Clifford–Fourier Transform for Color Image Processing

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Abstract The aim of this paper is to define a Clifford–Fourier transform that is suitable for color image spectral analysis. There have been many attempts to define such a transformation using quaternions or Clifford algebras. We focus here on a geometric approach using group actions. The idea is to generalize the usual definition based on the characters of abelian groups by considering group morphisms from $\mathbb{R}^2$ to spinor groups Spin(3) and Spin(4). The transformation we propose is parameterized by a bivector and a quadratic form, the choice of which is related to the application to be treated. A general definition for 4D signal defined on the plane is also given; for particular choices of spinors, it coincides with the definitions of S. Sangwine and T. Bülow.

1 Introduction

During the last years several attempts have been made to generalize the classical approach of scalar signal processing with the Fourier transform to higher-dimensional signals. The reader will find a detailed overview of the related works at the beginning of [1]. We only mention in this introduction some of the approaches investigated by several authors.

Motivated by the spectral analysis of color images, S. Sangwine and T. Ell have proposed in [13] and [5] a generalization based on the use of quaternions: a color corresponds to an imaginary quaternion, and the imaginary complex i is replaced by the unit quaternion μ coding the grey axis. A quaternionic definition is also given by T. Bülow and G. Sommer in the context of analytic signals, for signals defined on the plane and with values in the algebra $\mathbb{H}$ of quaternions [3]. Concerning analytic signals, M. Felsberg makes use of the Clifford algebras $\mathbb{R}_{2,0}$ and $\mathbb{R}_{3,0}$ to define an appropriate Clifford–Fourier transform [6].

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A generalization in the Clifford algebras context appears also in J. Ebling and G. Scheuermann [4]. The authors mainly use their transformation to analyze frequencies of vector fields. Using the same Fourier kernel, B. Mawardi and E. Hitzer obtain an uncertainty principle for $\mathbb{R}_{3,0}$ multivector functions [11]. The reader may find in [1] definitions of Clifford–Fourier transform and Clifford–Gabor filters based on the Dirac operator and Clifford analysis.

One could ask the reason why to propose a new generalization. An important thing when dealing with Fourier transform is its link with group representations. We then recall in Sect. 2 the usual definition of the Fourier transform of a function defined on an abelian Lie group by means of the characters of the group. The definition we propose in Sect. 3 relies mainly on the generalization of the notion of characters; that is why we study the group morphisms from $\mathbb{R}^2$ to Spin(3) and Spin(4). These morphisms help to understand the behavior of the Fourier transform with respect to well chosen spinors. We treat in Sect. 4 three applications corresponding to specific bivectors of $\mathbb{R}_{4,0}$. They consist in filtering frequencies according to color, hue, and chrominance part of a given color. In Sect. 5, we show that for particular choices of group morphisms and under well-chosen identification with quaternions, the Clifford–Fourier transform we propose coincides with the definitions of S. Sangwine [5] and T. Bülow [2].

2 Fourier Transform and Group Actions

Let us recall briefly some basic ideas related to the group approach of the definition of the Fourier transform. Details can be found in the Appendix; see also [15] for examples of applications to Fourier descriptors.

Let G be a Lie group. The Pontryagin dual of G, denoted $\widehat{G}$, is the set of equivalence classes of unitary irreducible representations of G. It appears that if G is abelian, every irreducible unitary representation of G is of dimension 1, i.e., is a continuous group morphism from G to S^1. This is precisely the definition of a character. It is well known that the characters of $\mathbb{R}^m$ are given by

$$(x_1, \dots, x_m) \longmapsto e^{i(u_1 x_1 + \cdots + u_m x_m)}$$

with real $u_1, u_2, \dots, u_m$. This shows that $\widehat{\mathbb{R}^m} = \mathbb{R}^m$. The characters of SO(2) are the group morphisms

$$\theta \longmapsto e^{in\theta}$$

for $n \in \mathbb{Z}$, and the corresponding Pontryagin dual is $\mathbb{Z}$. The characters of $\mathbb{Z}/n\mathbb{Z}$ are the group morphisms

$$u \longmapsto e^{i\frac{2\pi k u}{n}}$$

for $k \in \mathbb{Z}/n\mathbb{Z}$, from which we deduce that $\mathbb{Z}/n\mathbb{Z}$ is its own Pontryagin dual.

In the general case (provided that G is unimodular), the Fourier transform of a function $f \in L^2(G;\mathbb{C})$ is defined on $\widehat{G}$ by

$$\widehat{f}(\varphi) = \int_G f(x)\varphi\left(x^{-1}\right) d\nu(x)$$

(for ν a well-chosen invariant measure on G). Applying this formula to the case $G = \mathbb{R}^m$, resp. $G =$SO(2), resp. $G = \mathbb{Z}/n\mathbb{Z}$ leads to the usual definition of the Fourier transform, resp. Fourier coefficients, resp. discrete Fourier transform.

Traditionally, the Fourier transform in $L^2(\mathbb{R}^m, (\mathbb{R}^n, \| \|_2))$ is defined by n standard Fourier transforms in $L^2(\mathbb{R}^m, \mathbb{R})$ on each one of the components, embedding $\mathbb{R}$ into $\mathbb{C}$. Using group representations theory, we are able to define Fourier transforms that treat jointly the different components.

From now on, we deal with the abelian group $G = (\mathbb{R}^2, +)$ since this paper is devoted to image processing applications.

Let us make some crucial remarks about the case $n = 2$.

Let f be a real- or complex-valued function defined on $\mathbb{R}^2$. Its Fourier transform is given by

$$\widehat{f}(a, b) = \int_{\mathbb{R}^2} f(x, y)e^{-i(ax+by)}\, dx\, dy.$$

Identifying $\mathbb{C}$ with $(\mathbb{R}^2, \| \|_2)$, we have $S^1 = \mathrm{SO}(2)$, and the action of S^1 on $\mathbb{C}$, given by the complex multiplication, corresponds to the action of the group SO(2) on $(\mathbb{R}^2, \| \|_2)$. Hence, we can define a Fourier transform in $L^2(\mathbb{R}^2, (\mathbb{R}^2, \| \|_2))$ using the action of group morphisms from $\mathbb{R}^2$ to SO(2) on $(\mathbb{R}^2, \| \|_2)$. These ones are real unitary representations of the group $\mathbb{R}^2$ of dimension 2.

The Fourier transform of $f \in L^2(\mathbb{R}^2, (\mathbb{R}^2, \| \|_2))$ defined above can be written in the Clifford algebra language. Indeed, from the embedding of $(\mathbb{R}^2, \| \|_2)$ into $\mathbb{R}_{2,0}$, f may be viewed as an $\mathbb{R}^1_{2,0}$-valued function

$$f(x, y) = f_1(x, y)e_1 + f_2(x, y)e_2,$$

where $e_1^2 = e_2^2 = 1$ and $e_1e_2 = -e_2e_1$. From this point of view, the Fourier transform of f is given by

$$\begin{aligned}\widehat{f}(a, b) = \int_{\mathbb{R}^2} &\big[\cos\big((ax + by)/2\big)1 + \sin\big((ax + by)/2\big)e_1e_2\big] \\ &\times \big(f_1(x, y)e_1 + f_2(x, y)e_2\big) \\ &\times \big[\cos\big(-(ax + by)/2\big)1 + \sin\big(-(ax + by)/2\big)e_1e_2\big]\, dx\, dy\end{aligned}$$

using the fact that the action of Spin(2) on $\mathbb{R}^1_{2,0}$ corresponds to the action of SO(2) on $(\mathbb{R}^2, \| \|_2)$ (see Appendix). We can write this last formula in the following form:

$$\widehat{f}(a, b) = \int_{\mathbb{R}^2} \big(f_1(x, y)e_1 + f_2(x, y)e_2\big) \perp \varphi_{a,b}(-x, -y)\, dx\, dy,$$

where $\varphi_{a,b}$ is the morphism from $\mathbb{R}^2$ to Spin(2) that sends (x, y) to $\exp[((ax + by)/2)(e_1e_2)]$, and $\perp$ denotes the action $v \perp s = s^{-1}vs$ of Spin(2) on $\mathbb{R}^1_{2,0}$ and, more generally, the action of Spin(n) on $\mathbb{R}^1_{n,0}$.

Note that group morphisms from $\mathbb{R}^2$ to Spin(2) followed by the action on $\mathbb{R}^1_{2,0}$ correspond to the action of group morphisms from $\mathbb{R}^2$ to SO(2) on $(\mathbb{R}^2, \| \|_2)$. In other words, they are real unitary representations of $\mathbb{R}^2$ of dimension 2 too.

Remark 1 As in the standard case, where the Fourier transform of a real-valued function is defined by embedding $\mathbb{R}$ into $\mathbb{C}$, we define here the Fourier transform of a real-valued function by embedding $\mathbb{R}$ into $\mathbb{R}^2$.

Starting from these elementary observations, we now proceed to generalize this construction for $\mathbb{R}^n$-valued functions defined in $\mathbb{R}^2$. In other words, we are looking for a generalization of the action of group morphisms to SO(2) on the values of an $(\mathbb{R}^2, \| \|_2)$-valued function.

