Parameterization of 3D Conformal Transformations in Conformal Geometric Algebra

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Abstract Conformal geometric algebra is a powerful mathematical language for describing and manipulating geometric configurations and their conformal transformations. By providing a 5D algebraic representation of 3D geometric configurations, conformal geometric algebra proves to be very helpful in pose estimation, motion design, and neuron-based machine learning (Bayro-Corrochano et al., J. Math. Imaging Vis. 24(1):55–81, 2006; Dorst et al., Geometric Algebra for Computer Science, Morgan Kaufmann, San Mateo, 2007; Hildenbrand, Comput. Graph. 29(5):795–803, 2005; Lasenby, Computer Algebra and Geometric Algebra with Applications, LNCS, vol. 3519, pp. 298–328, Springer, Berlin, 2005; Li et al., Geometric Computing with Clifford Algebras, pp. 27–60, Springer, Heidelberg, 2001; Mourrain and Stolfi, Invariant Methods in Discrete and Computational Geometry, pp. 107–139, Reidel, Dordrecht, 1995; Rosenhahn and Sommer, J. Math. Imaging Vis. 22:27–70, 2005; Sommer et al., Computer Algebra and Geometric Algebra with Applications, pp. 278–297, Springer, Berlin, 2005). In this chapter, we present some theoretical results on conformal geometric algebra which should prove to be useful in computer applications. The focus is on parameterizing 3D conformal transformations with either quaternionic Vahlen matrices or polynomial Cayley transform from the Lie algebra to the Lie group of conformal transformations in space.

1 Terminology and Notations

By embedding Euclidean space $\mathbb{R}^n$ into the set of null vectors in $\mathbb{R}^{n+1,1}$ in a nonlinear manner, we get the *conformal model* of nD Euclidean geometry [2, 5, 6, 11]. In the Minkowski space $\mathbb{R}^{n+1,1}$, a nonzero vector is said to be *null* if its inner product with itself is zero and is said to be *positive* if so is the inner product.

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A point in $\mathbb{R}^n$ is represented by a null vector in $\mathbb{R}^{n+1,1}$, and the representation is unique up to scale. There is a unique-up-to-scale null vector $\mathbf{e} \in \mathbb{R}^{n+1,1}$ in the conformal model that does not represent any point in $\mathbb{R}^n$. We say that it represents the *conformal point at infinity*. A sphere or hyperplane in $\mathbb{R}^n$ is represented by a positive vector in $\mathbb{R}^{n+1,1}$, and the representation is unique up to scale. A positive vector represents a hyperplane if and only if its inner product with $\mathbf{e}$ equals zero.

The *Grassmann–Cayley algebra* [19] over $\mathbb{R}^{n+1,1}$, when equipped with the nD Euclidean geometric interpretations of the algebraic elements in this algebra, is called the *conformal Grassmann–Cayley algebra* of the nD space. The *Clifford algebra* over $\mathbb{R}^{n+1,1}$, when equipped with the nD conformal transformation interpretations of the algebraic elements in this algebra, is called the *conformal Clifford algebra* of the nD space. *Conformal geometric algebra* is an integration of conformal Grassmann–Cayley algebra and conformal Clifford algebra [9], the former representing geometric configurations, and the latter representing geometric transformations.

Terminology and notation:

1. *Tensor product*, denoted by "$\otimes$".
2. *Outer product*, denoted by "$\wedge$".
3. *Inner product*, denoted by "$\cdot$".
4. *Meet product*, denoted by "$\vee$".
5. *Geometric product*, denoted by juxtaposition of participating elements. The geometric product of r identical elements $\mathbf{A}$ is denoted by $\mathbf{A}^r$.
6. *Multivector*: any element in a Grassmann algebra (or Clifford algebra).
7. *Exponential* of a multivector $\mathbf{A}$: $\exp(\mathbf{A}) = e^{\mathbf{A}} = 1 + \mathbf{A} + \mathbf{A}^2/2! + \mathbf{A}^3/3! + \cdots$.
8. *Inverse* of a multivector $\mathbf{A}$, denoted by $\mathbf{A}^{-1}$.
9. *Blade*: a multivector which equals the outer product of several vectors.
10. *Grade*: the number of vector components in an outer product decomposition of a blade. A blade of grade r is called an r-blade.
11. *Homogeneous multivector*: a linear combination of blades of the same grade. The grade of a homogeneous multivector is that of any of the blade component. A homogeneous multivector of grade r is called an r-vector. A 2-vector is also called a bivector.
12. *r-graded part* of a multivector, denoted by "$\langle\ \rangle_r$".
13. *Scalar part* of a multivector: the 0-graded part, denoted by "$\langle\ \rangle$".
14. *Even (or odd) multivector*: a linear combination of homogeneous multivectors of even grades (or odd grades).
15. *Even-graded part* (or *odd-graded part*) of a multivector, denoted by "$\langle\ \rangle_+$" (or "$\langle\ \rangle_-$").
16. *Versor*: the geometric product of several invertible vectors.
17. *Rotor*: the geometric product of an even number of invertible vectors.
18. *Positive vector*: a vector whose inner product with itself is positive.
19. *Positive versor*: the geometric product of several positive vectors.
20. *Positive rotor*: the geometric product of an even number of positive vectors.
21. *Grassmann algebra over $\mathcal{V}^n$*, denoted by $\Lambda(\mathcal{V}^n)$.
22. *Clifford algebra over $\mathcal{V}^n$*, denoted by $C\ell(\mathcal{V}^n)$.

23. *Grassmann algebra* or *Clifford algebra* generated by a blade $\mathbf{I}_n$ and denoted by $\Lambda(\mathbf{I}_n)$ (or $C\ell(\mathbf{I}_n)$). The base vector space $\mathcal{V}^n$ of the Grassmann algebra (or Clifford algebra) is composed of all vectors whose outer product with $\mathbf{I}_n$ equals zero.
24. *Even Clifford subalgebra of* $C\ell(\mathcal{V}^n)$, composed of all even multivectors, denoted by $C\ell^+(\mathcal{V}^n)$.
25. *Odd vector subspace of* $C\ell(\mathcal{V}^n)$, composed of all odd multivectors, denoted by $C\ell^-(\mathcal{V}^n)$.
26. *Magnitude* of a multivector: denoted by "$|\ |$". The magnitude of a scalar is its absolute value. The magnitude of $\mathbf{A} \in C\ell(\mathcal{V}^n)$ is $|\mathbf{A}| = \sum_{i=0}^n \sqrt{|\langle \mathbf{A}\rangle_i \cdot \langle \mathbf{A}\rangle_i|}$.
27. *Pseudoscalar:* a blade of grade n in $\Lambda(\mathcal{V}^n)$ or $C\ell(\mathcal{V}^n)$.
28. *Dual operator* in a nondegenerate Clifford algebra, denoted by "$^\sim$". Fixing a pseudoscalar $\mathbf{I}_n$ of unit magnitude, for any multivector $\mathbf{A}$, $\mathbf{A}^\sim = \mathbf{A}\mathbf{I}_n^{-1}$.
29. *Reversion operator* in a Clifford algebra, denoted by "†". For vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_r$, $(\mathbf{a}_1\mathbf{a}_2 \cdots \mathbf{a}_r)^\dagger = \mathbf{a}_r \cdots \mathbf{a}_2\mathbf{a}_1$.
30. *Grade involution* in a Clifford algebra: denoted by overhat. For multivector $\mathbf{A}$, $\widehat{\mathbf{A}} = \langle \mathbf{A}\rangle_+ - \langle \mathbf{A}\rangle_-$.
31. *Conjugate operator* in a Clifford algebra: the composition of reversion and grade involution, denoted by overbar. For multivector $\mathbf{A}$, $\overline{\mathbf{A}} = \langle \mathbf{A}^\dagger\rangle_+ - \langle \mathbf{A}^\dagger\rangle_-$.
32. *Spin group over* $\mathcal{V}^n$, denoted by $\mathrm{Spin}(\mathcal{V}^n)$. It is composed of all rotors in $C\ell(\mathcal{V}^n)$ of unit magnitude together with the geometric product.

Example 1 In conformal Grassmann–Cayley algebra $C\ell(\mathbb{R}^{4,1})$, a circle passing through three points $\mathbf{1}, \mathbf{2}, \mathbf{3}$ in the space is represented by 3-blade $\mathbf{1} \wedge \mathbf{2} \wedge \mathbf{3}$, where $\mathbf{1}, \mathbf{2}, \mathbf{3}$ are null vectors of $\mathbb{R}^{4,1}$ representing points in $\mathbb{R}^3$. The blade is Minkowski, so its dual $(\mathbf{1} \wedge \mathbf{2} \wedge \mathbf{3})^\sim$ is a positive vector. Alternatively, the circle can be represented by a positive vector.

The set $\mathcal{N}$ of all null vectors in $\mathbb{R}^{n+1,1}$ has two connected components. In particular, null vectors $\pm\mathbf{a}$ are always in different connected components, as 0 is not a null vector. An orthogonal transformation in $\mathbb{R}^{n+1,1}$ keeping each component of $\mathcal{N}$ invariant is called a *positive orthogonal transformation*. All such transformations form a subgroup $O_+(n+1,1)$ of $O(n+1,1)$, called the *positive orthogonal group*. The orientation-preserving orthogonal transformations of $\mathbb{R}^{n+1,1}$ form another subgroup $SO(n+1,1)$ of $O(n+1,1)$, called the *special orthogonal group*. The intersection of the two subgroups, denoted by $SO_+(n+1,1)$, is called the *Lorentz group*, and its elements are called *Lorentz transformations*. Lorentz transformations are the linear isometries of $\mathbb{R}^{n+1,1}$ connected with the identity transformation $I_{\mathbb{R}^{n+1,1}}$.

