Chapter 4
Eigenvalues and Eigenvectors

This chapter provides an account of the max-algebraic eigenvalue-eigenvector theory for square matrices over $\overline{\mathbb{R}}$. The algorithms presented and proved here enable us to find all eigenvalues and bases of all eigenspaces of an $n \times n$ matrix in $O(n^3)$ time. These results are of fundamental importance for solving the reachability problems in Chap. 8 and elsewhere.

We start with definitions and basic properties of the eigenproblem, then continue by proving one of the most important results in max-algebra, namely that for every matrix the maximum cycle mean is the greatest eigenvalue, which motivates us to call it the principal eigenvalue. We then show how to describe the corresponding (principal) eigenspace. Next we present the Spectral Theorem, that enables us to find all eigenvalues of a matrix. It also makes it possible to characterize matrices with finite eigenvectors. Finally, we discuss how to efficiently describe all eigenvectors of a matrix.

4.1 The Eigenproblem: Basic Properties

Given $A \in \overline{\mathbb{R}}^{n\times n}$, the task of finding the vectors $x \in \overline{\mathbb{R}}^{n}$, $x \neq \varepsilon$ (*eigenvectors*) and scalars $\lambda \in \overline{\mathbb{R}}$ (*eigenvalues*) satisfying

$$A \otimes x = \lambda \otimes x \tag{4.1}$$

is called the (max-algebraic) *eigenproblem*. For some applications it may be sufficient to find one eigenvalue-eigenvector pair; however, in this chapter we show that all eigenvalues can be found and all eigenvectors can efficiently be described for any matrix.

The eigenproblem is of key importance in max-algebra. It has been studied since the 1960's [58] in connection with the analysis of the steady-state behavior of production systems (see Sect. 1.3.3). Full solution of the eigenproblem in the case of irreducible matrices has been presented in [60] and [98], see also [11, 61] and [144]. A general spectral theorem for reducible matrices has appeared in [84] and [12], and

P. Butkovič, *Max-linear Systems: Theory and Algorithms,*
Springer Monographs in Mathematics 151,
DOI 10.1007/978-1-84996-299-5_4,

partly in [48]. An application of the max-algebraic eigenproblem to the conventional eigenproblem and in music theory can be found in [79].

For $A \in \overline{\mathbb{R}}^{n\times n}$ and $\lambda \in \overline{\mathbb{R}}$ we denote by $V(A,\lambda)$ the set consisting of ε and all eigenvectors of A corresponding to λ, and by $\Lambda(A)$ the set of all eigenvalues of A, that is

$$V(A,\lambda) = \left\{x \in \overline{\mathbb{R}}^n; A \otimes x = \lambda \otimes x\right\}$$

and

$$\Lambda(A) = \left\{\lambda \in \overline{\mathbb{R}}; V(A,\lambda) \neq \{\varepsilon\}\right\}.$$

We also denote by $V(A)$ the set consisting of ε and all eigenvectors of A, that is

$$V(A) = \bigcup_{\lambda\in\Lambda(A)} V(A,\lambda).$$

Finite eigenvectors are of special significance for both theory and applications and we denote:

$$V^+(A,\lambda) = V(A,\lambda) \cap \mathbb{R}^n$$

and

$$V^+(A) = V(A) \cap \mathbb{R}^n.$$

We start by presenting basic properties of eigenvalues and eigenvectors. The set $\{\alpha \otimes x; x \in S\}$ for $\alpha \in \overline{\mathbb{R}}$ and $S \subseteq \overline{\mathbb{R}}^n$ will be denoted $\alpha \otimes S$.

Proposition 4.1.1 *Let $A, B \in \overline{\mathbb{R}}^{n\times n}$, $\alpha \in \mathbb{R}$, $\lambda, \mu \in \overline{\mathbb{R}}$ and $x, y \in \overline{\mathbb{R}}^n$. Then*

(a) $V(\alpha \otimes A) = V(A)$,
(b) $\Lambda(\alpha \otimes A) = \alpha \otimes \Lambda(A)$,
(c) $V(A,\lambda) \cap V(B,\mu) \subseteq V(A \oplus B, \lambda \oplus \mu)$,
(d) $V(A,\lambda) \cap V(B,\mu) \subseteq V(A \otimes B, \lambda \otimes \mu)$,
(e) $V(A,\lambda) \subseteq V(A^k, \lambda^k)$ *for all integers* $k \geq 0$,
(f) $x \in V(A,\lambda) \Longrightarrow \alpha \otimes x \in V(A,\lambda)$,
(g) $x, y \in V(A,\lambda) \Longrightarrow x \oplus y \in V(A,\lambda)$.

Proof If $A \otimes x = \lambda \otimes x$ then $(\alpha \otimes A) \otimes x = (\alpha \otimes \lambda) \otimes x$ which proves (a) and (b).

If $A \otimes x = \lambda \otimes x$ and $B \otimes x = \mu \otimes x$ then

$$\begin{aligned}(A \oplus B) \otimes x &= A \otimes x \oplus B \otimes x \\ &= \lambda \otimes x \oplus \mu \otimes x \\ &= (\lambda \oplus \mu) \otimes x\end{aligned}$$

and

$$(A \otimes B) \otimes x = A \otimes (B \otimes x)$$

$$= A \otimes \mu \otimes x$$
$$= \mu \otimes A \otimes x$$
$$= \mu \otimes \lambda \otimes x$$

which prove (c) and (d). Statement (e) follows by a repeated use of (d) and setting $A = B$.

If $A \otimes x = \lambda \otimes x$ then $A \otimes (\alpha \otimes x) = \lambda \otimes (\alpha \otimes x)$ which proves (f).

Finally, if $A \otimes x = \lambda \otimes x$ and $A \otimes y = \lambda \otimes y$ then

$$\begin{aligned} A \otimes (x \oplus y) &= A \otimes x \oplus A \otimes y \\ &= \lambda \otimes (x \oplus y) \end{aligned}$$

and (g) follows. □

It follows from Proposition 4.1.1 that $V(A, \lambda)$ is a subspace for every $\lambda \in \Lambda(A)$; it will be called an *eigenspace* (corresponding to the eigenvalue λ).

Remark 4.1.2 By (c) and (e) of Proposition 4.1.1 we have: If $A \in \overline{\mathbb{R}}^{n \times n}$ and $\varepsilon < \lambda(A) \leq 0$ then $V(A) \subseteq V(\Gamma(A))$. In particular,

$$V(A_\lambda, 0) \subseteq V(\Gamma(A_\lambda), 0).$$

The next statement summarizes spectral properties that are unaffected by a simultaneous permutation of the rows and columns.

Proposition 4.1.3 *Let $A, B \in \overline{\mathbb{R}}^{n \times n}$ and $B = P^{-1} \otimes A \otimes P$, where P is a permutation matrix. Then*

(a) *A is irreducible if and only if B is irreducible.*
(b) *The sets of cycle lengths in D_A and D_B are equal.*
(c) *A and B have the same eigenvalues.*
(d) *There is a bijection between $V(A)$ and $V(B)$ described by*:

$$V(B) = \left\{ P^{-1} \otimes x; x \in V(A) \right\}.$$

Proof To prove (a) and (b) note that B is obtained from A by simultaneous permutations of the rows and columns. Hence D_B differs from D_A by the numbering of the nodes only and the statements follow. For (c) and (d) we observe that $B \otimes z = \lambda \otimes z$ if and only if $A \otimes P \otimes z = \lambda \otimes P \otimes z$, that is, $z \in V(B)$ if and only if $z = P^{-1} \otimes x$ for some $x \in V(A)$. □

Remark 4.1.4 The eigenvectors as defined by (4.1) are also called right eigenvectors in contrast to left eigenvectors that are defined by the equation

$$y^T \otimes A = y^T \otimes \lambda.$$

By the rules for transposition we have that y is a left eigenvector of A if and only if y is a right eigenvector of A^T (corresponding to the same eigenvalue), and hence the task of finding left eigenvectors for A is converted to the task of finding right eigenvectors for A^T.

4.2 Maximum Cycle Mean is the Principal Eigenvalue

When solving the eigenproblem a crucial role is played by the concepts of the maximum cycle mean and that of a definite matrix. The aim of this section is to prove that the maximum cycle mean is an eigenvalue of every square matrix over $\overline{\mathbb{R}}$. We will first solve the extreme case when $\lambda(A) = \varepsilon$ and then we prove that the columns of $\Gamma(A_\lambda)$ with zero diagonal entries are eigenvectors corresponding to $\lambda(A)$ if $\lambda(A) > \varepsilon$.

Recall that the maximum cycle mean of $A = (a_{ij}) \in \overline{\mathbb{R}}^{n\times n}$ is

$$\lambda(A) = \max \frac{a_{i_1 i_2} + a_{i_2 i_3} + \cdots + a_{i_{k-1} i_k} + a_{i_k i_1}}{k}$$

where the maximization is taken over all (elementary) cycles $(i_1, \ldots, i_k, i_1)$ in D_A $(k = 1, \ldots, n)$, see Lemma 1.6.2. Due to the convention $\max \emptyset = \varepsilon$, it follows from this definition that $\lambda(A) = \varepsilon$ if and only if D_A is acyclic.

