Chapter 5
Plasma Kinetic Theory: Collective Equation in Phase Space

The motion of a large number of charged particles in plasma could be determined completely if the initial conditions are known since an individual particle follows Newton's equations of motion. The flow of the probability distribution function of the system in $6N$ phase space consisting of the position and momentum of N particles shows incompressibility (Liouville theorem). This property leads to an important theorem of the isolated dynamical system "Poincare's recurrence theorem," which guarantees that the system will return to be arbitrarily close to the initial state. The kinetics equation represented by the Boltzmann equation is derived from reversible mechanics equations, but is often irreversible. In the Boltzmann equation, a statistical assumption "Stosszahl Ansatz" leads to a collision term exhibiting the arrow of time. Thus, there is a fundamental difference between the reversible dynamical equation and the kinetic equation.

In the kinetic equation for high temperature plasma, a strange phenomenon (called Landau damping) occurs where the oscillating electric field damps with time even when collisions are negligible through the mechanism of "phase mixing" in the velocity space, since the operator of the kinetic equation $\boldsymbol{v} \cdot \partial f / \partial \boldsymbol{x}$ has a continuous spectrum. In this chapter, the basics of plasma kinetic equations including Coulomb collisions, the drift kinetic equation, and the gyro kinetic equations are introduced based on the orbit theories described in Chapter 4.

5.1 Phase Space: Liouville Theorem and Poincaré Recurrence Theorem

Plasma consists of many electrons and ions and the state of plasma motion is determined by their position and velocity. Once the initial values are determined, they are uniquely governed by the dynamical equation. The information necessary for each ion/electron is the position (x, y, z) and speed (v_x, v_y, v_z). Then, to specify the state of, for example, 10^{23} particles, a set of 6×10^{23} variables is necessary. This set of variables is regarded as "space" called "phase space" and the trajectory in

the phase space is considered. Visualization of a four or higher dimensional space is not possible, but it is easier to imagine the motion of one point in the 6×10^{23} dimensional virtual space (phase space) than to imagine the motion of 10^{23} particles in 3-dimensional space [1]. When N particles move according to Newton's equations of motion, the point representing the system state draws the trajectory according to the following Hamilton equation in the $6N$-dimensional phase space $\boldsymbol{Z} = (\boldsymbol{q}_{3N}, \boldsymbol{p}_{3N})$ (called Γ space in statistical mechanics).

$$\begin{aligned} \frac{\mathrm{d}q_j}{\mathrm{d}t} &= \frac{\partial H}{\partial p_j} , \\ \frac{\mathrm{d}p_j}{\mathrm{d}t} &= -\frac{\partial H}{\partial q_j} . \end{aligned} \qquad (j = 1, 3N) \tag{5.1}$$

According to the above considerations, the system state is determined as a single point in the phase space, according to Newton's equation, if the initial values are given. US physicist J. W. Gibbs (1839–1903) who constructed statistical mechanics [2] introduced the concept of "ensemble" (he considered that a measurable macroscopic state includes a large number of microscopic states and this set is called an "ensemble"). To define the probability of the system in a microscopic state in the ensemble, he introduced a probability density D. In other words, the description of the system was changed from a "deterministic view" to a non-deterministic "probabilistic view." The macroscopically identical system is assumed to have a smooth distribution in phase space and the distribution is given by the probability distribution function. By this "smoothness," the possibility of the direction of time is introduced. Combining the continuity equation in phase space and the Hamilton equation, the phase space flow of the probability density is shown to be incompressible. Arbitrary volume element Ω in the phase space changes its shape with time but conserves its volume. Phase space flow $\boldsymbol{v}$ of the probability density D in the $6N$-dimensional phase space satisfies $\nabla \cdot \boldsymbol{v} = 0$. In fact, using Equation 5.1,

$$\nabla \cdot \boldsymbol{v} = \sum_{j=1}^{3N} \left[\frac{\partial \dot{p}_j}{\partial p_j} + \frac{\partial \dot{q}_j}{\partial q_j} \right] = \sum_{j=1}^{3N} \left[\frac{\partial}{\partial p_j}(-\frac{\partial H}{\partial q_j}) + \frac{\partial}{\partial q_j}\frac{\partial H}{\partial p_j} \right] = 0 . \tag{5.2}$$

Substituting $\nabla \cdot \boldsymbol{v} = 0$ into the continuity equation in $6N$-dimensional phase space $\partial D/\partial t + \nabla \cdot (D\boldsymbol{v}) = 0$, we obtain $\mathrm{d}D/\mathrm{d}t = \partial D/\partial t + \boldsymbol{v} \cdot \nabla D = 0$.

$$\frac{\mathrm{d}D}{\mathrm{d}t} = \frac{\partial D}{\partial t} + \sum_{j=1}^{3N} \left[\frac{\partial D}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial D}{\partial p_j}\frac{\partial H}{\partial q_j} \right] = \frac{\partial D}{\partial t} + \{D, H\} = 0 . \tag{5.3}$$

Here, $\{D, H\}$ is called the Poisson bracket. The total derivative, $\mathrm{d}D/\mathrm{d}t$ is the time derivative of probability density along the $6N$-dimensional phase space flow, so probability density is conserved along the phase space flow. This is called the Liouville theorem [3]. Incompressibility of the phase space flow leads to an interesting

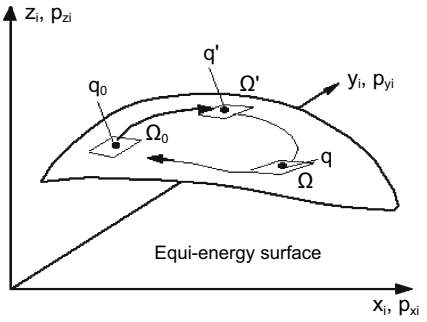


Figure 5.1 Phase space motion of dynamical system. System state is expressed by a point in the phase space. Starting from initial point q_0, q is always on the equi-energy surface

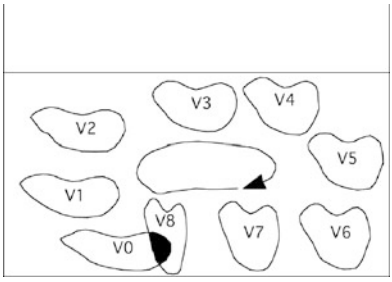


Figure 5.2 Water convection in a water tank. Initial volume element V_0 move as time goes $V_1, \ldots, V_8$. The incompressibility of water means all volumes are same

property of the "Poincaré recurrence theorem" [4] that an isolated dynamical system will return to be arbitrarily close to the starting point as time passes. This theorem plays an important role in the paradoxical discussion in deriving the laws of thermodynamics with an arrow of time from the time reversible dynamical equation [5].

Total energy is conserved in the isolated dynamical system consisting of N particles. The $6N$-dimensional phase space trajectory is shown in Figure 5.1 on the equi-energy surface. The Poincaré recurrence theorem claims that the system will return to an arbitrary close point from the initial point q_0 on the phase space (arbitrary neighborhood of q_0).

The Poincaré recurrence theorem is proved using reduction to absurdity. That is, contradiction occurs if the state does not return to be arbitrary close from the starting point. The motion on the phase space "behaves like a incompressible fluid," so the concept of the proof can be explained by using water as typical example of an incompressible fluid [6]. Figure 5.2 shows water convection in a finite volume water tank. The water occupying region V_0 at initial time t_0 moves to the new region V_1 at time t_1. The shape of V_1 may be different from that of V_0 but has same volume. Let V_2, V_3, ... be regions at later time t_2, t_3, ... If they never overlap, the volume of water becomes infinite and contradicts the initial assumption of the finite volume water tank. This shows that the initial assumption was wrong.

This Poincaré recurrence theorem predicts that in Figure 5.3 all gas molecules are in a left box (Figure 5.3 (a)) will expand to the full box after opening the shutter (Figure 5.3 (b)), but that a state arbitrary close to (Figure 5.3 (a)) can be realized some day (Figure 5.3 (c)).

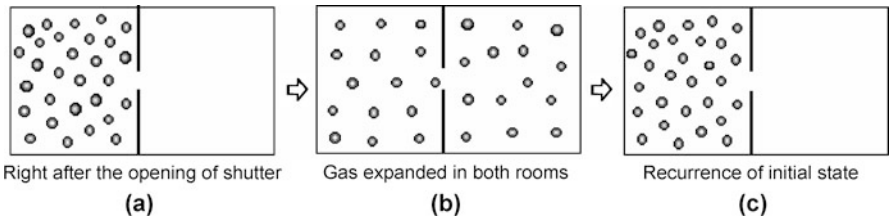


Figure 5.3 Molecular diffusion and prediction from the recurrence theorem

Note: The Poincaré Recurrence Theorem [7]

A mathematically more accurate definition and proof of the Poincaré theorem are given in Arnold [7].

Poincaré recurrence theorem: Let S be a bounded region in phase space and g a volume (measure) conserving one-to-one mapping from S to $S(gS = S)$. Then, in any neighborhood V of an arbitrary point in S, there is a point q that returns to the neighborhood V, $q \in V$. Namely, there exists $n > 0$ such that $g^n q \in V$.

