Chapter 3
Confinement Bottle: Topology of Closed Magnetic Field and Force Equilibrium

In the natural fusion reactor, the Sun, dense hot plasma is confined by a gravitational field. Characteristic of this force is that it is a central force field and acts in the direction of field line. For this reason, the confinement bottle has the topology of a sphere (Figure 3.1 (a)). In the man-made fusion reactor, high temperature plasma is confined by trapping charged particles with the Lorentz force in a magnetic field to sustain reaction in a small dimension of 100 millionth of that of the Sun. Characteristic of this force is that it acts in a direction perpendicular to the field line. For this reason, the confinement bottle has the topology of a torus (Figure 3.1 (b)). In this chapter, force equilibrium is treated to confine high temperature plasma in the topology of a torus. Practically, the magnetic field line dynamics is treated using the methodology of analytical mechanics and symmetry involved in the force equilibrium is discussed.

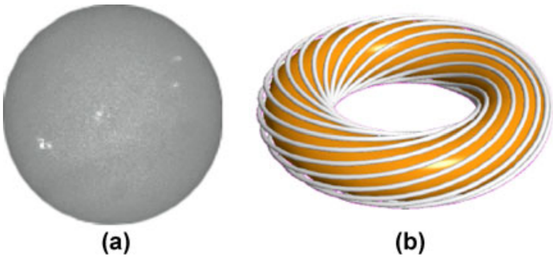

Figure 3.1 (a) Sphere and (b) torus

3.1 Field: Magnetic Field and Closed Magnetic Configuration

Danish physicist H. C. Oersted (1777–1851), during the years 1819–1820, discovered that a compass needle directs to a fixed point by some force when it is placed near a wire carrying a current from a battery invented by Italian physicist A. Volta (1745–1827) (Figure 3.2 (a)). This power is called the magnetic force. British physicist Michael Faraday (1791–1867) gave a novel interpretation of this phenomenon, suggesting that the space around the wire is in a special state by driving the current, instead of the explanation that a remote force works between the current and compass needle. The key here is the way of thinking "the space itself would be different." "Field" as a nature of the space was a concept that attracted Einstein's attention. "Vacuum" as state of nothing originated from Greek philosopher Democritus (460–370 BC). The vacuum can have a different state with a "magnetic field" and this state can have energy.

This magnetic field has a direction, and various phenomena can be explained if we assume that a virtual line, "magnetic line of force," exists along the direction of the magnetic N-pole as shown in Figure 3.2 (a). Magnetic field lines form a circle when circular coils are arranged in a donut shape as shown in Figure 3.2 (b). The donut-shaped space, the torus can be arranged inside the coil. Longitudinal current in the torus produces the magnetic field lines linked to the torus as shown in Figure 3.2 (c). A tokamak-type magnetic configuration which is said to be closest to the fusion reactor, is shown in Figure 3.2 (d), a superposition of the magnetic field lines shown in (b) and (c). The equation describing the magnetic field lines is $\mathrm{d}x/B_x = \mathrm{d}y/B_y = \mathrm{d}z/B_z$ or $\boldsymbol{B} \times \mathrm{d}\boldsymbol{x} = 0$. This equation is equivalent to the condition of the extremum of the path integral called the "action integral" (variational principle) as will be described in Section 3.4. In this case, the action integral is expressed by the path integral of the vector potential $\boldsymbol{A}$ (proof will be given in Section 3.4).

$$\delta \int \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{x} = 0 . \tag{3.1}$$

Faraday, in 1821, found that force acts on the current in the magnetic field ($\boldsymbol{F} = \boldsymbol{J} \times \boldsymbol{B}$, where $\boldsymbol{F}$ is force, $\boldsymbol{J}$ the current, $\boldsymbol{B}$ the magnetic field. Orientation is determined by Fleming's left-hand rule). Consider a charged particle with charge q (Coulombs) and velocity$\boldsymbol{v}$ (m/s) moving in a magnetic field $\boldsymbol{B}$ (Tesla), the current is given by $q\boldsymbol{v}$ and the force acting on the charged particle is $\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B}$. This is called the Lorentz force. Ions and electrons in the magnetic field have a circular motion (Larmor motion) with a radius defined by the balance between the centrifugal and Lorentz forces. On the other hand, they move in a straight line in the direction of the magnetic field. A combination of these two motions appears to be helical (Figure 3.2 (d)). It is difficult for the charged particles to escape perpendicular to the magnetic field. The method of confining hot plasma using this principle is called "magnetic confinement." Magnetic confinement becomes efficient if the field line is closed since charged particles move freely along the field.

The study of the structure of closed magnetic field lines to confine the hot plasma is called the theory of plasma equilibrium. Famous examples are the axisymmetric magnetic configuration called the tokamak invented by A. Sakharov (1921–1989) of Kurchatov Institute in the former Soviet Union [1] (Figure 3.2 (d)) and the stellarator (helical in general) invented by Lyman Spitzer Jr. (1914–1997) of Princeton University in the USA [2] (Figure 3.3).

In both configurations, there exists some region where field lines are closed and do not intersect with material walls and the high temperature plasma is confined.

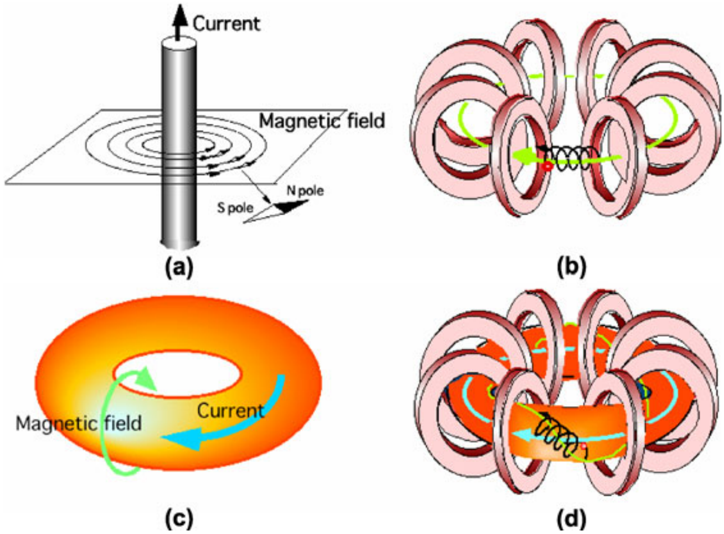

Figure 3.2 (a) When the current flows, a magnetic field is generated around the current. (b) Arranging circular coils around the torus and energizing the coil produces a magnetic field in the toroidal direction. (c) The toroidal plasma current produces a magnetic field linking the torus. (d) A combination of (b) and (c) creates twisted magnetic field lines and is called a tokamak

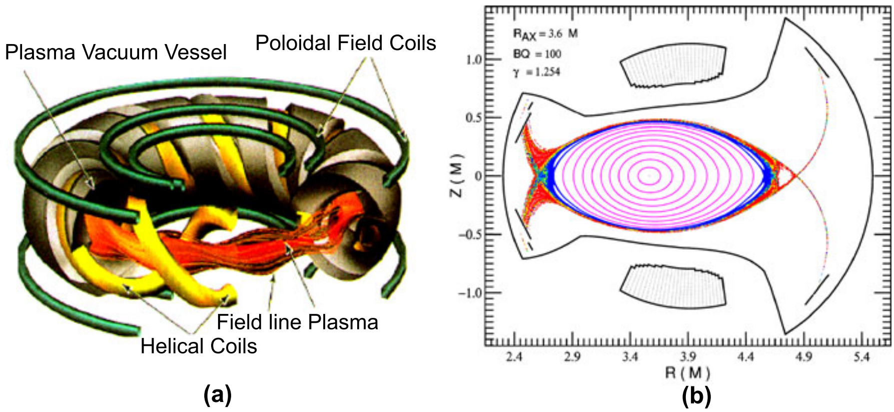

Figure 3.3 (a) Schematic view of helical device LHD showing how twisted magnetic field lines are formed without a toroidal plasma current using helical coils and (b) a typical example of flux surface shape in LHD [3]

Its shape is torus. It is difficult to create a closed magnetic field structure other than the torus. For closed surfaces other than the torus, a null-field point will exist according to the fixed-point theorem of mathematics. Important characteristic of the closed magnetic configuration is that a toroidal magnetic field line densely covers a constant pressure toroidal surface. A simple closed line cannot confine the plasma with a pressure difference. The closed surface should be formed by the magnetic field line and torus-shaped plasma has to be confined in it. Thus the problem of covering the surface with a magnetic field line becomes important. If the magnetic field line trajectory is on the surface, it is called "integrable."

Note: Integrable System [4–6]

"Integrable" is a term in classical mechanics, having its origin in the many-body-problem of celestial mechanics, and is plainly explained in Diacu and Homes [4]. French mathematician J. Liouville (1809–1882; Figure 3.4) gave its mathematical definition. For an equation of motion for particles in dynamical systems to be solvable or integrable, it is key to find some kind of symmetry (in dynamical systems, to find an ignorable coordinate). An N degrees of freedom system of Newtonian mechanics follows the Hamilton equation, and phase space flow is known as an incompressible flow. $N = 1$, *i.e.*, a one degree of freedom Hamilton system (x, v_x) is always integrable since the Hamiltonian of the system is always conserved. The system moves along $H(x, v_x) =$ constant contour.

Motion of $N = 2$, *i.e.*, a two degrees of freedom Hamilton system (x, y, v_x, v_y) is described as dynamics in 4D phase space (position $\boldsymbol{q} = (x, y)$ and momentum $\boldsymbol{p} = (p_x, p_y)$). Its solution is limited on the hyper-surface of $H(\boldsymbol{q}, \boldsymbol{p}) = \text{constant} = E$ (3-dimensional manifold with constant energy). If an additional first integral $\Phi(\boldsymbol{q}, \boldsymbol{p}) =$ constant exists, the flow line of phase space motion is on the curved surface (2-dimensional manifold) limited by $H(\boldsymbol{q}, \boldsymbol{p}) =$ constant and $\Phi(\boldsymbol{q}, \boldsymbol{p}) =$ constant. Such a case is called an "integrable system." Known 2-dimensional integrable systems are a 2-dimensional Kepler problem, 2-dimensional harmonic oscillator, and the movement of the mass point in the 2-dimensional central force field. If we cut this surface (2-dimensional manifold) in a plane (for example, the $x = 0$ plane), the flow line becomes a line in the plane. On the other hand, flow lines cut in a plane have a 2-dimensional spread for the non-integrable system.

Figure 3.4 J. Liouville who studied the mathematical nature of integrable systems. He derived the famous theorem, the "Liouville-Arnold theorem" [5]. He is more famous for his theorem, the "Liouville theorem," in phase space dynamics, which is given in Chapter 5

3.2 Topology: Closed Surface Without a Fixed Point

Considering the confinement of hot plasma in a region of 3-dimensional space, the boundary must be a closed surface. Figure 3.5 shows the characteristics of the flow on a torus and a sphere as typical closed surfaces. In the torus, there is no point where flow field vector becomes zero (called a “fixed point”) as shown in Figure 3.5 (a) and (b). On the other hand, the flow field on the sphere necessarily has a fixed point, as shown in (c) and (d). Since the hot plasma will leak from the fixed point where the magnetic field is zero, the sphere cannot be used for magnetic confinement.

We can provide a little more familiar example of this nature, “when the wind is blowing on the Earth, there is always somewhere to rest.” This is commonly true for the Earth, rugby balls and coffee in a cup.