Lemma 4.2.1 *Let* $A = (a_{ij}) \in \overline{\mathbb{R}}^{n\times n}$ *have columns* $A_1, A_2, \ldots, A_n$. *If* $\lambda(A) = \varepsilon$ *then* $\Lambda(A) = \{\varepsilon\}$, *at least one column of A is ε and the eigenvectors of A are exactly the vectors* $(x_1, \ldots, x_n)^T \in \overline{\mathbb{R}}^n$, $x \neq \varepsilon$ *such that* $x_j = \varepsilon$ *whenever* $A_j \neq \varepsilon$ $(j \in N)$. *Hence* $V(A, \varepsilon) = \{G \otimes z; z \in \overline{\mathbb{R}}^n\}$, *where* $G \in \overline{\mathbb{R}}^{n\times n}$ *has columns* $g_1, g_2, \ldots$ *and for all* $j \in N$:

$$g_j = \begin{cases} e^j, & \text{if } A_j = \varepsilon, \\ \varepsilon, & \text{if } A_j \neq \varepsilon. \end{cases}$$

Proof Suppose $\lambda(A) = \varepsilon$ and $A \otimes x = \lambda \otimes x$ for some $\lambda \in \mathbb{R}$, $x \neq \varepsilon$. Hence

$$\max_{j=1,\ldots,n} \left(a_{ij} + x_j\right) = \lambda + x_i \quad (i = 1, \ldots, n).$$

For every $i \in N$ there is a $j \in N$ such that

$$a_{ij} + x_j = \lambda + x_i.$$

Thus if, say $x_{i_1} > \varepsilon$, and $i = i_1$ then there are $i_2, i_3, \ldots$ such that

$$a_{i_i i_2} + x_{i_2} = \lambda + x_{i_1}$$

$$a_{i_2 i_3} + x_{i_3} = \lambda + x_{i_2}$$

$$\ldots .$$

where $x_{i_1}, x_{i_2}, x_{i_3}, \ldots > \varepsilon$. This process will eventually cycle. Let us assume without loss of generality that the cycle is $(i_1, \ldots, i_k, i_{k+1} = i_1)$. Hence the last equation in the above system is

$$a_{i_k i_1} + x_{i_1} = \lambda + x_{i_k}.$$

In all these equations both sides are finite. If we add them up and simplify, we get

$$a_{i_1 i_2} + a_{i_2 i_3} + \cdots + a_{i_{k-1} i_k} + a_{i_k i_1} = k\lambda$$

showing that a cycle in D_A exists, a contradiction to $\lambda(A) = \varepsilon$. Therefore $\Lambda(A) \cap \mathbb{R} = \emptyset$. At the same time A has an ε column by Lemma 1.5.3. If the jth column is ε then $A \otimes x = \lambda(A) \otimes x$ for any vector x whose components are all ε, except for the jth which may be of any finite value. Hence $\Lambda(A) = \{\varepsilon\}$ and the rest of the lemma follows. □

Since Lemma 4.2.1 completely solves the case $\lambda(A) = \varepsilon$, we may now assume that we deal with matrices whose maximum cycle mean is finite. Recall that the matrix $A_\lambda = (\lambda(A))^{-1} \otimes A$ is definite for any $A \in \overline{\mathbb{R}}^{n \times n}$ whenever $\lambda(A) > \varepsilon$ (Theorem 1.6.5).

Proposition 4.2.2 *Let $A \in \overline{\mathbb{R}}^{n \times n}$ and $\lambda(A) > \varepsilon$. Then*

$$V(A) = V(\lambda(A)^{-1} \otimes A).$$

Proof The statement follows from part (a) of Proposition 4.1.1. □

Thus by Lemma 4.2.1, Proposition 4.1.1 (parts (a) and (b)) and Proposition 4.2.2 the task of finding all eigenvalues and eigenvectors of a matrix has been reduced to the same task for definite matrices.

Recall that $\Gamma(A)$ was defined in Sect. 1.6.2 as the series $A \oplus A^2 \oplus A^3 \oplus \cdots$ and that

$$\Gamma(A) = A \oplus A^2 \oplus \cdots \oplus A^n$$

if and only if $\lambda(A) \le 0$ (Proposition 1.6.10).

Let us denote the columns of $\Gamma(A) = (\gamma_{ij})$ by $g_1, \ldots, g_n$. Recall that if A is definite then the values γ_{ij} $(i, j \in N)$ represent the weights of heaviest $i - j$ paths in D_A (Sect. 1.6.2). The significance of $\Gamma(A)$ for matrices with $\lambda(A) \le 0$ is indicated by the fact that for such matrices

$$A \otimes \Gamma(A) = A^2 \oplus \cdots \oplus A^{n+1} \le \Gamma(A)$$

due to (1.20), thus yielding

$$A \otimes g_j \le g_j \quad \text{for every } j \in N. \tag{4.2}$$

An important point of the max-algebraic eigenproblem theory is that in (4.2) actually equality holds whenever A is definite and $j \in N_c(A)$:

Lemma 4.2.3 *Let* $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$. *If* A *is definite,* $g_1, \ldots, g_n$ *are the columns of* $\Gamma(A)$ *and* $j \in N_c(A)$ *then* $A \otimes g_j = g_j$.

Proof Let $j \in N_c(A)$ and $i \in N$. Then by (4.2)

$$\max_{r=1,\ldots,n} (a_{ir} + \gamma_{rj}) \leq \gamma_{ij}$$

and we need to prove that actually equality holds. We may assume without loss of generality $\gamma_{ij} > \varepsilon$ (otherwise the wanted equality follows). Let $(i, k, \ldots, j)$ be a heaviest $i - j$ path. If $k = j$ then $\gamma_{ij} = a_{ij} = a_{ij} + \gamma_{jj}$. If $k \neq j$ then $\gamma_{ij} = a_{ik} + \gamma_{kj}$. In each case there is an r such that $a_{ir} + \gamma_{rj} = \gamma_{ij}$. □

Before we summarize our results in the main statement of this section, we give a practical description of the set of critical nodes $N_c(A)$. Since there are no cycles of weight more than 0 in D_A for definite matrices A but at least one has weight 0, we have then that for a definite matrix A at least one diagonal entry in $\Gamma(A)$ is 0 and all diagonal entries are 0 or less since the kth diagonal entry is the greatest weight of a cycle in D_A containing node k.

It also follows for any definite matrix A that zero diagonal entries in $\Gamma(A)$ exactly correspond to critical nodes, that is, we have

$$N_c(A) = \{j \in N; \gamma_{jj} = 0\}. \tag{4.3}$$

By Lemma 4.2.3 zero is an eigenvalue of every definite matrix. Hence Proposition 4.1.1 (part 2), Lemmas 4.2.1, 4.2.2, 1.6.6 and 4.2.3 and (4.3) imply:

Theorem 4.2.4 $\lambda(A)$ *is an eigenvalue for any matrix* $A \in \overline{\mathbb{R}}^{n \times n}$. *If* $\lambda(A) > \varepsilon$ *then up to* n *eigenvectors of* A *corresponding to* $\lambda(A)$ *can be found among the columns of* $\Gamma(A_\lambda)$. *More precisely, every column of* $\Gamma(A_\lambda)$ *with zero diagonal entry is an eigenvector of* A *with corresponding eigenvalue* $\lambda(A)$.

In view of Theorem 4.2.4 we will call $\lambda(A)$ the *principal eigenvalue* of A.

Note that when the result of Theorem 4.2.4 is generalized to matrices over linearly ordered commutative groups then the concept of radicability of the underlying group (see Sect. 1.4) is crucial, since otherwise it is not possible to guarantee the existence of the maximum cycle mean. Therefore in groups that are not radicable, such as the additive group of integers, an eigenvalue of a matrix may not exist.

4.3 Principal Eigenspace

The results of the previous section enable us to present a complete description of all eigenvectors corresponding to the principal eigenvalue. Such eigenvectors will be called *principal* and $V(A, \lambda(A))$ will be called the *principal eigenspace* of A. Our aim in this section is to describe bases of $V(A, \lambda(A))$.

The columns of $\Gamma(A_\lambda)$ with zero diagonal entry are principal eigenvectors by Theorem 4.2.4. We will call them the *fundamental eigenvectors* [60] of A (FEV). Clearly, every max-combination of fundamental eigenvectors is also a principal eigenvector.

We will use Theorem 4.2.4 and

- prove that there are no principal eigenvectors other than max-combinations of fundamental eigenvectors,
- identify fundamental eigenvectors that are multiples of the others, and
- prove that by removing fundamental eigenvectors that are multiples of the others we produce a basis of the principal eigenspace, that is, none of the remaining columns is a max-combination of the others.

We start with a technical lemma.

Lemma 4.3.1 [65] *Let* $A \in \overline{\mathbb{R}}^{n\times n}$, $\lambda(A) > \varepsilon$ *and* $g_1, \ldots, g_n$ *be the columns of* $\Gamma(A_\lambda) = (\gamma_{ij})$. *If* $x = (x_1, \ldots, x_n)^T \in V(A, \lambda(A))$ *and* $x_i > \varepsilon$ $(i \in N)$ *then there is an* $s \in N_c(A)$ *such that*

$$x_i = x_s + \gamma_{is}.$$

Proof Let $A_\lambda = (d_{ij})$ and $i \in N$, $x_i > \varepsilon$. Then $A_\lambda \otimes x = x$ by Proposition 4.1.1 (parts (a) and (b)) and $N_c(A) = N_c(A_\lambda)$ by Lemma 1.6.6. This implies that there is a sequence of indices $i_1 = i, i_2, \ldots$ such that

$$\begin{aligned} x_{i_1} &= d_{i_1 i_2} + x_{i_2} \\ x_{i_2} &= d_{i_2 i_3} + x_{i_3} \\ &\ldots \end{aligned} \tag{4.4}$$

This sequence will eventually cycle. Let us assume that the cycle is

$$(i_r, \ldots, i_k, i_{k+1} = i_r).$$

For this subsequence we have

$$\begin{aligned} x_{i_r} &= d_{i_r i_{r+1}} + x_{i_{r+1}} \\ &\ldots \\ x_{i_k} &= d_{i_k i_r} + x_{i_r}. \end{aligned}$$

In all these equations both sides are finite. If we add them up and simplify, we get

$$d_{i_r i_{r+1}} + \cdots + d_{i_k i_r} = 0$$

and hence $i_k \in N_c(A_\lambda) = N_c(A)$.

If we add up the first $k-1$ equations in (4.4) and simplify, we get

$$x_{i_1} = d_{i_1 i_2} + \cdots + d_{i_{k-1} i_k} + x_{i_k}.$$

Since $d_{i_1i_2} + \dots + d_{i_{k-1}i_k}$ is the weight of an $i_1 - i_k$ path in D_{A_λ} and $\gamma_{i_1i_k}$ is the weight of a heaviest $i_1 - i_k$ path, we have

$$x_{i_1} \le \gamma_{i_1i_k} + x_{i_k}.$$

At the same time $x \in V(\Gamma(A_\lambda))$ (see Remark 4.1.2) and so

$$x_{i_1} = \sum_{j\in N}^{\oplus} \gamma_{i_1j} \otimes x_j \ge \gamma_{i_1i_k} + x_{i_k}.$$

Hence i_k is the sought s. □

We are ready to prove that there are no principal eigenvectors other than max-combinations of fundamental eigenvectors:

Lemma 4.3.2 *Suppose that* $A = (a_{ij}) \in \overline{\mathbb{R}}^{n\times n}$, $\lambda(A) > \varepsilon$ *and* $g_1, \dots, g_n$ *are the columns of* $\Gamma(A_\lambda) = (\gamma_{ij})$. *If* $x = (x_1, \dots, x_n)^T \in V(A, \lambda(A))$ *then*

$$x = \sum_{j\in N_c(A)}^{\oplus} x_j \otimes g_j.$$

Proof Let $x = (x_1, \dots, x_n)^T \in V(A, \lambda(A))$. We have

$$A_\lambda \otimes x = x \tag{4.5}$$

by Proposition 4.1.1 (parts (a) and (b)) and $N_c(A) = N_c(A_\lambda)$ by Lemma 1.6.6. This implies (see Remark 4.1.2) that $x \in V(\Gamma(A_\lambda), 0)$, yielding

$$x = \sum_{j\in N}^{\oplus} x_j \otimes g_j \ge \sum_{j\in N_c(A)}^{\oplus} x_j \otimes g_j.$$

