Chapter 6
Magnetohydrodynamic Stability: Energy Principle, Flow, and Dissipation

It is not easy to discuss general plasma stability since plasma is a nonlinear and dissipative medium. In this chapter, after a survey of the general stability, linear stability, in particular, ideal magnetohydrodynamic stability with an Hermitian (self-adjoint) linear operator is discussed. Then, nonlinear tearing forming a magnetic island by magnetic reconnection caused by the dissipation, and the stability of plasma flow with a non-Hermitian operator are outlined.

6.1 Stability: Introduction

To confine high-temperature plasma in a torus is topologically reasonable, but it actually requires careful consideration. Plasma is "soft" matter, and often becomes unstable when the internal energy is large. In this section, we introduce a general definition of stability, "stability in the sense of Lyapunov" for the general evolution equation of the system. The property of the linear operator of the evolution equation are described as the basis of stability.

The mathematical theory of stability was developed through the investigation of stability in stellar dynamics by the French mathematician S. D. Poisson (1781–1840). A complete general mathematical definition of stability was given by the Russian mathematician A. M. Lyapunov (1857–1918) [1]. Assume that the behavior of the plasma is given by the following evolution equation.

$$\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}t} = N(\boldsymbol{X}) \ . \tag{6.1}$$

Here, it is important to note that time evolution is determined only by the present value of $\boldsymbol{X}$. Such a system is called a "dynamical system". The equilibrium point $\boldsymbol{X}_0(N(\boldsymbol{X}_0) = 0)$ is called "unstable in the sense of Lyapunov" if there is another solution that rapidly moves away from the first solution over time when a small change is applied to $\boldsymbol{X}$. Conversely, Lyapunov stability is given as follows.

Lyapunov stability: If there exists a neighborhood V for any neighborhood U of X so that orbit starting from inside of V stays within U, X is called stable in the sense of Lyapunov.

In other words, if the solution of $\mathrm{d}\Delta \boldsymbol{X}(t)/\mathrm{d}t = N(\boldsymbol{X}_0 + \boldsymbol{\xi})$ is always bounded, it is Lyapunov stable. Linearizing the evolution equation 6.1, we obtain the linearized equation $\mathrm{d}\boldsymbol{\xi}/\mathrm{d}t = L\boldsymbol{\xi}$. Here, $L = N'(\boldsymbol{X}_0)$ and $\boldsymbol{\xi} = \boldsymbol{X} - \boldsymbol{X}_0$. If the steady-state flow is zero, we have $\mathrm{d}\boldsymbol{\xi}/\mathrm{d}t = \partial\boldsymbol{\xi}/\partial t$, and the equation is led to an eigenvalue problem $L\boldsymbol{\xi} = \lambda\boldsymbol{\xi}$ by setting $\partial/\partial t = \lambda$. A linear operator L can be expressed by a finite dimensional matrix if the set of solutions of $L\boldsymbol{\xi} = \lambda\boldsymbol{\xi}$ is covered by a finite number of eigen functions. But, in general, an infinite number of eigen functions can exist and will form a "functional space" [2].

If the matrix L defined in the finite dimensional linear space is "regular" ($LL^* = L^*L$), a complete set of orthogonal eigen functions can be obtained. And the "unitary transformation" $U^{-1}LU$ diagonalizes the matrix L and eigenvalues appear as diagonal elements. If L is "self-adjoint" ($L^* = L$), the eigenvalues are all real. A negative eigenvalue means the system is unstable. The eigenvalue problem of linear operator L defined in the functional space (infinite dimensional linear space) is different in nature from that in the finite-dimensional linear space. An important difference is the existence of a "continuous spectrum." The solution of the eigenvalue problem in the functional space in general, consists of a discrete eigenvalue ("point spectrum") and continuous eigenvalue ("continuous spectrum") on the segment in the real axis. In quantum mechanics, the point spectrum appears in bound states, while the continuous spectrum appears in non-bound states. In plasma physics, the continuous spectrum appears in the Alfven waves and longitudinal waves in collisionless plasma (Section 5.3) [3].

Linear operators such as the linear Vlasov operator (see Section 5.3) and linear MHD (magnetohydrodynamic) operator F (see Section 6.2) appear in plasma physics. These linear operators are infinite dimensional linear operators and cause special behaviors such as Landau damping and Alfven continuum damping through the continuous spectrum.

The continuous spectrum has a singular eigen function (such as the Dirac δ function) not defined in the functional space for the linear operator ("Hilbert space"). Let us determine the operator to give the continuous spectrum. Consider position operator $Au(x) = xu(x)$, the eigenvalue problem for A is given by $xu = \lambda u$. From $(x - \lambda)u = 0$,we have $u = \delta(x - \lambda)$ (δ is the "Dirac delta function"). Since λ can take any real number, operator A gives a continuous spectrum.

Let A be the linear operator. The eigenvalue problem is to obtain eigenvalue $\lambda(\in C)$ and eigenvector u to satisfy $Au = \lambda u$. Rewriting this equation as $(\lambda I - A)u = 0$, the problem becomes finding null points for the linear operator $(\lambda I - A)$ or singular points of the operator $(\lambda I - A)^{-1}$. The theory requires a generalization of the concept of eigenvalue and eigen function for the operator in infinite dimensional linear space [2, 4].

Magnetohydrodynamic behavior of the plasma can be formulated in the form of a variational principle using the Lagrangian. If there is no dissipation, the total energy of the system (sum of potential energy and kinetic energy) is conserved. The system is unstable if a negative change occurs in the potential energy leading the kinetic energy to grow. Conversely, the system is stable if a positive change occurs in potential energy leading the kinetic energy to decrease. The method used to investigate the stability of the system through its potential energy in this way is called the "energy principle" [5].

The linear MHD equation can be expressed as $\rho\partial_t^2\boldsymbol{\xi} = F[\boldsymbol{\xi}]$. Here, F is the self-adjoint operator (Hermitian operator) as discussed in Section 6.2. The eigenvalue of the Hermitian operator (spectrum) is real. However, if the flow is included in the steady state, the linear MHD operator includes a non-Hermitian operator and is not easy to handle (see Section 6.7) [6].

The linear stability of the system is closely related to "bifurcation" in nonlinear phenomena. If the change of the system is described by a control parameter, bifurcation occurs when the eigenvalue of the linearized equation crosses the imaginary axis.

6.2 Ideal Magnetohydrodynamics: Action Principles and the Hermitian Operator

The conductivity of high temperature plasma is very similar to that of metals, and the motion of the magnetic field is strongly restricted. According to Alfven, the magnetic field is frozen into the plasma motion. Such plasma is treated in the continuum approximation, and is called "Ideal Magnetohydrodynamics" (Ideal MHD). As described in Goldstein [7], Lagrange mechanics in the continuum is reduced to the variational principle, whose action integral is given by the time and space integral of the Lagrangian density L. In ideal MHD [8], L is given by,

$$L = \frac{1}{2}\rho\boldsymbol{v}^2 - \frac{P}{\gamma - 1} - \frac{\boldsymbol{B}^2}{2\mu_0} \,. \tag{6.2}$$

Here, ρ, $\boldsymbol{v}$, P, $\boldsymbol{B}$ are the mass density, fluid velocity, plasma pressure, and the magnetic field, respectively. The first term of integral is the plasma kinetic energy, the second term is the plasma energy in the adiabatic approximation, and the third term is the magnetic energy. Using this Lagrangian, action S is represented by,

$$S = \int_{t_1}^{t_2} \mathrm{d}t \int L \mathrm{d}V \,. \tag{6.3}$$

Let $\boldsymbol{\xi}$ be the plasma displacement, variations of ρ, $\boldsymbol{v}$, P, and $\boldsymbol{B}$ are given by

$$\delta\boldsymbol{v} = \boldsymbol{v}\cdot\nabla\boldsymbol{\xi} - \boldsymbol{\xi}\cdot\nabla\boldsymbol{v} + \partial\boldsymbol{\xi}/\partial t \,, \tag{6.4}$$

$$\delta\rho = -\nabla\cdot(\rho\boldsymbol{\xi}) \,, \tag{6.5}$$

$$\delta P = -\gamma P\nabla\cdot\boldsymbol{\xi} - \boldsymbol{\xi}\cdot\nabla P \,, \tag{6.6}$$

$$\delta\boldsymbol{B} = \nabla\times(\boldsymbol{\xi}\times\boldsymbol{B}) \,. \tag{6.7}$$

