The Cylindrical Fourier Transform

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Abstract The aim of this paper is to show the application potential of the cylindrical Fourier transform, which was recently devised as a new integral transform within the context of Clifford analysis. Next to the approximation approach where, using density arguments, the spectrum of various types of functions and distributions may be calculated starting from the cylindrical Fourier images of the L_2-basis functions in $\mathbb{R}^m$, direct computation methods are introduced for specific distributions supported on the unit sphere, and an illustrative example is worked out.

1 Introduction

The Fourier transform is by far the most important integral transform. Since its introduction by Fourier in the early 1800s, it has remained an indispensable and stimulating mathematical concept that is at the core of the highly evolved branch of mathematics called Fourier analysis.

The second subject of great relevance for the paper is Clifford analysis, an elegant and powerful higher-dimensional generalization of the theory of holomorphic functions, which is moreover closely related but complementary to harmonic analysis. Clifford analysis also offers the possibility to generalize one-dimensional mathematical analysis to higher dimension in a rather natural way by encompassing all dimensions at once, as opposed to the usual tensorial approaches.

It is precisely this last qualification which has been exploited in [2] and [3] to construct a genuine multidimensional Fourier transform within the context of Clifford analysis. This so-called Clifford–Fourier transform is briefly discussed in Sect. 3.

In [4] and [5] we devised and thoroughly studied the so-called cylindrical Fourier transform within the Clifford analysis setting. The idea is the following: for a fixed

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E. Bayro-Corrochano, G. Scheuermann (eds.), *Geometric Algebra Computing*, DOI 10.1007/978-1-84996-108-0_6,

vector in the image space, the level surfaces of the traditional Fourier kernel are planes perpendicular to that fixed vector. For this Fourier kernel, we now substitute a new Clifford–Fourier kernel such that, again for a fixed vector in the image space, its phase is constant on co-axial cylinders w.r.t. that fixed vector. The point is that, when restricting to dimension two, this new cylindrical Fourier transform coincides with the earlier introduced Clifford–Fourier transform. We are now faced with the following situation: in dimension greater than two we have a first Clifford–Fourier transform with elegant properties but no kernel in closed form, and a second cylindrical one with a kernel in closed form but more complicated calculation formulae. In dimension two both transforms coincide.

The aim of this paper is to show the application potential of the cylindrical Fourier transform.

To make the paper self-contained, we have also included an introductory section (Sect. 2) on Clifford analysis.

2 The Clifford Analysis Toolkit

Clifford analysis (see, e.g., [1]) offers a function theory which is a higher-dimensional analogue of the theory of the holomorphic functions of one complex variable.

The functions considered are defined in $\mathbb{R}^m$ $(m > 1)$ and take their values in the Clifford algebra $\mathbb{R}_{0,m}$ or its complexification $\mathbb{C}_m = \mathbb{R}_{0,m} \otimes \mathbb{C}$. If $(e_1, \ldots, e_m)$ is an orthonormal basis of $\mathbb{R}^m$, then a basis for the Clifford algebra $\mathbb{R}_{0,m}$ or $\mathbb{C}_m$ is given by all possible products of basis vectors $(e_A : A \subset \{1, \ldots, m\})$, where $e_\emptyset = 1$ is the identity element. The noncommutative multiplication in the Clifford algebra is governed by the rules $e_j e_k + e_k e_j = -2\delta_{j,k}$ $(j, k = 1, \ldots, m)$.

Conjugation is defined as the anti-involution for which $\overline{e_j} = -e_j$ $(j = 1, \ldots, m)$. In case of $\mathbb{C}_m$, the Hermitian conjugate of an element $\lambda = \sum_A \lambda_A e_A$ $(\lambda_A \in \mathbb{C})$ is defined by $\lambda^\dagger = \sum_A \lambda_A^c \, \overline{e_A}$, where λ_A^c denotes the complex conjugate of λ_A. This Hermitian conjugation leads to a Hermitian inner product and its associated norm on $\mathbb{C}_m$ given respectively by

$$(\lambda, \mu) = \left[\lambda^\dagger \mu\right]_0 \quad \text{and} \quad |\lambda|^2 = \left[\lambda^\dagger \lambda\right]_0 = \sum_A |\lambda_A|^2,$$

where $[\lambda]_0$ denotes the scalar part of the Clifford element λ.

The Euclidean space $\mathbb{R}^m$ is embedded in the Clifford algebras $\mathbb{R}_{0,m}$ and $\mathbb{C}_m$ by identifying the point $(x_1, \ldots, x_m)$ with the vector variable $\underline{x}$ given by $\underline{x} = \sum_{j=1}^m e_j x_j$. The product of two vectors splits up into a scalar part (the inner product up to a minus sign) and a so-called bivector part (the wedge product):

$$\underline{x}\,\underline{y} = \underline{x}.\underline{y} + \underline{x} \wedge \underline{y},$$

where

$$\underline{x}.\underline{y} = -\langle \underline{x}, \underline{y} \rangle = -\sum_{j=1}^{m} x_j y_j \quad \text{and} \quad \underline{x} \wedge \underline{y} = \sum_{i=1}^{m} \sum_{j=i+1}^{m} e_i e_j (x_i y_j - x_j y_i).$$

Note that the square of a vector variable $\underline{x}$ is scalar-valued and equals the norm squared up to a minus sign: $\underline{x}^2 = -\langle \underline{x}, \underline{x} \rangle = -|\underline{x}|^2$.

