisson bracket given by

$$[J, \Omega] = \frac{\partial \Omega}{\partial \boldsymbol{k}} \cdot \frac{\partial J}{\partial \boldsymbol{x}} - \frac{\partial \Omega}{\partial \boldsymbol{x}} \cdot \frac{\partial J}{\partial \boldsymbol{k}} \ . \tag{7.28}$$

Equation 7.27 is called the "wave kinetic equation."

7.2 Lagrange Wave Dynamics: Ideal and Dissipative Systems

As described in Goldstein [4], Lagrange mechanics in a continuum are reduced to the variational principle with the action integral of time and space integration of the Lagrangian density L. Whitham formulated the Lagrange mechanics for dissipationless ideal plasma in 1965 [5,6], while the dynamics of the dissipative plasma wave have not been formulated [7]. The action integral S is given by

$$S = \int \mathrm{d}t \int L \mathrm{d}V \tag{7.29}$$

$$L = L_M(\boldsymbol{A}, \Phi) + \sum_a L_a(\boldsymbol{\xi}_a, \boldsymbol{A}, \Phi) \ , \tag{7.30}$$

$$L_M(\boldsymbol{A}, \Phi) = \varepsilon_0 \left[\frac{\partial \boldsymbol{A}}{\partial t} + \nabla \Phi \right]^2 - \frac{1}{\mu_0} (\nabla \times \boldsymbol{A})^2 \ , \tag{7.31}$$

$$L_a(\boldsymbol{\xi}_a, \boldsymbol{A}, \Phi) = n_a \left[\frac{m_a}{2} \dot{\boldsymbol{\xi}}_a^2 + e_a \left(\dot{\boldsymbol{\xi}}_a \cdot \boldsymbol{A}(\boldsymbol{x} + \boldsymbol{\xi}_a, t) - \Phi(\boldsymbol{x} + \boldsymbol{\xi}_a, t) \right) \right] \ . \tag{7.32}$$

Here, L_M (Equation 4.25) and L_a are Lagrangians for fields and particles, respectively. The Lagrangian of particle a can be expanded as a quadratic form as follows,

$$[L_a(\boldsymbol{\xi}_a, \boldsymbol{A}, \Phi)]_{\mathrm{lin}} = n_a \left[\frac{m_a}{2} \dot{\boldsymbol{\xi}}_a^2 + e_a \ \boldsymbol{\xi}_a \cdot (\dot{\boldsymbol{\xi}}_a \times \boldsymbol{B}_0) + e_a \dot{\boldsymbol{\xi}}_a \cdot \tilde{\boldsymbol{A}} - e_a \boldsymbol{\xi}_a \cdot \nabla \tilde{\Phi} \right] \ . \tag{7.33}$$

Lagrangian density can be rewritten in the following form by using the wave eikonal form $\boldsymbol{\xi} = \boldsymbol{\xi} \mathrm{e}^{\mathrm{i}\zeta(x,t)} + \mathrm{c.c.}$ and $\boldsymbol{E} = -\partial \boldsymbol{A}/\partial t - \nabla \Phi$ as follows,

$$[L]_{\mathrm{lin}} = \varepsilon_0 \hat{\boldsymbol{E}}^* \cdot \boldsymbol{M} \cdot \hat{\boldsymbol{E}} \ . \tag{7.34}$$

Here, cold plasma is a typical example of non-dissipative plasma and its $\boldsymbol{M}$ is given in next section. Variation of S with respect to $\hat{\mathbf{E}}^*$ gives the local dispersion relation,

$$\boldsymbol{M} \cdot \hat{\boldsymbol{E}} = 0 \ . \tag{7.35}$$

In addition, the minimization with respect to the eikonal ζ gives the following Euler–Lagrange equation.

$$\frac{\partial}{\partial t} \left[\frac{\partial L}{\partial \omega} \right] + \frac{\partial}{\partial \boldsymbol{x}} \cdot \left[\frac{\partial L}{\partial \boldsymbol{k}} \right] = 0 \ . \tag{7.36}$$

Here,

$$J = \frac{\partial L}{\partial \omega} = \frac{\partial L}{\partial \dot{\zeta}} = \varepsilon_0 \hat{\boldsymbol{E}}^* \cdot \frac{\partial \boldsymbol{M}}{\partial \omega} \cdot \hat{\boldsymbol{E}} \tag{7.37}$$

is the momentum conjugate to eikonal ζ and is the adiabatic invariant. Using $\partial L/\partial \boldsymbol{k} = (\partial L/\partial \omega)(\partial \omega/\partial \boldsymbol{k}) = \boldsymbol{v}_g(\partial L/\partial \omega) = \boldsymbol{v}_g J$, J corresponds to the number of photons in the wave packet and Equation 7.36 gives the conservation law for the number of photons.

$$\frac{\partial J}{\partial t} + \frac{\partial}{\partial \boldsymbol{x}} \cdot (\boldsymbol{v}_g J) = 0 \, . \tag{7.38}$$

Hamilton mechanics is a powerful technique in solving mathematical problems in plasma dynamics, but it cannot be applied to a dissipative system as it stands. Hamilton mechanics of dissipative systems can be formulated by the adjoint-variable method [8]. Consider ordinary differential equations in general with n-dimensional variable $\boldsymbol{x}$:

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{f}(\boldsymbol{x}, t) \, . \tag{7.39}$$

We introduce a new n dimensional variable $\boldsymbol{p}$, and define L as

$$L = \boldsymbol{p} \cdot \left(\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} - \boldsymbol{f} \right) = \boldsymbol{p} \cdot \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} - H \, . \tag{7.40}$$

Here $H = \boldsymbol{p} \cdot \boldsymbol{f}(\boldsymbol{x}, t)$ plays the role of the Hamiltonian. Also, it is easy to see that

$$\frac{\partial L}{\partial (\mathrm{d}\boldsymbol{x}/\mathrm{d}t)} = \boldsymbol{p} \, . \tag{7.41}$$

This means that $\boldsymbol{p}$ is momentum conjugate to $\boldsymbol{x}$. Then, we obtain the following Hamilton equation:

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \frac{\partial H}{\partial \boldsymbol{p}} = \boldsymbol{f}(\boldsymbol{x}, t) \, , \tag{7.42}$$

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = -\frac{\partial H}{\partial \boldsymbol{x}} = -\frac{\partial \boldsymbol{f}(\boldsymbol{x}, t)}{\partial \boldsymbol{x}} \cdot \boldsymbol{p} \, . \tag{7.43}$$

In other words, any system of ordinary differential equations including dissipation can be attributed to the Hamilton system by doubling the variables. Application of this formulation to dissipative plasma dynamics is left for future study.

7.3 Plasma as a Dielectric Medium: Cold and Hot Plasmas

Plasma is a dielectric media in which various waves can propagate. We assume that perturbed electromagnetic fields are given by a plane wave, $\boldsymbol{E}_1 = \boldsymbol{E} \exp(\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{x} - \mathrm{i}\omega t)$, $\boldsymbol{B}_1 = \boldsymbol{b} \exp(\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{x} - \mathrm{i}\omega t)$ and $\boldsymbol{J}_1 = \boldsymbol{j} \exp(\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{x} - \mathrm{i}\omega t)$. The Maxwell equation gives $\boldsymbol{k} \times \boldsymbol{B}_1 = -\mathrm{i}\omega\mu_0 \boldsymbol{J}_1 - \omega \boldsymbol{E}_1$. Combining this with Ohm's law $\boldsymbol{J}_1 = \boldsymbol{\sigma} \cdot \boldsymbol{E}_1$ leads to the relation $\boldsymbol{k} \times \boldsymbol{B}_1 = -(\omega/c^2)\boldsymbol{K} \cdot \boldsymbol{E}_1$, where $\boldsymbol{K} = \boldsymbol{I} + \mathrm{i}\boldsymbol{\sigma}/\varepsilon_0\omega$ is the dielectric tensor. Substitution of $\boldsymbol{k} \times \boldsymbol{B}_1 = -(\omega/c^2)\boldsymbol{K} \cdot \boldsymbol{E}_1$ into $\boldsymbol{k} \times \boldsymbol{E} = \omega \boldsymbol{B}_1$ gives,

$$\boldsymbol{M} \cdot \boldsymbol{E} = 0\,, \; \boldsymbol{M} = (\boldsymbol{kk} - \boldsymbol{I})\left(\frac{kc}{\omega}\right)^2 + \boldsymbol{K}\,. \tag{7.44}$$

The solvable condition is given as $M = \det(\boldsymbol{M}) = 0$ and is called the dispersion relation. The group velocity of the wave $\boldsymbol{v}_\mathrm{g}$ can be obtained from the $\boldsymbol{k}$ derivative of the dispersion relation $M(\omega, \boldsymbol{k}, \boldsymbol{x}, t) = \det(\boldsymbol{M}) = 0$ as $\boldsymbol{v}_\mathrm{g} = \partial\Omega/\partial\boldsymbol{k} = (\partial M/\partial\boldsymbol{k})/(\partial M/\partial\omega)$.

When the plasma temperature is low or the wave phase speed (ω/k) is much higher than the thermal speed ($v_\mathrm{th} \ll \omega/k$), the effect of thermal motion can be neglected and is called "cold plasma." When the plasma temperature is not too low, wave propagation is influenced by the sound wave and the pressure effect must be taken into account for the dielectric tensor. This is "warm plasma." When the resonant wave-particle interaction such as Landau damping plays some role, the Vlasov equation must be solved to obtain the dielectric tensor, and this is called "hot plasma."

In the cold plasma case, an important characteristic is that dielectric tensor $\boldsymbol{K}$ does not have any k dependence and is given as follows [9],

$$\boldsymbol{K} = \begin{bmatrix} S & -\mathrm{i}D & 0 \\ \mathrm{i}D & S & 0 \\ 0 & 0 & P \end{bmatrix}\,. \tag{7.45}$$

$$S = 1 - \sum_a \frac{\omega_{pa}^2}{\omega^2 - \Omega_a^2}\,, \quad D = \sum_a \frac{\Omega_a}{\omega}\frac{\omega_{pa}^2}{\omega^2 - \Omega_a^2}\,, \quad P = 1 - \sum_a \frac{\omega_{pa}^2}{\omega^2}\,,$$

$$\omega_{pa}^2 = \frac{n_a\, e_a^2}{\varepsilon_0 m_a}\,, \quad \Omega_a = \frac{e_a\, B}{m_a}\,.$$