Applying these relations, the action integral δS is given as follows,

$$\delta S = \int_{t_1}^{t_2} \mathrm{d}t \int \mathrm{d}V \left[\delta\rho \frac{\boldsymbol{v}^2}{2} + \rho \boldsymbol{v} \cdot \delta \boldsymbol{v} - \frac{\delta P}{\gamma - 1} - \frac{\boldsymbol{B} \cdot \delta \boldsymbol{B}}{\mu_0} \right]$$

$$= \int_{t_1}^{t_2} \mathrm{d}t \int \mathrm{d}V \Big[-\nabla \cdot (\rho \boldsymbol{\xi}) \frac{\boldsymbol{v}^2}{2} + \rho \boldsymbol{v} \cdot (\boldsymbol{v} \cdot \nabla \boldsymbol{\xi} - \boldsymbol{\xi} \cdot \nabla \boldsymbol{v} + \partial \boldsymbol{\xi}/\partial t)$$

$$+ \frac{\gamma P \nabla \cdot \boldsymbol{\xi} + \boldsymbol{\xi} \cdot \nabla P}{\gamma - 1} - \frac{\boldsymbol{B} \cdot \nabla \times (\boldsymbol{\xi} \times \boldsymbol{B})}{\mu_0} \Big] . \tag{6.8}$$

Partial integration for the displacement vector gives

$$\delta S = -\int_{t_1}^{t_2} \mathrm{d}t \int \mathrm{d}V \delta \boldsymbol{\xi} \cdot \left[\frac{\partial(\rho \boldsymbol{v})}{\partial t} + \nabla \cdot (\rho \boldsymbol{v} \boldsymbol{v}) + \nabla P - \boldsymbol{J} \times \boldsymbol{B} \right] . \tag{6.9}$$

Therefore, the variational principle $\delta S = 0$ is equivalent to the following equation:

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} + \rho \boldsymbol{v} \cdot \nabla \boldsymbol{v} = \boldsymbol{J} \times \boldsymbol{B} - \nabla P . \tag{6.10}$$

Here, the continuity equation for mass density $\partial\rho/\partial t + \nabla \cdot (\rho \boldsymbol{v}) = 0$ is used. If there is no flow ($\boldsymbol{v} = 0$), plasma is in static force equilibrium. The variational principle in this case was given by Kruskal–Krusrud in 1958 [9].

$$S = \int L \, \mathrm{d}V = \int \left[\frac{B^2}{2\mu_0} + \frac{P}{\gamma - 1} \right] \mathrm{d}V . \tag{6.11}$$

For this variational principle, substitution of Equations 6.6 and 6.7 into the above equation gives

$$\delta S = -\int \boldsymbol{\xi} \cdot [\mu_0^{-1} (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} - \nabla P] \, \mathrm{d}V . \tag{6.12}$$

Hence, the variational principle $\delta S = 0$ using Equation 6.11 is equivalent to the equilibrium condition $\boldsymbol{J} \times \boldsymbol{B} = \nabla P$. In the case of force equilibrium (the first order term of the action integral with respect to the displacement = 0), the variation δS is given by a quadratic form of the displacement. The stability of the equilibrium can be determined by its sign. Linearization of Equation 6.10 gives the following linear evolution equation considering $\rho \partial^2 \boldsymbol{\xi}/\partial t^2 = \delta \boldsymbol{J} \times \boldsymbol{B} + \boldsymbol{J} \times \delta \boldsymbol{B} - \nabla \delta P$, and (6.6), (6.7) and $\delta \boldsymbol{J} = \nabla \times \delta \boldsymbol{B}$.

$$\rho \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = \boldsymbol{F}(\boldsymbol{\xi}) , \tag{6.13}$$

$$\boldsymbol{F}(\boldsymbol{\xi}) = \mu_0^{-1} \{ \nabla \times [\nabla \times (\boldsymbol{\xi} \times \boldsymbol{B})] \} \times \boldsymbol{B}$$
$$+ \mu_0^{-1} (\nabla \times \boldsymbol{B}) \times [\nabla \times (\boldsymbol{\xi} \times \boldsymbol{B})] + \nabla [\gamma P \nabla \cdot \boldsymbol{\xi} + \boldsymbol{\xi} \cdot \nabla P] .$$

The linear operator $\boldsymbol{F}$ is characterized by its important Hermitian property (F is a self-adjoint operator). This property can be proved by a fairly complicated modification of Equation 6.13 [5], but simple derivation is possible by using the fact that energy is conserved in an ideal MHD fluid [8]. In fact, the energy E of the ideal MHD fluid is given by the sum of kinetic and potential energies and is constant from the equation of motion, Equation 6.10

$$E = \int \left[\frac{1}{2}\rho \boldsymbol{v}^2 + \frac{P}{\gamma - 1} + \frac{\boldsymbol{B}^2}{2\mu_0} \right] \mathrm{d}V \, . \tag{6.14}$$

The total energy can be expressed as a function of ξ

$$E = \int \frac{1}{2}\rho \left(\frac{\partial \boldsymbol{\xi}}{\partial t} \right)^2 \mathrm{d}V + W(\boldsymbol{\xi}, \boldsymbol{\xi}) \, . \tag{6.15}$$

Here, W is the quadratic form of displacement $\boldsymbol{\xi}$ and is actually quite complex. We proceed without the detailed structure of W and consider up to the second order expansion of W with respect to $\boldsymbol{\xi}$.

$$W(\boldsymbol{\xi}, \boldsymbol{\xi}) = W_0 + W_1(\boldsymbol{\xi}) + W_2(\boldsymbol{\xi}, \boldsymbol{\xi}) \, . \tag{6.16}$$

Since energy is conserved, $\mathrm{d}E/\mathrm{d}t = 0$. In other words,

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int \rho \frac{\partial \boldsymbol{\xi}}{\partial t}\frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} \mathrm{d}V + W_1\left(\frac{\partial \boldsymbol{\xi}}{\partial t} \right) + W_2\left(\frac{\partial \boldsymbol{\xi}}{\partial t}, \boldsymbol{\xi} \right) + W_2\left(\boldsymbol{\xi}, \frac{\partial \boldsymbol{\xi}}{\partial t} \right) = 0 \, . \tag{6.17}$$

Defining $\boldsymbol{\eta} = \partial \boldsymbol{\xi}/\partial t$ and substituting $\rho \partial^2 \boldsymbol{\xi}/\partial t^2 = F(\boldsymbol{\xi})$ into Equation 6.17 we obtain

$$\int \boldsymbol{\eta} \cdot F(\boldsymbol{\xi}) \, \mathrm{d}V + W_1(\boldsymbol{\eta}) + W_2(\boldsymbol{\eta}, \boldsymbol{\xi}) + W_2(\boldsymbol{\xi}, \boldsymbol{\eta}) = 0 \, . \tag{6.18}$$

Since the system is in equilibrium, $W_1(\boldsymbol{\eta}) = 0$ for arbitrary $\boldsymbol{\eta}$. Also, taking into account that $W_2(\boldsymbol{\xi}, \boldsymbol{\eta}) + W_2(\boldsymbol{\xi}, \boldsymbol{\eta})$ in the left hand of Equation 6.18 is symmetric with respect to the exchange of $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$, we obtain

$$\int \boldsymbol{\eta} \cdot F(\boldsymbol{\xi}) \, \mathrm{d}V = \int \boldsymbol{\xi} \cdot F(\boldsymbol{\eta}) \, \mathrm{d}V \, . \tag{6.19}$$

This property of the linear ideal MHD operator F is called the Hermitian (self-adjoint). The explicit expression of F as the Hermitian is given by Freidberg [10] as follows,

$$\begin{aligned} \int \boldsymbol{\eta} \cdot \boldsymbol{F}(\boldsymbol{\xi}) \, \mathrm{d}V = -\int \mathrm{d}V \Bigg[&\frac{1}{\mu_0} (\boldsymbol{B} \cdot \nabla \boldsymbol{\xi}_\perp) \cdot (\boldsymbol{B} \cdot \nabla \boldsymbol{\eta}_\perp) + \gamma P (\nabla \cdot \boldsymbol{\xi})((\nabla \cdot \boldsymbol{\eta}) \\ &+ \frac{B^2}{\mu_0} (\nabla \cdot \boldsymbol{\xi}_\perp + 2\boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa})(\nabla \cdot \boldsymbol{\eta}_\perp + 2\boldsymbol{\eta}_\perp \cdot \boldsymbol{\kappa}) \\ &- \frac{4B^2}{\mu_0} (\boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa})(\boldsymbol{\eta}_\perp \cdot \boldsymbol{\kappa}) + (\boldsymbol{\eta}_\perp \boldsymbol{\xi}_\perp : \nabla\nabla) \left(P + \frac{B^2}{2\mu_0} \right) \Bigg] . \end{aligned} \tag{6.20}$$