3 Clifford–Fourier Transform in $L^2(\mathbb{R}^2, (\mathbb{R}^n, Q))$

Let $f \in L^2(\mathbb{R}^2, (\mathbb{R}^n, Q))$ where Q is a positive definite quadratic form. We propose to associate the Fourier transform of f with the action of the following group morphisms on the values of f, depending on the parity of n.

If n is even, then we consider the morphisms

$$\varphi : \mathbb{R}^2 \longrightarrow \mathrm{SO}(Q).$$

If n is odd, then we embed $(\mathbb{R}^n, Q)$ into $(\mathbb{R}^{n+1}, Q \oplus 1)$ and consider the morphisms

$$\varphi : \mathbb{R}^2 \longrightarrow \mathrm{SO}(Q \oplus 1).$$

Thus the generalization we propose is based on the computation of real unitary representations of dimension n or $n+1$ of the abelian group $\mathbb{R}^2$. The main fact is that we no more consider equivalent classes of representations. This means in particular that the Fourier transform we define depends on the positive definite quadratic form of $\mathbb{R}^n$.

Remark 2 Recall that up to a change of the basis, a positive definite quadratic form is given by the identity matrix. Thus, f may always be viewed as an $(\mathbb{R}^p, \| \|_2)$-valued function (p denotes n if n is even and $n+1$ if n is odd). As a consequence of the change of the basis, SO(Q) become SO(p) and group morphisms from $\mathbb{R}^2$ to SO(Q) become group morphisms from $\mathbb{R}^2$ to SO(p).

As for the case of $\mathbb{R}^2$-valued functions, we can rewrite the Fourier transform in the Clifford algebra language, using the fact that the action of Spin(p) on $\mathbb{R}^1_{p,0}$ corresponds to the action of SO(p) on $\mathbb{R}^p$. Moreover, it appears to be much more easier to compute group morphisms to Spin(p) rather than group morphisms to the matrix group SO(p).

If n is even, then from the embedding of $\mathbb{R}^n$ into $\mathbb{R}_{n,0}$, f may be viewed as an $\mathbb{R}^1_{n,0}$-valued function:

$$f(x,y) = f_1(x,y)e_1 + f_2(x,y)e_2 + \cdots + f_n(x,y)e_n,$$

where $e_i^2 = 1$ and $e_ie_j = -e_je_i$. Denoting by φ a group morphism from $\mathbb{R}^2$ to Spin(n), we define the Clifford–Fourier transform of f by

$$\widehat{f}(\varphi) = \int_{\mathbb{R}^2} \varphi(x,y)f(x,y)\varphi(-x,-y)\,dx\,dy = \int_{\mathbb{R}^2} f(x,y)\perp\varphi(-x,-y)\,dx\,dy.$$

If n is odd, we first embed $\mathbb{R}^n$ into $\mathbb{R}^{n+1}$. Then, from the embedding of $\mathbb{R}^{n+1}$ into $\mathbb{R}_{n+1,0}$, f may be viewed as an $\mathbb{R}^1_{n+1,0}$-valued function:

$$f(x,y) = f_1(x,y)e_1 + f_2(x,y)e_2 + \cdots + f_n(x,y)e_n + 0e_{n+1},$$

where $e_i^2 = 1$ and $e_ie_j = -e_je_i$. Denoting by φ a group morphism from $\mathbb{R}^2$ to Spin($n+1$), we define the Clifford–Fourier transform of f by

$$\widehat{f}(\varphi) = \int_{\mathbb{R}^2} \varphi(x,y)f(x,y)\varphi(-x,-y)\,dx\,dy = \int_{\mathbb{R}^2} f(x,y)\perp\varphi(-x,-y)\,dx\,dy.$$

Remark 3 If n is even, the Clifford–Fourier transform of f is an $\mathbb{R}^1_{n,0}$-valued function. If n is odd, the Clifford–Fourier transform of f is an $\mathbb{R}^1_{n+1,0}$-valued function.

Remark 4 For $Q = 1$ on $\mathbb{R}$, the Fourier transforms we define correspond to the standard Fourier transforms of $\mathbb{R}$-valued functions.

From now on, we deal with the case $n = 3$ since this paper is devoted to color image processing. However, we have seen above that we treat the cases $n = 3$ and $n = 4$ in the same manner, by computing group morphisms from $\mathbb{R}^2$ to Spin(4).

3.1 *The Cases n* = 3, 4*: Group Morphisms from* $\mathbb{R}^2$ *to* Spin(4)

This part is devoted to the computation of group morphisms from $\mathbb{R}^2$ to Spin(4).

Using the fact that the group Spin(4) is isomorphic to the group Spin(3) × Spin(3), we first compute group morphisms from $\mathbb{R}^2$ to Spin(3).

One can verify that Spin(3) is the group

$$\mathrm{Spin}(3) = \left\{a1 + be_1e_2 + ce_2e_3 + de_3e_1,\ a^2 + b^2 + c^2 + d^2 = 1\right\}$$

and is isomorphic to the group of unit quaternions.

Proposition 1 *The group morphisms from* $\mathbb{R}^2$ *to* Spin(3) *are given by*

$$(x, y) \longmapsto e^{\frac{1}{2}(ux+vy)B},$$

where B belongs to $\mathbb{S}^2_{3,0}$, *the set of unit bivectors in* $\mathbb{R}_{3,0}$ (*see Appendix*), *and u and v are real.*

Proof We have to determine the abelian subalgebras of the Lie algebra $\mathfrak{spin}(3) = \mathbb{R}^2_{3,0}$ of the Lie group Spin(3). More precisely, as the exponential map of $\mathbb{R}^2$ is onto, group morphisms from $\mathbb{R}^2$ to Spin(3) are given by Lie algebra morphisms from the abelian Lie algebra $\mathfrak{R}^2$ of $\mathbb{R}^2$ to $\mathfrak{spin}(3)$. Taking two generators (f_1, f_2) of $\mathfrak{R}^2$, any morphism φ from $\mathfrak{R}^2$ to $\mathfrak{spin}(3)$ satisfies

$$\varphi(f_1) \times \varphi(f_2) = 0.$$

We deduce that $\mathrm{Im}(\varphi)$ is an abelian subalgebra of $\mathbb{R}^2_{3,0}$ whose dimension is inferior or equal to 2. If $a = a_1e_1e_2 + a_2e_3e_1 + a_3e_2e_3$ and $b = b_1e_1e_2 + b_2e_3e_1 + b_3e_2e_3$ satisfy $a \times b = 0$, then the structure relations of $\mathbb{R}^2_{3,0}$, i.e.,

$$e_1e_2 \times e_3e_1 = e_2e_3, \qquad e_3e_1 \times e_2e_3 = e_1e_2, \qquad e_2e_3 \times e_1e_2 = e_3e_1,$$

imply

$$(a_1b_2 - a_2b_1)e_2e_3 - (a_1b_3 - a_3b_1)e_3e_1 + (a_2b_3 - a_3b_2)e_1e_2 = 0.$$

This shows that two commuting elements of $\mathbb{R}^2_{3,0}$ are colinear and that the abelian subalgebras of $\mathbb{R}^2_{3,0}$ are of dimension 1. If we write $\varphi(f_1) = \frac{1}{2}uB$ and $\varphi(f_2) = \frac{1}{2}vB$ for some $u, v \in \mathbb{R}$ and $B \in \mathbb{S}^2_{3,0}$, we see that the morphisms from $\mathfrak{R}^2$ to $\mathbb{R}^2_{3,0}$ are parameterized by two real numbers and one unit bivector and are given by

$$\varphi_{u,v,B} : (x, y) \mapsto \frac{1}{2}(ux + vy)B.$$

Consequently, the group morphisms from $\mathbb{R}^2$ to Spin(3) are the morphisms $\widetilde{\varphi}_{u,v,B}$ with

$$\widetilde{\varphi}_{u,v,B} : (x, y) \mapsto e^{\frac{1}{2}(ux+vy)B}.$$

□

Let us recall what group is Spin(4). Every τ in Spin(4) is of the form

$$\begin{aligned}\tau &= u + Iv \\ &= (a1 + be_1e_2 + ce_2e_3 + de_3e_1) + I\big(a'1 + b'e_1e_2 + c'e_2e_3 + d'e_3e_1\big),\end{aligned}$$

where I denotes the pseudoscalar of $\mathbb{R}_{4,0}$, and the following relations hold:

$$u\overline{u} + v\overline{v} = 1, \qquad u\overline{v} + v\overline{u} = 0.$$

The morphism $\chi : \mathrm{Spin}(4) \longrightarrow \mathrm{Spin}(3) \times \mathrm{Spin}(3)$ with

$$\chi(u + Iv) = (u + v, u - v)$$

is an isomorphism. An alternative description of Spin(4) relies on the following fact: the morphism $\psi : \mathbb{H}_1 \times \mathbb{H}_1 \longrightarrow \mathrm{SO}(4)$ defined by

$$(\tau, \rho) \longmapsto (v \longmapsto \tau v \overline{\rho})$$

(where v is a vector of $\mathbb{R}^4$ considered as a quaternion) is a universal covering of SO(4) (see [12]). This means that Spin(4) is isomorphic to $\mathbb{H}_1 \times \mathbb{H}_1$. We will use this remark later on to compare our transform to Sangwine's and Bülow's ones.