Mathematically speaking, all surfaces “homeomorphic” to a sphere will have fixed points. This means that the sphere and torus have different topologies. The surface property of a sphere and torus does not change even if they are bent or stretched. A geometrical property, which is not changed by continuous deformations, is called the “topology” of the object. It should not vary during continuous deformations of bending and stretching.

French mathematician Henri Poincaré (1854–1912) proved the theorem: “A closed surface that can be covered with a vector field without a fixed point is restricted to a torus.” This is called the “Poincaré theorem” [7–10]. The Poincaré theorem is important for high temperature plasma confinement. Consider the boundary surface of the magnetic confinement, the plasma will leak from the zero point of magnetic field vector. To confine the hot plasma, the surface must be covered by a non-zero magnetic field. This is why we use toroidal geometry for magnetic confinement.

The people of ancient Greece aware that the regular polyhedrons are limited to regular (4) tetrahedron, regular (6) hexahedron, regular (8) octahedron, regular (12) dodecahedron, and regular (20) icosahedron. Let the number of vertices of a polyhedron be p, the number of sides q, and the number of the polygon r, the Swiss mathematician L. Euler (1707–1783) found the relation $p - q + r = 2$ for

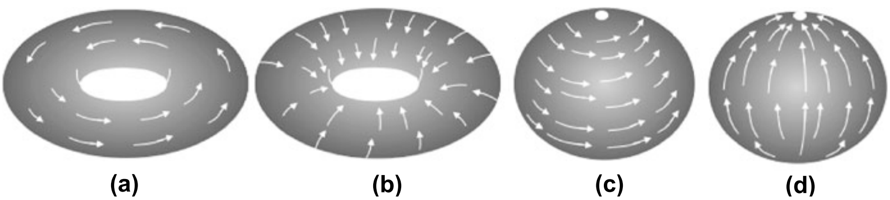

Figure 3.5 Topological properties of torus and sphere. (a) and (b) flow in the torus surface, where flow field without a fixed point can be formed. Flows in (a) and (b) are said to be commutable. (c) and (d) flow fields in a sphere always have a null-vector point (fixed point: ∘)

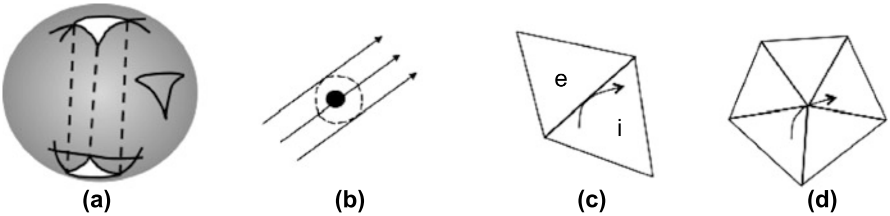

Figure 3.6 (a) Elimination of a triangular prism from a sphere produces a torus and fixed points can be eliminated. The number of vertices is the same as the sphere, the three sides (*the sides of the triangular prism*) increases by for each face. (b) A regular point and its enclosed loop (*dashed circle*). (c) Vector field touches a side from inside the triangle i. (d) Vector field through the vertex of the triangle

any regular polyhedron. For example, in a regular tetrahedron, the number of vertices $p = 4$, of sides $q = 6$, of polygon $r = 4$, gives the index $K = p - q + r = 2$. This relationship holds not only for regular polyhedrons but also for polyhedrons homeomorphic to a sphere, and is called "Euler's polyhedron theorem."

$$K = p - q + r \tag{3.2}$$

is the "Euler index." The relationship always holds irrespective of any division of the sphere by triangles.

Then, what will happen to the Euler index if the torus is covered with triangles? Drill the sphere from top to bottom to eliminate the triangular prism as in Figure 3.6 (a) it becomes homeomorphic to the torus. Then, the following relations hold between p, q, and r of the sphere and p', q', and r' of the drilled-sphere,

$$p' = p\,, \quad q' = q + 3\,, \quad r' = r - 2 + 3\,. \tag{3.3}$$

Thus, $p' + q' + r' = p - q + r - 2 = 0$. In other words, the Euler index of the torus is 0.

Poincaré showed that Euler index is related to the characteristics of the vector fields on the close surface. Poincaré defined the "index" of the vector field. The non-zero point shown in Figure 3.6 (b) is called the "regular point." The index value defined by Poincaré becomes zero for a regular point. Here, the index is defined by $k = (I - E)/2 + 1$, I is the number of vector lines from the inside in contact with a sufficiently small loop around the point (dashed line in Figure 3.6 (b)), E is the number of vector lines from the outside. For regular point, $I = 0$ and $E = 2$. So the index $k = 0$. The small loop in Figure 3.6 (b) can be continuously deformed to an infinitely small triangle. In this case, $I = 0$ and $E = 2$. On the other hand, the index at the singular point takes a non-zero value. The index of the flow surface is defined by the sum of the flow index of all points in the surface. Since the index of the regular point is zero, the index of the surface flow is the sum of the index of the singular points in the plane. The index of a closed surface becomes a sum of the index of each polygon when the surface is divided to several polygons (additivity).

Consider the flow contacting the side of a polygon in Figure 3.6 (c)), this flow has a positive contribution to the index of polygon i, which includes this flow but has a negative contribution to the index of polygon e. In the end, the flow tangent to the side does not contribute to the index of the closed surface (this discussion holds only for closed surfaces). So, the contribution to the index of the flow comes only from vertices of the polygon. Consider the vertex A associated with N sides, we see N polygons share the vertex A. The number of vector fields contacting A from outside is $N-2$ as can be seen in Figure 3.6 (d)). Now, the number of contacts from inside is 0. Thus, total number of contacts of the entire closed surface (external – internal) is the sum of $(N-2)$ for all vertices. By taking the sum for all the vertices, $\sum(E-I) = \sum_{\text{vertices}}(N-2) = (\sum_{\text{vertices}} N) - 2p = 2q - 2p$. Here, we use that sum of N for all vertices are twice the number of sides. Now, the sum of the flow index for the entire closed surface $\sum k = -\sum(E-I)/2 + \sum_{\text{polygon}} 1 = p - q + r$. Here,

$$K = p - q + r \tag{3.4}$$

is the Euler index, and flow index of the closed surface is equal to the Euler index. The Euler index is equal to the sum of the index of the closed surface, we can say that the necessary and sufficient condition that a flow without a fixed point can exist in a closed surface is that the Euler index of the closed surface is 0 and the surface is a torus. It is known that orientable 2-dimensional closed surfaces are limited to the sphere S^2, torus T^2, and n-holed torus $\sum_n$ $(n = 2, 3, \ldots)$. Poincaré's theorem tells us that the torus has a special nature as a 2-dimensional closed surface [11, 12].

Salon: L. Euler and H. Poincaré

Leonhard Euler (Figure 3.7 (a)) was a famous mathematician and physicist born in Swiss. He made a huge contribution to mathematics and physics. He solved the "Königsberg bridge problem" in 1736, starting graph theory as related to topology. He gave the so-called Euler identity $e^{i\pi} + 1 = 0$, which was described as "the most remarkable formula" by R. Feynman.

Henri Poincaré (Figure 3.7 (b)) was a famous French mathematician and physicist. His works appear in this book as "Poincaré theorem on topology" and also "Poincaré recurrence theorem" in Chapter 5. He also created a graphical method to analyze dynamical systems in which he discovered a phenomenon now called "Chaos." He is also famous for the "Poincaré conjecture," recently solved by Russian mathematician G. Perelman [12].

(a) (b)

Figure 3.7 (a) L. Euler and (b) H. Poincaré

3.3 Coordinates: Analytical Geometry of the Torus

Torus topology can be described without using "coordinates" as in Euclidean geometry. However, coordinates should be introduced to understand the physics of the torus, quantitatively. French philosopher René Descartes (1596–1650; Figure 3.8 (a)) published Discourse on Method in 1637, and in its appendix "Geometry," described how to assign numbers, called "Descartes coordinates," to a geometric shape. Descartes made the greatest contribution to the science by appointing numbers to all points in the plane by introducing $x - y$ coordinates (Figure 3.8 (b)).

Efforts to provide the most appropriate coordinates for the torus produced "Hamada coordinates" as a typical example (Section 3.6). Here, we consider general curvilinear coordinates, where space is expressed by 3-dimensional curvilinear coordinates [13, 14]. If we express the most fundamental Cartesian coordinates as (x, y, z) and, the position vector is given by $\boldsymbol{x} = x\boldsymbol{e}_x + y\boldsymbol{e}_y + z\boldsymbol{e}_z$. Let the general curvilinear coordinates (u^1, u^2, u^3) be given by the relations, $u^1 = u^1\ (x, y, z)$, $u^2 = u^2\ (x, y, z)$, $u^3 = u^3\ (x, y, z)$ (see Figure 3.9). The following relation is satisfied between the gradient vector ∇u^i and the tangent vector $\partial\boldsymbol{x}/\partial u^j$, and is called the "orthogonal relation."

Here

$$\nabla u^i \cdot \frac{\partial \boldsymbol{x}}{\partial u^j} = \frac{\partial u^i}{\partial u^j} = \delta_{ij}\ , \tag{3.5}$$

$$\nabla u^i = \left(\frac{\partial u^i}{\partial x}\right)\boldsymbol{e}_x + \left(\frac{\partial u^i}{\partial y}\right)\boldsymbol{e}_y + \left(\frac{\partial u^i}{\partial z}\right)\boldsymbol{e}_z\ , \tag{3.6}$$

$$\frac{\partial \boldsymbol{x}}{\partial u^j} = \frac{\partial x}{\partial u^j}\boldsymbol{e}_x + \frac{\partial y}{\partial u^j}\boldsymbol{e}_y + \frac{\partial z}{\partial u^j}\boldsymbol{e}_z\ . \tag{3.7}$$

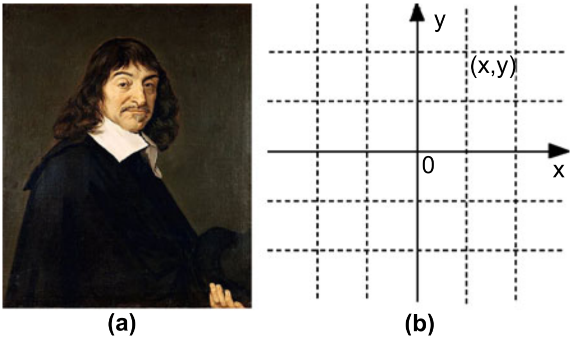

Figure 3.8 (a) René Descartes who introduced (b) "coordinates" in geometry. His most famous philosophical quotation is "cogito, ergo sum"

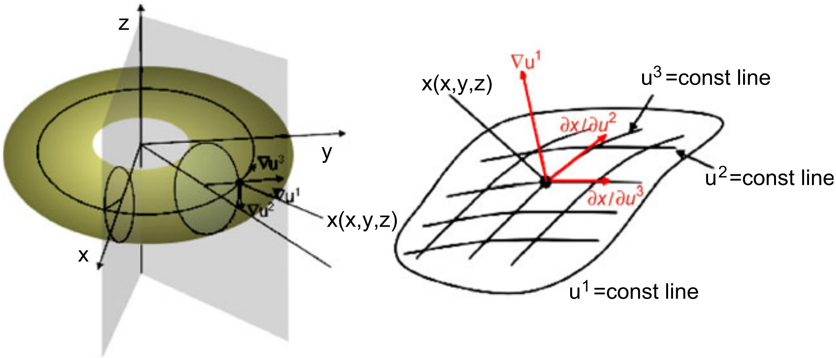

Figure 3.9 General curvilinear coordinate system in a torus. $\nabla \boldsymbol{u}^1$ is a gradient vector normal to $\boldsymbol{u}^1$ surface. $\partial \boldsymbol{x}/\partial \boldsymbol{u}^2$ and $\partial \boldsymbol{x}/\partial \boldsymbol{u}^3$ are tangent vector on the $\boldsymbol{u}^1$ surface and are perpendicular to $\nabla \boldsymbol{u}^1$