Proof: Consider an infinite series of the mapping of neighborhood $V, gV, g^2V, \ldots, g^nV, \ldots$. Since g is volume (measure) conserving mapping, these mappings have the same volume. If they do not intersect, volume (measure) of S becomes infinite. This is inconsistent with the assumption that S is a bounded region. Therefore, there is an intersection among mappings. Let g^kV and g^mV $(k > m \geq 0)$ intersect, this means $g^kV \cap g^mV \neq \emptyset$. Here $\cap$ and $\emptyset$ represent the intersection and empty sets, respectively. Then, we obtain $g^{(k-m)}V \cap V \neq \emptyset$. Therefore, selecting q from the intersection, we have $q \in V$ and $g^{(k-m)}q \in V$ and $n = k - m$.

5.2 Dynamics and Kinetics: Individual Reversible and Collective Non-reversible Equations

The concept of the "velocity distribution function" was introduced by British physicist J. C. Maxwell (1831–1879; Figure 5.4 (a)) in 1860 [8]. Instead of specifying the status of all the particles in a deterministic dynamical equation, the smooth function f is defined so that $f(\boldsymbol{x}, \boldsymbol{v}, t)\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{v}$ is the number of particles in position interval $\boldsymbol{x} \sim \boldsymbol{x} + \mathrm{d}\boldsymbol{x}$ and velocity interval $\boldsymbol{v} \sim \boldsymbol{v} + \mathrm{d}\boldsymbol{v}$. This smooth distribution function is constructed from the original discrete distribution function through some statistical operation. The deterministic reversible equation is converted to an irreversible collective equation. The exact velocity distribution function F, considering plasma is a group of discrete particles, is given in the following form by using a delta func-

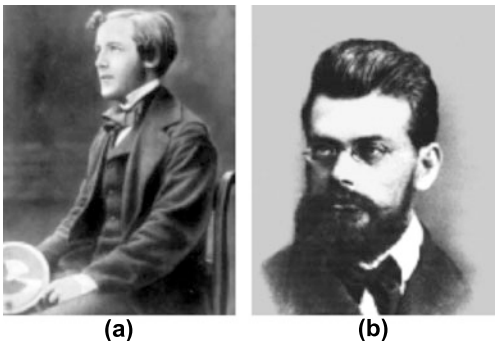


Figure 5.4 (a) James Clark Maxwell who invented the concept of velocity distribution function and (b) Ludwig Boltzmann who invented the Boltzmann equation and derived H-theorem to explain macroscopic irreversibility from microscopic law

tion, and it follows the "Klimontovich equation" in which particle density is conserved along the phase space $\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{v})$ (called γ space in statistical mechanics) trajectory [9].

$$F(\boldsymbol{x}, \boldsymbol{v}, t) = \sum_{i=1}^{N} \delta(\boldsymbol{x} - \boldsymbol{x}_i(t))\delta(\boldsymbol{v} - \boldsymbol{v}_i(t)) \,, \tag{5.4}$$

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\partial F}{\partial t} + \boldsymbol{v} \cdot \frac{\partial F}{\partial \boldsymbol{x}} + \boldsymbol{a} \cdot \frac{\partial F}{\partial \boldsymbol{v}} = 0 \,. \tag{5.5}$$

Here, the acceleration $\boldsymbol{a} = (e/m)(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$ includes the force of the average electromagnetic field and the Coulomb collisions between charged particles. Taking the ensemble average of F to get a smooth velocity distribution function $f = \langle F \rangle_{\text{ensemble}}$ and we obtain the following "collision term" [10],

$$C(f) = \bar{\boldsymbol{a}} \cdot \frac{\partial f}{\partial \boldsymbol{v}} - \left\langle \boldsymbol{a} \cdot \frac{\partial F}{\partial \boldsymbol{v}} \right\rangle_{\text{ensemble}} \tag{5.6}$$

Also, the Boltzmann-type transport equation is obtained.

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f}{\partial \boldsymbol{x}} + \bar{\boldsymbol{a}} \cdot \frac{\partial f}{\partial \boldsymbol{v}} = C(f) \tag{5.7}$$

$$C(f) = -\left\langle \tilde{\boldsymbol{a}} \cdot \frac{\partial \tilde{F}}{\partial \boldsymbol{v}} \right\rangle_{\text{ensemble}} \,. \tag{5.8}$$

Here, $\boldsymbol{a}$ and F are divided into an average part and a microscopic fluctuating part ($\boldsymbol{a} = \bar{\boldsymbol{a}} + \tilde{\boldsymbol{a}}$, $F = f + \tilde{F}$). We find that the collision term is a correlation of the acceleration by the microscopic Coulomb field and the associated velocity space gradient of the fluctuating distribution function. Acceleration by mean-field is treated deterministically and collision by the microscopic Coulomb field is treated statistically. The "smoothness" of the distribution function plays an important role in explaining collisionless damping (Landau damping) by "phase mixing." The collision term in the plasma is discussed in Section 5.6. The collision term for molecular gas was

derived by the Austrian physicist L. Boltzmann (1844–1906 Figure 5.4 (b)). He introduced "Stosszahl Ansatz" which states that that there will be no correlation between position and momentum of two colliding particles [1]. With this assumption, we can calculate the time variation of particle number into and out from phase space volume $\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{v}$ through collisional short-range force using f as follows,

$$C(f) = \int \left[f(\boldsymbol{v}') f(\boldsymbol{v}_1') - f(\boldsymbol{v}) f(\boldsymbol{v}_1) \right] |\boldsymbol{v}_1 - \boldsymbol{v}|\, \sigma\, \mathrm{d}\Omega\, \mathrm{d}\boldsymbol{v}_1 \ . \tag{5.9}$$

Here, $\boldsymbol{v}$ and $\boldsymbol{v}_1$ are the velocities before collision and $\boldsymbol{v}'$ and $\boldsymbol{v}_1'$ are the velocities after collision. Also, the first term on the right-hand side is incoming particles to the velocity interval $\mathrm{d}\boldsymbol{v}$ from the inverse collision of particles with $\boldsymbol{v}'$ and $\boldsymbol{v}_1'$, and the second term is outgoing particles from the velocity interval $\mathrm{d}\boldsymbol{v}$ from the collision. The Boltzmann equation is an irreversible equation constructed from reversible dynamics (see Salon).

Salon: Reversible Dynamical Equation and Irreversible Kinetic Equation

The Newton equation $m\,\mathrm{d}^2x/\mathrm{d}t^2 = F$ does not change by $t \to -t$ and is symmetric to time reversal. On the other hand, phenomena concerning heat are not time reversible, for example, when hot water gets cold or gas expands to a low-pressure region. Such phenomena are called irreversible processes. If the heat is from the microscopic motion of atom and molecule, question arises whether irreversible thermal phenomena such as the law of entropy increase can be explained by the reversible dynamical equation.

Boltzmann attacked this problem by applying a dynamical equation to the microscopic molecule. He constructed an equation (Boltzmann equation 5.7 and 5.9) governing the process whereby a non-equilibrium gas relaxes to equilibrium by collisions between molecules. The Boltzmann collision term shows that the Boltzmann H function $H = \int f \cdot \ln f \, \mathrm{d}\boldsymbol{v}$ decreases monotonically with time (the Boltzmann H theorem). He used a time-symmetric dynamical equation for the collision process, but the equation got a time arrow through the statistical operation of counting the number of colliding particles.

The German physicist E. Zermelo (1871–1953) pointed out that H theorem contradicts Poincaré recurrence theorem [5]. Boltzmann's H theorem is a statistical theorem and the very low probability of recurrence is neglected (or, the equivalently neglected case which takes a very long time from the deterministic dynamical equation). Boltzmann's equation eliminates the recurrence and describes evolution to the high probability state. The Japanese Nobel Prize in Physics winner S. Tomonaga discussed this paradox, which lies between the kinetics and dynamics, in detail [11]. This situation is the same for the Coulomb collision in plasma, and $H = \int f \cdot \ln f \, \mathrm{d}\boldsymbol{v}$ decreases monotonically with time.

Consider the small size of the realization probability of the Poincaré recurrence state from the example of particle diffusion in a box shown in Figure 5.3). The group of particles in the left box expands to both boxes but will revert to left box at some time according to the Poincaré recurrence theorem. Then, try to

evaluate the probability. The probability of having a particle in either box is the same and is $1/2$. Assuming the probability of each particle is independent, the probability of all particles entering the left box is $(1/2)^N$ (N is the number of particles). If $N = 10^{23}$, the chance to realize the state predicted by the Poincaré recurrence is very small. This means that it take an enormous amount of time to realize the recurrent state. From the viewpoint of the initial value, there is a set of initial values, with which particles can return to the left box in a relatively short timescale (for example, the time reversal solution of diffusion). But, this has a very low probability among all possible initial values for the particles spread in the whole box.

Interestingly, it seems apparent that the probability of having a particle in either box is the same, but it is not self-evident that the probability that exists in any location in the whole box is the same. This problem is an example in a dynamical system called an "ergodic problem" and is explained by "Weyl's billiards" [12]. The arrow of time in a many-particle system is caused by a macroscopic manipulation (for example, the operation of opening the shutter), which leads to a change in the number of possible motion states, and the system tends to the state of dominant probability. The key here is that the macro-operation can increase the number of microscopic states, but rarely reduce it. British astronomer Arthur Eddington (1882–1944) coined the phrase "arrow of time" in 1927. Some physical phenomena are difficult to reverse in time and this phrase indicated that "time" has direction related to the occurring phenomena. The arrow of time is discussed in detail by Davies in [13].

