Chapter 3
Circuit Theory and Other Discrete Physical Models

Abstract In this chapter we present linear electric circuit theory as the central physical model for performing calculus on networks. We adopt circuit theory as the central physical model for applying and understanding the concepts of discrete calculus on graphs for three reasons: because much of the progress in graph theory over the last century was created in the context of circuit theory; because of the early connection made between circuit theory and algebraic topology (Branin Jr. in Proc. of Conf. on Neural Networks, pp. 453–491, 1966; Roth in Proceedings of the National Academy of Sciences of the United States of America 41(7):518–521, 1955); and because circuits are physical, realizable systems which need not be seen as discretizations of an underlying continuous domain but rather as a domain unto themselves. Our focus in this chapter will be to cover the main concepts in linear circuit theory from an algebraic standpoint with a focus on operators. This will prepare the reader with notation and concepts that tie naturally to the previous chapter in which the discrete analogues of differential operators were introduced. We begin with definitions of the physical quantities and corresponding quantities in circuit theory, then proceed to define the laws that relate the quantities to each other, treat methods of solving for the unknowns, and end by connecting circuit theory to other discrete processes.

We recognize that our readers may have varying levels of familiarity with circuit theory, therefore no previous experience with circuit theory is assumed. However, regardless of the reader's familiarity with circuit theory, we believe that this chapter is useful for producing an intuitive understanding of the mathematical devices and discrete analogues of continuous differential operators that were developed in Chap. 2. In order to develop this intuition, our intention with this chapter is to establish notation, provide a physical model for the variables involved in the formalism presented in this work, and to gently introduce concepts that we will encounter again and again in this work. Therefore, we would strongly encourage the reader to at least briefly read this chapter regardless of the reader's knowledge level of circuit theory.

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We consider a circuit as a graph consisting of a set of nodes and edges. An edge in a circuit is interpreted physically as a wire that connects two nodes and which has some resistance to the flow of current. In circuit theory, edges are also known as **branches** or **arcs**. A node in a circuit is interpreted physically as any junction between two or more wires (edges) or as a terminal location that receives or sinks energy (more on this concept in a moment).

Viewing electrical circuits from a physics perspective, we are concerned with electrons and their motion through the circuit governed by an energy minimization principle. When there is an unequal distribution of electrons among different nodes in a circuit, then there is an **electric potential** for each node to equilibrate with a reference node, known as the **ground node**. The term *ground* is derived from the safe, neutral potential maintained naturally by the metallic content of the earth. The variable $x_i \in \mathbb{R}$ will be used to represent the electric potential at a single node v_i. The grounded node v_0 has a fixed potential $x_0 = 0$. Electric potential at a node is quantified in units of volts. If a potential x_i is negative, then there are more electrons present at node v_i then at the reference ground and if a potential x_i is positive, then there are fewer electrons present at node v_i then at the reference ground node. We will manipulate the node potentials as a single vector-valued variable $\mathbf{x} \in \mathbb{R}^{|\mathcal{V}|}$, which may be interpreted as a 0-cochain (node function).

The **current** that passes through a wire is a measure of the number of electrons that pass through the wire per unit time. Current is measured in units of amperes (i.e., coulombs per second) and is represented here by the vector $\mathbf{y} \in \mathbb{R}^{|\mathcal{E}|}$. The elements of $\mathbf{y}$, $y_i \in \mathbb{R}$, represent the current passing through the edges of the circuit $e_i \in \mathcal{E}$. Since $\mathbf{y}$ is defined on the edges, then this current vector is a 1-cochain. The value of any current variable y_{ij} through a wire may be positive or negative, and the sign indicates the direction y_i of positive current flow through edge e_i. In order to accommodate this interpretation of the sign of y_i, it is necessary to endow each edge with a sense of orientation (as seen in Chap. 2). The orientation for each edge variable may be assigned arbitrarily, thus a negative value is interpreted as a current flow *in the opposite direction* of the assigned orientation of the edge variable. Therefore, given a set of n nodes $\mathcal{V}$ with elements $v_i \in \mathcal{V}$, we represent each edge, e_{ij}, by the ordered pair $e_{ij} = \{v_i, v_j\}$. If current is flowing from node v_i to v_j the current will be *positive* and if the current is flowing from node v_j to v_i then the current will be *negative*. The designation of edge orientation is essential to solve problems in circuit theory, but the particular choice of e_{ij} compared to e_{ji} has no consequence for the solution to any circuit theory problem (except that the current calculated over edge e_{ij} will be the negative of the current calculated over e_{ji}). Given an edge with orientation e_{ij}, then the **voltage** across e_{ij} is defined by the difference of the potentials at the pair of nodes at the ends of the edge, $p_{ij} = v_i - v_j$.

The vectors of voltages $\mathbf{p}$ and currents $\mathbf{y}$ both take a value for every edge and each component of these two vectors is related to each other via the **resistance** of the edge, r_{ij} (in Sect. 3.3 the edges are generalized to represent capacitors and inductors as well). Resistance is measured in units of Ohms and is physically interpreted as the amount of work required to move electrons through a wire. For our purposes, we will assume that all $r_{ij} > 0$. A wire with higher resistance requires more work to

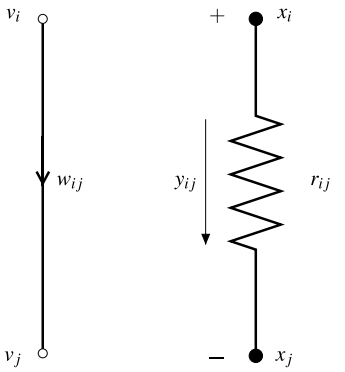

Fig. 3.1 Conventions for electric circuit theory. (*Left*) An edge in an oriented graph between nodes v_i and v_j with edge weight w_{ij} is equivalent to (*right*) a resistor in a circuit with node potentials x_i and x_j where the edge variable y_{ij} represents a current flow from node v_i to node v_j. The (distance) weight is provided by the resistance r_{ij}

move electrons through the wire and a wire with low resistance requires less work to move electrons through the wire. The resistance of each edge is a material property of each wire. The **conductance** at each edge, w_{ij}, is defined as the reciprocal of the resistance $w_{ij} = 1/r_{ij}$. Wire conductance is measured in units of Siemens and will emerge as the more convenient quantity to use in the theoretical development and applications in this work. In the context of Chap. 2, resistance represents a *distance* weight and conductance represents an *affinity* weight. As seen in the last chapter, the distance weight (resistance) is the reciprocal of the affinity weight (conductance). The collection of nodes, oriented edges, and resistors is the domain upon which all of the circuit laws and dynamics play out. Figure 3.1 summarizes the relationship between a resistive branch in an electric circuit and the equivalent graph edge.

3.1 Circuit Laws

Having defined the variables in the previous section, we now proceed to describe how these variables relate to each other.

Voltage across an edge and current through an edge are related through the conductance via **Ohm's Law** which states that

$$\mathbf{p} = \mathbf{G}\mathbf{y}, \tag{3.1}$$

where $\mathbf{p}$ represents the voltage each edge, $\mathbf{y}$ represents the current through each edge and $\mathbf{G}$ is a diagonal matrix containing the edge resistances along the diagonal, i.e., with $\mathbf{G}_{ii} = r_i$ for some $e_i \in \mathcal{E}$. Ohm's Law states that an edge with a linear resistor induces a linear relationship between voltage across an edge and the current through the edge.

Kirchhoff's Current Law (KCL) states that all of the current flowing into a node also flows out of the node. This law may be represented in matrix form via the equation

$$\mathbf{A}^{\mathsf{T}}\mathbf{y} = \mathbf{0}, \tag{3.2}$$

where $\mathbf{A}$ is the *edge–node* incidence matrix as defined in Chap. 2.

The relationship between electric potentials at nodes and voltages across edges may also be written in terms of the incidence matrix as

$$\mathbf{Ax} = \mathbf{p}, \tag{3.3}$$

which is known as **Kirchhoff's Voltage Law** (KVL). A consequence of KCL and KVL together is **Tellegen's Theorem** [305] which states that the vector of voltages is orthogonal to the vector of currents, i.e.,

$$\mathbf{p}^\mathsf{T}\mathbf{y} = \mathbf{x}^\mathsf{T}\mathbf{A}^\mathsf{T}\mathbf{y} = \mathbf{0}. \tag{3.4}$$

Tellegen's theorem has been employed several times in the literature as a tool of network analysis, e.g., [301, 302].

These three laws of circuit theory, Ohm's Law, Kirchhoff's Current Law and Kirchhoff's Voltage Law define all of the behavior of linear circuits. These three laws may be composed into one single law which states

$$\mathbf{A}^\mathsf{T}\mathbf{G}^{-1}\mathbf{Ax} = \mathbf{Lx} = \mathbf{0} \tag{3.5}$$

where $\mathbf{L}$ represents the Laplacian matrix from Chap. 2. It was shown in Chap. 2 that the nullspace of the $\mathbf{L} = \mathbf{A}^\mathsf{T}\mathbf{G}^{-1}\mathbf{A}$ matrix is dimension one and spanned by the vector for which each element is equal to the same constant. Therefore, the three laws compose to state that, in the absence of any other constraints, all of the node potentials will be equal. In the next section, we discuss the solution of the circuit equations in the presence of energy sources that force a nontrivial solution. In the context of circuit theory, we will consider a *solution* as a complete knowledge of the node potentials or, equivalently, edge voltages or edge currents for a specific circuit topology and a prescribed set of resistance values. Given any one of these sets of values, the others may be generated using the three circuit laws given above.

3.2 Steady-State Solutions

A **voltage source** or a **current source** provides a fixed source of energy to drive the circuit. Without these energy sources, the circuit would yield the trivial solution consisting of a constant potential at each node and no flow of current. Both of these sources supply a fixed voltage or current which is constant with respect to time and are traditionally known as direct current or DC sources. A voltage source forces the voltage between two nodes to be fixed and a current source forces the current through an edge to be fixed.

