8
Gaussian, mean and principal curvatures

In this chapter, we show how to extract geometric information from the second fundamental form of a surface or, equivalently, from its Weingarten map.

8.1 Gaussian and mean curvatures

We start by defining two new measures of the curvature of a surface.

Definition 8.1.1

Let $\mathcal{W}$ be the Weingarten map of an oriented surface $\mathcal{S}$ at a point $\mathbf{p} \in \mathcal{S}$. The Gaussian curvature K and mean curvature H of $\mathcal{S}$ at $\mathbf{p}$ are defined by

$$K = \det(\mathcal{W}), \quad H = \frac{1}{2}\text{trace}(\mathcal{W}).$$

Recall that the determinant and trace of a linear map (such as $\mathcal{W}$) can be computed as the determinant and the sum of the diagonal entries of the matrix of the linear map with respect to any basis (in this case of the tangent plane), and that they depend only on the linear map and not on the choice of basis.

When the sign of the unit normal of $\mathcal{S}$ is changed, the Weingarten map also changes sign (Exercise 7.2.2), thus leaving K unchanged. This implies that the Gaussian curvature is defined for *any* surface $\mathcal{S}$, orientable or not: to define K at a point $\mathbf{p} \in \mathcal{S}$, choose a surface patch $\boldsymbol{\sigma}$ with $\mathbf{p}$ in its image; this is an

Andrew Pressley, *Elementary Differential Geometry: Second Edition*,
Springer Undergraduate Mathematics Series, DOI 10.1007/978-1-84882-891-9_8,

oriented surface, which may be used to define K, and the result is independent of the choice of $\boldsymbol{\sigma}$. On the other hand, on a surface that is not necessarily orientable, H is in general only well defined up to sign.

To get explicit formulas for H and K, we work in a surface patch of $\mathcal{S}$. Let $\boldsymbol{\sigma}(u,v)$ be a surface patch with first and second fundamental forms

$$Edu^2 + 2Fdudv + Gdv^2 \quad \text{and} \quad Ldu^2 + 2Mdudv + Ndv^2,$$

respectively. Define symmetric 2×2 matrices $\mathcal{F}_I$ and $\mathcal{F}_{II}$ by

$$\mathcal{F}_I = \begin{pmatrix} E & F \\ F & G \end{pmatrix}, \quad \mathcal{F}_{II} = \begin{pmatrix} L & M \\ M & N \end{pmatrix}.$$

Proposition 8.1.2

Let $\boldsymbol{\sigma}$ be a surface patch of an oriented surface $\mathcal{S}$. Then, with the above notation, the matrix of $\mathcal{W}_{\mathbf{p},\mathcal{S}}$ with respect to the basis $\{\boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v\}$ of $T_{\mathbf{p}}\mathcal{S}$ is $\mathcal{F}_I^{-1}\mathcal{F}_{II}$.

Proof

By the proof of Proposition 7.2.2, $\mathcal{W}(\boldsymbol{\sigma}_u) = -\mathbf{N}_u$ and $\mathcal{W}(\boldsymbol{\sigma}_v) = -\mathbf{N}_v$, so the matrix of $\mathcal{W}$ is $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$, where

$$-\mathbf{N}_u = a\boldsymbol{\sigma}_u + b\boldsymbol{\sigma}_v, \quad -\mathbf{N}_v = c\boldsymbol{\sigma}_u + d\boldsymbol{\sigma}_v.$$

Take the dot product of each of these equations with $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$ and use Lemma 7.2.3; this gives

$$\begin{aligned} L &= aE + bF, & M &= cE + dF, \\ M &= aF + bG, & N &= cF + dG. \end{aligned}$$

These four scalar equations are equivalent to the single matrix equation

$$\begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$
$$\text{i.e.,} \quad \mathcal{F}_{II} = \mathcal{F}_I \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

Hence, the matrix of $\mathcal{W}$ with respect to the basis $\{\boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v\}$ is

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \mathcal{F}_I^{-1}\mathcal{F}_{II}. \qquad \square$$

Corollary 8.1.3

We have

$$H = \frac{LG - 2MF + NE}{2(EG - F^2)}, \quad K = \frac{LN - M^2}{EG - F^2}.$$

Proof

By Definition 8.1.1,

$$K = \det\left(\mathcal{F}_I^{-1}\mathcal{F}_{II}\right) = \frac{\det(\mathcal{F}_{II})}{\det(\mathcal{F}_I)} = \frac{LN - M^2}{EG - F^2}.$$

To compute H, we need the trace of the matrix

$$\begin{aligned} \mathcal{F}_I^{-1}\mathcal{F}_{II} &= \frac{1}{EG - F^2}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix}\begin{pmatrix} L & M \\ M & N \end{pmatrix} \\ &= \frac{1}{EG - F^2}\begin{pmatrix} LG - MF & MG - NF \\ ME - LF & NE - MF \end{pmatrix}. \end{aligned}$$

Thus,

$$2H = \operatorname{trace}\left(\mathcal{F}_I^{-1}\mathcal{F}_{II}\right) = \frac{LG - 2MF + NE}{EG - F^2}. \qquad \square$$

Example 8.1.4

In Examples 6.1.3 and 7.1.2 we considered the surface of revolution

$$\boldsymbol{\sigma}(u, v) = (f(u)\cos v, f(u)\sin v, g(u)),$$

where we can assume that $f > 0$ and $\dot{f}^2 + \dot{g}^2 = 1$ everywhere (a dot denoting d/du). We found that

$$E = 1,\ F = 0,\ G = f^2,\ \ L = \dot{f}\ddot{g} - \ddot{f}\dot{g},\ M = 0,\ N = f\dot{g}.$$

By Corollary 8.1.3, the Gaussian curvature is

$$K = \frac{LN - M^2}{EG - F^2} = \frac{(\dot{f}\ddot{g} - \ddot{f}\dot{g})f\dot{g}}{f^2}.$$

We can simplify this formula by noting that $\dot{f}^2 + \dot{g}^2 = 1$ implies (by differentiating with respect to u) that $\dot{f}\ddot{f} + \dot{g}\ddot{g} = 0$,

$$\begin{aligned} \therefore \quad (\dot{f}\ddot{g} - \ddot{f}\dot{g})\dot{g} &= -\dot{f}^2\ddot{f} - \ddot{f}\dot{g}^2 = -\ddot{f}(\dot{f}^2 + \dot{g}^2) = -\ddot{f}, \\ \therefore \quad K &= -\frac{\ddot{f}f}{f^2} = -\frac{\ddot{f}}{f}. \end{aligned}$$

We consider some special cases. If $\boldsymbol{\gamma}(u) = (u, 0, 0)$ is the x-axis, the corresponding surface of revolution is the xy-plane; since $f(u) = u$, we have $\dot{f} = 1$, $\ddot{f} = 0$, so $K = 0$. If $\boldsymbol{\gamma}(u) = (1, 0, u)$ is a straight line parallel to the z-axis, the corresponding surface is the unit cylinder; since $f(u) = 1$, $\ddot{f} = 0$, so $K = 0$. Finally, if $\boldsymbol{\gamma}(u) = (\cos u, 0, \sin u)$ is a circle of radius 1, the corresponding surface is the unit sphere; since $f(u) = \cos u$, $\dot{f} = -\sin u$, $\ddot{f} = -\cos u$ so $K = -\ddot{f}/f = -(-\cos u)/\cos u = 1$. Note that in each of these examples the curve $\boldsymbol{\gamma}$ is unit-speed.