5.3 Vlasov Equation: Invariants, Time-reversal Symmetry and Continuous Spectrum

"Collision" in terms of changes in the velocity distribution function due to the microscopic electric field of the Coulomb potential becomes negligible compared with the average force when the plasma temperature is higher and such a plasma state is called "collisionless plasma." In this case, the right-hand side of Equation 5.7 can be ignored. Russian physicist A. Vlasov (1908–1975; Figure 5.5 (a)) pointed out for the first time that the collision term can be ignored in high-temperature plasma and this equation is called Vlasov equation [14].

$$\frac{\mathrm{d}f_s}{\mathrm{d}t} = \frac{\partial f_s}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f_s}{\partial \boldsymbol{x}} + \bar{\boldsymbol{a}} \cdot \frac{\partial f_s}{\partial \boldsymbol{v}} = 0 \,. \tag{5.10}$$

Here, $\sum \bar{\boldsymbol{a}} = (e_s/m_s)(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$ is the average acceleration excluding the microscopic Coulomb field. The f_s in Vlasov equation is the ensemble-averaged "smooth" distribution function. Equation 5.10 means that the density f_s is conserved for the observer moving with particle trajectory in phase space $\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{v})$

Figure 5.5 (a) A. A. Vlasov who identified Vlasov equation as basic equation for collisionless plasma (with kind permission of Physics-Uspekhi journal) and (b) L. D. Landau who found Landau damping as collisionless damping mechanism

(density variation along phase space flow, $\mathrm{d}f_s/\mathrm{d}t$, the Lagrange differential, is zero). From this property, $f(\boldsymbol{x}, \boldsymbol{v}, t) > 0 (t > 0)$ if $f(\boldsymbol{x}, \boldsymbol{v}, t = 0) > 0$ are met. In other words, the trajectory of particle motion in phase space (characteristic curve) contours of constant f_s. If no sink and source exists in phase space, the particle conservation law in phase space is given by $\partial f_s/\partial t + \partial/\partial \boldsymbol{z} \cdot (\boldsymbol{u} f_s) = 0$ ($\boldsymbol{u} = (\boldsymbol{v}, \bar{\boldsymbol{a}})$ and comparison with Vlasov equation leads to $(\partial/\partial \boldsymbol{z}) \cdot \boldsymbol{u} = 0$. This means that the phase space flow is incompressible.

In a collision-dominated gas in equilibrium, entropy is conserved in the isolated system. In collisionless plasma satisfying the Vlasov equation, H defined by $H = \int G(f_s)\mathrm{d}\boldsymbol{z}$ is a conserved quantity ($\mathrm{d}H/\mathrm{d}t = 0$) for arbitrary function $G(f_s)$. In fact,

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \int \frac{\partial G(f_s)}{\partial t}\mathrm{d}\boldsymbol{z} = -\int G'(f_s)\boldsymbol{u}\cdot\frac{\partial f_s}{\partial \boldsymbol{z}}\mathrm{d}\boldsymbol{z} = -\int \frac{\partial \boldsymbol{u} G(f_s)}{\partial \boldsymbol{z}}\mathrm{d}\boldsymbol{z} = 0\,. \quad (5.11)$$

Since G is an arbitrary function, the Vlasov equation has an infinite number of invariants. If we choose $G = f_s$, it gives the conservation of particles. Also, $G = -f_s \ln f_s$ gives the conservation of entropy in collisionless plasma. This property of the Vlasov equation is also called the generalized entropy conservation law [15].

The Vlasov equation has interesting properties, although we need to note that the equation is valid in the zero-collision limit. One of them is time-reversal symmetry, the Boltzmann equation does not have such symmetry. If $\boldsymbol{\psi} = (f_s, \boldsymbol{E}, \boldsymbol{B})$ is a solution of the Vlasov equation, $T\boldsymbol{\psi}(-t)$ is also a solution and is called the time-reversal solution (here, T is the "time reversal operator," and requires a reversal of the magnetic field) [16]. In the Boltzmann equation, the distribution function will converge to an equilibrium solution due to the property of the collision term. On the other hand, the solution of the Vlasov equation does not necessarily converge to an equilibrium solution due to its conservation property and time-reversal symmetry. If $\boldsymbol{\psi}(t)$ is the converging solution to equilibrium, the time reversal solution $T\boldsymbol{\psi}(-t)$ is the solution away from equilibrium.

The Vlasov equation has the structure of a wave equation. In addition to the wave frequency determined by the dispersion properties of the system, there is a wave

with frequency $\omega_c = \boldsymbol{k} \cdot \boldsymbol{v}$ continuously changing with particle velocity $\boldsymbol{v}$ at fixed wave number $\boldsymbol{k}$, $(\exp(-\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{v}t))$, which is called free streaming solution [17].

In fact, if we expand the electrostatic wave solution of the Vlasov equation without magnetic field as $f_a = F_{a0} + f_{a1}$ ($f_{a1} \ll F_{a0}$), the following Poisson equation and the linearized Vlasov equation are obtained,

$$\frac{\partial f_{a1}}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f_{a1}}{\partial \boldsymbol{x}} = \frac{e_a}{m_a} \nabla\varphi \cdot \frac{\partial f_{a0}}{\partial \boldsymbol{v}} \,, \tag{5.12}$$

$$\varepsilon_0 \nabla^2 \varphi = -e_a \int_{-\infty}^{\infty} f_{a1} d\,\boldsymbol{v} \,. \tag{5.13}$$

For the case of $f_{a1} = f_{a1kw} \exp(\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{x} - \mathrm{i}\omega t)$ and $\boldsymbol{\varphi} = \boldsymbol{\varphi}_{kw} \exp(\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{x} - \mathrm{i}\omega t)$, we obtain

$$(\omega - \boldsymbol{k} \cdot \boldsymbol{v}) f_{a1\boldsymbol{k}\omega} = -\frac{e_a}{m_a} \varphi_{\boldsymbol{k}\omega} \boldsymbol{k} \cdot \frac{\partial f_{a0}}{\partial \boldsymbol{v}} \,. \tag{5.14}$$

Here we note that the homogeneous solution of (5.14) is free streaming solution $e^{(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{v}t)}$. Since the general solution of $xf(x) = g(x)$ is given by $f(x) = g(x)P[x^{-1}] + \lambda\delta(x)$ (P is a principal value, $\delta(x)$ is the Dirac delta function), we obtain

$$f_{a1\boldsymbol{k}\omega} = \left[-\frac{e_a}{m_a} \boldsymbol{k} \cdot \frac{\partial f_{a0}}{\partial \boldsymbol{v}} P \frac{1}{\omega - \boldsymbol{k} \cdot \boldsymbol{v}} + \lambda\delta(\omega - \boldsymbol{k} \cdot \boldsymbol{v}) \right] \varphi_{\boldsymbol{k}\omega} \,. \tag{5.15}$$

If we inverse-Fourier transform the delta function term of Equation 5.15, we obtain a free streaming solution $f_{a1} = \exp[-\mathrm{i}\boldsymbol{k} \cdot (\boldsymbol{x} - \boldsymbol{v}t)]$. The range of $\boldsymbol{k} \cdot \boldsymbol{v}$ spans $-\infty$ to $+\infty$ and the waves have a "continuous spectrum." In fact, from the Poisson equation 5.13, we obtain following [17],

$$\left[1 + \frac{e_a}{\varepsilon_0 k^2 m_a} \int_{-\infty}^{\infty} \frac{P}{\omega - \boldsymbol{k} \cdot \boldsymbol{v}} \boldsymbol{k} \cdot \frac{\partial f_{a0}}{\partial \boldsymbol{v}} \mathrm{d}\boldsymbol{v} \right] + \frac{e_a}{\varepsilon_0 k^2} \lambda = 0 \,. \tag{5.16}$$

This equation gives the relationship between two unknowns, ω and λ for a given wave number k. This means that the angular frequency ω is arbitrary for a given k, namely, the spectrum of eigenvalues is continuous. This property of linear Vlasov equations originates from the fact that operator $A = \boldsymbol{k} \cdot \boldsymbol{v}(\mathrm{d}f/\mathrm{d}t = -\mathrm{i}Af)$ is a linear operator with a "continuous spectrum" [18, 19]. A non-damping wave can exist with a real ω and λ can be determined using Equation 5.16 which gives the necessary resonant particles in Equation 5.15. This wave is called the "Van Kampen mode" [16]. A free-streaming solution coupled to Maxwell distribution $f_{a1} = \exp[\mathrm{i}kut - (u/u_{\mathrm{th}})^2/2]$ oscillates in the velocity space more violently with time, as shown in Figure 5.6, and actual physical quantities, such as the electric field calculated from the velocity integral, tends to zero with time. This "structure extinction" is called "phase mixing" since it occurs due to the phase overlapping of the wave. Mathematically, this structure extinction is guaranteed by the "Riemann–Lebesgue theorem" [16, 20]. Collisionless damping of the electric field caused by the contin-

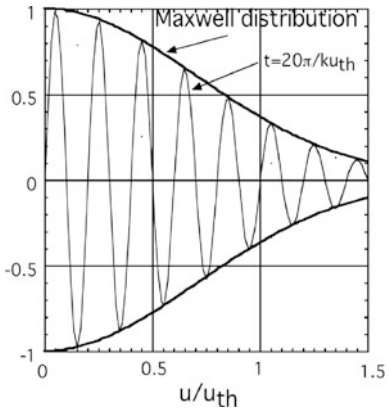


Figure 5.6 Maxwell distribution function $(\exp(-u^2/u_{\mathrm{th}}^2))$ and the phase mixing of perturbed distribution function $f_1 = \exp(\mathrm{i}kut - u^2/2u_{\mathrm{th}}^2)$ $(t = 20\pi/ku_{\mathrm{th}})$

uous spectrum of velocity space is called "Landau damping" as described in detail in Section 5.4 and was developed by L. D. Landau (1908–1968; Figure 5.5 (b)).