For example, the tangent vector $\partial \boldsymbol{x}/\partial u^2$ is a differentiation under u^1 and u^3 = constants, so it is tangent to u^3 = constant line on u^1 surface (see Figure 3.9). Naturally, $\partial \boldsymbol{x}/\partial u^2$ is orthogonal to ∇u^1 (and ∇u^3), which is perpendicular to the u^1 (and u^3) plane. A similar relation holds for $\partial \boldsymbol{x}/\partial u^3$. Then, a useful expression $\nabla u^1 = J^{-1}(\partial \boldsymbol{x}/\partial u^2 \times \partial \boldsymbol{x}/\partial u^3)$ is obtained (#1). Here, J is called the Jacobian (#2). Similar relation $\partial \boldsymbol{x}/\partial u^1 = J\nabla u^2 \times \nabla u^3$ are also obtained. Including a similar relationship for u^2 and u^3 yields, the following relations and are called "dual relations." Let $(i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2)$ (#3),

$$\nabla u^i = \frac{1}{J}\left(\frac{\partial \boldsymbol{x}}{\partial u^j} \times \frac{\partial \boldsymbol{x}}{\partial u^k}\right) \tag{3.8}$$

$$\frac{\partial \boldsymbol{x}}{\partial u^i} = J\nabla u^j \times \nabla u^k \tag{3.9}$$

here,

$$J \equiv \frac{\partial \boldsymbol{x}}{\partial u^1} \cdot \left(\frac{\partial \boldsymbol{x}}{\partial u^2} \times \frac{\partial \boldsymbol{x}}{\partial u^3}\right) \quad \text{(Jacobian)} \tag{3.10}$$

Orthogonal and dual relations are fundamental to the geometry of general curvilinear coordinates. Other formulas can be obtained from them. Any vector field (*e.g.*, magnetic field) can be expanded using the gradient and tangent vectors in general curvilinear coordinate system by using orthogonal relation

$$\boldsymbol{B} = \sum_i B^i \frac{\partial \boldsymbol{x}}{\partial u^i} \quad \text{(Contravariant form)} \tag{3.11}$$

$$\boldsymbol{B} = \sum_i B_i \nabla u^i \quad \text{(Covariant form)} \tag{3.12}$$

$$B^i = \boldsymbol{B} \cdot \nabla u^i \tag{3.13}$$

$$B_i = \boldsymbol{B} \cdot \frac{\partial \boldsymbol{x}}{\partial u^i} \ . \tag{3.14}$$

Here, B^i is called contravariant component and B_i is called covariant component.

Consider the trajectory of the magnetic lines of force in a general curvilinear coordinate system. Let s be the position coordinate along the magnetic field lines, the magnetic field orbit is given by $\mathrm{d}\boldsymbol{x}/\mathrm{d}s = \boldsymbol{b}$ ($\boldsymbol{b} = \boldsymbol{B}/|\boldsymbol{B}|$). Since $\mathrm{d}\boldsymbol{x}/\mathrm{d}s = \sum(\partial\boldsymbol{x}/\partial u^j)\mathrm{d}u^j/\mathrm{d}s$, the inner product between $\mathrm{d}\boldsymbol{x}/\mathrm{d}s = \boldsymbol{b}$ and ∇u^i and the orthogonal relation leads to,

$$\mathrm{d}u^i/\mathrm{d}s = \boldsymbol{b} \cdot \nabla u^i \ . \tag{3.15}$$

Since $\{\nabla u^j\}$ is not an orthogonal system in the general curvilinear coordinate system, the inner product of a vector is expressed as $\boldsymbol{A} \cdot \boldsymbol{B} = \sum A_i B^i = \sum A^i B_i$ using the orthogonal relation. Applying vector rotations $\nabla \times \nabla u^i = 0$ and $\nabla B_i = \sum \partial B_i/\partial u^j \nabla u^j$ to Equation 3.12, the following relation for the rotation of a vector is obtained using the dual relation (Equation 3.9) (summation runs for (i, j, k): right-handed),

$$\nabla \times \boldsymbol{B} = \sum_{i=1,3} \sum_{j=1,3} \frac{\partial B_i}{\partial u^j} \nabla u^j \times \nabla u^i = J^{-1} \sum_{k=1,3} \left[\frac{\partial B_j}{\partial u^i} - \frac{\partial B_i}{\partial u^j} \right] \frac{\partial \boldsymbol{x}}{\partial u^k} \ . \tag{3.16}$$

Applying the dual relation (Equation 3.9) to the vector expansion (Equation 3.11), the divergence of the vector is obtained taking $\nabla \cdot (\nabla a \times \nabla b) = 0$ into account,

$$\nabla \cdot \boldsymbol{B} = \nabla \cdot \sum_i B^i \frac{\partial \boldsymbol{x}}{\partial u^i} = J^{-1} \sum_i \frac{\partial J B^i}{\partial u^i} \ . \tag{3.17}$$

Here, $\nabla \cdot \boldsymbol{B} = \nabla \cdot \sum B^i \partial \boldsymbol{x}/\partial u^i = \nabla \cdot \sum J B^i \nabla u^j \times \nabla u^k = \sum(\partial J B^i/\partial u^i)[\nabla u^i \cdot \nabla u^j \times \nabla u^k]$. The relation between covariant component B_i and contravariant component B^j, $B_i = \sum g_{ij} B^j$ is obtained by substituting Equation 3.11 into Equation 3.14, where $g_{ij} = (\partial \boldsymbol{x}/\partial u^i) \cdot (\partial \boldsymbol{x}/\partial u^j)$ (#4). Substitution of Equation 3.13 to Equation 3.12 gives $B^i = \sum g^{ij} B_j$, where $g^{ij} = \nabla u^i \cdot \nabla u^j$. Matrix $[g_{ij}]$ is the inverse matrix of $[g^{jk}]$ as seen from $B_i = \sum g_{ij} B^j = \sum g_{ij} g^{jk} B_k = \sum \delta_{ik} B_k$. The formulas for line, surface and volume integrals are given by,

$$\int \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{x} = \int \boldsymbol{B} \cdot (\partial \boldsymbol{x}/\partial u^i) \mathrm{d}u^i \tag{3.18}$$

$$\int \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{a} = \int \boldsymbol{B} \cdot \nabla u^k J \,\mathrm{d}u^i \mathrm{d}u^j \tag{3.19}$$

$$\int f \,\mathrm{d}V = \int fJ \,\mathrm{d}u^1 \mathrm{d}u^2 \mathrm{d}u^3 \;. \tag{3.20}$$

Here, $\mathrm{d}\boldsymbol{x} = (\partial \boldsymbol{x}/\partial u^i)\mathrm{d}u^i$ for line integral, $\mathrm{d}\boldsymbol{a} = (\partial \boldsymbol{x}/\partial u^i)\times(\partial \boldsymbol{x}/\partial u^j)\mathrm{d}u^i \mathrm{d}u^j = \nabla u^k J \mathrm{d}u^i \mathrm{d}u^j$ for surface integral, $\mathrm{d}V = J\mathrm{d}u^1\mathrm{d}u^2\mathrm{d}u^3$ for the volume integral are considered.

#1: Expand ∇u^1 as $\nabla u^1 = a_1(\partial \boldsymbol{x}/\partial u^2) \times (\partial \boldsymbol{x}/\partial u^3) + a_2(\partial \boldsymbol{x}/\partial u^3) \times (\partial \boldsymbol{x}/\partial u^1) + a_3(\partial \boldsymbol{x}/\partial u^1)\times(\partial \boldsymbol{x}/\partial u^2)$ and take the inner product with $\cdot\partial \boldsymbol{x}/\partial u^1$, $\cdot\partial \boldsymbol{x}/\partial u^2$, $\cdot\partial \boldsymbol{x}/\partial u^3$.
#2: Jacobian J is originally defined to measure the volume of the coordinate system. The volume between (u^1, u^2, u^3) and $(u^1 + \mathrm{d}u^1, u^2 + \mathrm{d}u^2, u^3 + \mathrm{d}u^3)$ is given by $\mathrm{d}V = [(\partial \boldsymbol{x}/\partial u^1)\mathrm{d}u^1] \times [(\partial \boldsymbol{x}/\partial u^2)\mathrm{d}u^2] \times [\partial \boldsymbol{x}/\partial u^3 \mathrm{d}u^3] = J\mathrm{d}u^1\mathrm{d}u^2\mathrm{d}u^3$ consistent with its definition.
#3: $(i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2)$ is called right-handed.
#4: Metric tensor is originally defined to measure the distance of two points in the space. Infinitesimally small distance between two points is given by $\mathrm{d}\boldsymbol{x} = \sum(\partial \boldsymbol{x}/\partial u^i)\mathrm{d}u^i$. So, $(\mathrm{d}\boldsymbol{x})^2 = \sum g_{ij}\mathrm{d}u^i\mathrm{d}u^j$ consistent with the original definition of the metric g_{ij}.

3.4 Field Line Dynamics: Hamilton Dynamics of the Magnetic Field

A magnetic field is a vector field without source and sink, and therefore is incompressible as a flow field ($\nabla \cdot \boldsymbol{B} = 0$). For the incompressible flow, the volume of the fluid element is conserved along with the flow. The dynamic system is similar to incompressible flow in that the phase space flow is incompressible. From this similarity, the theory of magnetic field lines flow can be constructed using the Hamilton form in analytical mechanics (Note 1) [6, 15].

Let ζ be the toroidal angle of the torus and θ the poloidal angle (choice of ζ and θ is arbitrary). In general, the magnetic vector potential $\boldsymbol{A}(\nabla \times \boldsymbol{A} = \boldsymbol{B})$ is given by $\boldsymbol{A} = \phi\nabla\theta - \psi\nabla\zeta + \nabla G$ (G is the gauge transformation part) (#1), then, the magnetic field $\boldsymbol{B}$ can be expressed by following symplectic form.

$$\boldsymbol{B} = \nabla\phi \times \nabla\theta - \nabla\psi \times \nabla\zeta \;. \tag{3.21}$$

It is easy to show this expression satisfies $\nabla \cdot \boldsymbol{B} = 0$. Let us choose our coordinates (ϕ, θ, ζ) and find the orbit of the magnetic field along the toroidal angle ζ. Using Equations 3.15 and 3.21, the following are obtained:

$$\begin{aligned} \frac{\mathrm{d}\theta}{\mathrm{d}\zeta} &= \frac{\boldsymbol{B}\cdot\nabla\theta}{\boldsymbol{B}\cdot\nabla\zeta} = \frac{\partial\psi}{\partial\phi} \;, \\ \frac{\mathrm{d}\phi}{\mathrm{d}\zeta} &= \frac{\boldsymbol{B}\cdot\nabla\phi}{\boldsymbol{B}\cdot\nabla\zeta} = -\frac{\partial\psi}{\partial\theta} \;. \end{aligned} \tag{3.22}$$

This seems to be a Hamilton equation in the dynamical system if we regard ψ as the Hamiltonian, θ as the canonical coordinate, ϕ as the canonical angular momentum, and ζ as time. Thus, the magnetic field line has the same mathematical structure as the Hamilton system. This property is derived from the incompressibility of the magnetic field. In canonical Equation 3.22, ψ is, in general, not only a function of ϕ (*i.e.*, $\psi = \psi(\phi, \theta, \zeta)$), so the magnetic field lines are not necessarily integrable and its structure can be complex. Integrability in a dynamical system has a close relation with the existence of magnetic surfaces in magnetic confinement and chaos is closely related to the disruption of plasma.