The central notion in Clifford analysis is the notion of monogenicity, a notion which is the multidimensional counterpart to that of holomorphy in the complex plane. A function $F(x_1, \ldots, x_m)$ defined and continuously differentiable in an open region of $\mathbb{R}^m$ and taking values in $\mathbb{R}_{0,m}$ or $\mathbb{C}_m$ is called left monogenic in that region if $\partial_{\underline{x}}[F] = 0$. Here $\partial_{\underline{x}}$ is the Dirac operator in $\mathbb{R}^m$: $\partial_{\underline{x}} = \sum_{j=1}^{m} e_j \partial_{x_j}$, an elliptic, rotation-invariant, vector differential operator of the first order, which may be looked upon as the "square root" of the Laplace operator in $\mathbb{R}^m$: $\Delta_m = -\partial_{\underline{x}}^2$. This factorization of the Laplace operator establishes a special relationship between Clifford analysis and harmonic analysis in that monogenic functions refine the properties of harmonic functions.

In the sequel the monogenic homogeneous polynomials will play an important role. A left-monogenic homogeneous polynomial P_k of degree k $(k \geq 0)$ in $\mathbb{R}^m$ is called a left solid inner spherical monogenic of order k. The set of all left solid inner spherical monogenics of order k will be denoted by $M_\ell^+(k)$. The dimension of $M_\ell^+(k)$ is given by

$$\dim\big(M_\ell^+(k)\big) = \binom{m+k-2}{m-2} = \frac{(m+k-2)!}{(m-2)!k!}.$$

The set

$$\phi_{s,k,j}(\underline{x}) = \frac{2^{m/4}}{(\gamma_{s,k})^{1/2}} H_{s,k}\big(\sqrt{2}\underline{x}\big) P_k^{(j)}\big(\sqrt{2}\underline{x}\big) e^{(-|\underline{x}|^2/2)}, \tag{1}$$

$s, k \in \mathbb{N}$, $j \leq \dim(M_\ell^+(k))$, constitutes an orthonormal basis for the space $L_2(\mathbb{R}^m)$ of square-integrable functions. Here $\{P_k^{(j)}(\underline{x});\ j \leq \dim(M_\ell^+(k))\}$ denotes an orthonormal basis of $M_\ell^+(k)$, and $\gamma_{s,k}$ a real constant depending on the parity of s. The polynomials $H_{s,k}(\underline{x})$ are the so-called generalized Clifford–Hermite polynomials introduced by Sommen; they are a multidimensional generalization to Clifford analysis of the classical Hermite polynomials on the real line. Note that $H_{s,k}(\underline{x})$ is a polynomial of degree s in the variable $\underline{x}$ with real coefficients depending on k. Furthermore, $H_{2s,k}(\underline{x})$ only contains even powers of $\underline{x}$ and is hence scalar valued, while $H_{2s+1,k}(\underline{x})$ only contains odd ones and is thus vector valued.

A result, which will be frequently used in Sect. 4.3, is the following generalization of the classical Funk–Hecke theorem.

Theorem 1 (Funk–Hecke theorem in space) *Let S_k be a spherical harmonic of degree k, and $\underline{\eta}$ a fixed point on the unit sphere S^{m-1} in $\mathbb{R}^m$. Denote $\langle \underline{\omega}, \underline{\eta} \rangle =$*

$\cos(\widehat{\underline{\omega},\underline{\eta}}) = t_{\underline{\eta}}$ for $\underline{\omega} \in S^{m-1}$. Then

$$\int_{\mathbb{R}^m} g(r) f(t_{\underline{\eta}}) S_k(\underline{\omega})\, dV(\underline{x})$$
$$= A_{m-1}\left(\int_0^{+\infty} g(r) r^{m-1}\, dr\right)\left(\int_{-1}^{1} f(t)\big(1-t^2\big)^{(m-3)/2} P_{k,m}(t)\, dt\right) S_k(\underline{\eta}),$$

where $dV(\underline{x})$ denotes the Lebesgue measure on $\mathbb{R}^m$, $P_{k,m}(t)$ the Legendre polynomial of degree k in the m-dimensional Euclidean space, and $A_{m-1} = \frac{2\pi^{(m-1)/2}}{\Gamma(\frac{m-1}{2})}$ the surface area of the unit sphere S^{m-2} in $\mathbb{R}^{m-1}$.

As the Legendre polynomials are even or odd according to the parity of k, we can also state the following corollary.