We need to prove that the converse inequality holds too, that is, for every $i \in N$ there is an $s \in N_c(A)$ such that

$$x_i \le x_s + \gamma_{is}.$$

If $x_i = \varepsilon$ then this is trivially true. If $x_i > \varepsilon$ then it follows from Lemma 4.3.1. □

Clearly, when considering all possible max-combinations of a set of fundamental eigenvectors (or, indeed, of any vectors), we may remove from this set fundamental eigenvectors that are multiples of some other. To be more precise, we say that two fundamental eigenvectors g_i and g_j are *equivalent* if $g_i = \alpha \otimes g_j$ for some $\alpha \in \mathbb{R}$ and *nonequivalent* otherwise. We characterize equivalent fundamental eigenvectors using the equivalence of eigennodes in the next statement (note that the relation $i \sim j$ has been defined in Sect. 1.6.1):

Theorem 4.3.3 [60] *Suppose that* $A = (a_{ij}) \in \overline{\mathbb{R}}^{n\times n}$, $\lambda(A) > \varepsilon$ *and* $g_1, \dots, g_n$ *are the columns of* $\Gamma(A_\lambda) = (\gamma_{ij})$. *If* $i, j \in N_c(A)$ *then* $g_i = \alpha \otimes g_j$ *for some* $\alpha \in \mathbb{R}$ *if and only if* $i \sim j$.

Proof Recall that $N_c(A) = N_c(A_\lambda)$ by Lemma 1.6.6.

Let $i, j \in N_c(A_\lambda)$. If $g_i = \alpha \otimes g_j$, $\alpha \in \mathbb{R}$ then $\gamma_{ji} = \alpha \otimes \gamma_{jj} = \alpha$ and $\gamma_{ij} = \alpha^{-1} \otimes \gamma_{ii} = \alpha^{-1}$. Hence the heaviest $i - j$ path extended by the heaviest $j - i$ path is a cycle of weight $\alpha^{-1} \otimes \alpha = 0$, thus $i \sim j$. Conversely, let $i \sim j$ and α be the weight of the $j - i$ subpath of the critical cycle containing both i and j. Then for any $k \in N$ we have $\gamma_{ki} = \alpha \otimes \gamma_{kj}$ since $\geq$ follows from the definition of γ_{ki} and $>$ would imply $\alpha^{-1} \otimes \gamma_{ki} > \gamma_{kj}$. But α^{-1} is the weight of the $i - j$ subpath of the critical cycle containing both i and j and thus $\alpha^{-1} \otimes \gamma_{ki}$ is the weight of a $k - j$ path which is a contradiction with the maximality of γ_{kj}. Hence $g_i = \alpha \otimes g_j$. □

Note that if $i \sim j$ then we also write $g_i \sim g_j$.

From the last two theorems we can readily deduce:

Corollary 4.3.4 [60] *Suppose that* $A = (a_{ij}) \in \overline{\mathbb{R}}^{n\times n}$, $\lambda(A) > \varepsilon$ *and* $g_1, \dots, g_n$ *are the columns of* $\Gamma(A_\lambda)$. *Then*

$$V(A, \lambda(A)) = \left\{ \sum_{j \in N_c^*(A)}^{\oplus} \alpha_j \otimes g_j ; \alpha_j \in \overline{\mathbb{R}}, j \in N_c^*(A) \right\}$$

where $N_c^*(A)$ *is any maximal set of nonequivalent eigennodes of* A.

Clearly, any set $N_c^*(A)$ in Corollary 4.3.4 can be obtained by taking exactly one g_k for each equivalence class in $(N_c(A), \sim)$. The results on bases in Chap. 3 enable us now to easily describe bases of principal eigenspaces and, consequently, to define the principal dimension.

Theorem 4.3.5 [6] *Suppose that* $A = (a_{ij}) \in \overline{\mathbb{R}}^{n\times n}$, $\lambda(A) > \varepsilon$ *and* $g_1, \dots, g_n$ *are the columns of* $\Gamma(A_\lambda)$. *Then* $V(A, \lambda(A))$ *is a nontrivial subspace and we obtain a basis of* $V(A, \lambda(A))$ *by taking exactly one* g_k *for each equivalence class in* $(N_c(A), \sim)$.

Proof $V(A, \lambda(A))$ is a subspace by Proposition 4.1.1 (parts (f) and (g)). It is nontrivial due to (4.3) and Lemma 4.2.3. By Corollary 3.3.11 it remains to prove that every $g_k, k \in N_c(A)$, is an extremal.

Let $k \in N_c(A)$ be fixed and suppose that $g_k = u \oplus v$ where $u, v \in V(A, \lambda(A))$. Then by Lemma 4.3.2 we have:

$$u = \sum_{j \in N_c^*(A)}^{\oplus} \alpha_j \otimes g_j$$

and

$$v = \sum_{j \in N_c^*(A)}^{\oplus} \beta_j \otimes g_j$$

where $N_c^*(A)$ is a fixed maximal set of nonequivalent eigennodes of A and $\alpha_j, \beta_j \in \mathbb{R}$. We may assume without loss of generality that $g_k \in N_c^*(A)$ and thus $g_k \nsim g_h$ for any $h \in N_c^*(A), h \neq k$. Hence

$$g_k = \sum_{j \in N_c^*(A)}^{\oplus} \delta_j \otimes g_j$$

where $\delta_j = \alpha_j \oplus \beta_j$. Clearly $\delta_k \leq 0$. Suppose $\delta_k < 0$ then

$$g_k = \sum_{\substack{j \in N_c^*(A) \\ j \neq k}}^{\oplus} \delta_j \otimes g_j.$$

It follows that

$$0 = \gamma_{kk} = \sum_{\substack{j \in N_c^*(A) \\ j \neq k}}^{\oplus} \delta_j \otimes \gamma_{kj} = \delta_h \otimes \gamma_{kh}$$

for some $h \in N_c^*(A), h \neq k$. At the same time

$$\gamma_{hk} = \sum_{\substack{j \in N_c^*(A) \\ j \neq k}}^{\oplus} \delta_j \otimes \gamma_{hj} \geq \delta_h \otimes \gamma_{hh} = \delta_h.$$

Therefore

$$\gamma_{kh} \otimes \gamma_{hk} \geq \delta_h^{-1} \otimes \delta_h = 0.$$

The last inequality is in fact equality since there are no positive cycles in $D_{\Gamma(A_\lambda)}$, implying that $k \sim h$, a contradiction. Hence $\delta_k = 0$. Then (without loss of generality) $\alpha_k = 0$ implying $u \geq g_k = u \oplus v$ and thus $u = g_k$. □

The dimension of the principal eigenspace of A will be called the *principal dimension* of A and will be denoted pd(A). It follows from Theorems 4.3.3 and 4.3.5 that pd(A) is equal to the number of critical components of $C(A)$ or, equivalently, to the size of any basis of the column space of the matrix consisting of fundamental eigenvectors of A. Since this basis can be found in $O(n^3)$ time (Sect. 3.4), pd(A) can be found with the same computational effort.

Remark 4.3.6 It is easily seen that $\lambda(A^T) = \lambda(A)$, $\Gamma(A^T) = (\Gamma(A))^T$ and $N_c(A^T) = N_c(A)$. Hence an analogue of Theorem 4.3.5 in terms of rows of $\Gamma(A_\lambda)$ for left principal eigenvectors immediately follows. See also Remark 4.1.4.

Example 4.3.7 Consider the matrix

$$A = \begin{pmatrix} 7 & 9 & 5 & 5 & 3 & 7 \\ 7 & 5 & 2 & 7 & 0 & 4 \\ 8 & 0 & 3 & 3 & 8 & 0 \\ 7 & 2 & 5 & 7 & 9 & 5 \\ 4 & 2 & 6 & 6 & 8 & 8 \\ 3 & 0 & 5 & 7 & 1 & 2 \end{pmatrix}.$$

The maximum cycle mean is 8, attained by three critical cycles: $(1,2,1)$, $(5,5)$ and $(4,5,6,4)$. Thus $\lambda(A) = 8$, $\mathrm{pd}(A) = 2$ and

$$\Gamma(A_\lambda) = \begin{pmatrix} 0 & 1 & -1 & 0 & 1 & 1 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ -1 & 0 & -1 & 0 & 1 & 1 \\ -2 & -1 & -2 & -1 & 0 & 0 \\ -2 & -1 & -2 & -1 & 0 & 0 \end{pmatrix}.$$

Critical components have node sets $\{1,2\}$ and $\{4,5,6\}$. Hence the first and second columns of $\Gamma(A_\lambda)$ are multiples of each other and similarly the fourth, fifth and sixth columns. For the basis of $V(A,\lambda(A))$ we may take for instance the first and fourth columns.

Example 4.3.8 Consider the matrix

$$A = \begin{pmatrix} 0 & 3 & & \\ 1 & -1 & & \\ & & 2 & \\ & & & 1 \end{pmatrix},$$

where the missing entries are ε. Then $\lambda(A) = 2$, $N_c(A) = \{1,2,3\}$, critical components have node sets $\{1,2\}$ and $\{3\}$, $\mathrm{pd}(A) = 2$. We can compute

$$\Gamma(A_\lambda) = \begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & \\ & & & -1 \end{pmatrix},$$

hence a basis of the principal eigenspace is

$$\{g_2, g_3\} = \left\{ (1,0,\varepsilon,\varepsilon)^T, (\varepsilon,\varepsilon,0,\varepsilon)^T \right\}.$$

4.4 Finite Eigenvectors

The aim in this chapter is to show how to find all eigenvalues and describe all eigenvectors of a matrix. To achieve this goal, in this section we will study the set of finite eigenvectors. We will show how to efficiently describe all finite eigenvectors.

We will continue to use the notation $\Gamma(A_\lambda) = (\gamma_{ij})$ if $\lambda(A) > \varepsilon$. Recall that $N_c(A) = N_c(A_\lambda)$ by Lemma 1.6.6.

We will present the main results of this section in the following order:

- A proof that the maximum cycle mean is the only possible eigenvalue corresponding to finite eigenvectors.
- Criteria for the existence of finite eigenvectors.
- Description of all finite eigenvectors.
- A proof that irreducible matrices have only finite eigenvectors.

The first result shows that $\lambda(A)$ is the only possible eigenvalue corresponding to finite eigenvectors. Note that if $A = \varepsilon$ then every finite vector of a suitable dimension is an eigenvector of A and all correspond to the unique eigenvalue $\lambda(A) = \varepsilon$.