Let θ be the angle between $\boldsymbol{B}$ and $\boldsymbol{k}$ and the refractive index $n = kc/\omega$, the following dispersion relation is obtained:

$$[S\sin^2\theta + P\cos^2\theta]n^4 - [RL\sin^2\theta + PS(1+\cos^2\theta)]n^2 + PRL = 0 \tag{7.46}$$

where $R = (S + D)/2$ and $L = (S - D)/2$. The condition of the refractive index $n = 0$ (phase velocity $= \omega/k = \infty$) is called the "cut-off," and $n = \infty$ (phase velocity $= \omega/k = 0$) is called "resonance." From Equation 7.46

$$\text{Cut-off condition } (n = 0): \qquad PRL = 0\,, \tag{7.47}$$

$$\text{Resonance condition } (n = \infty): \qquad \tan^2\theta = -\frac{P}{S}\,. \tag{7.48}$$

If we have a cut-off layer in the plasma, the plasma wave can only propagate an evanescent wave. On the other hand, the resonance is important for plasma heating and as a damping mechanism for instability. Considering propagation parallel to the magnetic field ($\theta = 0$), S becomes ∞ at $\omega = \Omega_a$ and gives resonance. This is the cyclotron resonance. In the case of propagation perpendicular to magnetic field ($\theta = \pi/2$), the resonance condition is $S = 0$. Finally, it is worth noting that the cold plasma dielectric tensor has various symmetries [5].

Time symmetry: $\boldsymbol{K}(-\omega) = \boldsymbol{K}^*(\omega)$ by the time symmetry of the equation of motion.
Onsager Symmetry: $\boldsymbol{K}(-\boldsymbol{B}) = \boldsymbol{K}^t(\boldsymbol{B})$ corresponding to the Onsager theorem.
Hermitian: $\boldsymbol{K} = \boldsymbol{K}^+$ (+: complex conjugate) corresponding to energy conservation. If energy is not conserved, the system is not Hermitian.

In the hot plasma case, the plasma response as a dielectric medium can be expressed by the dielectric tensor $\boldsymbol{K}$, as for the cold plasma wave. But the structure of $\boldsymbol{K}$ is more complicated compared to that in cold plasma. Since $\boldsymbol{K} = \boldsymbol{I} + \mathrm{i}\boldsymbol{\sigma}/\varepsilon_0\omega$ and the conductivity tensor is related to the perturbed velocity distribution function $f_{a1k}(\boldsymbol{v})$ as $\boldsymbol{J} = \boldsymbol{\sigma}\boldsymbol{E}$ and $\boldsymbol{J} = \sum e_a \int \boldsymbol{v} f_{a1k}(\boldsymbol{v})\mathrm{d}\boldsymbol{v}$, we have to solve the following linearized Vlasov equation:

$$\frac{\partial f_{a1k}}{\partial t} + \boldsymbol{v}\cdot\frac{\partial f_{a1k}}{\partial \boldsymbol{x}} + \frac{e_a}{m_a}(\boldsymbol{v}\times\boldsymbol{B})\cdot\frac{\partial f_{a1k}}{\partial \boldsymbol{v}} = -\frac{e_a}{m_a}(\boldsymbol{E}_1 + \boldsymbol{v}\times\boldsymbol{B}_1)\cdot\frac{\partial f_{a0}}{\partial \boldsymbol{v}} . \tag{7.49}$$

Here, the suffix for equilibrium value for $\boldsymbol{B}$ is suppressed for simplicity. Since the left-hand side of Equation 7.49 is the Lagrange derivative Df_{a1k}/Dt along the unperturbed charged particle orbit, $f_{a1k}(\boldsymbol{v})$ is readily obtained as follows [9],

$$f_{a1k}(\boldsymbol{x},\boldsymbol{v},t) = -\frac{e_a}{m_a}\int_{-\infty}^{t}\left(\boldsymbol{E}_1(\boldsymbol{x}(t'),t') + \frac{1}{\omega}\boldsymbol{v}(t')\times(\boldsymbol{k}\times\boldsymbol{E}_1(\boldsymbol{x}(t'),t'))\right) \cdot\frac{\partial f_0(\boldsymbol{v}(t'))}{\partial \boldsymbol{v}}\mathrm{d}t' . \tag{7.50}$$

Here, $\boldsymbol{B}_1 = (\boldsymbol{k}\times\boldsymbol{E}_1)/\omega$ is used. The particle position at t', $\boldsymbol{x}(t')$ is given by a combination of cyclotron excursion and the original position as follows,

$$\begin{aligned} x(t') &= x(t) + \frac{v_\perp}{\Omega}\left(\sin(\theta + \Omega(t'-t)) - \sin\theta\right) , \\ y(t') &= y(t) - \frac{v_\perp}{\Omega}\left(\cos(\theta + \Omega(t'-t)) - \cos\theta\right) , \\ z(t') &= z(t) + v_z(t'-t) . \end{aligned} \tag{7.51}$$