In conformal Clifford algebra $C\ell(\mathbb{R}^{n+1,1})$, any positive rotor $\mathbf{U}$ induces a unique Lorentz transformation in $\mathbb{R}^{n+1,1}$ via the following *adjoint action*:

$$Ad_{\mathbf{U}}(\mathbf{x}) = \mathbf{U}\mathbf{x}\mathbf{U}^{-1} \quad \text{for all } \mathbf{x} \in \mathbb{R}^{n+1,1}. \tag{1}$$

Conversely, any Lorentz transformation in $\mathbb{R}^{n+1,1}$ is induced by a positive rotor that is unique up to scale.

In the conformal model, any Lorentz transformation in $\mathbb{R}^{n+1,1}$ induces a unique orientation-preserving conformal transformation in $\mathbb{R}^n$, and the converse is also true. Hence, any orientation-preserving conformal transformation in $\mathbb{R}^n$ is induced by a positive rotor in $C\ell(\mathbb{R}^{n+1,1})$ that is unique up to scale.

2 Exponential Map and Exterior Exponential Map

By a classical theorem of Riesz [14], any linear isometry of $\mathbb{R}^{n+1,1}$ connected with the identity is induced by a rotor of the exponential form. The group of rotors in $C\ell(\mathbb{R}^{n+1,1})$ differs from $\mathrm{Spin}(\mathbb{R}^{n+1,1})$ by a factor $\mathbb{R} - \{0\}$. Since the Lie algebra of the spin group is all bivectors in $\Lambda(\mathbb{R}^{n+1,1})$, any rotor connected with the identity can be expressed up to scale as the exponential $e^{\mathbf{B}_2}$ of a bivector $\mathbf{B}_2 \in \Lambda(\mathcal{V}^n)$. As a corollary, any positive rotor in $C\ell(\mathbb{R}^{n+1,1})$ is in the range of the exponential map.

Example 2 The Lie algebra representation of 3D rigid body motions via the exponential map.

Any rigid body motion in the space can be decomposed into a rotation followed by a translation. It can also be decomposed into a translation followed by a rotation. The decomposition is not unique without fixing the axis of rotation, which is a straight line in the space. However, there is a unique decomposition, in which the axis of rotation follows exactly the direction of translation. This is the *screw motion*, and the unique decomposition theorem is known as *Chasles' Theorem*.

In the conformal model of 3D Euclidean geometry, let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be an orthonormal basis of $\mathbb{R}^3$, and let $\mathbf{e}_0, \mathbf{e}$ be the pair of null vectors orthogonal to $\mathbb{R}^3$ in $\mathbb{R}^{4,1}$ and such that $\mathbf{e}_0 \cdot \mathbf{e} = -1$. The basis $(\mathbf{e}, \mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is called a *Witt basis* of $\mathbb{R}^{4,1}$. The vector $\mathbf{e}_0$ represents the origin of $\mathbb{R}^3$ in the conformal model, while the vector $\mathbf{e}$ represents the conformal point at infinity.

In conformal geometric algebra $C\ell(\mathbb{R}^{4,1})$, the Lie algebra of the spin group of rigid body motions is $\Lambda^2(\mathbf{e}^\sim)$, which is the bivector subspace of the Grassmann algebra generated by the vectors in $\mathbb{R}^{4,1}$ that are orthogonal to $\mathbf{e}$.

The vector space $\Lambda^2(\mathbf{e}^\sim)$ has an orthonormal basis $\mathbf{e}_1\mathbf{e}_2, \mathbf{e}_2\mathbf{e}_3, \mathbf{e}_1\mathbf{e}_3, \mathbf{e}\mathbf{e}_1, \mathbf{e}\mathbf{e}_2, \mathbf{e}\mathbf{e}_3$. Any nonzero element in $\Lambda^2(\mathbf{e}^\sim)$ can be written as

$$\mathbf{B}_2 = \frac{\mathbf{I}_2\theta + \mathbf{e}\mathbf{t}}{2}, \tag{2}$$

where 2-blade $\mathbf{I}_2 \in \Lambda(\mathbb{R}^3)$ is of unit magnitude, $\theta \in \mathbb{R}$, and $\mathbf{t} \in \mathbb{R}^3$.

If $\theta = 0$, then $e^{\mathbf{B}_2}$ induces the translation by the vector $\mathbf{t}$. If $\theta \neq 0$, then

$$\begin{aligned} e^{\mathbf{B}_2} &= e^{-\mathbf{e}(\mathbf{t}\cdot\mathbf{I}_2)/(2\theta)} e^{\mathbf{I}_2\theta/2} e^{\mathbf{e}(\mathbf{t}\cdot\mathbf{I}_2)/(2\theta)} e^{\mathbf{e}P^{\perp}_{\mathbf{I}_2}(\mathbf{t})/2} \\ &= \cos\frac{\theta}{2} + \mathbf{I}_2 \sin\frac{\theta}{2} + \frac{1}{\theta}\mathbf{e}P_{\mathbf{I}_2}(\mathbf{t})\sin\frac{\theta}{2} + \frac{1}{2}\mathbf{e}P^{\perp}_{\mathbf{I}_2}(\mathbf{t})\cos\frac{\theta}{2} \\ &\quad + \frac{1}{2}\mathbf{e}P^{\perp}_{\mathbf{I}_2}(\mathbf{t})\mathbf{I}_2 \sin\frac{\theta}{2}, \end{aligned} \tag{3}$$

where

$$\begin{aligned} P_{\mathbf{I}_2}(\mathbf{t}) &= (\mathbf{t}\cdot\mathbf{I}_2)\mathbf{I}_2^{-1}, \\ P_{\mathbf{I}_2}^{\perp}(\mathbf{t}) &= \mathbf{t} - P_{\mathbf{I}_2}(\mathbf{t}). \end{aligned} \tag{4}$$

Equation (3) induces a screw motion with the vector of translation $P_{\mathbf{I}_2}^{\perp}(\mathbf{t})$, the axis of rotation passing through the point $-\mathbf{t}\cdot\mathbf{I}_2/\theta \in \mathbb{R}^3$, and the angle of rotation $-\theta$.

In parameterizing 3D conformal transformations, the 10D vector space $\Lambda^2(\mathbb{R}^{4,1})$ provides an ideal parametric space for the group of 3D conformal transformations. Since the exponential map from $\Lambda^2(\mathbb{R}^{4,1})$ to the group of positive rotors is surjective, any orientation-preserving 3D conformal transformation can be parameterized via the exponential map, although the parameterization is not unique.

The problem of parameterizing with the exponential map lies in evaluating the map and computing its inverse. While the map can be evaluated when restricted to some vector subspaces such as $\Lambda^2(\mathbf{e}^{\sim})$, the evaluation for the general case is still not available. Even when the evaluation exists, in many cases such as (3), it is computationally expensive because the map is transcendental instead of algebraic. Furthermore, the exponential map is not an isometry, and its tangent map preserves volume only at the origin of the Lie algebra taken as a vector space. The exponential map has infinitely many inverses in general.

The first alternative of exponential map is the following exterior exponential map:

Definition 1 Let $\mathcal{V}^n$ be a vector space over a field $\mathbb{K}$. The *exterior exponential* is the following map from $\Lambda^2(\mathcal{V}^n)$ to $\Lambda(\mathcal{V}^n)$:

$$e^{\wedge \mathbf{B}_2} = 1 + \mathbf{B}_2 + \frac{\mathbf{B}_2 \wedge \mathbf{B}_2}{2!} + \cdots + \frac{\overbrace{\mathbf{B}_2 \wedge \mathbf{B}_2 \wedge \cdots \wedge \mathbf{B}_2}^{r}}{r!}, \tag{5}$$

where r is the greatest integer such that $\overbrace{\mathbf{B}_2 \wedge \mathbf{B}_2 \wedge \cdots \wedge \mathbf{B}_2}^{r} \neq 0$.

The exterior exponential has two obvious properties: first, the scalar part of $e^{\wedge \mathbf{B}_2}$ is 1; second, the mapping is injective because the bivector part of $e^{\wedge \mathbf{B}_2}$ is $\mathbf{B}_2$.

Since any bivector has a *completely orthogonal decomposition*, i.e., for a bivector $\mathbf{B}_2$, there exist vectors $\mathbf{a}_1, \mathbf{b}_1, \mathbf{a}_2, \mathbf{b}_2, \ldots, \mathbf{a}_r, \mathbf{b}_r$ such that

$$\mathbf{B}_2 = \lambda_1 \mathbf{a}_1 \wedge \mathbf{b}_1 + \lambda_2 \mathbf{a}_2 \wedge \mathbf{b}_2 + \cdots + \lambda_r \mathbf{a}_r \wedge \mathbf{b}_r, \tag{6}$$

where $\mathbf{a}_i \cdot \mathbf{b}_j = 0$ for any $1 \le i, j \le r$, and $\mathbf{a}_i \cdot \mathbf{a}_k = \mathbf{b}_i \cdot \mathbf{b}_k = 0$ for any $i \neq k$, we have

$$e^{\wedge \mathbf{B}_2} = (1 + \lambda_1 \mathbf{a}_1 \wedge \mathbf{b}_1)(1 + \lambda_2 \mathbf{a}_2 \wedge \mathbf{b}_2) \cdots (1 + \lambda_r \mathbf{a}_r \wedge \mathbf{b}_r).$$

So $e^{\wedge \mathbf{B}_2}$ is invertible if and only if each $\lambda_i \mathbf{a}_i \wedge \mathbf{b}_i$ is not a Minkowski blade of unit magnitude. When $\mathcal{V}^n$ is Minkowski or Euclidean, then if $e^{\wedge \mathbf{B}_2}$ is invertible, it must be a rotor connected with the identity, because so is each $1+\lambda_i \mathbf{a}_i \wedge \mathbf{b}_i$; furthermore,

$$\left(e^{\wedge \mathbf{B}_2}\right)^{-1} = \frac{e^{\wedge(-\mathbf{B}_2)}}{e^{\wedge \mathbf{B}_2} e^{\wedge(-\mathbf{B}_2)}}. \tag{7}$$

Below we assume that $\mathbf{B}_2$ is in the form of (6) and $e^{\wedge \mathbf{B}_2}$ is invertible, and analyze the range of the exterior exponential by restricting it to the conformal model $\mathbb{R}^{4,1}$ of 3D geometry.