Theorem 4.4.1 [60] *Let $A = (a_{ij}) \in \overline{\mathbb{R}}^{n\times n}$. If $A \neq \varepsilon$ and $V^+(A) \neq \emptyset$ then $\lambda(A) > \varepsilon$ and $A \otimes x = \lambda(A) \otimes x$ for every $x \in V^+(A)$.*

Proof Let $x = (x_1, \dots, x_n)^T \in V^+(A)$. We have

$$\max_{j=1,\dots,n} \left(a_{ij} + x_j\right) = \lambda + x_i \quad (i = 1, \dots, n)$$

for some $\lambda \in \overline{\mathbb{R}}$. Since $A \neq \varepsilon$ the LHS is finite for at least one i and thus $\lambda > \varepsilon$.

For every $i \in N$ there is a $j \in N$ such that

$$a_{ij} + x_j = \lambda + x_i.$$

Hence, if $i = i_1$ is any fixed index then there are indices $i_2, i_3, \dots$ such that

$$a_{i_1 i_2} + x_{i_2} = \lambda + x_{i_1},$$
$$a_{i_2 i_3} + x_{i_3} = \lambda + x_{i_2},$$
$$\dots.$$

This process will eventually cycle. Let us assume without loss of generality that the cycle is $(i_1, \dots, i_k, i_{k+1} = i_1)$, otherwise we remove the necessary first elements of this sequence. Hence the last equation in the above system is

$$a_{i_k i_1} + x_{i_1} = \lambda + x_{i_k}.$$

In all these equations both sides are finite. If we add them up and simplify, we get

$$\lambda = \frac{a_{i_1 i_2} + a_{i_2 i_3} + \dots + a_{i_{k-1} i_k} + a_{i_k i_1}}{k}.$$

At the same time, if $\sigma = (i_1, \dots, i_k, i_{k+1} = i_1)$ is an arbitrary cycle in D_A then it satisfies the system of inequalities obtained from the above system of equations after replacing $=$ by $\leq$. Hence

$$\lambda \geq \frac{a_{i_1 i_2} + a_{i_2 i_3} + \dots + a_{i_{k-1} i_k} + a_{i_k i_1}}{k} = \mu(\sigma, A).$$

It follows that $\lambda = \max_\sigma \mu(\sigma, A) = \lambda(A)$. □

Theorem 4.4.1 opens the possibility of answering questions such as the existence and description of finite eigenvectors.

Lemma 4.4.2 *Let* $A \in \overline{\mathbb{R}}^{n \times n}$. *If* $A \neq \varepsilon$ *and* $x = (x_1, \dots, x_n)^T \in V^+(A)$ *then for every* $i \in N$ *there is an* $s \in N_c(A)$ *such that*

$$x_i = x_s + \gamma_{is},$$

where $\Gamma(A_\lambda) = (\gamma_{ij})$.

Proof Since $\lambda(A) > \varepsilon$ and $x \in V(A, \lambda(A))$ by Theorem 4.4.1, the statement follows immediately from Lemma 4.3.1. □

We are ready to formulate the first criterion for the existence of finite eigenvectors.

Theorem 4.4.3 *Suppose that* $A \in \overline{\mathbb{R}}^{n \times n}$, $\lambda(A) > \varepsilon$ *and* $g_1, \dots, g_n$ *are the columns of* $\Gamma(A_\lambda) = (\gamma_{ij})$. *Then*

$$V^+(A) \neq \emptyset \iff \sum_{j \in N_c(A)}^{\oplus} g_j \in \mathbb{R}^n.$$

Proof Suppose $\sum_{j \in N_c(A)}^{\oplus} g_j \in \mathbb{R}^n$. Every g_j $(j \in N_c(A))$ is in $V(A, \lambda(A))$ by Lemma 4.2.3 and $\sum_{j \in N_c(A)}^{\oplus} g_j \in V(A)$ by Proposition 4.1.1. Hence $\sum_{j \in N_c(A)}^{\oplus} g_j \in V^+(A)$.

On the other hand, by Lemma 4.4.2, if $x = (x_1, \dots, x_n)^T \in V^+(A)$ then for every $i \in N$ there is an $s \in N_c(A)$ such that $\gamma_{is} \in \mathbb{R}$ and so $\sum_{j \in N_c(A)}^{\oplus} g_j \in \mathbb{R}^n$. □

We can now easily deduce a classical result:

Corollary 4.4.4 [60] *Suppose* $A \in \overline{\mathbb{R}}^{n \times n}$, $A \neq \varepsilon$. *Then* $V^+(A) \neq \emptyset$ *if and only if the following are satisfied*:

(a) $\lambda(A) > \varepsilon$.
(b) *In* D_A *there is*

$$(\forall i \in N)(\exists j \in N_c(A)) i \to j.$$

Proof By Theorem 4.4.1, $A \neq \varepsilon$ and $V^+(A) \neq \emptyset$ implies $\lambda(A) > \varepsilon$. Observe that

$$\sum_{j \in N_c(A)}^{\oplus} g_j \in \mathbb{R}^n \Longleftrightarrow \sum_{j \in N_c(A)}^{\oplus} \gamma_{ij} \in \mathbb{R} \quad \text{for all } i \in N.$$

Hence by Theorem 4.4.3 $V^+(A) \neq \emptyset$ if and only if

$$(\forall i \in N)(\exists j \in N_c(A))\gamma_{ij} \in \mathbb{R}.$$

However, γ_{ij} is the greatest weight of an $i - j$ path in D_{A_λ} or ε, if there is no such path, and the statement follows. □

The description of all finite eigenvectors can now easily be deduced:

Theorem 4.4.5 *Let $A \in \overline{\mathbb{R}}^{n \times n}$. If $\lambda(A) > \varepsilon$, $g_1, \ldots, g_n$ are the columns of $\Gamma(A_\lambda)$ and $V^+(A) \neq \emptyset$ then*

$$V^+(A) = \left\{ \sum_{j \in N_c^*(A)}^{\oplus} \alpha_j \otimes g_j; \alpha_j \in \mathbb{R} \right\}, \tag{4.6}$$

where $N_c^*(A)$ is any maximal set of nonequivalent eigennodes of A.

Proof $\supseteq$ follows from Lemma 4.2.3, Proposition 4.1.1 and Theorem 4.4.3 immediately. $\subseteq$ follows from Lemma 4.3.2. □

Remark 4.4.6 Note that (4.6) requires $\alpha_j \in \mathbb{R}$ and, in general, g_j may or may not be in $V^+(A)$. Therefore the subspace $V^+(A) \cup \{\varepsilon\}$ may or may not be finitely generated and hence, in general, there is no guarantee that it has a basis.

Example 4.4.7 Consider the matrix

$$A = \begin{pmatrix} 0 & 3 & & \\ 1 & -1 & & \\ & & 2 & \\ & & 0 & 1 \end{pmatrix},$$

where the missing entries are ε. Then $\lambda(A) = 2$, $N_c(A) = \{1, 2, 3\}$, critical components have node sets $\{1, 2\}$ and $\{3\}$, $\mathrm{pd}(A) = 2$. A finite eigenvector exists since an eigennode is accessible from every node (unlike in the slightly different Example 4.3.8). We can compute

$$\Gamma(A_\lambda) = \begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & \\ & & -2 & -1 \end{pmatrix},$$

hence a basis of the principal eigenspace is $\{(1,0,\varepsilon,\varepsilon)^T, (\varepsilon,\varepsilon,0,-2)^T\}$. All finite eigenvectors are max-combinations of the vectors in the basis provided that both coefficients are finite. However, $V^+(A) \cup \{\varepsilon\}$ has no basis.

The following classical complete solution of the eigenproblem for irreducible matrices is now easy to prove:

Theorem 4.4.8 (Cuninghame-Green [60]) *Every irreducible matrix* $A \in \overline{\mathbb{R}}^{n\times n}$ $(n > 1)$ *has a unique eigenvalue equal to* $\lambda(A)$ *and*

$$V(A) - \{\varepsilon\} = V^+(A) = \left\{ \sum_{j\in N_c^*(A)}^{\oplus} \alpha_j \otimes g_j ; \alpha_j \in \mathbb{R} \right\},$$

where $g_1, \ldots, g_n$ *are the columns of* $\Gamma(A_\lambda)$ *and* $N_c^*(A)$ *is any maximal set of nonequivalent eigennodes of* A.

Proof Let A be irreducible, thus $\lambda(A) > \varepsilon$. Also, $\Gamma(A_\lambda)$ is finite by Proposition 1.6.10. Every eigenvector of A is also an eigenvector of $\Gamma(A_\lambda)$ with eigenvalue 0 (Remark 4.1.2) but the product of a finite matrix and a vector $x \neq \varepsilon$ is finite. Hence an irreducible matrix can only have finite eigenvectors and thus its only eigenvalue is $\lambda(A)$ by Theorem 4.4.1.

On the other hand, due to the finiteness of all columns of $\Gamma(A_\lambda)$, by Theorem 4.4.3, $V^+(A) \neq \emptyset$ and the rest follows from Theorem 4.4.5. □

Remark 4.4.9 Note that every 1×1 matrix A over $\overline{\mathbb{R}}$ is irreducible and $V(A) - \{\varepsilon\} = V^+(A) = \mathbb{R}$.

The fact that $\lambda(A)$ is the unique eigenvalue of an irreducible matrix A was already proved in [58] and then independently in [144] for finite matrices. Since then it has been rediscovered in many papers worldwide. The description of $V^+(A)$ for irreducible matrices as given in Corollary 4.4.4 was also proved in [98].

Note that for an irreducible matrix A we have:

$$V(A) = V^+(A) \cup \{\varepsilon\} = \{\Gamma(A_\lambda) \otimes z; z \in \overline{\mathbb{R}}^n, z_j = \varepsilon \quad \text{for all } j \notin N_c(A)\}.$$

Remark 4.4.10 Since $\Gamma(A_\lambda)$ is finite for an irreducible matrix A, the generators of $V^+(A)$ are all finite if A is irreducible. Hence $V^+(A) \cup \{\varepsilon\} = V(A)$ has a basis in this case, which coincides with the basis of $V(A)$.