5.4 Landau Damping: Irreversible Phenomenon Caused by Reversible Equation

The Vlasov equation is symmetric to time reversal, and in order to satisfy causality (cause prior to the results or "arrow of time"), the Laplace transform on time (or an equivalent method) can be used. This corresponds to analyzing the problem as an initial value problem by restricting $t \geq 0$, in contrast to the "spectral analysis" detailed in Section 5.3. The electron plasma oscillation is described by considering that the electric field is determined by the perturbed plasma density according to the Poisson equation. The linear response of the system is a solution that meets both the Poisson equation and the Fourier transformed linearized Vlasov equation in space.

$$\frac{\partial f_{e1\boldsymbol{k}}}{\partial t} + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{v}\, f_{e1\boldsymbol{k}} = -\mathrm{i}\frac{e}{m_e}\varphi_{\boldsymbol{k}}\boldsymbol{k}\cdot\frac{\partial f_{e0}}{\partial \boldsymbol{v}}\ , \tag{5.17}$$

$$\varepsilon_0 k^2 \varphi_k = -e\int_{-\infty}^{\infty} f_{e1\boldsymbol{k}}\,\mathrm{d}\boldsymbol{v}\ . \tag{5.18}$$

A standard way to solve this equation correctly as an initial value problem is the Laplace transform in time (integration at $t \geq 0$).

$$f_{e1\boldsymbol{k}\omega}(\boldsymbol{v}) = \frac{1}{2\pi}\int_0^{\infty} f_{e1\boldsymbol{k}}(\boldsymbol{v},t)e^{\mathrm{i}\omega t}\,\mathrm{d}t\ , \tag{5.19}$$

$$\varphi_{\boldsymbol{k}\omega} = \frac{1}{2\pi}\int_0^{\infty} \varphi_{\boldsymbol{k}}(t)e^{\mathrm{i}\omega t}\,\mathrm{d}t\ . \tag{5.20}$$

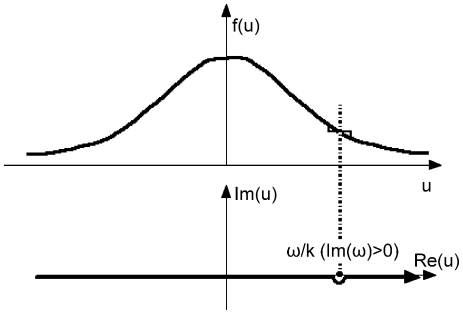


Figure 5.7 Maxwell distribution function ($\exp(-u^2/2u_{\rm th}^2)$) and the phase velocity of the wave that interacts with resonant particles with almost same velocity. Integration path in complex u plane for Landau problem

The key here is the Laplace transformation, Equations 5.19 and 5.20 are defined so that $\mathrm{Im}(\omega) = \omega_i$ (ω_i is a positive constant) is large enough to ensure the convergence of the integral. Causality is satisfied through this choice. The Fourier-Laplace transformation of Equations 5.17 and 5.18 are given as follows,

$$(\omega - \boldsymbol{k}\cdot\boldsymbol{v}) f_{e1\boldsymbol{k}\omega}(\boldsymbol{v}) = i f_{e1\boldsymbol{k}}(\boldsymbol{v}, t=0) + \frac{e}{m_e}\varphi_{\boldsymbol{k}\omega}\boldsymbol{k}\cdot\frac{\partial f_{e0}}{\partial \boldsymbol{v}}\ , \tag{5.21}$$

$$\mathrm{i}\varepsilon_0 k^2 \varphi_{\boldsymbol{k}\omega} = -e\int_{-\infty}^{\infty} f_{e1\boldsymbol{k}}(\boldsymbol{v},\omega)\mathrm{d}\boldsymbol{v}\ . \tag{5.22}$$

Substituting Equation 5.21 into 5.22 and eliminating $f_{e1k}(\boldsymbol{v}, t)$, we obtain

$$\varphi_{\boldsymbol{k}\omega} = -\frac{\mathrm{i}e}{\varepsilon_0 k^2 K(\omega,\boldsymbol{k})}\int_{-\infty}^{\infty}\frac{f_{e1k}(\boldsymbol{v}, t=0)}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}}\mathrm{d}\boldsymbol{v}\ , \tag{5.23}$$

$$K(\boldsymbol{k},\omega) = 1 + \frac{\omega_{pe}^2}{n_e k^2}\int\frac{\boldsymbol{k}\cdot\partial f_{e0}/\partial\boldsymbol{v}}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}}\mathrm{d}\boldsymbol{v} \tag{5.24}$$

where $\omega_{pe}^2 = e^2 n_e/\varepsilon_0 m_e$ and ω_{pe} is called plasma frequency.

The eigen modes are given by $K(\boldsymbol{k},\omega) = 0$, and this expression is called the "dispersion equation." In the integral of the dispersion equation, $u \equiv \boldsymbol{k}\cdot\boldsymbol{v}/k = \omega/k$ is a singular point (the denominator of the integral is zero: ω/k is the wave phase speed) and the integration method becomes the issue. Consider the initial value problem when a wave is excited in the plasma with a real wave number k. If the wave grows or decays, ω is complex. Therefore, the integral in Equation 5.24 has to be treated as a complex integration in the u plane. Vlasov took the Cauchy principal value. But Russian Nobel Prize winner L. D. Landau (1908–1968) realized that it should be treated as an initial value problem. This creates a term to circumvent singularity (the Landau damping term) [21]. In this case, the integration path of u is below the singular point since $\mathrm{Im}(\omega) > 0$ (see Figure 5.7). Separating the dispersion function K into real and imaginary parts with $\omega = \omega_r + \mathrm{i}\omega_i$, we obtain the following equations:

$$K(\boldsymbol{k},\omega) = K_r(\boldsymbol{k},\omega_r) + \mathrm{i}[K_i(\boldsymbol{k},\omega_r) + \omega_i\frac{\partial K_r(\boldsymbol{k},\omega_r)}{\partial\omega_r}] = 0\ , \tag{5.25}$$

$$\omega_i = -\frac{K_i(\boldsymbol{k}, \omega_r)}{\partial K_r(\boldsymbol{k}, \omega_r)/\partial \omega_r} , \tag{5.26}$$

$$K_r(\boldsymbol{k}, \omega) = 1 + \frac{\omega_{pe}^2}{n_e k^2} P \int \frac{\boldsymbol{k} \cdot \partial f_{e0}/\partial \boldsymbol{v}}{\omega_r - \boldsymbol{k} \cdot \boldsymbol{v}} \mathrm{d}\boldsymbol{v}, \; K_i(\boldsymbol{k}, \omega_r) = -\pi \frac{\omega_{pe}^2}{k^2} \frac{\partial f_{e0}}{\partial v} \Big|_{u=\omega_r/k} . \tag{5.27}$$

P indicates the Cauchy principal value integral. When electrons are Maxwellian and the wave phase velocity is much larger than the thermal velocity ($\omega_r/k \gg v_{te}$), we obtain $K_r(k, \omega) = 1 - (\omega_{pe}/\omega_r)^2 - 3(\omega_{pe}/\omega_r)^4 k^2 \lambda_{\mathrm{D}}^2$. So,

$$\omega_r = \omega_{pe}(1 + 1.5k^2\lambda_{\mathrm{D}}^2), \omega_i = -\sqrt{\frac{\pi}{8}} \frac{\omega_{pe}}{k^3 \lambda_{De}^3} \exp\left[-\left(\frac{1}{2k^2 \lambda De^2} + \frac{3}{2}\right)\right] . \tag{5.28}$$

Since $\omega_i < 0$, the wave will damp. This damping occurs without any dissipation of energy due to collision and is "collisionless damping." Landau was the first to identify this phenomenon [21] and so this is called "Landau damping". The physical mechanism of Landau damping is intuitively simple. First, since the decay rate comes from the residue at $u = \omega/k$, it is caused by the particle having almost the same speed of wave phase velocity (called the "resonant particle"). These particles can exchange energy with the wave creating an almost DC electric field, since particles move with the waves. In Landau damping, the number of particles gaining energy from the wave is larger than that losing energy to the wave as seen from the relation $\omega_i \sim \mathrm{d}f/\mathrm{d}v$ in Equation 5.27. Landau damping can be compared to surfing. If the surfboard is not on the wave, the wave simply passes and surfboard cannot gain energy. However, if the speed of the surfboard is the same as the wave, the board is pushed by the wave, giving it energy.