Here, the direction of $\boldsymbol{B}$ is chosen as z. Substitution of Equation 7.51 into Equation 7.50 gives the following form for $f_{a1k}(\boldsymbol{v})$:

$$\begin{aligned} f_{a1k}(\boldsymbol{x},\boldsymbol{v},t) = &-\frac{e_a}{m_a}\mathrm{e}^{\mathrm{i}(k_x x + k_z z - \omega t)} \\ &\times\int_{-\infty}^{t}\left(\left(1 - \frac{\boldsymbol{k}\cdot\boldsymbol{v}(t')}{\omega}\right)\boldsymbol{E} + (\boldsymbol{v}(t')\cdot\boldsymbol{E})\frac{\boldsymbol{k}}{\omega}\right)\cdot\frac{\partial f_0(\boldsymbol{v}(t'))}{\partial \boldsymbol{v}} \\ &\times\exp\left(\mathrm{i}\frac{k_x v_\perp}{\Omega}\left(\sin(\theta + \Omega(t'-t)) - \sin\theta\right) + i(k_z v_z - \omega)(t'-t)\right)\mathrm{d}t' . \end{aligned} \tag{7.52}$$

Here, the direction of the perpendicular wave vector is chosen as x. Using the Bessel function formula $\exp(\mathrm{i}a\sin\theta) = \sum J_m(a)\exp(\mathrm{i}m\theta)$ and $J_{-m}(a) = (-1)^m J_m(a)$, final form of the perturbed distribution function $f_{a1k}(\boldsymbol{x}, \boldsymbol{v}, t)$ is obtained and the dielectric tensor $\boldsymbol{K}$ is obtained by using $\boldsymbol{J} = \sum e_a \int \boldsymbol{v} f_{a1k}(\boldsymbol{v})\mathrm{d}\boldsymbol{v}$, $\boldsymbol{J} = \boldsymbol{\sigma E}$ and $\boldsymbol{K} = \boldsymbol{I} + \mathrm{i}\boldsymbol{\sigma}/\varepsilon_0\omega$ as follows for Maxwell distribution with $T_\parallel$ and $T_\perp$ [10] (note: the definition of v_{Ta} is different in [10]).

$$\boldsymbol{K} = \boldsymbol{I} + \sum_{a=i,e} \frac{\omega_{pa}^2}{\omega^2} \Bigg[2\eta_{0a}^2 \lambda_{Ta} \boldsymbol{L} + \sum_n \left(\zeta_{0a} Z(\zeta_{na}) - \left(1 - \frac{1}{\lambda_{Ta}}\right)(1 + \zeta_{na} Z(\zeta_{na})) \right) \times \exp(-b_a) \boldsymbol{X}_{na} \Bigg], \tag{7.53}$$

$$\boldsymbol{X}_{na} = \begin{bmatrix} (n^2/b_a) I_n & \mathrm{i}n(I_n' - I_n) & -2nI_n \lambda_{Ta}^{1/2} \eta_{na}/\alpha_a \\ -\mathrm{i}n(I_n' - I_n) & (n^2/b_a + 2b_a) I_n - 2b_a I_n' & \mathrm{i}\lambda_{Ta}^{1/2} \eta_{na} \alpha_a (I_n' - I_n) \\ -2nI_n \lambda_{Ta}^{1/2} \eta_{na}/\alpha_a & -\mathrm{i}\lambda_{Ta}^{1/2} \eta_{na} \alpha_a (I_n' - I_n) & 2\lambda_{Ta} \eta_{na}^2 I_n \end{bmatrix}, \tag{7.54}$$

$$\eta_{na} = \frac{\omega + n\Omega_a}{k_z v_{T\parallel a}}, \quad \lambda_{Ta} = \frac{T_{\parallel a}}{T_{\perp a}}, \quad b_a = \frac{1}{2}\left(\frac{k_\perp v_{T\perp a}}{\Omega_a}\right)^2, \quad v_{Ta} = \left(\frac{2T_a}{m_a}\right)^{1/2},$$

$$\zeta_{na} = \frac{\omega - k_\parallel u_{\parallel a} + n\Omega_a}{k_z v_{T\parallel a}}, \quad \alpha_a = \frac{k_x v_{T\perp a}}{\Omega_a}, \quad Z(\zeta) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{\exp(-\beta^2)}{\beta - \zeta},$$

Here, Z is called the plasma dispersion function and $u_{\parallel a}$ is the parallel flow velocity, the argument of the modified Bessel function is b_a. $\boldsymbol{L}$ is a tensor with $L_{zz} = 1$ and other components are 0.

Salon: Professor Thomas Stix

Professor Thomas Stix (Figure 7.1) was a plasma physicist. He wrote a pioneering text on plasma waves, *The Theory of Plasma Waves* in 1962 [9].