Example 8.1.5

For a ruled surface, take a patch

$$\boldsymbol{\sigma}(u, v) = \boldsymbol{\gamma}(u) + v\boldsymbol{\delta}(u),$$

(see Example 5.3.1). Denoting d/du by a dot, we have $\boldsymbol{\sigma}_u = \dot{\boldsymbol{\gamma}} + v\dot{\boldsymbol{\delta}}$, $\boldsymbol{\sigma}_v = \boldsymbol{\delta}$, so

$$\boldsymbol{\sigma}_{uv} = \dot{\boldsymbol{\delta}}, \quad \boldsymbol{\sigma}_{vv} = \mathbf{0}.$$

Hence, if $\mathbf{N} = (\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v)/ \parallel \boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v \parallel$ is the standard unit normal of $\boldsymbol{\sigma}$, then $M = \boldsymbol{\sigma}_{uv} \cdot \mathbf{N} = \dot{\boldsymbol{\delta}} \cdot \mathbf{N}$ and $N = 0$. So

$$K = \frac{LN - M^2}{EG - F^2} = \frac{-(\dot{\boldsymbol{\delta}} \cdot \mathbf{N})^2}{EG - F^2} \leq 0,$$

i.e., *the Gaussian curvature of a ruled surface is negative or zero.*

Gauss discovered a way to obtain the Gaussian curvature from the Gauss map itself, rather than from its derivative, the Weingarten map. His result is an analogue of Proposition 2.2.3, which shows that, if $\boldsymbol{\gamma}$ is a unit-speed plane curve, its signed curvature $\kappa_s = \dot{\varphi}$, where φ is the angle between its tangent vector $\dot{\boldsymbol{\gamma}}$ and a fixed direction, i.e., the (signed) *curvature is the rate of change of direction of the tangent vector of* $\boldsymbol{\gamma}$ *per unit length.* The 'direction' of the tangent plane to an oriented surface $\mathcal{S}$ is measured by its unit normal $\mathbf{N}$, so we might expect that a measure of the curvature of $\boldsymbol{\sigma}$ is the 'rate of change of $\mathbf{N}$ per unit area'. The values of $\mathbf{N}$ at points of $\mathcal{S}$ are recorded by the Gauss map $\mathcal{G}$, so if R is a small region on $\mathcal{S}$ containing a point $\mathbf{p}$, we should look at the ratio

$$\frac{\text{Area}(\mathcal{G}(R))}{\text{Area}(R)}$$

in the limit as the region R shrinks down to the point $\mathbf{p}$.

To make this idea precise, we work in a surface patch.

Theorem 8.1.6

Let $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ be a surface patch, let $(u_0, v_0) \in U$, and let $\delta > 0$ be such that the closed disc

$$R_\delta = \{(u, v) \in \mathbb{R}^2 \mid (u - u_0)^2 + (v - v_0)^2 \leq \delta^2\}$$

with centre (u_0, v_0) and radius δ is contained in U. Then,

$$\lim_{\delta \to 0} \frac{\mathcal{A}_{\mathbf{N}}(R_\delta)}{\mathcal{A}_{\boldsymbol{\sigma}}(R_\delta)} = |K|,$$

where K is the Gaussian curvature of $\boldsymbol{\sigma}$ at $\boldsymbol{\sigma}(u_0, v_0)$.

Note that a δ with the properties in the statement of the theorem exists because U is open.

Proof

By Definition 6.4.1,

$$\frac{\mathcal{A}_{\mathbf{N}}(R_\delta)}{\mathcal{A}_{\boldsymbol{\sigma}}(R_\delta)} = \frac{\int_{R_\delta} \| \mathbf{N}_u \times \mathbf{N}_v \| \; dudv}{\int_{R_\delta} \| \boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v \| \; dudv}. \tag{8.1}$$

In the notation of the proof of Proposition 8.1.2,

$$\begin{aligned}
\mathbf{N}_u \times \mathbf{N}_v &= (a\boldsymbol{\sigma}_u + b\boldsymbol{\sigma}_v) \times (c\boldsymbol{\sigma}_u + d\boldsymbol{\sigma}_v) \\
&= (ad - bc)\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v \\
&= \det(\mathcal{F}_I^{-1}\mathcal{F}_{II})\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v \\
&= \frac{\det(\mathcal{F}_{II})}{\det(\mathcal{F}_I)}\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v \\
&= \frac{LN - M^2}{EG - F^2}\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v \\
&= K\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v \quad \text{(by Corollary 8.1.3).}
\end{aligned} \tag{8.2}$$

Substituting in Eq. 8.1, we get

$$\frac{\mathcal{A}_{\mathbf{N}}(R_\delta)}{\mathcal{A}_{\boldsymbol{\sigma}}(R_\delta)} = \frac{\int_{R_\delta} |K| \, \| \boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v \| \; dudv}{\int_{R_\delta} \| \boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v \| \; dudv}.$$

Let ϵ be any positive number. Since $K(u, v)$ is a continuous function of (u, v) (see Exercise 8.1.3), we can choose $\delta > 0$ so small that

$$|K(u, v) - K(u_0, v_0)| < \epsilon$$

if $(u, v) \in R_\delta$. Since, for any real numbers a, b, $|a - b| \geq ||a| - |b||$, it follows that $||K(u, v)| - |K(u_0, v_0)|| < \epsilon$ if $(u, v) \in R_\delta$, i.e.,

$$|K(u_0, v_0)| - \epsilon < |K(u, v)| < |K(u_0, v_0)| + \epsilon$$

if $(u, v) \in R_\delta$. Multiplying through by $\| \boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v \|$ and integrating over R_δ, we get

$$\begin{aligned}
(|K(u_0, v_0)| - \epsilon)\int \| \boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v \| \, dudv &< \int |K(u, v)| \, \| \boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v \| \, dudv \\
&< (|K(u_0, v_0)| + \epsilon)\int \| \boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v \| \, dudv, \\
\therefore \quad |K(u_0, v_0)| - \epsilon < \frac{\mathcal{A}_{\mathbf{N}}(R_\delta)}{\mathcal{A}_{\boldsymbol{\sigma}}(R_\delta)} &< |K(u_0, v_0)| + \epsilon \quad \text{(using Eq. 8.1)} \\
\therefore \quad \left| \frac{\mathcal{A}_{\mathbf{N}}(R_\delta)}{\mathcal{A}_{\boldsymbol{\sigma}}(R_\delta)} - |K(u_0, v_0)| \right| &< \epsilon.
\end{aligned}$$

This proves the theorem. □

Although this proposition only gives the absolute value of the Gaussian curvature K, the sign can be recovered from the Gauss map if we define the *signed area* of $\mathcal{G}(R)$ to be $\pm\mathcal{A}_{\mathbf{N}}(R)$, where the sign is $+$ or $-$ according to whether $\mathbf{N}_u \times \mathbf{N}_v$ points in the same or the opposite direction as $\mathbf{N}$. By Eq. 8.3, this sign is that of K, so K is the limit of the ratio

$$\frac{\text{Signed area}(\mathcal{G}(R))}{\text{Area}(R)}$$

as the region R shrinks to the point $\mathbf{p}$.

As the following examples show, Theorem 8.1.6 sometimes allows one to find the Gaussian curvature of a surface with no calculation.

Example 8.1.7

For a plane, the unit normal is constant. Thus, for any R, $\mathcal{G}(R)$ is a single point, and thus has zero area. By the theorem, a plane has Gaussian curvature zero everywhere.

For a generalized cylinder, the unit normal is clearly always perpendicular to the rulings of the cylinder, so the image of the Gauss map is contained in the great circle on S^2 formed by intersecting S^2 with the plane passing through its centre perpendicular to the rulings of the cylinder. Any great circle obviously has zero area, so the cylinder has zero Gaussian curvature too.

Finally, for the unit sphere S^2 itself, the unit normal at a point $\mathbf{p}$ is clearly parallel to the radius vector from the centre of the sphere to $\mathbf{p}$. In other words, the Gauss map is the identity map or the antipodal map (depending on the choice of orientation). Both of these maps are obviously equiareal, so the absolute value of the Gaussian curvature of S^2 is 1. In fact, if $\boldsymbol{\sigma}$ is any surface patch of S^2, we have $\mathbf{N} = \pm\boldsymbol{\sigma}$ so with either choice of sign $\mathbf{N}_u \times \mathbf{N}_v = \boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v$ is a positive multiple of $\mathbf{N}$ and the Gaussian curvature is $+1$.

