Two-Dimensional Clifford Windowed Fourier Transform

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Abstract Recently several generalizations to higher dimension of the classical Fourier transform (FT) using Clifford geometric algebra have been introduced, including the two-dimensional (2D) Clifford–Fourier transform (CFT). Based on the 2D CFT, we establish the two-dimensional Clifford windowed Fourier transform (CWFT). Using the spectral representation of the CFT, we derive several important properties such as shift, modulation, a reproducing kernel, isometry, and an orthogonality relation. Finally, we discuss examples of the CWFT and compare the CFT and CWFT.

1 Introduction

One of the basic problems encountered in signal representations using the conventional Fourier transform (FT) is the ineffectiveness of the Fourier kernel to represent and compute location information. One method to overcome such a problem is the windowed Fourier transform (WFT). Recently, some authors [4, 7] have extensively studied the WFT and its properties from a mathematical point of view. In [6, 8] they applied the WFT as a tool of spatial-frequency analysis which is able to characterize the local frequency at any location in a fringe pattern.

On the other hand, Clifford geometric algebra leads to the consequent generalization of real and harmonic analysis to higher dimensions. Clifford algebra accurately treats geometric entities depending on their dimension as scalars, vectors, bivectors (oriented plane area elements), and trivectors (oriented volume elements), etc. Motivated by the above facts, we generalize the WFT in the framework of Clifford geometric algebra.

In the present paper we study the two-dimensional Clifford windowed Fourier transform (CWFT). A complementary motivation for studying this topic comes from

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the understanding that the 2D CWFT is in fact intimately related with Clifford–Gabor filters [1] and quaternionic Gabor filters [2, 3]. This generalization also enables us to establish the two-dimensional Clifford–Gabor filters.

2 Real Clifford Algebra $\mathcal{G}_2$

Let us consider an orthonormal vector basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ of the real 2D Euclidean vector space $\mathbb{R}^2 = \mathbb{R}^{2,0}$. The geometric algebra over $\mathbb{R}^2$ denoted by $\mathcal{G}_2$ then has the graded four-dimensional basis

$$\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_{12}\}, \tag{1}$$

where 1 is the real scalar identity element (grade 0), $\mathbf{e}_1, \mathbf{e}_2 \in \mathbb{R}^2$ are vectors (grade 1), and $\mathbf{e}_{12} = \mathbf{e}_1\mathbf{e}_2 = i_2$ defines the unit oriented pseudoscalar[1] (grade 2), i.e., the highest grade blade element in $\mathcal{G}_2$.

The associative geometric multiplication of the basis vectors obeys the following basic rules:

$$\mathbf{e}_1^2 = \mathbf{e}_2^2 = 1, \qquad \mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_2\mathbf{e}_1. \tag{2}$$

The general elements of a geometric algebra are called multivectors. Every multivector $f \in \mathcal{G}_2$ can be expressed as

$$f = \underbrace{\alpha_0}_{\text{scalar part}} + \underbrace{\alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2}_{\text{vector part}} + \underbrace{\alpha_{12}\mathbf{e}_{12}}_{\text{bivector part}} \qquad \forall \alpha_0, \alpha_1, \alpha_2, \alpha_{12} \in \mathbb{R}. \tag{3}$$

The grade selector is defined as $\langle f \rangle_k$ for the k-vector part of f. We often write $\langle \ldots \rangle = \langle \ldots \rangle_0$. Then (3) can be expressed as[2]

$$f = \langle f \rangle + \langle f \rangle_1 + \langle f \rangle_2. \tag{4}$$

The multivector f is called a parabivector if the vector part of (4) is zero, i.e.,

$$f = \alpha_0 + \alpha_{12}\mathbf{e}_{12}. \tag{5}$$

The reverse $\tilde{f}$ of a multivector $f \in \mathcal{G}_2$ is an anti-automorphism given by

$$\tilde{f} = \langle f \rangle + \langle f \rangle_1 - \langle f \rangle_2, \tag{6}$$

which fulfills $\widetilde{fg} = \tilde{g}\tilde{f}$ for every $f, g \in \mathcal{G}_2$. In particular, $\tilde{i}_2 = -i_2$.

The scalar product of two multivectors $f, \tilde{g}$ is defined as the scalar part of the geometric product $f\tilde{g}$,

$$f * \tilde{g} = \langle f\tilde{g} \rangle = \alpha_0\beta_0 + \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_{12}\beta_{12}, \tag{7}$$

[1] Other names in use are *bivector* or *oriented area element*.

[2] Note that (4) and (6) show grade selection and not component selection.

which leads to the cyclic product symmetry

$$\langle pqr\rangle = \langle qrp\rangle \quad \forall p, q, r \in \mathcal{G}_2. \tag{8}$$

For $f = g$ in (7), we obtain the modulus (or magnitude) $|f|$ of a multivector $f \in \mathcal{G}_2$ defined as

$$|f|^2 = f * \tilde{f} = \alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_{12}^2. \tag{9}$$

It is convenient to introduce an inner product for two multivector-valued functions $f, g : \mathbb{R}^2 \to \mathcal{G}_2$ as follows:

$$(f, g)_{L^2(\mathbb{R}^2;\mathcal{G}_2)} = \int_{\mathbb{R}^2} f(\mathbf{x})\widetilde{g(\mathbf{x})}\, d^2\mathbf{x}. \tag{10}$$

One can check that this inner product satisfies the following rules:

$$\begin{aligned} (f, g+h)_{L^2(\mathbb{R}^2;\mathcal{G}_2)} &= (f, g)_{L^2(\mathbb{R}^2;\mathcal{G}_2)} + (f, h)_{L^2(\mathbb{R}^2;\mathcal{G}_2)}, \\ (f, \lambda g)_{L^2(\mathbb{R}^2;\mathcal{G}_2)} &= (f, g)_{L^2(\mathbb{R}^2;\mathcal{G}_2)}\tilde{\lambda}, \\ (f\lambda, g)_{L^2(\mathbb{R}^2;\mathcal{G}_2)} &= (f, g\tilde{\lambda})_{L^2(\mathbb{R}^2;\mathcal{G}_2)}, \\ (f, g)_{L^2(\mathbb{R}^2;\mathcal{G}_2)} &= \widetilde{(g, f)}_{L^2(\mathbb{R}^2;\mathcal{G}_2)}, \end{aligned} \tag{11}$$

where $f, g \in L^2(\mathbb{R}^2; \mathcal{G}_2)$, and $\lambda \in \mathcal{G}_2$ is a multivector constant. The scalar part of the inner product gives the L^2-norm

$$\|f\|^2_{L^2(\mathbb{R}^2;\mathcal{G}_2)} = \big\langle (f, f)_{L^2(\mathbb{R}^2;\mathcal{G}_2)}\big\rangle. \tag{12}$$

