Chapter 9
Generalized Eigenproblem

This chapter deals with the *generalized eigenproblem* (GEP) in max-algebra defined as follows:

Given $A, B \in \overline{\mathbb{R}}^{m\times n}$, *find all* $\lambda \in \overline{\mathbb{R}}$ (*generalized eigenvalues*) *and* $x \in \overline{\mathbb{R}}^n$, $x \neq \varepsilon$ (*generalized eigenvectors*) *such that*

$$A \otimes x = \lambda \otimes B \otimes x. \tag{9.1}$$

When $\lambda \in \overline{\mathbb{R}}$ and $x \in \overline{\mathbb{R}}^n$, $x \neq \varepsilon$ satisfying (9.1) exist then we say that *GEP is solvable* or also that (A, B) *is solvable*. Obviously, the eigenproblem is obtained from the GEP when $B = I$ or $\lambda = \varepsilon$ and we will therefore assume in this chapter that $\lambda > \varepsilon$.

It is likely that GEP is much more difficult than the eigenproblem. This is indicated by the fact that the GEP for a pair of real matrices may have no generalized eigenvalue, a finite number or a continuum of generalized eigenvalues [70]. It is known [135] that the union of any system of closed (possibly one-element) intervals is the set of generalized eigenvalues for suitably taken A and B.

GEP has been studied in [15] and [70]. The first of these papers solves the problem completely when $m = 2$ and special cases for general m and n, the second solves some other special cases. No solution method seems to be known either for finding a λ or an $x \neq \varepsilon$ satisfying (9.1) for general real matrices. Obviously, once λ is fixed, the GEP reduces to a two-sided max-linear system (Chap. 7). We therefore concentrate on the question of finding the generalized eigenvalues. First we will study basic properties and solvable special cases of GEP. In Sect. 9.3 we then present a method for narrowing the search for generalized eigenvalues for a pair of real square matrices. It is based on the solvability conditions for two-sided systems formulated using symmetrized semirings (Sect. 7.5).

A motivation for the GEP is given in Sect. 1.3.2.

Given $A, B \in \overline{\mathbb{R}}^{m\times n}$ we denote the set of generalized eigenvalues by $\Lambda(A, B)$, the set containing ε and all generalized eigenvectors corresponding to $\lambda \in \overline{\mathbb{R}}$ by $V(A, B, \lambda)$ and the set of all generalized eigenvectors by $V(A, B)$, that is:

P. Butkovič, *Max-linear Systems: Theory and Algorithms,*
Springer Monographs in Mathematics 151,
DOI 10.1007/978-1-84996-299-5_9,

$$V(A, B, \lambda) = \left\{x \in \overline{\mathbb{R}}^n; A \otimes x = \lambda \otimes B \otimes x\right\}, \lambda \in \overline{\mathbb{R}},$$
$$V(A, B) = \left\{x \in \overline{\mathbb{R}}^n; A \otimes x = \lambda \otimes B \otimes x, \lambda \in \overline{\mathbb{R}}\right\}$$

and

$$\Lambda(A, B) = \left\{\lambda \in \overline{\mathbb{R}}; V(A, B, \lambda) \neq \{\varepsilon\}\right\}.$$

9.1 Basic Properties of the Generalized Eigenproblem

In this section we present some properties of the GEP provided that A and B are finite matrices [70]. We therefore assume that $A = (a_{ij})$, $B = (b_{ij}) \in \mathbb{R}^{m\times n}$ are given matrices and, as before, we denote $M = \{1, \dots, m\}$ and $N = \{1, \dots, n\}$. We will also denote:

$$C = (c_{ij}) = (a_{ij} \otimes b_{ij}^{-1})$$

and

$$D = (d_{ij}) = (b_{ij} \otimes a_{ij}^{-1}).$$

Theorem 9.1.1 *If (A, B) is solvable and $\lambda \in \Lambda(A, B)$ then C satisfies*

$$\max_{i\in M} \min_{j\in N} c_{ij} \leq \lambda \leq \min_{i\in M} \max_{j\in N} c_{ij}. \tag{9.2}$$

Proof No row of $\lambda \otimes B$ strictly dominates the corresponding row of A, so for every i there is a j such that $a_{ij} \geq \lambda \otimes b_{ij}$, i.e. $\lambda \leq c_{ij}$. Hence for all i we have $\lambda \leq \max_j c_{ij}$, thus $\lambda \leq \min_i \max_j c_{ij}$. Similarly, no row of A strictly dominates the corresponding row of $\lambda \otimes B$, yielding for all i: $\lambda \geq \min_j c_{ij}$, thus $\lambda \geq \max_i \min_j c_{ij}$. □

The interval $[\max_{i\in M} \min_{j\in N} c_{ij}, \min_{i\in M} \max_{j\in N} c_{ij}]$ is called the *feasible interval* for the generalized eigenproblem (9.1).

Example 9.1.2 If $A = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ then (A, B) is not solvable because $C = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$ does not satisfy (9.2).

Recall that for a square matrix A the symbol $\lambda(A)$ stands for the maximum cycle mean of A. We now also denote by $\lambda'(A)$ the *minimum cycle mean*.