By convention, a voltage source can be added in series to any resistor in the graph to insert an additional fixed voltage difference between the two nodes of the resistor [360, p. 149]. The value of the voltage added to edge e_k is denoted by b_k, and all such voltage sources can be assembled into a vector $\mathbf{b} \in \mathbb{R}^{|\mathcal{E}|}$. Note that the elements b_k may be positive or negative, and are zero-valued at edges where no voltage source is

present. Therefore, when voltage sources are present, the expression for the voltages in the circuit becomes

$$\mathbf{A}\mathbf{x} + \mathbf{b} = \mathbf{p} \tag{3.6}$$

with the edge–node incidence matrix $\mathbf{A}$ defined as above. The expression of Ohm's law, $\mathbf{y} = \mathbf{G}^{-1}\mathbf{p}$, is unchanged. Similarly, a current source with one end attached to ground can be attached to any other node in order to inject current into the node. If we represent the current inserted into node v_i as f_i, and assemble all additional current sources into a vector $\mathbf{f} \in \mathbb{R}^{|\mathcal{V}|}$, then the expression of KCL in the circuit becomes

$$\mathbf{A}^{\mathsf{T}}\mathbf{y} + \mathbf{f} = \mathbf{0}. \tag{3.7}$$

By combining these two laws and including the voltage and current source terms, we return to the single law summarizing the circuit behavior, given by

$$\mathbf{A}^{\mathsf{T}}\mathbf{G}^{-1}\mathbf{A}\mathbf{x} = \mathbf{L}\mathbf{x} = -\mathbf{A}^{\mathsf{T}}\mathbf{G}^{-1}\mathbf{b} - \mathbf{f}. \tag{3.8}$$

We let $\mathbf{q}$ represent the set of source terms on the right hand side of this equation, i.e., $\mathbf{q} = -\mathbf{A}^{\mathsf{T}}\mathbf{G}^{-1}\mathbf{b} - \mathbf{f}$.

Due to the nature of the circuit behavior, the above equations can not be solved uniquely for any given circuit configuration due to the inherent ambiguity in the node potentials—a constant potential can be added to each node without changing the voltages across the edges. In other words, the solution is not affected by adding a constant potential to all nodes. Therefore, a family of admissible solutions exists that are equivalent up to this constant potential. For this reason, it is often convenient to assign a single node v_0 as a reference or ground by eliminating one column of the edge–node incidence matrix $\mathbf{A}$. This **reduced incidence matrix** is often denoted as $\mathbf{A}_0$. The potential x_0 is set to zero and the corresponding node is eliminated from $\mathbf{x}$ and from $\mathbf{f}$. Once an arbitrary reference node is set to ground, then the solution can be calculated relative to the chosen reference.

Example 1 (Solving node potentials in a static circuit) To demonstrate the process of constructing the discrete operators for a circuit and using them to solve for a set of unknown node potentials or edge currents, consider the circuit diagram presented in Fig. 3.2. The edge–node incidence matrix $\mathbf{A}$ for this circuit can be constructed from the diagram after assigning an orientation for each edge. For this graph the incidence matrix is given by

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix}$$

based on an assumed orientation. The constitutive matrix consists of the affinity edge weights (conductances). Therefore, $\mathbf{G}^{-1} = \mathrm{diag}([2\ 1\ 1\ 4\ 1]^{\mathsf{T}})$ and the Laplacian

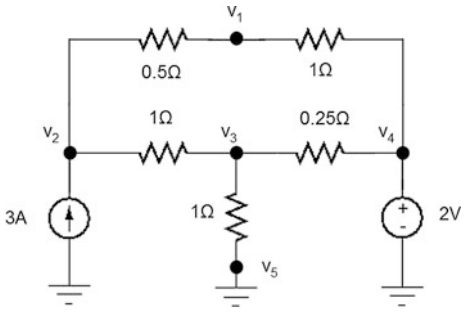

Fig. 3.2 Circuit diagram for Example 1. The circuit consists of five nodes and five edges, with a fixed current source injecting current into node v_2, a voltage source setting a fixed potential at node v_4, and node v_5 serving as the reference node for ground. Note that the standard symbol for resistance (Ohms) is given by Ω

matrix $\mathbf{L} = \mathbf{A}^{\mathsf{T}}\mathbf{G}^{-1}\mathbf{A}$ evaluates to

$$\mathbf{L} = \begin{bmatrix} 3 & -2 & 0 & -1 & 0 \\ -2 & 3 & -1 & 0 & 0 \\ 0 & -1 & 6 & -4 & -1 \\ -1 & 0 & -4 & 5 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix}.$$

In this example, the voltage source forces a potential at node v_4, indicating that this node voltage is not free (or unknown), and the voltage source does not coincide with an edge variable. For this reason the voltage source vector $\mathbf{b}$ is expressible in terms of the edge–node incidence matrix operating on the fixed voltages, i.e.,

$$\mathbf{b} = \mathbf{A}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 0 \\ -2 \\ 0 \end{bmatrix}.$$

The current source vector $\mathbf{f}$ is straightforward: $\mathbf{f} = [0\ 3\ 0\ 0\ 0]^{\mathsf{T}}$.

With the weighted Laplacian operator $\mathbf{L}$ defined for the circuit, and the vectors $\mathbf{b}$ and $\mathbf{f}$ representing the energy sources, the unknown node potentials $\mathbf{x}$ can be solved with any linear systems solver. However, the full Laplacian matrix is not full rank, and must be reduced to solve the system. Since nodes v_4 and v_5 are fixed, we can solve the reduced system $\mathbf{L}_0\mathbf{x}_0 = \mathbf{q}_0$, whose entries correspond only to the free nodes in the circuit, v_1, v_2, and v_3 (i.e., the '0' subscript indicates that the entries corresponding to fixed nodes have been removed). Therefore, for this example the node voltages evaluate to

$$\mathbf{x}_0 = \begin{bmatrix} 4.07 & 5.11 & 2.19 \end{bmatrix}^{\mathsf{T}}.$$

3.2.1 Dependent Sources

In the last section we considered voltage (current) sources which supplied a fixed voltage (current) across two branches. However, if the amount of voltage (current) supplied by the source is controlled by the voltage or current of a different branch, then the source is said to be a **dependent source**. In contrast, a fixed source considered above is said to be an **independent source**. Dependent sources can appear in the linear circuit equations as *off-diagonal elements of* $\mathbf{G}^{-1}$.

Before discussing the implications of the matrix $\mathbf{G}^{-1}$ with off-diagonal elements, we first illustrate the concept with an example. Consider the small circuit in Fig. 3.3 in which we employ the matrix

$$\mathbf{G}^{-1} = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \tag{3.9}$$

as the edge weighting matrix. Clearly all of the branch resistances are unit-weighted, but now there is a new relationship between edges 1 and 3. The nature of this relationship becomes apparent if we consider the set of branch voltages

$$\mathbf{p} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{3.10}$$

giving the currents

$$\mathbf{y} = \mathbf{G}^{-1}\mathbf{p} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}. \tag{3.11}$$

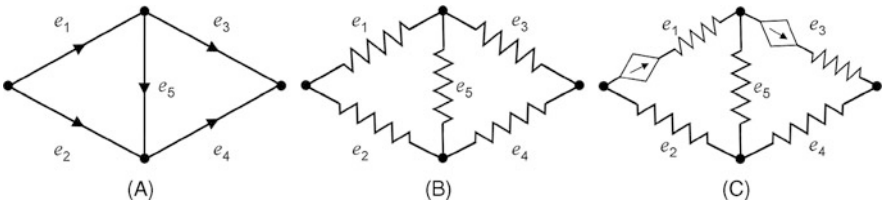

Fig. 3.3 A circuit with dependent current sources

Therefore, even though the voltage occurred over branch 1, a current was induced in branch 3, which allows us to interpret the off-diagonal entries as indicating the presence of *voltage-controlled current sources*, which are often used in circuit theory to model nonlinear elements such as transistors. This example is illustrated in Fig. 3.3.

In Chap. 2, we discussed how the metric tensor definition for a discrete space led to a diagonal matrix for $\mathbf{G}^{-1}$. Therefore, the appearance of off-diagonal elements in $\mathbf{G}^{-1}$ should be regarded only as a convenience that allows us to represent the dependent sources and/or nonlinear circuit elements.

By interpreting a $\mathbf{G}^{-1}$ having off-diagonal elements as representing dependent sources, we can derive an unusual interpretation of the discrete **biharmonic** operator. Recall that the biharmonic operator is defined in conventional mathematics as $\nabla^2\nabla^2$ and appears frequently in the description of the bending of thick-plate material (as opposed to the membrane deformation described by the standard Laplacian operator). Based on Chap. 2, we may rewrite the discrete biharmonic operator as

$$\mathbf{LL} = \mathbf{A}^{\mathsf{T}}\mathbf{G}_1^{-1}\mathbf{A}\mathbf{G}_0\mathbf{G}_0\mathbf{A}^{\mathsf{T}}\mathbf{G}_1^{-1}\mathbf{A}. \tag{3.12}$$

By letting

$$\tilde{\mathbf{G}}_1^{-1} = \mathbf{G}_1^{-1}\mathbf{A}\mathbf{G}_0\mathbf{G}_0\mathbf{A}^{\mathsf{T}}\mathbf{G}_1^{-1}, \tag{3.13}$$

then we can interpret the biharmonic operator as a Laplacian operator with a non-diagonal constitutive matrix $\tilde{\mathbf{G}}_1^{-1}$, i.e.,

$$\mathbf{L}^{\mathsf{T}}\mathbf{L} = \tilde{\mathbf{L}} = \mathbf{A}^{\mathsf{T}}\tilde{\mathbf{G}}_1^{-1}\mathbf{A}. \tag{3.14}$$

Since $\tilde{\mathbf{G}}_1^{-1}$ has off-diagonal elements, this interpretation allows us to further consider the biharmonic operator as introducing a set of *dependent sources* into the circuit.

As an example, consider the graph in Fig. 3.4 with unity edge and node weights. The biharmonic operator for the graph will consist of

$$\mathbf{L}^{\mathsf{T}}\mathbf{L} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 6 & -4 & -4 & 2 \\ -4 & 12 & -4 & -4 \\ -4 & -4 & 12 & -4 \\ 2 & -4 & -4 & 6 \end{bmatrix} \end{array}. \tag{3.15}$$

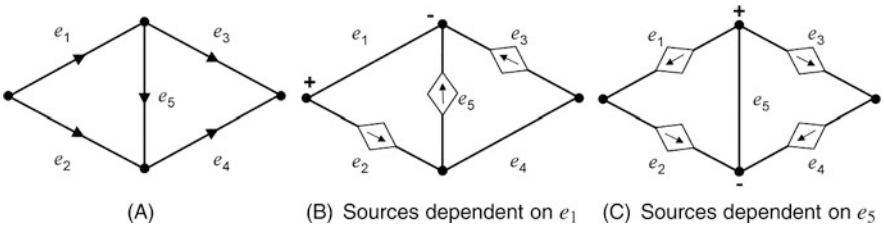

Fig. 3.4 Voltage-controlled current sources used to interpret the discrete biharmonic operator. Panels (**B**) and (**C**) display the current sources which respond to a voltage across e_1 and e_5, respectively, in the direction given by the '+' and '−' signs

This biharmonic operator may be viewed as the Laplacian, $\tilde{\mathbf{L}}$, for $\mathbf{L}^{\mathsf{T}}\mathbf{L} = \tilde{\mathbf{L}} = \mathbf{A}^{\mathsf{T}}\tilde{\mathbf{G}}_1^{-1}\mathbf{A}$, in which

$$\tilde{\mathbf{G}}_1^{-1} = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 2 & 1 & -1 & 0 & -1 \\ 1 & 2 & 0 & -1 & 1 \\ -1 & 0 & 2 & 1 & 1 \\ 0 & -1 & 1 & 2 & -1 \\ -1 & 1 & 1 & -1 & 2 \end{bmatrix} \end{array}. \tag{3.16}$$

The current sources which depend on voltages across various branches for this circuit are depicted in Fig. 3.4.