Proposition 2 *The group morphisms from* $\mathbb{R}^2$ *to Spin*(4) *are the morphisms* $\widetilde{\phi}_{u,v,B,w,z,C}$ *that send* (x, y) *to*

$$e^{\frac{1}{8}[x(u+w)+y(v+z)][B+C+I(B-C)]}\, e^{\frac{1}{8}[x(u-w)+y(v-z)][B-C+I(B+C)]}$$

with u, v, w, z *real and* B, C *two elements of* $\mathbb{S}^2_{3,0}$.

Proof The group law of $\mathrm{Spin}(3) \times \mathrm{Spin}(3)$ being

$$\big((a, b), (c, d)\big) \to (ac, bd),$$

the group morphisms from $\mathbb{R}^2$ to $\mathrm{Spin}(3) \times \mathrm{Spin}(3)$ are the morphisms $\widetilde{\varphi}_{u,v,B,w,z,C}$ defined by

$$\widetilde{\varphi}_{u,v,B,w,z,C} : (x, y) \mapsto \big(e^{\frac{1}{2}(ux+vy)B}, e^{\frac{1}{2}(wx+zy)C}\big)$$

with u, v, w, z real and B, C two elements of $\mathbb{S}^2_{3,0}$.

By χ^{-1}, the group morphisms from $\mathbb{R}^2$ to Spin(4) are the $\widetilde{\phi}_{u,v,B,w,z,C}$ that send (x, y) to

$$\frac{e^{\frac{1}{2}(ux+vy)B} + e^{\frac{1}{2}(wx+zy)C}}{2} + I\,\frac{e^{\frac{1}{2}(ux+vy)B} - e^{\frac{1}{2}(wx+zy)C}}{2}.$$

However, this writing is not convenient to determine group morphisms to SO(4) since it does not provide explicitly the rotations in $\mathbb{R}^4$ that $\widetilde{\phi}_{u,v,B,w,z,C}$ generates by its action on $\mathbb{R}^1_{4,0}$. The solution comes from an "orthogonalization" of the corresponding Lie algebras morphism from $\mathfrak{R}^2$ to $\mathbb{R}^2_{4,0}$, namely the linear map

$$\phi_{u,v,B,w,z,C}(X, Y) = T_{(0,0)}\, \widetilde{\phi}_{u,v,B,w,z,C}(X, Y),$$

where T denotes the linear tangent map. By definition,

$$\phi_{u,v,B,w,z,C}(X, Y) = \frac{d}{dt}\big(\widetilde{\phi}_{u,v,B,w,z,C}\big(\exp\big(t(X, Y)\big)\big)\big)\big|_{t=0}.$$

The exponential map of $\mathbb{R}^2$ being the identity map, we get

$$\begin{aligned}\phi_{u,v,B,w,z,C}(X,Y) &= \frac{d}{dt}\big(\widetilde{\phi}_{u,v,B,w,z,C}\big(t(X,Y)\big)\big)\Big|_{t=0} \\ &= \frac{d}{dt}\left(\frac{e^{\frac{1}{2}t(uX+vY)B}+e^{\frac{1}{2}t(wX+zY)C}}{2}\right. \\ &\quad \left. + I\,\frac{e^{\frac{1}{2}t(uX+vY)B}-e^{\frac{1}{2}t(wX+zY)C}}{2}\right)\Bigg|_{t=0} \\ &= \frac{(uX+vY)B+(wX+zY)C}{4} \\ &\quad + I\,\frac{(uX+vY)B-(wX+zY)C}{4}.\end{aligned}$$

The orthogonalization of the morphism $\phi_{u,v,B,w,z,C}$ consists in decomposing the bivector $\phi_{u,v,B,w,z,C}(X,Y)$ for each X,Y into commuting bivectors whose squares are real. The corresponding spinor is written as a product of commuting spinors of the form e^{F_i} with $F_i^2 < 0$. These ones represent rotations of angle $-F_i^2$ in the oriented planes given by the F_i's. In our case, the bivector $\phi_{u,v,B,w,z,C}(X,Y)$ is decomposed into $F_1 + F_2$ where

$$F_1 = \frac{1}{8}\big[\big(X(u+w)+Y(v+z)\big)\big(B+C+I(B-C)\big)\big],$$
$$F_2 = \frac{1}{8}\big[\big(X(u-w)+Y(v-z)\big)\big(B-C+I(B+C)\big)\big]$$

(see the Appendix for details). The group morphisms $\widetilde{\phi}_{u,v,B,w,z,C}$ from $\mathbb{R}^2$ to Spin(4) can then be written as

$$\begin{aligned}\widetilde{\phi}_{u,v,B,w,z,C}(x,y) &= e^{[\frac{(ux+vy)B+(wx+zy)C}{4}+I\,\frac{(ux+vy)B-(wx+zy)C}{4}]} \\ &= e^{\frac{1}{8}[(x(u+w)+y(v+z))(B+C+I(B-C))]} \\ &\quad \times e^{\frac{1}{8}[(x(u-w)+y(v-z))(B-C+I(B+C))]}.\end{aligned}$$ □

This is a convenient form to describe group morphisms from $\mathbb{R}^2$ to SO(4).

To conclude this part, let us remark that the expression of the morphisms $\widetilde{\phi}_{u,v,B,w,z,C}$ may be simplified. Indeed, when B and C describe $\mathbb{S}^2_{3,0} \subset \mathbb{R}_{4,0}$, the unit bivectors

$$D = \frac{1}{4}\big(B+C+I(B-C)\big) \quad \text{and} \quad ID = \frac{1}{4}\big(B-C+I(B+C)\big)$$

describe $\mathbb{S}^2_{4,0}$, the set of unit bivectors in $\mathbb{R}_{4,0}$.

Therefore, the morphisms $\widetilde{\phi}_{u,v,B,w,z,C}$ are parameterized by four real numbers and one unit bivector $D \in \mathbb{S}^2_{4,0}$ and may be written

$$\widetilde{\Phi}_{u,v,w,z,D}(x, y) = e^{\frac{1}{2}[(x(u+w)+y(v+z))D]}\, e^{\frac{1}{2}[(x(u-w)+y(v-z))ID]}.$$

3.2 *The Cases* $n = 3, 4$*: The Clifford–Fourier Transform*

From the computation of group morphisms from $\mathbb{R}^2$ to Spin(4), we give an explicit formula of the Clifford–Fourier transform $\widehat{f}$ of $f \in L^2(\mathbb{R}^2, (\mathbb{R}^3, Q))$ or $L^2(\mathbb{R}^2, (\mathbb{R}^4, Q))$.

Definition 1 Let $f \in L^2(\mathbb{R}^2, (\mathbb{R}^3, Q))$ resp. $L^2(\mathbb{R}^2, (\mathbb{R}^4, Q))$ and denote by f the embedding of f into the Clifford algebra $Cl(\mathbb{R}^4, Q \oplus 1)$ resp. $Cl(\mathbb{R}^4, Q)$. The Clifford–Fourier transform of f is given by

$$\begin{aligned}\widehat{f}(u, v, w, z, D) &= \int_{\mathbb{R}^2} f(x, y) \perp \widetilde{\Phi}_{u,v,w,z,D}(-x, -y)\, dx\, dy \\ &= \int_{\mathbb{R}^2} e^{\frac{1}{2}[(x(u+w)+y(v+z))D]} e^{\frac{1}{2}[(x(u-w)+y(v-z))ID]} f(x, y) \\ &\quad \times e^{-\frac{1}{2}[(x(u+w)+y(v+z))D]} e^{-\frac{1}{2}[(x(u-w)+y(v-z))ID]}\, dx\, dy.\end{aligned}$$

Decomposing f as $f_{||} + f_{\perp}$ with respect to the plane generated by the bivector D, we get

$$\begin{aligned}\widehat{f}(u, v, w, z, D) &= \int_{\mathbb{R}^2} f_{||}(x, y) e^{[-(x(u+w)+y(v+z))D]}\, dx\, dy \\ &\quad + \int_{\mathbb{R}^2} f_{\perp}(x, y) e^{[-(x(u-w)+y(v-z))ID]}\, dx\, dy.\end{aligned}$$

Indeed, the plane generated by ID represents the orthogonal of the plane generated by D in $\mathbb{R}^4$.

Proposition 3 *The Clifford–Fourier transform is left-invertible. Its inverse is the map* $\check{}$ *given by*

$$\check{g}(a, b) = \int_{\mathbb{R}^4 \times \mathbb{S}^2_{4,0}} g(u, v, w, z, D) \perp \widetilde{\Phi}_{u,v,w,z,D}(a, b)\, du\, dv\, dw\, dz\, d\nu,$$

where ν *is a unit measure on* $\mathbb{S}^2_{4,0}$.