EXERCISES

8.1.1 Show that the Gaussian and mean curvatures of the surface $z = f(x, y)$, where f is a smooth function, are

$$K = \frac{f_{xx}f_{yy} - f_{xy}^2}{(1 + f_x^2 + f_y^2)^2}, \quad H = \frac{(1 + f_y^2)f_{xx} - 2f_x f_y f_{xy} + (1 + f_x^2)f_{yy}}{2(1 + f_x^2 + f_y^2)^{3/2}}.$$

8.1.2 Calculate the Gaussian curvature of the helicoid and catenoid (Exercises 4.2.6 and 5.3.1).

8.1.3 Show that the Gaussian and mean curvatures of a surface $\mathcal{S}$ are smooth functions on $\mathcal{S}$.

8.1.4 In the notation of Example 8.1.5, show that if $\boldsymbol{\delta}$ is the principal normal $\mathbf{n}$ of $\boldsymbol{\gamma}$ or its binormal $\mathbf{b}$, then $K = 0$ if and only if $\boldsymbol{\gamma}$ is planar.

8.1.5 What is the effect on the Gaussian and mean curvatures of a surface $\mathcal{S}$ if we apply a dilation of $\mathbb{R}^3$ to $\mathcal{S}$?

8.1.6 Show that the Weingarten map $\mathcal{W}$ of a surface satisfies the quadratic equation

$$\mathcal{W}^2 - 2H\mathcal{W} + K = 0,$$

in the usual notation.

8.1.7 Show that the image of the Gauss map of a generalized cone is a *curve* on S^2, and deduce that the cone has zero Gaussian curvature.

8.1.8 Let $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ be a patch of a surface $\mathcal{S}$. Show that the image under the Gauss map of the part $\boldsymbol{\sigma}(R)$ of $\mathcal{S}$ corresponding to a region $R \subseteq U$ has area

$$\int_R |K| d\mathcal{A}_{\boldsymbol{\sigma}},$$

where K is the Gaussian curvature of $\mathcal{S}$.

8.1.9 Let $\mathcal{S}$ be the torus in Exercise 4.2.5. Describe the parts $\mathcal{S}^+$ and $\mathcal{S}^-$ of $\mathcal{S}$ where the Gaussian curvature K of $\mathcal{S}$ is positive and negative, respectively. Show, without calculation, that

$$\int_{\mathcal{S}^+} K\, d\mathcal{A} = -\int_{\mathcal{S}^-} K\, d\mathcal{A} = 4\pi.$$

It follows that $\int_{\mathcal{S}} K\, d\mathcal{A} = 0$, a result that will be 'explained' in Section 13.4.

8.1.10 Let $\mathbf{w}(u, v)$ be a *smooth tangent vector field on a surface patch* $\boldsymbol{\sigma}(u, v)$. This means that

$$\mathbf{w}(u, v) = \alpha(u, v)\boldsymbol{\sigma}_u + \beta(u, v)\boldsymbol{\sigma}_v$$

where α and β are smooth functions of (u, v). Then, if $\boldsymbol{\gamma}(t) = \boldsymbol{\sigma}(u(t), v(t))$ is any curve on $\boldsymbol{\sigma}$, $\mathbf{w}$ gives rise to the tangent vector field $\mathbf{w}|_{\boldsymbol{\gamma}}(t) = \mathbf{w}(u(t), v(t))$ along $\boldsymbol{\gamma}$. Let $\nabla_u \mathbf{w}$ be the covariant derivative of $\mathbf{w}|_{\boldsymbol{\gamma}}$ along a parameter curve $v =$ constant, and define $\nabla_v \mathbf{w}$ similarly. (Note that if $\boldsymbol{\sigma}$ is the uv-plane, then ∇_u and ∇_v become $\partial/\partial u$ and $\partial/\partial v$). Show that

$$\nabla_v(\nabla_u \mathbf{w}) - \nabla_u(\nabla_v \mathbf{w}) = (\mathbf{w}_v \cdot \mathbf{N})\mathbf{N}_u - (\mathbf{w}_u \cdot \mathbf{N})\mathbf{N}_v,$$

where $\mathbf{N}$ is the unit normal of $\boldsymbol{\sigma}$. Deduce that, if $\lambda(u, v)$ is a smooth function of (u, v), then

$$\nabla_v(\nabla_u(\lambda \mathbf{w})) - \nabla_u(\nabla_v(\lambda \mathbf{w})) = \lambda\left(\nabla_v(\nabla_u \mathbf{w}) - \nabla_u(\nabla_v \mathbf{w})\right).$$

Use Proposition 8.1.2 to show that

$$\nabla_v(\nabla_u \boldsymbol{\sigma}_u) - \nabla_u(\nabla_v \boldsymbol{\sigma}_u) = K(-F\boldsymbol{\sigma}_u + E\boldsymbol{\sigma}_v),$$

where

$$K = \frac{LN - M^2}{EG - F^2},$$

and find a similar expression for $\nabla_v(\nabla_u \boldsymbol{\sigma}_v) - \nabla_u(\nabla_v \boldsymbol{\sigma}_v)$. Deduce that

$$\nabla_v(\nabla_u \mathbf{w}) = \nabla_u(\nabla_v \mathbf{w})$$

for all tangent vector fields $\mathbf{w}$ if and only if $K = 0$ everywhere on the surface. (Note that this holds for the plane: $\mathbf{w}_{uv} = \mathbf{w}_{vu}$.) We shall see the significance of the condition $K = 0$ in Section 8.4.

8.2 Principal curvatures of a surface

We now examine the Weingarten map $\mathcal{W}_{\mathbf{p},\mathcal{S}}$ of a surface $\mathcal{S}$ at a point $\mathbf{p} \in \mathcal{S}$ in a little more detail (we shall usually omit the subscripts). The crucial point is that $\mathcal{W}$ is *self-adjoint* (Corollary 7.2.4). From Theorem A.0.3 we deduce the following proposition.

Proposition 8.2.1

Let $\mathbf{p}$ be a point of a surface $\mathcal{S}$. There are scalars κ_1, κ_2 and a basis $\{\mathbf{t}_1, \mathbf{t}_2\}$ of the tangent plane $T_{\mathbf{p}}\mathcal{S}$ such that

$$\mathcal{W}(\mathbf{t}_1) = \kappa_1 \mathbf{t}_1, \quad \mathcal{W}(\mathbf{t}_2) = \kappa_2 \mathbf{t}_2.$$

Moreover, if $\kappa_1 \neq \kappa_2$, then $\langle \mathbf{t}_1, \mathbf{t}_2 \rangle = 0$.

The real numbers κ_1 and κ_2 are the eigenvalues of $\mathcal{W}$, and $\mathbf{t}_1$ and $\mathbf{t}_2$ are corresponding eigenvectors. But in this situation, we adopt a special terminology: κ_1 and κ_2 are called the *principal curvatures* of $\mathcal{S}$, and $\mathbf{t}_1$ and $\mathbf{t}_2$ are called *principal vectors* corresponding to κ_1 and κ_2.