6.3 Energy Principle: Potential Energy and Spectrum

The energy conservation law for a small displacement can be given by the integration of $(\partial\boldsymbol{\xi}/\partial t)\cdot$ (Equation 6.13) over time using the Hermitian property, as follows,

$$\frac{1}{2}\int \rho\left(\frac{\partial\boldsymbol{\xi}}{\partial t}\right)^2 \mathrm{d}V = \frac{1}{2}\int \boldsymbol{\xi}\cdot F(\boldsymbol{\xi})\,\mathrm{d}V\ . \tag{6.21}$$

Here, $\delta K = (1/2)\int \rho(\partial\boldsymbol{\xi}/\partial t)^2\mathrm{d}V$ is the change of kinetic energy, $\delta W = -(1/2)\int \boldsymbol{\xi}\cdot F(\boldsymbol{\xi})\,\mathrm{d}V$ is the change of potential energy. Since total energy $E = K + W$ is conserved, a negative change in potential energy ($\delta W < 0$) gives an increase in the kinetic energy ($\delta K > 0$) and the system is unstable. Conversely, a positive change in potential energy ($\delta W > 0$) gives a reduction in kinetic energy ($\delta K < 0$) and the system is stable. In this way, the stability of the system can be examined by the potential energy and this method is called the "Energy Principle." Potential energy can be given by a quadratic form of $\boldsymbol{\xi}$ and Furth [11] gave such a form that is easy to understand as follows,

$$\delta W(\boldsymbol{\xi}) = \int \mathrm{d}V[\delta W_{\mathrm{SA}} + \delta W_{\mathrm{MS}} + \delta W_{\mathrm{SW}} + \delta W_{\mathrm{IC}} + \delta W_{\mathrm{KI}}]\ , \tag{6.22}$$

$$\begin{aligned}
\delta W_{\mathrm{SA}} &= B_1^2/2\mu_0\ , \quad \boldsymbol{B}_1 = \nabla\times(\boldsymbol{\xi}\times\boldsymbol{B})\ ,\\
\delta W_{\mathrm{MS}} &= \boldsymbol{B}^2(\nabla\cdot\boldsymbol{\xi}_\perp + 2\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa})^2/2\mu_0\ ,\\
\delta W_{\mathrm{SW}} &= \gamma P(\nabla\cdot\boldsymbol{\xi})^2/2\ , \quad \delta W_{\mathrm{EX}} = (\boldsymbol{\xi}_\perp\cdot\nabla P)(\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa})/2\ ,\\
\delta W_{\mathrm{KI}} &= -J_\parallel \boldsymbol{b}\cdot(\boldsymbol{B}_{1\perp}\times\boldsymbol{\xi}_\perp)/2\ .
\end{aligned}$$

Here, δW_{SA} is the "bending energy of the magnetic field" and is a source of "shear Alfven wave." δW_{MS} is the "compressing energy of the magnetic field" and is a source of "magnetosonic waves." δW_{SW} is the "compressing energy of the plasma" and a source of the "sound wave." All these terms are positive and stabilizing. Meanwhile, δW_{IC} is the "interchange energy" of plasma pressure in the curved magnetic field and can take positive or negative value. δW_{KI} is "kinking energy" of the current and can take a positive or negative value. Here, the curvature vector is given by $\boldsymbol{\kappa} = \boldsymbol{b}\cdot\nabla\boldsymbol{b}$. If $\boldsymbol{\kappa}\cdot\nabla P < 0$, the interchange energy is the source of instability.

Using F is Hermitian operator, we can show that the eigenvalue ω^2 is real. Setting $\boldsymbol{\xi} = \boldsymbol{\xi}\exp(i\omega t)$ in the linear MHD Equation 6.13 and taking the volume integral of $\boldsymbol{\xi}^*\cdot$ (Equation 6.13), we obtain

$$\omega^2\int \rho\,|\boldsymbol{\xi}|^2\,\mathrm{d}V = -\int \boldsymbol{\xi}^*\cdot\boldsymbol{F}(\boldsymbol{\xi})\,\mathrm{d}V\ . \tag{6.23}$$

Taking the difference with complex conjugate of Equation 6.23 and using the Hermitian relation $\int \boldsymbol{\xi}\cdot F(\boldsymbol{\xi}^*)\,\mathrm{d}V = \int \boldsymbol{\xi}^*\cdot F(\boldsymbol{\xi})\,\mathrm{d}V$, we obtain

$$\left(\omega^2 - \omega^{*2}\right)\int \rho\,|\boldsymbol{\xi}|^2\,\mathrm{d}V = 0\ . \tag{6.24}$$

Namely, eigenvalue ω^2 is real. The case of $\omega^2 > 0$ shows oscillation without damping and is stable, while the case of $\omega^2 < 0$ grows exponentially and is unstable. The transition from the stable to the unstable state occurs at $\omega^2 = 0$. The locus of the root moves on a real and imaginary axis in the complex plane.

Considering F is Hermitian, we can prove the orthogonality of the eigen functions weighted by ρ. Eigen functions $\boldsymbol{\xi}_m$ and $\boldsymbol{\xi}_n$ with different eigenvalues ω_m^2 and ω_n^2 satisfy $-\rho\omega_m^2\boldsymbol{\xi}_m = \boldsymbol{F}(\boldsymbol{\xi}_m)$ and $-\rho\omega_n^2\boldsymbol{\xi}_n = \boldsymbol{F}(\boldsymbol{\xi}_n)$. Taking the inner product with $\boldsymbol{\xi}_m$ and $\boldsymbol{\xi}_n$ and integrating over the volume, we obtain

$$(\omega_m^2 - \omega_n^2)\int \rho\boldsymbol{\xi}_m \cdot \boldsymbol{\xi}_n \mathrm{d}V = \int [\boldsymbol{\xi}_m \cdot F(\boldsymbol{\xi}_n) - \boldsymbol{\xi}_n \cdot F(\boldsymbol{\xi}_m)]\, \mathrm{d}V = 0\,. \quad (6.25)$$

If there is only a discrete spectrum, this orthogonality leads to the energy integral $\delta W = \sum a_n^2\omega_n^2$ for $\boldsymbol{\xi} = \sum a_n\boldsymbol{\xi}_n$. Hence, we may judge the stability by the sign of the minimum eigenvalue ω_j^2 $(j = 1, \ldots, n)$. However, the existence of a continuous spectrum in the linear MHD operator causes this argument to break down.

To examine the continuous spectrum case, we describe the general properties of linear MHD Equation 6.13. Setting $\omega^2 = -\lambda$ Equation 6.13 can be expressed as follows,

$$[\lambda - \boldsymbol{F}/\rho]\boldsymbol{\xi} = \boldsymbol{a}\,. \quad (6.26)$$

Here, $\boldsymbol{a}$ is either the initial value of the Laplace transform of Equation 6.13 or the external force which is not considered in Equation 6.13 (for example, the Alfven mode can be excited with external coils). Then,

$$\boldsymbol{\xi} = [\lambda - \boldsymbol{F}/\rho]^{-1}\boldsymbol{a}\,. \quad (6.27)$$

The linear MHD operator has an infinite number of independent eigen functions and eigenvalues (often it they are not countable and termed a "spectrum"). The spectrum of $\boldsymbol{F}$ corresponds to the singular points of $(\lambda - \boldsymbol{F}/\rho)^{-1}$. If $(\lambda - \boldsymbol{F}/\rho)\boldsymbol{x} = 0$ has a nontrivial solution, a point spectrum appears. If $(\lambda - \boldsymbol{F}/\rho)^{-1}$ exists but is unbounded, a continuous spectrum will appear (see Note).

In non-uniform plasma, MHD waves such as Alfven waves and slow and fast magnetosonic waves can have a continuous spectrum. For example, $\lambda - \boldsymbol{F}/\rho = \lambda - k_\parallel^2 V_A^2$ in a cylindrical inhomogeneous plasma and the Alfven wave has a phase velocity $V_p = V_A$ in the direction of a magnetic field. If the density changes in the direction perpendicular to the magnetic field, the Alfven wave will propagate with a different phase velocity to its local Alfven velocity for each layer of different density. The oscillation phase difference between adjacent layers increases and the arbitrary initial perturbation will decay with time. In non-uniform plasma, damping of waves occurs due to phase mixing in the radial direction, while Landau damping occurs by phase mixing in velocity space [3]. Thus this damping is called "continuum damping."

Note: Hermitian (self-adjoint) Operator and Spectral Theory [2,4]

During the construction phase of quantum mechanics, it was necessary to establish the spectral theory to generalize the concept of the eigenvalue problem. The operator in quantum mechanics is self-adjoint and J. von Neumann (1903–1957) created spectral theory in the functional analysis. However, the theory is limited to the self-adjoint operator and the general properties of the non-self-adjoint operator are not well understood. Among the operators in the infinite-dimensional space, the spectral resolution is possible in general only for self-adjoint operators (or unitary operators). Among the various functional spaces, the most frequently used space (or set) is the Hilbert space dubbed H-space of square integrable functions ([Chapter VIII of 2]).