Corollary 9.1.3 *If $m = n$, (A, B) is solvable and $\lambda \in \Lambda(A, B)$ then C satisfies*

$$\lambda'(C) \leq \lambda \leq \lambda(C).$$

Proof A cycle in D_C whose every arc has the weight equal to a row maximum in C exists. The arc weights on this cycle are all at least the smallest row maximum, thus $\lambda(C) \geq \min_{i \in M} \max_{j \in N} c_{ij}$. The second inequality now follows from Theorem 9.1.1 and the other inequality by swapping max and min. □

Recall that the conjugate of B is $B^* = (b^*_{ij}) = (b^{-1}_{ji})$. Then the ith element of the diagonal of $A \otimes B^*$ equals

$$\max_j (a_{ij} + b^*_{ji}) = \max_j (a_{ij} \otimes b^{-1}_{ij}) = \max_j c_{ij}.$$

Similarly, the ith element of the diagonal of $A \otimes' B^*$ equals $\min_j c_{ij}$. Hence by Theorem 9.1.1 we have:

Corollary 9.1.4 *If (A, B) is solvable then the greatest element of the diagonal of $A \otimes' B^*$ does not exceed the least element of the diagonal of $A \otimes B^*$.*

By Corollary 9.1.3 we also have:

Corollary 9.1.5 *If (A, B) is solvable and $\lambda \in \Lambda(A, B)$ then*

$$\lambda'(A \otimes' B^*) \leq \lambda \leq \lambda(A \otimes B^*).$$

The next statement is a remarkable observation on generalized eigenvalues, yet there is no description of the unique possible value for the eigenvalue.

Theorem 9.1.6 [15] *If both (A, B) and (A^T, B^T) are solvable then both these problems have a unique and identical eigenvalue, that is, there is a real number λ such that*

$$\Lambda(A, B) = \{\lambda\} = \Lambda(A^T, B^T)$$

provided that $\Lambda(A, B) \neq \emptyset$ and $\Lambda(A^T, B^T) \neq \emptyset$.

Proof Suppose that

$$A \otimes x = \lambda \otimes B \otimes x$$

and

$$A^T \otimes y = \mu \otimes B^T \otimes y$$

for some λ, μ, x, y. Then

$$\begin{aligned} \lambda \otimes y^T \otimes B \otimes x = y^T \otimes A \otimes x = x^T \otimes A^T \otimes y \\ = \mu \otimes x^T \otimes B^T \otimes y = \mu \otimes y^T \otimes B \otimes x. \end{aligned}$$

Since $y^T \otimes B \otimes x$ are finite it follows that $\lambda = \mu$. □

Corollary 9.1.7 *If* $A, B \in \mathbb{R}^{n\times n}$ *are symmetric then* $|\Lambda(A, B)| \leq 1$.

The following simple corollary provides in some cases a powerful tool of proving that the generalized eigenproblem is not solvable:

Corollary 9.1.8 *If* $A, B \in \mathbb{R}^{n\times n}$ *and* (A^T, B^T) *has more than one generalized eigenvalue then* (A, B) *is not solvable.*

9.2 Easily Solvable Special Cases

9.2.1 Essentially the Eigenproblem

If either A or B is a generalized permutation matrix then (9.1) is easily solvable. If (say) B is a generalized permutation matrix then B has the inverse B^{-1} and after multiplying (9.1) by B^{-1} the GEP is transformed to the eigenproblem. Unfortunately, since in max-algebra matrices other than generalized permutation matrices do not have an inverse (see Theorem 1.1.3), this case is fairly limited.

9.2.2 When A and B Have a Common Eigenvector

Proposition 9.2.1 [70] *A common eigenvector of* A *and* B *is a generalized eigenvector for* A *and* B; *more precisely, if* $A, B \in \overline{\mathbb{R}}^{n\times n}$, $\lambda \otimes \mu^{-1} \in \overline{\mathbb{R}}$, *then*

$$V(A, \lambda) \cap V(B, \mu) \subseteq V(A, B, \lambda \otimes \mu^{-1}).$$

Proof If $x \in V(A, \lambda) \cap V(B, \mu)$ and $\lambda > \varepsilon$ then $\mu \in \mathbb{R}$ and

$$A \otimes x = \lambda \otimes x = \lambda \otimes \mu^{-1} \otimes B \otimes x.$$

If $\lambda = \varepsilon$ then $\lambda \otimes \mu^{-1} = \varepsilon$ and the statement trivially follows. □

An example of pairs of matrices having a common eigenvector are commuting matrices (Theorem 4.7.2). Hence we have:

Theorem 9.2.2 *If* $A, B \in \mathbb{R}^{n\times n}$ *and* $A \otimes B = B \otimes A$ *then both* (A, B) *and* (A^T, B^T) *are solvable, with identical, unique generalized eigenvalue.*

Proof A and B have a common eigenvector corresponding to finite eigenvalues by Theorem 4.7.2 and so by Proposition 9.2.1 (A, B) is solvable. At the same time A^T and B^T are also commuting and by a repeated argument we have that (A^T, B^T) is solvable. The equality of all generalized eigenvalues now follows by Theorem 9.1.6. □