Corollary 1 *Let S_k be a spherical harmonic of degree k, and $\underline{\eta}$ a fixed point on the unit sphere S^{m-1}. Denote $\langle \underline{\omega}, \underline{\eta}\rangle = t_{\underline{\eta}}$ for $\underline{\omega} \in S^{m-1}$. Then the 3D-integral*

$$\int_{\mathbb{R}^m} g(r) f(t_{\underline{\eta}}) S_k(\underline{\omega})\, dV(\underline{x})$$

is zero whenever

- *f is an odd function, and k is even;*
- *f is an even function, and k is odd.*

3 The Clifford–Fourier Transform

In [2] a new multidimensional Fourier transform in the framework of Clifford analysis, the so-called Clifford–Fourier transform, is introduced. The idea behind its definition originates from an alternative representation for the standard tensorial multidimensional Fourier transform given by

$$\mathcal{F}[f](\underline{\xi}) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} e^{(-i\langle \underline{x}, \underline{\xi}\rangle)} f(\underline{x})\, dV(\underline{x}).$$

It is indeed so that this classical Fourier transform can be seen as the operator exponential

$$\mathcal{F} = e^{(-i\pi/2\mathcal{H})} = \sum_{k=0}^{\infty} \frac{1}{k!}\left(-i\frac{\pi}{2}\right)^k \mathcal{H}^k,$$

where $\mathcal{H}$ is the scalar-valued differential operator $\mathcal{H} = \frac{1}{2}(-\Delta_m + r^2 - m)$. Note that, due to the scalar character of the standard Fourier kernel, the Fourier spectrum inherits its Clifford algebra character from the original signal, without any interaction with the Fourier kernel. So in order to genuinely introduce the Clifford analysis character in the Fourier transform, the idea occurred to us to replace the scalar-valued operator $\mathcal{H}$ in the operator exponential by a Clifford algebra-valued one. To that end, we aimed at factorizing the operator $\mathcal{H}$, making use of the factorization of the Laplace operator by the Dirac operator. Splitting $\mathcal{H}$ into a sum of Clifford algebra-valued second-order operators leads in a natural way to a *pair* of transforms $\mathcal{F}_{\mathcal{H}^{\pm}}$, the harmonic average of which is precisely the standard Fourier transform $\mathcal{F}$, i.e., $\mathcal{F}^2 = \mathcal{F}_{\mathcal{H}^+}\mathcal{F}_{\mathcal{H}^-}$.

The two-dimensional case of this Clifford–Fourier transform is special in that we are able to find a closed form for the kernel of the integral representation. Indeed, the two-dimensional Clifford–Fourier transform takes the form

$$\mathcal{F}_{\mathcal{H}^{\pm}}[f](\underline{\xi}) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{(\pm(\underline{\xi}\wedge\underline{x}))} f(\underline{x})\, dV(\underline{x}).$$

This closed form enables us to generalize the well-known results for the standard Fourier transform both in the L_1- and L_2-contexts (see [3]). Note that we have not succeeded yet in obtaining such a closed form in arbitrary dimension. For a detailed account of the Clifford–Fourier transform, we refer the reader to the survey paper [5].

4 The Cylindrical Fourier Transform

4.1 Definition

The cylindrical Fourier transform is obtained by taking the multidimensional generalization of the two-dimensional Clifford–Fourier kernel.

Definition 1 The cylindrical Fourier transform of a function f is given by

$$\mathcal{F}_{\mathrm{cyl}}[f](\underline{\xi}) = \frac{1}{(2\pi)^{m/2}}\int_{\mathbb{R}^m} e^{(\underline{x}\wedge\underline{\xi})} f(\underline{x})\, dV(\underline{x})$$

with $e^{(\underline{x}\wedge\underline{\xi})} = \sum_{r=0}^{\infty} \frac{(\underline{x}\wedge\underline{\xi})^r}{r!}$.

The integral kernel of this cylindrical Fourier transform can be rewritten in terms of the cosine and the sinc function, which also reveals its form of a scalar plus a bivector, i.e., a so-called parabivector.

Proposition 1 *The kernel of the cylindrical Fourier transform can be rewritten as*

$$e^{(\underline{x}\wedge\underline{\xi})} = \cos\big(|\underline{x}\wedge\underline{\xi}|\big) + (\underline{x}\wedge\underline{\xi})\mathrm{sinc}\big(|\underline{x}\wedge\underline{\xi}|\big)$$

where $\mathrm{sinc}(x) := \frac{\sin(x)}{x}$ *is the unnormalized sinc function.*

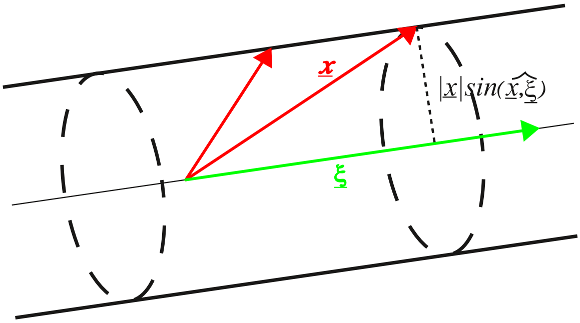

Fig. 1 In case of the cylindrical Fourier transform, for fixed $\underline{\xi}$, the phase $|\underline{x} \wedge \underline{\xi}|$ is constant on coaxial cylinders