If $\mathbf{B}_2$ is a nonzero blade, then $e^{\wedge \mathbf{B}_2} = 1 + \lambda_1 \mathbf{a}_1 \wedge \mathbf{b}_1$. When λ_1 varies, the range of $e^{\wedge \mathbf{B}_2}$ modulo scale contains all rotors in $\Lambda(\mathbf{a}_1 \wedge \mathbf{b}_1)$ whose 0-graded part and 2-graded part are both nonzero.

If $\mathbf{B}_2$ is not a blade, then

$$e^{\wedge \mathbf{B}_2} = 1 + \lambda_1 \mathbf{a}_1 \wedge \mathbf{b}_1 + \lambda_2 \mathbf{a}_2 \wedge \mathbf{b}_2 + \lambda_1 \lambda_2 \mathbf{a}_1 \wedge \mathbf{b}_1 \wedge \mathbf{a}_2 \wedge \mathbf{b}_2, \tag{8}$$

whose 0-graded part and 4-graded part are both nonzero. The range of $e^{\wedge \mathbf{B}_2}$ modulo scale is all rotors whose 0-graded part and 4-graded part are both nonzero.

Proposition 1 *When the range of the exterior exponential is restricted to rotors in* $C\ell(\mathbb{R}^{4,1})$, *the domain of definition is all bivectors in* $\Lambda(\mathbb{R}^{4,1})$ *satisfying*

$$(\mathbf{B}_2 \wedge \mathbf{B}_2)^2 \neq 4(\mathbf{B}_2 \cdot \mathbf{B}_2 - 1) \tag{9}$$

and is a set $\mathbb{R}^{10} - V^9$, *where* V^9 *is a* 9*D algebraic variety in* $\mathbb{R}^{10}$. *The image space modulo scale is all rotors whose scalar parts are nonzero*; *topologically*, *it is the remainder of the positive orthogonal group* $O_+(4,1)$, *which is a* 10*D Lie group with two connected components*, *after removal of a* 9*D closed subset*.

Proof By

$$e^{\wedge \mathbf{B}_2} = 1 + \mathbf{B}_2 + \frac{\mathbf{B}_2 \wedge \mathbf{B}_2}{2}, \tag{10}$$

we get $e^{\wedge \mathbf{B}_2} e^{\wedge(-\mathbf{B}_2)} = 1 - \mathbf{B}_2 \cdot \mathbf{B}_2 + (\mathbf{B}_2 \wedge \mathbf{B}_2)^2/4$, and (9) follows. □

Similar to the exponential map, the exterior exponential provides half-scaled bivector representations for rotations, translations, and dilations.

Given a rotor $\mathbf{A}$ in the image space of the exterior exponential, let $\mathbf{B}_2$ be a bivector whose exterior exponential equals $\mathbf{A}$ up to scale. Then

$$1 + \mathbf{B}_2 + \frac{\mathbf{B}_2 \wedge \mathbf{B}_2}{2} = \frac{\mathbf{A}}{\langle \mathbf{A} \rangle}, \tag{11}$$

so

$$\mathbf{B}_2 = \frac{\langle \mathbf{A} \rangle_2}{\langle \mathbf{A} \rangle}. \tag{12}$$

Example 3 Let $e^{\mathbf{I}_2 \frac{\theta}{2}}$ be a rotor inducing the rotation in space with axis $\mathbf{I}_2^{\sim}$ and angle $-\theta \neq \pi \mod 2\pi$. Then up to scale,

$$e^{\mathbf{I}_2 \frac{\theta}{2}} = e^{\wedge \mathbf{I}_2 \tan(\frac{\theta}{2})}. \tag{13}$$

Let $e^{\frac{1}{2}\mathbf{et}} = 1 + \mathbf{et}/2$ be a rotor inducing the translation by a vector $\mathbf{t}$. Then

$$e^{\frac{1}{2}\mathbf{et}} = e^{\wedge \frac{1}{2}\mathbf{et}}. \tag{14}$$

Let $e^{\frac{\theta}{2}\mathbf{I}_2}$ be a rotor inducing the dilation of scale $e^{-\theta}$ centering at point $\mathbf{I}_2$ (affine representation, cf. [9]). Then up to scale,

$$e^{\mathbf{I}_2 \frac{\theta}{2}} = e^{\wedge \mathbf{I}_2 \tanh(\frac{\theta}{2})}. \tag{15}$$

Let $\mathbf{I}_2 e^{\frac{\theta}{2}\mathbf{I}_2}$ be a rotor inducing a dilation of scale $-e^{-\theta} \neq -1$. Then up to scale,

$$\mathbf{I}_2 e^{\mathbf{I}_2 \frac{\theta}{2}} = e^{\wedge \mathbf{I}_2 \operatorname{ctanh}(\frac{\theta}{2})}. \tag{16}$$

On one hand, the exterior exponential is an injective quadratic map, which is superior to the exponential map algebraically. On the other hand, the exterior exponential has two severe drawbacks: first, the domain of definition is decomposed into several disconnected regions, which blocks the construction of large-scope bivector parameters in the design of continuous conformal transformations; second, the image space is also decomposed into several disconnected regions, making it impossible to represent rotors of large-scale continuous conformal transformations.

3 Twisted Vahlen Matrices and Quaternionic Vahlen Matrices

Recall that in complex analysis, any 2D conformal transformation can be represented by a fractional linear map from the Riemann sphere to itself, the sphere being the complex plane plus the complex point at infinity. In Clifford analysis, there is a similar representation for any nD conformal transformation. This is the so-called *Vahlen matrix representation* [1, 11, 12, 16].

In this section, we introduce the classical work of Vahlen (1902) on representing nD conformal transformations projectively by 2×2 matrices whose components are in $C\ell(\mathbb{R}^n)$, by means of introducing two alternatives of Vahlen's matrix representation:

- *Twisted Vahlen matrices*: the product of two 2×2 matrices of multivector components is no longer the usual matrix product, but a "twisted" one. The twisted matrix product is exactly the geometric product in the conformal geometric algebra $C\ell(\mathbb{R}^{n+1,1})$.
- *Quaternionic Vahlen matrices*: When $n = 3$, any twisted Vahlen matrix can be written as a 2×2 quaternionic matrix, and projectively, any 3D conformal transformation can be represented by such a quaternionic matrix. The composition of 3D conformal transformations is just the usual matrix product.

For 3D conformal transformations, twisted Vahlen matrices, Vahlen matrices, and quaternionic Vahlen matrices provide three equivalent transcendental parameterizations in the Lie algebra $\Lambda^2(\mathbb{R}^{4,1})$. The evaluation of each parameterization and the computing of the inverse are both very easy.

With respect to the Witt basis $(\mathbf{e}, \mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n)$ of $\mathbb{R}^{n+1,1}$, any vector has the decomposition $\mathbf{a} + \lambda\mathbf{e} + \mu\mathbf{e}_0$, where $\mathbf{a} \in \mathbb{R}^n$ is a linear combination of $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$. By $\mathbf{e}\mathbf{e}_0\mathbf{e} = -2\mathbf{e}$ and $\mathbf{e}_0\mathbf{e}\mathbf{e}_0 = -2\mathbf{e}_0$, any versor $\mathbf{M} = (\mathbf{a}_1 + \lambda_1\mathbf{e} + \mu_1\mathbf{e}_0)(\mathbf{a}_2 + \lambda_2\mathbf{e} + \mu_2\mathbf{e}_0)\cdots(\mathbf{a}_r + \lambda_r\mathbf{e} + \mu_r\mathbf{e}_0)$, where $\mathbf{a}_i \in \mathbb{R}^n$ and $\lambda_i, \mu_i \in \mathbb{R}$, after multilinear expansion, is changed into the following form:

$$\mathbf{M} = -\frac{\mathbf{A}}{2}\mathbf{e}\mathbf{e}_0 - \frac{\mathbf{B}}{2}\mathbf{e} + \mathbf{C}\mathbf{e}_0 - \frac{\mathbf{D}}{2}\mathbf{e}_0\mathbf{e}, \tag{17}$$

where $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \in C\ell(\mathbb{R}^n)$. More generally, by means of linearity any multivector $\mathbf{M} \in C\ell(\mathbb{R}^{n+1,1})$ has the unique decomposition (17).