Definition 1 (Clifford module) Let $\mathcal{G}_2$ be the real Clifford algebra of 2D Euclidean space $\mathbb{R}^2$. The Clifford algebra module $L^2(\mathbb{R}^2; \mathcal{G}_2)$ is defined by

$$L^2\big(\mathbb{R}^2; \mathcal{G}_2\big) = \big\{ f : \mathbb{R}^2 \longrightarrow \mathcal{G}_2 \mid \|f\|_{L^2(\mathbb{R}^2;\mathcal{G}_2)} < \infty \big\}. \tag{13}$$

3 Clifford Fourier Transform (CFT)

It is natural to extend the FT to the Clifford algebra $\mathcal{G}_2$. This extension is often called the Clifford–Fourier transform (CFT). For detailed discussions on the properties of the CFT and their proofs, see, e.g., [1, 5]. In the following we briefly review the 2D CFT.

Definition 2 The CFT of $f \in L^2(\mathbb{R}^2; \mathcal{G}_2) \cap L^1(\mathbb{R}^2; \mathcal{G}_2)$ is the function $\mathcal{F}\{f\}: \mathbb{R}^2 \to \mathcal{G}_2$ given by

$$\mathcal{F}\{f\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2} f(\mathbf{x}) e^{-i_2 \boldsymbol{\omega}\cdot\mathbf{x}}\, d^2\mathbf{x}, \tag{14}$$

where we can write $\omega = \omega_1 \mathbf{e}_1 + \omega_2 \mathbf{e}_2$ and $\mathbf{x} = x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2$. Note that

$$d^2\mathbf{x} = \frac{d\mathbf{x}_1 \wedge d\mathbf{x}_2}{i_2} \tag{15}$$

is scalar valued ($d\mathbf{x}_k = dx_k \mathbf{e}_k$, $k = 1, 2$, no summation). Notice that the Clifford–Fourier kernel $e^{-i_2 \boldsymbol{\omega} \cdot \mathbf{x}}$ does not commute with every element of the Clifford algebra $\mathcal{G}_2$. Furthermore, the product has to be performed in a fixed order.

Theorem 1 *Suppose that $f \in L^2(\mathbb{R}^2; \mathcal{G}_2)$ and $\mathcal{F}\{f\} \in L^1(\mathbb{R}^2; \mathcal{G}_2)$. Then the CFT is an invertible transform, and its inverse is calculated by*

$$\mathcal{F}^{-1}\big[\mathcal{F}\{f\}(\boldsymbol{\omega})\big](\mathbf{x}) = f(\mathbf{x}) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \mathcal{F}\{f\}(\boldsymbol{\omega})\, e^{i_2 \boldsymbol{\omega} \cdot \mathbf{x}}\, d^2\boldsymbol{\omega}. \tag{16}$$

4 2D Clifford Windowed Fourier Transform

In [1, 5] the 2D CFT has been introduced. This enables us to establish the 2D CWFT. We will see that several properties of the WFT can be established in the new construction with some modifications. We begin with the definition of the 2D CWFT.

4.1 Definition of the CWFT

Definition 3 A Clifford window function is a function $\phi \in L^2(\mathbb{R}^2; \mathcal{G}_2) \setminus \{0\}$ such that $|\mathbf{x}|^{1/2}\phi(\mathbf{x}) \in L^2(\mathbb{R}^2; \mathcal{G}_2)$.

$$\phi_{\boldsymbol{\omega},\mathbf{b}}(\mathbf{x}) = \frac{e^{i_2 \boldsymbol{\omega} \cdot \mathbf{x}} \phi(\mathbf{x} - \mathbf{b})}{(2\pi)^2} \tag{17}$$

denote the so-called Clifford window daughter functions.

Definition 4 (Clifford windowed Fourier transform) The Clifford windowed Fourier transform (CWFT) $G_\phi f$ of $f \in L^2(\mathbb{R}^2; \mathcal{G}_2)$ is defined by

$$\begin{aligned} f(\mathbf{x}) \longrightarrow G_\phi f(\omega, \mathbf{b}) &= (f, \phi_{\omega,\mathbf{b}})_{L^2(\mathbb{R}^2; \mathcal{G}_2)} \\ &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} f(\mathbf{x}) \left\{ e^{i_2 \boldsymbol{\omega} \cdot \mathbf{x}} \phi(\mathbf{x} - \mathbf{b}) \right\}^{\sim} d^2\mathbf{x} \\ &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} f(\mathbf{x})\, \widetilde{\phi(\mathbf{x} - \mathbf{b})}\, e^{-i_2 \boldsymbol{\omega} \cdot \mathbf{x}}\, d^2\mathbf{x}. \end{aligned} \tag{18}$$

This shows that the CWFT can be regarded as the CFT of the product of a Clifford-valued function f and a shifted and reversed Clifford window function ϕ, or as an inner product (10) of a Clifford-valued function f and the Clifford window daughter functions $\phi_{\boldsymbol{\omega},\mathbf{b}}$.

Taking the Gaussian function as the window function of (17) with fixed $\boldsymbol{\omega} = \boldsymbol{\omega}_0 = \omega_{0,1}\mathbf{e}_1 + \omega_{0,2}\mathbf{e}_2$, we obtain Clifford Gabor filters, i.e.,

$$g_c(\boldsymbol{x}, \sigma_1, \sigma_2) = \frac{1}{(2\pi)^2} e^{i_2 \boldsymbol{\omega}_0 \cdot \mathbf{x}} e^{-[(x_1/\sigma_1)^2 + (x_2/\sigma_2)^2]/2}, \tag{19}$$

where σ_1 and σ_2 are standard deviations of the Gaussian functions, and the translation parameters are $b_1 = b_2 = 0$.