The Vlasov equation describing collisionless plasma does not have irreversibility such as that due to the collision term in the Boltzmann equation, but, it has an arrow of time through the damping of the wave called "Landau damping." Irreversibility of the Boltzmann equation was created by the "Stosszahl Ansatz," while Landau damping originated from "phase mixing" in the processes described in Section 5.3. The inverse operator of the linear operator $L = \omega - \boldsymbol{k} \cdot \boldsymbol{v}$ is $L^{-1} = P[1/(\omega - \boldsymbol{k} \cdot \boldsymbol{v})] + \lambda\delta(\omega - \boldsymbol{k} \cdot \boldsymbol{v})$ (λ is an arbitrary constant). If we impose the condition that the velocity distribution function is "smooth" at $t = 0$, λ needs to take a specific value $\lambda = \mathrm{i}\pi$. An inverse Laplace transformation of Equation 5.23 gives,

$$\boldsymbol{E}_{\boldsymbol{k}}(t) = -\frac{e\boldsymbol{k}}{2\pi\varepsilon_0 k^2} \int_{-\infty}^{\infty} \mathrm{d}\boldsymbol{v} \, f_{e1\boldsymbol{k}}(\boldsymbol{v}, t = 0) \int_{-\infty+\mathrm{i}\omega_i}^{\infty+\mathrm{i}\omega_i} \frac{\exp(-\mathrm{i}\omega t)\mathrm{d}\omega}{K(\omega, \boldsymbol{k})(\omega - \boldsymbol{k} \cdot \boldsymbol{v})} . \tag{5.29}$$

The free streaming term $\exp(-\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{v}t)$ is produced from the pole $\omega = \boldsymbol{k} \cdot \boldsymbol{v}$ of ω integration of Equation 5.29, and the density perturbation in velocity space oscillates more strongly with time. The resulting density perturbation n_1 and electric field $\boldsymbol{E}$ after integration in velocity space will damp with time due to this phase mixing. It might be thought that this collisionless damping by phase mixing would be incon-

sistent with the reversibility of the Vlasov equation (similar to the discussion with Loschmidt's "reversibility paradox" against the Boltzmann equation). For the solution $f_{e1k}(\boldsymbol{v}, t)$, the time reversal solution $f_{e1k}(\boldsymbol{x}, -\boldsymbol{v}, -t)$ is also a solution of the Vlasov equation and the density perturbation $n_1(\boldsymbol{x}, t)$ increases exponentially with time. However, this time reversal solution may have an initial value $f_{e1k}(\boldsymbol{x}, \boldsymbol{v}, t_1)$ and is not smooth since it includes $\exp(-i\boldsymbol{k} \cdot \boldsymbol{v}t_1)$. After t_1 passes, the density perturbation reaches a maximum where f_{e1k} is a smooth function. Then, the density will decay again with time due to phase mixing [16].

5.5 Coulomb Logarithm: Collective Behavior in the Coulomb Field

Now, let us consider collisions in the Coulomb field ignored in previous sections. Since plasma consists of charged particles (ions and electrons), a repulsion force acts between similarly charged particles and an attractive force acts between differently charged particles. Shielding of the electric field occurs in the same way as Debye shielding in electrolytes. Incidentally, this shielding phenomenon does not occur in many body systems under gravitational force, which have only attractive forces. When a potential is formed by density changes in the plasma, electrons and ions will follow a Boltzmann distribution $n_e = n_{e0}\exp(e\phi/kT)$ and $n_i = n_{i0}\exp(-eZ_i\phi/kT)$. Consider the potential around the ion. Assuming the thermal energy kT is much larger than the potential energy $e\phi$ ($e\phi \ll kT$), the solution of the Poisson equation for ϕ is obtained as follows,

$$\phi = \frac{e}{4\pi\varepsilon_0 r}\mathrm{e}^{-r/\lambda_\mathrm{D}} . \tag{5.30}$$

Here, $\lambda_\mathrm{D}^{-2} = \lambda_{De}^{-2} + \sum\lambda_{Di}^{-2}$, $\lambda_{De}^2 = (\varepsilon_0 kT/e^2 n_e)^{0.5} (= 7.43 \times 10^3 [T_e\,(\mathrm{eV})/n_e(m^{-3})]^{0.5}$ [m]), $\lambda_{Di}^2 = (\varepsilon_0 kT/e^2 Z_i^2 n_i)^{0.5}$. This shielding effect is called Debye shielding and ϕ is the Debye potential. For this relationship to be valid and statistically meaningful many particles must exist in the potential well (in the Debye sphere), $n\lambda_\mathrm{D}^3 \gg 1$ must be met. When this condition is met, the collective shielding effect of the Coulomb field works. This condition can be modified to $kT \gg e^2/4\pi\varepsilon_0 d$ ($d = n^{-1/3}$ is inter-electron distance), which means that kinetic energy is sufficiently larger than the potential energy between electrons (close to the ideal gas). Coulomb collisions in the plasma generally occur within the Debye radius, but the degree of scattering due to collision varies greatly depending on the value of the impact parameter. The impact parameter b in the center-of-mass system in Coulomb scattering is related to the scattering angle θ as follows (see Figure 5.8) [3],

$$b = b_0 \cot\left(\frac{\theta}{2}\right) . \tag{5.31}$$

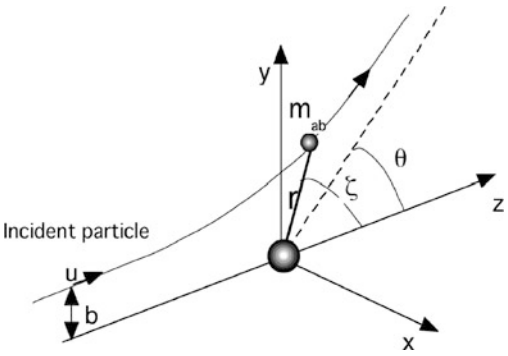


Figure 5.8 Scattering geometry for Coulomb collision. Scattering angle θ is defined as $\zeta(r \to \infty)$

Here, $b_0 = e_a e_b/(4\pi e_0 m_{ab} u^2) = 7.2 \times 10^{-10} Z_a Z_b / E_r$, (eV) (m) is the impact parameter at 90 degrees scattering and is called the Landau parameter. Here, $m_{ab} = m_a m_b/(m_a + m_b)$ is reduced mass, u is the relative velocity, $E_r = m_{ab} u^2/2$ is the particle energy in the center-of-mass system. Substituting Equation 5.31 into the differential cross section $\sigma(\theta) = b(\mathrm{d}b/\mathrm{d}\theta)/\sin\theta$ scattered into differential solid angle $\mathrm{d}\Omega = 2\pi \sin\theta \mathrm{d}\theta$, we obtain following well-known Rutherford scattering cross section [3].

$$\sigma(\theta) = \frac{b_0^2}{\sin^4(\theta/2)} \ . \tag{5.32}$$

The velocity of particle species a, $\boldsymbol{v}_a$ is given by the velocity of the center of mass $\boldsymbol{V}$ and the relative velocity $\boldsymbol{u}_{ab} = \boldsymbol{v}_a - \boldsymbol{v}_b$ as $\boldsymbol{v}_a = \boldsymbol{V} + m_b \boldsymbol{u}_{ab}/(m_a + m_b)$ and the change in velocity of particle a is given by $\Delta \boldsymbol{v}_a = (m_{ab}/m_a)\Delta \boldsymbol{u}_{ab}$. Since the relative speed, u_{ab} is conserved for the elastic collisions, a change in u can be obtained as $\Delta \boldsymbol{u}_{ab} = u_{ab} \sin\theta \boldsymbol{n} - 2\sin^2(\theta/2)\boldsymbol{u}_{ab}$ using the formula for two isosceles triangle $(\sin^2(\theta/2) = b_0^2/(b_0^2 + b^2))$. Here, $\boldsymbol{n}$ is a unit vector perpendicular to $\boldsymbol{u}_{ab}$. Considering that the interaction occurs in the Debye area $(\pi\lambda_\mathrm{D}^2)$ and the particle flux of species b with velocity $\boldsymbol{v}_b$ passing through the Debye area in the time interval Δt is given by $\Delta\phi_b = \delta n_b(\boldsymbol{v}_b) u \Delta t$, the velocity change of species a by a collision with b is given by,

$$\Delta \boldsymbol{v}_a = \Delta\phi_b \frac{m_{ab}}{m_a} \int_0^{\lambda_\mathrm{D}} \Delta \boldsymbol{u}_{ab} 2\pi b \, \mathrm{d}b = -4\pi b_0^2 \Delta\phi_b \boldsymbol{u}_{ab} \frac{m_{ab}}{m_a} \int_0^{\lambda_\mathrm{D}} \frac{b}{b^2 + b_0^2} \mathrm{d}b \ . \tag{5.33}$$

Here, $b = 0$ corresponds to head-on collision $(\theta = \pi)$ and $b = \lambda_\mathrm{D}$ corresponds to the scattering angle $\theta_{\min} \sim b_0/\lambda_\mathrm{D}$. The component of $\Delta \boldsymbol{v}_a$ perpendicular to $\boldsymbol{u}_{ab}$ disappears due to rotational symmetry. The integral term is $(1/2)\ln(1+(\lambda_\mathrm{D}/b_0)^2) \sim \ln(\lambda_\mathrm{D}/b_0)$, and $\ln\Lambda \equiv \ln(\lambda_\mathrm{D}/b_0)$ is called Coulomb logarithm.