Under the following correspondence of bases:

$$\begin{aligned} 1 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad & \mathbf{e} &= \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}, \quad & \mathbf{e}_0 &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \\ \mathbf{e}\mathbf{e}_0 &= \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}, \quad & \mathbf{e}_0\mathbf{e} &= \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}, \quad & \mathbf{e} \wedge \mathbf{e}_0 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \end{aligned} \tag{18}$$

(17) becomes

$$\mathbf{M} = -\frac{\mathbf{A}}{2}\mathbf{e}\mathbf{e}_0 - \frac{\mathbf{B}}{2}\mathbf{e} + \mathbf{C}\mathbf{e}_0 - \frac{\mathbf{D}}{2}\mathbf{e}_0\mathbf{e} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}. \tag{19}$$

It is a classical result [11], which is also easy to verify, that (19) provides an algebraic isomorphism between $C\ell(\mathbb{R}^{n+1,1})$ and the following 2×2 twisted Clifford matrix algebra $\hat{M}_{2\times 2}(C\ell(\mathbb{R}^n))$:

Definition 2 The 2×2 *twisted Clifford matrix algebra* over $\mathbb{R}^n$ is the linear space of 2×2 matrices whose components are in $C\ell(\mathbb{R}^n)$, equipped with the *twisted multiplication* defined as follows: for any 2×2 matrices $\mathbf{M}_1, \mathbf{M}_2$ whose components are in $C\ell(\mathbb{R}^n)$,

$$\mathbf{M}_1\mathbf{M}_2 = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}\begin{pmatrix} \mathbf{A}' & \mathbf{B}' \\ \mathbf{C}' & \mathbf{D}' \end{pmatrix} := \begin{pmatrix} \mathbf{A}\mathbf{A}' + \mathbf{B}\hat{\mathbf{C}}' & \mathbf{A}\mathbf{B}' + \mathbf{B}\hat{\mathbf{D}}' \\ \mathbf{C}\hat{\mathbf{A}}' + \mathbf{D}\mathbf{C}' & \mathbf{C}\hat{\mathbf{B}}' + \mathbf{D}\mathbf{D}' \end{pmatrix}. \tag{20}$$

In each component on the right side of (20), the overhat (grade involution) is always added to the element of the second matrix that is not in the same row with the corresponding element of the first matrix multiplied with it. For example, in the first component $\mathbf{A}\mathbf{A}' + \mathbf{B}\hat{\mathbf{C}}'$, $\mathbf{A}, \mathbf{A}'$ are each in the first row of the corresponding matrix, while $\mathbf{B}, \mathbf{C}'$ are in different rows, so the overhat is added to $\mathbf{C}'$.

Let $\mathbf{I}_n$ be the unit pseudoscalar representing $\mathbb{R}^n$ with its positive orientation. The following formulas can be easily derived from (19):

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}^{\dagger} = \begin{pmatrix} \mathbf{D}^{\dagger} & \overline{\mathbf{B}} \\ \overline{\mathbf{C}} & \mathbf{A}^{\dagger} \end{pmatrix}, \qquad \widehat{\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}} = \begin{pmatrix} \hat{\mathbf{A}} & -\hat{\mathbf{B}} \\ -\hat{\mathbf{C}} & \hat{\mathbf{D}} \end{pmatrix},$$
$$\overline{\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}} = \begin{pmatrix} \overline{\mathbf{D}} & -\mathbf{B}^{\dagger} \\ -\mathbf{C}^{\dagger} & \overline{\mathbf{A}} \end{pmatrix}, \qquad \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}^{\sim} = \begin{pmatrix} -\mathbf{A}\mathbf{I}_n^{-1} & \mathbf{B}\mathbf{I}_n^{-1} \\ -\mathbf{C}\mathbf{I}_n^{-1} & \mathbf{D}\mathbf{I}_n^{-1} \end{pmatrix}. \tag{21}$$

Under the correspondence (18), any vector $\mathbf{a} \in \mathbb{R}^{n+1,1}$ corresponds to the following matrix:

$$\begin{pmatrix} \mathbf{x} & \alpha \\ \beta & \mathbf{x} \end{pmatrix}, \tag{22}$$

where

$$\begin{aligned} \mathbf{x} &= P^{\perp}_{\mathbf{e}\wedge\mathbf{e}_0}(\mathbf{a}), \\ \alpha &= 2\mathbf{a}\cdot\mathbf{e}_0, \\ \beta &= -\mathbf{a}\cdot\mathbf{e}. \end{aligned} \tag{23}$$

In particular, the null vector $\mathbf{e}_0 + \mathbf{x} + \mathbf{e}\mathbf{x}^2/2$, where $\mathbf{x} \in \mathbb{R}^n$, corresponds to the matrix

$$\begin{pmatrix} \mathbf{x} & -\mathbf{x}^2 \\ 1 & \mathbf{x} \end{pmatrix}. \tag{24}$$

Of particular interest are the matrices corresponding to versors in $C\ell(\mathbb{R}^{n+1,1})$. We first take a look at some examples. Let $\mathbf{I}_2 \in \Lambda^2(\mathbb{R}^n)$ and $\mathbf{t} \in \mathbb{R}^n$.

- The rotor of rotation $e^{\theta\mathbf{I}_2/2}$ corresponds to $\begin{pmatrix} e^{\theta\mathbf{I}_2/2} & 0 \\ 0 & e^{\theta\mathbf{I}_2/2} \end{pmatrix}$.
- The rotor of dilation $e^{\theta\mathbf{e}\wedge\mathbf{e}_0/2}$ corresponds to $\begin{pmatrix} e^{-\theta/2} & 0 \\ 0 & e^{\theta/2} \end{pmatrix}$.
- The rotor of dilation $(\mathbf{e}\wedge\mathbf{e}_0)e^{\theta\mathbf{e}\wedge\mathbf{e}_0/2}$ corresponds to $\begin{pmatrix} -e^{-\theta/2} & 0 \\ 0 & e^{\theta/2} \end{pmatrix}$.
- The rotor of translation $1 + \mathbf{e}\mathbf{t}/2$ corresponds to $\begin{pmatrix} 1 & \mathbf{t} \\ 0 & 1 \end{pmatrix}$.
- The rotor of transversion $1 - \mathbf{e}_0\mathbf{t}$ corresponds to $\begin{pmatrix} 1 & 0 \\ \mathbf{t} & 1 \end{pmatrix}$.

Definition 3 A 2×2 matrix $\mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}$ over $C\ell(\mathbb{R}^n)$ is called a *twisted Vahlen matrix* if

1. $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ are either versors or zero.
2. $\mathbf{A}\mathbf{B}^{\dagger}, \mathbf{B}\mathbf{D}^{\dagger}, \mathbf{D}\mathbf{C}^{\dagger}, \mathbf{C}\mathbf{A}^{\dagger}$ are vectors.
3. $\Delta = \mathbf{A}\mathbf{D}^{\dagger} + \mathbf{B}\mathbf{C}^{\dagger}$ is a nonzero scalar.

In the above definition, Condition 1 guarantees $\mathbf{B}^{\dagger}\mathbf{A} = \mathbf{A}^{-1}(\mathbf{A}\mathbf{B}^{\dagger})\mathbf{A} \in \mathbb{R}^n$ if $\mathbf{A}\mathbf{B}^{\dagger} \in \mathbb{R}^n$. So in Condition 2, $\mathbf{A}\mathbf{B}^{\dagger}$ can be replaced by any of $\mathbf{B}\mathbf{A}^{\dagger}, \mathbf{B}^{\dagger}\mathbf{A}, \mathbf{A}^{\dagger}\mathbf{B}$, and so for the other three elements in Condition 2. By Conditions 1 and 2,

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}^{\dagger} = \begin{pmatrix} \mathbf{A}\mathbf{D}^{\dagger} + \mathbf{B}\mathbf{C}^{\dagger} & \mathbf{A}\overline{\mathbf{B}} + \mathbf{B}\overline{\mathbf{A}} \\ \mathbf{C}\overline{\mathbf{D}} + \mathbf{D}\overline{\mathbf{C}} & \mathbf{C}\mathbf{B}^{\dagger} + \mathbf{D}\mathbf{A}^{\dagger} \end{pmatrix}$$
$$= \begin{pmatrix} \mathbf{A}\mathbf{D}^{\dagger} + \mathbf{B}\mathbf{C}^{\dagger} & 0 \\ 0 & (\mathbf{A}\mathbf{D}^{\dagger} + \mathbf{B}\mathbf{C}^{\dagger})^{\dagger} \end{pmatrix}, \tag{25}$$

so Condition 3 is equivalent to $\mathbf{M}\mathbf{M}^{\dagger}$ being a nonzero scalar.

Theorem 1 *In twisted Vahlen matrix* $\mathbf{M} = \left(\begin{smallmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{smallmatrix}\right)$, *when* $\mathbf{A} \neq 0$, *there exist* $\lambda \in \mathbb{R} - \{0\}$ *and* $\mathbf{b}, \mathbf{c} \in \mathbb{R}^n$ *such that*

$$\mathbf{M} = \mathbf{A} \begin{pmatrix} 1 & \mathbf{b} \\ \mathbf{c} & \lambda - \mathbf{c}\mathbf{b} \end{pmatrix}. \tag{26}$$

When $\mathbf{A} = 0$, *there exist* $\mu \in \mathbb{R} - \{0\}$ *and* $\mathbf{d} \in \mathbb{R}^n$ *such that*

$$\mathbf{M} = \mathbf{B} \begin{pmatrix} 0 & 1 \\ \mu & \mathbf{d} \end{pmatrix}. \tag{27}$$

Proof (i) If $\mathbf{A} \neq 0$ and $\mathbf{B} \neq 0$, by denoting

$$\mathbf{A}^{\dagger}\mathbf{B} = \mathbf{b}, \qquad \mathbf{A}^{\dagger}\mathbf{C} = \mathbf{c}, \qquad \mathbf{B}^{\dagger}\mathbf{D} = \mathbf{d}, \qquad \mathbf{A}^{\dagger}\mathbf{A} = \lambda^{-1}, \qquad \mathbf{B}^{\dagger}\mathbf{B} = \mu^{-1},$$

the matrix $\mathbf{M}$ can be written as

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} = \mathbf{A} \begin{pmatrix} 1 & \lambda\mathbf{b} \\ \lambda\mathbf{c} & \lambda\mu\mathbf{b}\mathbf{d} \end{pmatrix},$$

where $\mathbf{d}$ satisfies

$$\mathbf{d} = \mu^{-1}\Delta\mathbf{b}^{-1} - \mu^{-1}\mathbf{b}\mathbf{c}\mathbf{b}^{-1}. \tag{28}$$

By (28), $\lambda\mu\mathbf{b}\mathbf{d} = \lambda\Delta - \lambda^2\mathbf{c}\mathbf{b}$, so $\mathbf{M}$ can be written as

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} = \mathbf{A} \begin{pmatrix} 1 & \lambda\mathbf{b} \\ \lambda\mathbf{c} & \lambda\Delta - \lambda^2\mathbf{c}\mathbf{b} \end{pmatrix}.$$

(ii) If $\mathbf{A} \neq 0$ but $\mathbf{B} = 0$, then $\mathbf{M}$ can be written as

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} = \mathbf{A} \begin{pmatrix} 1 & 0 \\ \lambda\mathbf{c} & \lambda\Delta \end{pmatrix},$$

which is a special case of (i) where $\mathbf{b} = 0$.