In terms of the $\mathcal{G}_2$ Clifford–Fourier transform, (19) can be expressed as

$$\mathcal{F}\{g_c\}(\boldsymbol{\omega}) = \frac{1}{\pi \sigma_1 \sigma_2} e^{-\frac{1}{2}[(\sigma_1^2(\omega_1 - \omega_{0,1})^2 + \sigma_2^2(\omega_2 - \omega_{0,2})^2]}. \tag{20}$$

From (19) and (20) we see that Clifford–Gabor filters are well localized in the spatial and Clifford–Fourier domains.

The energy density is defined as the square modulus of the CWFT (18) given by

$$\left|G_\phi f(\boldsymbol{\omega}, \mathbf{b})\right|^2 = \frac{1}{(2\pi)^4} \left| \int_{\mathbb{R}^2} f(\mathbf{x}) \widetilde{\phi(\mathbf{x} - \mathbf{b})} e^{-i_2 \boldsymbol{\omega} \cdot \mathbf{x}} \, d^2\mathbf{x} \right|^2. \tag{21}$$

Equation (21) is often called a spectrogram which measures the energy of a Clifford-valued function f in the position–frequency neighborhood of $(\mathbf{b}, \boldsymbol{\omega})$.

In particular, when the Gaussian function (19) is chosen as the Clifford window function, the CWFT (18) is called the Clifford–Gabor transform.

4.2 Properties of the CWFT

We will discuss the properties of the CWFT. We find that many of the properties of the WFT are still valid for the CWFT, however, with certain modifications.

Theorem 2 (Left linearity) *Let $\phi \in L^2(\mathbb{R}^2; \mathcal{G}_2)$ be a Clifford window function. The CWFT of $f, g \in L^2(\mathbb{R}^2; \mathcal{G}_2)$ is a left linear operator,[3] which means that*

$$\left[G_\phi(\lambda f + \mu g)\right](\boldsymbol{\omega}, \mathbf{b}) = \lambda G_\phi f(\boldsymbol{\omega}, \mathbf{b}) + \mu G_\phi g(\boldsymbol{\omega}, \mathbf{b}) \tag{22}$$

with Clifford constants $\lambda, \mu \in \mathcal{G}_2$.

Proof Using the definition of the CWFT, the proof is obvious. □

Remark 1 Since the geometric multiplication is noncommutative, the right-linearity property of the CWFT does not hold in general.

[3]The CWFT of f is a *linear* operator for real constants $\mu, \lambda \in \mathbb{R}$.

Theorem 3 (Reversion) *Let $f \in L^2(\mathbb{R}^2; \mathcal{G}_2^+)$ be a parabivector-valued function. For a parabivector-valued window function ϕ, we have*

$$G_{\tilde{\phi}} \tilde{f}(\boldsymbol{\omega}, \mathbf{b}) = \left\{G_\phi f(-\boldsymbol{\omega}, \mathbf{b})\right\}^{\sim}. \tag{23}$$

Proof Application of Definition 4 to the left-hand side of (23) gives

$$\begin{aligned} G_{\tilde{\phi}} \tilde{f}(\boldsymbol{\omega}, \mathbf{b}) &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \widetilde{f(\mathbf{x})} \phi(\mathbf{x}-\mathbf{b})\, e^{-i_2 \boldsymbol{\omega}\cdot\mathbf{x}}\, d^2\mathbf{x} \\ &= \frac{1}{(2\pi)^2} \left\{ \int_{\mathbb{R}^2} e^{i_2 \boldsymbol{\omega}\cdot\mathbf{x}} \widetilde{\phi(\mathbf{x}-\mathbf{b})}\, f(\mathbf{x})\, d^2\mathbf{x} \right\}^{\sim} \\ &= \frac{1}{(2\pi)^2} \left\{ \int_{\mathbb{R}^2} f(\mathbf{x})\, \widetilde{\phi(\mathbf{x}-\mathbf{b})}\, e^{i_2 \boldsymbol{\omega}\cdot\mathbf{x}}\, d^2\mathbf{x} \right\}^{\sim}. \end{aligned} \tag{24}$$

This finishes the proof of the theorem. □

Theorem 4 (Switching) *If $|\mathbf{x}|^{1/2} f(\mathbf{x}) \in L^2(\mathbb{R}^2; \mathcal{G}_2)$ and $|\mathbf{x}|^{1/2} \phi(\mathbf{x}) \in L^2(\mathbb{R}^2; \mathcal{G}_2)$ are parabivector-valued functions, then we obtain*

$$G_\phi f(\boldsymbol{\omega}, \mathbf{b}) = e^{-i_2 \boldsymbol{\omega}\cdot\mathbf{b}} \left\{G_f \phi(-\boldsymbol{\omega}, -\mathbf{b})\right\}^{\sim}. \tag{25}$$

Proof We have, by the CWFT definition,

$$\begin{aligned} G_\phi f(\boldsymbol{\omega}, \mathbf{b}) &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} f(\mathbf{x}) \widetilde{\phi(\mathbf{x}-\mathbf{b})}\, e^{-i_2 \boldsymbol{\omega}\cdot\mathbf{x}}\, d^2\mathbf{x} \\ &= \frac{1}{(2\pi)^2} \left\{ \int_{\mathbb{R}^2} \phi(\mathbf{x}-\mathbf{b})\, \widetilde{f(\mathbf{x})}\, e^{i_2 \boldsymbol{\omega}\cdot\mathbf{x}}\, d^2\mathbf{x} \right\}^{\sim}. \end{aligned} \tag{26}$$