There are some subtleties in Debye shielding as the origin of the Coulomb logarithm. Consider the example of magnetic fusion plasma in Figure 5.9. A huge number of charged particles are contained in the Debye sphere $(n\lambda_\mathrm{D}^3 = 4 \times 10^7)$, but the integration of Debye potential in Equation 5.30 gives a charge number for the total

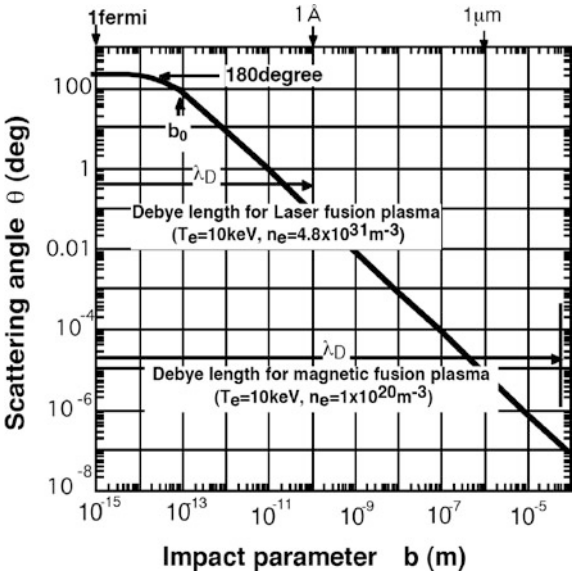


Figure 5.9 Relation between impact parameter b and scattering angle θ in the Coulomb scattering for relative energy $E_r = 10\,\text{keV}$ ($b_0 = 7.2 \times 10^{-14}\,\text{m}$). Parameters are magnetic fusion parameters ($T_e = 10\,\text{keV}$, $n_e = 1 \times 10^{20}\,\text{m}^{-3}$, $d = n_e^{-1/3} = 2 \times 10^{-7}\,\text{m}$, $\lambda_D = 7.4 \times 10^{-5}\,\text{m}$) and laser fusion parameters ($T_e = 10\,\text{keV}$, $n_e = 4.8 \times 10^{31}\,\text{m}^{-3}$, $d = n_e^{-1/3} = 2.8 \times 10^{-11}\,\text{m}$, $\lambda_D = 1.1 \times 10^{-10}\,\text{m}$)

electron cloud as only 1. This value is much smaller than the fluctuations in the number of electrons in the sphere ($(n\lambda_{De}^3)^{1/2} = 6000$). Thus, Van Kampen expressed the Debye sphere as "somewhat ghost-like existence." The Coulomb logarithm is a logarithmic integral, and the circumstances of collision depend on the particle distance. For $0 \le b < 3b_0 = 2 \times 10^{-13}\,\text{m}$ ($0.2\pi < \theta \le \pi$), the scattering between particles is large-angle scattering. The logarithmic integral in this regime is 1.15. For $3b_0 \le b < \lambda_D/3.7 = 2 \times 10^{-5}\,\text{m}$, approximation of small-angle scattering is valid and the Debye shielding effect ($e^{-1/3.7} = 0.76$) can also be neglected. The logarithmic integral is 18.4 in this region. The logarithmic integral where Debye potential is effective (ghost region of Van Kampen), has the small value of 1.3, which validates the rough approximation to cut at Debye length. On the other hand, the two-body correlation or interaction with other particle cannot be ignored for $d(= n^{-1/3} = 2 \times 10^{-7}\,\text{m}) < b$ and the question of the validity of two-body collision remains. The two-body correlation is assumed to be negligible statistically. A brainstorming discussion on the Coulomb logarithm in a plasma is given by Van Kampen [16].

Here we discuss the difference between the collision process in plasma and the molecular collision considered by Boltzmann. In molecular collisions, interaction occurs only when a molecule reaches the molecular radius ($r_0 \sim 10^{-10}\,\text{m} = 1\,\text{Å}$). Since the molecular radius r_0 is much smaller than the inter-molecule distance $n^{-1/3}$ ($r_0 \ll n^{-1/3}$), the two-body correlation is expected to be small. On the other hand, since Coulomb force is a long-range force, collision time t_c is relatively long ($t_c \sim \lambda_D/v_{th} = 10^{-10}\,\text{s}$) and is longer than the time taken to approach next target for collision ($t_{mfp} \sim 1/n^{1/3}v_{th} = 10^{-13}\,\text{s}$). Namely, the relation t_c (collision time) $\ll \Delta t \ll t_{mfp}$ (mean free time) valid for molecular collision does not hold in plasmas. The general collision theory of a many-body system is discussed by Balescu [22].

5.6 Fokker–Planck Equation: Statistics of Soft Coulomb Collision

As mentioned in the previous section, the momentum change in Coulomb interaction is quite small and small-angle scattering is dominant except for the small region of the impact parameter b close to the Landau parameter b_0. Small-angle scattering process is called the Fokker–Planck process and the form of the collision term can be determined without detailed information about collision dynamics. If we assume the timescale Δt to be much longer than the correlation time of random force t_c ($\Delta t \gg t_c$), the force can be assumed to be statistically independent of the previous value. Under this circumstance, the state at $t + \Delta t$ is determined only by the state at t, independent of past history. This is the "Markov process." If we define $P(\boldsymbol{v}; \Delta\boldsymbol{v}, \Delta t)$ as the probability of a particle changing velocity $\boldsymbol{v}$ by $\Delta\boldsymbol{v}$ in the time interval Δt, the velocity distribution function $f(\boldsymbol{v}, t)$ is given as follows,

$$f_a(\boldsymbol{v}, t) = \int \mathrm{d}\Delta\boldsymbol{v}\, f_a(\boldsymbol{v}-\Delta\boldsymbol{v}, t-\Delta t) P(\boldsymbol{v}-\Delta\boldsymbol{v}; \Delta\boldsymbol{v}, \Delta t)\ . \tag{5.34}$$

Here, $P(\boldsymbol{v}; \Delta\boldsymbol{v}, \Delta t)$ is rapidly decreasing function with $|\Delta\boldsymbol{v}|$. Then, taking up to the second terms of the Taylor expansion, the following equation is obtained for $C(f_s) = \Delta f_s/\Delta t$.

$$C(f_a) = -\frac{\partial}{\partial\boldsymbol{v}} \cdot \left(\frac{\langle \Delta\boldsymbol{v} \rangle}{\Delta t} f_a \right) + \frac{\partial^2}{\partial\boldsymbol{v}\partial\boldsymbol{v}} : \left(\frac{\langle \Delta\boldsymbol{v}\Delta\boldsymbol{v} \rangle}{2\Delta t} f_a \right)\ . \tag{5.35}$$

This is called the Fokker–Planck collision term. If we take the first coordinate along $\boldsymbol{u}_{ab} = \boldsymbol{v}_a - \boldsymbol{v}_b$, coefficients of the Fokker–Planck collision term, $\langle \Delta\boldsymbol{v} \rangle/\Delta t$ and $\langle \Delta\boldsymbol{v}\Delta\boldsymbol{v} \rangle/2\Delta t$ are given as follows due to the symmetry around $\boldsymbol{u}_{ab}$ axis,

$$\langle \Delta\boldsymbol{v} \rangle/\Delta t = \begin{bmatrix} \langle \Delta v_\parallel \rangle/\Delta t \\ 0 \\ 0 \end{bmatrix}, \tag{5.36}$$

$$\frac{\langle \Delta\boldsymbol{v}\Delta\boldsymbol{v} \rangle}{2\Delta t} = \begin{bmatrix} \langle \Delta v_\parallel^2/\Delta t \rangle & 0 & 0 \\ 0 & \langle \Delta v_\perp^2/2\Delta t \rangle & 0 \\ 0 & 0 & \langle \Delta v_\perp^2/2\Delta t \rangle \end{bmatrix}.$$

Substitution of $\delta n_b = f_b(\boldsymbol{v}_b)\mathrm{d}\boldsymbol{v}_b$ into Equation 5.33 and integration by $\boldsymbol{v}_b$ gives,

$$\left\langle \frac{\Delta\boldsymbol{v}_a}{\Delta t} \right\rangle = -\sum_b \frac{e_a^2 e_b^2 \ln\Lambda}{4\pi m_a^2 \varepsilon_0^2} \left(1 + \frac{m_a}{m_b} \right) \int \frac{\boldsymbol{u}_{ab}}{u_{ab}^3} f_b(\boldsymbol{v}_b)\mathrm{d}\boldsymbol{v}_b\ . \tag{5.37}$$

Here, it should be remembered that the contribution at small b in the integral expression of Equation 5.33 is not small-angle scattering although it is small. Similarly, using the relation $\sin^2\theta = 2[1 - b_0^2/(b_0^2 + b^2)]b_0^2/(b_0^2 + b^2)$, we obtain,

$$\left\langle \frac{\Delta v_\perp^2}{\Delta t} \right\rangle = \sum_b \frac{e_a^2 e_b^2}{4\pi m_a^2 \varepsilon_0^2} \left(\frac{1}{2} + \ln\Lambda \right) \int \frac{1}{u_{ab}} f_b(\boldsymbol{v}_b)\mathrm{d}\boldsymbol{v}_b\ , \tag{5.38}$$

$$\left\langle \frac{\Delta v_{\parallel}^2}{\Delta t} \right\rangle = \sum_b \frac{e_a^2 e_b^2}{16\pi m_a^2 \varepsilon_0^2} \int \frac{1}{u} f_b(\boldsymbol{v}_b) \mathrm{d}\boldsymbol{v}_b \ . \tag{5.39}$$