3.2.2 Energy Minimization

The solution for the node potentials given by (3.5) may be considered as the minimum energy solution for a variational problem. Specifically, if we define the total energy for a configuration of voltages $\mathbf{x}$ as

$$\mathcal{E}[\mathbf{x}] = \mathbf{x}^{\mathsf{T}}\mathbf{L}\mathbf{x}, \tag{3.17}$$

then, because $\mathbf{L}$ is both symmetric and positive semi-definite, the steady-state solution

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}\mathbf{x}} = \mathbf{L}\mathbf{x} = \mathbf{0}, \tag{3.18}$$

represents a minimum of the energy in (3.17). Note that it is possible to introduce current sources into the above energy by letting $\mathcal{E}[\mathbf{x}] = \mathbf{x}^{\mathsf{T}}\mathbf{L}\mathbf{x} + \mathbf{x}^{\mathsf{T}}\mathbf{f}$. Likewise, it is possible to introduce voltage sources by fixing the potentials $\mathbf{x}$ and taking the gradient of the energy with respect to $\mathbf{b}$. The energy in (3.17) represents electrical *power dissipation* of the circuit and may be rewritten in several equivalent ways.

For example, the power can be phrased as: (a) as the sum of voltages multiplied by currents across all edges,

$$\mathcal{E}[\mathbf{x}] = \mathbf{x}^{\mathsf{T}}\mathbf{L}\mathbf{x} = \mathbf{p}^{\mathsf{T}}\mathbf{y} = \sum_{e_{ij}} p_{ij}\, y_{ij}, \tag{3.19a}$$

(b) as the sum of squared voltages divided by resistances across all edges,

$$\mathcal{E}[\mathbf{x}] = \mathbf{x}^{\mathsf{T}}\mathbf{L}\mathbf{x} = \mathbf{p}^{\mathsf{T}}\mathbf{G}^{-1}\mathbf{p} = \sum_{e_{ij}} \frac{p_{ij}^2}{r_{ij}}, \tag{3.19b}$$

or (c) as the sum of squared currents multiplied by resistances across all edges,

$$\mathcal{E}[\mathbf{x}] = \mathbf{x}^{\mathsf{T}}\mathbf{L}\mathbf{x} = \mathbf{y}^{\mathsf{T}}\mathbf{G}\mathbf{y} = \sum_{e_{ij}} y_{ij}^2\, r_{ij}. \tag{3.19c}$$

These three expressions for power dissipated by each edge are classical. The energy minimization principles expressed above in terms of circuit theory are equivalent to the conventional Dirichlet's principle on a graph that was discussed in Chap. 2.

Using the above expressions for power dissipation, we now show that the energy minimization formulation may also be written in terms of the current variable. Specifically, the currents distribute themselves to optimize

$$\begin{aligned} &\min_{\mathbf{y}} \mathcal{E}[\mathbf{y}] = \min_{\mathbf{y}} \mathbf{y}^{\mathsf{T}}\mathbf{G}\mathbf{y}, \\ &\text{s.t.} \quad \mathbf{A}^{\mathsf{T}}\mathbf{y} = \mathbf{0}. \end{aligned} \tag{3.20}$$

This constrained optimization may be converted to an unconstrained optimization (see Appendix B) by noting that any solution that satisfies the constraints must lie in the nullspace of $\mathbf{A}^{\mathsf{T}}$. If we represent a basis to span the nullspace of $\mathbf{A}^{\mathsf{T}}$ by the matrix $\mathbf{B}$, then we may write $\mathbf{y} = \mathbf{B}\mathbf{z}$ in terms of a new variable $\mathbf{z}$. Adding this change of variable into the optimization of the currents gives us the unconstrained optimization problem

$$\min_{\mathbf{z}} \mathcal{E}[\mathbf{z}] = \min_{\mathbf{z}} \mathbf{z}^{\mathsf{T}}\mathbf{B}^{\mathsf{T}}\mathbf{G}\mathbf{B}\mathbf{z}. \tag{3.21}$$

Once again, current sources or voltage sources provide constraints that, when included in the energy formulation in terms of $\mathbf{z}$, produce a nontrivial solution. The matrix $\mathbf{B}$ has the convenient form of the *face–edge* incidence matrix and the $\mathbf{z}$ variables are associated with the faces (known as "mesh variables" in the circuit literature). In the context of circuit theory, solution for the circuit variables via optimization of the first energy minimization problem (3.17) in terms of the node variables is known as **node analysis**. Similarly, optimization of the second energy minimization problem of (3.21) in terms of the mesh variables is known as **mesh analysis**. In some circumstances, mesh analysis is much easier or has many fewer variables (e.g., if the circuit consists of a single long cycle). However, in typical circuits, the extra

computation required to find cycles and thus generate $\mathbf{B}$ is not justified by any computational gain in performing mesh analysis instead of node analysis. Consequently, the practical solution of circuit problems is predominated by node analysis.

3.2.2.1 Power Minimization with Nonlinear Resistors

So far we have considered graph edges to represent resistors which act to translate differences in quantities defined at the nodes (electric potentials) into flows through the edges (currents). Linear resistors create flows which are proportional to the potential differences, but we may also consider edges which act as nonlinear resistances as well. We may view the linear resistor circuit as minimizing the *square* of the difference of node variables by writing the circuit power in (3.17) as a summation over edges

$$\mathcal{E}[\mathbf{x}] = \mathbf{x}^{\mathsf{T}}\mathbf{L}\mathbf{x} = \mathbf{x}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}\mathbf{G}^{-1}\mathbf{A}\mathbf{x} = \sum_{e_{ij}} w_{ij}(x_i - x_j)^2. \tag{3.22}$$

Viewing the circuit power dissipation in this form, we can see that other functions of the difference between nodes may be optimized in the circuit by introducing *nonlinear resistors*. For example, by using resistors with the value $r_{ij} = \frac{1}{w_{ij}}(x_i - x_j)^{2-p}$, the power dissipated by the circuit becomes

$$\mathcal{E}[\mathbf{x}] = \sum_{e_{ij}} w_{ij}|x_i - x_j|^p. \tag{3.23}$$

Therefore, by using a nonlinear resistor we may build a circuit for which the minimum power distribution minimizes the measurement of the potential differences under any p-norm. It has been shown [350] that if $p = 1$ (i.e., using nonlinear resistors with value $r_{ij} = \frac{1}{w_{ij}}|x_i - x_j|$) then the minimum power solution may be found by solving a *maximum-flow/minimum-cut problem* for a particular configuration of sources. Specifically, if all power sources either ground some nodes or fix node potentials to unit voltage above ground, then any partitioning of the nodes obtained by thresholding the electrical potentials at 0.5 represents a minimum cut between the nodes of unit voltage and the grounded nodes. In the second part of this book, many applications will require an optimization of an energy in the form (3.23), which will be called the *Basic Energy Model* in the context of these applications. Therefore, each of the applications using he Basic Energy Model may be interpreted as finding the minimum power dissipation of a circuit with nonlinear resistors (3.23).

3.3 AC Circuits

Both linear and nonlinear resistors map potential differences directly to flows. We now consider edge elements that instead translate between flows and differences via

the *change* in these quantities over time. Such a circuit element may be physically realized in the form of the standard circuit elements of a **capacitor** and **inductor**. In this book, we will not consider any of the details of real capacitors and inductors, but rather work with their idealized mathematical behavior.

A capacitor translates the temporal change in potential difference between nodes into a current flow through the edge connecting the nodes. Specifically, the relationship is given by

$$y_{ij} = c_{ij} \frac{\mathrm{d}p_{ij}}{\mathrm{d}t}, \tag{3.24}$$

where the variable c is used to represent **capacitance**, and $\mathrm{d}t$ is the time differential. We may write the capacitance equation (3.24) in integral form for the voltage as

$$p_{ij} = \frac{1}{c_{ij}} \int y_{ij} \, \mathrm{d}t. \tag{3.25}$$

The inductor element has the opposite effect as the capacitor in the sense that the inductor translates the change in current flow through an edge into a potential differences across the edge. Specifically, the relationship is given by

$$p_{ij} = u_{ij} \frac{\mathrm{d}y_{ij}}{\mathrm{d}t}, \tag{3.26}$$

where u_{ij} represents the inductance of the edge e_{ij}.

With these additional elements in a circuit, we may consider the response of the current flow through an edge when the voltage across the edge varies periodically with time (i.e., an **alternating voltage source**). Because the circuits we consider represent linear and time invariant systems, these time varying quantities can be conveniently represented with complex exponentials (or superpositions thereof). If we represent the voltage as the real part of $\mathrm{Re}\{p_{ij}\, \mathrm{e}^{\mathrm{i}\omega t}\}$, then we may substitute the current $\mathrm{Re}\{y_{ij}\, \mathrm{e}^{\mathrm{i}\omega t}\}$ into (3.25) and (3.26) to produce the current flow through an edge containing an inductor, resistor and capacitor in series

$$\left(\mathrm{i}\omega u_{ij} + r_{ij} + \frac{1}{\mathrm{i}\omega c_{ij}}\right) y_{ij}\, \mathrm{e}^{\mathrm{i}\omega t} = p_{ij}\, \mathrm{e}^{\mathrm{i}\omega t}. \tag{3.27}$$

Note that ω represents the oscillation frequency of the sources. Since the factor $\mathrm{e}^{\mathrm{i}\omega t}$ appears on both sides of the equation, it may be ignored to give us a relationship which is *independent of time*

$$\left(\mathrm{i}\omega u_{ij} + r_{ij} + \frac{1}{\mathrm{i}\omega c_{ij}}\right) y_{ij} = p_{ij}. \tag{3.28}$$

The factor applied to the current is a constant which depends on the material properties of the edge (the resistance, inductance and capacitance of the edge). Therefore, we may replace this factor with

$$z_{ij} = \left(\mathrm{i}\omega u_{ij} + r_{ij} + \frac{1}{\mathrm{i}\omega c_{ij}}\right), \tag{3.29}$$

which is known as the **impedance** of the edge and may be considered as a generalized conception of the edge resistance. Writing the time-independent equation for an alternating voltage/current relationship in terms of the impedance gives

$$z_{ij} y_{ij} = p_{ij}, \tag{3.30}$$

which may be viewed as a generalized form of Ohm's Law (3.1) that reflects the "complex resistor" represented by the impedance. Similar to the relationship between resistance and conductance, the reciprocal of the impedance is called the **admittance**, $\psi_{ij} = 1/z_{ij}$.