In analytical mechanics, the variational principle is formulated using Hamilton's action integral $S = \int [\boldsymbol{p}\cdot \mathrm{d}\boldsymbol{x}/\mathrm{d}t - H]\mathrm{d}t$ leading to the Hamilton equation (Note 2). If we use relations $\boldsymbol{p} \to \phi$, $\mathrm{d}\boldsymbol{x}/\mathrm{d}t \to \mathrm{d}\theta/\mathrm{d}\zeta$, $H \to \psi$ introduced for Equation 3.22, we reach $S = \int [\phi \mathrm{d}\theta/\mathrm{d}\zeta - \psi]\mathrm{d}\zeta = \int [\phi\nabla\theta - \psi\nabla\zeta]\cdot \mathrm{d}\boldsymbol{x} = \int \boldsymbol{A}\cdot \mathrm{d}\boldsymbol{x}$ (gauge part of vector potential ∇G does not contribute to the integral since it becomes the difference in boundary values after integration, that is zero). Thus, the variational principle to give the magnetic field line orbit is,

$$\delta S = \delta \int \boldsymbol{A}\cdot \mathrm{d}\boldsymbol{x} = 0\,. \tag{3.23}$$

Actually, since

$$\delta S(\theta,\phi) = \int \left[\left(\frac{\mathrm{d}\theta}{\mathrm{d}\zeta} - \frac{\partial\psi}{\partial\phi}\right)\delta\phi - \left(\frac{\mathrm{d}\phi}{\mathrm{d}\zeta} + \frac{\partial\psi}{\partial\theta}\right)\delta\theta + \frac{\mathrm{d}(\phi\delta\theta)}{\mathrm{d}\zeta}\right]\mathrm{d}\zeta\,. \tag{3.24}$$

$\delta\int \boldsymbol{A}\cdot \mathrm{d}\boldsymbol{x} = 0$ gives Equation 3.22 (total derivative, 3rd term of right-hand side is zero after integration since boundary value is fixed in the variational principle). This coordinate system (ϕ, θ, ζ) is termed the "magnetic coordinates."

#1: Any vector $\boldsymbol{A}$ can be expressed as $\boldsymbol{A} = A_u\nabla u + A_\theta\nabla\theta + A_\zeta\nabla\zeta$ in the general curvilinear coordinates (u, θ, ζ). If we define a scalar G by $G = \int A_u\,\mathrm{d}u$ ($\partial G/\partial u = A_u$) and consider $\nabla G = \partial G/\partial u\nabla u + \partial G/\partial\theta\nabla\theta + \partial G/\partial\zeta\nabla\zeta$, $\boldsymbol{A}$ can be expressed as $\boldsymbol{A} = \nabla G + (A_\theta - \partial G/\partial\theta)\nabla\theta + (A_\zeta - \partial G/\partial\zeta)\nabla\zeta$. If we define $\phi = A_\theta - \partial G/\partial\theta$ and $\psi = -A_\zeta + \partial G/\partial\zeta$, we reach general expression for the vector potential $\boldsymbol{A} = \phi\nabla\theta - \psi\nabla\zeta + \nabla G$.

Note 1: Hamilton Equations in Dynamical Systems [15]

British physicist Isaac Newton (1642–1727) showed in *Principia* (1687, 1723) that the motion of the object can be described by Newton's equations of motion, $\mathrm{d}p_i/\mathrm{d}t = \partial V/\partial x_i$, $\mathrm{d}x_i/\mathrm{d}t = p_i/m$. Then about 100 years later, another British physicist, W. Hamilton (1805–1865) in 1835 derived the following equation from Newton's equation, now known as the Hamilton equation.

$$\begin{aligned} \mathrm{d}x_i/\mathrm{d}t &= \partial H/\partial p_i \\ \mathrm{d}p_i/\mathrm{d}t &= -\partial H/\partial x_i \end{aligned} \tag{3.25}$$

Here, H is the Hamiltonian, the sum of the kinetic energy T and the potential energy V ($H = T + V$). p_i and x_i are called the canonical momentum and canonical coordinate, respectively.

Note 2: Variational Principle in Hamilton Form (see Section 4.1)

Lagrangian function L is defined by $L = T - V$. Define generalized momentum p_i by $p_i \equiv \partial L/\partial \dot{q}_i$ and the Hamiltonian by $H(\boldsymbol{q}, \boldsymbol{p}, t) = \sum p_i \dot{q}_i - L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$, the variational principle is expanded from position space ($\boldsymbol{q}$) to phase space ($\boldsymbol{q}, \boldsymbol{p}$), where the action integral S is defined under independent variables q_i and p_i,

$$S(\boldsymbol{q}, \boldsymbol{p}) = \int_{t_1}^{t_2} \left[\sum p_i \dot{q}_i - H(\boldsymbol{q}, \boldsymbol{p}, t)\right] \mathrm{d}t \ . \tag{3.26}$$

Taking its variation leads to $\delta S = \int \sum [\delta p_i \{\mathrm{d}q_i/\mathrm{d}t - \partial H/\partial p_i\} - \delta q_i \{\mathrm{d}p_i/\mathrm{d}t + \partial H/\partial q_i\}]\mathrm{d}t$. Since variation δp_i and δq_i are independent, the following Hamiltonian equation is obtained.

$$\frac{\mathrm{d}q_j}{\mathrm{d}t} = \frac{\partial H}{\partial p_j} \ , \quad \frac{\mathrm{d}p_j}{\mathrm{d}t} = -\frac{\partial H}{\partial q_j} \ . \tag{3.27}$$

One should be aware that there is other variational principle, where only position coordinate $\boldsymbol{q}$ is an independent variable.

3.5 Magnetic Surface: Integrable Magnetic Field and Hidden Symmetry

In plasma force equilibrium, the plasma's expansion force (∇P) is balanced with the Lorentz force ($\boldsymbol{J} \times \boldsymbol{B}$). Here, $\boldsymbol{J}$ is the current flowing in the plasma, $\boldsymbol{B}$ is the magnetic field, P is the pressure of the plasma. This is the basic principle of the magnetic confinement fusion.

$$\boldsymbol{J} \times \boldsymbol{B} = \nabla P \ . \tag{3.28}$$

From this equation, we obtain,

$$\boldsymbol{B} \cdot \nabla P = 0 \ , \tag{3.29}$$

$$\boldsymbol{J} \cdot \nabla P = 0 \ . \tag{3.30}$$

In other words, the magnetic field lies on a constant pressure surface (P = constant) in force equilibrium. This constant pressure surface is called the "magnetic surface." Similarly, current field also lies on a constant pressure surface. The magnetic surface is a surface formed by independent vectors $\boldsymbol{B}$ and $\boldsymbol{J}$. It is a special state that the field line orbit always lies on a surface. We choose coordinates (u^1, u^2, u^3) such that $u^1 = u$ is the label of magnetic surface, and u^2 ($= \theta$) and u^3 ($= \zeta$) are arbitrary poloidal and toroidal angles, respectively. In the (u, θ, ζ) coordinates, the magnetic field is expressed by the linear combination of tangent vectors, $\partial \boldsymbol{x}/\partial \theta$ and $\partial \boldsymbol{x}/\partial \zeta$. Using the dual relation ($\partial \boldsymbol{x}/\partial u^i = J \nabla u^j \times \nabla u^k$),

$$\boldsymbol{B} = b_2 \nabla \zeta \times \nabla u + b_3 \nabla u \times \nabla \theta \ . \tag{3.31}$$

Substituting this into $\nabla \cdot \boldsymbol{B} = 0$ and using vector formula $\nabla \cdot (\nabla a \times \nabla b) = 0$, we obtain $\partial b_2/\partial\theta + \partial b_3/\partial\zeta = 0$, which leads to the existence of a "stream function" h for the flow $\boldsymbol{B}$ on the magnetic surface.

$$b_2 = -\frac{\partial h}{\partial \zeta}\,, \quad b_3 = \frac{\partial h}{\partial \theta}\,. \tag{3.32}$$

On the other hand, b_2 and b_3 are periodic function of (θ, ζ), so the flow function h is given by

$$h(u,\theta,\zeta) = h_2(u)\theta + h_3(u)\zeta + \tilde{h}(u,\theta,\zeta)\,. \tag{3.33}$$

Here, $\tilde{h}(u,\theta,\zeta)$ is a periodic function of θ and ζ. Define $\lambda = \tilde{h}(u,\theta,\zeta)/h_2(u)$ and introduce new variable $\theta_{\mathrm{m}} = \theta + \lambda$, we obtain $h(u,\theta_{\mathrm{m}},\zeta) = h_2(u)\theta_{\mathrm{m}} + h_3(u)\zeta$. Flow coefficients $h_2(u)$ and $h_3(u)$ are related to the toroidal magnetic flux $2\pi\phi(u) = \int \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{a}_\zeta$, and the poloidal magnetic flux $2\pi\psi(u) = -\int \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{a}_\theta$ as follows:

$$\frac{\mathrm{d}\psi}{\mathrm{d}u} = -h_3(u)\,, \quad \frac{\mathrm{d}\phi}{\mathrm{d}u} = h_2(u)\,. \tag{3.34}$$

Then,

$$\boldsymbol{B} = \nabla\phi \times \nabla\theta_{\mathrm{m}} - \nabla\psi \times \nabla\zeta = \nabla\phi \times \nabla(\theta_{\mathrm{m}} - \zeta/q)\,. \tag{3.35}$$

Here, $q = \mathrm{d}\phi(u)/\mathrm{d}\psi(u)$ is called a safety factor. The first expression of Equation 3.35 coincides with Equation 3.21, but there is an essential difference in that ϕ and ψ are functions only of the magnetic surface label u. $\alpha = \theta_{\mathrm{m}} - \zeta/q$ is called the "surface potential." The expression $\boldsymbol{B} = \nabla\phi \times \nabla\alpha$ is called "Clebsch form." The coordinate u is equivalent to the toroidal magnetic flux ϕ and $(u, \theta_{\mathrm{m}}, \zeta)$ is called the flux coordinates. In the flux coordinates $(u, \theta_{\mathrm{m}}, \zeta)$, the magnetic field lines on the magnetic surface become a straight line in the $(\theta_{\mathrm{m}}\ \zeta)$ coordinates, whose gradient is given by

$$\frac{\mathrm{d}\theta_{\mathrm{m}}}{\mathrm{d}\zeta} = \frac{\boldsymbol{B} \cdot \nabla\theta_{\mathrm{m}}}{\boldsymbol{B} \cdot \nabla\zeta} = \frac{1}{q(\psi)}\,. \tag{3.36}$$

This gradient can be regarded as the "oscillation frequency" of angle variable θ_{m} when we regard ζ as the "time" variable. In fact, the magnetic field in the force equilibrium is given by Equation 3.35 and the vector potential $\boldsymbol{A}$ is given by $\boldsymbol{A} = \phi\nabla\theta_{\mathrm{m}} - \psi\nabla\zeta$. The action integral S to give the magnetic field line trajectory is given by

$$S = \int \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{x} = \int [\phi\, \mathrm{d}\theta_{\mathrm{m}} - \psi\, \mathrm{d}\zeta]\,. \tag{3.37}$$