Usually, terms that do not include the Coulomb integral (1/2 of Equations 5.38 and 5.39) are neglected. Then, we obtain

$$\left\langle \frac{\Delta \boldsymbol{v}_a}{\Delta t} \right\rangle = \sum_b \frac{e_a^2 e_b^2 \ln \Lambda}{4\pi m_a^2 \varepsilon_0^2} \frac{\partial h_{ab}(\boldsymbol{v}_a)}{\partial \boldsymbol{v}_a} \ , \tag{5.40}$$

$$\left\langle \frac{\Delta \boldsymbol{v}_a \Delta \boldsymbol{v}_a}{2\Delta t} \right\rangle = \sum_b \frac{e_a^2 e_b^2 \ln \Lambda}{8\pi m_a^2 \varepsilon_0^2} \int \frac{u_{ab}^2 \boldsymbol{I} - \boldsymbol{u}_{ab}\boldsymbol{u}_{ab}}{u_{ab}^3} f_b(\boldsymbol{v}_b) \mathrm{d}\boldsymbol{v}_b \tag{5.41}$$

$$= \sum_b \frac{e_a^2 e_b^2 \ln \Lambda}{8\pi m_a^2 \varepsilon_0^2} \frac{\partial^2 g_{ab}(\boldsymbol{v}_a)}{\partial \boldsymbol{v}_a \partial \boldsymbol{v}_a} \ ,$$

$$h_{ab}(\boldsymbol{v}_a) = \left(1 + \frac{m_a}{m_b}\right) \int \frac{f_b(\boldsymbol{v}_b)}{u_{ab}} \mathrm{d}\boldsymbol{v}_b \ , \tag{5.42}$$

$$g_{ab}(\boldsymbol{v}_a) = \int u_{ab} f_b(\boldsymbol{v}_b) \mathrm{d}\boldsymbol{v}_b \ . \tag{5.43}$$

Here, $\partial u_{ab}^{-1}/\partial \boldsymbol{v}_a = -\boldsymbol{u}_{ab}/u_{ab}^3$, $\partial^2 u_{ab}/\partial \boldsymbol{v}_a \partial \boldsymbol{v}_a = u_{ab}^{-3}(u_{ab}^2 \boldsymbol{I} - \boldsymbol{u}_{ab}\boldsymbol{u}_{ab}) = \boldsymbol{U}_{ab}$ are used, where h_{ab}, g_{ab} is called the Rosenbluth potential [23]. Substituting these into Equation 5.35 gives,

$$C(f_a) = \sum_b \frac{e_a^2 e_b^2 \ln \Lambda}{4\pi m_a^2 \varepsilon_0^2} \left[-\frac{\partial}{\partial \boldsymbol{v}_a} \cdot \left(\frac{\partial h_{ab}}{\partial \boldsymbol{v}_a} f_a \right) + \frac{1}{2} \frac{\partial^2}{\partial \boldsymbol{v}_a \partial \boldsymbol{v}_a} : \left(\frac{\partial^2 g_{ab}}{\partial \boldsymbol{v}_a \partial \boldsymbol{v}_a} f_a \right) \right] . \tag{5.44}$$

Taking the partial integral for $\boldsymbol{v}_b$ integral and using the relation $\partial \boldsymbol{U}_{ab}/\partial \boldsymbol{v}_a = -2\boldsymbol{u}_{ab}/u_{ab}^3$, we obtain $2\partial h_{ab}/\partial \boldsymbol{v}_a = -(1 + m_a/m_b) \int (\partial \boldsymbol{U}_{ab}/\partial \boldsymbol{v}_b) f_b(\boldsymbol{v}_b) \mathrm{d}\boldsymbol{v}_b = (1+m_a/m_b) \int (\boldsymbol{U}_{ab} \partial f_b/\partial \boldsymbol{v}_b) \mathrm{d}\boldsymbol{v}_b$. Also, $\partial/\partial \boldsymbol{v}_a \cdot [(\partial^2 g_{ab}/\partial \boldsymbol{v}_a \partial \boldsymbol{v}_a) f_a(\boldsymbol{v}_a)] = \partial/\partial \boldsymbol{v}_a \cdot [\int \boldsymbol{U}_{ab} f_b(\boldsymbol{v}_b) \mathrm{d}\boldsymbol{v}_b f_a(\boldsymbol{v}_a)] = \int \boldsymbol{U}_{ab} [-\partial f_b/\partial \boldsymbol{v}_b f_a(\boldsymbol{v}_a) + f_b(\boldsymbol{v}_b) \partial f_a/\partial \boldsymbol{v}_a] \mathrm{d}\boldsymbol{v}_b$. Substituting these equations into Equation 5.44 gives,

$$C(f_a) = \sum_b \frac{e_a^2 e_b^2 \ln \Lambda}{8\pi \varepsilon_0^2 m_a} \frac{\partial}{\partial \boldsymbol{v}} \cdot \int \mathrm{d}\boldsymbol{v}_b \boldsymbol{U} \cdot \left[\frac{f_b(\boldsymbol{v}_b)}{m_a} \frac{\partial f_a(\boldsymbol{v})}{\partial \boldsymbol{v}} - \frac{f_a(\boldsymbol{v})}{m_b} \frac{\partial f_b(\boldsymbol{v}_b)}{\partial \boldsymbol{v}_b} \right] . \tag{5.45}$$

This form of collision term is given by Landau in 1936 [5–24]. He obtained this collision term from the Boltzmann collision term, which is not valid for plasma but he obtained a correct result. Indeed, the Boltzmann collision term can be applied outside the collision between molecules as discussed by Balescu in detail [22].

Let us return to the discussion on Debye shielding. As discussed in Section 5.2, the Coulomb collision term of particle species s, $C(f_s)$ is the ensemble average of the gradient in the velocity space of the fluctuating part of the discrete distribution function and acceleration by a microscopic Coulomb field.

$$C(f_a) = -\left\langle \tilde{\boldsymbol{a}} \cdot \frac{\partial \tilde{F}_a}{\partial \boldsymbol{v}} \right\rangle_{\text{ensemble}} \tag{5.46}$$

Substitution of $\tilde{\boldsymbol{a}} = -(e_s/m_s)\nabla\phi$ (ϕ: electrostatic potential) into Equation 5.5 gives $\partial\tilde{F}_s/\partial t + \boldsymbol{v}\cdot\partial\tilde{F}_s/\partial\boldsymbol{x} = (e_s/m_s)\nabla\phi\cdot\partial f_s/\partial\boldsymbol{v}$ and considering $-\nabla^2\phi = \sum(e_s/\varepsilon_0)\int\tilde{F}_s\mathrm{d}\boldsymbol{x}$, we obtain $\tilde{F}_s(\boldsymbol{x},\boldsymbol{v},t) = \tilde{F}_s(\boldsymbol{x}-\boldsymbol{v}t,\boldsymbol{v},0) + (e_s/m_s)\int_0^\infty \mathrm{d}\tau\nabla\phi(\boldsymbol{x}-\boldsymbol{v}\tau,t-\tau)\cdot\partial f_s/\partial\boldsymbol{v}$ from the convolution integral. Substitution into Equation 5.46 gives the Balescu–Lenard collision term in Landau form after some assumptions and manipulations [10, 25].

$$C(f_a) = -\frac{e_a^2 e_b^2}{8\pi\varepsilon_0^2 m_a}\frac{\partial}{\partial\boldsymbol{v}_a}\cdot\int \mathrm{d}\boldsymbol{v}_b \boldsymbol{K}_{ab}(\boldsymbol{v}_a,\boldsymbol{v}_b)\cdot\left[f_a\frac{1}{m_b}\frac{\partial f_b}{\partial\boldsymbol{v}_b} - f_b\frac{1}{m_a}\frac{\partial f_a}{\partial\boldsymbol{v}_a}\right], \tag{5.47}$$

$$\boldsymbol{K}_{ab}(\boldsymbol{v}_a,\boldsymbol{v}_b) = \int \mathrm{d}\boldsymbol{k}\,\delta(\boldsymbol{k}\cdot(\boldsymbol{v}_a-\boldsymbol{v}_b))\frac{\boldsymbol{k}\boldsymbol{k}}{k^4\left|\kappa(\boldsymbol{k},\boldsymbol{k}\cdot\boldsymbol{v}_a)\right|^2}, \tag{5.48}$$

$$\kappa(\boldsymbol{k},\omega) = 1 + \frac{e_b^2}{\varepsilon_0 m_b k^2}\int \mathrm{d}\boldsymbol{v}\frac{\boldsymbol{k}\cdot\partial f_b/\partial\boldsymbol{v}}{\omega-\boldsymbol{k}\cdot\boldsymbol{v}}. \tag{5.49}$$

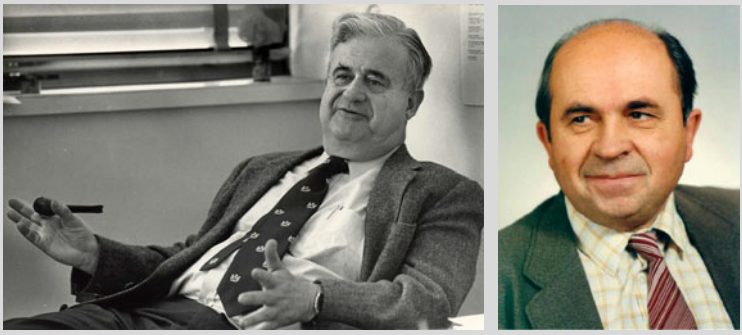
Approximating $\kappa = (k^2+k_\mathrm{D}^2)/k^2 (k_\mathrm{D} = 1/\lambda_\mathrm{D})$ (corresponding to Debye potential), K_{ab} is given by,

$$K_{ab} \sim \int \mathrm{d}\boldsymbol{k}\frac{\boldsymbol{k}\boldsymbol{k}\delta(\boldsymbol{k}\cdot(\boldsymbol{v}_a-\boldsymbol{v}_b))}{\left|k^2+k_\mathrm{D}^2\right|^2} = \frac{1}{u^3}[u^2\boldsymbol{I}-\boldsymbol{u}\boldsymbol{u}]\int\frac{\mathrm{d}k}{k}\frac{k^4}{\left|k^2+k_\mathrm{D}^2\right|^2}. \tag{5.50}$$

Divergence in the short wave number (long wave length) regime is suppressed by the Debye shielding. If we take $k_\mathrm{max} = 1/b_0$ where b_0 is Landau parameter, integration in wave number gives the Coulomb logarithm $\ln(\lambda_\mathrm{D}/b_0)$. In this approximation the Balescu–Lenard collision term is consistent with the Landau collision term.