Proof Splitting the defining series expansion of $e^{(\underline{x}\wedge\underline{\xi})}$ into its even and odd part and taking into account that $(\underline{x} \wedge \underline{\xi})^2 = -|\underline{x} \wedge \underline{\xi}|^2$ yields

$$\begin{aligned} e^{(\underline{x}\wedge\underline{\xi})} &= \sum_{\ell=0}^{\infty}(-1)^{\ell}\frac{|\underline{x}\wedge\underline{\xi}|^{2\ell}}{(2\ell)!} + (\underline{x}\wedge\underline{\xi})\sum_{\ell=0}^{\infty}(-1)^{\ell}\frac{|\underline{x}\wedge\underline{\xi}|^{2\ell}}{(2\ell+1)!} \\ &= \cos\big(|\underline{x}\wedge\underline{\xi}|\big) + (\underline{x}\wedge\underline{\xi})\,\mathrm{sinc}\big(|\underline{x}\wedge\underline{\xi}|\big). \end{aligned}$$ □

Let us now explain why we have chosen the name "cylindrical" for our new Fourier transform. From

$$|\underline{x}\wedge\underline{\xi}|^2 = |\underline{x}|^2|\underline{\xi}|^2 - \big(\langle\underline{x},\underline{\xi}\rangle\big)^2 = |\underline{x}|^2|\underline{\xi}|^2\big(1-\cos(\widehat{\underline{x},\underline{\xi}})^2\big) = |\underline{x}|^2|\underline{\xi}|^2\sin(\widehat{\underline{x},\underline{\xi}})^2$$

it is clear that for $\underline{\xi}$ fixed, the "phase" $|\underline{x}\wedge\underline{\xi}|$ is constant if and only if $|\underline{x}|\sin(\widehat{\underline{x},\underline{\xi}})$ is constant. In other words, for a fixed vector $\underline{\xi}$ in the image space, the phase $|\underline{x}\wedge\underline{\xi}|$ is constant on co-axial cylinders w.r.t. that fixed vector (see Fig. 1). For comparison, for a fixed vector $\underline{\xi}$ in the image space, the level surfaces of the traditional Fourier kernel are planes perpendicular to that fixed vector, since $\langle\underline{x},\underline{\xi}\rangle = |\underline{x}||\underline{\xi}|\cos(\widehat{\underline{x},\underline{\xi}})$ (see Fig. 2).

4.2 Properties

The cylindrical Fourier transform is well defined for each integrable function.

Theorem 2 *Let $f \in L_1(\mathbb{R}^m)$. Then $\mathcal{F}_{\mathrm{cyl}}[f] \in L_\infty(\mathbb{R}^m) \cap C_0(\mathbb{R}^m)$, and, moreover,*

$$\big\|\mathcal{F}_{\mathrm{cyl}}[f]\big\|_\infty \le 2\left(\frac{2}{\pi}\right)^{m/2}\|f\|_1.$$

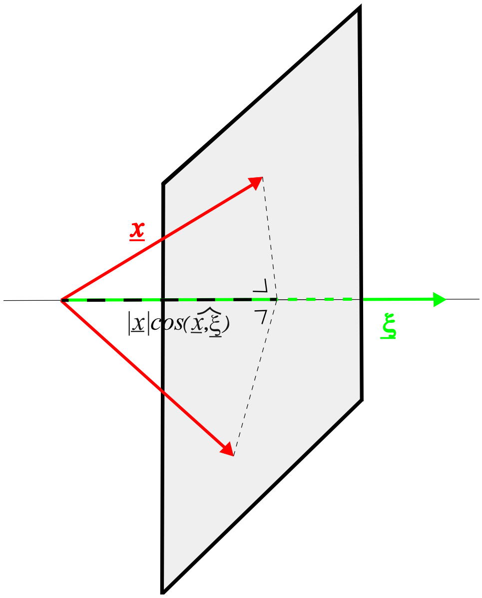

Fig. 2 In case of the classical Fourier transform, for fixed $\underline{\xi}$, the phase $\langle \underline{x}, \underline{\xi} \rangle$ is constant on planes perpendicular to $\underline{\xi}$

Proof Taking into account Proposition 1, we have that

$$\begin{aligned}\left|e^{(\underline{x}\wedge\underline{\xi})}\right| &= \left|\cos\big(|\underline{x}\wedge\underline{\xi}|\big) + \frac{(\underline{x}\wedge\underline{\xi})}{|\underline{x}\wedge\underline{\xi}|}\sin\big(|\underline{x}\wedge\underline{\xi}|\big)\right| \\ &\leq \left|\cos\big(|\underline{x}\wedge\underline{\xi}|\big)\right| + \left|\sin\big(|\underline{x}\wedge\underline{\xi}|\big)\right| \leq 2,\end{aligned}$$

which leads to the desired result. □

Although the cylindrical Fourier transform has a "simple" integral kernel, it satisfies calculation formulae that are more complicated than those of the multidimensional Clifford–Fourier transform (see [5]). For example, we state the differentiation and multiplication rules that nicely show that the two-dimensional case, in which the cylindrical Fourier transform and the Clifford–Fourier transform coincide, is special.