Proof We have to verify that $\check{} \circ \hat{}(f)(\lambda, \mu) = f(\lambda, \mu)$ for all $(\lambda, \mu) \in \mathbb{R}^2$.

$$\check{} \circ \hat{}(f)(\lambda, \mu) = \int_{\mathbb{R}^4 \times \mathbb{S}^2_{4,0}} \left[\int_{\mathbb{R}^2} f_{||}(x, y) e^{[-(x(u+w)+y(v+z))D]}\, dx\, dy \right] \times e^{[(\lambda(u+w)+\mu(v+z))D]}\, du\, dv\, dw\, dz\, d\nu \quad (1)$$

$$+ \int_{\mathbb{R}^4 \times \mathbb{S}^2_{4,0}} \left[\int_{\mathbb{R}^2} f_{\perp}(x, y) e^{[-(x(u-w)+y(v-z))ID]}\, dx\, dy \right] \times e^{[(\lambda(u-w)+\mu(v-z))ID]}\, du\, dv\, dw\, dz\, d\nu. \quad (2)$$

It is sufficient to prove that (1) $= f_{||}(\lambda, \mu)$.

$$(1) = \int_{\mathbb{R}^4 \times \mathbb{S}^2_{4,0}} \int_{\mathbb{R}^2} f_{||}(x, y) e^{[(\lambda-x)(u+w)+(\mu-y)(v+z)]D}\, dx\, dy\, du\, dv\, dw\, dz\, d\nu$$

$$= \int_{\mathbb{R}^4 \times \mathbb{S}^2_{4,0}} \int_{\mathbb{R}^2} f_{||}(x, y) e^{u(\lambda-x)D} e^{w(\lambda-x)D} e^{v(\mu-y)D} \times e^{z(\mu-y)D}\, dx\, dy\, du\, dv\, dw\, dz\, d\nu$$

$$= \int_{\mathbb{R}^2} \int_{\mathbb{R}^3 \times \mathbb{S}^2_{4,0}} f_{||}(x, y) \left(\int_{\mathbb{R}} e^{u(\lambda-x)D}\, du \right) e^{w(\lambda-x)D} e^{v(\mu-y)D} \times e^{z(\mu-y)D}\, dw\, dv\, dz\, d\nu\, dx\, dy$$

$$= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2 \times \mathbb{S}^2_{4,0}} f_{||}(x, y) \delta_{\lambda,x} \left(\int_{\mathbb{R}} e^{w(\lambda-x)D}\, dw \right) e^{v(\mu-y)D} \times e^{z(\mu-y)D}\, dv\, dz\, d\nu\, dx\, dy$$

$$= \int_{\mathbb{R}^2} \int_{\mathbb{R} \times \mathbb{S}^2_{4,0}} f_{||}(x, y) \delta_{\lambda,x} \delta_{\lambda,x} \left(\int_{\mathbb{R}} e^{v(\mu-y)D} dv \right) dz\, d\nu\, dx\, dy$$

$$= \int_{\mathbb{R}^2} \int_{\mathbb{S}^2_{4,0}} f_{||}(x, y) \delta_{\lambda,x} \delta_{\lambda,x} \delta_{\mu,y} \left(\int_{\mathbb{R}} e^{z(\mu-y)D}\, dz \right) d\nu\, dx\, dy$$

$$= \int_{\mathbb{R}^2} \int_{\mathbb{S}^2_{4,0}} f_{||}(x, y) \delta_{\lambda,x} \delta_{\lambda,x} \delta_{\mu,y} \delta_{\mu,y}\, d\nu\, dx\, dy$$

$$= \int_{\mathbb{R}^2} f_{||}(x, y) \delta_{\lambda,x} \delta_{\lambda,x} \delta_{\mu,y} \delta_{\mu,y}\, dx\, dy = f_{||}(\lambda, \mu).$$

□

4 Application to Color Image Filtering

4.1 Clifford–Fourier Transform of Color Images

For the applications we have in mind to color image filtering, we define a partial Clifford–Fourier transform, i.e., we deal with a subset of the set of unitary group representations of $\mathbb{R}^2$ of dimension 4. The subset we consider will depend of the colors we aim at filtering.

More precisely, we restrict Definition 1 to the set of group morphisms $\widetilde{\Phi}_{u,v,0,0,D}$ where the bivector D is fixed.

Definition 2 (Clifford–Fourier transform with respect to a bivector) Let $f \in L^2(\mathbb{R}^2, (\mathbb{R}^3, Q))$ resp. $L^2(\mathbb{R}^2, (\mathbb{R}^4, Q))$ and denote by f the embedding of f into the Clifford algebra $Cl(\mathbb{R}^4, Q \oplus 1)$ resp. $Cl(\mathbb{R}^4, Q)$. The Clifford–Fourier transform of f with respect to the bivector D is defined by

$$\begin{aligned}\widehat{f}_D(u,v) &= \int_{\mathbb{R}^2} f(x,y) \perp \widetilde{\Phi}_{u,v,0,0,D}(-x,-y)\,dx\,dy \\ &= \int_{\mathbb{R}^2} e^{\frac{1}{2}(xu+yv)ID} e^{\frac{1}{2}(xu+yv)D} f(x,y) e^{-\frac{1}{2}(xu+yv)D} e^{-\frac{1}{2}(xu+yv)ID}\,dx\,dy.\end{aligned}$$

It follows the definition of the Clifford–Fourier transform of a color image.

Definition 3 (Clifford–Fourier transform of a color image) Let I be a color image. We associate to I a function $f \in L^2(\mathbb{R}^2, (\mathbb{R}^3, Q))$ defined by

$$f(x,y) = r(x,y)e_1 + g(x,y)e_2 + b(x,y)e_3 + 0e_4,$$

where r, g, and b correspond to the red, green, and blue levels.

The Clifford–Fourier transform of I with respect to Q and D is the $Cl(\mathbb{R}^4, Q \oplus 1)$-valued function $\widehat{I}_{Q,D}$ defined by

$$\widehat{I}_{Q,D}(u,v) = \widehat{f}_D(u,v) = \int_{\mathbb{R}^2} f(x,y) \perp \widetilde{\Phi}_{u,v,0,0,D}(-x,-y)\,dx\,dy.$$

Thus, given a color image, we define a set of associated Clifford–Fourier transforms parameterized by the set of positive definite quadratic forms on $\mathbb{R}^3$ and unit bivectors in $\mathbb{R}_{4,0}$.

As the Clifford–Fourier transform in $L^2(\mathbb{R}^3, Q)$ and $L^2(\mathbb{R}^4, Q)$, we can show that the Clifford–Fourier transform of a color image is invertible.

Proposition 4 *Let $f \in L^2(\mathbb{R}^2, (\mathbb{R}^3, Q))$, and D be a unit bivector in $Cl(\mathbb{R}^4, Q \oplus 1)$. Then, the Clifford–Fourier transform of f with respect to D is invertible. Its inverse is the map $\check{}$ defined by*

$$\check{g}(x,y) = \int_{\mathbb{R}^2} g(u,v) \perp \widetilde{\Phi}_{u,v,0,0,D}(x,y)\,du\,dv.$$

Proof Decomposing f with respect to the plane generated by D as $f = f_{||} + f_{\perp}$, we have

$$\widehat{f_D}(u,v) = \int_{\mathbb{R}^2} \big(f_{||}(x,y) + f_{\perp}(x,y)\big) \perp \widetilde{\Phi}_{u,v,0,0,D}(-x,-y)\,dx\,dy.$$

This can be written

$$\widehat{f_D}(u,v) = \widehat{f_{D_{\|}}}(u,v) + \widehat{f_{D_{\perp}}}(u,v),$$

where

$$\begin{aligned}\widehat{f_{D_{\|}}}(u,v) &= \int_{\mathbb{R}^2} f_{||}(x,y) \perp \widetilde{\Phi}_{u,v,0,0,D}(-x,-y)\,dx\,dy \\ &= \int_{\mathbb{R}^2} f_{||}(x,y) e^{-(ux+vy)D}\,dx\,dy\end{aligned}$$

and

$$\begin{aligned}\widehat{f_{D_{\perp}}}(u,v) &= \int_{\mathbb{R}^2} f_{\perp}(x,y) \perp \widetilde{\Phi}_{u,v,0,0,D}(-x,-y)\,dx\,dy \\ &= \int_{\mathbb{R}^2} f_{\perp}(x,y) e^{-(ux+vy)ID}\,dx\,dy.\end{aligned}$$

Let us remark that each of the two integrals may be identified with the Fourier transform of a function from $\mathbb{R}^2$ to $\mathbb{C}$. Then, we deduce that there exists an inversion formula (left and right) for the Clifford–Fourier transform $\widehat{f_D}$ given by

$$f(x,y) = \int_{\mathbb{R}^2} \widehat{f_D}(u,v) \perp \widetilde{\Phi}_{u,v,0,0,D}(x,y)\,du\,dv.$$

Indeed, the right term equals

$$\begin{aligned}&\int_{\mathbb{R}^2} \big(\widehat{f_{D_{||}}}(u,v) + \widehat{f_{D_{\perp}}}(u,v)\big) \perp \widetilde{\Phi}_{u,v,0,0,D}(x,y)\,du\,dv \\ &\quad= \int_{\mathbb{R}^2} \widehat{f_{D_{||}}}(u,v) e^{(ux+vy)D}\,du\,dv + \int_{\mathbb{R}^2} \widehat{f_{D_{\perp}}}(u,v) e^{(ux+vy)ID}\,du\,dv. \qquad (3)\end{aligned}$$

Each of these integrals may be identified with the inversion formula of the Fourier transform of a function from $\mathbb{R}^2$ to $\mathbb{C}$; hence,

$$(3) = f_{||}(x,y) + f_{\perp}(x,y) = f(x,y). \qquad \square$$

The following proposition is useful for applications and in particular for applications to the frequencies filtering developed in the next section. It gives an integral representation of any 3D-valued signal defined on the plane by 3D-valued cosinusoidal signals. This representation is obtained from the Clifford–Fourier transform with respect to some bivector. In this proposition we show that the representation is invariant with respect to the choice of the bivector. In the discrete case, we obtain a decomposition of the signal as a sum of cosinusoidal signals.