The introduction of capacitors, inductors and alternating energy sources affects only the relationship of the voltage across an edge to the current through the edge. However, the voltages are still generated as the differences of the electrical potentials, (3.3), and the sum of the currents into a node still equals the currents leaving a node, (3.2). Consequently, if all of the sources have the same frequency of oscillation, ω, then we may write the three laws of circuit theory for an alternating current (AC) circuit as

$$\mathbf{A}\mathbf{x} = \mathbf{p}, \tag{3.31}$$

$$\mathbf{\Psi}\mathbf{p} = \mathbf{y}, \tag{3.32}$$

$$\mathbf{A}^{\mathsf{T}}\mathbf{y} = \mathbf{0}, \tag{3.33}$$

where $\mathbf{\Psi}$ represents the diagonal matrix containing the admittance of each edge along the diagonal. As before, these laws combine to give

$$\mathbf{A}^{\mathsf{T}}\mathbf{\Psi}\mathbf{A}\mathbf{x} = \mathbf{0}, \tag{3.34}$$

which governs the steady-state distribution of currents and potentials in a circuit where $\mathbf{x}$ and $\mathbf{y}$ are understood to be multiplied by $\mathrm{Re}\{e^{i\omega t}\}$.

We may now adopt the viewpoint that the steady state behavior of a circuit with alternating sources is equivalent to the steady state behavior of a circuit with constant sources that has "complex resistors". Given this viewpoint, we may follow the procedure above to produce the response of the circuit to voltage and current sources of various magnitudes (but the same oscillation frequency) to give

$$\mathbf{L}\mathbf{x} = -(\mathbf{A}^{\mathsf{T}}\mathbf{\Psi}\mathbf{A}\mathbf{x} + \mathbf{f}). \tag{3.35}$$

Similarly, we may consider this equation as the solution to a minimization of the power dissipated by the circuit given by

$$\mathcal{E}[\mathbf{x}] = \mathbf{x}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}\mathbf{\Psi}\mathbf{A}\mathbf{x}. \tag{3.36}$$

Since $\mathbf{\Psi}$ is complex valued, the power dissipation in the above equation is also a complex-valued quantity, which is referred to in the circuit literature as the **complex power**. The magnitude, $|\mathcal{E}[\mathbf{x}]|$, is called the **apparent power**, while the real component $\mathrm{Re}\{\mathcal{E}[\mathbf{x}]\}$ is called the **real power** and the imaginary part $\mathrm{Im}\{\mathcal{E}[\mathbf{x}]\}$ is called the **reactive power**. In a physical circuit, the real power is the only energy dissipated by the system.

3.4 Connections Between Circuit Theory and Other Discrete Domains

Our main purpose for reviewing circuit theory is to provide a physical, realizable model for discrete mechanics that provides a natural analogy for the continuous calculus developed in the last chapter. The utility of seeing circuit theory from the standpoint of linear algebra is that it focuses attention on the *operators* (defined by incidence matrices) that define the circuit laws and the energy minimization problems behind these laws.

Although circuit theory is a natural setting to explore the physical interpretation of the discrete calculus machinery developed in Chap. 2, circuit theory is by no means the only discipline in which these mechanics appear. In fact, just like conventional vector calculus is used to describe the structure of many physical theories, so too is discrete calculus. Specifically, we will use the treatment of circuit theory above to examine the topics of spring networks, random walks on a graph, Markov Random Fields, the use of tree counting in graph theory and linear algebra.

3.4.1 Spring Networks

Spring–mass networks offer another example of a physically realizable system that employs the same equations as we reviewed in circuit theory. A spring–mass network consists of a series of masses connected to each other via springs, where the quantity of each mass may be different and each spring may have a different spring constant. The equations that govern spring networks are used not only to calculate quantities of physical systems but also as computational models for wide ranging applications (e.g., [208, 210, 270]).

The elements of spring–mass networks have been mapped to elements of circuit theory in several different ways. In fact, there are some parts of the literature which map force to current and others which map force to voltage, with accompanying arguments about the utility of one electrical analogy over the other [202, 357]. A standard reference for these analogies is Shearer et al. [341]. The typical focus of attention in these electrical-mechanical analogies is study of the system *dynamics*. Therefore, the analogy is drawn between the mechanical system and an RLC circuit in order to study the system dynamics. These dynamical systems form the basis for studies in the synchronization of oscillators [12, 26], with a strong emphasis on the role of graph topology in the behavior of the system.

In contrast to these dynamical systems, the primary focus of the later chapters of this book are on how energy minimization techniques may be used to find solutions to problems from several different applications. Therefore, since our focus will generally be on *steady-state* problems, we will present a simplified spring-circuit analogy of a steady-state system, following Strang [359]. The equations that govern the steady-state of a spring–mass model differ from the equations that govern a DC electric circuit mainly in the interpretation of the variables. The setting that we consider for a spring–mass model is depicted in Fig. 3.5. Each mass is associated with

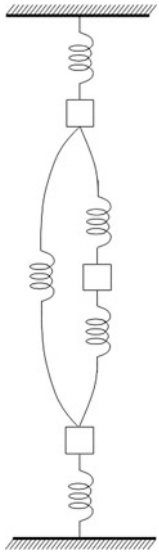

Fig. 3.5 A spring–mass system in steady-state

a node and each spring between nodes with an edge. The displacement (location) of each node, v_i, along the line is represented by the variable x_i. Some node, v_0, is fixed at a reference location such that $x_0 = 0$. When two connected nodes are separated, they stretch the spring between them with an **elongation**, p_{ij}. This elongation for each edge may be represented in matrix form via

$$\mathbf{Ax} = \mathbf{p}. \tag{3.37}$$

When a spring is elongated, Hooke's Law states that it responds with a force proportional to the elongation. The constant of proportionality is determined individually for each spring and is known as the **spring constant** of the spring, w_{ij}. Written in matrix form, the constitutive relationship represented by Hooke's Law states that

$$\mathbf{G}^{-1}\mathbf{p} = \mathbf{y}, \tag{3.38}$$

where y_{ij} represents the force exerted by the spring on edge e_{ij} in response the elongation. Each spring incident on a node exerts a force, but the node may also have an external force applied to it, such as gravity. In a steady-state system, these internal and external forces must balance each other, which may be written as

$$\mathbf{A}^{\mathsf{T}}\mathbf{y} - \mathbf{f} = \mathbf{0}, \tag{3.39}$$

where $\mathbf{f}$ represents the external force (e.g., $f_i = m_i\gamma$ where m_i is the mass of node v_i and γ is the gravitational constant). These three equations combine to allow us to solve for the steady-state solution of the displacements via

$$\mathbf{A}_0^{\mathsf{T}}\mathbf{G}^{-1}\mathbf{A}_0 = \mathbf{L}_0\mathbf{x} = \mathbf{f}. \tag{3.40}$$

Table 3.1 List of equivalent quantities between linear electric circuits and spring–mass networks

Symbolic	Electrical circuit	Mass–spring network
$\mathbf{x}$	electric potential	mass displacement
$\mathbf{p}$	voltage (potential difference)	elongation
$\mathbf{y}$	current	responding spring force
$\mathbf{w}$	conductance (1/resistance)	spring constant
v_0	grounded node	reference node
$\mathbf{f}$	currents injected into nodes from ground	external forces on masses
$\mathbf{G}^{-1}\mathbf{p} = \mathbf{y}$	Ohm's Law	Hooke's Law
$\mathbf{A}^{\mathsf{T}}\mathbf{y} = \mathbf{0}$	Kirchhoff's Current Law	conservation of forces

The matrix $\mathbf{L}_0$ in this context is known as the **stiffness matrix**.

By using a consistent notation, it is straightforward to see in Table 3.1 the corresponding variables in a spring network and an electrical network. Consequently, the (one dimensional) spring network behaves *exactly* like an electrical circuit when both systems are in the steady state.

3.4.2 Random Walks

The connection between random walks on a graph and electrical circuits is unexpected. Although this connection has been recognized for a long time [221, 227, 228], the book by Doyle and Snell [115] provided a clear and introductory exposition that propelled subsequent interest in this connection.

We define a random walk on a graph as an iterative process that tracks a random walker located at a node as it moves from node to node along edges. At each iteration of the random walk, a random walker located at node v_i will transition to one of its neighbor nodes, v_j with probability equal to $q_{ij} = w_{ij}/d_i$, where d_i is the (weighted) degree of node v_i. If we let x_i represent the probability that the random walker is present at node v_i, then we may write the transition rule in matrix form as

$$\mathbf{x}^{[k+1]} = \mathbf{D}^{-1}\mathbf{W}\mathbf{x}^{[k]} \tag{3.41}$$

where $\mathbf{x}^{[k]}$ indicates the k-th iteration step and $\mathbf{W}$ is the adjacency matrix.

Given this definition of a random walk on a graph, we may consider the following problem: What is the probability that a random walker starting at node v_i reaches node v_1 before it reaches node v_0? Let x_i represent the probability that a random walker leaving node v_i reaches node v_1 before it reaches node v_0. By definition of the problem, we know that $x_0 = 0$ and $x_1 = 1$. For the remaining nodes, we can imagine that if we knew the probability that each of the neighbors of node v_i sent a random walker to v_1 before v_0 then we know that x_i is just the sum of these probabilities weighted by the likelihood that the random walker transitions to each

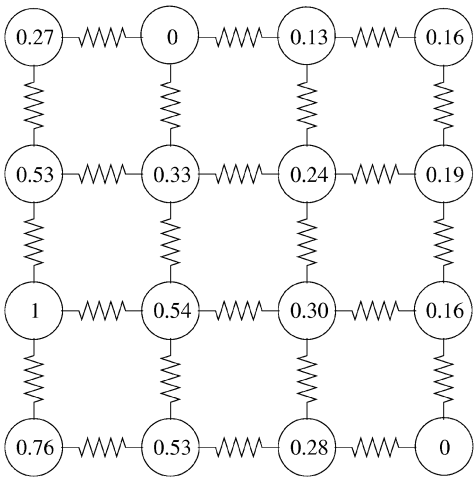

Fig. 3.6 Equivalent circuit to calculate the probability that a random walker leaving each node first arrives at the '1' node before arriving at the '0' nodes. The circuit is a passive resistive network in which the '0' nodes are connected to ground and the '1' node is connected to a unit voltage source with ground. The electrical potential established by this circuit at each node (printed inside the node) equals the probability that a random walker leaving each node arrives at the '1' node before arriving at any '0' node

neighbor. Formally, we can write that

$$x_i = \sum_j q_{ij}\, x_j = \frac{1}{d_i} \sum_j w_{ij}\, x_j, \tag{3.42}$$

or in matrix form

$$\mathbf{x} = \mathbf{D}^{-1}\mathbf{W}\mathbf{x}, \tag{3.43}$$

$$\mathbf{D}\mathbf{x} = \mathbf{W}\mathbf{x}, \tag{3.44}$$

$$\mathbf{L}\mathbf{x} = \mathbf{0}. \tag{3.45}$$

Therefore, this random walk problem amounts to solving $\mathbf{Lx} = \mathbf{0}$ subject to $x_0 = 0$ and $x_1 = 1$. This is exactly the same circuit theory problem that we encountered before when trying to solve for the electrical potentials $\mathbf{x}$ that were induced by grounding v_0 and establishing a unit voltage source between v_1 and ground, i.e., in (3.8). If we replace the unit voltage source by a voltage source of arbitrary magnitude k, then the equivalent random walk problem adjusts by calculating the expected payoff of a game in which a player is paid k if the random walker arrives first at v_1 and is paid zero if the random walker arrives first at v_0. Figure 3.6 illustrates the circuit construction that produces node potentials equal to the probability that a random walker leaving each node is more likely to arrive at node v_1 before arriving at node v_0.