Example 4.4.11 Consider the irreducible matrix

$$A = \begin{pmatrix} 0 & 3 & & 0 \\ 1 & -1 & 0 & \\ & 0 & 2 & \\ & & 0 & 1 \end{pmatrix},$$

where the missing entries are ε. Then $\lambda(A) = 2$, $N_c(A) = \{1, 2, 3\}$, critical components have node sets $\{1, 2\}$ and $\{3\}$, $\text{pd}(A) = 2$. We can compute

$$\Gamma(A_\lambda) = \begin{pmatrix} 0 & 1 & -4 & -2 \\ -1 & 0 & -5 & -3 \\ -3 & -2 & 0 & -5 \\ -5 & -4 & -2 & -1 \end{pmatrix},$$

hence a basis of the principal eigenspace is

$$\left\{(1, 0, -2, -4)^T,\ (-4, -5, 0, -2)^T\right\}.$$

4.5 Finding All Eigenvalues

Our next step is to describe all eigenvalues of square matrices over $\overline{\mathbb{R}}$. The information about principal eigenvectors obtained in the previous sections will be substantially used.

We have already seen in Sect. 1.5 that if $A, B \in \overline{\mathbb{R}}^{n\times n}$ are equivalent ($A \equiv B$), then D_A can be obtained from D_B by a renumbering of the nodes and that $B = P^{-1} \otimes A \otimes P$ for some permutation matrix P. Hence if $A \equiv B$ then A is irreducible if and only if B is irreducible. We also know by Proposition 4.1.3 that $V(A)$ and $V(B)$ are essentially the same (the eigenvectors of A and B only differ by the order of their components).

It follows from Theorem 4.4.8 that a matrix with a nonfinite eigenvector cannot be irreducible. The following lemma provides an alternative and somewhat more detailed explanation of this simple but remarkable property. It may also be useful for a good understanding of the structure of the set $V(A)$ for a general matrix A.

Lemma 4.5.1 *Let $A = (a_{ij}) \in \overline{\mathbb{R}}^{n\times n}$ and $\lambda \in \Lambda(A)$. If $x \in V(A, \lambda) - V^+(A, \lambda)$, $x \neq \varepsilon$, then $n > 1$,*

$$A \equiv \begin{pmatrix} A^{(11)} & \varepsilon \\ A^{(21)} & A^{(22)} \end{pmatrix},$$

$\lambda = \lambda(A^{(22)})$, and hence A is reducible.

Proof Permute the rows and columns of A simultaneously so that the vector arising from x by the same permutation of its components is $x' = \binom{x^{(1)}}{x^{(2)}}$, where $x^{(1)} = \varepsilon \in \overline{\mathbb{R}}^p$ and $x^{(2)} \in \mathbb{R}^{n-p}$ for some p $(1 \leq p < n)$. Denote the obtained matrix by A' (thus $A \equiv A'$) and let us write blockwise

$$A' = \begin{pmatrix} A^{(11)} & A^{(12)} \\ A^{(21)} & A^{(22)} \end{pmatrix},$$

where $A^{(11)}$ is $p \times p$. The equality $A' \otimes x' = \lambda \otimes x'$ now yields blockwise:

$$A^{(12)} \otimes x^{(2)} = \varepsilon,$$

$$A^{(22)} \otimes x^{(2)} = \lambda \otimes x^{(2)}.$$

Since $x^{(2)}$ is finite, it follows from Theorem 4.4.4 that $\lambda = \lambda(A^{(22)})$; also clearly $A^{(12)} = \varepsilon$. □

We already know (Theorem 4.4.8) that all eigenvectors of an irreducible matrix are finite. We now can prove that only irreducible matrices have this property.

Theorem 4.5.2 *Let* $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$. *Then* $V(A) - \{\varepsilon\} = V^+(A)$ *if and only if* A *is irreducible.*

Proof It remains to prove the "only if" part since the "if" part follows from Theorem 4.4.8. If A is reducible then $n > 1$ and $A \equiv \begin{pmatrix} A^{(11)} & \varepsilon \\ A^{(21)} & A^{(22)} \end{pmatrix}$, where $A^{(22)}$ is irreducible. By setting $\lambda = \lambda(A^{(22)}), x^{(2)} \in V^+(A_{22}), x = \begin{pmatrix} \varepsilon \\ x^{(2)} \end{pmatrix} \in \overline{\mathbb{R}}^n$ we see that $x \in V(A) - V^+(A), x \neq \varepsilon$. □

Theorem 4.5.2 does not exclude the possibility that a reducible matrix has finite eigenvectors. The following spectral theory will, as a by-product, enable us to characterize all situations when this occurs.

Every matrix $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$ can be transformed in linear time by simultaneous permutations of the rows and columns to a *Frobenius normal form* (FNF) [11, 18, 126]

$$\begin{pmatrix} A_{11} & \varepsilon & \cdots & \varepsilon \\ A_{21} & A_{22} & \cdots & \varepsilon \\ \cdots & \cdots & \cdots & \cdots \\ A_{r1} & A_{r2} & \cdots & A_{rr} \end{pmatrix} \tag{4.7}$$

where $A_{11}, \ldots, A_{rr}$ are irreducible square submatrices of A. The diagonal blocks are determined uniquely up to a simultaneous permutation of their rows and columns: however, their order is not determined uniquely. Since any such form is essentially determined by strongly connected components of D_A, an FNF can be found in $O(|V| + |E|)$ time [18, 142]. It will turn out later in this section that the FNF is a particularly convenient form for studying spectral properties of matrices. Since these are essentially preserved by simultaneous permutations of the rows and columns (Proposition 4.1.3) we will often assume, without loss of generality, that the matrix under consideration already is in an FNF.

If A is in an FNF then the corresponding partition of the node set N of D_A will be denoted as $N_1, \ldots, N_r$ and these sets will be called *classes* (of A). It follows that each of the induced subgraphs $D_A[N_i]$ $(i = 1, \ldots, r)$ is strongly connected and an arc from N_i to N_j in D_A exists only if $i \geq j$. Clearly, every A_{jj} has a unique eigenvalue $\lambda(A_{jj})$. As a slight abuse of language we will, for simplicity, also say that $\lambda(A_{jj})$ is the eigenvalue of N_j.

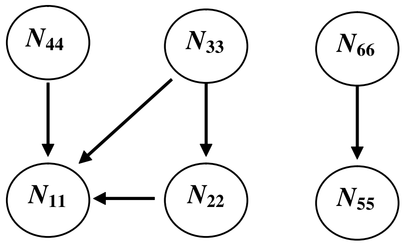

Fig. 4.1 Condensation digraph (6 classes)

If A is in an FNF, say (4.7), then the *condensation digraph*, notation C_A, is the digraph

$$(\{N_1, \ldots, N_r\}, \{(N_i, N_j); (\exists k \in N_i)(\exists l \in N_j)a_{kl} > \varepsilon\}).$$

Observe that C_A is acyclic.

Recall that the symbol $N_i \to N_j$ means that there is a directed path from a node in N_i to a node in N_j in C_A (and therefore from each node in N_i to each node in N_j in D_A).

If there are neither outgoing nor incoming arcs from or to an induced subgraph $C_A[\{N_{i_1}, \ldots, N_{i_s}\}]$ $(1 \le i_1 < \cdots < i_s \le r)$ and no proper subdigraph has this property then the submatrix

$$\begin{pmatrix} A_{i_1 i_1} & \varepsilon & \cdots & \varepsilon \\ A_{i_2 i_1} & A_{i_2 i_2} & \cdots & \varepsilon \\ \cdots & \cdots & \cdots & \cdots \\ A_{i_s i_1} & A_{i_s i_2} & \cdots & A_{i_s i_s} \end{pmatrix}$$

is called an *isolated superblock* (or just *superblock*). The nodes of C_A (that is, classes of A) with no incoming arcs are called the *initial classes*, those with no outgoing arcs are called the *final classes*. Note that an isolated superblock may have several initial and final classes.

For instance the condensation digraph for the matrix

$$\begin{pmatrix} A_{11} & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ * & A_{22} & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ * & * & A_{33} & \varepsilon & \varepsilon & \varepsilon \\ * & \varepsilon & \varepsilon & A_{44} & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & A_{55} & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & * & A_{66} \end{pmatrix} \tag{4.8}$$

can be seen in Fig. 4.1 (note that in (4.8) and elsewhere $*$ indicates a submatrix different from ε). It consists of two superblocks and six classes including three initial and two final ones.

Lemma 4.5.3 *If $x \in V(A)$, $N_i \to N_j$ and $x[N_j] \neq \varepsilon$ then $x[N_i]$ is finite. In particular, $x[N_j]$ is finite.*

Proof Suppose that $x \in V(A, \lambda)$ for some $\lambda \in \overline{\mathbb{R}}$. Fix $s \in N_j$ such that $x_s > \varepsilon$. Since $N_i \to N_j$ we have that for every $r \in N_i$ there is a positive integer q such that $b_{rs} > \varepsilon$ where $B = A^q = (b_{ij})$. Since $x \in V(B, \lambda^q)$ by Proposition 4.1.1 we also have $\lambda^q \otimes x_r \geq b_{rs} \otimes x_s > \varepsilon$. Hence $x_r > \varepsilon$. □

We are now able to describe all eigenvalues of any square matrix over $\overline{\mathbb{R}}$.

Theorem 4.5.4 (Spectral Theorem) *Let* (4.7) *be an FNF of a matrix* $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$. *Then*

$$\Lambda(A) = \left\{\lambda(A_{jj}); \lambda(A_{jj}) = \max_{N_i \to N_j} \lambda(A_{ii})\right\}.$$

Proof Note that

$$\lambda(A) = \max_{i=1,\ldots,r} \lambda(A_{ii}) \tag{4.9}$$

for a matrix A in FNF (4.7).

First we prove the inclusion $\supseteq$. Suppose

$$\lambda(A_{jj}) = \max\{\lambda(A_{ii}); N_i \to N_j\}$$

for some $j \in R = \{1, \ldots, r\}$. Denote

$$S_2 = \{i \in R; N_i \to N_j\},$$
$$S_1 = R - S_2$$

and

$$M_p = \bigcup_{i \in S_p} N_i \quad (p = 1, 2).$$

Then $\lambda(A_{jj}) = \lambda(A[M_2])$ and

$$A \equiv \begin{pmatrix} A[M_1] & \varepsilon \\ * & A[M_2] \end{pmatrix}.$$

If $\lambda(A_{jj}) = \varepsilon$ then at least one column, say the lth in A is ε. We set x_l to any real number and $x_j = \varepsilon$ for $j \neq l$. Then $x \in V(A, \lambda(A_{jj}))$.

If $\lambda(A_{jj}) > \varepsilon$ then $A[M_2]$ has a finite eigenvector by Theorem 4.4.4, say $\tilde{x}$. Set $x[M_2] = \tilde{x}$ and $x[M_1] = \varepsilon$. Then $x = (x[M_1], x[M_2]) \in V(A, \lambda(A_{jj}))$.

Now we prove $\subseteq$. Suppose that $x \in V(A, \lambda), x \neq \varepsilon$, for some $\lambda \in \overline{\mathbb{R}}$.