Proposition 5 *Using the previous notation, if B and D are elements of $\mathbb{S}^2_{4,0}$, we have*

$$\widehat{f_B}(u,v)\perp\widetilde{\Phi}_{u,v,0,0,B}(x,y)+\widehat{f_B}(-u,-v)\perp\widetilde{\Phi}_{-u,-v,0,0,B}(x,y)$$
$$=\widehat{f_D}(u,v)\perp\widetilde{\Phi}_{u,v,0,0,D}(x,y)+\widehat{f_D}(-u,-v)\perp\widetilde{\Phi}_{-u,-v,0,0,D}(x,y).$$

Moreover, the e_4 component of this expression is null.

Proof Simple computations show that

$$\begin{aligned}
&\widehat{f_B}(u,v)\perp\widetilde{\Phi}_{u,v,0,0,B}(x,y)+\widehat{f_B}(-u,-v)\perp\widetilde{\Phi}_{-u,-v,0,0,B}(x,y)\\
&=\int_{\mathbb{R}^2} e^{-\frac{xu+yv}{2}(B+IB)}e^{\frac{\lambda u+\mu v}{2}(B+IB)}f(\lambda,\mu)\,e^{-\frac{\lambda u+\mu v}{2}(B+IB)}e^{\frac{xu+yv}{2}(B+IB)}\,d\lambda\,d\mu\\
&\quad+\int_{\mathbb{R}^2} e^{\frac{xu+yv}{2}(B+IB)}e^{-\frac{\lambda u+\mu v}{2}(B+IB)}f(\lambda,\mu)\,e^{\frac{\lambda u+\mu v}{2}(B+IB)}e^{-\frac{xu+yv}{2}(B+IB)}\,d\lambda\,d\mu\\
&=\int_{\mathbb{R}^2} 2\cos\big(u(x-\lambda)+v(y-\mu)\big)f_{||}(\lambda,\mu)\,d\lambda\,d\mu\\
&\quad+\int_{\mathbb{R}^2} 2\cos\big(u(x-\lambda)+v(y-\mu)\big)f_{\perp}(\lambda,\mu)\,d\lambda\,d\mu.
\end{aligned}\tag{4}$$

Hence,

$$(4)=\int_{\mathbb{R}^2} 2\cos\big(u(x-\lambda)+v(y-\mu)\big)f(\lambda,\mu)\,d\lambda\,d\mu. \qquad \square$$

This proposition justifies the fact that these filters are symmetric with respect to the transformation $(u,v)\mapsto(-u,-v)$.

4.2 Color Image Filtering

We now present applications to color image filtering. The use of the Fourier transform is motivated by the well-known fact that nontrivial filters in the spatial domain may be implemented efficiently with masks in the Fourier domain. Although it seems natural to believe that the results on grey level images may be generalized, there are not so many works dedicated to the specific case of color images. Let us mention [14], where an attempt is made through the use of an ad hoc quaternionic transform. The mathematical construction we propose appears to be well founded since it explains the fundamental role of bivectors and scalar products in terms of group actions. As explained before, the possibility to choose the bivector D and the quadratic form Q is an asset allowing a wider range of applications. Indeed, Sangwine et al. proposal can be written in our formalism by considering appropriate D and Q.

The applications proposed in this paper are based on the following fact:

$$(\widehat{f_D})_{||} = \widehat{(f_{||})_D} \quad \text{and} \quad (\widehat{f_D})_{\perp} = \widehat{(f_{\perp})_D}.$$

In other words, the part of the Clifford–Fourier transform of f that is parallel to D corresponds to the standard Fourier transform of the part of f that is parallel to D. The same principle holds for the orthogonal part.

We use low pass, high pass, and directional filters on the D-parallel part resp. D-orthogonal part, leaving the D-parallel part resp. D-orthogonal unmodified. The choice of the bivector D and the quadratic form Q (that determines the D-orthogonal part) will depend on the colors we aim at filtering. Then, we show the action of such filters using the inversion formula of the Clifford–Fourier transform.

There is another way to decompose a color $\alpha = (r, g, b)$, that is, with respect to its luminance and chrominance parts, respectively denoted by l_α and v_α. Embedding the color space RGB into the Clifford algebra $\mathbb{R}_{4,0}$ by

$$i_\alpha = r\, e_1 + g\, e_2 + b\, e_3 + 0\, e_4,$$

the former corresponds to the projection of i_α on the axis generated by the unit vector $(e_1 + e_2 + e_3)/\sqrt{3}$; the latter its projection on the orthogonal plane in $e_1e_2e_3$, called the chrominance plane, represented by the unit bivector $(e_1e_2 - e_1e_3 + e_2e_3)/3$. In what follows we make use of the following fact too: every hue can be represented as an equivalence class of bivectors of $\mathbb{R}_{4,0}$. More precisely, we have the following result.

Proposition 6 *Let T be the set of bivectors*

$$T = \left\{(e_1 + e_2 + e_3) \wedge i_\alpha,\ \alpha \in \text{RGB}\right\}$$

with the following equivalence relation:

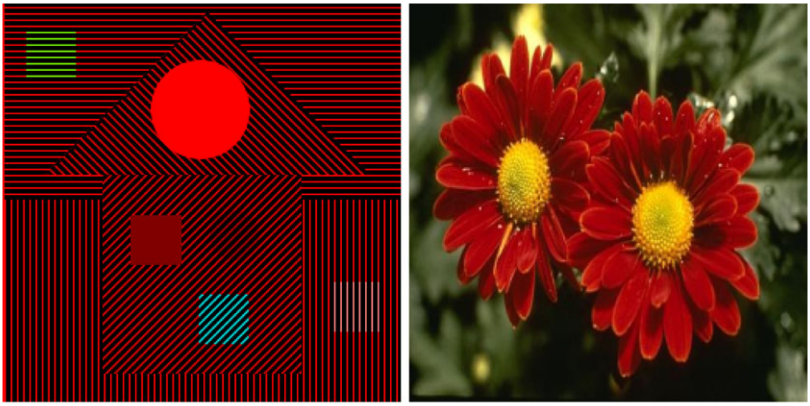
$$B \simeq C \quad \Longleftrightarrow \quad B = \lambda C \quad \textit{for } \lambda > 0.$$

Then, there is a bijection between $T/\simeq$ and the set of hues.

Proof We have

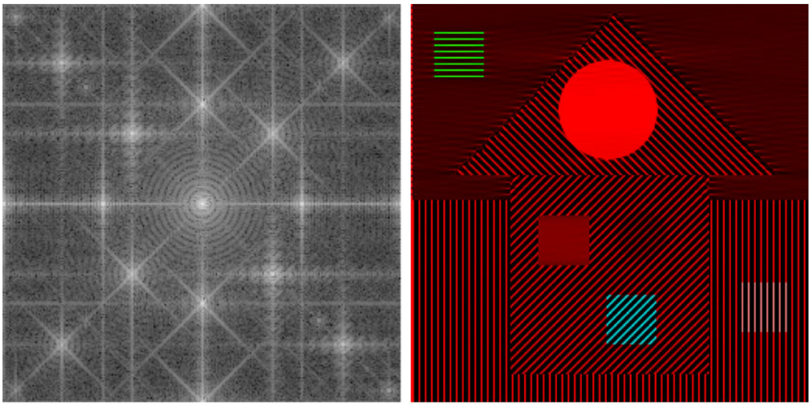
$$(e_1 + e_2 + e_3) \wedge i_\alpha = (e_1 + e_2 + e_3)v_\alpha.$$

Then, there is a bijection between $T/\simeq$ and the set $(e_1+e_2+e_3)\,v$ for v a unit vector in the chrominance plane. This latter being in bijection with the set of different hues, we conclude that there exists a bijection between $T/\simeq$ and the set of hues. □



(a) Fourier house (b) Flowers

Fig. 1 Original images



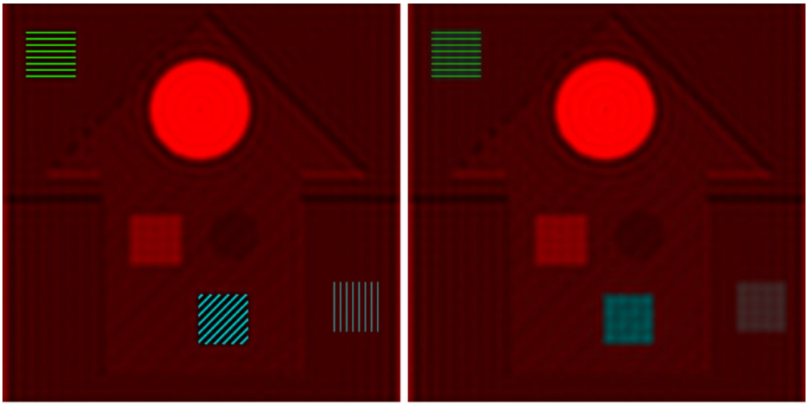
(a) Log-modulus of the parallel part (b) Directional cut filtering for the red color

Fig. 2 The Clifford–Fourier transform $\widehat{H}_{Q_1,e_1e_4}$

Figure 1 shows the original images used for these experiments.[1] Figure 1(a) H is a modified color version of the Fourier house containing red, desatured red, green, cyan stripes in various directions, a uniform red circle and a red square with lower luminance. Figure 1(b) F is a natural image taken from the Berkeley image segmentation database [10].