Other random walker problems also have equivalent circuit problems. For example, the **hitting time** of a random walk from node v_a to node v_0, $h(v_a, v_0)$, is defined as the expected number of steps taken by a random walker to travel from v_a to v_0. As before, we can solve for the hitting time by considering the relationship of the hitting time of every node to reach v_0 in relationship to its neighbors. Specifically, if we let x_i represent the hitting time from node v_i to some reference node v_0, then

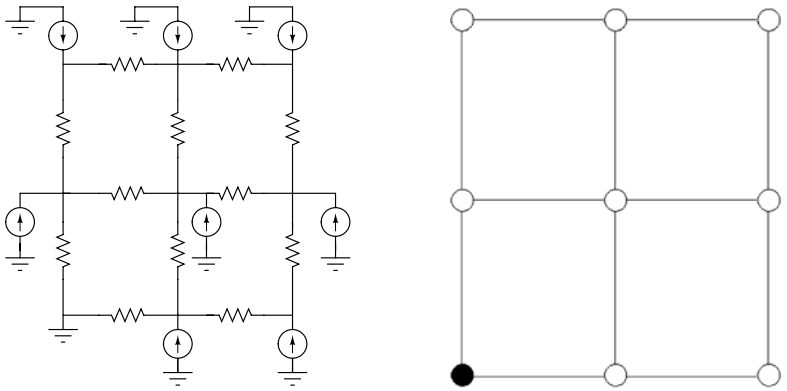

Fig. 3.7 Equivalent circuit to compute the hitting times of a graph. The electrical potential induced at a node in the circuit in (**A**) equals the hitting time for a random walker leaving that node to reach the grounded node. (**B**) The same graph in which the reference (grounded) node is indicated as the filled node. Note that the magnitude of each current source equals the degree of the node

the hitting time at v_i will be equal to the hitting time of its neighbors multiplied by the probability of transitioning to each neighbor, plus one (to account for the extra step taken to a neighbor), i.e.,

$$\mathbf{x} = \mathbf{D}^{-1}\mathbf{W}\mathbf{x} + \mathbf{1}, \tag{3.46}$$

$$\mathbf{D}\mathbf{x} = \mathbf{W}\mathbf{x} + \mathbf{d}, \tag{3.47}$$

$$\mathbf{L}\mathbf{x} = \mathbf{d}, \tag{3.48}$$

where $x_0 = 0$ is fixed. From the standpoint of circuits, the above equation (3.48) may be interpreted as a circuit problem in which node v_0 is grounded, an amount of current equal to each node degree d_i is injected into every node $v_i \neq v_0$ and the resulting potentials x_i equal the hitting time from node v_i to v_0. Figure 3.7 illustrates the circuit construction that produces node potentials equal to the hitting time to pass to node v_0 from every other node [115].

The hitting time $h(v_i, v_j)$ does not necessarily equal $h(v_j, v_i)$, since some nodes in a graph are simply faster to reach via a random walk process than other nodes (e.g., nodes with high degree are easier to reach). Therefore the **commute time** between two nodes is defined as the expected number of steps a random walker will traverse between nodes v_i and v_j and then back again from v_j to v_i, i.e., $c(v_i, v_j) = h(v_i, v_j) + h(v_j, v_i) = c(v_j, v_i)$. It is possible to show that the commute time between two nodes is proportional to the **effective resistance** between the nodes when the graph is viewed as an electrical circuit [72]. The effective resistance between two nodes v_i and v_0, $R_{\text{eff}}(v_i, v_0)$, is defined as the electrical potential induced by injecting one unit of current into node v_i and grounding node v_0. Formally, let the vector $r_k = 1$ if $k = i$ and $r_k = 0$ otherwise. The electrical potentials

established by injecting one unit of current into v_i and grounding node v_0 satisfies

$$\mathbf{L}_0\bar{\mathbf{x}} = \mathbf{r}. \tag{3.49}$$

The quantity $\bar{x}_i = R_{\text{eff}}(v_i, v_0)$. We may then calculate the hitting time for all nodes to reach v_i from (3.48) as

$$\mathbf{L}\tilde{\mathbf{x}} = \begin{bmatrix} \mathbf{d} \\ \sum_k d_k - d_i \end{bmatrix} \tag{3.50}$$

(where the right hand side is set such that it sums to zero) and the hitting time for all nodes to reach v_0 as

$$\mathbf{L}\hat{\mathbf{x}} = \begin{bmatrix} \sum_k d_k - d_0 \\ \mathbf{d} \end{bmatrix}. \tag{3.51}$$

Then, we may note that

$$\mathbf{L}(\tilde{\mathbf{x}} - \hat{\mathbf{x}}) = \begin{bmatrix} +\sum_k d_k \\ 0 \\ -\sum_k d_k \end{bmatrix}, \tag{3.52}$$

For convenience we may assume that the graph is unweighted (all edges are unity weighted), meaning that

$$\begin{bmatrix} +\sum_k d_k \\ 0 \\ -\sum_k d_k \end{bmatrix} = \begin{bmatrix} +2m \\ 0 \\ -2m \end{bmatrix}, \tag{3.53}$$

where $m = |\mathcal{E}|$. The entry $(\tilde{\mathbf{x}} - \hat{\mathbf{x}})_i$ represents $h(v_j, v_i)$ and the entry $(\tilde{\mathbf{x}} - \hat{\mathbf{x}})_j$ represents $-h(v_i, v_0)$. Consequently, by adding the value $h(v_i, v_0)$ to each entry of $(\tilde{\mathbf{x}} - \hat{\mathbf{x}})$, the entry belonging to v_0 becomes zero and the ith entry represents $h(v_0, v_i) + h(v_i, v_0) = c(v_i, v_0)$. We may therefore consider the v_0 node grounded and restrict our attention of $\mathbf{L}$ and the vector $(\tilde{\mathbf{x}} - \hat{\mathbf{x}})$ to the portion excluding v_0. With this restriction, the difference satisfies

$$\mathbf{L}_0(\tilde{\mathbf{x}} - \hat{\mathbf{x}}) = \begin{bmatrix} 2m \\ 0 \end{bmatrix} = 2m\mathbf{r} = 2m\bar{\mathbf{x}}. \tag{3.54}$$

Therefore, for any pair of nodes in any graph, $c(v_i, v_0) = 2m\, R_{\text{eff}}(v_i, v_0)$.

In their book, Doyle and Snell also give an interpretation of the current through each edge when one unit of current is injected into node v_i and out of v_j [115], i.e., in the circuit with potentials and currents satisfying (3.49). The current through edge e_{pq} represents the expected number of times that a random walker will pass through edge e_{pq} when the walker is inserted at v_i and exits at v_j. Note that in this interpretation of the current, the current represents the expected number of times that the random walker pass through the edge in the v_p to v_q direction *subtracted from* the expected number of times that the random walker passes through the edge in the v_q to v_p direction. The proof of this statement is somewhat involved, so the interested reader is referred to [115].

We have seen that several problems concerning random walks on a graph have interpretations as problems in circuit theory and that the quantities of voltage and current in a circuit may be interpreted as expectations of certain random walks. This connection has been very helpful for analyzing random walk problems on graphs. One problem in particular for which this connection has been useful is to provide a much simpler proof of Polya's theorem [308], which states that a random walk on an infinite lattice will always return to its starting point when the lattice is either one-dimensional or two-dimensional, but has a finite probability of never returning to its starting point for lattices in dimensions three or greater [115]. In the next section, we leave random walks and consider the Gaussian Markov Random Field (GMRF), a different probabilistic model that leads back to the circuit equations.

3.4.3 Gaussian Markov Random Fields

A Markov Random Field (MRF) describes any collection of random variables defined on nodes $v_i \in \mathcal{V}$ for which the value of each individual variable x_i is conditionally dependent only on the values of its neighbors. This conditional probability can be represented by a simple graph structure recorded by the set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. In this structure, each node represents a random variable and each edge represents conditional dependence between the variables. Node (variables) which are independent are not connected by an edge, i.e., if v_i and v_j are independent, then $p(x_i|x_j) = p(x_i)$. In this short section, we will show that the MAP estimate of a certain class of Gaussian MRFs may be interpreted as the electrical potentials of a specific circuit construction.

The classic MRF is the Ising Model for ferromagnetic material in which each variable can assume one of two states, $x_i \in \{-1, +1\}$. In contrast to the two-state model, a *Gaussian* Markov Random Field model assumes that each variable $x_i \in \mathbb{R}$ represents a random variable with a Gaussian distribution, $x_i \sim \mathrm{N}(\mu_i, \sigma_i)$, and all individual x_i are correlated with covariance $\mathbf{\Sigma}$. The probability distribution for any particular state of the entire MRF (i.e., a particular set of values for each of the random variables) is typically given by the Gibbs distribution

$$P(X = \mathbf{x}) = \frac{1}{Z} \exp\left(-\frac{1}{2} (\mathbf{x} - \boldsymbol{\mu})^{\mathsf{T}} \mathbf{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right), \tag{3.55}$$

where $\mathbf{\Sigma}$ is the covariance matrix, and Z is the *partition function* that serves as a normalization constant, which is set such that the integral of P over $\mathbf{x}$ is unity. The covariance matrix captures the coupling between each pair of nodes, yet represents both direct covariance between neighboring nodes and indirect coupling that results from a propagation of covariance to remote pairs of nodes. The inverse of the covariance matrix, $\mathbf{\Sigma}^{-1}$, has several interpretations and is sometimes called the *potential matrix*, the *precision matrix* or the **information matrix** in the literature. The information matrix can provide a more compact representation of the coupling between the random variables x_i and more succinctly captures the neighborhood relations in the GMRF [224].