9.2.3 When One of A, B Is a Right-multiple of the Other

Theorem 9.2.3 [70] *If one of $A, B \in \overline{\mathbb{R}}^{m \times n}$ is a right-multiple of the other then (A, B) is solvable.*

Proof Suppose e.g. $A = B \otimes P$, where $P \in \overline{\mathbb{R}}^{n \times n}$. Let $\lambda \in \Lambda(P)$ and $x \in V(P, \lambda)$, $x \neq \varepsilon$. Then

$$A \otimes x = B \otimes P \otimes x = B \otimes (\lambda \otimes x) = \lambda \otimes B \otimes x. \qquad \square$$

Example 9.2.4 Suppose

$$A = \begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ 3 & 1 \end{pmatrix}, \qquad P = \begin{pmatrix} 4 & 6 \\ -2 & 0 \end{pmatrix}.$$

Then $\lambda(P) = 4$,

$$\Gamma(\lambda^{-1} \otimes P) = \begin{pmatrix} 0 & 2 \\ -6 & -4 \end{pmatrix}$$

and

$$x = \begin{pmatrix} 0 \\ -6 \end{pmatrix}, \qquad A \otimes x = \begin{pmatrix} 4 \\ 7 \end{pmatrix}, \qquad B \otimes x = \begin{pmatrix} 0 \\ 3 \end{pmatrix}.$$

We can also prove a sufficient condition for λ to attain the upper bound in (9.2) when (say) A is a right-multiple of B and $A, B \in \mathbb{R}^{m \times n}$. Recall that $C = (c_{ij})$ is the matrix $(a_{ij} \otimes b_{ij}^{-1})$, $D = (d_{ij}) = (b_{ij} \otimes a_{ij}^{-1})$ and let us denote

$$L = \max_i \min_j c_{ij}$$

and

$$U = \min_i \max_j c_{ij}.$$

It follows from the proof of Theorem 9.2.3 and from Theorem 9.1.1 that $\lambda(P) \in [L, U]$ for every P satisfying $A = B \otimes P$. If $A = B \otimes P$ then we have:

$$A = B \otimes (B^* \otimes' A).$$

Let us denote $B^* \otimes' A$ by $\overline{P} = (\overline{p}_{ij})$ and $\overline{\lambda} = \lambda(\overline{P})$; thus $L \le \overline{\lambda} \le U$. The following technical lemma will help us to characterize in Theorem 9.2.6 when the upper bound U is attained.

Lemma 9.2.5 *If $A, B \in \mathbb{R}^{m \times n}$ and $L' = \max_j \min_i c_{ij}$ then $L' \le \overline{\lambda}$.*

Proof

$$\overline{\lambda} = \lambda(\overline{P}) \geq \max_i \overline{p}_{ii} = \max_i \min_j (b^*_{ij} \otimes a_{ji})$$
$$= \max_i \min_j (a_{ji} \otimes b^{-1}_{ji}) = \max_i \min_j c_{ji} = \max_j \min_i c_{ij} = L'. \qquad \square$$

Theorem 9.2.6 [70] *If $A, B \in \mathbb{R}^{m \times n}$, D has a saddle point and there is a matrix P such that $A = B \otimes P$ then $\overline{\lambda} = U$ where $\overline{\lambda} = \lambda(\overline{P}) = \lambda(B^* \otimes' A)$.*

Proof $D = (d_{ij})$ has a saddle point means

$$\max_i \min_j d_{ij} = \min_j \max_i d_{ij}.$$

Therefore the inverses of both sides are equal:

$$U = \min_i \max_j c_{ij} = \max_j \min_i c_{ij} = L'.$$

Hence by Lemma 9.2.5: $L' = \overline{\lambda} = U$. $\square$

The following dual statement is proved in a dual way:

Theorem 9.2.7 [70] *Let $A, B \in \mathbb{R}^{m \times n}$. If there is a matrix P such that $A = B \otimes' P$ and C has a saddle point then $\overline{\lambda}' = L$ where $\overline{\lambda}' = \lambda'(\overline{P}) = \lambda'(B^* \otimes' A)$.*

Even if one of A, B is a right-multiple of the other, the eigenvalue may not be unique as the following example shows.

Example 9.2.8 With A, B as in Example 9.2.4, we find for the principal solution matrix $\overline{P}$:

$$\overline{P} = \begin{pmatrix} 4 & 6 \\ 3 & 5 \end{pmatrix}, \qquad \lambda(\overline{P}) = 5,$$

$$\Gamma(\lambda^{-1} \otimes \overline{P}) = \begin{pmatrix} -1 & 1 \\ -2 & 0 \end{pmatrix},$$

$$A \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 9 \end{pmatrix}$$

and

$$B \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}.$$

Hence for the same A, B we find two solutions to (9.1), with different values of λ.

9.3 Narrowing the Search for Generalized Eigenvalues

9.3.1 Regularization

In the absence of any method, exact or approximate, for finding generalized eigenvalues for a general pair of matrices, we concentrate now on narrowing the set containing all generalized eigenvalues (if there are any) for finite A and B.