Figure 7.1 Prof. T. H. Stix (1924–2001) (Courtesy of Princeton Plasma Physics Laboratory)

Note: Causality and translational symmetry in plasma waves

Plasma exhibits characteristics of a dielectric medium and excites the "wave" as a collective motion [11]. Let $\boldsymbol{E}(\boldsymbol{x}', t')$ be an excited perturbation at $\boldsymbol{x}'$ in the time t'. In response to this excitation, the current $\boldsymbol{J}(\boldsymbol{x}, t)$ is created at another location in the plasma $\boldsymbol{x}$ at time t

$$\boldsymbol{J}(\boldsymbol{x}, t) = \boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{x}', t, t')\boldsymbol{E}(\boldsymbol{x}', t') \; . \tag{7.55}$$

Since $\boldsymbol{J}$ will not be created before the excitation of $\boldsymbol{E}$, $\boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{x}'; t, t') = 0$ for $t < t'$ (causality requirement). In the usual Ohm's law, $\boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{x}'; t, t') = \delta(\boldsymbol{x} - \boldsymbol{x}')\delta(t - t')$. Considering the linear response, the superposition principle is valid and the induced current is given as follows,

$$\boldsymbol{J}(\boldsymbol{x}, t) = \frac{1}{2\pi} \iint \mathrm{d}\boldsymbol{x}'\mathrm{d}t' \boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{x}', t, t')\boldsymbol{E}(\boldsymbol{x}', t') \; . \tag{7.56}$$

Moreover, if we assume "translational symmetry" in space and time (medium is uniform and stationary), $\boldsymbol{\sigma}$ becomes a function of $\boldsymbol{x} - \boldsymbol{x}'$ and $t - t'$.

$$\boldsymbol{J}(\boldsymbol{x}, t) = \frac{1}{2\pi} \iint \mathrm{d}\boldsymbol{x}'\mathrm{d}t' \boldsymbol{\sigma}(\boldsymbol{x} - \boldsymbol{x}', t - t')\boldsymbol{E}(\boldsymbol{x}', t') \; . \tag{7.57}$$

It is important to show that Ohm's law $\boldsymbol{J}(\omega, \boldsymbol{k}) = \boldsymbol{\sigma}(\omega, \boldsymbol{k})\boldsymbol{E}(\omega, \boldsymbol{k})$ is equivalent to the electrical conductivity depending only on $\boldsymbol{x} - \boldsymbol{x}'$ and $t - t'$. The Fourier transform and its inverse Fourier transform of the electric field perturbation $\boldsymbol{E}(\omega, \boldsymbol{k})$, $\boldsymbol{E}(\boldsymbol{x}, t)$ and $\boldsymbol{J}(\boldsymbol{x}, t)$ are given by the following equations:

$$\boldsymbol{E}(\omega, \boldsymbol{k}) = \frac{1}{2\pi} \iint \mathrm{d}\boldsymbol{x}\mathrm{d}t \mathrm{e}^{-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)} \boldsymbol{E}(\boldsymbol{x}, t) \; , \tag{7.58}$$

$$\boldsymbol{E}(\boldsymbol{x}, t) = \frac{1}{2\pi} \iint \mathrm{d}\boldsymbol{k}\mathrm{d}\omega \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)} \boldsymbol{E}(\omega, \boldsymbol{k}) \; . \tag{7.59}$$

$$\begin{aligned}
\boldsymbol{J}(\boldsymbol{x}, t) &= \frac{1}{2\pi} \iint \mathrm{d}\omega\mathrm{d}\boldsymbol{k} \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)} \boldsymbol{\sigma}(\omega, \boldsymbol{k})\boldsymbol{E}(\omega, \boldsymbol{k}) \\
&= \frac{1}{2\pi} \iint \mathrm{d}\omega\mathrm{d}\boldsymbol{k} \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)} \left[\frac{1}{2\pi} \iint \mathrm{d}\boldsymbol{x}''\mathrm{d}t'' \mathrm{e}^{-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}'' - \omega t'')} \boldsymbol{\sigma}(\boldsymbol{x}'', t'')\right] \\
&\quad \times \left[\frac{1}{2\pi} \iint \mathrm{d}\boldsymbol{x}'\mathrm{d}t' e^{-i(\boldsymbol{k}\cdot\boldsymbol{x}' - \omega t')} \boldsymbol{E}(\boldsymbol{x}', t')\right] \\
&= \frac{1}{(2\pi)^3} \iint \mathrm{d}\omega\mathrm{d}\boldsymbol{k} \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot(\boldsymbol{x} - \boldsymbol{x}' - \boldsymbol{x}'') - \omega(t - t' - t''))} \iint \mathrm{d}\boldsymbol{x}''\mathrm{d}t'' \boldsymbol{\sigma}(\boldsymbol{x}'', t'') \\
&\quad \times \iint \mathrm{d}\boldsymbol{x}'\mathrm{d}t' \boldsymbol{E}(\boldsymbol{x}', t') \\
&= \frac{1}{2\pi} \iint \mathrm{d}\boldsymbol{x}'\mathrm{d}t' \boldsymbol{\sigma}(\boldsymbol{x} - \boldsymbol{x}', t - t')\boldsymbol{E}(\boldsymbol{x}', t') \; .
\end{aligned} \tag{7.60}$$

Here, the properties of the Dirac delta function $\delta(\boldsymbol{x} - \boldsymbol{x}' - \boldsymbol{x}'') = (2\pi)^{-1} \int \mathrm{d}k \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}'-\boldsymbol{x}'')}$, $\delta(t - t' - t'') = (2\pi)^{-1} \int \mathrm{d}\omega \mathrm{e}^{-\mathrm{i}\omega(t-t'-t'')}$ are used. The causality saying "the response in the stable medium always appears after its excitation" requires that the response should follow the excitation and not come before.