For the linear operator A, the eigenvalue problem is to obtain the eigenvalues $\lambda(\in C)$ and eigenvectors u to satisfy $Au = \lambda u$. This can be rewritten as $(\lambda I - A)u = 0$ and the problem is to find a set of null points of the linear operator $(\lambda I - A)$. In the operator in infinite dimensional linear space, spectrum analysis is used to investigate singularity of $(\lambda I - A)^{-1}$. For complex values of λ, the following three classes of the spectrum arise [4].

1. Point spectrum: In the case where $(\lambda I - A)^{-1}$ does not exist since $(\lambda I - A)u = 0$ has a non-trivial u, the corresponding set of λ is called a "point spectrum."
   Example: $A = -\partial_x^2$, solves the eigenvalue problem $(\lambda I - A)u = 0$ are $\lambda = \{(n\pi)^2; n = 1, 2, \ldots\}$.
2. Continuous spectrum: In this case, the unbounded inverse $(\lambda I - A)^{-1}$ exists, the corresponding set of λ is called a "continuous spectrum."
   Example: $A = x$, the solution for $(\lambda x)u = 0$ is $u = \delta(x - \lambda)$. This Dirac delta function is not square integrable and does not belong to Hilbert space.
3. Residual spectrum: In the case where inverse $(\lambda I - A)^{-1}$ exists and is bounded, the corresponding set of λ is called a "residual spectrum." It is important to note that if λ is in the residual spectrum of A, λ is in the point spectrum of the adjoint operator A^*. So, there is no residual spectrum in Hermitian operator.

Here, a linear operator A is said to be "bounded" if there exists a constant N for all $u \in H$ such that

$$\|Au\| \leq N \|u\| \ . \tag{6.28}$$

6.4 Newcomb Equation: Euler–Lagrange Equation of Ideal MHD

Minimization of the energy integral of the linear ideal MHD equation in cylindrical plasma and axisymmetric plasma can be reduced to the Euler–Lagrange equation of the radial coordinate. This is called the "Newcomb equation." Newcomb [12] derived the equation for cylindrical plasma and Tokuda [13] derived the equation for axisymmetric plasma.

Cylindrical plasma

In the case of cylindrical symmetry, ξ_r, $\mathrm{i}\xi_\theta$, $\mathrm{i}\xi_z$ can be expressed as the real normal mode $\exp(\mathrm{i}m\theta + \mathrm{i}kz)$ without loss of generality considering the symmetry in the cylindrical coordinates (r, θ, z). The stability condition can be given for a pair (m, k). Minimization of energy integral Equation 6.22 for $\mathrm{i}\xi_\theta$ and $\mathrm{i}\xi_z$ gives incompressibility of displacement $\nabla \cdot \boldsymbol{\xi} = 0$ and $v = \mathrm{i}(\xi_\theta B_z - \xi_z B_\theta) = \zeta_0(\xi_r, \mathrm{d}\xi_r, /\mathrm{d}r)$, and the energy integral W for unit length along z direction is given using $\xi = \xi_r$ as follows,

$$W = \frac{\pi}{2\mu_0} \int_0^a \left[f \left| \frac{\mathrm{d}\xi}{\mathrm{d}r} \right|^2 + g\,|\xi|^2 \right] \mathrm{d}r + W_a + W_v\ , \tag{6.29}$$

$$g = \frac{1}{r} \frac{(kB_z - (m/r)B_\theta)^2}{k^2 + (m/r)^2} + r(kB_z + (m/r)B_\theta)^2 - \frac{2B_\theta}{r} \frac{\mathrm{d}(rB_\theta)}{\mathrm{d}r}$$
$$- \frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{k^2 B_z^2 - (m/r)^2 B_\theta^2}{k^2 + (m/r)^2} \right)\ ,$$

$$f = \frac{r(kB_z + (m/r)B_\theta)^2}{k^2 + (m/r)^2}\ ,$$

$$\zeta_0\left(\xi, \frac{\mathrm{d}\xi}{\mathrm{d}r}\right) = \frac{r}{k^2 r^2 + m^2} \left[(krB_\theta - mB_z) \frac{\mathrm{d}\xi}{\mathrm{d}r} - (krB_\theta + mB_z) \frac{\xi}{r} \right]\ .$$

Here, W_a and W_v are the surface terms from the partial integration and the energy integral in the vacuum, respectively. The Euler–Lagrange equation to minimize Equation 6.29 is given by the following equation:

$$\frac{\mathrm{d}}{\mathrm{d}r} \left(f \frac{\mathrm{d}\xi}{\mathrm{d}r} \right) - g\xi = 0\ . \tag{6.30}$$

This equation is known as the "Newcomb equation." A significant feature of the Newcomb equation is that it becomes singular at the rational surface given by $f(r) = 0$. Since $f \geq 0$, the $f(\mathrm{d}\xi/\mathrm{d}r)^2$ term in (6.29) is stabilizing. At the rational surface, the condition of the local solution to be non-oscillatory (oscillatory solution is unstable) gives the "Suydam condition" for local mode stability $(q'(r)/q(r))^2 + 8\mu_0 P'(r)/rB_z^2 > 0$ with $q = rB_z/RB_q$ (the stability condition in the torus is given by $r(\mathrm{d}\ln q/\mathrm{d}r)^2/4 + 2\mu_0(\mathrm{d}P/\mathrm{d}r)(1 - q^2)/B_z^2 > 0$ and is usually satisfied since $\mathrm{d}P/\mathrm{d}r(1-q^2) > 0$ in the $q > 1$ regime, even if $\mathrm{d}P/\mathrm{d}r < 0$ is a large negative value, the "Mercier stability criteria" [14]). The $q'(r)/q(r)$ term is stabilized by the magnetic shear. Considering the case with multiple singularities in the plasma $(r_1, r_2, \ldots)$, the Euler–Lagrange solution is separated at the singular point and the energy integral between adjacent singular points can be minimized independently. In this case, the energy integral of the Euler–Lagrange solution between the singular points r_1 and r_2 is given by $W = (\pi/2\mu_0)[f\xi \mathrm{d}\xi/\mathrm{d}r]_{r_1}^{r_2}$. For $x = r - r_s$, the solution near the singular points is given by two eigen solutions $x \sim x_1^{-n}$ and

x_2^{-n}, where n_1 and n_2 are solutions of $n^2 - n + \gamma = 0$ ($\gamma = -rP'(r)/(B_z^2/2\mu_0)s^2$: $s = r(\mathrm{d}q/\mathrm{d}r)$ is the magnetic shear). Assuming $n_1 < n_2$, $x \sim x_1^{-n}$ is called the "small solution" and $x \sim x_2^{-n}$ is called the "large solution." Newcomb derived 14 theorems of the Euler–Lagrange solution of Equation 6.30 [12]. Theorem 10 is particularly important.

Newcomb's theorem 10: For specific values of m and k, cylindrical plasma is stable in an independent interval I if and only if (1) Suydam's condition is fulfilled at the left endpoint if the point is singular, and (2) the Euler–Lagrange solutions that are small at the left endpoint never vanish in the interior of I. In marginal cases, the solution is also small at the right endpoint.

If the numerical integration of this Euler–Lagrange equation using, for example, the Runge-Kutta method with a boundary condition $\xi = 0$, $\mathrm{d}\xi/\mathrm{d}r = 1$ at the left edge gives a crossing $\xi = 0$ within the interval, the plasma is unstable according to this theorem.