Figure 2(a) is the centered log-modulus of the D-parallel part of $\widehat{H}_{Q_1,D}$, where Q_1 is the quadratic form such that $Q_1 \oplus 1$ is given by the identity matrix I_4 in the basis (e_1, e_2, e_3, e_4), and D is the bivector e_1e_4. Figure 2(b) is the result of a directional cut filter around $\pi/2$ which removes of vertical frequencies. Let us point

[1] Available at http://mia.univ-larochelle.fr/ → Production → Démos.



(a) red color (b) red hue

Fig. 3 Low pass filtering

out that horizontal green stripes are not altered since green color 255 e_2 belongs to $ID = e_2e_3$.

Figure 3 shows the difference between a low pass filter in the D-parallel part of $\widehat{H}_{Q_1,e_1e_4}$ (Fig. 3(a)) and the D-parallel part $\widehat{H}_{Q_1,\frac{1}{\sqrt{2}}(e_2+e_3)e_1}$ (Fig. 3(b)). The first one consists in removing high frequencies of the red components of the image, whereas the second one consists in removing high frequencies of the red hue part of the image.

In Fig. 3(a), we can see that both green and cyan stripes are not modified. As in the previous case, this comes from the fact that both green color and cyan color 255 e_2 + 255 e_3 belong to ID. The result is different in Fig. 3(b). The unit bivector $\frac{1}{\sqrt{2}}(e_2 + e_3)e_1 = \frac{1}{\sqrt{2}}(e_1 + e_2 + e_3) \wedge e_1$ represents the red hue, involving that the cyan stripes are blurred. Indeed, unit bivectors representing cyan and red hues are opposite, and therefore they generate the same plane. Green stripes are no more invariant to the low pass filter since the green axis e_2 is not orthogonal to the bivector $\frac{1}{\sqrt{2}}(e_2 + e_3)e_1$.

In Fig. 4, the color α has been chosen to match with the color of the background green leaves. As the low pass filter (Fig. 4(a)) removes green high frequencies, the center of flowers containing yellow high frequencies turns red. In Fig. 4(b), background pixels corresponding to green low frequencies appear almost grey.

To conclude this part, we propose to compare the results of two low pass filters on the D-orthogonal part with respect to the same bivector $D = e_1e_4$ but changing the quadratic form. As a consequence, the bivector ID differs in the two cases. For the first one (Fig. 5(a)), we take Q_1, whereas for the second one (Fig. 5(b)), we construct the quadratic form Q_2 such that Q_2 is given by I_4 in the basis $(e_1, \frac{1}{\sqrt{2}}(e_1 + e_2), \frac{i_\alpha}{\|i_\alpha\|}, e_4)$. In other words, we orthogonalize the red, the yellow, and the color of leaves which are the main colors in the image.

In Fig. 5(a), the unit bivector ID is e_2e_3. Hence, the low pass filter removes green and blue high frequencies but preserves red high frequencies. This explains

(a) Low pass filter in the parallel part (b) High pass filter in the parallel part

Fig. 4 The Clifford–Fourier transform $\widehat{F}_{Q_1,\frac{v_\alpha \wedge e_4}{\|v_\alpha \wedge e_4\|}}$ with $\alpha = (96, 109, 65)$

(a) The Clifford-Fourier transform $\widehat{F}_{Q_1,e_1e_4}$ (b) The Clifford-Fourier transform $\widehat{F}_{Q_2,e_1e_4}$

Fig. 5 Low pass filters in the ID part

why the image turns red. In Fig. 5(b), the unit bivector ID is $\frac{(e_1+e_2)}{\sqrt{2}} \frac{i_\alpha}{\|i_\alpha\|}$; it contains the colors of the background and inside the flowers. Therefore, the low pass filter removes all the high frequencies in the image except the ones of the red petals.

For some specific applications, a fine tuning of the quadratic form Q should give better results.

5 Related Works

To conclude this paper, we show how to recover the hypercomplex Fourier transform of S. Sangwine and the quaternionic Fourier transform of T. Bülow in the Clifford algebras context, using the appropriate morphism from $\mathbb{R}^2$ to Spin(4). First of all, let us recall the definitions of these Fourier transforms.

5.1 The Hypercomplex Fourier Transform of Sangwine et al.

In [5], the authors define the discrete hypercomplex Fourier transform. It can be extended to $\mathbb{R}^2$ as follows. Let $f:\mathbb{R}^2 \to \mathbb{H}$; then its hypercomplex Fourier transform is given by

$$F(u,v) = \int_{\mathbb{R}^2} e^{-\mu(xu+yv)} f(x,y)\,dx\,dy,$$

where $\mu \in \mathbb{H}_0 \cap \mathbb{H}_1$.

There is a freedom in the choice of μ in the hypercomplex Fourier transform as we have a freedom in the choice of the bivector D in the Clifford–Fourier transform for color images. In fact, they have the same role, i.e., they decompose the four-dimensional space $\mathbb{R}^4$ into two orthogonal two-dimensional subspaces and decompose the Fourier transform into two standard Fourier transforms.

This is shown in the following proposition.

Proposition 7 *Let $\mu = \mu_1 i + \mu_2 j + \mu_3 k$ be a unit quaternion. Let $f \in L^2(\mathbb{R}^2, (\mathbb{R}^4, Q))$ where Q is the quadratic form represented by I_4 in the basis (e_1, e_2, e_3, e_4), and let C be the unit bivector $e_4 \wedge (\mu_1 e_1 + \mu_2 e_2 + \mu_3 e_3)$. Then, $\widehat{f_C}$ given by*

$$\begin{aligned}\widehat{f_C}(u,v) &= \int_{\mathbb{R}^2} f(x,y) \rfloor \widetilde{\Phi}_{u,v,0,0,C}(-x,-y)\,dx\,dy \\ &= \int_{\mathbb{R}^2} e^{\frac{1}{2}(xu+yv)IC} e^{\frac{1}{2}(xu+yv)C} f(x,y) e^{-\frac{1}{2}(xu+yv)C} e^{-\frac{1}{2}(xu+yv)IC}\,dx\,dy\end{aligned}$$

corresponds to the hypercomplex Fourier transform of f seen as an $\mathbb{H}$-valued function under the identification[2]

$$e_1 \leftrightarrow i, \qquad e_2 \leftrightarrow j, \qquad e_3 \leftrightarrow k, \qquad e_4 \leftrightarrow 1.$$

Proof We have to determine the four-dimensional rotation that is generated by the action of the unit quaternion $e^{\mu\phi}$ on $\mathbb{H}$ given by

$$q \longmapsto e^{\mu\phi} q.$$

It is explained in [5] that this rotation may be decomposed as the sum of two two-dimensional rotations of angle $-\phi$ in the planes generated by $(1,\mu)$ and its orthogonal (with respect to the euclidean quadratic form).

[2] The product law needs not to be respected since we just use an isomorphism of vector spaces.

Therefore, we can identify this rotation with the action of the spinor

$$e^{-\frac{\phi}{2}(C+IC)}$$

on the four-dimensional space $\mathbb{R}^1_{4,0}$. As a consequence, the action of group morphisms $(x,y) \longmapsto e^{\mu(xu+yv)}$ from $\mathbb{R}^2$ to $\mathbb{H}_1$ on $\mathbb{H}$ corresponds to the action of group morphisms $(x,y) \longmapsto e^{-\frac{1}{2}(xu+yv)(C+IC)}$ from $\mathbb{R}^2$ to Spin(4) on $\mathbb{R}^1_{4,0}$. □

Remark 5 To the best of our knowledge, the authors restrict for their applications to μ taken as the grey axis, i.e.,

$$\mu = \frac{1}{\sqrt{3}}(i+j+k).$$

In other words, the Fourier transform they propose is decomposed as a standard Fourier transform of the luminance part and a standard Fourier transform of the chrominance part.

5.2 The Quaternionic Fourier Transform of Bülow

The quaternionic Fourier transform [2] of a function $f : \mathbb{R}^2 \to \mathbb{R}$ is the quaternion-valued function $\mathcal{F}(f)$ defined by

$$\mathcal{F}(f)(y_1,y_2) = \int_{\mathbb{R}^2} \exp(-2\pi i y_1 x_1) f(x_1,x_2) \exp(-2\pi j y_2 x_2)\, dx_1\, dx_2.$$

The link between this Fourier transform and the one proposed here is given by the next result.