Let $C = (c_{ij}), D = (d_{ij}) \in \mathbb{R}^{m\times n}$. The system

$$C \otimes x = D \otimes x \tag{9.3}$$

is called *regular* if

$$c_{ij} \neq d_{ij}$$

for all i, j. The aim of the method we will present in this section is to identify as closely as possible the set of generalized eigenvalues for which (9.1) is regular.

Let us first briefly discuss the values of λ for which this requirement is not satisfied. There are at most mn such values of λ. We will call these values *extreme* and the set of extreme values will be denoted by L. More precisely, for $A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}^{m\times n}$ we set

$$L = \{\lambda \in \mathbb{R}; a_{ij} = \lambda \otimes b_{ij} \text{ for some } i, j\}.$$

Note that the elements of L are entries of the matrix $A - B$. Obviously,

$$|L| \leq mn \tag{9.4}$$

and (9.1) is regular for all $\lambda \in \mathbb{R} - L$. Recall that solvability of (9.1) can be checked for each fixed and in particular extreme value of λ using, say, the Alternating Method.

Remark 9.3.1 The upper bound in (9.4) can slightly be improved: If for some i we have $c_{ij} > d_{ij}$ for all j then (9.3) has no nontrivial solution. Therefore (9.1) has no nontrivial solution if λ is too big or too small, in particular for $\lambda > \max L$ and $\lambda < \min L$. These two conditions may be slightly refined as follows: $a_{ij} > \lambda \otimes b_{ij}$ for all j or $a_{ij} < \lambda \otimes b_{ij}$ for all j must not hold for any $i = 1, \dots, m$. Hence (9.1) has no nontrivial solution for $\lambda < \lambda'$ and $\lambda > \lambda''$ where λ' is the mth smallest value in L and λ'' is the mth greatest value in L (both considered with multiplicities). So actually only at most $mn - 2m$ extreme values of λ need to be checked individually by the Alternating Method.

Let us denote the extreme values described in Remark 9.3.1 by $\lambda_1, \dots, \lambda_t$, where $\lambda_1 < \cdots < \lambda_t$ and $t \leq mn - 2m$. All these values can easily be found among the entries of $A - B$ and checked individually for being generalized eigenvalues. Thus we may now concentrate on the real numbers in open intervals $(\lambda_j, \lambda_{j+1})$, $j = 1, \dots, t - 1$. We will call these intervals *regular* and we will also call every real number *regular* if it belongs to a regular interval. It follows that there are at most $mn - 2m - 1$ regular intervals to be considered. In the rest of this section we assume that one such interval, say J, has been fixed, and we consider (9.1) only for $\lambda \in J$.

9.3.2 A Necessary Condition for Generalized Eigenvalues

Symmetrized semirings have been introduced in Sect. 7.5 and they have been used to derive necessary conditions for the existence of a nontrivial solution to two-sided systems. We now reformulate this to obtain a necessary condition for generalized eigenvalues.

Recall first that $\mathbb{S} = \overline{\mathbb{R}} \times \overline{\mathbb{R}}$ and the operations $\oplus$ and $\otimes$ are extended to $\mathbb{S}$ as follows:

$$(a, a') \oplus (b, b') = (a \oplus b, a' \oplus b'),$$
$$(a, a') \otimes (b, b') = (a \otimes b \oplus a' \otimes b', a \otimes b' \oplus a' \otimes b).$$

Also, $\ominus(a, a') = (a', a)$ and (a, a') is called balanced if $a = a'$. The determinant of $A = (a_{ij}) \in \mathbb{S}^{n\times n}$ has been defined as

$$\det(A) = \sum_{\sigma \in P_n}^{\oplus} \left(\operatorname{sgn}(\sigma) \otimes \prod_{i\in N}^{\otimes} a_{i,\sigma(i)} \right),$$

and we know that

$$|\det(A)| = \operatorname{maper}|A|,$$

see Proposition 7.5.6.

The next statement follows from Theorem 7.5.4 and Corollary 7.5.5. We denote here and in the rest of this section

$$C(\lambda) = A \ominus \lambda \otimes B.$$

Corollary 9.3.2 *Let $A, B \in \mathbb{R}^{n\times n}$ and $\lambda \in \mathbb{R}$. Then a necessary condition that the system $A \otimes x = \lambda \otimes B \otimes x$ have a nontrivial solution is that $C(\lambda)$ has balanced determinant.*

The idea of narrowing the search for the eigenvalues is based on Corollary 9.3.2: We show how to find *all* λ for which $C(\lambda)$ has balanced determinant. It turns out that this can be done using a polynomial number of operations in terms of n. This method may in some cases identify all eigenvalues, see Examples 9.3.7 and 9.3.8. In general however, it finds only a superset of generalized eigenvalues, see Example 9.3.9.