Points of the surface at which the two principal curvatures are equal (to κ, say) are called *umbilics*. At an umbilic, the equations $\mathcal{W}(\mathbf{t}_1) = \kappa \mathbf{t}_1$ and $\mathcal{W}(\mathbf{t}_2) = \kappa \mathbf{t}_2$ imply that $\mathcal{W}(\mathbf{t}) = \kappa \mathbf{t}$ if $\mathbf{t}$ is any linear combination of $\mathbf{t}_1$ and $\mathbf{t}_2$. Thus, $\mathbf{p}$ *is an umbilic if and only if* $\mathcal{W}_{\mathbf{p},\mathcal{S}}$ *is a scalar multiple of the identity map*, and in that case every tangent vector is principal. On the other hand, if $\mathbf{p} \in \mathcal{S}$ is not an umbilic, Proposition 8.2.1 tells us that principal vectors corresponding to the two principal curvatures are necessarily orthogonal (Theorem A.0.3). Thus, whether or not $\mathbf{p}$ is an umbilic we can always find two orthogonal principal vectors in $T_{\mathbf{p}}\mathcal{S}$, and we obtain:

Corollary 8.2.2

If $\mathbf{p}$ is a point of a surface $\mathcal{S}$, there is an orthonormal basis of the tangent plane $T_{\mathbf{p}}\mathcal{S}$ consisting of principal vectors.

The principal curvatures are related in a simple way to the mean and Gaussian curvatures:

Proposition 8.2.3

If κ_1 and κ_2 are the principal curvatures of a surface, the mean and Gaussian curvatures are given by

$$H = \frac{1}{2}(\kappa_1 + \kappa_2), \quad K = \kappa_1\kappa_2.$$

Proof

The determinant and trace of the Weingarten map $\mathcal{W}$ can be computed using the matrix of $\mathcal{W}$ with respect to *any* basis of the tangent plane. Using the basis formed by the principal vectors, the matrix is

$$\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}.$$

The proposition now follows immediately from Definition 8.1.1. □

One reason for introducing the principal curvatures and principal vectors is contained in the following result, which shows that, if we know the principal curvatures and principal vectors of a surface, it is easy to calculate the normal curvature of any curve on the surface:

Euler's Theorem 8.2.4

Let $\boldsymbol{\gamma}$ be a curve on an oriented surface $\mathcal{S}$, and let κ_1 and κ_2 be the principal curvatures of $\boldsymbol{\sigma}$, with non-zero principal vectors $\mathbf{t}_1$ and $\mathbf{t}_2$. Then, the normal curvature of $\boldsymbol{\gamma}$ is

$$\kappa_n = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta,$$

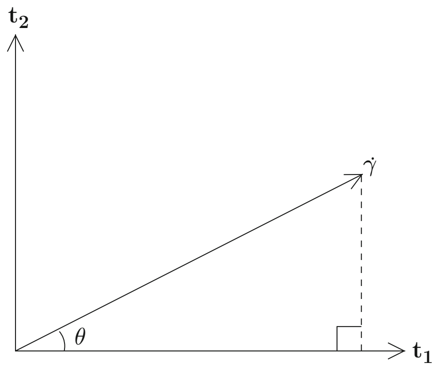
where θ is the oriented angle $\widehat{\mathbf{t}_1\dot{\boldsymbol{\gamma}}}$.

Proof

Let $\mathbf{p} \in \mathcal{S}$, let κ_1 and κ_2 be the principal curvatures of $\mathcal{S}$ at $\mathbf{p}$, and let $\mathbf{t}_1$ and $\mathbf{t}_2$ be corresponding principal vectors. By Corollary 8.2.2, we can assume that $\{\mathbf{t}_1, \mathbf{t}_2\}$ is an orthonormal basis of $T_\mathbf{p}\mathcal{S}$. Moreover, by replacing $\mathbf{t}_2$ by $-\mathbf{t}_2$ if necessary, we can assume that the oriented angle $\widehat{\mathbf{t}_1\mathbf{t}_2} = +\pi/2$.

With these assumptions, we have

$$\dot{\boldsymbol{\gamma}} = \cos\theta\mathbf{t}_1 + \sin\theta\mathbf{t}_2.$$



By Proposition 7.3.5,

$$\kappa_n = \langle\langle\dot{\boldsymbol{\gamma}}, \dot{\boldsymbol{\gamma}}\rangle\rangle = \cos^2\theta\langle\langle\mathbf{t}_1, \mathbf{t}_1\rangle\rangle + 2\sin\theta\cos\theta\langle\langle\mathbf{t}_1, \mathbf{t}_2\rangle\rangle + \sin^2\theta\langle\langle\mathbf{t}_2, \mathbf{t}_2\rangle\rangle.$$

Now, for $i, j = 1, 2$,

$$\langle\langle\mathbf{t}_i, \mathbf{t}_j\rangle\rangle = \langle\mathcal{W}(\mathbf{t}_i), \mathbf{t}_j\rangle = \langle\kappa_i\mathbf{t}_i, \mathbf{t}_j\rangle = \begin{cases}\kappa_i & \text{if } i = j\\ 0 & \text{if } i \neq j\end{cases}.$$

Hence the result. □

Corollary 8.2.5

The principal curvatures at a point of a surface are the maximum and minimum values of the normal curvature of all curves on the surface that pass through the point. Moreover, the principal vectors are the tangent vectors of the curves giving these maximum and minimum values.

Proof

If the principal curvatures κ_1 and κ_2 are different, we might as well suppose that $\kappa_1 > \kappa_2$. Let κ_n be the normal curvature of a curve $\boldsymbol{\gamma}$ on the surface. Then, since

$$\kappa_n = \kappa_1\cos^2\theta + \kappa_2\sin^2\theta = \kappa_1 - (\kappa_1 - \kappa_2)\sin^2\theta,$$

it is clear that $\kappa_n \leq \kappa_1$ with equality if and only if $\theta = 0$ or π, i.e., if and only if the tangent vector $\dot{\boldsymbol{\gamma}}$ of $\boldsymbol{\gamma}$ is parallel to the principal vector $\mathbf{t}_1$. Similarly, one shows that $\kappa_n \geq \kappa_2$ with equality if and only if $\dot{\boldsymbol{\gamma}}$ is parallel to $\mathbf{t}_2$.

If $\kappa_1 = \kappa_2$, the normal curvature of every curve is equal to κ_1 by Euler's Theorem and every tangent vector to the surface is a principal vector. □

To compute the principal curvatures, we work in a surface patch $\boldsymbol{\sigma}(u,v)$; let

$$Edu^2 + 2Fdudv + Gdv^2 \quad \text{and} \quad Ldu^2 + 2Mdudv + Ndv^2$$

be its first and second fundamental forms. In the notation of Section 8.1, the matrix of the Weingarten map $\mathcal{W}$ with respect to the basis $\{\boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v\}$ of the tangent plane is $\mathcal{F}_I^{-1}\mathcal{F}_{II}$. Hence, the principal curvatures are the roots κ of the equation

$$\det(\mathcal{F}_I^{-1}\mathcal{F}_{II} - \kappa I) = 0,$$

and a tangent vector $\mathbf{t} = \xi\boldsymbol{\sigma}_u + \eta\boldsymbol{\sigma}_v$ is a principal vector if

$$(\mathcal{F}_I^{-1}\mathcal{F}_{II} - \kappa I)\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Writing $\mathcal{F}_I^{-1}\mathcal{F}_{II} - \kappa I$ as $\mathcal{F}_I^{-1}(\mathcal{F}_{II} - \kappa\mathcal{F}_I)$, we obtain the following.

Proposition 8.2.6

In the above notation, the principal curvatures are the roots of the equation

$$\begin{vmatrix} L - \kappa E & M - \kappa F \\ M - \kappa F & N - \kappa G \end{vmatrix} = 0,$$

and the principal vectors corresponding to the principal curvature κ are the tangent vectors $\mathbf{t} = \xi\boldsymbol{\sigma}_u + \eta\boldsymbol{\sigma}_v$ such that

$$\begin{pmatrix} L - \kappa E & M - \kappa F \\ M - \kappa F & N - \kappa G \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Example 8.2.7

It is intuitively clear that a sphere curves the same amount in every direction, and at every point of the sphere. Thus, we expect that the principal curvatures of a sphere are equal to each other at every point, and are constant over the sphere. To confirm this by calculation, we work with the unit sphere S^2 and use the latitude longitude parametrization as usual. We found in Example 6.1.3 that $E = 1, F = 0, G = \cos^2\theta$ and in Example 7.1.2 that $L = 1, M = 0, N = \cos^2\theta$. So the principal curvatures are the roots of

$$\begin{vmatrix} 1 - \kappa & 0 \\ 0 & \cos^2\theta - \kappa\cos^2\theta \end{vmatrix} = 0,$$

i.e., $\kappa = 1$ (repeated root), as we expected. Every tangent vector is a principal vector.