Proposition 2 (Differentiation and multiplication rule) *Let* $f, g \in L_1(\mathrm{R}^m)$. *The cylindrical Fourier transform satisfies*:

(i) *the differentiation rule*

$$\begin{aligned}\mathcal{F}_{\mathrm{cyl}}\big[\partial_{\underline{x}}\big[f(\underline{x})\big]\big](\underline{\xi}) = -\underline{\xi}\mathcal{F}_{\mathrm{cyl}}\big[f(\underline{x})\big](-\underline{\xi}) + \frac{(2-m)}{(2\pi)^{m/2}}\underline{\xi} \\ \times \int_{\mathbb{R}^m} \mathrm{sinc}\big(|\underline{x}\wedge\underline{\xi}|\big) f(\underline{x})\, dV(\underline{x})\end{aligned}$$

with $\mathrm{sinc}(x) := \frac{\sin(x)}{x}$ *the unnormalized sinc function*;

(ii) *the multiplication rule*

$$\mathcal{F}_{\mathrm{cyl}}\big[\underline{x}\, f(\underline{x})\big](\underline{\xi}) = -\partial_{\underline{\xi}}\big[\mathcal{F}_{\mathrm{cyl}}\big[f(\underline{x})\big](-\underline{\xi})\big] + \frac{(2-m)}{(2\pi)^{m/2}} \times \int_{\mathbb{R}^m} \operatorname{sinc}\big(|\underline{x} \wedge \underline{\xi}|\big)\underline{x}\, f(\underline{x})\, dV(\underline{x}).$$

4.3 Spectrum of the L_2-Basis Consisting of Generalized Clifford–Hermite Functions

Let us now calculate the cylindrical Fourier spectrum of the L_2-basis (1). As these basis elements belong to the space of rapidly decreasing functions $\mathcal{S}(\mathbb{R}^m) \subset L_1(\mathbb{R}^m)$, their cylindrical Fourier image should be a bounded and continuous function. The calculation method is based on the Funk–Hecke theorem in space (see Theorem 1) and the following cylindrical Fourier kernel decomposition (see Proposition 1):

$$e^{(\underline{x}\wedge\underline{\xi})} = \cos\Big(r\rho\sqrt{1-t_{\underline{\eta}}^2}\Big) - r\rho t_{\underline{\eta}} \operatorname{sinc}\Big(r\rho\sqrt{1-t_{\underline{\eta}}^2}\Big) - r\rho\underline{\eta}\,\underline{\omega}\operatorname{sinc}\Big(r\rho\sqrt{1-t_{\underline{\eta}}^2}\Big), \quad (2)$$

where we have introduced the spherical coordinates

$$\underline{x} = r\underline{\omega}, \qquad \underline{\xi} = \rho\underline{\eta}, \qquad r = |\underline{x}|, \qquad \rho = |\underline{\xi}|, \qquad \underline{\omega}, \underline{\eta} \in S^{m-1}$$

and the notation $t_{\underline{\eta}} = \langle\underline{\omega}, \underline{\eta}\rangle$. For convenience, we denote the three terms in the decomposition (2) by A, B, and C.

As a first example, let us now calculate the cylindrical Fourier transform of the basis function $\phi_{0,k,j}(\underline{x})$ which is given, up to constants, by $P_k(\underline{x})e^{(-|\underline{x}|^2/2)}$ with P_k a left solid inner spherical monogenic of order k. By Corollary 1 it is obvious that we must make a distinction between k even and odd.

(A) *k even*

In the case where k is even, as a consequence of Corollary 1, the integrals containing the B- and C-terms of the kernel decomposition (2) reduce to zero. Furthermore, applying the Funk–Hecke theorem in space (see Theorem 1), we have that

$$\begin{aligned}\mathcal{F}_{\mathrm{cyl}}\big[e^{(-|\underline{x}|^2/2)}\, P_k(\underline{x})\big](\underline{\xi}) &= \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} e^{(-r^2/2)} r^k \cos\Big(r\rho\sqrt{1-t_{\underline{\eta}}^2}\Big) P_k(\underline{\omega})\, dV(\underline{x}) \\ &= \frac{A_{m-1}}{(2\pi)^{m/2}} P_k(\underline{\eta}) \bigg(\int_0^{+\infty} e^{(-r^2/2)} r^{k+m-1}\, dr\bigg) \\ &\quad \times \bigg(\int_{-1}^{1} \cos\Big(r\rho\sqrt{1-t^2}\Big)\big(1-t^2\big)^{(m-3)/2} P_{k,m}(t)\, dt\bigg).\end{aligned}$$