(iii) If $\mathbf{A} = 0$, then $\mathbf{B} \neq 0$, and $\mathbf{M}$ can be written as

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} = \mathbf{B} \begin{pmatrix} 0 & 1 \\ \mu\Delta & \mu\mathbf{d} \end{pmatrix}.$$

□

Theorem 2 *Any versor in* $C\ell(\mathbb{R}^{n+1,1})$ *corresponds via* (19) *to a twisted Vahlen matrix.*

Proof Let $\mathbf{M}$ be a versor. When $\mathbf{M}$ is a vector, then it is of the form (22) and is a twisted Vahlen matrix if and only if it is neither zero nor null. To prove the theorem by induction, we need only prove that for any versor $\mathbf{M}$ and invertible vector $\mathbf{M}'$,

$$\mathbf{M}\mathbf{M}' = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \begin{pmatrix} \mathbf{x} & \alpha \\ \beta & \mathbf{x} \end{pmatrix} = \begin{pmatrix} \mathbf{A}\mathbf{x} + \beta\mathbf{B} & \alpha\mathbf{A} - \mathbf{B}\mathbf{x} \\ -\mathbf{C}\mathbf{x} + \beta\mathbf{D} & \alpha\mathbf{C} + \mathbf{D}\mathbf{x} \end{pmatrix} \tag{29}$$

is a twisted Vahlen matrix.

By (26) and (27), we only need to consider two cases:

$$\text{(i)} \quad \mathbf{M} = \begin{pmatrix} 1 & \lambda\mathbf{b} \\ \lambda\mathbf{c} & \lambda\Delta - \lambda^2\mathbf{c}\mathbf{b} \end{pmatrix}, \qquad \text{(ii)} \quad \mathbf{M} = \begin{pmatrix} 0 & 1 \\ \mu\Delta & \mu\mathbf{d} \end{pmatrix}.$$

The corresponding matrix $\mathbf{M}\mathbf{M}'$ is respectively

$$\text{(i)} \quad \begin{pmatrix} \mathbf{x} + \lambda\beta\mathbf{b} & \alpha - \lambda\mathbf{b}\mathbf{x} \\ \lambda\Delta\beta - \lambda\mathbf{c}(\mathbf{x} + \lambda\beta\mathbf{b}) & \lambda\Delta\mathbf{x} + \lambda\mathbf{c}(\alpha - \lambda\mathbf{b}\mathbf{x}) \end{pmatrix},$$

$$\text{(ii)} \quad \begin{pmatrix} \beta & -\mathbf{x} \\ \mu(\beta\mathbf{d} - \Delta\mathbf{x}) & \mu(\Delta\alpha + \mathbf{d}\mathbf{x}) \end{pmatrix},$$

and it can be easily verified that each matrix is a twisted Vahlen matrix. □

Theorem 3 (Twisted version of Vahlen's Theorem) *Any twisted Vahlen matrix* $\mathbf{M}$ *generates the following conformal transformation in* $\mathbb{R}^n$:

$$\mathbf{x} \longmapsto \mathbf{M}(\mathbf{x}) = (\mathbf{A}\mathbf{x} + \mathbf{B})\big(\hat{\mathbf{C}}\mathbf{x} + \hat{\mathbf{D}}\big)^{-1} \quad \forall \mathbf{x} \in \mathbb{R}^n. \tag{30}$$

Conversely, any conformal transformation in $\mathbb{R}^n$ *has such a twisted fractional linear representation.*

Proof In the conformal model, a point $\mathbf{x} \in \mathbb{R}^n$ is represented by the null vector $\mathbf{e}_0 + \mathbf{x} + \mathbf{e}\mathbf{x}^2/2$ whose twisted Vahlen matrix representation is (24). The *graded adjoint action* of versor $\mathbf{M}$ on the null vector is, up to scale,

$$\begin{aligned} &\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \begin{pmatrix} \mathbf{x} & -\mathbf{x}^2 \\ 1 & \mathbf{x} \end{pmatrix} \overline{\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}} \\ &= \begin{pmatrix} \mathbf{A}\mathbf{x}\mathbf{D}^\dagger + \mathbf{B}\mathbf{D}^\dagger + \mathbf{x}^2\mathbf{A}\overline{\mathbf{C}} + \mathbf{B}\mathbf{x}\overline{\mathbf{C}} & -\mathbf{A}\mathbf{x}\mathbf{B}^\dagger - \mathbf{B}\mathbf{B}^\dagger - \mathbf{x}^2\mathbf{A}\mathbf{A}^\dagger - \mathbf{B}\mathbf{x}\mathbf{A}^\dagger \\ -\mathbf{C}\mathbf{x}\overline{\mathbf{D}} + \mathbf{D}\overline{\mathbf{D}} + \mathbf{x}^2\mathbf{C}\mathbf{C}^\dagger - \mathbf{D}\mathbf{x}\mathbf{C}^\dagger & \mathbf{C}\mathbf{x}\overline{\mathbf{B}} - \mathbf{D}\overline{\mathbf{B}} - \mathbf{x}^2\mathbf{C}\overline{\mathbf{A}} + \mathbf{D}\mathbf{x}\overline{\mathbf{A}} \end{pmatrix} \\ &= \begin{pmatrix} (\mathbf{A}\mathbf{x} + \mathbf{B})(\mathbf{D}^\dagger + \mathbf{x}\overline{\mathbf{C}}) & -(\mathbf{A}\mathbf{x} + \mathbf{B})(\mathbf{B}^\dagger + \mathbf{x}\mathbf{A}^\dagger) \\ -(\mathbf{C}\mathbf{x} - \mathbf{D})(\overline{\mathbf{D}} - \mathbf{x}\mathbf{C}^\dagger) & (\mathbf{C}\mathbf{x} - \mathbf{D})(\overline{\mathbf{B}} - \mathbf{x}\overline{\mathbf{A}}) \end{pmatrix}. \end{aligned}$$

So in $\mathbb{R}^n$, $\mathbf{M}$ changes a vector $\mathbf{x}$ to the vector

$$-\frac{(\mathbf{Ax}+\mathbf{B})(\mathbf{D}^\dagger+\mathbf{x}\overline{\mathbf{C}})}{(\mathbf{Cx}-\mathbf{D})(\overline{\mathbf{D}}-\mathbf{x}\mathbf{C}^\dagger)} = -(\mathbf{Ax}+\mathbf{B})\big(\mathbf{D}^\dagger+\mathbf{x}\overline{\mathbf{C}}\big)\big(\widehat{\mathbf{D}^\dagger+\mathbf{x}\overline{\mathbf{C}}}\big)^{-1}(\mathbf{Cx}-\mathbf{D})^{-1}$$
$$= -(\mathbf{Ax}+\mathbf{B})\big(\widehat{\mathbf{Cx}-\mathbf{D}}\big)^{-1}$$
$$= (\mathbf{Ax}+\mathbf{B})\big(\hat{\mathbf{C}}\mathbf{x}+\hat{\mathbf{D}}\big)^{-1}. \qquad \square$$

When $C\ell(\mathbb{R}^n)$ is represented by a matrix algebra, the twisted matrix multiplication is very inconvenient and needs to be revised to usual matrix multiplication. The work was done by Vahlen in 1902.

Definition 4 The *algebra of* 2×2 *Clifford matrices* over $C\ell(\mathbb{R}^n)$, denoted by $M_{2\times 2}(C\ell(\mathbb{R}^n))$, is the linear space of matrices of the form $\mathbf{M}=\begin{pmatrix}\mathbf{A}&\mathbf{B}\\ \mathbf{C}&\mathbf{D}\end{pmatrix}$, where $\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}\in C\ell(\mathbb{R}^n)$, equipped with the usual matrix multiplication

$$\begin{pmatrix}\mathbf{A}&\mathbf{B}\\ \mathbf{C}&\mathbf{D}\end{pmatrix}\begin{pmatrix}\mathbf{A}'&\mathbf{B}'\\ \mathbf{C}'&\mathbf{D}'\end{pmatrix}=\begin{pmatrix}\mathbf{AA}'+\mathbf{BC}'&\mathbf{AB}'+\mathbf{BD}'\\ \mathbf{CA}'+\mathbf{DC}'&\mathbf{CB}'+\mathbf{DD}'\end{pmatrix}. \tag{31}$$

Definition 5 2×2 matrix $\mathbf{M}=\begin{pmatrix}\mathbf{A}&\mathbf{B}\\ \mathbf{C}&\mathbf{D}\end{pmatrix}$ over $C\ell(\mathbb{R}^n)$ is called a *Vahlen matrix* if

1. $\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}$ are either versors or zero.
2. $\mathbf{AB}^\dagger,\mathbf{BD}^\dagger,\mathbf{DC}^\dagger,\mathbf{CA}^\dagger$ are vectors.
3. $\Delta=\mathbf{AD}^\dagger-\mathbf{BC}^\dagger$ is a nonzero scalar.