The substitution $\mathbf{y} = \mathbf{x} - \mathbf{b}$ into the above expression gives

$$\begin{aligned} G_\phi f(\boldsymbol{\omega}, \mathbf{b}) &= \frac{1}{(2\pi)^2} \left\{ \int_{\mathbb{R}^2} \phi(\mathbf{y})\, \widetilde{f(\mathbf{y}+\mathbf{b})}\, e^{i_2 \boldsymbol{\omega}\cdot(\mathbf{y}+\mathbf{b})}\, d^2\mathbf{y} \right\}^{\sim} \\ &= \frac{1}{(2\pi)^2} e^{-i_2 \boldsymbol{\omega}\cdot\mathbf{b}} \left\{ \int_{\mathbb{R}^2} \phi(\mathbf{y})\, \widetilde{f(\mathbf{y}+\mathbf{b})}\, e^{i_2 \boldsymbol{\omega}\cdot\mathbf{y}}\, d^2\mathbf{y} \right\}^{\sim} \\ &= \frac{1}{(2\pi)^2} e^{-i_2 \boldsymbol{\omega}\cdot\mathbf{b}} \left\{ \int_{\mathbb{R}^2} \phi(\mathbf{y})\, \widetilde{f\big(\mathbf{y}-(-\mathbf{b})\big)}\, e^{-i_2 (-\boldsymbol{\omega})\cdot\mathbf{y}}\, d^2\mathbf{y} \right\}^{\sim}, \end{aligned} \tag{27}$$

which proves the theorem. □

Theorem 5 (Parity) *Let $\phi \in L^2(\mathbb{R}^2; \mathcal{G}_2)$ be a Clifford window function. If P is the parity operator defined as $P\phi(\mathbf{x}) = \phi(-\mathbf{x})$, then we have*

$$G_{P\phi}\{Pf\}(\boldsymbol{\omega}, \mathbf{b}) = G_\phi f(-\boldsymbol{\omega}, -\mathbf{b}). \tag{28}$$

Proof Direct calculations give, for every $f \in L^2(\mathbb{R}^2; \mathcal{G}_2)$,

$$\begin{aligned} G_{P\phi}\{Pf\}(\boldsymbol{\omega}, \mathbf{b}) &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} f(-\mathbf{x}) \{\phi(-\mathbf{x} + \mathbf{b})\}^{\sim} e^{-i_2(-\boldsymbol{\omega})\cdot(-\mathbf{x})} \, d^2\mathbf{x} \\ &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} f(-\mathbf{x}) \{\phi(-\mathbf{x} - (-\mathbf{b}))\}^{\sim} e^{-i_2(-\boldsymbol{\omega})\cdot(-\mathbf{x})} \, d^2\mathbf{x} \\ &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} f(\mathbf{x}) \{\phi(\mathbf{x} - (-\mathbf{b}))\}^{\sim} e^{-i_2(-\boldsymbol{\omega})\cdot\mathbf{x}} \, d^2\mathbf{x}, \end{aligned} \tag{29}$$

which completes the proof. □

Theorem 6 (Shift in space domain, delay) *Let ϕ be a Clifford window function. Introducing the translation operator $T_{\mathbf{x}_0} f(\mathbf{x}) = f(\mathbf{x} - \mathbf{x}_0)$, we obtain*

$$G_\phi\{T_{\mathbf{x}_0} f\}(\boldsymbol{\omega}, \mathbf{b}) = \left(G_\phi f(\boldsymbol{\omega}, \mathbf{b} - \mathbf{x}_0)\right) e^{-i_2 \boldsymbol{\omega}\cdot\mathbf{x}_0}. \tag{30}$$

Proof We have by using (18)

$$G_\phi\{T_{\mathbf{x}_0} f\}(\boldsymbol{\omega}, \mathbf{b}) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} f(\mathbf{x} - \mathbf{x}_0) \widetilde{\phi(\mathbf{x} - \mathbf{b})} \, e^{-i_2 \boldsymbol{\omega}\cdot\mathbf{x}} \, d^2\mathbf{x}. \tag{31}$$

We substitute $\mathbf{t} = \mathbf{x} - \mathbf{x}_0$ into the above expression and get, with $d^2\mathbf{x} = d^2\mathbf{t}$,

$$\begin{aligned} G_\phi\{T_{\mathbf{x}_0} f\}(\boldsymbol{\omega}, \mathbf{b}) &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} f(\mathbf{t}) \{\phi(\mathbf{t} - (\mathbf{b} - \mathbf{x}_0))\}^{\sim} e^{-i_2 \boldsymbol{\omega}\cdot(\mathbf{t} + \mathbf{x}_0)} \, d^2\mathbf{t} \\ &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \left[f(\mathbf{t}) \{\phi(\mathbf{t} - (\mathbf{b} - \mathbf{x}_0))\}^{\sim} e^{-i_2 \boldsymbol{\omega}\cdot\mathbf{t}} \right] d^2\mathbf{t} \, e^{-i_2 \boldsymbol{\omega}\cdot\mathbf{x}_0}. \end{aligned} \tag{32}$$

This ends the proof of (30). □

Theorem 7 (Shift in frequency domain, modulation) *Let ϕ be a parabivector-valued Clifford window function. If $\boldsymbol{\omega}_0 \in \mathbb{R}^2$ and $f_0(\mathbf{x}) = f(\mathbf{x}) e^{i_2 \boldsymbol{\omega}_0\cdot\mathbf{x}}$, then*

$$G_\phi f_0(\boldsymbol{\omega}, \mathbf{b}) = G_\phi f(\boldsymbol{\omega} - \boldsymbol{\omega}_0, \mathbf{b}). \tag{33}$$

Proof Using Definition 4 and simplifying it, we get

$$\begin{aligned} G_\phi f_0(\boldsymbol{\omega}, \mathbf{b}) &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} f(\mathbf{x}) e^{i_2 \boldsymbol{\omega}_0\cdot\mathbf{x}} \, \widetilde{\phi(\mathbf{x} - \mathbf{b})} \, e^{-i_2 \boldsymbol{\omega}\cdot\mathbf{x}} \, d^2\mathbf{x} \\ &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} f(\mathbf{x}) \widetilde{\phi(\mathbf{x} - \mathbf{b})} \, e^{-i_2 (\boldsymbol{\omega} - \boldsymbol{\omega}_0)\cdot\mathbf{x}} \, d^2\mathbf{x}, \end{aligned} \tag{34}$$

which proves the theorem. □

Theorem 8 (Reconstruction formula) *Let ϕ be a Clifford window function. Then every 2D Clifford signal $f \in L^2(\mathbb{R}^2; \mathcal{G}_2)$ can be fully reconstructed by*

$$f(\mathbf{x}) = (2\pi)^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} G_\phi f(\boldsymbol{\omega}, \mathbf{b}) \phi_{\boldsymbol{\omega},\mathbf{b}}(\mathbf{x}) \big(\tilde{\phi}, \tilde{\phi}\big)^{-1}_{L^2(\mathbb{R}^2; \mathcal{G}_2)} d^2\mathbf{b}\, d^2\boldsymbol{\omega}. \tag{35}$$