Proposition 8 *Let $f \in L^2(\mathbb{R}^2; \mathbb{R}e_4)$ where (e_1,e_2,e_3,e_4) is the basis of $\mathbb{R}^4$ that generates $\mathbb{R}_{4,0}$. The Clifford–Fourier transform of f defined by*

$$\widehat{f_C}(2\pi y_1, 0, 0, 2\pi y_2) = \int_{\mathbb{R}^2} f(x_1,x_2) \perp \widetilde{\Phi}_{2\pi y_1,0,0,2\pi y_2,C}(-x_1,-x_2)\, dx_1\, dx_2,$$

where C is the bivector

$$-\frac{1}{4}(e_1+e_2)(e_3-e_4),$$

corresponds to the quaternionic Fourier transform of f seen as an $\mathbb{H}$-valued function under the following identification:[3]

$$e_1 \leftrightarrow i, \qquad e_2 \leftrightarrow j, \qquad e_3 \leftrightarrow k, \qquad e_4 \leftrightarrow 1.$$

[3] The product law needs not to be respected since we just use an isomorphism of vector spaces.

Proof We have to determine one of the two elements of Spin(3) × Spin(3) that generate the following rotation in $\mathbb{H}$:

$$f(x_1, x_2) \mapsto \exp(-2\pi i y_1 x_1) f(x_1, x_2) \exp(-2\pi j y_2 x_2).$$

Simple computations show that the rotation

$$f(x_1, x_2) \mapsto \exp(-2\pi i y_1 x_1) f(x_1, x_2)$$

can be written in $\mathbb{R}^1_{4,0}$ as

$$f(x_1, x_2) \mapsto e^{-\pi x_1 y_1 (e_4 e_1 + e_2 e_3)} f(x_1, x_2) e^{\pi x_1 y_1 (e_4 e_1 + e_2 e_3)}.$$

In the same way,

$$f(x_1, x_2) \mapsto f(x_1, x_2) \exp(-2\pi j y_2 x_2)$$

corresponds to

$$f(x_1, x_2) \mapsto e^{-\pi x_2 y_2 (e_4 e_2 + e_1 e_3)} f(x_1, x_2)\, e^{\pi x_2 y_2 (e_4 e_2 + e_1 e_3)}.$$

By associativity, this shows that

$$\exp(-2\pi i y_1 x_1)\, f(x_1, x_2) \exp(-2\pi j y_2 x_2) = e^{-\tau} e^{-\rho} f(x_1, x_2) e^{\rho} e^{\tau},$$

where

$$\tau = \pi x_2 y_2 (e_4 e_2 + e_1 e_3)$$

and

$$\rho = \pi x_1 y_1 (e_4 e_1 + e_2 e_3).$$

By definition,

$$\chi\left(e^{\rho} e^{\tau}\right) = \chi\left(e^{\pi x_1 y_1\, e_4 e_1}\right) \chi\left(e^{\pi x_1 y_1\, e_2 e_3}\right) \chi\left(e^{\pi x_2 y_2\, e_4 e_2}\right) \chi\left(e^{\pi x_2 y_2\, e_1 e_3}\right).$$

By simple computations we get

$$\chi\left(e^{\rho} e^{\tau}\right) = \left(e^{2\pi x_1 y_1\, e_2 e_3}, e^{2\pi x_2 y_2\, e_1 e_3}\right)$$

and conclude therefore that

$$(x_1, x_2) \mapsto \left(e^{2\pi x_1 y_1\, e_2 e_3}, e^{2\pi x_2 y_2\, e_1 e_3}\right)$$

is the morphism $\widetilde{\phi}_{2\pi y_1, 0, e_2 e_3, 0, 2\pi y_2, e_1 e_3}$.

From Sect. 3, this latter may be rewritten $\widetilde{\Phi}_{2\pi y_1, 0, 0, 2\pi y_2, \frac{1}{4}(e_1 + e_2)(e_3 - e_4)}$.

Indeed, we have

$$\begin{aligned}\frac{1}{4}\big(e_2e_3+e_1e_3+I(e_2e_3-e_1e_3)\big)&=\frac{1}{4}(e_2e_3+e_1e_3-e_1e_4-e_2e_4)\\&=\frac{1}{4}\big(e_1(e_3-e_4)+e_2(e_3-e_4)\big)\\&=\frac{1}{4}\big((e_1+e_2)(e_3-e_4)\big).\end{aligned}$$ □

6 Conclusion

We proposed in this paper a definition of Clifford–Fourier transform that is motivated by group actions considerations. We defined a Clifford–Fourier transform that is associated with the action of all the group morphisms $\widetilde{\Phi}_{u,v,w,z,D}$ from $\mathbb{R}^2$ to Spin(4), parameterized by four real numbers and one unit bivector. This transform has the property of being left invertible. For the particular case of a color image, we associate the Clifford–Fourier transform with the action of group morphisms $\widetilde{\Phi}_{u,v,0,0,D}$, specified by only two real numbers (the frequencies) and where the bivector D is fixed. This transform is parameterized by a quadratic form on $\mathbb{R}^4$ and a unit bivector in the corresponding Clifford algebra. Some previous Fourier transforms based on quaternions are proved to be particular settings of ours. We have treated in this context an application to color image filtering. Future works will be devoted to find applications of the general transform that should easily deal with relations between colors in the image. Applications to multispectral images such as color/infrared images will be also investigated.

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Appendix

A.1 Lie Groups Representations and Fourier Transforms

From the group theory approach, the basic structure we need to define Fourier transforms is locally compact unimodular groups. Let us start by the definition of the dual of a topological group G, that is, the set of the equivalence classes of its unitary irreducible representations, denoted by $\widehat{G}$. We refer to [16] for details.

Definition 4 (Group representation) Let G be a topological group, and V be a topological vector space over $\mathbb{R}$ or $\mathbb{C}$.

A continuous linear representation (φ, V) from G to V is a group morphism

$$\varphi : g \mapsto \varphi(g)$$

from G to $GL(V)$ such that the map

$$(a, g) \mapsto \varphi(g)(a)$$

from $V \times G$ to V is continuous.

In general, V is a Hilbert space. If V is finite-dimensional, then the representation is said to be finite, and the dimension of V is called the degree of the representation.

Definition 5 (Irreducible representation) A subspace W of V is said to be invariant by φ if $\varphi(g)(W) \subset W \ \forall g \in G$.

Then, the representation φ is said to be irreducible if W, and $\{0\}$ are the only subspaces of V that are invariant by φ.

Definition 6 (Equivalent representations) Let (φ_1, V_1) and (φ_2, V_2) be two linear representations of the same group G. We say that they are equivalent if there exists an isomorphism $\gamma : V_1 \to V_2$ such that

$$\gamma \circ \varphi_1(g) = \varphi_2(g) \circ \gamma \quad \forall g \in G.$$

From now on, V is a $\mathbb{C}$-vector space equipped with a hermitian form $\langle\, , \rangle$.

Definition 7 (Unitary representation) The representation φ is unitary with respect to $\langle\, , \rangle$ if

$$\big\langle \varphi(g)(a), \varphi(g)(b) \big\rangle = \langle a, b \rangle \quad \forall a, b \in V,\ \forall g \in G.$$

We now restrict to locally compact unimodular groups. On such groups, we can construct a measure that is invariant with respect to both left and right translations. It a called a Haar measure. From a Haar measure a Haar integral of the group is defined.

Proposition 9 *Let G be a locally compact unimodular group, and let ν denote a Haar measure. Then, for $f \in L^2(G; \mathbb{C})$ and $h \in G$, we have*

$$\int_G f(g)\, d\nu(g) = \int_G f(gh)\, d\nu(g) = \int_G f(hg)\, d\nu(g).$$

Remark 6 Locally compact abelian groups and compact groups are unimodular.

Definition 8 (Fourier transform on locally compact unimodular groups) Let G be a locally compact unimodular group with Haar measure ν. The Fourier transform of $f \in L^2(G; \mathbb{C})$ is the map $\hat{f}$ defined on $\widehat{G}$ by

$$\hat{f}(\varphi) = \int_G f(g)\varphi\big(g^{-1}\big)\, d\nu(g).$$

Theorem 1 (Inversion formula of the Fourier transform) *$\hat{f}(\varphi)$ is a Hilbert–Schmidt operator over the space of the representation φ. There is a measure over $\widehat{G}$ denoted by $\widehat{\nu}$ such that $\hat{f} \in L^2(\widehat{G};\mathbb{C})$ and $f \mapsto \hat{f}$ is an isometry. Moreover, the following inverse formula holds*:

$$f(g) = \int_{\widehat{G}} \operatorname{Trace}\big(\hat{f}(\varphi)\varphi(g)\big)\, d\,\widehat{\nu}(\varphi).$$

Let us now have a closer look on Lie groups. We refer to [7] for an introduction to differential geometry.

Definition 9 (Lie group and Lie algebra) A real C^∞ Lie group is a topological group endowed with a structure of real C^∞-manifold. The Lie algebra of G is (isomorphic to) the tangent space of G at the neutral element e: T_eG. It is usually denoted by $\mathfrak{g}$. It can be made into an algebra over $\mathbb{R}$ by considering the Lie bracket $[\,,\,]$ that satisfies: $(X,Y) \mapsto [X,Y]$ from $\mathfrak{g} \times \mathfrak{g}$ to $\mathfrak{g}$ is $\mathbb{R}$-bilinear. Moreover, it satisfies

$$[X,X] = 0 \quad \forall\, X \in \mathfrak{g}$$

and

$$\big[X,[Y,Z]\big] + \big[Y,[Z,X]\big] + \big[Z,[X,Y]\big] = 0 \quad \forall X,Y,Z \in \mathfrak{g}.$$

Definition 10 (Exponential map) Let G be a C^∞ Lie group. The exponential map of G is the map from $\mathfrak{g}$ to G

$$\exp : X \longmapsto f(1),$$

where $f : \mathbb{R} \to G$ satisfies

$$f(t+s) = f(t)f(s) \quad \forall t,s \in \mathbb{R}$$

and

$$f'(0) = X.$$

f is called a one-parameter subgroup.