It is easy to verify that Condition 3 in the above definition is equivalent to matrix $\mathbf{M}$ being invertible.

Under the following correspondence, any twisted Clifford matrix corresponds to a unique Clifford matrix, and vice versa:

$$\text{twisted Clifford matrix }\begin{pmatrix}\mathbf{A}&\mathbf{B}\\ \mathbf{C}&\mathbf{D}\end{pmatrix}\longleftrightarrow\text{Clifford matrix }\begin{pmatrix}\mathbf{A}&\mathbf{B}\\ \hat{\mathbf{C}}&\hat{\mathbf{D}}\end{pmatrix}. \tag{32}$$

The above correspondence is in fact an algebraic isomorphism. All the previous results presented in the form of twisted Clifford matrices can be translated easily into Clifford matrices. For example, the following is a translation of Theorem 3.

Theorem 4 (Vahlen's Theorem) *Any Vahlen matrix* $\mathbf{M}$ *generates the following conformal transformation in* $\mathbb{R}^n$:

$$\mathbf{x}\longmapsto\mathbf{M}(\mathbf{x})=(\mathbf{Ax}+\mathbf{B})(\mathbf{Cx}+\mathbf{D})^{-1}\quad\forall\mathbf{x}\in\mathbb{R}^n; \tag{33}$$

and any conformal transformation has such a fractional linear representation.

Consider the special case where $n=3$. Any 3D conformal transformation is induced by the adjoint action of a rotor in $C\ell(\mathbb{R}^{4,1})$, and the rotor is unique up to scale.

A rotor in $C\ell(\mathbb{R}^{4,1})$ corresponds to a twisted Vahlen matrix $\mathbf{M} = \left(\begin{smallmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{smallmatrix}\right)$, where $\mathbf{A}, \mathbf{D}$ are even and $\mathbf{B}, \mathbf{C}$ are odd. Such a matrix is called an *even twisted Vahlen matrix*.

Fix a Witt basis $(\mathbf{e}, \mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ of $\mathbb{R}^{4,1}$. Under the well-known correspondence

$$\begin{aligned} i &= \mathbf{e}_2 \wedge \mathbf{e}_3, \\ j &= \mathbf{e}_1 \wedge \mathbf{e}_3, \\ k &= \mathbf{e}_1 \wedge \mathbf{e}_2, \end{aligned} \tag{34}$$

the algebra of quaternions $\mathbb{Q}$ is isomorphic to the even subalgebra $C\ell^+(\mathbb{R}^3)$. Any nonzero element of $C\ell^+(\mathbb{R}^3)$ is a rotor, and by duality, any nonzero element of $C\ell^-(\mathbb{R}^3)$ is an odd versor.

Definition 6 A 2×2 quaternionic matrix $\mathbf{M} = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$ is called a *quaternionic Vahlen matrix* if

1. $\alpha\overline{\beta},\ \beta\overline{\delta},\ \delta\overline{\gamma},\ \gamma\overline{\alpha}$ are all pure imaginary, or equivalently, $\langle\alpha\overline{\beta}\rangle = \langle\beta\overline{\delta}\rangle = \langle\delta\overline{\gamma}\rangle = \langle\gamma\overline{\alpha}\rangle = 0$;
2. The *determinant* $\Delta = \alpha\overline{\delta} + \beta\overline{\gamma} \neq 0$ is real.

It can be easily proved that a quaternionic Vahlen matrix is invertible if and only if its determinant is nonzero.

Theorem 5 *The following correspondence, together with* (34), *provides an algebraic isomorphism between the group of even twisted Vahlen matrices and the group of quaternionic Vahlen matrices*:

$$\mathbf{M} = \begin{pmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{pmatrix} \longmapsto \begin{pmatrix}\mathbf{A} & \mathbf{B}\mathbf{I}_3^{-1}\\ \mathbf{C}\mathbf{I}_3^{-1} & \mathbf{D}\end{pmatrix}. \tag{35}$$

Proof First, in $C\ell(\mathbf{I}_3)$, $\mathbf{A}\mathbf{B}^\dagger$ being a vector is equivalent to $\mathbf{A}(\overline{\mathbf{B}\mathbf{I}_3^{-1}})$ being a bivector. Second,

$$\Delta = \mathbf{A}\mathbf{D}^\dagger + \mathbf{B}\mathbf{C}^\dagger = \mathbf{A}\overline{\mathbf{D}} + \big(\mathbf{B}\mathbf{I}_3^{-1}\big)\big(\mathbf{C}\mathbf{I}_3^{-1}\big)^\dagger = \mathbf{A}\overline{\mathbf{D}} + \big(\mathbf{B}\mathbf{I}_3^{-1}\big)\big(\overline{\mathbf{C}\mathbf{I}_3^{-1}}\big).$$

Third, by usual matrix multiplication,

$$\begin{aligned} \begin{pmatrix}\mathbf{A} & \mathbf{B}\mathbf{I}_3^{-1}\\ \mathbf{C}\mathbf{I}_3^{-1} & \mathbf{D}\end{pmatrix}\begin{pmatrix}\mathbf{A}' & \mathbf{B}'\mathbf{I}_3^{-1}\\ \mathbf{C}'\mathbf{I}_3^{-1} & \mathbf{D}'\end{pmatrix} &= \begin{pmatrix}\mathbf{A}\mathbf{A}' - \mathbf{B}\mathbf{C} & (\mathbf{A}\mathbf{B}' + \mathbf{B}\mathbf{D})\mathbf{I}_3^{-1}\\ (\mathbf{C}\mathbf{A}' + \mathbf{D}\mathbf{C}')\mathbf{I}_3^{-1} & \mathbf{D}\mathbf{D}' - \mathbf{C}\mathbf{B}'\end{pmatrix} \\ &= \begin{pmatrix}\mathbf{A}\mathbf{A}' + \mathbf{B}\hat{\mathbf{C}} & (\mathbf{A}\mathbf{B}' + \mathbf{B}\hat{\mathbf{D}})\mathbf{I}_3^{-1}\\ (\mathbf{C}\hat{\mathbf{A}}' + \mathbf{D}\mathbf{C}')\mathbf{I}_3^{-1} & \mathbf{D}\mathbf{D}' + \mathbf{C}\hat{\mathbf{B}}'\end{pmatrix}. \end{aligned}$$

□

A point $\mathbf{x} \in \mathbb{R}^3$ is represented by the pure imaginary quaternion $\mathbf{x}\mathbf{I}_3^{-1}$ under the correspondence (34), or in the 2D right-linear quaternionic vector space $\mathbb{Q}^2$ realizing the 1D projective space $\mathbb{QP}^1$, is represented by the vector $(\mathbf{x}\mathbf{I}_3^{-1}\ 1)^T$.

The matrix multiplication of a quaternionic Vahlen matrix $\mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{B}\mathbf{I}_3^{-1} \\ \mathbf{C}\mathbf{I}_3^{-1} & \mathbf{D} \end{pmatrix}$ with $(\mathbf{x}\mathbf{I}_3^{-1} \; 1)^T$ results in

$$\begin{pmatrix} \mathbf{A} & \mathbf{B}\mathbf{I}_3^{-1} \\ \mathbf{C}\mathbf{I}_3^{-1} & \mathbf{D} \end{pmatrix} \begin{pmatrix} \mathbf{x}\mathbf{I}_3^{-1} \\ 1 \end{pmatrix} = \begin{pmatrix} (\mathbf{A}\mathbf{x} + \mathbf{B})\mathbf{I}_3^{-1} \\ -\mathbf{C}\mathbf{x} + \mathbf{D} \end{pmatrix} = \begin{pmatrix} (\mathbf{A}\mathbf{x} + \mathbf{B})\mathbf{I}_3^{-1} \\ \hat{\mathbf{C}}\mathbf{x} + \hat{\mathbf{D}} \end{pmatrix}. \tag{36}$$

Combining the above result with (30), we get the following:

Theorem 6 (Vahlen's Theorem in quaternionic form) *Any quaternionic Vahlen matrix* $\mathbf{M} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ *generates the following* 3*D conformal transformation*: *for any pure imaginary quaternion* υ *representing a point in space*,

$$\upsilon \longmapsto \mathbf{M}(\upsilon) = (\alpha\upsilon + \beta)(\gamma\upsilon + \delta)^{-1}; \tag{37}$$

or equivalently, *in* $\mathbb{QP}^1$ *where the point is represented homogeneously by* $(\upsilon : 1)$, *the conformal transformation is just the projectivity induced by the following invertible right-linear transformation over* $\mathbb{Q}$:

$$\begin{pmatrix} \upsilon \\ 1 \end{pmatrix} \longmapsto \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \upsilon \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha\upsilon + \beta \\ \gamma\upsilon + \delta \end{pmatrix}. \tag{38}$$

Conversely, *any* 3*D conformal transformation has such a quaternionic fractional linear representation*.