Taking into account the series expansion of the cosine function, this result becomes

$$\begin{aligned}\mathcal{F}_{\mathrm{cyl}}\big[e^{(-|\underline{x}|^2/2)}P_k(\underline{x})\big](\underline{\xi}) &= \frac{k!(m-3)!}{(k+m-3)!}\frac{A_{m-1}}{(2\pi)^{m/2}}P_k(\underline{\eta})\sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{(2\ell)!}\rho^{2\ell}\\ &\quad\times\left(\int_0^{+\infty}e^{(-r^2/2)}r^{2\ell+k+m-1}\,dr\right)\\ &\quad\times\left(\int_{-1}^{1}\big(1-t^2\big)^{(2\ell+m-3)/2}C_k^{(m-2)/2}(t)\,dt\right),\end{aligned} \quad (3)$$

where we have also used the expression

$$P_{k,m}(t)=\frac{k!(m-3)!}{(k+m-3)!}C_k^{(m-2)/2}(t)$$

of the Legendre polynomials in $\mathbb{R}^m$ in terms of the Gegenbauer polynomials $C_k^\lambda(t)$. As these Gegenbauer polynomials C_k^λ are orthogonal on $]-1,1[$ w.r.t. the weight function $(1-t^2)^{\lambda-1/2}$ $(\lambda>-\frac{1}{2})$, it is easily seen that, for $\ell\le\frac{k}{2}-1$,

$$\int_{-1}^{1}\big(1-t^2\big)^\ell\big(1-t^2\big)^{(m-3)/2}C_k^{(m-2)/2}(t)\,dt=0.$$

Moreover, combining the integral formula (see [7], p. 826, formula (4) with $\alpha=\beta$)

$$\begin{aligned}&\int_{-1}^{1}\big(1-t^2\big)^\alpha C_k^\lambda(t)\,dt\\ &\quad=\frac{2^{2\alpha+1}(\Gamma(\alpha+1))^2\Gamma(k+2\lambda)}{k!\Gamma(2\lambda)\Gamma(2\alpha+2)}\,{}_3F_2\left(-k,k+2\lambda,\alpha+1;\lambda+\frac{1}{2},2\alpha+2;1\right),\end{aligned}$$

where $\mathrm{Re}(\alpha)>-1$, and ${}_3F_2(a,b,c;d,e;z)$ denotes the generalized hypergeometric series, with Watson's theorem (see, e.g., [6])

$${}_3F_2\left(a,b,c;\frac{a+b+1}{2},2c;1\right)=\frac{\sqrt{\pi}\,\Gamma(c+\frac{1}{2})\Gamma(\frac{a+b+1}{2})\Gamma(\frac{1-a-b+2c}{2})}{\Gamma(\frac{a+1}{2})\Gamma(\frac{b+1}{2})\Gamma(\frac{1-a+2c}{2})\,\Gamma(\frac{1-b+2c}{2})}$$

results into

$$\begin{aligned}&\int_{-1}^{1}(1-t^2)^\alpha\;C_k^\lambda(t)\,dt\\ &\quad=\frac{\sqrt{\pi}2^{2\alpha+1}(\Gamma(\alpha+1))^2\Gamma(k+2\lambda)\Gamma(\alpha+\frac{3}{2})\Gamma(\lambda+\frac{1}{2})\Gamma(\frac{2\alpha-2\lambda+3}{2})}{k!\Gamma(2\lambda)\Gamma(2\alpha+2)\Gamma(\frac{-k+1}{2})\Gamma(\frac{k+2\lambda+1}{2})\Gamma(\frac{2\alpha+3+k}{2})\Gamma(\frac{2\alpha-2\lambda+3-k}{2})}.\end{aligned} \quad (4)$$

Applying the above result and taking into account that $\Gamma(2z)=\pi^{-1/2}2^{2z-1}\times\Gamma(z)\Gamma(z+1/2)$, (3) can be simplified to

$$\mathcal{F}_{\text{cyl}}\big[e^{(-|\underline{x}|^2/2)}P_k(\underline{x})\big](\underline{\xi})$$
$$= \frac{2^{k/2}\sqrt{\pi}}{\Gamma(\frac{-k+1}{2})\Gamma(\frac{k+m-1}{2})} P_k(\underline{\xi}) \sum_{\ell=k/2}^{\infty} \frac{(-1)^\ell 2^\ell \ell!\,\Gamma(\frac{2\ell+m-1}{2})}{(2\ell)!\;\Gamma(\frac{2\ell+2-k}{2})}\;|\underline{\xi}|^{2\ell-k}$$
$$= {}_1F_1\left(1-\frac{m}{2};\frac{k+1}{2};\frac{|\underline{\xi}|^2}{2}\right)e^{(-|\underline{\xi}|^2/2)}P_k(\underline{\xi})$$

with ${}_1F_1(a;c;z)$ Kummer's function, also called confluent hypergeometric function.