Example 8.2.8

We consider the unit cylinder parametrized in the usual way:

$$\boldsymbol{\sigma}(u,v) = (\cos v, \sin v, u).$$

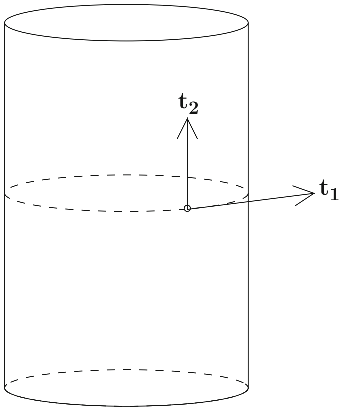
We found in Example 6.1.4 that $E=1, F=0, G=1$ and in Example 7.1.2 that $L=0, M=0, N=1$. So the principal curvatures are the roots of

$$\begin{vmatrix} 0-\kappa & 0 \\ 0 & 1-\kappa \end{vmatrix} = 0,$$

i.e., $\kappa = 0$ or 1. Any principal vector $\mathbf{t}_1$ corresponding to $\kappa_1 (=1)$ satisfies

$$\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix} = 0,$$

so $\xi_1 = 0$ and $\mathbf{t}_1$ is a multiple of $\boldsymbol{\sigma}_v = (-\sin v, \cos v, 0)$. Similarly, one finds that any principal vector corresponding to κ_2 $(=0)$ is a multiple of $\boldsymbol{\sigma}_u = (0,0,1)$.



Example 8.2.7 proves the intuitively obvious fact that on a sphere every point is an umbilic. The same is clearly true for a plane, since in that case both principal curvatures are zero everywhere. Remarkably, there are no other surfaces with this property:

Proposition 8.2.9

Let $\mathcal{S}$ be a (connected) surface of which every point is an umbilic. Then, $\mathcal{S}$ is an open subset of a plane or a sphere.

Proof

For every tangent vector $\mathbf{t}$, we have $\mathcal{W}(\mathbf{t}) = \kappa\mathbf{t}$ where κ is the principal curvature. Let $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ be a surface patch of $\mathcal{S}$ with U a (connected) open subset of $\mathbb{R}^2$. Taking $\mathbf{t} = \boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$ and recalling from the proof of Proposition 7.2.2 that $\mathcal{W}(\boldsymbol{\sigma}_u) = -\mathbf{N}_u, \mathcal{W}(\boldsymbol{\sigma}_v) = -\mathbf{N}_v$, we get

$$\mathbf{N}_u = -\kappa\boldsymbol{\sigma}_u, \quad \mathbf{N}_v = -\kappa\boldsymbol{\sigma}_v. \tag{8.3}$$

Hence,

$$(\kappa\boldsymbol{\sigma}_u)_v = -(\mathbf{N}_u)_v = -(\mathbf{N}_v)_u = (\kappa\boldsymbol{\sigma}_v)_u,$$

so

$$\kappa_v\boldsymbol{\sigma}_u = \kappa_u\boldsymbol{\sigma}_v.$$

Since $\boldsymbol{\sigma}$ is regular, $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$ are linearly independent, so the last equation implies that $\kappa_u = \kappa_v = 0$. Thus, κ is constant.

There are now two cases to consider. If $\kappa = 0$, Eqs. 8.3 show that $\mathbf{N}$ is constant. Then,

$$(\mathbf{N}\cdot\boldsymbol{\sigma})_u = \mathbf{N}\cdot\boldsymbol{\sigma}_u = 0, \quad (\mathbf{N}\cdot\boldsymbol{\sigma})_v = \mathbf{N}\cdot\boldsymbol{\sigma}_v = 0,$$

so $\mathbf{N}\cdot\boldsymbol{\sigma}$ is a constant, say c. Then $\boldsymbol{\sigma}(U)$ is an open subset of the plane $\mathbf{v}\cdot\mathbf{N} = c$.

If $\kappa \neq 0$, Eq. 8.3 shows that

$$\mathbf{N} = -\kappa\boldsymbol{\sigma} + \mathbf{a},$$

where $\mathbf{a}$ is a constant vector. Hence,

$$\left\|\boldsymbol{\sigma} - \frac{1}{\kappa}\mathbf{a}\right\|^2 = \left\|-\frac{1}{\kappa}\mathbf{N}\right\|^2 = \frac{1}{\kappa^2},$$

so $\boldsymbol{\sigma}(U)$ is an open subset of the sphere with centre $\kappa^{-1}\mathbf{a}$ and radius κ^{-1}.

We have now proved the proposition when $\mathcal{S}$ is covered by a single surface patch. For an arbitrary surface $\mathcal{S}$, the preceding argument shows that each patch in the atlas of $\mathcal{S}$ is contained in a plane or a sphere. But if the images of two patches intersect they must clearly be part of the same plane or the same sphere. It follows that the whole of $\mathcal{S}$ is contained in a plane or a sphere. □

Note that this proposition is an analogue for surfaces of Example 2.2.7, which tells us that a plane curve with constant curvature is part of a circle.

We conclude this section by showing how the values of the principal curvatures at a point $\mathbf{p}$ of a surface $\mathcal{S}$ provide information about the shape of $\mathcal{S}$ near $\mathbf{p}$. To simplify the situation, we assume that $\mathbf{p}$ is the origin and that $T_{\mathbf{p}}\mathcal{S}$ is the xy-plane: this can be arranged by applying a suitable isometry of $\mathbb{R}^3$ to $\mathcal{S}$ (which does not change its shape). By a further rotation around the z-axis, we can also assume that the tangent vectors $\mathbf{t}_1 = (1, 0, 0)$ and $\mathbf{t}_2 = (0, 1, 0)$ are principal, and correspond to principal curvatures κ_1 and κ_2. Finally, by reflecting in the xy-plane if necessary, we can assume that the unit normal of $\mathcal{S}$ at $\mathbf{p}$ is $\mathbf{N} = (0, 0, 1)$.

Let $\boldsymbol{\sigma}$ be a surface patch of $\mathcal{S}$ with $\boldsymbol{\sigma}(0, 0) = \mathbf{0}$. For any $x, y \in \mathbb{R}$, there are unique $s, t \in \mathbb{R}$ such that

$$(x, y, 0) = s\boldsymbol{\sigma}_u + t\boldsymbol{\sigma}_v$$

(here and below, the derivatives of $\boldsymbol{\sigma}$ are evaluated at $(0, 0)$). By Taylor's theorem,

$$\boldsymbol{\sigma}(s, t) = \boldsymbol{\sigma}(0, 0) + s\boldsymbol{\sigma}_u + t\boldsymbol{\sigma}_v + \frac{1}{2}(s^2\boldsymbol{\sigma}_{uu} + 2st\boldsymbol{\sigma}_{uv} + t^2\boldsymbol{\sigma}_{vv})$$

if we neglect terms involving higher powers of s and t. Hence, if x and y (and hence s and t) are small, we have $\boldsymbol{\sigma}(s, t) = (x, y, z)$, where

$$z = \frac{1}{2}(s^2\boldsymbol{\sigma}_{uu} + 2st\boldsymbol{\sigma}_{uv} + t^2\boldsymbol{\sigma}_{vv}) \cdot \mathbf{N} = \frac{1}{2}(Ls^2 + 2Mst + Nt^2)$$

approximately, where $Ldu^2 + 2Mdudv + Ndv^2$ is the second fundamental form of $\boldsymbol{\sigma}$ at the origin. If $\mathbf{t} = s\boldsymbol{\sigma}_u + t\boldsymbol{\sigma}_v$, then by Proposition 7.3.3,

$$Ls^2 + 2Mst + Nt^2 = \langle\langle \mathbf{t}, \mathbf{t} \rangle\rangle = \langle \mathcal{W}(\mathbf{t}), \mathbf{t} \rangle.$$