The integrant of this action integral has a form of action-angle variables in classical mechanics $(J\mathrm{d}q - H(J)\mathrm{d}t)$ where ϕ and θ_{m} play the roles of "action" and "angle," respectively. Similar to previous section, ψ and ζ play the roles of the

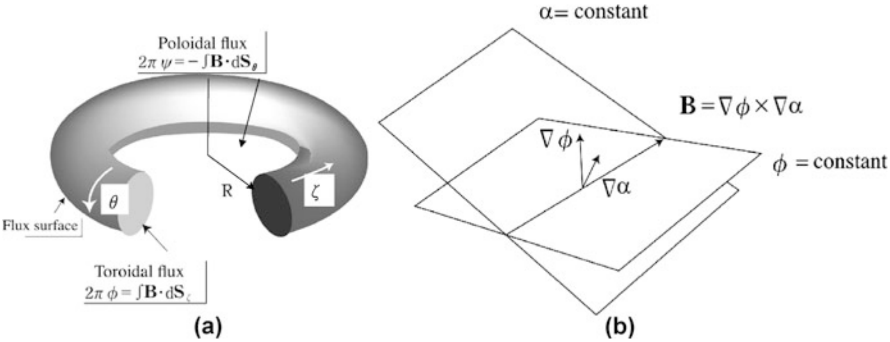

Figure 3.10 (a) Definition of the magnetic surface and fluxes of toroidal plasma and (b) the geometric meaning of the Clebsch expression of the magnetic field

Hamiltonian and time, respectively. In classical mechanics as the time t advances, $\mathrm{d}H/\mathrm{d}J$ gives the "oscillation frequency of the motion" if the system have periodic motion in θ direction. In magnetic field line dynamics, the oscillation frequency of the motion is $\mathrm{d}\psi/\mathrm{d}\phi = 1/q$ [16]. The system Lagrangian L is given in the flux coordinates $(\phi, \theta_\mathrm{m}, \zeta)$ from Equation 3.37 by

$$L = \phi \frac{\mathrm{d}\theta_\mathrm{m}}{\mathrm{d}\zeta} - \psi(\phi) \; . \tag{3.38}$$

The coordinate θ_m becomes a cyclic coordinate. The Hamiltonian ψ and canonical momentum ϕ conjugate to θ_m, and the surface function α is conserved along the magnetic lines of force (independent of "time").

$$\frac{\mathrm{d}\psi}{\mathrm{d}\zeta} = \frac{\boldsymbol{B} \cdot \nabla\psi}{\boldsymbol{B} \cdot \nabla\zeta} = 0 \tag{3.39}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}\zeta} = \frac{\boldsymbol{B} \cdot \nabla\phi}{\boldsymbol{B} \cdot \nabla\zeta} = 0 \tag{3.40}$$

$$\frac{\mathrm{d}\alpha}{\mathrm{d}\zeta} = \frac{\boldsymbol{B} \cdot \nabla\alpha}{\boldsymbol{B} \cdot \nabla\zeta} = 0 \; . \tag{3.41}$$

Geometrically, the magnetic field $\boldsymbol{B}$ is perpendicular to gradients of both ϕ and α ($\boldsymbol{B} \cdot \nabla\phi = 0$, $\boldsymbol{B} \cdot \nabla\alpha = 0$) as seen from Figure 3.10 (b). The magnetic field line trajectory lies on constant ϕ surface. The conserved quantity along the trajectory is called the "first integral" in the dynamical system, and we call such a case integrable if the first integral exists.

In the derivation of the flux coordinate system, no geometrical symmetry is assumed for the torus plasma. But if we assume the existence of force equilibrium, double periodicity of the torus leads to "hidden symmetry" and hence becomes integrable. In analytical mechanics and the gauge field theory of elementary particles, classic methodology to find the conservation law from a cyclic coordinate is extended to "Noether's theorem" (see Section 4.1) which is independent of the choice of coordinates [17].

Salon: Hidden Symmetry in Algebraic Equation [18, 19]

It is well known that an nth ($n \geq 5$) order algebraic equation does not have a general solution as proved by Norwegian mathematician N. H. Abel (1802–1829). If there is a solution through root of power and arithmetic operations, there should be symmetry against the exchange of solutions as investigated by French mathematician J. L. Lagrange (1736–1813). This hidden symmetry in the algebraic equation led to the foundation of group theory by French mathematician Evariste Galois (1811–1832). By using group theory, he identified the solvable condition of 5th order algebraic equations.

3.6 Flux Coordinates: Hamada and Boozer Coordinates

The flux coordinates in Section 3.5 impose only one constraint $\theta_{\mathrm{m}} = \theta + \lambda$ to the two arbitrary angle variables θ and ζ. So, there is one more arbitrary factor to add to the angle variables. In fact, Equation 3.35 is invariant under the coordinate transformation $\theta_{\mathrm{m}1} = \theta_{\mathrm{m}} + \eta(\phi, \theta_{\mathrm{m}}, \zeta)$ and $\zeta_1 = \zeta + q(\phi)\eta(\phi, \theta_{\mathrm{m}}, \zeta)$ for arbitrary function $\eta(\phi, \theta_{\mathrm{m}}, \zeta)$. Thus freedom remains for the combination $(\theta_{\mathrm{m}}, \zeta)$. Using this arbitrariness, Hamada and Boozer coordinates are defined in this section.

Discussion of the flow functions in Section 3.5 can be extended to the current density vector as well as the magnetic field. Using flux coordinates (u, θ, ζ) with u as a label of the magnetic surface, $\boldsymbol{B}$ and $\boldsymbol{J}$ tangent to the magnetic surface can be expressed by the tangent vectors on u plane, $\partial \boldsymbol{x}/\partial\theta$ and $\partial \boldsymbol{x}/\partial\zeta$. Using the dual relation ($\partial \boldsymbol{x}/\partial u^i = J \nabla u^j \times \nabla u^k$), $\boldsymbol{B}$ and $\boldsymbol{J}$ are given by

$$\boldsymbol{a} = a_2 \nabla\zeta \times \nabla u + a_3 \nabla u \times \nabla\theta \quad (\boldsymbol{a} = \boldsymbol{B}, \boldsymbol{J})\ . \tag{3.42}$$

For the equilibrium state, the current density $\boldsymbol{J}$ is incompressible as well as the magnetic field, $\nabla \cdot \boldsymbol{a} = 0$ ($\boldsymbol{a} = \boldsymbol{B}, \boldsymbol{J}$). From the vector formula $\nabla \cdot (\phi \boldsymbol{a}) = \phi \nabla \cdot \boldsymbol{a} + \boldsymbol{a} \cdot \nabla\phi$ and $\nabla \cdot (\nabla F \times \nabla G) = 0$, we obtain $\partial a_2/\partial\theta + \partial a_3/\partial\zeta = 0$. So flow field $\boldsymbol{a}$ has the stream function h on a magnetic surface,

$$a_2 = \frac{\partial h}{\partial \zeta}\ , \quad a_3 = \frac{\partial h}{\partial \theta}\ . \tag{3.43}$$

Since a_2 and a_3 are periodic functions of (θ, ζ), stream functions for the magnetic field ($h = b$) and current density ($h = j$) are given by

$$\begin{aligned} b(u, \theta, \zeta) &= b_2(u)\theta + b_3(u)\zeta + \tilde{b}(u, \theta, \zeta) \\ j(u, \theta, \zeta) &= j_2(u)\theta + j_3(u)\zeta + \tilde{j}(u, \theta, \zeta)\ . \end{aligned} \tag{3.44}$$

Here $\tilde{b}(u, \theta, \phi)$, $\tilde{j}(u, \theta, \phi)$ are periodic functions of θ and ζ. Coordinate transformations to remove them are given by $\theta_h = \theta + \theta_1$ and $\zeta_h = \zeta + \zeta_1$ where θ_1 and ζ_1

are given by

$$\theta_1 = \frac{\tilde{b} j_3 - \tilde{j} b_3}{b_2 j_3 - b_3 j_2} , \quad \zeta_1 = \frac{-\tilde{b} j_2 + \tilde{j} b_2}{b_2 j_3 - b_3 j_2} . \tag{3.45}$$

The flux coordinates thus obtained (u, θ_h, ζ_h) are called (in a broad sense) Hamada coordinates. Coefficients of the stream functions of the magnetic field and current density, $b_2(u)$, $b_3(u)$, $j_2(u)$, $j_3(u)$ are related to the toroidal magnetic flux within the magnetic surface $2\pi\phi(u) = \int \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{a}_\zeta$, poloidal flux $2\pi\psi(u) = -\int \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{a}_\theta$, the toroidal current flux $2\pi f(u) = \int \boldsymbol{J} \cdot \mathrm{d}\boldsymbol{a}_\zeta$, and the poloidal current flux $2\pi g(u) = \int \boldsymbol{J} \cdot \mathrm{d}\boldsymbol{a}_\theta$ by the relationships $\psi'(u) = -b_3(u)$, $\phi'(u) = b_2(u)$, $g'(u) = j_3(u)$, $f'(u) = j_2(u)$ as follows,

$$\boldsymbol{B} = \nabla\phi \times \nabla(\theta_h - \zeta_h/q) , \tag{3.46}$$

$$\boldsymbol{J} = \nabla f \times \nabla(\theta_h - \zeta_h/q_J) . \tag{3.47}$$

Here, $q = \mathrm{d}\phi/\mathrm{d}\psi(u)$ and $q_J = -\mathrm{d}f(u)/\mathrm{d}g(u)$. $\boldsymbol{B}$ and $\boldsymbol{J}$ have to be linearly independent to enable coordinate transformation to the Hamada coordinates, $b_2 j_3 - b_3 j_2 \neq 0$ $(q \neq q_J)$. Defining surface functions by $\alpha = \theta_h - \zeta_h/q$ and $\alpha_J = \theta_h - \zeta_h/q_J$, $\boldsymbol{B}$ and $\boldsymbol{J}$ can be expressed by $\boldsymbol{B} = \nabla\phi \times \nabla\alpha$ and $\boldsymbol{J} = \nabla f \times \nabla\alpha_J$, which leads to $\boldsymbol{B} \cdot \nabla\alpha = 0$ and $\boldsymbol{J} \cdot \nabla\alpha_J = 0$. Both magnetic field and current density are given by straight lines in Hamada coordinates. This Hamada coordinates were derived by a Japanese physicist, Shigeo Hamada, (1931–2001; Figure 3.11 (a)) in 1962 [20]. Hamada called it the natural coordinate system and they are now called "Hamada coordinates." Consider the expression of magnetic field and current density in the flux coordinates (ϕ, θ_m, ζ). Using Equation 3.44, $\boldsymbol{B}$ and $\boldsymbol{J}$ are given by,

$$\boldsymbol{B} = \nabla\phi \times \nabla\theta_\mathrm{m} + q^{-1}\nabla\zeta \times \nabla\phi \tag{3.48}$$

$$\mu_0 \boldsymbol{J} = -\frac{\partial h}{\partial \zeta}\nabla\zeta \times \nabla\phi + \frac{\partial h}{\partial \theta_\mathrm{m}}\nabla\phi \times \nabla\theta_\mathrm{m} \tag{3.49}$$

$$h = f'(\phi)\theta_\mathrm{m} + g'(\phi)\zeta + \nu(\phi, \theta_\mathrm{m}, \zeta) . \tag{3.50}$$