If λ is regular then $C = A \ominus \lambda \otimes B$ has no balanced entry. The following statement is a reformulation of Theorem 7.5.7 (note that the matrix $\widetilde{C}$ has been defined just before that theorem):

Corollary 9.3.3 *Let $A, B \in \mathbb{R}^{n\times n}$, λ be regular. Then $C(\lambda)$ has balanced determinant if and only if $\widetilde{C(\lambda)}$ is not SNS.*

The problem of checking whether a $(0, 1, -1)$ matrix is SNS or not is equivalent to the even cycle problem in digraphs [18] and therefore polynomially solvable (Remark 1.6.45). Therefore the necessary solvability condition in Corollary 9.3.3 can be checked in polynomial time for any fixed regular value of λ. This will be used later in Sect. 9.3.4. However, $C(\lambda)$ may have balanced determinant for a continuum of values of λ (see Example 9.3.8) and therefore we also need a tool which enables us to make the same decision for an interval. This tool will be presented in Sect. 9.3.4. As a preparation we first show in Sect. 9.3.3 how to find $\text{maper}|C(\lambda)|$ as a function of $\lambda \in J$.

9.3.3 *Finding* maper$|C(\lambda)|$

In this subsection we show how to efficiently find the function

$$f(\lambda) = \text{maper}|C(\lambda)|.$$

This will be used in the next section to produce a method for finding all regular values of $\lambda \in J$ for which $\widetilde{C(\lambda)}$ is not SNS.

Recall first that $|C(\lambda)| = (a_{ij} \oplus \lambda \otimes b_{ij}) = (c_{ij}(\lambda))$ and for every $\lambda \in J$ we have

$$a_{ij} \neq \lambda \otimes b_{ij}$$

for all $i, j \in N$. Therefore for every $\lambda \in J$ and for all $i, j \in N$ the entry $c_{ij}(\lambda) = a_{ij} \oplus \lambda \otimes b_{ij}$ is equal to exactly one of a_{ij} and $\lambda \otimes b_{ij}$. Observe that $f(\lambda) = \text{maper}|C(\lambda)|$ is the maximum of $n!$ terms. Each term is a $\otimes$ product of n entries $c_{ij}(\lambda)$, hence of the form $b \otimes \lambda^k$, where $b \in \mathbb{R}$ and k is a natural number between 0 and n. Since $b \otimes \lambda^k$ in conventional notation is simply $k\lambda + b$, we deduce that $f(\lambda)$ is the maximum of a finite number of linear functions and therefore a piecewise linear convex function. Note that the slopes of all linear pieces of $f(\lambda)$ are natural numbers between 0 and n. Recall that $f(\lambda)$ for any particular λ can easily be found by solving the assignment problem for $|C(\lambda)|$. It follows that all linear pieces can therefore efficiently be identified. We now describe one possible way of finding these linear functions: Assume for a while that the linear pieces of smallest and greatest slope are known, let us denote them $f_l(\lambda) = a_l \otimes \lambda^l$ and $f_h(\lambda) = a_h \otimes \lambda^h$, respectively. If $l = h$ then there is nothing to do, so assume $l \neq h$. We start by finding the intersection point of f_l and f_h, that is, say, λ_1 satisfying $f_l(\lambda_1) = f_h(\lambda_1)$. Calculate $f(\lambda_1) = \text{maper}|C(\lambda_1)|$. If $f(\lambda_1) = f_l(\lambda_1) = f_h(\lambda_1)$ then there is no linear piece other than f_l and f_h. Otherwise $f(\lambda_1) > f_l(\lambda_1) = f_h(\lambda_1)$. Let r be the number of λ terms appearing in an optimal permutation (if there are several optimal permutations with various numbers of λ appearances then take any). Since r is the slope of the linear piece we have $l < r < h$. Then $a_r = f(\lambda_1) - r\lambda_1$ and $f_r(\lambda) = a_r \otimes \lambda^r$. This term is a new linear piece and we then repeat this procedure with f_l and f_r and f_r and f_h, and so on. At every step a new linear piece is discovered unless all linear pieces have already been found. Hence the number of iterations is at most $n-1$.

For finding f_l and f_h it will be convenient to use the *independent ones problem* (IOP) for $0-1$ square matrices:

Given a $0-1$ *matrix* $M=(m_{ij})\in\mathbb{R}^{n\times n}$, *find the greatest number of ones in M so that no two are from the same row or column or, equivalently, so that there is a* $\pi\in P_n$ *selecting all these ones.*

Clearly, IOP is a special case of the assignment problem, and therefore easily solvable. Note that in combinatorial terminology IOP is known as the maximum cardinality bipartite matching problem solvable in $O(n^{2.5})$ time [22]. In general we say that a set of positions in a matrix are independent if no two of them belong to the same row or column.

Now we discuss how to find f_l and f_h. The values of l and h are obviously the smallest and biggest number of independent entries in $|C(\lambda)|$ containing λ and these can be found by solving the corresponding IOP. For h this problem can be described by the matrix $M=(m_{ij})$ with $m_{ij}=1$ when $|c_{ij}(\lambda)|=\lambda\otimes b_{ij}$ and 0 otherwise and for l by $E-M$, where E is the all-one matrix.