Any bivector $\mathbf{B}_2 \in \Lambda^2(\mathbb{R}^{4,1})$ has the following decomposition:

$$\mathbf{B}_2 = \mathbf{A}_2 + \mathbf{b} \wedge \mathbf{e} + \mathbf{c} \wedge \mathbf{e}_0 + \lambda \mathbf{e} \wedge \mathbf{e}_0, \tag{39}$$

where $\mathbf{A}_2 \in \Lambda^2(\mathbb{R}^3)$, $\lambda \in \mathbb{R}$, and $\mathbf{b}, \mathbf{c} \in \mathbb{R}^3$. The following map from the Lie algebra $\Lambda^2(\mathbb{R}^{4,1})$ to the group of quaternionic Vahlen matrices under the correspondence (34) provides a transcendental parameterization of 3D conformal transformations, called *quaternionic Vahlen parameterization*:

$$\mathbf{B}_2 = \mathbf{A}_2 + \mathbf{b} \wedge \mathbf{e} + \mathbf{c} \wedge \mathbf{e}_0 + \lambda \mathbf{e} \wedge \mathbf{e}_0 \longmapsto e^{\mathbf{A}_2} \begin{pmatrix} 1 & \mathbf{b}\mathbf{I}_3^{-1} \\ \mathbf{c}\mathbf{I}_3^{-1} & \lambda - \mathbf{c}\mathbf{b} \end{pmatrix} \quad \text{if } \lambda \neq 0. \tag{40}$$

The Jacobian of the above parameterization is that of the exponential map $\mathbf{A}_2 \mapsto e^{\mathbf{A}_2}$ from $\Lambda^2(\mathbb{R}^3)$ to $\mathrm{Spin}(\mathbb{R}^3)$. It is always bounded. Those not in the range of the parameterization are conformal transformations induced by twisted Vahlen matrices of the form (27). The effect of such a transformation is

$$\mathbf{x} \in \mathbb{R}^3 \longmapsto Ad_{\mathbf{B}\mathbf{I}_3^{-1}}\left((\mu\mathbf{x} - \mathbf{d})^{-1}\right) \in \mathbb{R}^3, \tag{41}$$

which is the composition of the translation by vector $-\mathbf{d}/\mu$, the inversion with respect to the sphere centering at the origin and of radius $\mu^{-1/2}$, and a rotation whose axis passes through the origin of $\mathbb{R}^3$. The dimension of the conformal transformations outside the range of the parameterization is 7.

Collecting results from the above two paragraphs, we get the following:

Proposition 2 *The domain of definition of quaternionic Vahlen parameterization is a set* $\mathbb{R}^{10} - \mathbb{R}^9$ *parameterized by* $(\mathbf{A}_2, \mathbf{b}, \mathbf{c}, \lambda)$ *according to* (39), *where* $\lambda \neq 0$. *The image space is all* 3*D conformal transformations whose fractional linear representation* (30) *has the property that* $\mathbf{A} = 0$; *it is the remainder of* $O_+(4, 1)$ *after removal of a* 7*D closed subset and is topologically* $\mathbb{S}^3 \times (\mathbb{R} - \{0\}) \times \mathbb{R}^6$.

Compared with the exterior exponential, quaternionic Vahlen parameterization has the drawback that it is transcendental, and generally there are infinitely many inverses, but has the significant advantage that its domain of definition is simpler, and its image space is larger.

Example 4 Let there be a rotation in the space with fixed axis $\mathbf{I}_2^\sim$ and angle of rotation $\theta = \theta(t)$, where t is the time variable, and the range of θ is an interval of $\mathbb{R}$. The parameterization of the motion by outer exponential is $e^{\wedge \mathbf{I}_2 \tan(\frac{\theta(t)}{2})}$ and is invalid when $\theta(t) = \pi \mod 2\pi$.

In contrast, in the special case where the axis passes through the origin, the parameterization of the motion by quaternionic matrix is $e^{\mathbf{I}_2 \frac{\theta(t)}{2}} \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. In the general case, let $\mathbf{I}_2^\sim$ represent the line passing through point $\mathbf{p} \in \mathbb{R}^3$ and following unit direction $\mathbf{n} \in \mathbb{R}^3$, i.e.,

$$\mathbf{I}_2^\sim = \mathbf{e} \wedge (\mathbf{e}_0 + \mathbf{p}) \wedge \mathbf{n}, \tag{42}$$

then the parameterization of the motion by quaternionic matrix is

$$\begin{aligned} &\begin{pmatrix} 1 & -\mathbf{p}\mathbf{I}_3^{-1} \\ 0 & 1 \end{pmatrix} e^{\mathbf{n}\mathbf{I}_3^{-1} \frac{\theta(t)}{2}} \begin{pmatrix} 1 & \mathbf{p}\mathbf{I}_3^{-1} \\ 0 & 1 \end{pmatrix} \\ &= e^{\mathbf{n}\mathbf{I}_3^{-1} \frac{\theta(t)}{2}} \begin{pmatrix} 1 & (\mathbf{p} - e^{-\mathbf{n}\mathbf{I}_3^{-1} \frac{\theta(t)}{2}} \mathbf{p} e^{\mathbf{n}\mathbf{I}_3^{-1} \frac{\theta(t)}{2}})\mathbf{I}_3^{-1} \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{43}$$

It is valid for all $\theta(t) \in \mathbb{R}$.

4 Cayley Transform

In application, rational polynomial functions are much simpler than exponentials or trigonometric functions. For the special orthogonal group $SO(p, q)$, whose Lie algebra $so(p, q)$ is the set of antisymmetric linear transformations in $\mathbb{R}^{p,q}$, besides the exponential map, there is also a classical rational polynomial map from the Lie algebra to the Lie group, called *Cayley transform* [11]:

$$\begin{aligned} so(p, q) &\longrightarrow SO(p, q), \\ g &\longmapsto (I_{\mathbb{R}^{p,q}} + g)(I_{\mathbb{R}^{p,q}} - g)^{-1}, \quad \text{where } I_{\mathbb{R}^{p,q}} - g \text{ is invertible.} \end{aligned} \tag{44}$$

The map is injective but generally not surjective.

In the conformal model of 3D space, a natural idea is to consider simplifying the exponential map from $\Lambda^2(\mathbb{R}^{4,1})$ to the group of rotors by a fractional linear map similar to (44). The following mapping C:

$$\begin{aligned} \Lambda^2(\mathbb{R}^{4,1}) &\longrightarrow C\ell(\mathbb{R}^{4,1}), \\ \mathbf{B}_2 &\longmapsto (1+\mathbf{B}_2)(1-\mathbf{B}_2)^{-1}, \quad \text{where } 1-\mathbf{B}_2 \text{ is invertible}, \end{aligned} \tag{45}$$

is called the *Cayley transform* from Lie algebra $\Lambda^2(\mathbb{R}^{4,1})$ to the group of rotors in $C\ell(\mathbb{R}^{4,1})$.

The Cayley transform in terms of dual quaternions has been an important tool in describing and manipulating 3D rigid-body motions [17]. In this section, we enlarge the scope to 3D conformal transformations, explore the range and domain of definition of the Cayley transform, and present a degree-4 polynomial form of it, together with several neat formulas for the inverse of Cayley transform.

By computing the inverse $(1-\mathbf{B}_2)^{-1}$, we get that for any $\mathbf{B}_2 \in \Lambda^2(\mathbb{R}^{4,1})$ such that $\mathbf{B}_2^2 \neq 1$, the following equality holds up to scale:

$$C(\mathbf{B}_2) = (1+\mathbf{B}_2)^2(1-\mathbf{B}_2\cdot\mathbf{B}_2+\mathbf{B}_2\wedge\mathbf{B}_2). \tag{46}$$

If $C(\mathbf{B}_2)$ is required to be of unit magnitude, then

$$C(\mathbf{B}_2) = \frac{(1+\mathbf{B}_2)^2(1-\mathbf{B}_2\cdot\mathbf{B}_2+\mathbf{B}_2\wedge\mathbf{B}_2)}{(1-\mathbf{B}_2\cdot\mathbf{B}_2)^2-(\mathbf{B}_2\wedge\mathbf{B}_2)^2}. \tag{47}$$

Equation (46) can be used as an alternative definition of the Cayley transform. From this aspect, the Cayley transform is just a polynomial of degree 4 in $\mathbf{B}_2$, with values in the group of positive rotors of $C\ell(\mathbb{R}^{4,1})$; or equivalently, it is a rational polynomial of degree 4, with values in $\mathrm{Spin}_+(4,1)$.

Theorem 7 [9] *The domain of definition of the Cayley transform C is all bivectors except the Minkowski blades of unit magnitude and is a set $\mathbb{R}^{10}-V^5$, where V^5 is a 5D algebraic variety in $\mathbb{R}^{10}$. The image space of C modulo scale is all positive rotors except those of the form $\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_4$, where the $\mathbf{a}_i$ are pairwise orthogonal positive vectors.*

Geometrically, the image space modulo scale is composed of positive rotors generating all orientation-preserving conformal transformations except the antipodal inversions, as shown in Fig. 1, each of which is the composition of an inversion with respect to a sphere and the reflection with respect to the center of the sphere. Topologically, the image space modulo scale is the remainder of the Lorentz group of $\mathbb{R}^{4,1}$, which is a 10D connected Lie group, after removal of a 4D open disk.

In the following, we present the "inverse" of the Cayley transform by finding all the preimages of a rotor in its range. Given a positive rotor $\mathbf{A}$ such that $\mathbf{A}\neq 1$ up to scale, let $\mathbf{B}_2$ be a bivector whose Cayley transform equals $\mathbf{A}$ up to scale.