Here, we replace the notation to $j_2 = f'(\phi)$, $j_3 = g'(\phi)$ and $\tilde{j} = \nu$ to match notation of Boozer [13]. Substituting Equations 3.48, 3.49 and $\nabla P = \mathrm{d}P/\mathrm{d}\phi \, \nabla\phi$ into Equation 3.28, we obtain a relation of stream function $\partial h/\partial\zeta + q^{-1}\,\partial h/\partial\theta_m = -\mu_0 J \mathrm{d}P/\mathrm{d}\phi$. Taking flux surface average $(2\pi)^{-2}\int \mathrm{d}\zeta \mathrm{d}\theta_m$ and using the relation $\mathrm{d}V/\mathrm{d}\phi = \int J \mathrm{d}\theta_m \mathrm{d}\zeta$ (volume enclosed by ϕ is given by $V = \int J\, \mathrm{d}\phi \mathrm{d}\theta_m \mathrm{d}\zeta$), we obtain $g'(\phi) + f'(\phi)/q = -\mu_0 V'(\phi) P'(\phi)$. Taking difference of two equations, we obtain

$$\left(\frac{\partial}{\partial\zeta} + \frac{1}{q}\frac{\partial}{\partial\theta_m}\right)\nu = \mu_0 \left(\frac{\mathrm{d}V}{\mathrm{d}\phi} - (2\pi)^2 J\right)\frac{\mathrm{d}P}{\mathrm{d}\phi} . \tag{3.51}$$

Case for $\nu = 0$ corresponds to Hamada coordinates. Therefore, Jacobian of Hamada coordinates is given by $J = (2\pi)^{-2}\mathrm{d}V/\mathrm{d}\phi$ and is a flux function. If we change coordinate ϕ to $v = V/4\pi^2$, Jacobian of Hamada coordinates is given by $J = 1$.

From Equation 3.49, $\boldsymbol{B}$ satisfying $\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$ is given by,

$$\boldsymbol{B} = g(\phi)\nabla\zeta + f(\phi)\nabla\theta_\mathrm{m} - \nu(\phi, \theta_\mathrm{m}, \zeta)\nabla\phi + \nabla F(\phi, \theta_\mathrm{m}, \zeta) . \tag{3.52}$$

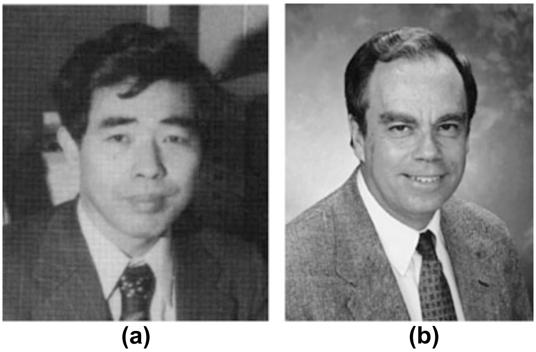

Figure 3.11 (a) Shigeo Hamada who invented Hamada coordinates (with kind permision of Nihon University) and (b) Alan Boozer who invented Boozer coordinates (with kind permission of Prof. Boozer)

Here, F is the magnetic scalar potential satisfying $\nabla \times \boldsymbol{B} = 0$. The case where $\nu = 0$ in Equation 3.51 corresponds to the Hamada coordinates. US physicist A. Boozer (Figure 3.11 (b)) found another set of flux coordinates in 1981 [21]. In the Boozer coordinates, the gauge term ∇F in Equation 3.51 is eliminated by the transformation. Here, we show that such a coordinate transformation exists by using the remaining freedom of coordinate transformation. Boozer coordinates are given by the coordinate transformation $(\theta_b, \zeta_b) = (\theta_\mathrm{m} + \eta, \zeta + q(\phi)\eta)$ and Equation 3.52 reads,

$$\boldsymbol{B} = g(\phi)\nabla\zeta_b + f(\phi)\nabla\theta_b + \beta_*\nabla\phi, \quad \boldsymbol{B} = \nabla\phi \times \nabla\alpha \ , \tag{3.53}$$

$$\eta(\phi, \theta_\mathrm{m}, \zeta) = \frac{F(\phi, \theta_\mathrm{m}, \zeta)}{g(\phi)q(\phi) + f(\phi)} \ , \quad \alpha = \theta_b - \zeta_b/q \ , \tag{3.54}$$

$$\beta_* = \eta(\phi, \theta_\mathrm{m}, \zeta)(q(\phi)g'(\phi) + f'(\phi)) - \nu(\phi, \theta_\mathrm{m}, \zeta) \ . \tag{3.55}$$

Two-way expressions of $\boldsymbol{B}$ in Equation 3.53 is especially useful to simplify the particle orbit equation (see Chapter 4). The magnetic field in Boozer coordinates $(\phi, \theta_b, \zeta_b)$ can be transformed into the following form,

$$\boldsymbol{B} = \nabla\chi + \beta\nabla\phi \ , \qquad \boldsymbol{B} = \nabla\phi \times \nabla\alpha \ , \tag{3.56}$$

$$\chi = g(\phi)\zeta_b + f(\phi)\theta_b \ , \quad \beta = \beta_* - g'(\phi)\zeta_b - f'(\phi)\theta_b \ . \tag{3.57}$$

Corresponding to this form, (ϕ, α, χ) are called Boozer–Grad coordinates and are one of the variants of Clebsch coordinates (coordinates using the two Euler potentials ϕ and α are called Clebsch coordinates). It is important to note that $\boldsymbol{B}$ is expressed in two ways (covariant form and Clebsch form) in two Boozer coordinates.

3.7 Ergodicity: A Field Line Densely Covers the Torus

Figure 3.12 shows the magnetic surface of the torus plasma in the cylindrical coordinate system (R, ζ, Z). We use the flux coordinate system (ϕ, θ_m, ζ) in which $\boldsymbol{B}$ is expressed by $\boldsymbol{B} = \nabla\phi \times \nabla(\theta_m - \zeta/q)$ and its trajectory along toroidal direction $\mathrm{d}\theta_m/\mathrm{d}\zeta = 1/q(\phi)$ is a straight line with a gradient $1/q$. We choose the toroidal angle of cylindrical coordinates for ζ of flux coordinates.

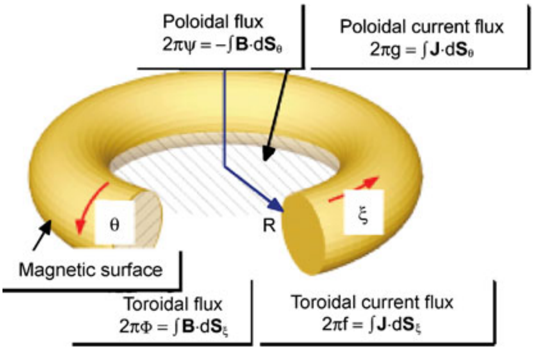

Figure 3.12 The definition of magnetic field and current density fluxes in a torus

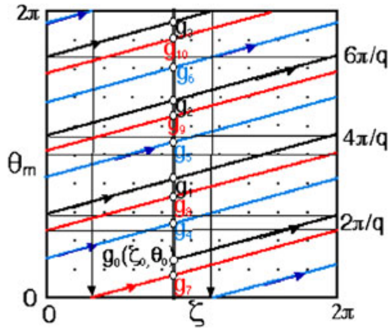

Figure 3.13 Locus of the magnetic field in the flux coordinates $(\phi, \theta_{\rm m}, \zeta)$ and the mapping series of $g_0(\zeta_0, \theta_0), g_1, g_2, \ldots$

Consider the magnetic field starting from the point on the magnetic surface $(\zeta, \theta) = (\zeta_0, \theta_0)$, poloidal rotation angle per one toroidal rotation is given by $\Delta\theta = 2\pi/q$ and field line returns to the point $\theta = 2\pi/q + \theta_0$ at $\zeta = \zeta_0$. If $\theta \geq 2\pi$, 2π is subtracted so that θ is within $[0, 2\pi)$. Repeating this procedure, the sequence of points $g\theta_0, g^2\theta_0, g^3\theta_0, g^4\theta_0, \ldots, g^j\theta_0$ are drawn on the $\zeta = \zeta_0$ plane. The poloidal angle of the sequence of points $\{g^j\theta_0\}$ is given by $\theta^j = 2\pi j/q + \theta_0$. This mapping to some plane ($\zeta = \zeta_0$ in this case) is called "Poincaré mapping." Let real semi open set $\Theta = [0, 2\pi)$, this mapping is the mapping from Θ to Θ itself ($g : \Theta \to \Theta$, $g\theta_0 = \theta_0 + 2\pi/q$ for $\theta_0 \in \Theta$). Now, when q is a rational number, *i.e.*, integers m and n exist to satisfy $q = m/n$, rotation by mapping g^m is given by $\theta^m = 2\pi m/q + \theta_0 = 2n\pi + \theta_0$ and returns to the original position (ζ_0, θ_0) ("identity mapping"). See Figure 3.13.

However, if q is an irrational number, it can be proved by using "reduction to absurdity (reductio ad absurdum)," originating from Aristotle (384–322 BC), that the magnetic field line will not return to the original point after any toroidal circulations. In fact, if we assume that it returns to the original position, it contradicts the assumption of an irrational number, and it goes around the torus infinitely. Then, it can be shown by Poincaré mapping that the sequence of points $\{g^j\theta_0\}$ will densely cover the poloidal circumference. The magnetic field line is a 1-dimensional line. The line is a 1-dimensional set and the width of the line is zero according to Euclid's definition. If we place two lines side by side, there will still be a gap between them and we cannot form a continuous surface. However, we can reduce the gap infinitely. Then, we can form a magnetic field line originating from (ζ_0, θ_0) passes at any closest distance of any position of the torus.

Considering the magnetic field lines on the torus surface, the magnetic field is said to "densely cover the torus" if there always exists a field line in any neighborhood (closer distance if we set arbitrary $\varepsilon > 0$) of any point on the torus. In general, the set A (set of points of a magnetic field line, in this case) is said to densely cover the set B (torus magnetic surface, in this case) if there exists a point of the set A at any neighborhood of any point in set B.

The fact that the magnetic field densely covers the torus when q is irrational can be proved by using reduction to absurdity [5]. Consider an arbitrary poloidal angle θ_0 at $\zeta = \zeta_0$ and its neighborhood U. Since mapping points $\{g^i \theta_0\}$ continue indefinitely (do not return to any previous point), the mapping series $\{g^i U\}$ also continues indefinitely. If there is no intersection among mapping series, the poloidal length of the mapping series becomes infinity and contradicts a finite poloidal circumference. Therefore, this mapping series of the neighborhood should have a common set. This means that there are integers $k \geq 1$ and $l \geq 1$ $(k > l)$ such that $g^k U \cap g^l U \neq \emptyset$, then $g^{k-l} U \cap U \neq \emptyset$, and $\theta = g^{k-l}\theta_0$ should be in the neighborhood of θ_0. Since the choice of ζ_0 is arbitrary, there is a field line point $(\zeta_0, g^{k-l}\theta_0)$ in the neighborhood of arbitrary point (ζ_0, θ_0). "Densely covered" is also termed "ergodically covered." This stems from the "ergodic hypothesis" in the phase space to derive the "principle of equal weight" by L. Boltzmann.