In many applications [9, 224], the information matrix is assigned to a sum of a weighted graph Laplacian matrix with a non-negative diagonal matrix $\mathbf{T}$, i.e., $\boldsymbol{\Sigma}^{-1} = \lambda\mathbf{L} + \mathbf{T}$, where λ is the weighting parameter.

Using this choice for the information matrix, the maximum likelihood or ML estimate of the state $\mathbf{x}$ of the Gaussian MRF under the Gibbs distribution (3.55) is given by

$$\widehat{\mathbf{x}}_{\mathrm{ML}} = \underset{\mathbf{x}}{\operatorname{argmax}}\{P\} = \underset{\mathbf{x}}{\operatorname{argmax}}\big\{(\mathbf{x}-\boldsymbol{\mu})^{\mathsf{T}}(\lambda\mathbf{L}+\mathbf{T})(\mathbf{x}-\boldsymbol{\mu})\big\}.$$

Consequently, the ML estimate may be generated by solving the linear system

$$(\lambda\mathbf{L}+\mathbf{T})\widehat{\mathbf{x}}_{\mathrm{ML}} = (\lambda\mathbf{L}+\mathbf{T})\boldsymbol{\mu}. \tag{3.56}$$

Similarly, if we adopt an observation model of the Gaussian MRF where the state $\mathbf{x}$ is observed in the presence of noise, the observed state $\mathbf{y}$ is expressed as

$$\mathbf{y} = \mathbf{x} + \mathbf{n}, \tag{3.57}$$

where the noise is zero-mean and Gaussian distributed, $\mathbf{n} \sim \mathrm{N}(0, \mathbf{R}^{-1})$ with covariance matrix $\mathbf{R}^{-1}$. For simplicity here we assume that $\mathbf{n}$ is independent for each variable v_i (i.e., $\mathbf{R}$ is diagonal), and $\mathbf{x}$ is zero mean, $\mathbf{x} \sim \mathrm{N}(0, \boldsymbol{\Sigma})$. If again the information matrix $\boldsymbol{\Sigma}^{-1}$ is captured by the weighted graph Laplacian of the MRF graph, $\boldsymbol{\Sigma}^{-1} = \lambda\mathbf{L} + \mathbf{T}$, then

$$P(\mathbf{y}|\mathbf{x}) = \frac{1}{Z}\exp\left(-\frac{1}{2}(\mathbf{y}-\mathbf{x})^{\mathsf{T}}\mathbf{R}(\mathbf{y}-\mathbf{x})\right). \tag{3.58}$$

Therefore, the maximum *a posteriori* or MAP estimate of

$$P(\mathbf{x}|\mathbf{y}) \propto P(\mathbf{x})P(\mathbf{y}|\mathbf{x}), \tag{3.59}$$

is given by

$$(\lambda\mathbf{L}+\mathbf{T}-\mathbf{R})\widehat{\mathbf{x}}_{\mathrm{MAP}} = -\mathbf{R}\mathbf{y}. \tag{3.60}$$

Viewed as a circuit, the MAP estimate in (3.60) may be interpreted as the electrical potentials established by a circuit where (a) every pair of neighboring nodes is connected by a resistor with value $1/\lambda$; (b) a resistor connects each node v_i to ground via a resistor with value $1/(T_{ii} - R_{ii})$; and (c) a current source injects $-R_{ii}y_i$ current into each node v_i. A similar circuit interpretation may be given to the ML estimate in (3.56) in which the current sources are given instead by $(\lambda\mathbf{L}+\mathbf{T})\boldsymbol{\mu}$.

We can now examine how this circuit model reflect various conditions in the Markov Random Field through considering three examples. First, if the weighting parameter $\lambda = 0$, then each variable is independent and the MAP estimate is simply $\widehat{x}_i = -R_{ii}y_i/(T_{ii} - R_{ii})$. From a circuit standpoint, this result is explained by the fact that each node is connected to ground by an edge with resistance $R_{ii}/(T_{ii} - R_{ii})$ and that $-R_{ii}y_i$ current is injected into each node. Thus, the resulting voltages are apparent from Ohm's law. As a second example, consider the case of an observed

variable at some node v_i such that $x_i = x_{\text{obs}}$. From a circuit perspective, this requirement that $x_i = x_{\text{obs}}$ is equivalent to imposing a voltage source between v_i and ground which establishes the potential x_{obs} at v_i. Consequently, the remaining potentials may be calculated using the steady-state solution for a voltage source (as in (3.8)) where the observed variables form the set of constrained voltages $\mathcal{D}$. As a final example, we can give the circuit interpretation of the probability for any state in the as given by the Gibbs energy in (3.4.3). If $\boldsymbol{\mu} = \mathbf{0}$ then, given any state of the MRF, $\mathbf{x}$, the power dissipated by the circuit equivalent model is given in (3.17) by $\mathcal{E}[\mathbf{x}] = \mathbf{x}^{\mathsf{T}}(\lambda\mathbf{L} + \mathbf{T})\mathbf{x}$ and therefore the probability of state $\mathbf{x}$ may be interpreted as

$$P(X = \mathbf{x}) = \frac{1}{Z}\exp\left(-\frac{1}{2}\mathcal{G}[\mathbf{x}]\right). \tag{3.61}$$

In other words, the probability of any state in the Gaussian MRF is proportional to the exponential of half the power dissipated by the equivalent circuit. Thus configurations of electrical potentials in the circuit that yield larger power dissipation are less likely to occur under the MRF model. Therefore, the minimum power distribution for the circuit which is produced by nature is the maximum likelihood solution for the equivalent MRF.

In this section we considered a second probabilistic model that could also be interpreted as an electrical circuit. Although circuit interpretations of Gaussian MRFs are not common in the literature, this connection provides another example of an area in which the same variables, operators and equations all participate.

3.4.4 Tree Counting

Almost any quantity in circuit theory may be calculated by counting subtrees of the graph. Counting subtrees is very impractical for computation in most circumstances (although some treatments of computational circuit theory have pursued exactly this approach [75]), but it is sometimes useful to examine a circuit theory problem (or a partial differential equation) in terms of counting subtrees. One utility of the tree viewpoint can be that certain properties of the behavior of a variable under perturbation or noise are simpler to prove by using trees [161]. We believe that the tree interpretations of the circuit equations are particularly valuable because it is difficult to see an analogue for this viewpoint in the continuum equations. In other words, random walks are straightforward to define in the continuum, and electromagnetism provides an intuitive analogy for electrical circuits, yet we are not aware of any equivalent to tree counting in the continuum. Therefore, since this method is available in the discrete domain, we devote some space here to computing solutions to the circuit equations by counting subtrees of the graph. In this section we largely follow the results and notation of Biggs [34], which provides support for all of the results reviewed here.

Define a tree $\mathcal{T}$ as a graph on n nodes and m edges for which there is a single connected component such that $m = n - 1$. Note that some authors refer to this type

of tree as a *spanning tree*. One important property of any tree is that a tree consists of no cycles (by contradiction, if a tree did contain a cycle then it would be possible to remove one edge without losing connectivity). A second property of a spanning tree that we will use in this section is that within a spanning tree, there exists a unique path that connects any two nodes. The number of subtrees in a graph is the number of unique subgraphs of the graph that are trees. In this section, any use of the term *tree* should be interpreted as referring to a spanning tree.

By definition of the edge–node incidence matrix, any vector in the right nullspace must take the same value for each pair of nodes connected by an edge. Therefore, if the graph is connected, the right nullspace of the edge–node incidence matrix is spanned by the set of constant-valued functions, i.e., $\mathbf{A1} = \mathbf{0}$. Consequently, the *reduced incidence matrix*, $\mathbf{A}_0$, formed by removing the column corresponding to any node, has a right nullspace of dimension zero. Since the left nullspace of the edge–node incidence matrix has already been discussed as being comprised of the graph cycles, it is clear that if $\mathbf{A}$ represents the incidence matrix of a tree, there can be no such cycles, and thus the left nullspace of the incidence matrix of a tree is also dimension zero. Furthermore, for any tree, the reduced incidence matrix $\mathbf{A}_0$ is both a square matrix and nonsingular. Put differently, any $n \times n$ submatrix of an incidence matrix $\mathbf{A}_0$ is nonsingular if and only if this submatrix represents a tree. The invertibility of this matrix indicates that we may solve for a set of node potentials x_i such that the differences across the edges of a tree fit prescribed values, i.e., a solution $\mathbf{x}$ will always exist for the equation

$$\mathbf{Ax} = \mathbf{p}. \tag{3.62}$$

In summary, if our graph is a tree, then *any* flow in the graph may be represented via potential differences for some set of node potentials due to the guaranteed existence of a solution. Additionally, if we set $x_i = 0$ for some node v_i, then the solution is unique.[1]

The connection between circuit theory and tree counting goes back to the earliest days of circuit theory. Kirchhoff himself proved that the number of trees in a graph, κ, can be calculated as the determinant of the *reduced Laplacian matrix*, $\mathbf{L}_0 = \mathbf{A}_0^\mathsf{T}\mathbf{A}_0$ for a reduced incidence matrix $\mathbf{A}_0$ created by removing any node [232], i.e., $\kappa = |\mathbf{A}_0^\mathsf{T}\mathbf{A}_0|$. This result is known as the **Matrix-Tree Theorem**. When the Laplacian matrix represents a weighted graph, $\mathbf{L}_0 = \mathbf{A}_0^\mathsf{T}\mathbf{G}^{-1}\mathbf{A}_0$, then the determinant $\omega = |\mathbf{A}_0^\mathsf{T}\mathbf{G}^{-1}\mathbf{A}_0|$ represents the weighted sum of distinct subgraphs within the graph that are trees, i.e.,

$$\omega = |\mathbf{L}_0| = \sum_{T \in \mathcal{T}} w[T], \tag{3.63}$$

[1] In fact, Branin [58] showed that the inverse of the reduced incidence matrix $\mathbf{S} = \mathbf{A}_0^{-1}$ has a semantic interpretation:

$$S(v_i, e_{jk}) = \begin{cases} +1 & \text{if edge } e_{jk} \text{ is traversed in the path from } v_i \text{ to } v_0 \text{ in the positive direction,} \\ -1 & \text{if edge } e_{jk} \text{ is traversed in the path from } v_i \text{ to } v_0 \text{ in the negative direction,} \\ 0 & \text{otherwise.} \end{cases}$$