Now, $\mathbf{t} = x\mathbf{t}_1 + y\mathbf{t}_2$ so

$$\mathcal{W}(\mathbf{t}) = x\mathcal{W}(\mathbf{t}_1) + y\mathcal{W}(\mathbf{t}_2) = \kappa_1 x\mathbf{t}_1 + \kappa_2 y\mathbf{t}_2 = (\kappa_1 x, \kappa_2 y, 0).$$

Hence,

$$Ls^2 + 2Mst + Nt^2 = (\kappa_1 x, \kappa_2 y, 0) \cdot (x, y, 0) = \kappa_1 x^2 + \kappa_2 y^2.$$

Hence, near the point $\mathbf{p}$, $\mathcal{S}$ is approximated by the quadric surface

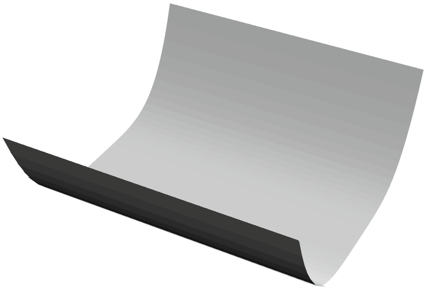
$$z = \frac{1}{2}(\kappa_1 x^2 + \kappa_2 y^2). \tag{8.4}$$

We distinguish four cases:

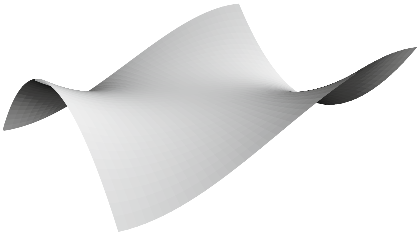
(i) *κ_1 and κ_2 are both > 0 or both < 0.* Then, (8.4) is the equation of an elliptic paraboloid (see Theorem 5.2.2) and one says that $\mathbf{p}$ is an *elliptic point* of the surface.

(ii) *κ_1 and κ_2 are of opposite sign (both non-zero).* Then, (8.4) is the equation of a hyperbolic paraboloid and one says that $\mathbf{p}$ is a *hyperbolic point* of the surface.

(iii) *One of* κ_1 *and* κ_2 *is zero, the other is non-zero.* Then, (8.4) is the equation of a parabolic cylinder and one says that $\mathbf{p}$ is a *parabolic point* of the surface.



$z = y^4$



$z = x^3 - 3xy^2$

(iv) *Both principal curvatures are zero at* $\boldsymbol{p}$. Then, (8.4) is the equation of a plane, and one says that $\mathbf{p}$ is a *planar point* of the surface. In this case, one cannot determine the shape of the surface near $\mathbf{p}$ without examining derivatives of order higher than the second (in the non-planar case, these terms are small compared to $\kappa_1 x^2 + \kappa_2 y^2$ when x and y are small). For example, the surfaces above both have the origin as a planar point, but they have quite different shapes. (The surface on the right is called the *monkey saddle* as it is the right shape for the saddle on a bicycle ridden by a monkey: two ways down for the two legs and a third for the tail.)

The classification of points of a surface as elliptic, hyperbolic, parabolic or planar is independent of the surface patch $\boldsymbol{\sigma}$, since reparametrizing either leaves the principal curvatures unchanged or changes the sign of both of them (Exercise 8.2.8).

Example 8.2.10

On S^2, $\kappa_1 = \kappa_2 = \pm 1$ (the sign depending on the parametrization) so all points are elliptic (and umbilics). On a circular cylinder, $\kappa_1 = \pm 1, \kappa_2 = 0$, so every point is parabolic (and there are no umbilics). On a plane, $\kappa_1 = \kappa_2 = 0$ so all points are planar (!) (and umbilics).

Example 8.2.11

For the torus $\boldsymbol{\sigma}(\theta, \varphi) = ((a+b\cos\theta)\cos\varphi, (a+b\cos\theta)\sin\varphi, b\sin\theta)$ (see Exercise 4.2.5), we find that the first and second fundamental forms are

$$b^2 d\theta^2 + (a + b\cos\theta)^2 d\varphi^2 \quad \text{and} \quad b\, d\theta^2 + (a + b\cos\theta)\cos\theta\, d\varphi^2,$$

respectively, so the principal curvatures are

$$\kappa_1 = \frac{1}{b}, \quad \kappa_2 = \frac{\cos\theta}{a + b\cos\theta}.$$

Since $\kappa_1 > 0$ (everywhere), the point $\boldsymbol{\sigma}(\theta, \varphi)$ of the torus is elliptic, parabolic or hyperbolic according to κ_2 is > 0, $= 0$ or < 0, respectively; from the formula for κ_2, these are the regions of the torus given by $-\pi/2 < \theta < \pi/2$, $\theta = \pm\pi/2$ and $\pi/2 < \theta < 3\pi/2$, respectively. Pictures of the elliptic and hyperbolic regions can be found in the solution to Exercise 8.1.9 (where they are labelled $\mathcal{S}^+$ and $\mathcal{S}^-$, respectively); the parabolic region consists of two circles of radius a centred on the z-axis.

EXERCISES

8.2.1 Calculate the principal curvatures of the helicoid and the catenoid, defined in Exercises 4.2.6 and 5.3.1, respectively.

8.2.2 A curve $\boldsymbol{\gamma}$ on a surface $\mathcal{S}$ is called a *line of curvature* if the tangent vector of $\boldsymbol{\gamma}$ is a principal vector of $\mathcal{S}$ at all points of $\boldsymbol{\gamma}$ (a 'line' of curvature need not be a straight line!). Show that $\boldsymbol{\gamma}$ is a line of curvature if and only if

$$\dot{\mathbf{N}} = -\lambda\dot{\boldsymbol{\gamma}},$$

for some scalar λ, where $\mathbf{N}$ is the standard unit normal of $\boldsymbol{\sigma}$, and that in this case the corresponding principal curvature is λ. (This is called *Rodrigues' formula.*)

8.2.3 Show that a curve $\boldsymbol{\gamma}(t) = \boldsymbol{\sigma}(u(t), v(t))$ on a surface patch $\boldsymbol{\sigma}$ is a line of curvature if and only if (in the usual notation)

$$(EM - FL)\dot{u}^2 + (EN - GL)\dot{u}\dot{v} + (FN - GM)\dot{v}^2 = 0.$$

Deduce that all parameter curves are lines of curvature if and only if either

(i) the second fundamental form of $\boldsymbol{\sigma}$ is proportional to its first fundamental form, or

(ii) $F = M = 0$.

For which surfaces does (i) hold? Show that the meridians and parallels of a surface of revolution are lines of curvature.

8.2.4 In the notation of Example 8.1.5, show that if $\boldsymbol{\gamma}$ is a curve on a surface $\mathcal{S}$ and $\boldsymbol{\delta}$ is the unit normal of $\mathcal{S}$, then $K = 0$ if and only if $\boldsymbol{\gamma}$ is a line of curvature of $\mathcal{S}$.

8.2.5 Suppose that two surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$ intersect in a curve $\mathcal{C}$ that is a line of curvature of $\mathcal{S}_1$. Show that $\mathcal{C}$ is a line of curvature of $\mathcal{S}_2$ if and only if the angle between the tangent planes of $\mathcal{S}_1$ and $\mathcal{S}_2$ is constant along $\mathcal{C}$.