(B) *k odd*

For k odd, the integral containing the A-term of the kernel decomposition is zero, again as a consequence of Corollary 1. By means of the Funk–Hecke theorem in space we obtain

$$\mathcal{F}_{\text{cyl}}\big[e^{(-|\underline{x}|^2/2)}P_k(\underline{x})\big](\underline{\xi})$$
$$= \rho\frac{A_{m-1}}{(2\pi)^{m/2}}P_k(\underline{\eta})\left(\int_0^{+\infty} e^{(-r^2/2)}r^{k+m}\,dr\right)$$
$$\times\left(\int_{-1}^{1}\text{sinc}\Big(r\rho\sqrt{1-t^2}\Big)\big(1-t^2\big)^{(m-3)/2}\big(P_{k+1,m}(t)-t\,P_{k,m}(t)\big)\,dt\right).$$

Now, taking into account the Gegenbauer recurrence relation

$$(k+2\lambda)tC_k^\lambda(t)-(k+1)C_{k+1}^\lambda(t)=2\lambda\big(1-t^2\big)C_{k-1}^{\lambda+1}(t),$$

we have that

$$P_{k+1,m}(t)-t\,P_{k,m}(t) = -\frac{k!(m-2)!}{(k+m-2)!}\big(1-t^2\big)C_{k-1}^{m/2}(t),$$

which in turn yields

$$\mathcal{F}_{\text{cyl}}\big[e^{(-|\underline{x}|^2/2)}P_k(\underline{x})\big](\underline{\xi}) = -\frac{k!(m-2)!}{(k+m-2)!}\;\frac{A_{m-1}}{(2\pi)^{m/2}}\rho P_k(\underline{\eta})$$
$$\times\left(\int_0^{+\infty} e^{(-r^2/2)}r^{k+m}\,dr\right)$$
$$\times\left(\int_{-1}^{1}\text{sinc}\Big(r\rho\sqrt{1-t^2}\Big)\big(1-t^2\big)^{(m-1)/2}C_{k-1}^{m/2}(t)\,dt\right).$$

Next, applying consecutively the series expansion of the sinc function, the orthogonality of the Gegenbauer polynomials and expression (4), we find

$$\mathcal{F}_{\text{cyl}}\big[e^{(-|\underline{x}|^2/2)}P_k(\underline{x})\big](\underline{\xi}) = -{}_1F_1\left(1-\frac{m}{2};\frac{k+2}{2};\frac{|\underline{\xi}|^2}{2}\right)e^{(-|\underline{\xi}|^2/2)}P_k(\underline{\xi}).$$

So note that the cylindrical Fourier transform reproduces the Gaussian times the spherical monogenic up to a Kummer's function factor.

A second example is provided by the cylindrical Fourier transform of the basis function $\phi_{1,k,j}$ given, up to constants, by $e^{(-|\underline{x}|^2/2)}\underline{x}\, P_k(\underline{x})$. Its calculation runs along similar lines. Making again a distinction between k even and k odd, we find:

(A) *for k even*,

$$\mathcal{F}_{\text{cyl}}\left[e^{(-|\underline{x}|^2/2)}\ \underline{x}\ P_k(\underline{x})\right](\underline{\xi})$$
$$= \frac{(k+m-1)}{(k+1)}\ {}_1F_1\left(1-\frac{m}{2};\frac{k+3}{2};\frac{|\underline{\xi}|^2}{2}\right)e^{(-|\underline{\xi}|^2/2)}\underline{\xi}\ P_k(\underline{\xi});$$

(B) *for k odd*,

$$\mathcal{F}_{\text{cyl}}\left[e^{(-|\underline{x}|^2/2)}\ \underline{x}\ P_k(\underline{x})\right](\underline{\xi}) = {}_1F_1\left(1-\frac{m}{2};\frac{k+2}{2};\frac{|\underline{\xi}|^2}{2}\right)e^{(-|\underline{\xi}|^2/2)}\underline{\xi}\ P_k(\underline{\xi}),$$

showing again the reproducing property up to a Kummer's function factor.

For the calculation of the cylindrical Fourier spectrum of a general basis element $\phi_{s,k,j}$, we refer the reader to [4] and the survey paper [5].