Axisymmetric plasma

In the case of an axisymmetric torus, the energy integral is minimized under the incompressibility condition $\nabla \cdot \boldsymbol{\xi} = 0$ as in the case of cylindrical symmetry. The magnetic field is expressed by Equation 3.59 in an axisymmetric torus, and the Grad–Shafranov equation is given in the flux coordinates (r, θ, ζ) with $r = [2R_0 \int_0^{\psi} (q/F)\mathrm{d}\psi]^{1/2}$ and Jacobian $J = g^{1/2} = R^2 r/R_0$ as follows,

$$\frac{\partial}{\partial r}\left[r\frac{\mathrm{d}\psi}{\mathrm{d}r}|\nabla r|^2\right] + \frac{\partial(\nabla r \cdot \nabla\theta)}{\partial\theta}\frac{\mathrm{d}\psi}{\mathrm{d}r} = -\mu_0 R^2 \frac{\mathrm{d}P}{\mathrm{d}\psi} - F\frac{\mathrm{d}F}{\mathrm{d}\psi} \,. \tag{6.31}$$

By using $X = \boldsymbol{\xi} \cdot \nabla r$ and $V = r\boldsymbol{\xi} \cdot \nabla(\theta - \zeta/q)$ in the flux coordinates (r, θ, ζ), the energy integral W under $\nabla \cdot \boldsymbol{\xi} = 0$ can be expressed in a following form,

$$W_p = \frac{\pi}{2\mu_0}\int_0^a \mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta L\left(X, \frac{\partial X}{\partial\theta}, \frac{\partial X}{\partial r}, V, \frac{\partial V}{\partial\theta}\right) \tag{6.32}$$

where $r = a$ is the plasma surface. Minimization of the energy integral with respect to V is easy in the cylindrical plasma. In the axisymmetric case, minimization with respect to V is a bit more complicated since the energy integral contains the $\partial V/\partial\theta$ term but the absence of the $\partial V/\partial r$ term in the energy integral leads to following Euler equation [13],

$$\frac{\partial}{\partial\theta}\left[\frac{\partial L}{\partial(\partial V/\partial\theta)}\right] - \frac{\partial L}{\partial V} = 0 \,. \tag{6.33}$$

The solvability of Equation 6.33 imposes a condition for L, called the "solvable condition." By integrating of Equation 6.33, $\theta = 0 - 2\pi$, $\partial L/\partial(\partial V/\partial\theta)$ must have the same value at $\theta = 0$ and 2π (periodic boundary conditions). The solvable condition becomes,

$$\int_0^{2\pi} \frac{\partial L}{\partial V} \mathrm{d}\theta = 0 \,. \tag{6.34}$$

Fourier expansion of V and X for θ is defined as follows,

$$X(r,\theta) = \sum_{m=-\infty}^{m=\infty} X_m(r) \exp(\mathrm{i}m\theta) \,, \quad V(r,\theta) = -\mathrm{i} \sum_{m=-\infty}^{m=\infty} V_m(r) \exp(\mathrm{i}m\theta) \,. \tag{6.35}$$

Substitution of these equations into Equation 6.34 gives linear equations for V_m and the solution is substituted into Equation 6.32. The integrant L is now given by $\boldsymbol{X} = (\ldots, X_{-2}, X_{-1}, X, X_1, X_2, \ldots)^t$ (t is transposed) and $\mathrm{d}\boldsymbol{X}/\mathrm{d}r$ and the Euler–Lagrange equation is obtained [6–13].

$$W_p = \frac{\pi^2}{\mu_0} \int_0^a L\left(\boldsymbol{X}, \frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}r}\right) \mathrm{d}r$$

$$\frac{\mathrm{d}}{\mathrm{d}r} \frac{\partial L}{\partial(\mathrm{d}\boldsymbol{X}/\mathrm{d}r)} - \frac{\partial L}{\partial \boldsymbol{X}} = 0 \,. \tag{6.36}$$

Since L is given by a quadratic form of $\boldsymbol{X}, \mathrm{d}\boldsymbol{X}/\mathrm{d}r$, the Euler–Lagrange equation follows the form of the second order ordinary differential equation,

$$\frac{\mathrm{d}}{\mathrm{d}r} \boldsymbol{f} \frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}r} + \boldsymbol{g} \frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}r} + \boldsymbol{h}\boldsymbol{X} = 0 \,. \tag{6.37}$$

where $\boldsymbol{f}, \boldsymbol{g}$, and $\boldsymbol{h}$ are matrices. This is called the "two-dimensional Newcomb equation." Diagonal elements of $\boldsymbol{f}$ have $(n/m - 1/q)^2$ dependence similar to the one-dimensional Newcomb equation and the radius of $q = m/n$ is the singular point. Small and large solutions exist near the singular point and the Mercier condition is derived as the local stability condition. Once the Mercier condition is met, a similar method can be applied as Newcomb's theorem 10 to determine the stability. Also, "kink" and "peeling modes*" can be studied using the two-dimensional Newcomb equation.

6.5 Tension of Magnetic Field: Kink and Tearing

As described in Chapter 3, the magnetic field is bent helically and densely covers the torus to confine high temperature plasma. As Maxwell's equations teach us, the

* Peeling mode: Finite edge current can drive external modes localized near the plasma edge. This mode is called the peeling mode. The peeling mode becomes most unstable when a rational surface is located just outside the plasma surface. This mode can be coupled to the pressure driven ballooning mode and is thought to be a cause of ELM (Edge Localized Modes) in tokamak.

tension of the magnetic field works in the direction of the magnetic field and works to make the field lines straight. When the field becomes straight, plasma is deformed helically. This is the generation mechanism of instabilities called the "kink" mode and "tearing" mode. Kink is the deformation in the limit of zero plasma resistivity (ideal MHD plasma), while tearing is the deformation allowed by the magnetic reconnection with the change in magnetic field topology. This reconnection occurs at the rational surface, which is a singular point of the Newcomb equation of ideal MHD. There is an "external kink mode" and "internal kink mode" in the kink mode. The energy integral $W = W_p + W_v$ in the cylindrical plasma approximation (low beta (beta is the ratio of volume average plasma pressure $\langle P \rangle$ to magnetic pressure $B^2/2\mu_0$), large aspect ratio, circular cross section tokamak approximation) is obtained from Equation 6.29 as follows,

$$W_p = \frac{\pi^2 B_\xi^2}{\mu_0 R_0} \left\{ \int_0^a \left[\left(r \frac{\mathrm{d}\xi}{\mathrm{d}r} \right)^2 + (m^2 - 1)\xi^2 \right] \left(\frac{n}{m} - \frac{1}{q} \right)^2 r \mathrm{d}r \right\} , \tag{6.38}$$

$$W_v = \frac{\pi^2 B_\xi^2}{\mu_0 R_0} \left[\frac{2}{q_a} \left(\frac{n}{m} - \frac{1}{q_a} \right) + (1 + m\lambda) \left(\frac{n}{m} - \frac{1}{q_a} \right)^2 \right] a^2 \xi_a^2 \, . \tag{6.39}$$

Here, $\lambda = (1+(a/b)^{2m})/(1-(a/b)^{2m})$, a and b are the plasma minor radius and the radius of ideally conducting wall, respectively. Although the resistance of the wall is finite, the wall can be regarded as an ideal wall for timescales shorter than the wall time constant τ_{wall}. The energy integral inside the plasma is non-negative ($W_p \geq 0$), but the energy integral of the vacuum W_v can be negative when $(m/n)(1-2/(m\lambda+1)) < q_a < m/n$. The external kink is unstable for $q_a < m/n$ if the energy integral inside the plasma is small.

An unstable plasma mode with only internal displacement is possible, even if the surface displacement is zero, $\xi_a = 0$. This is called the internal kink mode. If $\xi_a = 0$, vacuum energy is zero, $W_v = 0$. Also, if a $q = 1$ surface exists in the plasma ($q(0) < 1$), internal energy can be zero ($W_p = 0$) for the non-trivial solution for $m = 1$ and $\mathrm{d}\xi/\mathrm{d}r = 0$, that means that the plasma is in neutral stability. This mode becomes weakly unstable if the poloidal beta value is above ~ 0.3 if we take into account the destabilizing effect of pressure by the toroidal effect.

The instability of practical importance is the tearing mode associated with the reconnection of the magnetic field at the resonant rational surface. This mode is destabilized by changing the topology of the magnetic field, while it is stable within the ideal MHD context. The linear growth rate of this mode is given as $\gamma \sim \eta^{3/5}$ but it soon goes into the nonlinear region. The nonlinear regime is the "Rutherford regime," derived by P. H. Rutherford [15]. Substituting Ohm's law $\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} = \eta \boldsymbol{J}$ into $\partial \boldsymbol{B}/\partial t = \nabla \times \boldsymbol{E}$, we can write down the major terms in r direction as follows,

$$\gamma B_r - \frac{B_\theta}{r}(m - nq)\mathrm{i}v_r = \frac{\eta}{\mu_0} \frac{\mathrm{d}^2 B_r}{\mathrm{d}r^2} \, . \tag{6.40}$$

In Equation 6.40, the second term on the left-hand side originates from $\boldsymbol{v} \times \boldsymbol{B}$ and becomes zero at the resonant surface. Then, the resistive diffusion term becomes important. Conversely, the effect of resistivity is not important except near the resonant surface. Defining $\psi = \mathrm{i} r B_r / m$, the following magnetic diffusion equation governs the dynamics near the rational surface,

$$\frac{\partial \psi}{\partial t} = \frac{\eta}{\mu_0} \frac{\partial^2 \psi}{\partial r^2} . \tag{6.41}$$

Integration of this equation within the "magnetic island" width w gives,

$$w \frac{\partial \psi}{\partial t} = \frac{\eta}{\mu_0} \left[\frac{\partial \psi}{\partial r} \left(r_\mathrm{s} + \frac{w}{2} \right) - \frac{\partial \psi}{\partial r} \left(r_\mathrm{s} - \frac{w}{2} \right) \right] . \tag{6.42}$$

where, r_s is a singular radius. Since w and ψ are related as $w = 4(q\psi / q' B_\theta)^{1/2}$,