Proof It follows from the CWFT defined by (18) that

$$G_\phi f(\boldsymbol{\omega}, \mathbf{b}) = \frac{1}{(2\pi)^2} \mathcal{F}\big\{ f(\mathbf{x}) \widetilde{\phi(\mathbf{x} - \mathbf{b})} \big\}(\boldsymbol{\omega}). \tag{36}$$

Taking the inverse CFT of both sides of (36), we obtain

$$\begin{aligned} f(\mathbf{x}) \widetilde{\phi(\mathbf{x} - \mathbf{b})} &= (2\pi)^2 \mathcal{F}^{-1}\big\{ G_\phi f(\boldsymbol{\omega}, \mathbf{b}) \big\}(\mathbf{x}) \\ &= \frac{(2\pi)^2}{(2\pi)^2} \int_{\mathbb{R}^2} G_\phi f(\boldsymbol{\omega}, \mathbf{b})\, e^{i_2 \boldsymbol{\omega}\cdot\mathbf{x}}\, d^2\boldsymbol{\omega}. \end{aligned} \tag{37}$$

Multiplying both sides of (37) by $\phi(\mathbf{x} - \mathbf{b})$ and then integrating with respect to $d^2\mathbf{b}$, we get

$$f(\mathbf{x}) \int_{\mathbb{R}^2} \widetilde{\phi(\mathbf{x} - \mathbf{b})} \phi(\mathbf{x} - \mathbf{b})\, d^2\mathbf{b} = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} G_\phi f(\boldsymbol{\omega}, \mathbf{b})\, e^{i_2 \boldsymbol{\omega}\cdot\mathbf{x}} \phi(\mathbf{x} - \mathbf{b})\, d^2\boldsymbol{\omega}\, d^2\mathbf{b} \tag{38}$$

or, equivalently,

$$f(\mathbf{x}) \big(\tilde{\phi}, \tilde{\phi}\big)_{L^2(\mathbb{R}^2; \mathcal{G}_2)} = (2\pi)^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} G_\phi f(\boldsymbol{\omega}, \mathbf{b})\, \phi_{\boldsymbol{\omega},\mathbf{b}}(\mathbf{x})\, d^2\boldsymbol{\omega}\, d^2\mathbf{b}, \tag{39}$$

which gives (35). □

It is worth noting here that if the Clifford window function is a parabivector-valued function, then the reconstruction formula (35) can be written in the form

$$f(\mathbf{x}) = \frac{(2\pi)^2}{\|\phi\|^2_{L^2(\mathbb{R}^2; \mathcal{G}_2)}} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} G_\phi f(\boldsymbol{\omega}, \mathbf{b}) \phi_{\boldsymbol{\omega},\mathbf{b}}(\mathbf{x})\, d^2\mathbf{b}\, d^2\boldsymbol{\omega}. \tag{40}$$

Theorem 9 (Orthogonality relation) *Assume that the Clifford window function ϕ is a parabivector-valued function. If two Clifford functions $f, g \in L^2(\mathbb{R}^2; \mathcal{G}_2)$, then we have*

$$\begin{aligned} &\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (f, \phi_{\boldsymbol{\omega},\mathbf{b}})_{L^2(\mathbb{R}^2; \mathcal{G}_2)} \widetilde{(g, \phi_{\boldsymbol{\omega},\mathbf{b}})}_{L^2(\mathbb{R}^2; \mathcal{G}_2)}\, d^2\boldsymbol{\omega}\, d^2\mathbf{b} \\ &\quad = \frac{\|\phi\|^2_{L^2(\mathbb{R}^2; \mathcal{G}_2)}}{(2\pi)^2} (f, g)_{L^2(\mathbb{R}^2; \mathcal{G}_2)}. \end{aligned} \tag{41}$$

Proof By inserting (18) into the left side of (41), we obtain

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}(f,\phi_{\boldsymbol{\omega},\mathbf{b}})_{L^2(\mathbb{R}^2;\mathcal{G}_2)}\widetilde{(g,\phi_{\boldsymbol{\omega},\mathbf{b}})}_{L^2(\mathbb{R}^2;\mathcal{G}_2)}\,d^2\boldsymbol{\omega}\,d^2\mathbf{b}$$

$$=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}(f,\phi_{\boldsymbol{\omega},\mathbf{b}})_{L^2(\mathbb{R}^2;\mathcal{G}_2)}\left(\int_{\mathbb{R}^2}\frac{1}{(2\pi)^2}\,e^{i_2\boldsymbol{\omega}\cdot\mathbf{x}}\phi(\mathbf{x}-\mathbf{b})\widetilde{g(\mathbf{x})}\,d^2\mathbf{x}\right)d^2\boldsymbol{\omega}\,d^2\mathbf{b}$$

$$=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{1}{(2\pi)^4}\,f(\mathbf{x}')\,\widetilde{\phi(\mathbf{x}'-\mathbf{b})}\,e^{i_2\boldsymbol{\omega}\cdot(\mathbf{x}-\mathbf{x}')}d^2\boldsymbol{\omega}\,d^2\mathbf{x}'\right)$$

$$\times\,\phi(\mathbf{x}-\mathbf{b})\widetilde{g(\mathbf{x})}\,d^2\mathbf{x}\,d^2\mathbf{b}$$

$$=\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}f(\mathbf{x}')\widetilde{\phi(\mathbf{x}'-\mathbf{b})}\delta(\mathbf{x}-\mathbf{x}')\phi(\mathbf{x}-\mathbf{b})\,d^2\mathbf{x}'\right)\widetilde{g(\mathbf{x})}\,d^2\mathbf{b}\,d^2\mathbf{x}$$

$$=\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}f(\mathbf{x})\int_{\mathbb{R}^2}\underbrace{\widetilde{\phi(\mathbf{x}-\mathbf{b})}\phi(\mathbf{x}-\mathbf{b})}_{\phi\text{ parabiv. funct.}}\,d^2\mathbf{b}\,\widetilde{g(\mathbf{x})}\,d^2\mathbf{x}$$

$$=\frac{1}{(2\pi)^2}\|\phi\|^2_{L^2(\mathbb{R}^2;\mathcal{G}_2)}\int_{\mathbb{R}^2}f(\mathbf{x})\widetilde{g(\mathbf{x})}\,d^2\mathbf{x},\tag{42}$$