Now we show how to find a_l and a_h. Let $d_{ij}=b_{ij}$ if $c_{ij}(\lambda)=\lambda\otimes b_{ij}$ and $d_{ij}=a_{ij}$ if $c_{ij}(\lambda)=a_{ij}$ (note that by regularity of λ only one of these two possibilities occurs for $\lambda\in J$). For finding a_l and a_h we need to determine permutations π and σ that maximize $\sum_{i\in N}d_{i,\pi(i)}$ and $\sum_{i\in N}d_{i,\sigma(i)}$ and select l and h entries containing λ, respectively. To achieve this we interpret the two above mentioned IOPs as assignment problems and describe their solution sets using matrices M_h and M_l obtained by the Hungarian method (that is, nonpositive matrices whose max-algebraic permanent is zero). It remains then to replace all entries in $D=(d_{ij})$ corresponding to nonzero entries in M_h and M_l by $-\infty$ and solve the assignment problem for the obtained matrices.

9.3.4 Narrowing the Search

In this subsection we show how to efficiently find the set of all regular values of λ for which $\det(C(\lambda))$ is balanced. This set will be denoted by S. We use essentially the fact that the decision whether $\det(C(\lambda))$ is balanced can be made efficiently for any individual value of λ (Corollary 9.3.3). The following will be useful:

Lemma 9.3.4 *Let* $f(x),g(x),h(x)$ *be piecewise linear convex functions on* $\mathbb{R}$, $f(x)=g(x)\oplus h(x)$ *for all* $x\in\mathbb{R}$. *Suppose* $a,b\in\mathbb{R}$ *are such that* f *is linear on* $[a,b]$. *If* $g(x)=h(x)$ *for at least one* $x\in(a,b)$ *then* $g(x)=h(x)$ *for all* $x\in[a,b]$.

Proof Suppose $g(x_0)=h(x_0),x_0\in(a,b)$. Hence $g(x_0)=h(x_0)=f(x_0)$. If $g(x)<f(x)$ for an $x\in[a,b]$, without loss of generality for $x\in[a,x_0)$, then by convexity of g and linearity of f we have that $g(x)>f(x)$ for all $x\in(x_0,b)$, a contradiction. Therefore $g(x) = f(x)$ for all $x \in [a, b]$ and similarly $h(x) = f(x)$ for all $x \in [a, b]$. □

Recall that as before J is a regular interval. Let us denote

$$\det(C(\lambda)) = \left(d^+(C(\lambda)), d^-(C(\lambda))\right),$$

or just $(d^+(\lambda), d^-(\lambda))$. Then $C(\lambda)$ for $\lambda \in J$ has balanced determinant if and only if

$$d^+(\lambda) = d^-(\lambda). \tag{9.5}$$

It follows from the results of the previous section that the piecewise linear convex function

$$|\det(C(\lambda))| = d^+(\lambda) \oplus d^-(\lambda) = \text{maper}|C(\lambda)|$$

can efficiently be found. By the same argument as for maper$|C(\lambda)|$ we see that both $d^+(\lambda)$ and $d^-(\lambda)$ are max-algebraic polynomials in λ (hence piecewise linear and convex functions) containing at most $n + 1$ powers of λ between 0 and n. No method other than exhaustive search (requiring $n!$ permutation evaluations) seems to be known for finding $d^+(\lambda)$ and $d^-(\lambda)$ separately for any particular λ [29]; however, for a fixed $\lambda \in \mathbb{R} - L$ by Corollary 9.3.3 we can decide in polynomial time whether $d^+(\lambda) = d^-(\lambda)$ or not. Since $d^+(\lambda) \oplus d^-(\lambda) = \text{maper}|C(\lambda)|$ then if maper$|C(\lambda)|$ is known, using Lemma 9.3.4 we can easily find *all* values of $\lambda \in J$ satisfying $d^+(\lambda) = d^-(\lambda)$ by checking this equality for any point strictly between any two consecutive breakpoints and for the breakpoints of maper$|C(\lambda)|$. We summarize these observations in the following:

Theorem 9.3.5 *If the set $S = \{\lambda \in J; d^+(\lambda) = d^-(\lambda)\}$ is nonempty then it consists of some of the breakpoints of* maper$|C(\lambda)|$ *and a number (possibly none) of closed intervals whose endpoints are pairs of adjacent breakpoints of* maper$|C(\lambda)|$. *All these can be identified in $O(n^3)$ time.*

Proof The statement is essentially proved by Lemma 9.3.4. We only need to add that each interval whose endpoints are adjacent breakpoints of maper$|C(\lambda)|$ can be decided by checking $d^+(\lambda) = d^-(\lambda)$ for one (arbitrary) internal point of the interval and that the number of breakpoints is at most n and therefore the number of intervals is at most $n - 1$. The equality $d^+(\lambda) = d^-(\lambda)$ for a fixed λ can be decided in polynomial time by Theorem 9.3.3. □

We summarize our work in the following procedure for finding all regular values of λ for which $\det(C(\lambda))$ is balanced:

Algorithm 9.3.6 NARROWING THE EIGENVALUE SEARCH
Input: $A, B \in \mathbb{R}^{n \times n}$ and a regular interval J.
Output: The set $S = \{\lambda \in J; d^+(\lambda) = d^-(\lambda)\}$.