If $\lambda = \varepsilon$ then A has an ε column, say the kth, thus $a_{kk} = \varepsilon$. Hence the 1×1 submatrix (a_{kk}) is a diagonal block in an FNF of A. In the corresponding decomposition of N one of the sets, say N_j, is $\{k\}$. The set $\{i; N_i \to N_j\} = \{j\}$ and the theorem statement follows.

If $\lambda > \varepsilon$ and $x \in V^+(A)$ then $\lambda = \lambda(A)$ (cf. Theorem 4.4.1) and the statement now follows from (4.9).

If $\lambda > \varepsilon$ and $x \notin V^+(A)$ then similarly as in the proof of Lemma 4.5.1 permute the rows and columns of A simultaneously so that

$$x = \begin{pmatrix} x^{(1)} \\ x^{(2)} \end{pmatrix},$$

where $x^{(1)} = \varepsilon \in \overline{\mathbb{R}}^p$, $x^{(2)} \in \mathbb{R}^{n-p}$ for some p $(1 \le p < n)$. Hence

$$A \equiv \begin{pmatrix} A^{(11)} & \varepsilon \\ A^{(21)} & A^{(22)} \end{pmatrix}$$

and we can assume without loss of generality that both $A^{(11)}$ and $A^{(22)}$ are in an FNF and therefore also

$$\begin{pmatrix} A^{(11)} & \varepsilon \\ A^{(21)} & A^{(22)} \end{pmatrix}$$

is in an FNF. Let

$$A^{(11)} = \begin{pmatrix} A_{i_1 i_1} & \varepsilon & \cdots & \varepsilon \\ A_{i_2 i_1} & A_{i_2 i_2} & \cdots & \varepsilon \\ \cdots & \cdots & \cdots & \cdots \\ A_{i_s i_1} & A_{i_s i_2} & \cdots & A_{i_s i_s} \end{pmatrix}$$

and

$$A^{(22)} = \begin{pmatrix} A_{i_{s+1} i_{s+1}} & \varepsilon & \cdots & \varepsilon \\ A_{i_{s+2} i_{s+1}} & A_{i_{s+2} i_{s+2}} & \cdots & \varepsilon \\ \cdots & \cdots & \cdots & \cdots \\ A_{i_q i_{s+1}} & A_{i_q i_{s+2}} & \cdots & A_{i_q i_q} \end{pmatrix}.$$

We have

$$\lambda = \lambda(A^{(22)}) = \lambda(A_{jj}) = \max_{i=s+1,\dots,q} \lambda(A_{ii}),$$

where $j \in \{s+1, \dots, q\}$. It remains to say that if $N_i \to N_j$ then $i \in \{s+1, \dots, q\}$. □

The Spectral Theorem has been proved in [84] and, independently, also in [12]. Spectral properties of reducible matrices have also been studied in [10] and [145]. Significant correlation exists between the max-algebraic spectral theory and that for nonnegative matrices in linear algebra [13, 128], see also [126]. For instance the Frobenius normal form and accessibility between classes play a key role in both theories. The maximum cycle mean corresponds to the Perron root for irreducible (nonnegative) matrices and finite eigenvectors in max-algebra correspond to positive eigenvectors in the spectral theory of nonnegative matrices. However there are also differences, see Remark 4.6.8.

Let A be in the FNF (4.7). If

$$\lambda(A_{jj}) = \max_{N_i \to N_j} \lambda(A_{ii})$$

then A_{jj} (and also N_j or just j) will be called *spectral*. Thus $\lambda(A_{jj}) \in \Lambda(A)$ if j is spectral but not necessarily the other way round.

Corollary 4.5.5 *All initial classes of C_A are spectral.*

Proof Initial classes have no predecessors and so the condition of the theorem is satisfied. □

Recall that $\lambda(A) = \min\{\lambda; (\exists x \in \mathbb{R}^n) A \otimes x \leq \lambda \otimes x\}$ if $\lambda(A) > \varepsilon$ (Theorem 1.6.29). In contrast we have:

Corollary 4.5.6

$$\begin{aligned}\lambda(A) &= \max \Lambda(A) \\ &= \max \left\{\lambda; \left(\exists x \in \overline{\mathbb{R}}^n, x \neq \varepsilon\right) A \otimes x = \lambda \otimes x\right\}\end{aligned}$$

for every matrix $A \in \overline{\mathbb{R}}^{n\times n}$.

Proof If A is in an FNF, say (4.7), then $\lambda(A) = \max_{i=1,\dots,r} \lambda(A_{ii}) \geq \lambda(A_{jj})$ for all j. □

We easily deduce two more useful statements:

Corollary 4.5.7 $1 \leq |\Lambda(A)| \leq n$ *for every $A \in \overline{\mathbb{R}}^{n\times n}$.*

Proof Follows from the previous corollary and from the fact that the number of classes of A is at most n. □

Corollary 4.5.8 $V(A) = V(A, \lambda(A))$ *if and only if all initial classes have the same eigenvalue $\lambda(A)$.*

Proof The eigenvalues of all initial classes are in $\Lambda(A)$ since all initial classes are spectral, hence all must be equal to $\lambda(A)$ if $\Lambda(A) = \{\lambda(A)\}$. On the other hand, if all initial classes have the same eigenvalue $\lambda(A)$, and λ is the eigenvalue of any spectral class then

$$\lambda \geq \lambda(A) = \max_i \lambda(A_{ii})$$

since there is a path from some initial class to this class and thus $\lambda = \lambda(A)$. □

Figure 4.2 shows a condensation digraph with 14 classes including two initial classes and four final ones. The integers indicate the eigenvalues of the corresponding classes. The six bold classes are spectral, the others are not.

Note that the unique eigenvalues of all classes (that is, of diagonal blocks of an FNF) can be found in $O(n^3)$ time by applying Karp's algorithm (see Sect. 1.6) to

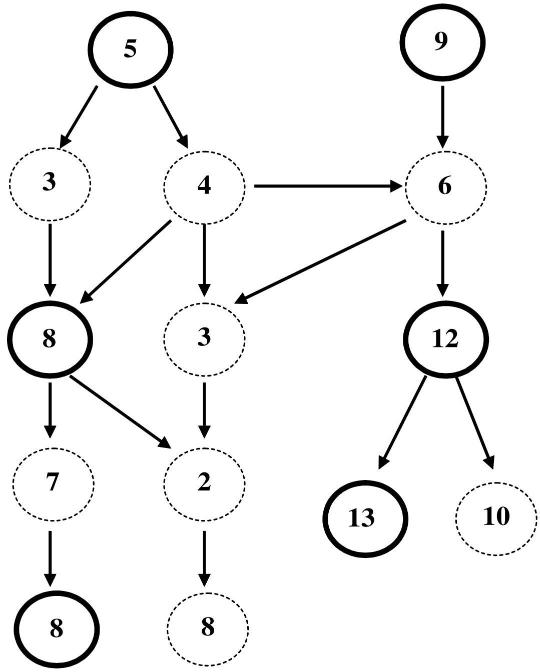

Fig. 4.2 Condensation digraph

each block. The condition for identifying all spectral submatrices in an FNF provided in Theorem 4.5.4 enables us to find them in $O(r^2) \leq O(n^2)$ time by applying standard reachability algorithms to C_A.

Example 4.5.9 Consider the matrix

$$A = \begin{pmatrix} 0 & 3 & & & & \\ 1 & 1 & & & & \\ & & 4 & & & \\ & & 0 & 3 & 1 & \\ & & & -1 & 2 & \\ & & & 1 & & 5 \end{pmatrix},$$

where the missing entries are ε. Then $\lambda(A_{11}) = 2$, $\lambda(A_{22}) = 4$, $\lambda(A_{33}) = 3$, $\lambda(A_{44}) = 5$, $r = 4$; $\Lambda(A) = \{2, 5\}$, $\lambda(A) = 5$, initial classes are N_1 and N_4 and there are no other spectral classes. Final classes are N_1 and N_2.

We will now use the Spectral Theorem to prove two results, Theorems 4.5.10 and 4.5.14, whose proofs are easier when the Spectral Theorem is available. The first of them has been known for certain types of matrices for some time [65, 102]: however, using Theorem 4.5.4 we are able to prove it conveniently for any matrix:

Theorem 4.5.10 *Let* $A \in \overline{\mathbb{R}}^{n\times n}$. *Then*

$$\lambda(A^k) = (\lambda(A))^k$$

holds for all integers $k \geq 0$.

Proof The proof is trivial if $n = 1$ or $k = 0$, so assume $n \geq 2, k \geq 1$.

Suppose first that A is irreducible. Let $x \in V^+(A) = V(A, \lambda(A)) - \{\varepsilon\}$. By Proposition 4.1.1 we have $x \in V(A^k, \lambda(A^k))$ and thus by Theorem 4.4.1 $(\lambda(A))^k = \lambda(A^k)$. It also follows that $(\lambda(A))^k$ is the greatest principal eigenvalue of a diagonal block in any FNF of (possibly reducible) A^k.

Now suppose that A is reducible and without loss of generality let A be in the FNF (4.7). Then $\lambda(A) = \lambda(A_{ii})$ for some i, $1 \leq i \leq r$. The matrix A^k is again lower blockdiagonal and has diagonal blocks $A^k_{11}, \ldots, A^k_{ii}, \ldots, A^k_{rr}$. These blocks may or may not be irreducible. However $(\lambda(A))^k = (\lambda(A_{ii}))^k$ is the greatest principal eigenvalue of a diagonal block in any FNF of A^k_{ii} (by the first part of this proof since A_{ii} is irreducible) and therefore also in any FNF of A^k. This completes the proof. □

For the second result we need two lemmas.

Lemma 4.5.11 *Let* $A \in \overline{\mathbb{R}}^{n\times n}$. *Then* $\varepsilon \in \Lambda(A)$ *if and only if* A *has an* ε *column.*

Proof If $A \otimes x = \varepsilon$ and $x_k \neq \varepsilon$ then the kth column of A is ε. A similar argument is used for the converse. □

Lemma 4.5.12 *Let* $A \in \overline{\mathbb{R}}^{n\times n}$ *be irreducible. If* $A \otimes x \leq \lambda \otimes x, x \neq \varepsilon, \lambda \in \overline{\mathbb{R}}$ *then* $x \in \mathbb{R}^n$.