where the sum is understood to be taken over all such trees $\mathcal{T}$ [36], and we define the weight of each individual tree as the product of the edge weights comprising that tree

$$w[T] \equiv \prod_{e_{ij} \in T} w(e_{ij}). \tag{3.64}$$

We now consider how we may use tree counting to solve problems in circuit theory. The first problem we consider is the solution of the electrical potentials when a voltage source is applied between nodes v_0 and v_1, where v_0 is tied to ground. Consequently, $x_0 = 0$ and $x_1 = 1$. In this case, we may calculate the voltage x_i for any other node via

$$x_i = \frac{1}{\chi} \sum_{F \in \mathcal{F}_{(0|1,i)}} w[F], \tag{3.65}$$

where the sum is over all 2-trees, $\mathcal{F}_{(0|1,i)}$, for which v_1 and v_i are in one component and v_0 is in another component. Recall that a **2-tree** is defined as any graph that is the result of removing one edge from a spanning subtree of $\mathcal{G}$, and that a 2-tree therefore need not be a tree. As before, we let

$$w[F] = \prod_{e \in F} w(e_{ij}), \tag{3.66}$$

and define the scale factor

$$\chi \equiv \sum_{F \in \mathcal{F}_{(0|1)}} w[F], \tag{3.67}$$

where this sum is taken over all 2-trees in the graph in which v_0 is in one component and v_1 is in the other. When the graph is unweighted, $w[\mathcal{F}_{(0|1)}] = w[\mathcal{T}] = 1$, meaning that χ is simply the number of 2-trees separating v_0 and v_1. In this case, (3.65) simply says that the potential at node v_i is given by the percentage of 2-trees separating v_0 and v_1 that groups v_i with v_1. This result may therefore be interpreted as the *expected value* of a random variable that selects 2-trees with uniform probability and gives $x_i = 1$ if v_i is grouped with v_1 and $x_i = 0$ otherwise [267].[2] If the graph is weighted, then process is modified such that the probability of selecting a 2-tree $\mathcal{F}$ is proportional to $w[\mathcal{F}]$.

If we replace our voltage source between v_0 and v_1 with a unit current source such that v_0 is still tied to ground, then the induced potential at node v_i is given

[2]It is possible to interpret the formulae concerning 2-trees by considering a second graph $\mathcal{G}^+$ formed from $\mathcal{G}$ by adding an edge between v_0 and v_1. In this case, the 2-trees of $\mathcal{G}$ are the same as the trees of $\mathcal{G}$ that contain the edge spanning v_0 and v_1. Therefore, all of the formulae concerning 2-trees could be written in terms of the trees of $\mathcal{G}^+$ that contain the new edge. This development in terms of the trees of $\mathcal{G}^+$ was the view adopted by Kirchhoff in his original work on this topic [232].

instead by

$$x_i = \frac{1}{\omega} \sum_{F \in \mathcal{F}_{(0|1,i)}} w[F], \tag{3.68}$$

where, as before, the sum is over all 2-trees, $\mathcal{F}_{(0|1,i)}$, for which v_1 and v_i are in one component and v_0 is in another component. Therefore, the only difference between the potentials induced by a voltage source and a current source are the denominator χ or ω. Consequently, it immediately follows that we may calculate the effective resistance between nodes v_0 and v_1 by

$$R_{\text{eff}}(v_0, v_1) = \frac{\omega}{\chi}. \tag{3.69}$$

The current through any edge which is induced by a unit current source between v_1 and v_0 also admits an interpretation in terms of trees. Specifically if the unit current is supplied at v_1 and drawn from v_0, then the current y_{ij} through edge e_{ij} is given by

$$y_{ij} = \frac{1}{\omega} \left(\sum_{T^+ \in \mathcal{T}^+_{\{0,1|i,j\}}} w[T^+] - \sum_{T^- \in \mathcal{T}^-_{\{0,1|i,j\}}} w[T^-] \right), \tag{3.70}$$

where $\mathcal{T}^+$ is a tree in which the path from v_1 to v_0 passes through e_{ij} in the positive direction and $\mathcal{T}^-$ is a tree in which the path from v_1 to v_0 passes through e_{ij} in the negative direction. In the first term the sum is taken over all $\mathcal{T}^+$ trees in the graph in the sum in the second term is taken over all $\mathcal{T}^-$ trees in the graph.

The reader may be concerned that our treatment has addressed only the interpretation of potentials and currents induced by unit voltage sources or current sources. Real circuit problems often involve multiple sources or source magnitudes which are not equal to unity. However, it is important to remember that we are considering *linear* circuits. Therefore, we may decompose any set of sources into a sum of unit sources weighted appropriately by a set of coefficients. Given this decomposition, we could count the trees for each source independently (multiplied by the factor) and sum the results to obtain the solution for the original set of sources. This approach to decomposing sources is common in circuit theory and is known as the **method of superposition**. Consequently, it is sufficient to consider only the solutions for unit voltage and current sources, which form the building blocks for solving more complicated problems.

Example 2 (The Wheatstone bridge) To demonstrate the utility of tree counting for solving for unknowns in an electrical circuit, we will consider the classic *Wheatstone bridge circuit*. The Wheatstone bridge is a circuit that enables the determination of an unknown resistor value r_x. The circuit consists of two known fixed resistors r_1 and r_2 plus a variable resistor or *potentiometer* r_3, together with a fixed voltage source V and a simple current meter or galvanometer. The configuration is shown in Fig. 3.8. To measure r_x, the resistance r_3 of the potentiometer is adjusted

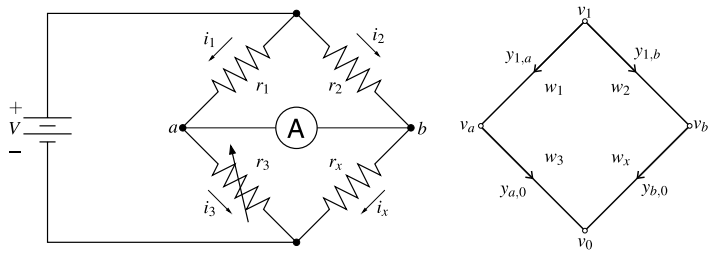

Fig. 3.8 The Wheatstone bridge. The unknown resistance r_x is derived from fixed resistors r_1 and r_2 after the variable resistor r_3 is set so that the current between nodes a and b is measured to be zero by the ammeter. The corresponding weighted, oriented graph is presented *on the right*, with edge weights w_1, w_2, w_3, and w_x on edges $y_{1,a}$, $y_{1,b}$, $y_{a,0}$, and $y_{b,0}$, respectively. Recall that, by convention, the edge weights are conductances and therefore $w_i = 1/r_i$

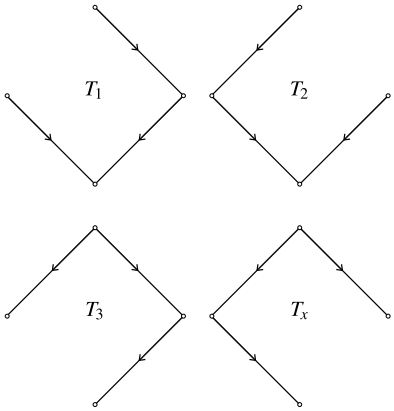

Fig. 3.9 The four subgraphs of the Wheatstone bridge graph that are trees

until the current measured between nodes a and b is zero. Nodal analysis shows that the resulting resistance is given by

$$r_x = \frac{r_2}{r_1} r_3. \tag{3.71}$$

To re-derive this result from tree counting, we begin by identifying the graph corresponding to the circuit, which is shown in Fig. 3.8, and the four trees of the graph, provided in Fig. 3.9. The determinant of the weighted reduced Laplacian ω can then be computed, and in this example evaluates to $\omega = w_2 w_3 w_x + w_1 w_3 w_x + w_1 w_2 w_x + w_1 w_2 w_3$.

The next step is to identify the 2-trees $\mathcal{T}_{(0|1)}$ within the graph that separate nodes v_0 and v_1, shown in Fig. 3.10. From these 2-trees the scale factor χ as defined in (3.67) can be calculated and evaluates to $\chi = 2w_1 w_2 + 2w_2 w_3 + 2w_3 w_x + 2w_x w_1$. The effective resistance can then be calculated from these two scale factors, however the value of w_x is still unknown.

The voltage x_a is given in terms of the 2-trees $\mathcal{T}_{(0|1,a)}$ (in which one component contains v_0 and the other component contains both v_a and v_1) identified from the

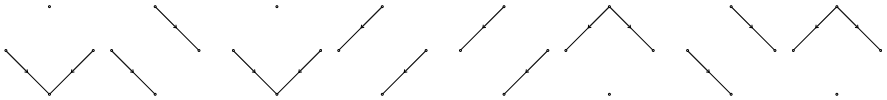

Fig. 3.10 The eight 2-trees separating v_1 and v_0

three trees containing a connection between nodes v_a and v_1, and similarly for x_b. From (3.65), we see that

$$x_a = \frac{1}{\chi}(2w_1w_x + 2w_1w_2)$$

and

$$x_b = \frac{1}{\chi}(2w_2w_3 + 2w_1w_2).$$

Therefore when the bridge is balanced by the potentiometer, $x_a = x_b$ and thus

$$w_x = \frac{w_2}{w_1}w_3.$$

Converting the edge weights into resistances we arrive at the final expression for the unknown r_x given by

$$r_x = \frac{r_2}{r_1}r_3.$$

3.4.5 Linear Algebra Applied to Circuit Analysis

In our exposition of linear circuit theory, all of the operators were represented by matrices and the solutions for circuit variables were obtained via linear algebra. Therefore, it should be no surprise that some of the common techniques in linear algebra also have interpretations in terms of circuit theory. Specifically, we will address here the formation of an **equivalent circuit** via a series of circuit transforms.

3.4.5.1 The Delta–Wye and Star–Mesh Transforms

The circuit transform we will consider first is the **delta–wye** transform or, more generally, the **star–mesh** transform. In the context of linear algebra, we will show that one can view an application of these transforms as one step of Gauss–Jordan elimination on the Laplacian matrix.