7.4 Non-uniform Plasma: Alfven Wave Resonance and Continuous Spectrum

We describe here the Alfven resonance in non-uniform plasma since it plays an important role in the damping of the Alfven Eigen (AE) mode or resistive wall mode in toroidal confinement plasmas. Defining $n_\parallel = n \cos\theta$ and $n_\perp = n \sin\theta$, we can rewrite Equation 7.46 as follows,

$$n_\perp^2 = \frac{(R - n_\parallel^2)(L - n_\parallel^2)}{S - n_\parallel^2} \, . \tag{7.61}$$

In this case, the resonance condition is $S = n_\parallel^2$. Since $S < 0$ at $\Omega_i < \omega < \omega_{LH}$ ($\omega_{LH} = 1/[(\Omega_i^2 + \omega_{pe}^2)^{-1} + 1/|\Omega_i \Omega_e|]^{1/2}$ is the lower hybrid frequency), the resonance occurs ω below the ion cyclotron frequency. Since $S \sim 1 + (c^2/V_A^2)[\Omega_i^2/(\Omega_i^2 - \omega^2)] \sim c^2/V_A^2$ at $\omega \ll \Omega_i$, the Alfven resonance condition is given as follows,

$$\omega = k_\parallel V_A \, . \tag{7.62}$$

Figure 7.2 shows the schematics of Alfven resonance in high-density confined plasma. In the Alfven resonance, triple layer of cut-off–resonance–cut-off appears spatially close to each other. From the outward side this is the cut-off of the shear Alfven wave ($L - n_\parallel = 0$: $n_\perp = 0$ at $r = r_s$), Alfven resonance ($S - n_\parallel = 0$: $n_\perp = \infty$ at $r = r_A$), the cut-off of compressional Alfven wave ($R - n_\parallel = 0$: $n_\perp = 0$ at $r = r_s$). The wave equation in the vicinity of the resonance can be obtained by replacing $n_\perp$ of Equation 7.61 to $-\mathrm{i}(c/\omega)\mathrm{d}/\mathrm{d}x (x = r - r_A)$ as follows,

$$\frac{c^2}{\omega^2}\frac{\mathrm{d}^2 E}{\mathrm{d}x^2} + \frac{(R - n_\parallel^2)(L - n_\parallel^2)}{S - n_\parallel^2} E = 0 \, . \tag{7.63}$$

Assuming that density near the resonance points is proportional to x including the cut-off point, we define $y = xS'(0)/D(0)$ to convert Equation 7.63 to the following singular turning-point equation [9]:

$$\frac{\mathrm{d}^2 E}{\mathrm{d}y^2} + \frac{\lambda^2(y^2 - 1)}{y + \mathrm{i}\varepsilon} E = 0 \, , \tag{7.64}$$

$$\lambda^2 = \left| \frac{D^3 \omega^2}{c^2 (\mathrm{d}S(0)/\mathrm{d}x)^2} \right| \tag{7.65}$$

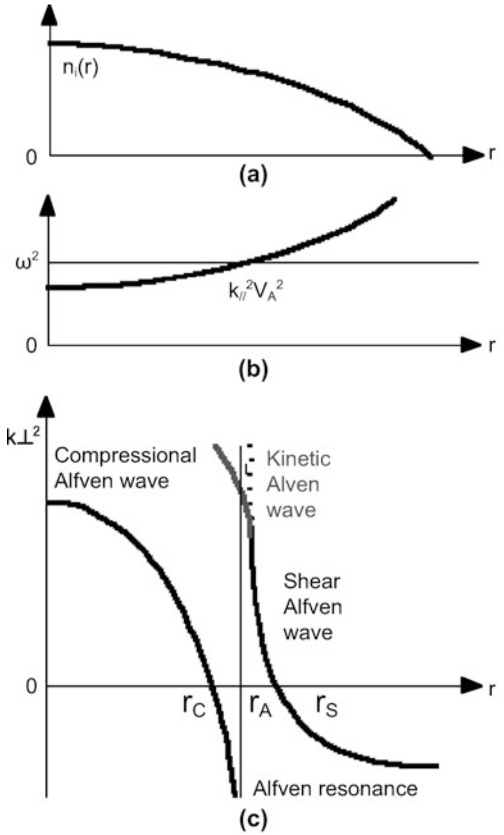

Figure 7.2 Schematics of Alfven resonance in tokamak plasma (cold plasma approximation) taking into account the effect of kinetic Alfven waves [13]. (a) Density profile; (b) Alfven frequency; (c) perpendicular wave number

where ε is a negative infinitesimal constant. As Budden showed in [12], complete absorption of the plasma wave occurs at the singular turning point. Singularity at $y = 0$ produces phase mixing discussed in Section 6.2 and Alfven wave energy is absorbed by the particles.

Near the Alfven resonance, kinetic effects become significant and the mode is converted to the Kinetic Alfven wave (KAW) [13]. Derivation of the kinetic dielectric tensor is lengthy. The dispersion relation of the KAW is given as follows at $(k_\perp \rho_i)^2 \ll 1$:

$$\omega^2 = k_\parallel^2 V_A^2 \left[1 + k_\perp^2 \rho_i^2 \left(\frac{3}{4} + \frac{T_e}{T_i} \right) \right] . \tag{7.66}$$

After conversion to the KAW mode, the Alfven wave damps via electron Landau damping. This wave damping mechanism is called "continuum damping" since it is caused by the existence of the continuous spectrum of the Alfven wave. This mechanism is important as a stabilizing mechanism of the resistive wall mode (RWM) destabilized by the finite electrical resistivity of the wall around the plasma.