which completes the proof of (41). □

Theorem 10 (Reproducing kernel) *For a parabivector-valued Clifford window function* $|\mathbf{x}|^{1/2}\phi\in L^2(\mathbb{R}^2;\mathcal{G}_2)$, *if*

$$\mathbb{K}_\phi\big(\boldsymbol{\omega},\mathbf{b};\boldsymbol{\omega}',\mathbf{b}'\big)=\frac{(2\pi)^2}{\|\phi\|^2_{L^2(\mathbb{R}^2;\mathcal{G}_2)}}(\phi_{\boldsymbol{\omega},\mathbf{b}},\phi_{\boldsymbol{\omega}',\mathbf{b}'})_{L^2(\mathbb{R}^2;\mathcal{G}_2)},\tag{43}$$

then $\mathbb{K}_\phi(\boldsymbol{\omega},\mathbf{b};\boldsymbol{\omega}',\mathbf{b}')$ *is a reproducing kernel, i.e.,*

$$G_\phi f\big(\boldsymbol{\omega}',\mathbf{b}'\big)=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}G_\phi f(\boldsymbol{\omega},\mathbf{b})\mathbb{K}_\phi\big(\boldsymbol{\omega},\mathbf{b};\boldsymbol{\omega}',\mathbf{b}'\big)\,d^2\boldsymbol{\omega}\,d^2\mathbf{b}.\tag{44}$$

Proof By inserting the inverse CWFT (40) into the definition of the CWFT (18) we easily obtain

$$G_\phi f\big(\boldsymbol{\omega}',\mathbf{b}'\big)$$

$$=\int_{\mathbb{R}^2}f(\mathbf{x})\widetilde{\phi_{\boldsymbol{\omega}',\mathbf{b}'}(\mathbf{x})}\,d^2\mathbf{x}$$

$$=\int_{\mathbb{R}^2}\left(\frac{(2\pi)^2}{\|\phi\|^2_{L^2(\mathbb{R}^2;\mathcal{G}_2)}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}G_\phi f(\boldsymbol{\omega},\mathbf{b})\,\phi_{\boldsymbol{\omega},\mathbf{b}}(\mathbf{x})\,d^2\mathbf{b}\,d^2\boldsymbol{\omega}\right)\widetilde{\phi_{\boldsymbol{\omega}',\mathbf{b}'}(\mathbf{x})}\,d^2\mathbf{x}$$

$$= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} G_\phi f(\boldsymbol{\omega},\mathbf{b}) \frac{(2\pi)^2}{\|\phi\|^2_{L^2(\mathbb{R}^2;\mathcal{G}_2)}} \left(\int_{\mathbb{R}^2} \phi_{\boldsymbol{\omega},\mathbf{b}}(\mathbf{x})\widetilde{\phi_{\boldsymbol{\omega}',\mathbf{b}'}(\mathbf{x})}\, d^2\mathbf{x}\right) d^2\mathbf{b}\, d^2\boldsymbol{\omega}$$

$$= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} G_\phi f(\boldsymbol{\omega},\mathbf{b}) \mathbb{K}_\phi\big(\boldsymbol{\omega},\mathbf{b};\boldsymbol{\omega}',\mathbf{b}'\big)\, d^2\mathbf{b}\, d^2\boldsymbol{\omega}, \tag{45}$$

which finishes the proof. □

Remark 2 Formulas (40), (41), and (43) also hold if the Clifford window function is a vector-valued function, i.e., $\phi(\mathbf{x}) = \phi_1(\mathbf{x})\mathbf{e}_1 + \phi_2(\mathbf{x})\mathbf{e}_2$.

The above properties of the CWFT are summarized in Table 1.

Table 1 Properties of the CWFT of $f, g \in L^2(\mathbb{R}^2;\mathcal{G}_2)$, $L^2 = L^2(\mathbb{R}^2;\mathcal{G}_2)$, where $\lambda,\mu \in \mathcal{G}_2$ are constants, $\omega_0 = \omega_{0,1}\mathbf{e}_1 + \omega_{0,2}\mathbf{e}_2 \in \mathbb{R}^2$, and $\mathbf{x}_0 = x_0\mathbf{e}_1 + y_0\mathbf{e}_2 \in \mathbb{R}^2$

Property	Clifford-valued function	2D CWFT
Left linearity	$\lambda f(\boldsymbol{x}) + \mu g(\boldsymbol{x})$	$\lambda G_\phi f(\boldsymbol{\omega},\mathbf{b}) + \mu G_\phi g(\boldsymbol{\omega},\mathbf{b})$
Delay	$f(\mathbf{x}-\mathbf{x}_0)$	$(G_\phi f(\boldsymbol{\omega},\mathbf{b}-\mathbf{x}_0))\, e^{-i_2\boldsymbol{\omega}\cdot\mathbf{x}_0}$
Modulation	$f(\mathbf{x})e^{i_2\boldsymbol{\omega}_0\cdot\mathbf{x}}$	$G_\phi f(\boldsymbol{\omega}-\boldsymbol{\omega}_0,\mathbf{b})$ if ϕ is a parabivector-valued function
Formulas		
Reversion	$G_{\tilde{\phi}}\tilde{f}(\boldsymbol{\omega},\mathbf{b}) =$	$\{G_\phi f(-\boldsymbol{\omega},\mathbf{b})\}^\sim$ if f and ϕ are parabivector-valued functions
Switching	$G_\phi f(\boldsymbol{\omega},\mathbf{b}) =$	$e^{-i_2\boldsymbol{\omega}\cdot\mathbf{b}}\{G_f\phi(-\boldsymbol{\omega},-\mathbf{b})\}^\sim$ if f and ϕ are parabivector-valued functions
Parity	$G_{P\phi}\{Pf\}(\boldsymbol{\omega},\mathbf{b}) =$	$G_\phi f(-\boldsymbol{\omega},-\mathbf{b})$
Orthogonality	$\frac{1}{(2\pi)^2}\|\phi\|^2_{L^2}(f,g)_{L^2} =$	$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}(f,\phi_{\boldsymbol{\omega},\mathbf{b}})_{L^2(\mathbb{R}^2;\mathcal{G}_2)} \times \widetilde{(g,\phi_{\boldsymbol{\omega},\mathbf{b}})}_{L^2(\mathbb{R}^2;\mathcal{G}_2)} d^2\boldsymbol{\omega}\, d^2\mathbf{b}$ if ϕ is a parabivector-valued function
Reconstruction	$f(\mathbf{x}) =$	$(2\pi)^2\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} G_\phi f(\boldsymbol{\omega},\mathbf{b})\phi_{\boldsymbol{\omega},\mathbf{b}}(\mathbf{x}) \times (\tilde{\phi},\tilde{\phi})^{-1}_{L^2(\mathbb{R}^2;\mathcal{G}_2)} d^2\mathbf{b}\, d^2\boldsymbol{\omega}$, $\frac{(2\pi)^2}{\|\phi\|^2_{L^2(\mathbb{R}^2;\mathcal{G}_2)}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} G_\phi f(\boldsymbol{\omega},\mathbf{b}) \times \phi_{\boldsymbol{\omega},\mathbf{b}}(\mathbf{x})\, d^2\mathbf{b} d^2\boldsymbol{\omega}$, if ϕ is a parabivector-valued function
Reproducing kernel	$G_\phi f(\boldsymbol{\omega}',\mathbf{b}') =$	$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} G_\phi f(\boldsymbol{\omega},\mathbf{b})\mathbb{K}_\phi(\boldsymbol{\omega},\mathbf{b};\boldsymbol{\omega}',\mathbf{b}')\, d^2\boldsymbol{\omega}\, d^2\mathbf{b}$, $\mathbb{K}_\phi(\boldsymbol{\omega},\mathbf{b};\boldsymbol{\omega}',\mathbf{b}') = \frac{(2\pi)^2}{\|\phi\|^2_{L^2(\mathbb{R}^2;\mathcal{G}_2)}}(\phi_{\boldsymbol{\omega},\mathbf{b}},\phi_{\boldsymbol{\omega}',\mathbf{b}'})_{L^2(\mathbb{R}^2;\mathcal{G}_2)}$, if ϕ is a parabivector-valued function