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\eta}{2\mu_0} \Delta'(w) . \tag{6.43}$$

Here, $\Delta'(w) = [\mathrm{d}\psi/\mathrm{d}r(r_\mathrm{s} + w/2) - \mathrm{d}\psi/\mathrm{d}r(r_\mathrm{s} - w/2)]/\psi(r_\mathrm{s})$. According to the White's detailed calculation [16], the time evolution of the island width is given by,

$$\frac{\mathrm{d}w}{\mathrm{d}t} = 1.66 \frac{\eta}{\mu_0} (\Delta'(w) - \alpha w) . \tag{6.44}$$

Here, $\Delta'(w)$ is the solution ignoring resistive diffusion (external solution) and α is a constant. The external ψ can be obtained from the helically perturbed equilibrium equation from the cylindrical one as shown below. Except for the case $\psi = 0$ at the resonant surface for $m = 1$, ψ can be assumed to be constant near the resonant surface. This is essentially the same as the Newcomb equation. Near the resonant surface, the derivative diverges logarithmically and this term must be separated for the accurate evaluation of $\Delta'(w)$.

$$\frac{1}{r} \frac{\mathrm{d}}{\mathrm{d}r} \left(r \frac{\mathrm{d}\psi}{\mathrm{d}r} \right) - \frac{m^2}{r^2} \psi - \frac{\mu_0 \mathrm{d}J/\mathrm{d}r}{B_\theta (1 - nq/m)} \psi = 0 \tag{6.45}$$

As seen from Figure 6.1, the perturbed current inside the magnetic island is anti-parallel to the equilibrium plasma current forming counter-clockwise field lines around the island for the case of positive "magnetic shear" $s > 0$ ($s = (r/q)\mathrm{d}q/\mathrm{d}r$). The formation of magnetic islands reduces the pressure gradient and the reduction of the "bootstrap current" (see Section 8.5) occurs and accelerates the growth of magnetic islands. This mode is called the "neoclassical tearing mode" (NTM). On the other hand, the perturbed current is parallel to the equilibrium plasma current and reduction of the bootstrap current reduces the magnetic island for $s < 0$.

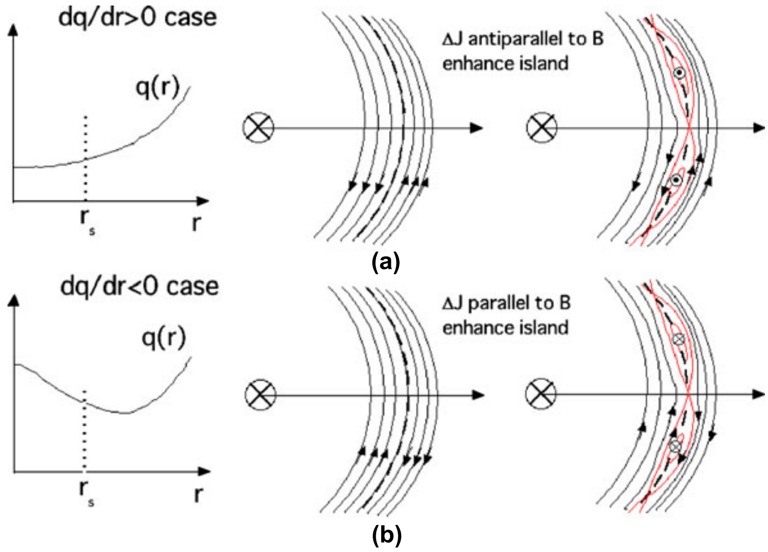

Figure 6.1 Relative magnetic field line flow inside and outside of the resonant surface in equilibrium (*left*) and the formation of magnetic island by the magnetic reconnection shown by *red line* (*right*) for (a) positive magnetic shear $s = r\,\mathrm{d}q/\mathrm{d}r/q > 0$ case and (b) negative magnetic shear $s = r\,\mathrm{d}q/\mathrm{d}r/q < 0$ case. For the positive shear case, perturbed current, which is antiparallel to the equilibrium current enhances the formation of a magnetic island and the perturbed field line encircles the magnetic island in a counter-clockwise direction. For the negative magnetic shear ($s < 0$) case, the perturbed current parallel to the equilibrium current enhances the formation of a magnetic island and the perturbed field line encircles the magnetic island in a clockwise direction

Salon: Harold Furth

Professor Harold Furth (1930–2002; Figure 6.2) was a US fusion physicist and was Director of Princeton Plasma Physics Laboratory (PPPL) between 1981 and 1990. Before coming to PPPL, he wrote a pioneering paper on resistive instabilities using matched asymptotic expansion [17].

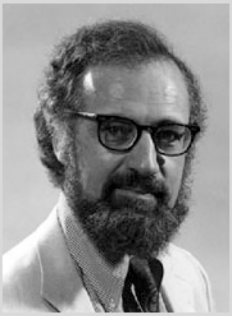

Figure 6.2 H. P. Furth (1930–2002) (Courtesy of Princeton Plasma Physics Laboratory)

6.6 Curvature of Magnetic Field: Ballooning and Quasi-mode Expansion

The local mode without the amplitude variation along $\boldsymbol{B}$ is stabilized by the average minimum B effect ($(q^2 - 1)$ term in the Mercier condition). But the ballooning mode can be unstable when the amplitude along $\boldsymbol{B}$ is larger in the weak magnetic field regime (outside of the torus). This mode has a long wavelength along $\boldsymbol{B}$ ($\lambda_\parallel \sim Rq$), and short wavelength perpendicular to $\boldsymbol{B}$. It must satisfy the periodic boundary conditions in both poloidal and toroidal directions. The magnetic field in Clebsch coordinates (r, θ, α) is given by $\boldsymbol{B} = \nabla\alpha \times \nabla\psi$ ($\alpha = \zeta - q\theta$). The displacement perpendicular to the magnetic field $\boldsymbol{\xi}_\perp$ is given by using the stream function Φ as follows,

$$\boldsymbol{\xi}_\perp = \frac{\mathrm{i}\boldsymbol{B} \times \nabla\Phi}{B^2} , \tag{6.46}$$

$$\Phi = F(r, \theta) \exp(\mathrm{i}S(r, \alpha)) . \tag{6.47}$$

We use eikonal S depending on r and α that are perpendicular to $\boldsymbol{B}$, since ballooning has a short wavelength perpendicular to $\boldsymbol{B}$. Here, $r \equiv a(\phi/\phi_a)^{1/2}$ is the radius defined using the toroidal flux. Φ is a slowly varying function of r and θ. Toroidal symmetry allows Fourier expansion in the toroidal direction, as $\mathrm{i}S \sim -\mathrm{i}n\zeta$. Since $\alpha = \zeta - q\theta$, a possible form of S is $S(r, \alpha) = -n(\alpha + \alpha_0(r))$. Considering the relation $S(r, \theta + 2\pi, \zeta) = S(r, \theta, \zeta) + 2\pi q$, S will not satisfy the periodic condition on θ. This is expected from the nature of the magnetic field lines on the magnetic surface mentioned in Section 3.7. Wave number perpendicular to the magnetic field is given from the following expression,

$$\boldsymbol{k}_\perp = \nabla S = n\left[\nabla\alpha + \alpha_0'(r)\nabla r\right] . \tag{6.48}$$

With this wave number, the phase along $\boldsymbol{B}$ becomes uniform but the wave cannot be completed within $[0, 2\pi]$ for θ and should be extended to $\pm\infty$ since $\boldsymbol{B}$ is wound endlessly around the torus as discussed in Section 3.7. In other words, as in "Riemann surfaces," the solution must be obtained to infinity by inserting the cut for every poloidal circulation. Here, in relation to $\theta \in [0, 2\pi]$, $y \in [-\infty, \infty]$ is called the "covering space." Using the arbitrariness of $\alpha_0(r)$, we can construct a solution Φ satisfying the periodic boundary condition of θ from the solution in the covering space. Let $\Phi(y, r) = \varphi(y, r) \exp[-\mathrm{i}n(\alpha + \alpha_0(r))]$ as the function defined in the covering space, where $\varphi(y, r)$ is the non-periodic square integrable function defined in $[-\infty, \infty]$ ($\varphi \in L_2$). It can be seen that the sum of $\Phi(y, r)$ shifted $2\pi j$ ($j = -\infty, +\infty$) will satisfy the periodic condition $\Phi(\theta + 2\pi, r) = \Phi(\theta, r)$ (this is called a "quasi-mode expansion"),

$$\begin{aligned}\Phi(\theta, r) &= \sum_{j=-\infty}^{\infty} \Phi_0(\theta + 2\pi j, r) = \sum_{j=-\infty}^{\infty} \varphi(\theta + 2\pi j, r)\mathrm{e}^{\mathrm{i}nq(\theta - \theta_0 + 2\pi j)}\mathrm{e}^{-\mathrm{i}n\zeta} \\ &= F(\theta, r)\mathrm{e}^{-\mathrm{i}n\alpha} \\ \theta_0 &= \frac{\alpha_0(r)}{q} .\end{aligned} \tag{6.49}$$