Proof The statement is trivial for $n = 1$. Let $n > 1$, then $\lambda(A) > \varepsilon$. Without loss of generality we assume that A is definite. Then we have

$$\begin{aligned}\Gamma(A) \otimes x &= A \otimes x \oplus A^2 \otimes x \oplus \cdots \oplus A^n \otimes x \\ &\leq \lambda \otimes x \oplus \lambda^2 \otimes x \oplus \cdots \oplus \lambda^n \otimes x \\ &= \left(\lambda \oplus \cdots \oplus \lambda^n\right) \otimes x.\end{aligned}$$

The LHS is finite since $\Gamma(A)$ is finite (Proposition 1.6.10) and $x \neq \varepsilon$, hence both λ and x are finite. □

Corollary 4.5.13 *Let* $A \in \overline{\mathbb{R}}^{n\times n}$ *be irreducible. Then*

$$\begin{aligned}\lambda(A) &= \min\{\lambda; \left(\exists x \in \mathbb{R}^n\right) A \otimes x \leq \lambda \otimes x\} \\ &= \min\left\{\lambda; \left(\exists x \in \overline{\mathbb{R}}^n, x \neq \varepsilon\right) A \otimes x \leq \lambda \otimes x\right\}.\end{aligned}$$

Proof The statement is trivial for $n = 1$. If $n > 1$ then $\lambda(A) > \varepsilon$ and the first equality follows from Theorem 1.6.29. The second follows from Lemma 4.5.12. □

We now make another use of Theorem 4.5.4 and prove a more general version of Theorem 1.6.29:

Theorem 4.5.14 *If $A \in \overline{\mathbb{R}}^{n\times n}$ then*

$$\min\left\{\lambda;\left(\exists x\in\overline{\mathbb{R}}^{n}, x\neq\varepsilon\right)A\otimes x\leq\lambda\otimes x\right\}=\min\Lambda(A).$$

Proof Without loss of generality let A be in the FNF (4.7) and as before $R = \{1, \ldots, r\}$. Let

$$L=\inf\left\{\lambda;\left(\exists x\in\overline{\mathbb{R}}^{n}, x\neq\varepsilon\right)A\otimes x\leq\lambda\otimes x\right\}.$$

Clearly $L \leq \min \Lambda(A)$ since for x we may take any eigenvector of A. If $\varepsilon \in \Lambda(A)$ then using $x \in V(A, \varepsilon) - \{\varepsilon\}$ we deduce that $L = \varepsilon$. We will therefore assume in the rest of the proof that $\varepsilon \notin \Lambda(A)$.

Let $x \in \overline{\mathbb{R}}^{n}$, $x \neq \varepsilon$, $\lambda \in \mathbb{R}$ and $A\otimes x \leq \lambda\otimes x$. We need to show that $\lambda \geq \min\Lambda(A)$. Observe that $\lambda > \varepsilon$ since otherwise $x \in V(A, \varepsilon) - \{\varepsilon\}$, a contradiction with $\varepsilon \notin \Lambda(A)$. Let us denote

$$K=\{k\in R; x[N_k]\neq\varepsilon\}.$$

Take any $k \in K$. We have

$$A[N_k]\otimes x[N_k]\leq(A\otimes x)[N_k]\leq\lambda\otimes x[N_k].$$

Then $x[N_k]$ is finite by Lemma 4.5.12 and so $\lambda \geq \lambda(A[N_k])$ by Theorem 1.6.18.

If $a_{st} = \varepsilon$ for all $s \in N_i$, $i \in R$ and $t \in N_k$, then N_k is spectral and the statement follows.

If $a_{st} > \varepsilon$ for some $s \in N_i$, $i \in R$ and $t \in N_k$, then $x_s \geq \lambda^{-1} \otimes a_{st} \otimes x_t > \varepsilon$. Therefore $i \in K$ and again, as above, by Lemma 4.5.12 $x[N_i]$ is finite. C_A is acyclic and finite, hence after a finite number of repetitions we will reach an $i \in R$ such that N_i is initial, and hence also spectral, yielding $\lambda(A[N_i]) > \varepsilon$ (since $\varepsilon \notin \Lambda(A)$) and $\lambda(A[N_i]) \geq \min \Lambda(A)$.

At the same time

$$A[N_i]\otimes x[N_i]\leq(A\otimes x)[N_i]\leq\lambda\otimes x[N_i].$$

Therefore $x[N_i]$ is finite by Lemma 4.5.12 and by Theorem 1.6.18 we have:

$$\lambda\geq\lambda(A[N_i]),$$

from which the statement follows. □

4.6 Finding All Eigenvectors

Our final effort in this chapter is to show how to efficiently describe all eigenvectors of a matrix.

Let $A \in \overline{\mathbb{R}}^{n\times n}$ be in the FNF (4.7), $N_1, \ldots, N_r$ be the classes of A and $R = \{1, \ldots, r\}$. For the following discussion suppose that $\lambda \in \Lambda(A)$ is a fixed eigenvalue, $\lambda > \varepsilon$, and denote $I(\lambda) = \{i \in R; \lambda(N_i) = \lambda, N_i \text{ spectral}\}$.

We denote by $g_1, \ldots, g_n$ the columns of $\Gamma(\lambda^{-1} \otimes A) = (\gamma_{ij})$. Note that $\lambda(\lambda^{-1} \otimes A) = \lambda^{-1} \otimes \lambda(A)$ may be positive since $\lambda \leq \lambda(A)$ and thus $\Gamma(\lambda^{-1} \otimes A)$ may include entries equal to $+\infty$ (Proposition 1.6.10). However, for $i \in I(\lambda)$ we have

$$\lambda\left(\lambda^{-1} \otimes A_{ii}\right) = \lambda^{-1} \otimes \lambda(A_{ii}) \leq 0$$

by Theorem 4.5.4 and hence $\Gamma(\lambda^{-1} \otimes A_{ii})$ is finite for $i \in I(\lambda)$.

Let us denote

$$N_c(\lambda) = \bigcup_{i \in I(\lambda)} N_c(A_{ii}) = \left\{ j \in N; \gamma_{jj} = 0, j \in \bigcup_{i \in I(\lambda)} N_i \right\}.$$

Two nodes i and j in $N_c(\lambda)$ are called λ-*equivalent* (notation $i \sim_\lambda j$) if i and j belong to the same cycle whose mean is λ. Note that if $\lambda = \lambda(A)$ then $\sim_\lambda$ coincides with $\sim$.

Theorem 4.6.1 [44] *Suppose* $A \in \overline{\mathbb{R}}^{n\times n}$ *and* $\lambda \in \Lambda(A), \lambda > \varepsilon$. *Then* $g_j \in \overline{\mathbb{R}}^{n}$ (*that is*, g_j *does not contain* $+\infty$) *for all* $j \in N_c(\lambda)$ *and a basis of* $V(A, \lambda)$ *can be obtained by taking one* g_j *for each* $\sim_\lambda$ *equivalence class.*

Proof Let us denote $M = \bigcup_{i \in I(\lambda)} N_i$. By Lemma 4.1.3 we may assume without loss of generality that A is of the form

$$\begin{pmatrix} \bullet & \varepsilon \\ \bullet & A[M] \end{pmatrix}.$$

Hence $\Gamma(\lambda^{-1} \otimes A)$ is

$$\begin{pmatrix} \bullet & \varepsilon \\ \bullet & C \end{pmatrix}$$

where $C = \Gamma((\lambda(A[M]))^{-1} \otimes A[M])$, and the statement now follows by Proposition 1.6.10 and Theorem 4.3.5 since $\lambda = \lambda(A[M])$ and thus $\sim_\lambda$ equivalence for A is identical with $\sim$ equivalence for $A[M]$. □

Corollary 4.6.2 *A basis of* $V(A, \lambda)$ *for* $\lambda \in \Lambda(A), \lambda > \varepsilon$, *can be found using* $O(k^3)$ *operations, where* $k = |I(\lambda)|$ *and we have*

$$V(A, \lambda) = \{\Gamma(\lambda^{-1} \otimes A) \otimes z; z \in \overline{\mathbb{R}}^{n}, z_j = \varepsilon \textit{ for all } j \notin N_c(\lambda)\}.$$

Consequently, the bases of all eigenspaces can be found in $O(n^3)$ *operations.*

Using Lemma 4.2.1 and Corollary 4.6.2 we get:

Corollary 4.6.3 *If* $A \in \overline{\mathbb{R}}^{n\times n}$, $\lambda \in \Lambda(A)$ *and the dimension of* $V(A,\lambda)$ *is* r_λ *then there is a column* $\mathbb{R}$*-astic matrix* $G_\lambda \in \overline{\mathbb{R}}^{n\times r_\lambda}$ *such that*

$$V(A,\lambda) = \left\{G_\lambda \otimes z; z \in \overline{\mathbb{R}}^{r_\lambda}\right\}.$$

It follows from the proofs of Lemma 4.5.1 and Theorem 4.5.4 that $V(A,\lambda)$ can also be found as follows: If $I(\lambda) = \{j\}$ then define

$$M_2 = \bigcup_{N_i \to N_j} N_i, \qquad M_1 = N - M_2.$$

Hence

$$V(A,\lambda) = \{x; x[M_1] = \varepsilon, x[M_2] \in V^+(A[M_2])\}.$$

If the set $I(\lambda)$ consists of more than one index then the same process has to be repeated for each nonempty subset of $I(\lambda)$, that is, for each $J \subseteq I(\lambda)$, $J \neq \emptyset$, we set $S = \bigcup_{j\in J} N_j$ and

$$M_2 = \bigcup_{N_i \to S} N_i, \qquad M_1 = N - M_2.$$

Obviously, this is not a practical way of finding all eigenvectors as considering all subsets would be computationally infeasible, but it enables us to conveniently prove another criterion for the existence of finite eigenvectors:

Theorem 4.6.4 [10] $V^+(A) \neq \emptyset$ *if and only if* $\lambda(A)$ *is the eigenvalue of all final classes* (*in all superblocks*).

Proof The set M_1 in the above construction must be empty to obtain a finite eigenvector, hence a class in S must be reachable from every class of its superblock. This is only possible if S is the set of all final classes since no class is reachable from a final class (other than the final class itself). Conversely, if all final classes have the same eigenvalue $\lambda(A)$ then for $\lambda = \lambda(A)$ the set S contains all the final classes, they are reachable from all classes of their superblocks, and consequently $M_1 = \emptyset$, yielding a finite eigenvector. □

Corollary 4.6.5 $V^+(A) = \emptyset$ *if and only if a final class has eigenvalue less than* $\lambda(A)$.

Example 4.6.6 For the matrix A of Example 4.5.9 each of the two eigenspaces has dimension 1. Since

$$\Gamma((A_{11})_\lambda) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$V(A,2)$ is the set of multiples of $(1,0,\varepsilon,\varepsilon,\varepsilon,\varepsilon)^T$, similarly $V(A,5)$ is the set of multiples of $(\varepsilon,\varepsilon,\varepsilon,\varepsilon,\varepsilon,0)^T$. There are no finite eigenvectors since for the final class N_2 we have $\lambda(A_{22}) < 5$.