To compute group morphisms from $\mathbb{R}^2$ to Spin(3) and Spin(4), we use the following result on Lie groups morphisms.

Proposition 10 *Let G and H be two C^∞ Lie groups, and* $\exp_G$, $\exp_H$ *be the corresponding exponential maps. Let $\phi : G \to H$ be a Lie group morphism. The linear tangent map of ϕ at g, denoted by $T_g\phi$, is the linear map from T_gG to $T_{\phi(g)}H$ given by*

$$T_g\phi(X) = \frac{d}{dt}\phi\big(g \exp_G(tX)\big)|_{t=0}.$$

Then, if we note e the neutral element of G, we have

$$\phi\big(\exp_G(X)\big) = \exp_H\big(T_e\phi(X)\big). \tag{5}$$

The map $T_e\phi$ is a Lie algebra morphism, i.e., it satisfies

$$T_e\phi\big([X,Y]\big) = \big[T_e\phi(X), T_e\phi(Y)\big] \quad \forall X, Y \in \mathfrak{g}.$$

From (5) we deduce that if the group G is connected and the exponential map of G is onto, then the Lie group morphisms from G to H are determined by Lie algebras morphisms from $\mathfrak{g}$ to $\mathfrak{h}$.

A.2 Clifford Algebras

Let V be a vector space of finite dimension n over $\mathbb{R}$ equipped with a quadratic form Q. Formally speaking, the Clifford algebra $Cl(V,Q)$ is the solution of the following universal problem. We search a couple $(Cl(V,Q), i_Q)$ where $Cl(V,Q)$ is an $\mathbb{R}$-algebra and $i_Q : V \longrightarrow Cl(V,Q)$ is $\mathbb{R}$-linear satisfying

$$\big(i_Q(v)\big)^2 = Q(v).1$$

for all v in V (1 denotes the unit of $Cl(V,Q)$) such that, for each $\mathbb{R}$-algebra A and each $\mathbb{R}$-linear map $f : V \longrightarrow A$ with

$$\big(f(v)\big)^2 = Q(v).1$$

for all v in V (1 denotes the unit of A), then there exists a unique morphism

$$g : Cl(V,Q) \longrightarrow A$$

of $\mathbb{R}$-algebras such that $f = g \circ i_Q$.

The solution is unique up to isomorphisms and is given as the (noncommutative) quotient

$$T(V)/\big(v \otimes v - Q(v).1\big)$$

of the tensor algebra of V by the ideal generated by $v \otimes v - Q(v).1$, where v belongs to V (see [12] for a proof).

It is well known that there exists a unique anti-automorphism t on $Cl(V,Q)$ such that

$$t\big(i_Q(v)\big) = i_Q(v)$$

for all v in V. It is called reversion and usually denoted by $x \longmapsto x^\dagger$, x in $Cl(V,Q)$. In the same way there exists a unique automorphism α on $Cl(V,Q)$ such that

$$\alpha\big(i_Q(v)\big) = -i_Q(v)$$

for all v in V. In this paper we write v for $i_Q(v)$ (according to the fact that i_Q embeds V in $Cl(V, Q)$).

As a vector space, $Cl(V, Q)$ is of dimension 2^n on $\mathbb{R}$ and a basis is given by the set

$$\left\{e_{i_1}e_{i_2}\cdots e_{i_k},\ i_1 < i_2 < \cdots < i_k,\ \ k \in \{1, \ldots, n\}\right\}$$

and the unit 1. An element of degree k

$$\sum_{i_1<\cdots<i_k} \alpha_{i_1\ldots i_k} e_{i_1}e_{i_2}\cdots e_{i_k}$$

is called a k-vector. A 0-vector is a scalar, and $e_1e_2\cdots e_n$ is called the pseudoscalar. We denote $\langle x\rangle_k$ the component of degree k of an element x of $Cl(V, Q)$.

The inner product of x_r of degree r and y_s of degree s is defined by

$$x_r \cdot y_s = \langle x_r y_s\rangle_{|r-s|}$$

if r and s are positive and by

$$x_r \cdot y_s = 0$$

otherwise.

The outer product of x_r of degree r and y_s of degree s is defined by

$$x_r \wedge y_s = \langle x_r y_s\rangle_{r+s}.$$

These products extend by linearity on $Cl(V, Q)$. Clearly, if a and b are vectors of V, then the inner product of a and b coincides with the scalar product defined by Q. When it is defined (for example, when x is a versor and Q is positive), we denote

$$\|x\| = \sqrt{xx^\dagger}$$

and say that x is a unit if $xx^\dagger = \pm 1$.

In this paper, we deal in particular with the Clifford algebra of the Euclidean $\mathbb{R}^n$ denoted by $\mathbb{R}_{n,0}$. $\mathbb{R}^k_{n,0}$ is the subspace of elements of degree k, and $\mathbb{R}^*_{n,0}$ is the group of elements that admit an inverse in $\mathbb{R}_{n,0}$. We denote by $\mathbb{S}^2_{n,0}$ the set of elements of $\mathbb{R}^2_{n,0}$ of norm 1.

Let a be a vector in $\mathbb{R}_{n,0}$, and B be the k-vector $a_1 \wedge a_2 \wedge \cdots \wedge a_k$. Then the orthogonal projection of a on the k-plane generated by the a_i's is the vector

$$P_B(a) = (a \cdot B)B^{-1}.$$

The vector

$$a - (a \cdot B)B^{-1} = (a \wedge B)B^{-1}$$

is called the rejection of a on B.

A.3 The Spinor Group Spin(n)

It is defined by

$$\mathrm{Spin}(n) = \left\{ \prod_{i=1}^{2k} a_i,\ a_i \in \mathbb{R}^1_{n,0},\ \|a_i\| = 1 \right\}$$

or equivalently

$$\mathrm{Spin}(n) = \left\{ x \in \mathbb{R}_{n,0},\ \alpha(x) = x,\ xx^\dagger = 1, xvx^{-1} \in \mathbb{R}^1_{n,0}\ \forall v \in \mathbb{R}^1_{n,0} \right\}.$$

It is well known that Spin(n) is a connected compact Lie group that universally covers SO(n) ($n \geq 3$). One can verify that Spin(3) is the group

$$\left\{ a1 + be_1e_2 + ce_2e_3 + de_3e_1,\ a^2 + b^2 + c^2 + d^2 = 1 \right\}$$

and is isomorphic to the group $\mathbb{H}^1$ of unit quaternions. It is also a classical result that Spin(4) is isomorphic to Spin(3) $\times$ Spin(3) (see [9] for more information on spinors in $\mathbb{R}^3$ and $\mathbb{R}^4$).

The Lie algebra of Spin(n) is $\mathbb{R}^2_{n,0}$ with Lie bracket

$$A \times B = AB - BA.$$

As the exponential map from its Lie algebra to Spin(n) is onto (see [7] for a proof), every spinor can be written as

$$S = \sum_{i=0}^{\infty} \frac{1}{i!} A^i$$

for some bivector A.

From Hestenes and Sobczyk [8] we know that every A in $\mathbb{R}^2_{n,0}$ can be written as

$$A = A_1 + A_2 + \cdots + A_m,$$

where $m \leq n/2$, and

$$A_j = \|A_j\| a_j b_j, \quad j \in \{1, \ldots, m\}$$

with

$$\{a_1, \ldots, a_m, b_1, \ldots, b_m\}$$

a set of orthonormal vectors. Thus,

$$A_j A_k = A_k A_j = A_k \wedge A_j$$

whenever $j \neq k$ and

$$A_k^2 = -\|A_k\|^2 < 0.$$

This means that the planes encoded by A_k and A_j are orthogonal and implies that

$$e^{A_1+A_2+\cdots+A_m} = e^{A_{\sigma(1)}} e^{A_{\sigma(2)}} \ldots e^{A_{\sigma(m)}}$$

for all σ in the permutation group $\mathfrak{S}(m)$. Actually, as A_k^2 is negative, we have

$$e^{A_i} = \cos\big(\|A_i\|\big) + \sin\big(\|A_i\|\big)\frac{A_i}{\|A_i\|}.$$

The corresponding rotation

$$R_i : x \longmapsto e^{-A_i} x e^{A_i}$$

acts in the oriented plane defined by A_i as a plane rotation of angle $2\|A_i\|$. The vectors orthogonal to A_i are invariant under R_i.

It then appears that any element R of SO(n) is a composition of commuting simple rotations, in the sense that they have only one invariant plane. The vectors left invariant by R are those of the orthogonal subspace to A. If $m = n/2$, this latter is trivial. The previous decomposition is not unique if $\|A_k\| = \|A_j\|$ for some j and k with $j \neq k$. In this case infinitely many planes are left invariant by R.

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