When the mode is expressed by Equation 6.49, the major terms of energy integral δW (Equation 6.22) are given by,

$$\delta W_{\mathrm{SA}} = \frac{B_1^2}{2\mu_0} \sim \frac{(\nabla\alpha)^2}{2\mu_0 B^2} |\boldsymbol{B}\cdot\nabla F|^2 \tag{6.50}$$

$$\delta W_{\mathrm{EX}} = (\boldsymbol{\xi}_\perp\cdot\nabla P)(\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa})/2 \sim P'(\psi)[(\boldsymbol{B}\times\nabla\alpha)\cdot\boldsymbol{\kappa}/B^2]\,|F|^2 \ . \tag{6.51}$$

The other terms are $O(1/n)$ and can be ignored in large n approximation [18]. Physically, ballooning mode stability is determined by the balance between the bending energy of the magnetic field and the interchange energy of plasma. Then,

$$W_p = \frac{1}{2\mu_0}\int\left[\frac{|\nabla\alpha|^2}{B^2}(\boldsymbol{B}\cdot\nabla F)^2 - 2\mu_0 P'(\psi)\kappa_w F^2\right]\mathrm{d}V \ . \tag{6.52}$$

Here $\kappa_w = (\boldsymbol{B}\times\nabla\alpha)\cdot\boldsymbol{\kappa}/B^2$ is negative with bad magnetic curvature and the term $\kappa_w P' F^2$ is destabilizing. The Euler–Lagrange equation to minimize the energy integral is obtained considering $\boldsymbol{B}\cdot\nabla = \boldsymbol{B}\cdot((\nabla\psi)\partial/\partial\psi + (\nabla\theta)\partial/\partial\theta + (\nabla\alpha)\partial/\partial\alpha) = J^{-1}\partial/\partial\theta$ as follows,

$$J^{-1}\frac{\partial}{\partial\theta}\left[\frac{|\nabla\alpha|^2}{JB^2}\frac{\partial F}{\partial\theta}\right] + \mu_0 P'(\psi)\kappa_w F = 0 \ . \tag{6.53}$$

This equation written in Clebsch coordinates (ψ, θ, α) is the same as Equation 9 of Connor and Taylor [19], who first derived a correct ballooning equation with orthogonal coordinates (ψ, χ, ζ) (ψ: poloidal flux RA_ζ, χ: poloidal angle, ζ: toroidal angle) [20]. Since Equation 6.53 does not include ψ derivative, we can solve it without considering the radial structure as given by $\Phi = F(\theta)\mathrm{e}^{-\mathrm{i}n\alpha}$. However, its meaning is given in the note. Considering Equation 6.53 is a linear equation and the quasi-mode expansion $F = \sum_j \varphi(\theta + 2\pi j)\mathrm{e}^{\mathrm{i}nq(2\pi j - \theta_0)} = \sum_j F_1(\theta + 2\pi j)$, $F_1(y)$ satisfies the same Euler–Lagrange equation for F but with its domain $(-\infty, \infty)$.

$$J^{-1}\frac{\partial}{\partial y}\left[\frac{|\nabla\alpha|^2}{JB^2}\frac{\partial F_1(y)}{\partial y}\right] + \mu_0 P'(\psi)\kappa_w F_1(y) = 0 \ . \tag{6.54}$$

The stability condition is $F_1(y)$ should not cross zero, as in Newcomb's theorem 10 and $F_1(y) \to 0$ at $y \to \pm\infty$ for marginal stability.

Note: Radial Structure of Quasi-modes [21]

Zakharov [21] gave a physical explanation showing that the quasi-mode is a superposition of infinite radially (perpendicular to flux surface) overlapping modes (see Figure 6.3). We start from the following Fourier expansion of F, since any periodic function of θ can be expanded in the Fourier series,

$$\Phi = \sum_{k=-\infty}^{\infty} \Phi_k(q) e^{i(m+k)\theta} e^{-in\zeta} \tag{6.55}$$

where $q = m/n$ (since we consider the case of $n \to \infty$ limit, we can use the fact that any irrational number can be given as the limit of a rational number).

It should be noted that the Fourier spectrum of Equation 6.55 resonates at a different safety factor (or radial position) $q(r) = (m+k)/n$. Namely, $\Phi_k(q)$ is a resonant mode at the rational surface $q + \Delta q = (m+k)/n$. Since radial variation of equilibrium quantities is weak, we can assume the translational symmetry for $\Phi_k(q)$ with the amplitude envelope $a(\Delta q)$. Namely, $\Phi_k(q) = a(\Delta q)\Phi_0(q - \Delta q)(\Delta q = k/n)$, $\Phi_0(q)$ is an eigen function for $k = 0$).

For the $n \to \infty$ ballooning mode, we can set $a(\Delta q) = 1$, since $\Delta q = k/n \to 0$. If we consider the expression of $\Phi_0(q)$ by the Fourier transform $\Phi_0(q) = (2\pi)^{-1} \int F_0(s) \exp(isnq)\, ds$ in the infinite domain of $nq \in (-\infty, \infty)$, we obtain following form of Φ,

$$\begin{aligned}
\Phi &= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} e^{ik\theta} e^{-in\alpha} \int_{-\infty}^{\infty} F_0(s) e^{is(nq-k)}\, ds \\
&= e^{-in\alpha} \int_{-\infty}^{\infty} F_0(s) e^{isnq} \sum_{j=-\infty}^{\infty} \delta(\theta - s + 2\pi j)\, ds \\
&= -e^{-in\alpha} \sum_{j=-\infty}^{\infty} F_0(\theta + 2\pi j) e^{inq(\theta+2\pi j)} = F(\theta, r) e^{-in\alpha} .
\end{aligned} \tag{6.56}$$

Here, we defined $F = -\sum_j F(\theta + 2\pi j) e^{2\pi i n j} = \sum_j F_1(\theta + 2\pi j)$ and used the following delta function formula,

$$\frac{1}{2\pi} \sum_{k=-\infty}^{\infty} e^{ik(\theta-s)} = \sum_{j=-\infty}^{\infty} \delta(\theta - s + 2\pi j) .$$

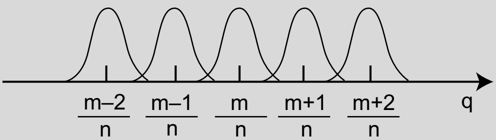

Figure 6.3 Radial mode overlap of ballooning modes in Equation 6.55

6.7 Flow: Non-Hermitian Frieman–Rotenberg Equation

In an axisymmetric system such as tokamak, the neoclassical viscosity in the toroidal direction is small and the toroidal rotation at a fraction of the speed of sound can be induced. In this case, we need to consider the flow in the force equilibrium as follows [22],

$$\rho(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla P - \boldsymbol{J}\times\boldsymbol{B} = 0 \tag{6.57}$$

$$\nabla\times(\boldsymbol{u}\times\boldsymbol{B}) = 0 \tag{6.58}$$

$$\boldsymbol{B} = \nabla\zeta\times\nabla\psi + F\nabla\zeta\ . \tag{6.59}$$

From Equation 6.58, we obtain $\boldsymbol{u}\times\boldsymbol{B} = -\nabla\Phi$ and considering $\boldsymbol{B}\cdot\nabla\Phi = 0$,

$$\nabla\Phi = \Omega(\psi)\nabla\psi\ . \tag{6.60}$$

Flow $\boldsymbol{u}$ on the flux surface can be expressed as

$$\boldsymbol{u} = \frac{\Phi_M}{\rho}\boldsymbol{B} + R^2\Omega\nabla\zeta\ . \tag{6.61}$$

In a tokamak, the poloidal rotation is small for neoclassical viscosity, so we consider the case of pure toroidal rotation $\Phi_M = 0$. In this case, we obtain $\rho(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\rho R\Omega^2\nabla R$ (the centrifugal force term) from $\boldsymbol{u} = R^2\Omega\nabla\zeta$. Substituting this equation into Equation 6.57 and taking ζ component, we get $(\boldsymbol{J}\times\boldsymbol{B})\cdot\nabla\zeta = 0$ because of axisymmetry. The following relation can be obtained by taking $\nabla\times$ Equation 6.59.

$$\boldsymbol{J} = \mu_0^{-1}[\nabla F\times\nabla\zeta + \Delta^*\psi\nabla\zeta]\ . \tag{6.62}$$