Remark 4.6.7 Note that a final class with eigenvalue less than $\lambda(A)$ may not be spectral and so $\Lambda(A) = \{\lambda(A)\}$ is possible even if $V^+(A) = \emptyset$. For instance in the case of

$$A = \begin{pmatrix} 1 & \varepsilon & \varepsilon \\ \varepsilon & 0 & \varepsilon \\ 0 & 0 & 1 \end{pmatrix}$$

we have $\lambda(A) = 1$, but $V^+(A) = \emptyset$.

Remark 4.6.8 Following the terminology of nonnegative matrices in linear algebra we say that a class is basic if its eigenvalue is $\lambda(A)$. It follows from Theorem 4.6.4 that $V^+(A) \neq \emptyset$ if basic classes and final classes coincide. Obviously this requirement is not necessary for $V^+(A) \neq \emptyset$, which is in contrast to the spectral theory of nonnegative matrices where for A to have a positive eigenvector it is necessary and sufficient that basic classes (that is, those whose eigenvalue is the Perron root) are exactly the final classes [126].

Remark 4.6.9 The principal eigenspace of any matrix may contain either finite eigenvectors only (for instance when the matrix is irreducible) or only nonfinite eigenvectors (see Remark 4.6.7), or both finite and non-finite eigenvectors, for instance when $A = I$.

4.7 Commuting Matrices Have a Common Eigenvector

The theory of commuting matrices in max-algebra seems to be rather modest at the time when this book goes to print: however, it is known that any two commuting matrices have a common eigenvector. This will be useful in the theory of two-sided max-linear systems (Chap. 7) and for solving some special cases of the generalized eigenproblem (Chap. 9).

Lemma 4.7.1 [70] *Let* $A, B \in \overline{\mathbb{R}}^{n\times n}$ *and* $A \otimes B = B \otimes A$. *If* $x \in V(A,\lambda)$, $\lambda \in \overline{\mathbb{R}}$, *then* $B \otimes x \in V(A,\lambda)$.

Proof We have $A \otimes x = \lambda \otimes x$ and thus

$$A \otimes (B \otimes x) = B \otimes (A \otimes x) = B \otimes \lambda \otimes x = \lambda \otimes (B \otimes x). \qquad \square$$

Theorem 4.7.2 (Schneider [107]) *If* $A, B \in \overline{\mathbb{R}}^{n\times n}$ *and* $A \otimes B = B \otimes A$ *then* $V(A) \cap V(B) \neq \{\varepsilon\}$, *more precisely, for every* $\lambda \in \Lambda(A)$ *there is a* $\mu \in \Lambda(B)$ *such that*

$$V(A,\lambda) \cap V(B,\mu) \neq \{\varepsilon\}.$$

Proof Let $\lambda \in \Lambda(A)$ and r_λ be the dimension of $V(A,\lambda)$. By Corollary 4.6.3 there is a matrix $G_\lambda \in \overline{\mathbb{R}}^{n\times r_\lambda}$ such that

$$V(A,\lambda) = \left\{ G_\lambda \otimes z; z \in \overline{\mathbb{R}}^{r_\lambda} \right\}.$$

Clearly, $A \otimes G_\lambda = \lambda \otimes G_\lambda$. It follows from Lemma 4.7.1 that all columns of $B \otimes G_\lambda$ are in $V(A,\lambda)$ and hence

$$B \otimes G_\lambda = G_\lambda \otimes C$$

for some $r_\lambda \times r_\lambda$ matrix C. Let $v \in V(C)$, $v \neq \varepsilon$, thus $v \in V(C,\mu)$ for some $\mu \in \overline{\mathbb{R}}$, and set $u = G_\lambda \otimes v$. Then $u \neq \varepsilon$ since G_λ is column $\mathbb{R}$-astic and we have:

$$A \otimes u = A \otimes G_\lambda \otimes v = \lambda \otimes G_\lambda \otimes v = \lambda \otimes u$$

and

$$B \otimes u = B \otimes G_\lambda \otimes v = G_\lambda \otimes C \otimes v = \mu \otimes G_\lambda \otimes v = \mu \otimes u.$$

Hence $u \in V(A,\lambda) \cap V(B,\mu)$ and $u \neq \varepsilon$. □

The proof of Theorem 4.7.2 is constructive and enables us to find a common eigenvector of commuting matrices: The system $B \otimes G_\lambda = G_\lambda \otimes C$ is a one-sided system for C and since a solution exists, the principal solution $\overline{C} = G_\lambda^* \otimes' (B \otimes G_\lambda)$ is a solution (Corollary 3.2.4).

Note that [107] contains more information on commuting matrices in max-algebra.

4.8 Exercises

Exercise 4.8.1 Find the eigenvalue, $\Gamma(A_\lambda)$ and the scaled basis of the unique eigenspace for each of the matrices below:

(a) $A = \begin{pmatrix} 3 & 6 \\ 2 & 1 \end{pmatrix}$. [$\lambda(A) = 4$;

$$\Gamma(A_\lambda) = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix},$$

the scaled basis is $\{(0,-2)^T\}$.]

(b) $A = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$. [$\lambda(A) = 0$; $\Gamma(A_\lambda) = A$, the scaled basis is $\{(0,-1)^T, (0,0)^T\}$.]

(c) $A = \begin{pmatrix} 1 & 0 & 4 & 3 \\ 0 & 1 & -3 & 3 \\ 0 & 1 & 0 & 2 \\ -3 & -1 & 0 & 1 \end{pmatrix}$. [$\lambda(A) = 2$;

$$\Gamma(A_\lambda) = \begin{pmatrix} 0 & 1 & 2 & 2 \\ -2 & -1 & 0 & 1 \\ -2 & -1 & 0 & 0 \\ -4 & -3 & -2 & -1 \end{pmatrix},$$

the scaled basis is $\{(0, -2, -2, -4)^T\}$.]

(d) Find the eigenvalue, $\Gamma(A_\lambda)$ and the scaled basis of the unique eigenspace of the matrix

$$A = \begin{pmatrix} 4 & 4 & 3 & 8 & 1 \\ 3 & 3 & 4 & 5 & 4 \\ 5 & 3 & 4 & 7 & 3 \\ 2 & 1 & 2 & 3 & 0 \\ 6 & 6 & 4 & 8 & 1 \end{pmatrix}.$$

[$\lambda(A) = 5$;

$$\Gamma(A_\lambda) = \begin{pmatrix} 0 & -1 & 0 & 3 & -2 \\ 0 & 0 & 0 & 3 & -1 \\ 0 & -1 & 0 & 3 & -2 \\ -3 & -4 & -3 & 0 & -5 \\ 1 & 1 & 1 & 4 & 0 \end{pmatrix},$$

the scaled basis is $\{(-1, -1, -1, -4, 0)^T, (-2, -1, -2, -5, 0)^T\}$.]

Exercise 4.8.2 Find all eigenvalues and the scaled bases of all eigenspaces of the matrix

$$A = \begin{pmatrix} 3 & 2 & & & & & & & \\ 2 & 3 & & & & & & & \\ & & 4 & & & & & & \\ & & & 3 & 4 & & & & \\ & & & 6 & 1 & & & & \\ & & & 4 & 1 & 7 & 2 & & \\ & & & & & 3 & 0 & & \\ & & & & & & & 1 & 4 \\ 0 & & & & & & & 0 & 2 \end{pmatrix},$$

where the missing entries are ε. [$\Lambda(A) = \{3, 4, 7, 2\}$, the scaled basis of $V(A, 3)$ is

$$\left\{(0, -1, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, -2, -3)^T, (-1, 0, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, -3, -4)^T\right\},$$

the scaled basis of $V(A, 4)$ is

$$\left\{(\varepsilon, \varepsilon, 0, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon)^T\right\},$$

the scaled basis of $V(A, 7)$ is

$$\left\{(\varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, 0, -4, \varepsilon, \varepsilon)^T\right\},$$

the scaled basis of $V(A, 2)$ is

$$\left\{(\varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, 0, -2)^T\right\}.]$$

Exercise 4.8.3 In the matrix A below the sign $\times$ indicates a finite entry, all other off-diagonal entries are ε. Find all spectral indices and all eigenvalues of A, and decide whether this matrix has finite eigenvectors.

$$A = \begin{pmatrix} 4 & & & & & & \\ \times & 3 & & & & & \\ & \times & 5 & & & & \\ & & & 7 & & & \\ & & & \times & 8 & & \\ & \times & \times & & & 2 & \\ & & \times & \times & & & 4 \end{pmatrix}$$

[Spectral indices: 3, 5, 6, 7, $\Lambda(A) = \{5, 8, 2, 4\}$, no finite eigenvectors.]

Exercise 4.8.4 Prove that $\lambda(A) = \lambda(A^T)$, $\Gamma(A^T) = (\Gamma(A))^T$ and $N_c(A) = N_c(A^T)$ for every square matrix A. Then prove or disprove that $\Lambda(A) = \Lambda(A^T)$. [false]

Exercise 4.8.5 Prove or disprove each of the following statements:

(a) If $A \in \mathbb{Z}^{n\times n}$ then A has an integer eigenvector if and only if $\lambda(A) \in \mathbb{Z}$. [true]
(b) If $A \in \mathbb{R}^{n\times n}$ then A has an integer eigenvector if and only if $\lambda(A) \in \mathbb{Z}$. [false]
(c) If $A \in \mathbb{R}^{n\times n}$ then A has an integer eigenvalue and an integer eigenvector if and only if $A \in \mathbb{Z}^{n\times n}$. [false]

Exercise 4.8.6 We say that $T = (t_{ij}) \in \mathbb{R}^{n\times n}$ is triangular if it satisfies the condition $t_{ij} < \lambda(T)$ for all $i, j \in N$, $i \leq j$. Prove the statement: If $A \in \mathbb{R}^{n\times n}$ then $\lambda(A) = \lambda(B)$ for every B equivalent to A if and only if A is not equivalent to a triangular matrix. [See [39]]

Exercise 4.8.7 Show that the maximum cycle mean and an eigenvector for $0-1$ matrices can be found using $O(n^2)$ operations. [See [33, 66]]

Exercise 4.8.8 Prove that the following problem is NP-complete: Given $A \in \overline{\mathbb{R}}^{n\times n}$ and $x \in \overline{\mathbb{R}}^n$, decide whether it is possible to permute the components of x so that the obtained vector is an eigenvector of A. [See [31]]

Exercise 4.8.9 Let A and B be square matrices of the same order. Prove then that the set of finite eigenvalues of $A \otimes B$ is the same as the set of finite eigenvalues of $B \otimes A$.