If energetic particles exist in the plasma with velocities close to the Alfven velocity, the Alfven wave tends to be destabilized by getting energy from the energetic particles, but is stabilized if there is Alfven resonance in the plasma due to its strong continuum damping. On the other hand, if the Alfven resonance does not exist for some frequency band due to the toroidal coupling of the modes, etc., the Alfven wave becomes unstable due to a lack of continuum damping and is called the Alfven Eigen mode [14].

7.5 Drift Waves: Universal Waves in Confined Plasma

In confined plasma, there is a radial temperature gradient (∇T), density gradient (∇n), and a electrostatic potential gradient ($\nabla \Phi$) (see also Chapter 8). These gradients create first order flows on the flux surface and the coupling of these flows (drift) with the ion sound wave produces a drift wave. We only consider the density gradient (namely $\nabla T_i = \nabla T_e = 0$, in this case). Assuming collisionless ($\eta = 0$) and uniform T_e along $\boldsymbol{B}(\nabla_\parallel T_e = 0)$, particle conservation ($\partial n/\partial t + \nabla \cdot (n\boldsymbol{V}) = 0$), Ohm's law parallel to $\boldsymbol{B}(en_e E_\parallel + \nabla_\parallel P_e = 0)$, and the momentum balance equation parallel to $\boldsymbol{B}(m_i n \partial V_\parallel / \partial t = -\nabla_\parallel P)$ are given as follows,

Particle balance: $$\mathrm{i}\omega \tilde{n}_e + \mathrm{i}k_\parallel n_e \tilde{V}_\parallel + \mathrm{i}\omega_* n_e (e\tilde{\Phi}/T_e) = 0 \, , \tag{7.67}$$

Ohm's Law: $$-\mathrm{i}k_\parallel e n_e \tilde{\Phi} + \mathrm{i}k_\parallel T_e \tilde{n}_e = 0 \, , \tag{7.68}$$

Momentum balance: $$-\mathrm{i}\omega m_i n_i \tilde{V}_\parallel + \mathrm{i}k_\parallel (\tilde{p}_e + \tilde{p}_i) = 0 \, . \tag{7.69}$$

Here, $\omega_* = -(k_\perp T/eB)(\mathrm{d}\ln n_e/\mathrm{d}r)$ is the electron drift wave frequency. Equation 7.68 ($\tilde{n}_e/n_e = e\tilde{\Phi}/T_e$) is the "Boltzmann condition." If ions satisfy the adiabatic law $\tilde{p}_i = \gamma_i T_i \tilde{n}_i$, a combination of these equations leads to the following simple dispersion relation using $C_s = ((Z_i T_e + \gamma_i T_i)/m_i)^{1/2}$:

$$\omega(\omega - \omega_*) = k_\parallel^2 C_s^2 \, . \tag{7.70}$$

This drift wave is stable (ω is real) but is destabilized by the inclusion of the ion temperature gradient ($\nabla T_i \neq 0$). The ion energy equation $((3/2)(\partial p_i/\partial t + V_E \cdot \nabla p_i + (5/2) p_i \nabla_\parallel V_\parallel) = 0)$ is given by the following equation with $\gamma_i = 5/2$,

$$-\frac{3}{2}\mathrm{i}\omega \tilde{P}_i + \frac{3}{2}\mathrm{i}\omega_* \tau (1 + \eta_i) e n_e \tilde{\Phi} + \mathrm{i}\gamma_i n_i T_i k_\parallel \tilde{V}_\parallel = 0 \, . \tag{7.71}$$

Here $\tau = T_i/T_e, \eta_i = \mathrm{d}\ln T_i / \mathrm{d}\ln n_e$. Combining this with Equations 7.67, 7.68, and 7.69 including the effect of the ion temperature gradient, we obtain the dispersion relation of the ion temperature gradient (ITG) mode as follows,

$$\omega(\omega - \omega_*) = k_\parallel^2 C_s^2 \left[1 + \frac{\omega_*}{\omega} \frac{Z_i}{\gamma_i \tau + Z_i} \left(\eta_i - \frac{\gamma_i - Z_i}{Z_i} \right) \right] \, . \tag{7.72}$$

For the case of $Z_i = 1$, $\gamma_i = 5/2$, $\omega \sim k_\parallel C_s \ll \omega_*$, we obtain

$$\omega^2 \sim -\frac{k_\parallel^2 C_s^2}{2.5\tau + 1}\left(\eta_i - \frac{2}{3}\right) . \tag{7.73}$$

So, this drift wave becomes unstable when the ion temperature gradient exceeds a critical value, $\mathrm{d}T_i^{\mathrm{Crit}}/\mathrm{d}r$. In the real torus, the mode structure becomes more complex and is not so simple, but the threshold in the ion temperature gradient for destabilization exists [15].