1. $S := \emptyset$.
2. $C(\lambda) := A \ominus \lambda \otimes B$.
3. Find $f(\lambda) = \text{maper}|C(\lambda)|$ as a function of λ, that is, find all breakpoints and linear pieces of $f(\lambda)$.
4. For every breakpoint λ_0 of $f(\lambda)$ do: If $\widetilde{C(\lambda_0)}$ is not SNS then $S := S \cup \{\lambda_0\}$.
5. For any two consecutive breakpoints a, b and arbitrarily taken $\lambda_0 \in (a, b)$ do: If $\widetilde{C(\lambda_0)}$ is not SNS then $S := S \cup (a, b)$.

9.3.5 Examples

In the first two examples below we demonstrate that the described method for narrowing the search for eigenvalues may actually find all eigenvalues. Note that in these examples all matrices are of small sizes and therefore the functions $d^+(\lambda)$ and $d^-(\lambda)$ are explicitly evaluated; however, for bigger matrices this would not be practical and the method described in Sect. 9.3.4 would be used as an efficient tool for finding all regular values of λ for which $d^+(\lambda) = d^-(\lambda)$.

The third example illustrates the situation when the algorithm narrows the feasible interval containing the eigenvalues but a significant proportion of the final interval still consists of real numbers that are not eigenvalues.

Example 9.3.7 Let

$$A = \begin{pmatrix} 3 & 8 & 2 \\ 7 & 1 & 4 \\ 0 & 6 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} 4 & 4 & 3 \\ 2 & 3 & 4 \\ 3 & 2 & 1 \end{pmatrix}.$$

Then

$$A - B = \begin{pmatrix} -1 & 4 & -1 \\ 5 & -2 & 0 \\ -3 & 4 & 2 \end{pmatrix}$$

and $L = \{-3, -2, -1, 0, 2, 4, 5\}$. For $\lambda < -1$ all terms on the RHS of the first equation in $A \otimes x = \lambda \otimes B \otimes x$ are strictly less than the corresponding terms on the left and therefore there is no nontrivial solution to $A \otimes x = \lambda \otimes B \otimes x$. Similarly, for $\lambda > 4$ all these terms are greater than their counterparts on the left. Hence we only need to investigate regular intervals $(-1, 0)$, $(0, 2)$ and $(2, 4)$ and extreme points $-1, 0, 2, 4$.

For $\lambda \in (-1, 0)$ we have

$$|C(\lambda)| = \begin{pmatrix} 4+\lambda & 8 & 3+\lambda \\ 7 & 3+\lambda & 4 \\ 3+\lambda & 6 & 3 \end{pmatrix},$$

$$\begin{aligned} d^{+}(\lambda) &= \max(10+2\lambda, 14+\lambda, 9+3\lambda), \\ d^{-}(\lambda) &= \max(16+\lambda, 15+\lambda, 18), \\ \text{maper}|C(\lambda)| &= 18. \end{aligned}$$

Since $d^{+}(\lambda) \neq d^{-}(\lambda)$ for $\lambda \in (-1,0)$, there are no eigenvalues in this interval.

For $\lambda \in (0,2)$ we have

$$|C(\lambda)| = \begin{pmatrix} 4+\lambda & 8 & 3+\lambda \\ 7 & 3+\lambda & 4+\lambda \\ 3+\lambda & 6 & 3 \end{pmatrix},$$

$$\begin{aligned} d^{+}(\lambda) &= \max(10+2\lambda, 15+2\lambda, 9+3\lambda), \\ d^{-}(\lambda) &= \max(16+\lambda, 14+2\lambda, 18), \\ \text{maper}|C(\lambda)| &= \max(18, 16+\lambda, 15+2\lambda, 9+3\lambda). \end{aligned}$$

For $\lambda \in (0,2)$ there is only one breakpoint for maper$|C(\lambda)|$ at $\lambda_0 = 3/2$. Since $d^{+}(\lambda) = d^{-}(\lambda)$ for $\lambda = \lambda_0$, this value is the only candidate for an eigenvalue in $(0,2)$. It is not difficult to verify that $x = (2,0,3.5)^T$ is a corresponding eigenvector.

For $\lambda \in (2,4)$ we have

$$|C(\lambda)| = \begin{pmatrix} 4+\lambda & 8 & 3+\lambda \\ 7 & 3+\lambda & 4+\lambda \\ 3+\lambda & 6 & 1+\lambda \end{pmatrix},$$

$$\begin{aligned} d^{+}(\lambda) &= \max(15+2\lambda, 16+\lambda, 9+3\lambda), \\ d^{-}(\lambda) &= \max(16+\lambda, 14+2\lambda, 8+3\lambda), \\ \text{maper}|C(\lambda)| &= 15+2\lambda. \end{aligned}$$

Since $d^{+}(\lambda) \neq d^{-}(\lambda)$ for $\lambda \in (2,4)$, there are no eigenvalues in this interval.