The delta–wye transform can be viewed as a method for producing a second graph that is missing one node, but for which the electrical connections between all remaining nodes are the same. Formally, we use the delta–wye transform to produce

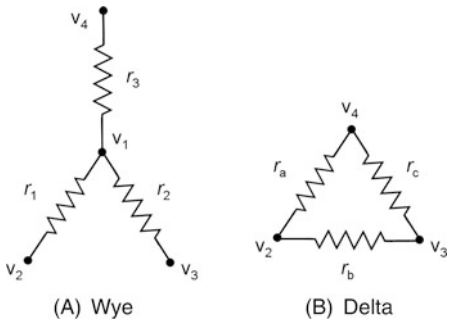

Fig. 3.11 The original 'Y' graph and the 'Δ' graph obtained by eliminating the central node, v_1, from the 'Y' graph. The delta–wye transform consists of a mapping of resistor values that ensures that the voltages and currents at the terminal nodes v_2, v_3, and v_4 are identical in the two configurations

a new graph $\mathcal{G}_1$ from an original graph $\mathcal{G}_0$ with the node v_a removed, thus $\mathcal{V}_1 = \mathcal{V}_0 - v_a$, and such that the effective resistance $R_{\text{eff}}(v_i, v_j)$ between all remaining nodes is unchanged after the transformation. Consider the graphs in Fig. 3.11. The delta–wye transform gives us a method for calculating the resistances r_a, r_b and r_c from the original resistors r_1, r_2 and r_3 such that the effective resistances (and thus the voltages and currents at the terminals of the circuit) are preserved. The transform is called "delta–wye" due to the shape of the two graphs (a 'Δ' and 'Y'). It is known from basic circuit theory [294] that the resistances of the 'Δ' circuit can be phrased in terms of resistances of an equivalent 'Y' circuit as

$$r_a = \frac{r_1 r_2 + r_2 r_3 + r_3 r_1}{r_2}, \tag{3.72}$$

$$r_b = \frac{r_1 r_2 + r_2 r_3 + r_3 r_1}{r_3}, \tag{3.73}$$

$$r_c = \frac{r_1 r_2 + r_2 r_3 + r_3 r_1}{r_1}. \tag{3.74}$$

Let us consider the Laplacian matrix for the original 'Y' graph

$$\mathbf{L} = \begin{bmatrix} w_1 + w_2 + w_3 & -w_1 & -w_2 & -w_3 \\ -w_1 & w_1 & 0 & 0 \\ -w_2 & 0 & w_2 & 0 \\ -w_3 & 0 & 0 & w_3 \end{bmatrix}, \tag{3.75}$$

where the central node, v_1, in the 'Y' graph is given by the first row/column. Recall that the edge weights in the Laplacian matrix, w_i, are affinity weights which represent the conductances $w_i = 1/r_i$ (see Chap. 2). If we perform Gauss–Jordan elimination of the central 'Y' node, v_1, in $\mathbf{L}$, then we are left with the matrix

$$\mathbf{L} = \frac{1}{w_1 + w_2 + w_3} \times \begin{bmatrix} w_1 + w_2 + w_3 & 0 & 0 & 0 \\ 0 & w_1 w_2 + w_1 w_3 & -w_1 w_2 & -w_1 w_3 \\ 0 & -w_1 w_2 & w_1 w_2 + w_2 w_3 & -w_2 w_3 \\ 0 & -w_1 w_3 & -w_2 w_3 & w_1 w_3 + w_2 w_3 \end{bmatrix}.$$

The lower-right submatrix of this matrix now corresponds to a graph whose topology matches that of the 'Δ' graph. The resulting conductances in the graph corresponding to this submatrix are given by

$$w_a = \frac{w_1 w_3}{w_1 + w_2 + w_3}, \tag{3.76a}$$

$$w_b = \frac{w_1 w_2}{w_1 + w_2 + w_3}, \tag{3.76b}$$

$$w_c = \frac{w_2 w_3}{w_1 + w_2 + w_3}, \tag{3.76c}$$

which yield the same resistance values given for the delta–wye transform. Consequently, the delta–wye transform is equivalent to removing the node, v_1, in the center of the 'Y' via Gauss–Jordan elimination on the Laplacian matrix.

More generally, a Gauss–Jordan elimination on the Laplacian matrix may be used to eliminate any node and produce a second circuit with the node removed but otherwise equivalent electrical properties. This more general elimination procedure is called a "star–mesh" transform, since the center node of a "star" is replaced by a mesh of resistors that connect all nodes incident on the star. Since each elimination of a node in this manner causes all nodes that were previously connected to the removed node to now be connected to each other, the Gauss–Jordan elimination of a node often adds more edges to the graph than there were originally. These additional edges may cause computational problems since the new graph may require more memory to store and process than the old one. This problem can be addressed by eliminating nodes in a particular order.

3.4.5.2 Minimum-Degree Orderings

Specifically, when using Gaussian elimination to produce "LU" factors (or Cholesky factors) of a sparse Laplacian matrix, the amount of "fill-in" of zeros in the Laplacian matrix caused by the creation of new edges after each elimination may be disastrous for the memory required to store the matrix factors. However, elimination of the nodes in a different order may result in a different level of fill-in. This viewpoint on the elimination of nodes was the primary motivation behind the development of **minimum-degree orderings** that are used to factor sparse matrices [271, 322, 376].

One graph for which the ordering is particularly convenient is any tree. When the graph is a tree, there exists an ordering of the elimination that produces the creation of *zero* new edges and therefore does not create any fill-in for the factored matrix. The principle behind such an ordering is to order the elimination of nodes from the leaf nodes inward, which causes the creation of no new edges in the process. Figure 3.12 gives an example of this elimination procedure on small tree. Algorithm 3.1 accomplishes the ordering in linear time, where the array `tree` contains, for each node, the index of one neighbor (with no edges overrepresented) and the array `degree` contains the degree of each node in the tree. This representation is possible

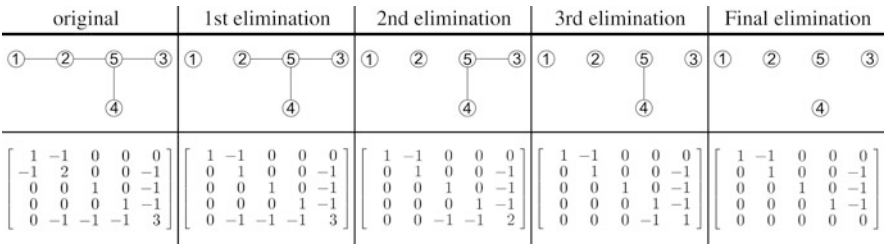

Fig. 3.12 Gaussian elimination of the Laplacian matrix of a tree with ordering given by the numbers inside the nodes. Note that the resulting Gaussian elimination has the same sparsity structure as the original matrix when a no-fill ordering is used (e.g., as computed by Algorithm 3.1). Note that the Laplacian matrix is singular—the last elimination produces a row of all zeros. Once the graph has been reduced (grounded), as in (3.77), this is no longer a concern e.g., if node 5 were grounded, the elimination would stop after the third elimination and $x_5 = 0$ would be used to recover the remaining values of the solution. *Top row*: Elimination of the tree—the figures depict the graph represented by the lower triangle of the matrix. *Bottom row*: Laplacian matrix of the tree after each elimination step

Algorithm 3.1 Produce a no-fill ordering of a tree

```
1:  void compute_ordering(degree, tree, ground, ordering)
2:  k ⇐ 0
3:  degree[root] ⇐ 0 {Fixed so that ground is not eliminated}
4:  ordering[N − 1] ⇐ ground
5:  for each node in the graph do
6:      while degree[current_node] equals 1 do
7:          ordering[k] ⇐ current_node
8:          degree[current_node] ⇐ degree[current_node]−1
9:          current_node ⇐ tree[current_node]
10:         degree[current_node] ⇐ degree[current_node]−1
11:         k ⇐ k + 1
12:     end while
13:     k ⇐ k + 1
14: end for
```

since a tree has $n - 1$ edges (where the root would contain a '0'). Therefore, one could solve the linear system of equations

$$\mathbf{L}_0\mathbf{x} = \mathbf{f}, \tag{3.77}$$

in linear time when $\mathbf{L}$ represents a tree. Algorithm 3.2 finds a solution to (3.77) in linear time, given a tree and an elimination ordering. For ease of exposition, we assume that all $w_{ij} = 1$, but the algorithm could be easily modified to handle an arbitrary set of nonnegative weights. Because the full Gaussian elimination has the same sparsity structure as the original matrix when a no-fill ordering is used we need only compute the no-fill ordering, and not the full Gaussian elimination, in order to solve the linear system.

Algorithm 3.2 Given a tree, solve (3.77)

```
1: solve_system(ordering, f, tree, r, output)
2: {Forward pass}
3: k ⇐ 0
4: for each non-ground node do
5:    r[tree[ordering[k]]] ⇐ r[tree[ordering[k]]] + r[ordering[k]]/f[ordering[k]]
6:    f[tree[ordering[k]]] ⇐ f[tree[ordering[k]]] − 1/f[ordering[k]]
7:    k ⇐ k + 1
8: end for
9:
10: output[ordering[N − 1]] ⇐ r[ordering[N − 1]]/f[ordering[N − 1]]
11:
12: {Backward pass}
13: k ⇐ N − 2 {Last non-ground node}
14: for each non-ground node do
15:    output[ordering[k]] ⇐ output[tree[ordering[k]]] +
       r[ordering[k]]/f[ordering[k]]
16:    k ⇐ k − 1
17: end for
```

In this section, we showed that another aspect of circuit theory also appears in other areas of finite mathematics. Specifically, we showed that the methods for circuit transformations and equivalent circuits arise from Gauss–Jordan elimination on the Laplacian matrix. This insight is helpful for understanding the motivation behind matrix ordering algorithms (specifically minimum-degree orderings). When the graph is a tree, we showed that the graph has a straightforward ordering that allows linear systems that represent trees to be solved in linear time. This linear time solution has been exploited before in applications to computer vision (see [160]).

3.5 Conclusion

In his Lecture Notes on Physics, Richard Feynman paused to consider why the same operators and equations recur throughout seemingly different areas of physics [132]. Specifically, Feynman focused on the pervasiveness of the Laplacian operator and questioned whether it was possible that the commonality of the governing equations indicated that all of these physical phenomena were actually the same underlying process on some fundamental level. After examining these different phenomena individually, Feynman concludes that it is highly unlikely that each of these processes are actually the same on a fundamental level and instead offers an alternative explanation for the commonality of the governing equations throughout physics. Feynman's explanation is that the common thread tying together these physical phenomena is that the processes *all occur in the same space* and that this fact alone practically forces a certain relationship between variables.

Similar to Feynman's inquiry, we can seek an explanation of why circuit theory, graph theory, mass-spring networks, random walks and Markov Random Fields (among others) all involve the same equations. In our discrete setting, Feynman's explanation would suggest that the reason for the recurrence of these equations is that all of the phenomena are defined on a set of locations which are connected via some neighborhood structure (i.e., a graph). Therefore, if we are going to describe a relationship between variables defined at the nodes, our basic tool set will have to reflect the *space* upon which these variables are defined. The operators of graph theory—the Laplacian matrix, adjacency matrix and incidence matrix—explicitly represent the space and therefore it should come as no surprise that these operators recur throughout any equations that govern the relationship between variables defined on a graph.