8.2.6 Let $\boldsymbol{\Sigma} : W \to \mathbb{R}^3$ be a smooth function defined on an open subset W of $\mathbb{R}^3$ such that, for each fixed value of u (resp. v, w), $\boldsymbol{\Sigma}(u, v, w)$ is a (regular) surface patch. Assume also that

$$\boldsymbol{\Sigma}_u \cdot \boldsymbol{\Sigma}_v = \boldsymbol{\Sigma}_v \cdot \boldsymbol{\Sigma}_w = \boldsymbol{\Sigma}_w \cdot \boldsymbol{\Sigma}_u = 0. \qquad (8.5)$$

This means that the three families of surfaces formed by fixing the values of u, v or w constitute a triply orthogonal system (see Section 5.5).

(i) Show that $\boldsymbol{\Sigma}_u \cdot \boldsymbol{\Sigma}_{vw} = \boldsymbol{\Sigma}_v \cdot \boldsymbol{\Sigma}_{uw} = \boldsymbol{\Sigma}_w \cdot \boldsymbol{\Sigma}_{uv} = 0$.

(ii) Show that, for each of the surfaces in the triply orthogonal system, the matrices $\mathcal{F}_I$ and $\mathcal{F}_{II}$ are diagonal.

(iii) Deduce that the intersection of any surface from one family of the triply orthogonal system with any surface from another family is a line of curvature on both surfaces. (This is called *Dupin's Theorem*.)

8.2.7 Show that, if p, q and r are distinct positive numbers, there are exactly four umbilics on the ellipsoid

$$\frac{x^2}{p^2} + \frac{y^2}{q^2} + \frac{z^2}{r^2} = 1.$$

What happens if p, q and r are not distinct?

8.2.8 Show that the principal curvatures of a surface patch $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ are smooth functions on U provided that $\boldsymbol{\sigma}$ has no umbilics. Show also that the principal curvatures either stay the same or both change sign when $\boldsymbol{\sigma}$ is reparametrized.

8.3 Surfaces of constant Gaussian curvature

We have seen in the examples in Section 8.1 some surfaces of zero and constant positive curvature. For an example of a surface with *constant negative* Gaussian curvature, however, we have to construct a new surface. To this end, we examine again the surface of revolution

$$\boldsymbol{\sigma}(u, v) = (f(u)\cos v, f(u)\sin v, g(u))$$

obtained by rotating the unit-speed curve $u \mapsto (f(u), 0, g(u))$ in the xz-plane around the z-axis. We found in Example 8.1.4 that its Gaussian curvature is

$$K = -\frac{\ddot{f}}{f}. \tag{8.6}$$

Suppose first that $K = 0$ everywhere. Then, Eq. 8.6 gives $\ddot{f} = 0$, so $f(u) = au + b$ for some constants a and b. Since $\dot{f}^2 + \dot{g}^2 = 1$, we get $\dot{g} = \pm\sqrt{1 - a^2}$ (so we must have $|a| \leq 1$) and hence $g(u) = \pm\sqrt{1 - a^2}u + c$, where c is another constant. By applying a translation along the z-axis we can assume that $c = 0$, and by applying a rotation by π about the x-axis, if necessary, we can assume that the sign is $+$. This gives the ruled surface

$$\boldsymbol{\sigma}(u, v) = (b\cos v, b\sin v, 0) + u(a\cos v, a\sin v, \sqrt{1 - a^2}).$$

If $a = 0$ this is a circular cylinder; if $|a| = 1$ it is the xy-plane; and if $0 < |a| < 1$ it is a circular cone (to see this, put $\tilde{u} = au + b$).

Now suppose that $K > 0$, say $K = 1/R^2$, where $R > 0$ is a constant. Then, Eq. 8.6 becomes

$$\ddot{f} + \frac{f}{R^2} = 0,$$

which has the general solution

$$f(u) = a\cos\left(\frac{u}{R} + b\right),$$

where a and b are constants. We can assume that $b = 0$ by performing a reparametrization $\tilde{u} = u + Rb, \tilde{v} = v$. Then, up to a change of sign and adding a constant,

$$g(u) = \int \sqrt{1 - \frac{a^2}{R^2}\sin^2\frac{u}{R}}\,du.$$

The integral in the formula for $g(u)$ can be evaluated in terms of 'elementary' functions only when $a = 0$ or $\pm R$. The case $a = 0$ does not give a surface, and if $a = R$ then $f(u) = R\cos\frac{u}{R}$, $g(u) = R\sin\frac{u}{R}$, and we have a sphere of radius R (the case $a = -R$ can be reduced to this by rotating the surface by π around the z-axis).

Suppose finally that $K < 0$. We can restrict ourselves to the case $K = -1$, as the general case can be obtained from this by applying a dilation of $\mathbb{R}^3$ (see Exercise 8.1.5). In view of the preceding case, we can think of a surface with $K = -1$ as a 'sphere of imaginary radius' $\sqrt{-1}$, or a 'pseudosphere'.

When $K = -1$ the general solution of Eq. 8.6 is

$$f(u) = ae^u + be^{-u},$$

where a and b are arbitrary constants. The function $g(u)$ can be expressed in terms of elementary functions only if one of a or b is zero. If $b = 0$ we can assume that $a = 1$ by a reparametrization $u \mapsto u +$ constant, and the case in which $a = 0$ can be reduced to the case $b = 0$ by the reparametrization $u \mapsto -u$. Suppose then that $a = 1$ and $b = 0$; then, $f(u) = e^u$ and we can take

$$g(u) = \int \sqrt{1 - e^{2u}}\, du. \tag{8.7}$$

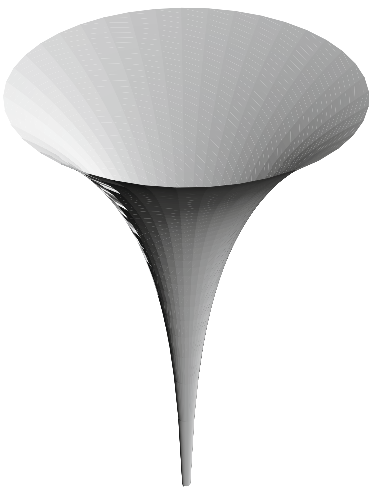
Note that we must have $u \le 0$ for the integral in Eq. 8.7 to make sense, since otherwise $1 - e^{2u}$ would be negative. The integral can be evaluated by putting $\cos\theta = e^u$. Then,

$$\begin{aligned}\int \sqrt{1 - e^{2u}}\, du = -\int \frac{\sin^2\theta}{\cos\theta}\, d\theta &= \sin\theta - \ln(\sec\theta + \tan\theta)\\ &= \sqrt{1 - e^{2u}} - \ln(e^{-u} + \sqrt{e^{-2u} - 1}).\end{aligned}$$

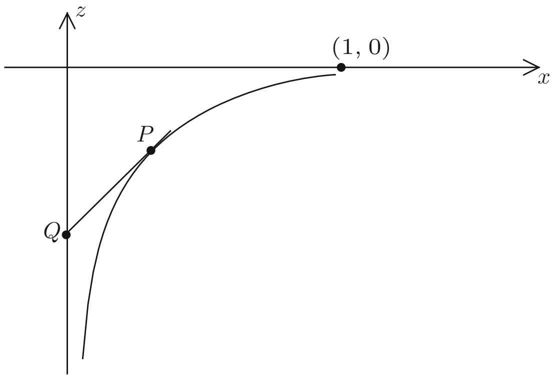
We have omitted the arbitrary constant, but we can take it to be zero by a suitable translation of the surface parallel to the z-axis. Putting $x = f(u)$, $z = g(u)$, and noting that $\cosh^{-1}(v) = \ln(v + \sqrt{v^2 - 1})$, we see that the profile curve in the xz-plane has equation

$$z = \sqrt{1 - x^2} - \cosh^{-1}\left(\frac{1}{x}\right). \tag{8.8}$$

Rotating this curve around the z -axis thus gives a surface which has Gaussian curvature -1 everywhere. Note that, since $u \le 0$, $x = e^u$ is restricted to the range $0 < x \le 1$.