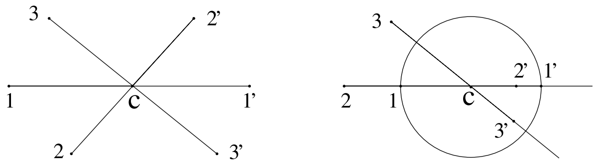

Fig. 1 Antipodal inversion: composition of a reflection with respect to a point and an inversion centering at the same point

A bivector is said to be *entangled*, or *coherent*, if in its completely orthogonal decomposition there are two components having equal square. It can be proved that for a bivector $\langle\mathbf{A}\rangle_2 \in \Lambda^2(\mathbb{R}^{4,1})$ to be entangled, it is necessary and sufficient that

$$\big(\langle\mathbf{A}\rangle_2 \cdot \langle\mathbf{A}\rangle_2\big)^2 = \big(\langle\mathbf{A}\rangle_2 \wedge \langle\mathbf{A}\rangle_2\big)^2. \tag{48}$$

Theorem 8 [9] *A positive rotor* $\mathbf{A}$ *in the range of the Cayley transform has exactly one bivector preimage if and only if either it is in* $\Lambda(\mathbf{C}_2)$, *where* $\mathbf{C}_2$ *is a* 2*-blade of degenerate signature*, *or its bivector part is entangled. The unique solution is*

$$\frac{\langle\mathbf{A}\rangle_2}{\langle\mathbf{A}\rangle_4 + 2\langle\mathbf{A}\rangle + |\langle\mathbf{A}\rangle\langle\mathbf{A}\rangle_4|/\langle\mathbf{A}\rangle}. \tag{49}$$

Any other positive rotor $\mathbf{A}$ *in the range of the Cayley transform has two bivector preimages*, *and they are inverse to each other*:

$$\frac{\langle\mathbf{A}\rangle_2}{\langle\mathbf{A}\rangle_4 + \langle\mathbf{A}\rangle \pm |\mathbf{A}|}. \tag{50}$$

Example 5 In $C\ell(\mathbb{R}^{4,1})$, let $\mathbf{A} = e^{\mathbf{I}_2\frac{\theta}{2}}$ be a rotor inducing a 2D rotation, where $\mathbf{I}_2 \in \Lambda(\mathbf{e}^{\sim})$ is a Euclidean 2-blade of unit magnitude such that $\mathbf{I}_2^{\sim}$ is the axis of rotation, and $-\theta$ is the angle of rotation. Then

$$\mathbf{B}_2 = \frac{e^{\mathbf{I}_2\frac{\theta}{2}} - e^{-\mathbf{I}_2\frac{\theta}{2}}}{e^{\mathbf{I}_2\frac{\theta}{2}} + e^{-\mathbf{I}_2\frac{\theta}{2}} + 2} = \mathbf{I}_2 \tan\frac{\theta}{4}, \qquad \mathbf{B}_2^{-1} = -\mathbf{I}_2/\tan\frac{\theta}{4}, \tag{51}$$

and both generate $\mathbf{A}$ by the Cayley transform.

While the bivector representation of a rotation via the exponential map is a half-angle representation, the bivector representation via the Cayley transform is a quarter-angle representation.

Example 6 In $C\ell(\mathbb{R}^{4,1})$, let $\mathbf{A} = 1 + \mathbf{e}\mathbf{t}/2$ be a rotor realizing a translation, where $\mathbf{t} \in \mathbf{e}^{\sim}$ is a positive vector. Then

$$\mathbf{B}_2 = \frac{\mathbf{e}\mathbf{t}}{4} \tag{52}$$

generates $\mathbf{A}$ by the Cayley transform. So the Cayley transform provides a quarter-distance bivector representation of the translation.

Example 7 Let $\mathbf{A} = e^{\frac{\theta}{2}\mathbf{e}\wedge\mathbf{a}}$ be a rotor realizing a dilation (or scaling), where $\theta \in \mathbb{R}$, $\mathbf{a}$ is a null vector representing a point, and $\mathbf{a}\cdot\mathbf{e} = -1$. Rotor $\mathbf{A}$ generates the dilation centering at point $\mathbf{a}$ and with scale $e^{-\theta}$. Denote $\mathbf{I}_2 = \mathbf{e}\wedge\mathbf{a}$. Then

$$\mathbf{B}_2 = \frac{e^{\mathbf{I}_2\frac{\theta}{2}} - e^{-\mathbf{I}_2\frac{\theta}{2}}}{e^{\mathbf{I}_2\frac{\theta}{2}} + e^{-\mathbf{I}_2\frac{\theta}{2}} + 2} = \mathbf{I}_2 \tanh\frac{\theta}{4}, \qquad \mathbf{B}_2^{-1} = \mathbf{I}_2/\tanh\frac{\theta}{4}, \tag{53}$$

and both generate $\mathbf{A}$ by the Cayley transform. So the Cayley transform provides a quarter-scale bivector representation of the dilation.

All orientation-preserving similarity transformations in $\mathbb{R}^3$ can be induced by bivectors in $\Lambda^2(\mathbb{R}^{4,1})$ through the Cayley transform and adjoint action. A translation is induced by the Cayley transform of a unique bivector. Any other orientation-preserving similarity transformation is induced by the Cayley transform of exactly two bivectors.

When choosing between the two bivector preimages $\mathbf{B}_2$ and $\mathbf{B}_2^{-1}$ of a rotor, since $|\mathbf{B}_2||\mathbf{B}_2^{-1}| = 1$, one of $|\mathbf{B}_2|$ and $|\mathbf{B}_2^{-1}|$ is greater than or equal to 1. By (50),

$$\left|\mathbf{B}_2 - \mathbf{B}_2^{-1}\right| = 2\frac{|\mathbf{A}|}{|\langle\mathbf{A}\rangle_2|} \geq 2. \tag{54}$$

So for two rotors that are close to each other, we can always choose their bivector preimages to be close to each other. If their magnitudes are greater than 1, we can choose their inverses so that the magnitudes become smaller than 1.

Then what is the use of having two bivector preimages for the same rotor? Take, as an example, the 2D rotation in Example 5. It is well known that $SO(2)$ is a circle which has the following rational parameterization induced by the stereographic projection from the north pole N:

$$e^{\mathbf{I}_2\theta} = \cos\theta + \mathbf{I}_2\sin\theta = \frac{2t}{1-t^2} + \mathbf{I}_2\frac{1+t^2}{1-t^2}, \tag{55}$$

where, as shown in Fig. 2, $t = \tan(\theta/2)$ is half the signed distance from the south pole S to point X in the horizontal direction, and point $e^{i\theta}$ in the complex plane is represented by the intersection R of line NX with the unit circle.

Since the map $\theta \mapsto e^{\mathbf{I}_2\theta}$ is an isometric immersion, the Jacobian of the rational parameterization (55) equals

$$\frac{d\theta}{dt} = 1\Big/\left(\frac{dt}{d\theta}\right) = 2\cos^2\frac{\theta}{2}. \tag{56}$$

When point R moves from the south pole S to point T, the Jacobian decreases from 2 to 1, while when point R continues to move from point T to the north pole N, the

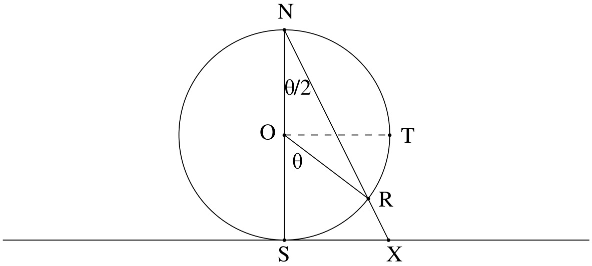

Fig. 2 Rational parametrization of a 2D rotation

Jacobian decreases from 1 to 0. The parameterization becomes practically useless nearby the north pole.

The two preimages in (51) are both mapped to the rotor $e^{\mathbf{I}_2\theta/2}$ by the Cayley transform. So the Cayley transform provides a generalization of the rational parameterization of a circle taken as the 2D rotation group. In fact, it serves as two rational parameterizations of the circle derived from two different stereographic projections: one from the north pole, and the other from the south pole.

Since the two maps

$$\theta \mapsto \mathbf{I}_2 \tan\frac{\theta}{4} \quad \text{and} \quad \theta \mapsto -\mathbf{I}_2 \,\text{ctan}\,\frac{\theta}{4} \tag{57}$$

have Jacobians $1/(4\cos^2(\theta/4))$ and $1/(4\sin^2(\theta/4))$, respectively, the maps

$$\begin{aligned} t = \tan\frac{\theta}{4} &\overset{\text{Cayley}}{\longmapsto} e^{\mathbf{I}_2\theta/2}, \\ t = -\text{ctan}\frac{\theta}{4} &\overset{\text{Cayley}}{\longmapsto} e^{\mathbf{I}_2\theta/2}, \end{aligned} \tag{58}$$

have Jacobians $J_1 = 2\cos^2\frac{\theta}{4}$ and $J_2 = 2\sin^2\frac{\theta}{4}$, respectively.

- When $(4k-1)\pi \le \theta \le (4k+1)\pi$, then $1 \le J_1 \le 2$ and $0 \le J_2 \le 1$.
- When $(4k+1)\pi \le \theta \le (4k+3)\pi$, then $1 \le J_2 \le 2$ and $0 \le J_1 \le 1$.

Hence the two maps in (58) cover different zones of the parameter $\theta \in \mathbb{R}$ for the Jacobians to be effective between 1 and 2. They serve as two different local coordinate charts whose union covers the whole Lie group.

The Cayley transform as a polynomial map of degree 4 is computationally superior; its inverse map has two branches and involves only one square-root computing. Its domain of definition and its image space, when restricted to orientation-preserving conformal transformations, are both larger than those of exterior differential and quaternionic Vahlen parameterization. Thus the Cayley transform and its inverse are an ideal tool for motion planning, interpolating, and fitting in the Lie algebra of 3D conformal transformations.

5 Conclusion

This chapter explores the issue of parameterizing 3D conformal transformations. Two new results are presented, one on quaternionic Vahlen parameterization, the other on the polynomial 3D Cayley transform. They provide compact representations of 3D conformal transformations and should prove to be useful in geometric applications.

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