In the force equilibrium of the plasma, the safety factor q continuously changes with different magnetic surfaces, and the range of q is a real closed interval. In this real closed interval, the number of irrational number is uncountable infinity. On the other hand, the number of the rational number is countable infinity, and the so-called "measure" is zero (see the salon).

Salon: Wonder of Infinity [22]

The German mathematician Georg Cantor (1845–1918; Figure 3.14), a famous founder of set theory, investigated the "number" or "number line" which relates the point in a line to a number. For example, the number of natural numbers is infinity, but they can be counted as $1, 2, 3, \ldots$ and are said to be countable infinity ("denumerable"). Counting infinity is different from counting the finite number. In 1874, Cantor showed that a set of rational numbers has the same number as a set of natural numbers using a "diagonal argument" (the "same number," to be precise, means there is one-to-one mapping between a set of natural numbers and a set of rational numbers). Natural numbers are discrete and rational numbers are dense on the number line. Rational numbers always exist in any neighborhood of any point in the number line (a set of real numbers), by which the set of rational numbers is said to be dense everywhere in the set of real numbers. It is the nature of infinity that such different sets of rational and natural numbers have "equal numbers."

In 1874, Cantor also showed that real numbers are uncountable using "reduction to absurdity" In fact, assuming real numbers of set [0, 1] are countable, he expressed a real number by an infinite decimal, arranged in a series in a vertical column with numbering $(1, 2, 3, \ldots, n)$ from the top. Then, he picked up each digit at n decimal places to form a new number (the diagonal number) and added 1 to each digit of the new number. It is easy to prove that this number is not included in the series since it differs at least at the n decimal place, which is a contradiction (absurdity). The "infinity" of real numbers is in a higher order than the "infinity" of rational and natural numbers [23].

Real number R has a one-to-one correspondence with a number line, and a non-negative real number, "length," can be defined. R^2 and R^3 have one-to-one correspondences with plane and space, respectively. And the corresponding nonnegative real numbers "area" and "volume" can be defined. A set of rational numbers is denumerable and its "length" is zero even if it densely covers the number line. Such length, area, and volume are generalized to a concept of "measure" [23]. This is a nonnegative real number set to meet the complete additivity.

Figure 3.14 G. Cantor is a founder of set theory who investigated the nature of "infinity" in depth

3.8 Apparent Symmetry: Force Equilibrium of Axisymmetric Torus

Let us consider the axisymmetric torus, which is a major object of present fusion research. In the cylindrical coordinate system, (R, ζ, Z), the ζ is a cyclic coordinate and $\partial/\partial\zeta = 0$ (Figure 3.15). If we define a flux function by using the ζ component of the vector potential $\boldsymbol{A}$ as $\psi = RA_\zeta(R, Z)$, B_R and B_Z in the poloidal cross section are given as follows:

$$\begin{aligned} RB_R &= -\frac{\partial \psi}{\partial Z} \, , \\ RB_Z &= \frac{\partial \psi}{\partial R} \, . \end{aligned} \tag{3.58}$$

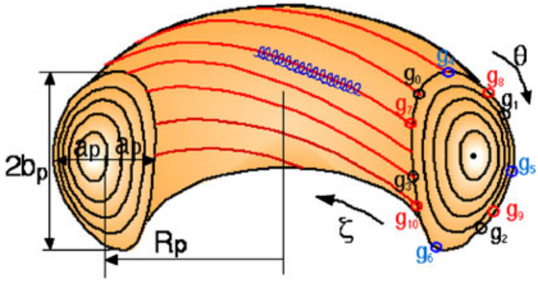

Figure 3.15 Locus of the magnetic field in cylindrical coordinates (R, ζ, Z)

Equation 3.58 satisfies the basic nature of $\boldsymbol{B}$, the incompressibility condition $\nabla \cdot \boldsymbol{B} = 0$ $((1/R)\partial(RB_R)/\partial R + \partial B_Z/\partial Z = 0)$ $(\nabla \cdot \boldsymbol{B} = 0 \rightarrow \boldsymbol{B}_p = \nabla\zeta \times \nabla\psi)$. Also, $\boldsymbol{B} \cdot \nabla\psi = 0$ can be checked easily. Then the magnetic field trajectory is on the ψ = constant surface. In terms of terminology in dynamical system, the system has a first integral and the orbit is "integrable." The constant ψ surface is called a "magnetic surface," or "magnetic surface ψ." In terms of Hamilton dynamics in Section 3.4, the system is independent of "time" ζ and the Hamiltonian ψ becomes an invariant.

Axisymmetry guarantees that B_R and B_Z can be expressed by the first integral, but it does not impose any constraint on the toroidal magnetic field B_ζ. The constraint on B_ζ comes from the equilibrium condition $\boldsymbol{J} \times \boldsymbol{B} = \nabla P$. In fact, $\boldsymbol{B} \cdot \nabla P = 0$ reads $\partial(\psi, P)/\partial(R, Z) = 0$ and $P = P(\psi)$ is derived. Then, $\boldsymbol{J} \cdot \nabla P = 0$ reads $\partial(RB_\zeta P)/\partial(R, Z) = 0$ and $RB_\zeta = F(P) = F(\psi)$ is derived. Such functions of flux function ψ only are called a magnetic "flux function." From the above, the following relationships can be obtained:

$$\boldsymbol{B} = \nabla\zeta \times \nabla\psi + F\nabla\zeta\,, \tag{3.59}$$

$$\boldsymbol{J} = \mu_0^{-1}\left[\nabla F \times \nabla\zeta + \Delta^*\psi\nabla\zeta\right]\,. \tag{3.60}$$

Here, $\Delta^* = R\partial/\partial R(R^{-1}\partial/\partial R) + \partial^2/\partial R^2$ is called the "Grad–Shafranov operator." $F(\psi)$ plays the role of stream function for $\boldsymbol{J}_p = -\mu_0^{-1}\nabla\zeta \times \nabla F$. Substituting Equations 3.59 and 3.60 into $\boldsymbol{J} \times \boldsymbol{B} = \nabla P$ yields,

$$\Delta^*\psi = -\mu_0 R^2 P'(\psi) - FF'(\psi)\,. \tag{3.61}$$

This nonlinear elliptic partial differential equation is called the "Grad–Shafranov equation" [24, 25]. The functional form of $P(\psi)$ and $F(\psi)$ cannot be determined by the force equilibrium (determined by the transport equations of current and temperature/density). In general, the Grad–Shafranov equation is solved numerically by giving the functional form of $P(\psi)$ and $F(\psi)$. The Grad-Shafranov equation can be derived using the variational principle $\delta S = 0$ [25].

$$S = \int L(\psi, \psi_R, \psi_Z, R)\,\mathrm{d}R\,\mathrm{d}Z\,. \tag{3.62}$$

Here, $\psi_R = \partial\psi/\partial R$ and $\psi_Z = \partial\psi/\partial Z$. The Lagrangian L is given by,

$$L = R\left(\frac{B_p^2}{2\mu_0} - \frac{B_\zeta^2}{2\mu_0} - P\right) \tag{3.63}$$

where, $B_p = |\nabla\psi|/R$ and $B_z = F(\psi)/R$. The Euler–Lagrange equation derived from variational principle $\delta S = 0$ is

$$\frac{\partial L}{\partial\psi} - \frac{\partial}{\partial R}\frac{\partial L}{\partial\psi_R} - \frac{\partial}{\partial Z}\frac{\partial L}{\partial\psi_Z} = 0\ . \tag{3.64}$$

And the Equation 3.61 can be obtained. $B_p^2/2\mu_0$ plays the role of the kinetic energy of the Lagrangian, and $B_\zeta^2/2\mu_0 + P$ plays the role of the effective potential energy. It may be natural to question why the roles of the toroidal and poloidal magnetic field energies are different? In this variational principle, the toroidal magnetic field and pressure are already given by $B_\zeta = F(\psi)/R$ and $P = P(\psi)$ and the problem is reduced to obtain a solution of "motion" of ψ.

The "flux surface average" of a physical quantity $\langle A\rangle$ is defined by the volume integral in an infinitesimal small shell in $(\psi, \psi + \mathrm{d}\psi)$. Using $\mathrm{d}V = J\mathrm{d}\psi\,\mathrm{d}\theta\,\mathrm{d}\zeta$,

$$\langle A\rangle = \frac{\int_\psi^{\psi+\mathrm{d}\psi} AJ\mathrm{d}\psi\,\mathrm{d}\theta\,\mathrm{d}\zeta}{\int_\psi^{\psi+\mathrm{d}\psi} J\mathrm{d}\psi\,\mathrm{d}\theta\,\mathrm{d}\zeta} = \frac{\int_0^{2\pi}\frac{A\,\mathrm{d}\theta}{\boldsymbol{B}_p\cdot\nabla\theta}}{\int_0^{2\pi}\frac{\mathrm{d}\theta}{\boldsymbol{B}_p\cdot\nabla\theta}}\ . \tag{3.65}$$

Here, $J = 1/(\nabla\zeta\times\nabla\psi)\cdot\nabla\theta = 1/\boldsymbol{B}_p\cdot\nabla\theta$ is used. The flux surface average annihilates the differential operator $\boldsymbol{B}\cdot\nabla = J^{-1}\partial/\partial\theta$. The differential equation along the magnetic field appears frequently in the magnetic confinement theory and was named the "magnetic differential equation" by the famous MHD stability theoretician W. Newcomb.

$$\boldsymbol{B}\cdot\nabla h = S\ . \tag{3.66}$$

Here, h and S are single-valued. In a closed magnetic configuration, integrability of the magnetic field sets a constraint on h and S, called the "solvable condition." Equation 3.66 in the flux coordinates (ϕ, θ, ζ) becomes

$$(q\nabla\psi\times\nabla\theta - \nabla\psi\times\nabla\zeta)\left[\frac{\partial h}{\partial\psi}\nabla\psi + \frac{\partial h}{\partial\theta}\nabla\theta + \frac{\partial h}{\partial\zeta}\nabla\zeta\right] = S\ . \tag{3.67}$$

Using $\boldsymbol{B}\cdot\nabla\psi = 0$, axisymmetric condition $\partial h/\partial\zeta = 0$, and $J^{-1} = \nabla\psi\times\nabla\theta\cdot\nabla\zeta = \boldsymbol{B}\cdot\nabla\theta = \boldsymbol{B}_p\cdot\nabla\theta$,

$$\frac{\partial h}{\partial\theta} = \frac{S}{\boldsymbol{B}_p\cdot\nabla\theta}\ . \tag{3.68}$$

Since h is single-valued, it must be the same value at $\theta = 0$ and 2π. Integration of Equation 3.67 for θ gives the following "solvable condition."

$$\int_0^{2\pi}\frac{S}{\boldsymbol{B}_p\cdot\nabla\theta}\mathrm{d}\theta = 0\ . \tag{3.69}$$

This means $\langle S \rangle = 0$ and application of the flux surface averaging operator to the differential operator $\boldsymbol{B} \cdot \nabla = J^{-1}\partial/\partial\theta$ gives $\langle \boldsymbol{B} \cdot \nabla \rangle \equiv 0$. This is the origin of the naming of the annihilator of $\boldsymbol{B} \cdot \nabla = J^{-1}\partial/\partial\theta$.