4.3 Examples of the CWFT

For illustrative purposes, we shall discuss examples of the CWFT. We then compute their energy densities.

Example 1 Consider the Clifford–Gabor filters (see Fig. 1) defined by ($\sigma_1 = \sigma_2 = 1/\sqrt{2}$)

$$f(\mathbf{x}) = \frac{1}{(2\pi)^2} e^{-\mathbf{x}^2 + i_2 \boldsymbol{\omega}_0 \cdot \mathbf{x}}. \tag{46}$$

Obtain the CWFT of f with respect to the Gaussian window function $\phi(\mathbf{x}) = e^{-\mathbf{x}^2}$.

By the definition of the CWFT (18), we have

$$G_\phi f(\boldsymbol{\omega}, \mathbf{b}) = \frac{1}{(2\pi)^4} \int_{\mathbb{R}^2} e^{-\mathbf{x}^2 + i_2 \boldsymbol{\omega}_0 \cdot \mathbf{x}} e^{-(\mathbf{x}-\mathbf{b})^2} e^{-i_2 \boldsymbol{\omega} \cdot \mathbf{x}} \, d^2\mathbf{x}. \tag{47}$$

Substituting $\mathbf{x} = \mathbf{y} + \mathbf{b}/2$, we can rewrite (47) as

$$\begin{aligned}
G_\phi f(\boldsymbol{\omega}, \mathbf{b}) &= \frac{1}{(2\pi)^4} \int_{\mathbb{R}^2} e^{-(\mathbf{y}+\mathbf{b}/2)^2 + i_2 \boldsymbol{\omega}_0 \cdot (\mathbf{y}+\mathbf{b}/2)} e^{-(\mathbf{y}-\mathbf{b}/2)^2} e^{-i_2 \boldsymbol{\omega} \cdot (\mathbf{y}+\mathbf{b}/2)} \, d^2\mathbf{y} \\
&= \frac{e^{-\mathbf{b}^2/2}}{(2\pi)^4} \int_{\mathbb{R}^2} e^{-2\mathbf{y}^2} e^{-i_2 \boldsymbol{\omega} \cdot \mathbf{y}} e^{i_2 \boldsymbol{\omega}_0 \cdot \mathbf{y}} \, d^2\mathbf{y} \, e^{-i_2 (\boldsymbol{\omega}-\boldsymbol{\omega}_0) \cdot \mathbf{b}/2} \\
&= \frac{e^{-\mathbf{b}^2/2}}{(2\pi)^4} \int_{\mathbb{R}^2} e^{-2\mathbf{y}^2} e^{-i_2 (\boldsymbol{\omega}-\boldsymbol{\omega}_0) \cdot \mathbf{y}} \, d^2\mathbf{y} \, e^{-i_2 (\boldsymbol{\omega}-\boldsymbol{\omega}_0) \cdot \mathbf{b}/2} \\
&= \frac{e^{-\mathbf{b}^2/2}}{(2\pi)^4} \frac{\pi}{2} e^{-(\boldsymbol{\omega}-\boldsymbol{\omega}_0)^2/8} e^{-i_2 (\boldsymbol{\omega}-\boldsymbol{\omega}_0) \cdot \mathbf{b}/2} \\
&= \frac{e^{-\mathbf{b}^2/2}}{32\pi^3} e^{-(\boldsymbol{\omega}-\boldsymbol{\omega}_0)^2/8} e^{-i_2 (\boldsymbol{\omega}-\boldsymbol{\omega}_0) \cdot \mathbf{b}/2}.
\end{aligned} \tag{48}$$

The energy density is given by

$$\left| G_\phi f(\boldsymbol{\omega}, \mathbf{b}) \right|^2 = \frac{e^{-\mathbf{b}^2}}{(32\pi^3)^2} e^{-(\boldsymbol{\omega}-\boldsymbol{\omega}_0)^2/4}. \tag{49}$$

Example 2 Consider the first-order two-dimensional B-spline window function defined by

$$\phi(\mathbf{x}) = \begin{cases} 1 & \text{if } 0 \le x_1 \le 1 \text{ and } 0 \le x_2 \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{50}$$

Obtain the CWFT of the function defined as follows:

$$f(\mathbf{x}) = \begin{cases} \mathbf{x} & \text{if } 0 \le x_1 \le 1 \text{ and } 0 \le x_2 \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{51}$$