Note: M. N. Rosenbluth and B. B. Kadomtsev

M. N. Rosenbluth (1927–2003; Figure 5.10 (a)) and B. B. Kadomtsev (1928–1998 Figure 5.10 (b)) were great US and Russian theoreticians in plasma physics, respectively. They made significant contributions to the development of plasma physics for fusion research.



(a) (b)

Figure 5.10 (a) M. N. Rosenbluth (Courtesy of the University of Texas at Austin) and (b) B. B. Kadomtsev (with kind permission of Physics-Uspekhi journal)

5.7 Gyro-center Kinetic Theory: Drift and Gyro Kinetic Theory

The plasma kinetic equation including the collision is given by,

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f}{\partial \boldsymbol{x}} + (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot \frac{\partial f}{\partial \boldsymbol{v}} = C(f) \, . \tag{5.51}$$

Here, C is the collision term. This equation is not appropriate for studying collisional transport or turbulent transport due to drift wave turbulence with the wavelength near the ion gyro radius ρ_i since this equation includes both the slow drift motion and fast gyro motion. So, we derive the kinetic equation for the guiding center (called drift kinetic equation) using the guiding center equations, Equations 4.39 and 4.40 derived in Section 4.3. We define the guiding center Poisson brackets $\{,\}$ as follows,

$$\begin{aligned} \{X, Y\} = \frac{e_a}{m_a} \left(\frac{\partial X}{\partial \theta} \frac{\partial Y}{\partial \mu} - \frac{\partial X}{\partial \mu} \frac{\partial Y}{\partial \theta} \right) - \frac{\boldsymbol{b}}{e_a \, B_\parallel^*} \cdot \nabla X \times \nabla Y \\ + \frac{\boldsymbol{B}^*}{m_a \, B_\parallel^*} \left(\nabla X \frac{\partial Y}{\partial v_\parallel} - \frac{\partial X}{\partial v_\parallel} \nabla Y \right) \, . \end{aligned} \tag{5.52}$$

Here, X and Y are arbitrary functions of $\boldsymbol{z} = (\boldsymbol{r}, v_\parallel, \mu, \theta)$. Our target is to transform this without ε dependence into new coordinates $\bar{\boldsymbol{z}} = \{\bar{z}^\mu\} = \{t, \bar{z}^i\}$ as $L\mathrm{d}t = \Gamma = \Gamma_i(\bar{\boldsymbol{z}})\mathrm{d}\bar{z}^i - H(\bar{\boldsymbol{z}})\mathrm{d}t + \mathrm{d}S(\bar{\boldsymbol{z}})$ (see Equation 4.91). Then using the Hamiltonian Equation 4.30, the magnetic moment and gyro angle evolution equation and Equations 4.39 and 4.40 are given as follows,

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = \{\mu, H\} = 0 \, , \quad \frac{\mathrm{d}\theta}{\mathrm{d}t} = \{\theta, H\} \, , \tag{5.53}$$

$$\frac{\mathrm{d}v_\parallel}{\mathrm{d}t} = \{v_\parallel, H\} = -\frac{\boldsymbol{B}^*}{m_a \, B_\parallel^*} \nabla H \, , \tag{5.54}$$

$$\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \{\boldsymbol{r}, H\} = \frac{\boldsymbol{b}}{e_a \, B_\parallel^*} \times \nabla H + \frac{\boldsymbol{B}^*}{m_a \, B_\parallel^*} \frac{\partial H}{\partial v_\parallel} \, . \tag{5.55}$$

The drift kinetic equation for guiding center velocity distribution function F to treat slow drift motion is given as follows by considering $\partial f / \partial \theta = 0$ and $\mathrm{d}\mu/\mathrm{d}t = 0$.

$$\frac{\partial F}{\partial t} + \dot{\boldsymbol{z}} \cdot \frac{\partial F}{\partial \boldsymbol{z}} = \frac{\partial F}{\partial t} + \{F, H\} = \frac{\partial F}{\partial t} + \dot{\boldsymbol{r}} \cdot \frac{\partial F}{\partial \boldsymbol{r}} + \dot{v}_\parallel \frac{\partial F}{\partial v_\parallel} = C(F) \, . \tag{5.56}$$

Turbulent fluctuation in plasma has been observed at wavelengths near the ion gyro radius. In order to treat the electromagnetic fluctuation of the order of the ion gyro radius, the gyro kinetic equation is developed by formulating the motion in time and varying electromagnetic field. In particular, the polarization drift (see Section 7.3) must be considered in the drift wave turbulence. For the set of electrostatic and vector potential fluctuation, $(\delta\varphi, \delta\boldsymbol{A})$, the perturbation Lagrangian δL is given

by

$$\delta L\,\mathrm{d}t = e_a\delta_*\boldsymbol{A}\cdot(\mathrm{d}\boldsymbol{r}+\mathrm{d}\boldsymbol{\rho}) - e_a\delta_*\varphi\,\mathrm{d}t = -\delta H\,\mathrm{d}t\ , \qquad (5.57)$$
$$\delta_*\boldsymbol{A} = \delta\boldsymbol{A}(\boldsymbol{r}+\boldsymbol{\rho}), \delta_*\varphi = \delta\varphi(\boldsymbol{r}+\boldsymbol{\rho})\ .$$

The perturbed Hamiltonian δH is given by,

$$\delta H = e_a\delta_*\varphi - e_a\delta_*\boldsymbol{A}\cdot\boldsymbol{v}\ . \qquad (5.58)$$

Since Equation 5.58 has gyro radius and $\boldsymbol{v}_\perp$ dependences, perturbed Hamiltonian depends on gyro phase θ. Therefore, the magnetic moment μ is no longer a conserved quantity ($\{\mu, \delta H\} \neq 0$). So, we construct new coordinates $\boldsymbol{z} = (\boldsymbol{r}, v_\parallel, \mu, \theta) \Rightarrow \bar{\boldsymbol{z}} = (\bar{\boldsymbol{r}}, \bar{v}_\parallel, \bar{\mu}, \bar{\theta})$ where the newly defined magnetic moment becomes a conserved quantity, and a mathematical tool called Lie perturbation theory is used. An important point here is the gauge arbitrariness of the Lagrangian ($L + \mathrm{d}S/\mathrm{d}t$) gives the same equation of motion as that of L, see Section 4.1). Using the perturbation expansion of Hamiltonian $\bar{H} = \bar{H}_0 + \bar{H}_1 + \bar{H}_2 + \ldots$ ($H_0 =$ Equation 4.31 formula), we obtain following formula similar to Equations (4.100) and (4.105).

$$\bar{H}_1 = \delta H - \frac{\mathrm{d}S_1}{\mathrm{d}t}\ , \quad \bar{H}_2 = \frac{e_a^2}{2m_a}|\delta_*\boldsymbol{A}|^2 - \frac{1}{2}\{S_1, \delta H\} - \frac{\mathrm{d}S_2}{\mathrm{d}t} \qquad (5.59)$$

and are solved by evaluating S_1 and S_2. Then, the coordinates after transformation are given by,

$$\bar{z}_a = z_a + \{S_1, z_a\} + e_a\delta_*\boldsymbol{A}\cdot\{\boldsymbol{r}+\boldsymbol{\rho}, z_a\} + \ldots \qquad (5.60)$$

Using this new coordinate system with the Hamiltonian obtained in this way, the gyrokinetic equation is given as follows,

$$\frac{\partial\bar{F}}{\partial t} + \left\{\bar{F}, \bar{H}\right\} = \bar{C}(\bar{F})\ \left(\text{or } \frac{\partial\bar{F}}{\partial t} + \dot{\bar{\boldsymbol{r}}}\cdot\frac{\partial\bar{F}}{\partial\bar{\boldsymbol{r}}} + \dot{\bar{v}}_\parallel\frac{\partial\bar{F}}{\partial\bar{v}_\parallel} = \bar{C}(\bar{F})\right)\ . \qquad (5.61)$$

The Hamilton equation of motion for gyro-center is given by,

$$\frac{\mathrm{d}\bar{v}_\parallel}{\mathrm{d}t} = \left\{\bar{v}_\parallel, \bar{H}\right\} = -\frac{\boldsymbol{B}^*}{m_a\ B_\parallel^*}\bar{\nabla}\bar{H}\ , \qquad (5.62)$$

$$\frac{\mathrm{d}\bar{\boldsymbol{r}}}{\mathrm{d}t} = \left\{\bar{\boldsymbol{r}}, \bar{H}\right\} = \frac{\boldsymbol{b}}{e_a\ B_\parallel^*}\times\bar{\nabla}\bar{H} + \frac{\boldsymbol{B}^*}{m_a\ B_\parallel^*}\frac{\partial\bar{H}}{\partial\bar{v}_\parallel}\ . \qquad (5.63)$$

Details are given in Brizard–Hahm [26] and more plainly in Brizard [27].

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