From $(\boldsymbol{J}\times\boldsymbol{B})\cdot\nabla\zeta = 0$, $(\nabla\psi\times\nabla F)\cdot\nabla\zeta = 0$ is obtained by using the vector formula and we obtain $F = F(\psi)$:

$$\mu_0\boldsymbol{J}\times\boldsymbol{B} = -\frac{FF'(\psi) + \Delta^*\psi}{R^2}\nabla\psi\ . \tag{6.63}$$

Therefore, the following relation is obtained from Equation 6.57:

$$-\rho R\Omega^2\nabla R = -\nabla P - \frac{FF'(\psi) + \Delta^*\psi}{\mu_0 R^2}\nabla\psi\ . \tag{6.64}$$

From the centrifugal force term in the left-hand side, the pressure is no longer a flux function. Taking Equation 6.64 $\cdot\partial\boldsymbol{x}/\partial R$, and considering the orthogonality relation (Equation 3.5), we get the following relation:

$$\rho R\Omega^2 = \left.\frac{\partial P}{\partial R}\right|_{\psi}\ . \tag{6.65}$$

Namely, the centrifugal force term is compensated by the radial pressure gradient. Furthermore, taking Equation 6.64 $\cdot\partial\boldsymbol{x}/\partial\psi$ and considering Equation 3.5, we obtain the following Grad–Shafranov equation with toroidal flow:

$$\Delta^* \psi = -\mu_0 R^2 \partial P(\psi, R)/\partial \psi - FF'(\psi) \,. \tag{6.66}$$

Assuming $T = T(\psi)$ and defining the ion mass $M = \rho/n$, R integration of Equation 6.6) gives the following formula:

$$P(\psi, R) = P_0(\psi) \exp\left[\frac{M}{2T} R^2 \Omega^2\right] \,. \tag{6.67}$$

The action principle of magnetic fluid flow in the plasma, Frieman–Rotenberg [6] is given as follows:

$$S = \int L \,\mathrm{d}V \mathrm{d}t \tag{6.68}$$

$$L = \frac{1}{4}\rho\dot{\boldsymbol{\xi}}^2 - \rho\boldsymbol{\xi}\cdot(\boldsymbol{u}\cdot\nabla)\dot{\boldsymbol{\xi}} + \frac{1}{2}\rho\boldsymbol{\xi}\cdot\boldsymbol{F}(\boldsymbol{\xi}) \,. \tag{6.69}$$

From the Lagrangian L, the generalized momentum is given by $\boldsymbol{p} \equiv \partial L/\partial\boldsymbol{\xi} = \rho(\partial\boldsymbol{\xi}/\partial t) + \rho\boldsymbol{u}\cdot\nabla\boldsymbol{\xi}$ and the Hamiltonian $H(= \boldsymbol{p}\cdot\boldsymbol{\xi} - L)$ is given below:

$$H = \frac{1}{2\rho}[\boldsymbol{p} - \rho\boldsymbol{u}\cdot\nabla\boldsymbol{\xi}]^2 - \frac{1}{2}\rho\boldsymbol{\xi}\cdot\boldsymbol{F}(\boldsymbol{\xi}) \,. \tag{6.70}$$

The Hamilton equation $\mathrm{d}\boldsymbol{p}/\mathrm{d}t = -\partial H/\partial\boldsymbol{\xi}$ gives $\mathrm{d}\boldsymbol{p}/\mathrm{d}t = \boldsymbol{F}(\boldsymbol{\xi}) - \rho\boldsymbol{u}\cdot\nabla[(\boldsymbol{p}/\rho) - \boldsymbol{u}\cdot\nabla\boldsymbol{\xi}]$. From this equation, the following Frieman–Rotenberg equation is obtained as the linearized equation of motion with the magnetic fluid flow [6]:

$$\begin{aligned}
\rho\frac{\partial^2\boldsymbol{\xi}}{\partial t^2} + 2\rho(\boldsymbol{u}\cdot\nabla)\frac{\partial\boldsymbol{\xi}}{\partial t} &= \boldsymbol{F}(\boldsymbol{\xi}) \\
\boldsymbol{F}(\boldsymbol{\xi}) &= \boldsymbol{F}_s(\boldsymbol{\xi}) + \boldsymbol{F}_d(\boldsymbol{\xi}) \\
\boldsymbol{F}_s(\boldsymbol{\xi}) &= \nabla[\boldsymbol{\xi}\cdot\nabla P + \gamma P\nabla\cdot\boldsymbol{\xi}] + (\nabla\times\boldsymbol{B}_1)\times\boldsymbol{B} + \boldsymbol{J}\times\boldsymbol{B}_1 \\
\boldsymbol{F}_d(\boldsymbol{\xi}) &= \nabla\cdot[\rho\boldsymbol{\xi}(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \rho\boldsymbol{u}(\boldsymbol{u}\cdot\nabla)\boldsymbol{\xi}] \\
\boldsymbol{B}_1 &= \nabla\times(\boldsymbol{\xi}\times\boldsymbol{B}) \,.
\end{aligned} \tag{6.71}$$

$\boldsymbol{F}_s(\boldsymbol{\xi})$ and $\boldsymbol{F}_d(\boldsymbol{\xi})$ are the static and dynamic operators, respectively, and both are Hermitian operators [22]. This Hermitian property of $\boldsymbol{F}$ is consistent with the energy conservation equation as given by

$$H = \frac{1}{2}\int\left[\rho\left(\frac{\partial\boldsymbol{\xi}}{\partial t}\right)^2 - \boldsymbol{\xi}\cdot\boldsymbol{F}\cdot\boldsymbol{\xi}\right]\mathrm{d}V = \text{const.} \tag{6.72}$$

On the other hand, the convective term $L = 2\rho(\boldsymbol{u}\cdot\nabla)\partial_t\boldsymbol{\xi}$ is an anti-Hermitian operator ($L(\boldsymbol{\zeta},\boldsymbol{\xi}) = -L^*(\boldsymbol{\xi},\boldsymbol{\zeta})$), and the system is not self-adjoint as a whole. It is difficult to solve the equation as an eigenvalue problem. Therefore, the Frieman–Rotenberg equation is solved as the initial value problem [23] or by the Laplace transform technique [24]. For example, the Laplace transform, $\boldsymbol{\xi}(t) \to \boldsymbol{\xi}(\omega)$ $(t \in R, \omega \in C)$ gives

$$L\boldsymbol{\xi}(\omega) = \boldsymbol{m}_0(\omega) \ . \tag{6.73}$$

Here, $L = \omega^2\rho + 2i\omega\rho(\boldsymbol{u}\cdot\nabla) + \boldsymbol{F}$ and $\boldsymbol{m}_0(\omega) = i\omega\rho\boldsymbol{\xi}_0 + \rho\boldsymbol{u}\times(\nabla\times\boldsymbol{\xi}_0) + \boldsymbol{B}\times(\nabla\times\boldsymbol{\eta}_0) + \rho\nabla\alpha - \beta\nabla s$. Let the eigenvalue of this equation (the spectrum) ω_j $(j = 1,\ldots)$, and the continuous eigenvalue (continuous spectrum) $\omega \in \sigma_c$, the eigen mode decomposition of the solution is given by

$$\xi(t) = \sum_j \xi(\omega_j)\exp(-i\omega_j t) + \int_{\sigma_c} \xi(\omega)\exp(-i\omega t)\,d\omega \ . \tag{6.74}$$

Here, the eigen function $\boldsymbol{\xi}(\omega_j)$ for the point spectrum ω_j $(j = 1,\ldots)$, and the singular eigen function $\boldsymbol{\xi}(\omega)$ corresponding to the continuous spectrum are given as follows,

$$\boldsymbol{\xi}(\omega_j) = -(1/2\pi)\int_{\Gamma(\omega_j)} \boldsymbol{\xi}_\omega d\omega \ , \tag{6.75}$$

$$\boldsymbol{\xi}(\omega) = (1/2\pi)[\boldsymbol{\xi}(\omega + i0) - \boldsymbol{\xi}(\omega - i0)] \ . \tag{6.76}$$

For cylindrical plasma, Newcomb equation 6.30 has to be modified to include Doppler shift $\Delta\omega = \boldsymbol{k}\cdot\boldsymbol{u}$ due to plasma flow. This Doppler shift splits the singular point of the Newcomb equation from a rational surface $(\boldsymbol{k}\cdot\boldsymbol{B} = 0)$ to two singular points $(\boldsymbol{k}\cdot\boldsymbol{v}_A = \pm\boldsymbol{k}\cdot\boldsymbol{u}$ $(\boldsymbol{v}_A = \boldsymbol{B}/(\mu_0\rho)^{1/2})$ for Alfven and slow magnetosonic resonances [25].

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