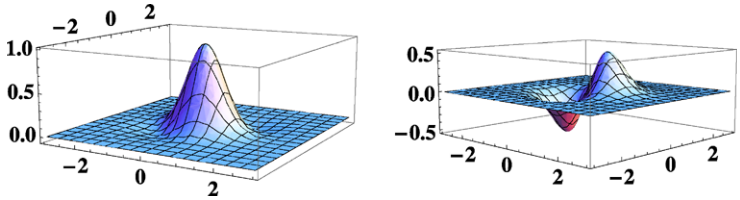

Fig. 1 The scalar part (*left*) and bivector part (*right*) of Clifford–Gabor filter for the parameters $\omega_{0,1}=\omega_{0,2}=1, b_1=b_2=0, \sigma_1=\sigma_2=1/\sqrt{2}$ in the spatial domain using Mathematica 6.0

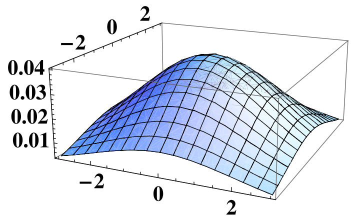

Fig. 2 Plot of the CWFT of Clifford–Gabor filter of Example 1 using Mathematica 6.0. Note that it is scalar valued for the parameters $b_1=b_2=0$

Applying Definition 4 and simplifying it, we obtain

$$
\begin{aligned}
&G_\phi f(\boldsymbol{\omega},\mathbf{b})\\
&\quad=\frac{1}{(2\pi)^2}\int_{b_1}^{1+b_1}\int_{b_2}^{1+b_2}\mathbf{x}\,e^{-i_2\boldsymbol{\omega}\cdot\mathbf{x}}\,dx_1\,dx_2\\
&\quad=\frac{1}{(2\pi)^2}\int_{b_1}^{1+b_1}\int_{b_2}^{1+b_2}(x_1\mathbf{e}_1+x_2\mathbf{e}_2)\left(e^{-i_2\omega_1x_1}e^{-i_2\omega_2x_2}\right)dx_1\,dx_2\\
&\quad=\frac{1}{(2\pi)^2}\,\mathbf{e}_1\int_{b_1}^{1+b_1}x_1e^{-i_2\omega_1x_1}\,dx_1\int_{b_2}^{1+b_2}e^{-i_2\omega_2x_2}\,dx_2\\
&\qquad+\frac{1}{(2\pi)^2}\,\mathbf{e}_2\int_{b_1}^{1+b_1}e^{-i_2\omega_1x_1}dx_1\int_{b_2}^{1+b_2}x_2e^{-i_2\omega_2x_2}\,dx_2\\
&\quad=\big\{\mathbf{e}_2\omega_2\big[(1+i_2\omega_1b_1)\big(e^{-i_2\omega_1}-1\big)+i_2\omega_1e^{-i_2\omega_1}\big]\big(e^{-i_2\omega_2}-1\big)\\
&\qquad-\mathbf{e}_1\omega_1\big[(1+i_2\omega_2b_2)\big(e^{-i_2\omega_2}-1\big)+i_2\omega_2e^{-i_2\omega_2}\big]\big(e^{-i_2\omega_1}-1\big)\big\}\\
&\qquad\times\frac{e^{-i_2\boldsymbol{\omega}\cdot\mathbf{b}}}{(2\pi\omega_1\omega_2)^2}
\end{aligned}
\tag{52}
$$

with

$$
\mathbf{b}=b_1\mathbf{e}_1+b_2\mathbf{e}_2.
$$

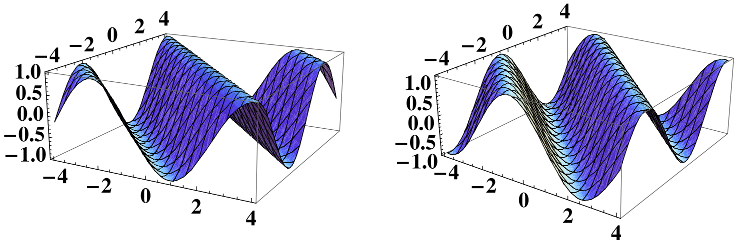

Fig. 3 Representation of the CFT basis for $\omega_1 = \omega_2 = 1$ with scalar part (*left*) and bivector part (*right*) using Mathematica 6.0

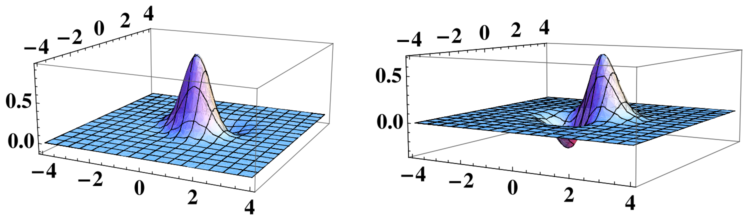

Fig. 4 Representation of the CWFT basis of a Gaussian window function for the parameters $\omega_{0,1} = \omega_{0,2} = 1, b_1 = b_2 = 0.2$ with scalar part (*left*) and bivector part (*right*) using Mathematica 6.0

5 Comparison of CFT and CWFT

Since the Clifford–Fourier kernel $e^{-i_2 \boldsymbol{\omega} \cdot \mathbf{x}}$ is a global function, the CFT basis has an infinite spatial extension as shown in Fig. 3. In contrast, the CWFT basis $\phi(\mathbf{x} - \mathbf{b})\, e^{-i_2 \boldsymbol{\omega} \cdot \mathbf{x}}$ has a limited spatial extension due to the local Clifford window function $\phi(\mathbf{x} - \mathbf{b})$ (see Fig. 4). This means that the CFT analysis cannot provide information about the signal with respect to position and frequency, so that we need the CWFT to fully describe the characteristics of the signal simultaneously in both spatial and frequency domains.

6 Conclusion

Using the basic concepts of Clifford geometric algebra and the CFT, we introduced the CWFT. Important properties of the CWFT were demonstrated. This generalization enables us to work with 2D Clifford–Gabor filters, which can extend the applications of the 2D complex Gabor filters.

Because the CWFT represents a signal in a joint space–frequency domain, it can be applied in many fields of science and engineering, such as image analysis and image compression, object and pattern recognition, computer vision, optics and filter banks.

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