Let us consider the extreme point $\lambda = 0$: In this small example we solve the system $A \otimes x = B \otimes x$ by direct analysis but note that in general the Alternating Method would be used. By the cancellation law (Lemma 7.4.1) the two-sided system $A \otimes x = B \otimes x$ is equivalent to the one with

$$A = \begin{pmatrix} \varepsilon & 8 & \varepsilon \\ 7 & \varepsilon & 4 \\ \varepsilon & 6 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} 4 & \varepsilon & 3 \\ \varepsilon & 3 & 4 \\ 3 & \varepsilon & \varepsilon \end{pmatrix}.$$

Here from the first equation either $x_2 = -4 + x_1$ or $x_2 = -5 + x_3$. In the first case the third equation yields $\max(2+x_1, 3+x_3) = 3+x_1$, thus $x_1 = x_3$. By substituting into the second equation then $x_1 = -4 + x_2$, a contradiction. In the second case the third equation yields again $x_1 = x_3$, which implies a contradiction in the same way. Hence $\lambda = 0$ is not an eigenvalue and a similar analysis would show that neither are the remaining three extreme values.

We conclude that $\Lambda(A, B) = \{3/2\}$.

Example 9.3.8 Let $A = \begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 3 & 1 \end{pmatrix}$. It is easily seen that $J = (4, 5)$ is the unique regular interval. For $\lambda \in (4, 5)$ we have

$$|C(\lambda)| = \begin{pmatrix} \lambda & 6 \\ 3 + \lambda & 9 \end{pmatrix}$$

and

$$\text{maper}|C(\lambda)| = \max(9 + \lambda, 9 + \lambda) = 9 + \lambda = d^-(\lambda) = d^+(\lambda).$$

Hence every $\lambda \in J$ satisfies the necessary condition. In fact all these values are eigenvalues as $x = (6, \lambda)^T$ is a corresponding eigenvector (for every $\lambda \in J$). This vector is also an eigenvector for $\lambda \in \{4, 5\}$ and thus $\Lambda(A, B) = [4, 5]$.

Example 9.3.9 [132] Let

$$A = \begin{pmatrix} 0 & 1/2 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & -2 & -2 \\ -2 & 0 & 0 \\ 0 & -2 & -2 \end{pmatrix}.$$

Consider only the regular interval $J = (0, 2)$. For $\lambda \in J$ we have

$$|C(\lambda)| = \begin{pmatrix} \lambda & 1/2 & 1 \\ 1 & \lambda & \lambda \\ \lambda & 0 & 1 \end{pmatrix},$$

$$d^+(\lambda) = \max(1 + 2\lambda, 2),$$

and

$$d^-(\lambda) = \max(1 + 2\lambda, 5/2).$$

We deduce that $d^-(\lambda) = d^+(\lambda)$ if and only if $\lambda \geq 3/4$. Hence the algorithm returns $S = [3/4, 2]$. However, there are no eigenvalues in $(1, 2)$. To see this, realize that for $\lambda \in J$ the system (9.1) simplifies using the cancellation rules and then by setting $x_1 = 0$ to:

$$(1/2) \otimes x_2 \oplus 1 \otimes x_3 = \lambda,$$

$$1 = \lambda \otimes x_2 \oplus \lambda \otimes x_3,$$

$$x_2 \oplus 1 \otimes x_3 = \lambda.$$

The second equation is equivalent to $x_2 \oplus x_3 = 1 - \lambda$. Hence, if $\lambda > 1$ and $x = (0, x_2, x_3)^T$ is a solution then both x_2 and x_3 are negative, thus $x_2 \oplus 1 \otimes x_3 < 1 < \lambda$, a contradiction. Note that all $\lambda \in [3/4, 1]$ are eigenvalues since for such λ the vector $(0, 1 - \lambda, \lambda - 1)^T$ is a solution to (9.1).

9.4 Exercises

Exercise 9.4.1 Use Theorem 9.3.3 to give an alternative proof that $\lambda(A)$ is the unique eigenvalue for any irreducible matrix A.

Exercise 9.4.2 Show that the generalized eigenproblem has no nontrivial solution for the matrices

$$A = \begin{pmatrix} 3 & 5 & 4 \\ 7 & 9 & 8 \end{pmatrix}, \qquad B = \begin{pmatrix} 7 & 4 & 1 \\ 3 & 5 & 2 \end{pmatrix}.$$

[The feasible interval is empty]

Exercise 9.4.3 Find all extreme values in the feasible interval for the generalized eigenproblem with matrices

$$A = \begin{pmatrix} 3 & 5 & 4 \\ 0 & 3 & 7 \end{pmatrix}, \qquad B = \begin{pmatrix} 7 & 4 & 1 \\ 3 & 5 & 2 \end{pmatrix}.$$

[$(-3, -2, 1, 3)^T$]

Exercise 9.4.4 Prove the following: Let $A, B \in \mathbb{R}^{n \times n}$. Then (A, B) is solvable if and only if there exist P, Q such that $A \otimes P = B \otimes Q$ and (P, Q) is solvable.

Exercise 9.4.5 Prove or disprove: If $A, B \in \mathbb{R}^{n \times n}$ and $A = B \otimes Q$ then $\lambda(B)$ is the greatest corner of the maxpolynomial maper$(A \oplus \lambda \otimes B)$. [false]

Exercise 9.4.6 Find all generalized eigenvalues if

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 4 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}.$$

[0, 1, 2]