Note: Symmetry and Invariant of the Dynamical System [15]

For the case where the system has symmetry and some position coordinate q_s is not included in Lagrangian $L(q_i, \dot{q}_i)$ (means $\partial L/\partial q_s = 0$, but assume $\dot{q}_s$ is included in L since q_s is not a dynamical variable if both are not included). Then, the Lagrange equation (see Section 4.1)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s} = 0 \tag{3.70}$$

reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_s}\right) = 0\ . \tag{3.71}$$

Thus, generalized momentum $p_s = \partial L/\partial \dot{q}_s$ is conserved (invariant). Such a coordinate is called a "cyclic" coordinate. In the system of rotational symmetry (axisymmetric system), ζ is not included in L and the generalized angular momentum $p_\zeta = \partial L/\partial \dot{\zeta}$ is conserved. Symmetry in dynamical systems is closely related to the existence of the invariant (integrability).

3.9 3-dimensional Force Equilibrium: Search for Hidden Symmetry

A typical example of force equilibrium without apparent geometrical symmetry is the stellarator concept originated by Spitzer [2], see Figure 3.16. The 3-dimensional equilibrium may not have global equilibrium in some cases, in contrast to the tokamak equilibrium. There are few mathematical theories of 3-dimensional equilibrium except for KAM theory, which treats slight symmetry breaking. The variational principle may be useful to examine 3-dimensional equilibrium since it is independent of the coordinate system. In 1958, H. Grad derived a variational principle (δS = 0) equivalent to the plasma equilibrium condition $\boldsymbol{J} \times \boldsymbol{B} = \nabla P$ by defining the action integral S with a variable to $\boldsymbol{B}$ and P satisfying $\nabla \cdot \boldsymbol{B} = 0$ and $\boldsymbol{B} \cdot \nabla P = 0$ [24,27].

$$S(\boldsymbol{B}, P) = \int_v L\,\mathrm{d}V = \int_v \left[\frac{B^2}{2\mu_0} - P\right]\mathrm{d}V\ , \tag{3.72}$$

$$\delta S(\boldsymbol{B}, P) = \int_v \left[\frac{1}{\mu_0}\boldsymbol{B} \cdot \delta\boldsymbol{B} - \delta P\right]\mathrm{d}V\ ,$$

where V is the plasma volume and $\boldsymbol{B} \cdot \boldsymbol{n} = 0$ at the plasma surface. The Lagrangian $L = B^2/2\mu_0 - P$ is the difference between magnetic pressure and plasma pressure,

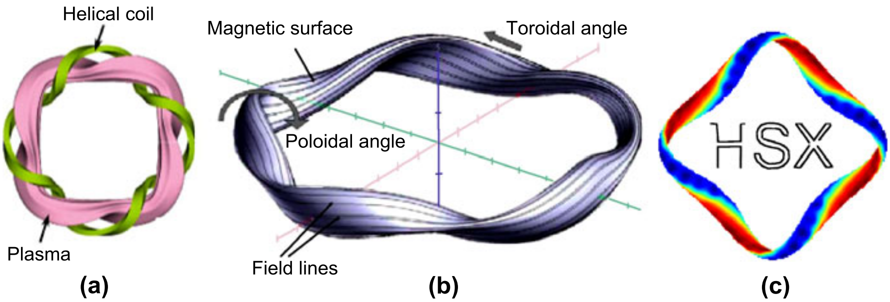

Figure 3.16 Typical 3-dimensional toroidal equilibrium configurations with periodic symmetry. (a) Heliotron-J (by kind permission of IAE Kyoto University) with 4-fold symmetry, (b) Wendelstein 7-X (by courtesy of the Max Planck Institute for plasma physics) with 5-fold symmetry and (c) HSX (with kind permission by Prof. D. Anderson, Wisconsin University) with 4-fold symmetry

implying magnetic field energy is "kinetic energy," plasma pressure is "potential energy." The magnetic field satisfying $\boldsymbol{B} \cdot \nabla P = 0$ and $\nabla \cdot \boldsymbol{B} = 0$ are given by,

$$\boldsymbol{B} = \nabla P \times \nabla \omega \, . \tag{3.73}$$

Here, the flow function ω is expected to be a multi-valued function containing angle variables from the discussion of surface function in Section 3.5. Under this strong constraint of $\boldsymbol{B}$ on the constant pressure surface, action integral S becomes a function of ω and P. Using Equation 3.73,

$$\boldsymbol{B} \cdot \delta \boldsymbol{B} = (\nabla \omega \times \boldsymbol{B}) \cdot \nabla \delta P + (\boldsymbol{B} \times \nabla P) \cdot \nabla \delta \omega \, . \tag{3.74}$$

Applying vector formula $\nabla \cdot (\phi \boldsymbol{a}) = \phi \nabla \cdot \boldsymbol{a} + \boldsymbol{a} \cdot \nabla \phi$ to Equation 3.74 and transforming $\nabla \cdot (\delta P \nabla \omega \times \boldsymbol{B} + \delta \omega \boldsymbol{B} \times \nabla P)$ into surface integral by the Gauss's theorem and set zero by the boundary conditions $\delta \psi = \delta \omega = 0$, the remaining term of volume integral of $\boldsymbol{B} \cdot \delta \boldsymbol{B}$ is given by,

$$\boldsymbol{B} \cdot \delta \boldsymbol{B} = -\left[\nabla \cdot (\nabla \omega \times \boldsymbol{B})\right] \delta P - \left[\nabla \cdot (\boldsymbol{B} \times \nabla P)\right] \delta \omega \, . \tag{3.75}$$

Then the following form of δS is obtained by using the vector formula $\nabla \cdot (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{b} \cdot \nabla \times \boldsymbol{a} - \boldsymbol{a} \cdot \nabla \times \boldsymbol{b}$,

$$\delta S(\omega, P) = \int \left[(\boldsymbol{J} \cdot \nabla \omega - 1)\delta P - \boldsymbol{J} \cdot \nabla P \delta \omega\right] \mathrm{d}V \, . \tag{3.76}$$

Then,

$$\boldsymbol{J} \cdot \nabla \omega = 1 \, , \tag{3.77}$$

$$\boldsymbol{J} \cdot \nabla P = 0 \tag{3.78}$$

are obtained as the Euler equations to extremize S. Since $\boldsymbol{J} \times \boldsymbol{B} - \nabla P = (\boldsymbol{J} \cdot \nabla \omega - 1)\nabla P - (\boldsymbol{J} \cdot \nabla P)\nabla \omega = 0$, the variational principle $\delta S = 0$ is equivalent to $\boldsymbol{J} \times \boldsymbol{B} = \nabla P$. As is clear from the proof of the above, the plasma pressure P plays a role of "potential energy" and strongly constrains the magnetic field ($\boldsymbol{B} \cdot \nabla P = 0$). Magnetic energy plays the role of "kinetic energy" and the variational principle becomes an extremal problem on the stream function ω under the strong constraint of P.

If a small displacement $\delta \boldsymbol{x}$ of the plasma induces a pressure change δP and the change in the stream function $\delta\omega$, we obtain $\delta P = -\delta \boldsymbol{x} \cdot \nabla P$ and $\delta\omega = -\delta \boldsymbol{x} \cdot \nabla\omega$. The variational principle, Equation 3.76, can be rewritten in the following form given by Kruskal–Krusrud [28]:

$$\delta S(\omega, P) = -\int \delta \boldsymbol{x} \cdot [\boldsymbol{J} \times \boldsymbol{B} - \nabla P]\, \mathrm{d}V \,. \tag{3.79}$$

From $\nabla \cdot \boldsymbol{J} = 0$ and $\boldsymbol{J} \cdot \nabla P = 0$, $\boldsymbol{J}$ is given by,

$$\boldsymbol{J} = \nabla P \times \nabla \omega_J \,. \tag{3.80}$$

Here, ω_J is a flow function for $\boldsymbol{J}$. Using $\boldsymbol{B} \cdot \nabla P = 0$, we see $\boldsymbol{J} \times \boldsymbol{B} = (\nabla P \times \nabla\omega_J) \times \boldsymbol{B} = -(\boldsymbol{B} \cdot \nabla\omega_J)\nabla P$ to reach,

$$\boldsymbol{B} \cdot \nabla \omega_J = -1 \,. \tag{3.81}$$

Substituting Equations 3.73 and 3.80 into the force equilibrium $\boldsymbol{J} \times \boldsymbol{B} = \nabla P$, we obtain the following equation, equivalent to Equation 3.78,

$$(\nabla\omega_J \times \nabla\omega) \cdot \nabla P = 1 \,. \tag{3.82}$$

"3-dimensional force equilibrium" with no apparent geometric symmetry, must have a coordinate transformation from the flux coordinates with hidden symmetry to the real coordinate system (such as cylindrical coordinate system). The equilibrium problem is understood as finding "inverse mapping" $\boldsymbol{x} = \boldsymbol{x}(u, \theta, \zeta)$ from the flux coordinates (u, θ, ζ) to the cylindrical coordinates (R, ζ, Z). The variational principle has merit in the simplicity of its coordinate-independent formulation as an extremal problem of a single scalar function rather than vector differential equations.

Such a variational principle for 3-dimensional plasma equilibrium is implemented in Hirshman's VMEC code [28]. Introducing the virtual time "t" into Equation 3.79 and changing δS to $\mathrm{d}S/\mathrm{d}t$, we have,

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -\int [\boldsymbol{J} \times \boldsymbol{B} - \nabla P] \cdot \frac{\partial \delta \boldsymbol{x}}{\partial t}\, \mathrm{d}V \,. \tag{3.83}$$

This is an evolution of the equation from the state of $\boldsymbol{F} = \boldsymbol{J} \times \boldsymbol{B} - \nabla P \neq 0$ to the state of $\boldsymbol{F} = 0$ with virtual displacement $\delta \boldsymbol{x}$. Applying Cauchy-Schwartz inequality, $|\int \boldsymbol{A}^* \boldsymbol{B}\, \mathrm{d}u|^2 \le \int |\boldsymbol{A}|^2 \mathrm{d}u \int |\boldsymbol{B}|^2 \mathrm{d}u$ to Equation 3.83, we have

$$\left| \int [\boldsymbol{J} \times \boldsymbol{B} - \nabla P] \cdot \frac{\partial \delta \boldsymbol{x}}{\partial t}\, \mathrm{d}V \right|^2 \le \int |\boldsymbol{J} \times \boldsymbol{B} - \nabla P|^2\, \mathrm{d}V \int \left| \frac{\partial \delta \boldsymbol{x}}{\partial t} \right|^2 \mathrm{d}V \,. \tag{3.84}$$

Here, the equality holds only when $\partial \delta \boldsymbol{x}/\partial t = c(\boldsymbol{J} \times \boldsymbol{B} - \nabla P)$ is satisfied (c can be set 1 since $\delta \boldsymbol{x}$ is the virtual displacement). Convergence of this equation can be accelerated by adding the second order time derivative (second order Richardson scheme). By setting unknown constants in the coordinate transformation equation from flux coordinates $\{u_i\} = (\phi, \theta_m, \zeta)$ to cylindrical coordinates (R, ζ, Z), we treat this problem as an extremal problem of $(\delta S)^2$. The toroidal angular variable ζ in the flux coordinate is chosen to be same as that of the cylindrical coordinates. Poloidal angle θ is determined by the condition of fast convergence of the Fourier expansion in the plasma surface, the unknown functions are $\boldsymbol{x} = (R, \lambda, Z)$. Assuming F_R, F_λ, F_Z are virtual forces which are zero in equilibrium, each Fourier component F_j^{mn} $(j = R, \lambda, Z)$ should be zero in equilibrium. However, equilibrium with separatrix cannot be reconstructed if we expand in Fourier series. Minimization of the action integral is possible numerically, but it must be noted that this does not mean that the 3-dimensional equilibrium is obtained [30].

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