5 Application Potential of the Cylindrical Fourier Transform

In the foregoing section we established the image under the cylindrical Fourier transform of an L_2-basis for the space of all L_2-functions in $\mathbb{R}^m$. Using density arguments, these results may be used to approximate the cylindrical Fourier image of various types of functions and distributions in $\mathbb{R}^m$. However, for certain types of functions or distributions, direct calculation methods are available on top of this approximation approach. A typical example is provided by the case of distributions concentrated on the unit sphere, which are of the form

$$F(\underline{x}) = \delta(r-1)\, f(\underline{\omega}), \quad \underline{x} = r\underline{\omega},\ r = |\underline{x}| \in [0,\infty[\ ,\ \underline{\omega} \in S^{m-1}.$$

The corresponding cylindrical Fourier transform is given by

$$\mathcal{F}_{\text{cyl}}[F](\underline{\xi}) = \mathcal{F}_{\text{cyl}}[f](\underline{\xi}) = \frac{1}{(2\pi)^{m/2}}\int_{S^{m-1}} \exp\left(\underline{\omega}\wedge\underline{\xi}\right) f(\underline{\omega})\, dS(\underline{\omega}).$$

Hereby $\underline{\xi}$ still belongs to the whole space $\mathbb{R}^m$, while the data $f(\underline{\omega})$ are defined on the unit sphere, a codimension one surface of $\mathbb{R}^m$. It is hence expected that the data $f(\underline{\omega})$ are already determined by the cylindrical Fourier image restricted to a suitable codimension one surface as well, typical examples being:

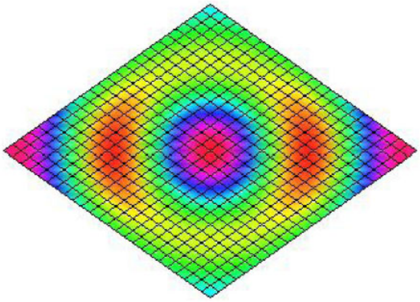

Fig. 3 The real part of the cylindrical Fourier spectrum of the characteristic function of a geodesic triangle on S^2

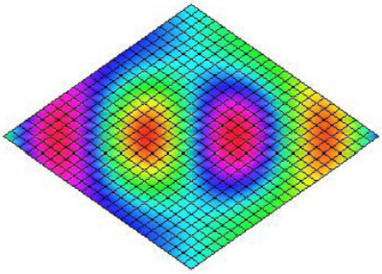

Fig. 4 The e_1e_2-component of the cylindrical Fourier spectrum of the characteristic function of a geodesic triangle on S^2

(i) $\underline{\xi} = \underline{\eta} \in S^{m-1}$, leading to an integral transform from S^{m-1} to S^{m-1},
(ii) $\underline{\xi} = \xi_1 e_1 + \cdots + \xi_{m-1} e_{m-1}$, i.e., $\underline{\xi}$ belongs to the affine subspace given by $\xi_m = 0$.

To evaluate the cylindrical Fourier transform explicitly, it suffices in both cases to express the function $f(\underline{\omega})$ as a series of spherical monogenics and to apply a Funk–Hecke argument on the spherical monogenics. This may lead to correspondences between function spaces on S^{m-1} and isomorphisms between them including inversion methods. The establishment of direct inversion formulae remains an independent and interesting problem for future research.

As an example (see Figs. 3, 4, 5, and 6), we have computed directly the cylindrical Fourier image of the characteristic function of a geodesic triangle on the two sphere S^2 that may be expressed in spherical coordinates by the integral

$$\frac{1}{(2\pi)^{3/2}} \int_0^{\pi/2} \int_0^{\pi/2} \exp(\underline{\omega} \wedge \underline{\xi}) \sin(\theta)\, d\theta\, d\phi$$

with $\underline{\omega} = \sin(\theta)\cos(\phi) e_1 + \sin(\theta)\sin(\phi) e_2 + \cos(\theta) e_3$ and $\underline{\xi} = a e_1 + b e_2$.

6 Conclusion

There is a recent increasing interest in integral transforms and in particular Fourier transforms which take advantage of the algebraic structure inherent in hypercomplex function theories, especially quaternionic and Clifford analysis. In this pa-

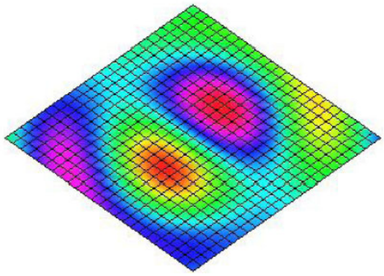

Fig. 5 The e_1e_3-component of the cylindrical Fourier spectrum of the characteristic function of a geodesic triangle on S^2

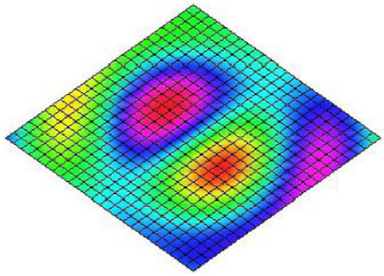

Fig. 6 The e_2e_3-component of the cylindrical Fourier spectrum of the characteristic function of a geodesic triangle on S^2

per we have shown that the recently developed cylindrical Fourier transform of Clifford analysis in Euclidean space of arbitrary dimension is a promising higher-dimensional integral transform with application potential. We have introduced a few methods for the practical computation of the corresponding spectra and illustrated one of these methods by working out an explicit example. For the theory underlying the cylindrical Fourier transform and similar integral transforms in Clifford analysis, we refer the reader to, e.g., [2–4] and in particular to the survey paper [5], and the references contained therein.

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