The most basic equation describing the drift wave turbulence is the Hasegawa–Mima equation [16]. We derive this equation to see the fundamental process of drift wave turbulence. The guiding center motion is used for simplicity. Since the electrostatic approximation ($\boldsymbol{E} = -\nabla\Phi$) is applicable for the drift wave, we have

$$\boldsymbol{v}_{i\perp} = -\frac{\nabla\tilde{\Phi} \times \boldsymbol{B}}{B^2} - \frac{m_i}{eB^2}\frac{\mathrm{d}}{\mathrm{d}t}\nabla_\perp\tilde{\Phi} . \tag{7.74}$$

Here, the first term is $\boldsymbol{E} \times \boldsymbol{B}$ drift and the second term is the polarization drift. The origin of the polarization drift is as follows. Consider the particle motion when the electric field is suddenly applied. The particle moves in the direction of the electric field $\boldsymbol{E}$ first. When $\boldsymbol{E}$ becomes stationary, the particle moves with $\boldsymbol{E} \times \boldsymbol{B}$ drift. This transient drift by $\mathrm{d}\boldsymbol{E}/\mathrm{d}t$ is called the "polarization drift" and given as follows,

$$\boldsymbol{v}_{pa} = -\frac{m_a}{e_a B^2}\frac{\mathrm{d}\boldsymbol{E}}{\mathrm{d}t} . \tag{7.75}$$

The direction of this drift is opposite for ions and electrons and causes charge separation unlike the $\boldsymbol{E} \times \boldsymbol{B}$ drift. The ion polarization drift is important while the electron one is negligible since this drift is proportional to the mass. The electron and ion motion along the magnetic field compensates the charge separation by the polarization drift. Since the electron has a small mass, we can assume the Boltzmann relation $\tilde{n}_e/n_e = e\tilde{\Phi}/T_e$ for the electron. The ion equation of motion and particle conservation law are given as,

$$m_i n_i \frac{\mathrm{d}\boldsymbol{v}_{i\parallel}}{\mathrm{d}t} = -\nabla_\parallel(en\tilde{\Phi} + \gamma n_i T_i) , \tag{7.76}$$

$$\frac{\partial n_i}{\partial t} + \nabla\cdot(n\boldsymbol{v}_i) = 0 . \tag{7.77}$$

Here, we give the electrostatic potential as follows,

$$\tilde{\Phi} = \Phi_{\boldsymbol{k},\omega}\exp[\mathrm{i}k_\perp y + k_\parallel z - \mathrm{i}\omega t] . \tag{7.78}$$

From Equations 7.74 and 7.76, we can obtain an expression for $v_{\parallel i}$ and $v_{\perp i}$ and substituting them into the particle conservation law taking the quasi-neutrality condition $\tilde{n}_e = \tilde{n}_i$ into account, the drift wave dispersion relation is obtained from the linear term:

$$\omega^2(1 + \tau k_\perp^2 \rho_i^2/2) - \omega\omega_* = k_\parallel^2 C_s^2 \,. \tag{7.79}$$

Here, $C_s = [(T_e + \gamma T_i)/m_i]^{1/2}$ and ρ_i is the Larmor radius. The difference from Equation 7.71 comes from the polarization drift. In any case, the drift wave is a kind of ion-acoustic wave in non-uniform plasma. Let the spatial Fourier expansion of the electrostatic potential be $\tilde{\Phi} = \Phi_{\boldsymbol{k}}(t) \exp[\mathrm{i}k_\perp y + k_\parallel z]$. Selecting the wave number k from the nonlinear wave–wave coupling term in the $n_i \boldsymbol{v}_i$ term of Equation 7.77, we reach the Hasegawa-Mima equation:

$$\frac{\partial \Phi_{\boldsymbol{k}}(t)}{\partial t} + \mathrm{i}\omega_{\boldsymbol{k}*}\Phi_{\boldsymbol{k}}(t) = \sum_{\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2} V_{\boldsymbol{k}_1,\boldsymbol{k}_2}\Phi_{\boldsymbol{k}1}(t)\Phi_{\boldsymbol{k}2}(t) \,, \tag{7.80}$$

$$V_{\boldsymbol{k}_1,\boldsymbol{k}_2} = \frac{\rho_s^2}{(1 + \tau k^2 \rho_s^2)B}(\boldsymbol{k}_1 \times \boldsymbol{k}_2) \cdot \boldsymbol{e}_z [k_2^2 - k_1^2] \,. \tag{7.81}$$

Here, $\omega_{\boldsymbol{k}}{}^* = \omega_*/(1 + \tau k_\perp^2 \rho_i^2/2)$, and $\rho_s = (T_e/m_i)^{1/2}/\Omega_i$.

The nonlinear term of Hasegawa–Mima Equation 7.80 comes from the polarization drift and is used as the most basic equation for plasma turbulence.

Salon: Hasegawa–Mima Equation

Professor A. Hasegawa (Figure 7.3 (a)) and Professor K. Mima (Figure 7.3 (b)) who derived the Hasegawa-Mima equation. The Hasegawa–Mima equation produces zonal flow through the inverse cascade, which is essentially the same as the zonal flow in Jovian atmosphere [17]. At the time, Professor Hasegawa was at the Bell Laboratories and Professor Mima was a visitor from ILE-Osaka. Professor Mima became a director of ILE-Osaka, which promotes laser fusion.


(a)


(b)

Figure 7.3 (a) Professor A. Hasegawa (with kind permission of Prof. Hasegawa) and (b) Professor K. Mima (with kind permission of Prof. Mima)

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