The curve defined by Eq. 8.8 is called the *tractrix*, and it has an interesting geometrical property. Consider the tangent line at a point P of its graph, and suppose that it intersects the z-axis at the point Q. Let us compute the distance from P to Q.



Suppose that P is the point (x_0, z_0). Either by a direct calculation or by inspecting the calculation of the integral (8.7), one finds that

$$\frac{dz}{dx} = \frac{\sqrt{1-x^2}}{x}.$$

Hence, the tangent line at P has equation

$$z - z_0 = \frac{\sqrt{1-x_0^2}}{x_0}(x - x_0).$$

This meets the z-axis at the point $(0, z_1)$, where

$$z_1 - z_0 = \frac{\sqrt{1-x_0^2}}{x_0}(0 - x_0) = -\sqrt{1-x_0^2}.$$

Hence, the square of the distance from P to Q is

$$x_0^2 + (z_1 - z_0)^2 = x_0^2 + 1 - x_0^2 = 1,$$

so the distance from P to Q is *constant* and equal to 1.

This means that the tractrix has the following description. Let a donkey pull a box of stones by a rope of length 1. Suppose that the donkey is initially at $(0, 0)$, the box is initially at $(1, 0)$, and let the donkey walk slowly along the negative z-axis. Then, the box of stones moves along the tractrix.

EXERCISES

8.3.1 Show that:

(i) Setting $w = e^{-u}$ gives a reparametrization $\boldsymbol{\sigma}_1(v, w)$ of the pseudosphere with first fundamental form

$$\frac{dv^2 + dw^2}{w^2}$$

(called the *upper half-plane model*).

(ii) Setting

$$V = \frac{v^2 + w^2 - 1}{v^2 + (w+1)^2}, \quad W = \frac{-2v}{v^2 + (w+1)^2}$$

defines a reparametrization $\boldsymbol{\sigma}_2(V, W)$ of the pseudosphere with first fundamental form

$$\frac{4(dV^2 + dW^2)}{(1 - V^2 - W^2)^2}$$

(called the *Poincaré disc model*: the region $w > 0$ of the vw-plane corresponds to the disc $V^2 + W^2 < 1$ in the VW-plane).

(iii) Setting

$$\bar{V} = \frac{2V}{V^2 + W^2 + 1}, \quad \bar{W} = \frac{2W}{V^2 + W^2 + 1}$$

defines a reparametrization $\boldsymbol{\sigma}_2(\bar{V}, \bar{W})$ of the pseudosphere with first fundamental form

$$\frac{(1 - \bar{W}^2)d\bar{V}^2 + 2\bar{V}\bar{W}d\bar{V}d\bar{W} + (1 - \bar{V}^2)d\bar{W}^2}{(1 - \bar{V}^2 - \bar{W}^2)^2}$$

(called the *Beltrami-Klein model*: the region $w > 0$ of the vw-plane again corresponds to the disc $\bar{V}^2 + \bar{W}^2 < 1$ in the $\bar{V}\bar{W}$-plane).

In cases (i) and (ii), find the open subsets of the vw- and VW-plane, respectively, corresponding to the open set

$$\{(u, v) \mid u < 0, -\pi < v < \pi\}$$

in the parametrization of the pseudosphere given in the text.

These models are discussed in much more detail in Chapter 11.

8.4 Flat surfaces

In Section 8.3, we gave some examples of surfaces of constant Gaussian curvature K, but this certainly falls well short of a complete classification of such surfaces. It is possible, however, to give a fairly complete description of *flat surfaces*, i.e., surfaces for which $K = 0$ everywhere. To do so, we shall make use of a special parametrization, valid for any surface, described in the following proposition.

Proposition 8.4.1

Let $\mathbf{p}$ be a point of a surface $\mathcal{S}$, and suppose that $\mathbf{p}$ is not an umbilic. Then, there is a surface patch $\boldsymbol{\sigma}(u, v)$ of $\mathcal{S}$ containing $\mathbf{p}$ whose first and second fundamental forms are

$$Edu^2 + Gdv^2 \quad \text{and} \quad Ldu^2 + Ndv^2,$$

respectively, for some smooth functions *E,G,L* and *N*.

We recall that a point $\mathbf{p}$ of a surface $\mathcal{S}$ is an umbilic if the two principal curvatures of $\mathcal{S}$ at $\mathbf{p}$ are equal. From Section 8.2, we see that for the patch $\boldsymbol{\sigma}$ in the statement of the proposition, $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$ are principal vectors with corresponding principal curvatures L/E and N/G. We call $\boldsymbol{\sigma}$ a *principal patch*.

We assume Proposition 8.4.1 for the moment, and use it to give the proof of

Proposition 8.4.2

Let $\mathbf{p}$ be a point of a flat surface $\mathcal{S}$, and assume that $\mathbf{p}$ is not an umbilic. Then, there is a patch of $\mathcal{S}$ containing $\mathbf{p}$ that is a ruled surface.

Proof

We take a principal patch $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ containing $\mathbf{p}$ as in Proposition 8.4.1, say $\mathbf{p} = \boldsymbol{\sigma}(u_0, v_0)$. By Corollary 8.1.3, the Gaussian curvature $K = LN/EG$. Since the Gaussian curvature is zero everywhere, either $L = 0$ or $N = 0$ at each point of U, and since $\mathbf{p}$ is not an umbilic L and N are not both zero. Suppose that $L(u_0, v_0) \neq 0$, say. Then, $L(u, v) \neq 0$ for (u, v) in some open subset of U containing (u_0, v_0). Hence, by shrinking U if necessary, we can assume that $L \neq 0$ at every point of U. Then, $N = 0$ everywhere, and the second fundamental form of $\boldsymbol{\sigma}$ is Ldu^2.

We shall prove that the parameter curves $u =$ constant are straight lines. Such a curve can be parametrized by $v \mapsto \boldsymbol{\sigma}(u_0, v)$, where u_0 is the constant value of u. A unit tangent vector to this curve is $\mathbf{t} = \boldsymbol{\sigma}_v/G^{1/2}$, so by Proposition 1.1.6 what we have to prove is that $\mathbf{t}_v = \mathbf{0}$.

By the proof of Proposition 8.1.2, the derivatives of the unit normal are

$$\mathbf{N}_u = -E^{-1}L\boldsymbol{\sigma}_u, \quad \mathbf{N}_v = \mathbf{0}. \tag{8.9}$$

Hence, $\mathbf{t}_v \cdot \boldsymbol{\sigma}_u = -EL^{-1}\mathbf{t}_v \cdot \mathbf{N}_u$. Now, $\mathbf{t} \cdot \mathbf{N}_u = 0$ and $\mathbf{N}_{uv} = \mathbf{0}$ by Eq. 8.9, so $\mathbf{t}_v \cdot \mathbf{N}_u = -\mathbf{t} \cdot \mathbf{N}_{uv} = 0$. Hence, $\mathbf{t}_v \cdot \boldsymbol{\sigma}_u = 0$. Next, $\mathbf{t}_v \cdot \mathbf{t} = 0$ since $\mathbf{t}$ is a unit vector by construction, so $\mathbf{t}_v \cdot \boldsymbol{\sigma}_v = 0$. Finally, $\mathbf{t}_v \cdot \mathbf{N} = -\mathbf{t} \cdot \mathbf{N}_v = 0$ by Eq. 8.9 again. Since the vectors $\boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v$ and $\mathbf{N}$ form a basis of $\mathbb{R}^3$, we have proved that $\mathbf{t}_v = \mathbf{0}$. □

Our task, then, is to describe the structure of flat ruled surfaces. We parametrize the ruled surface as in Example 8.1.5:

$$\boldsymbol{\sigma}(u, v) = \boldsymbol{\gamma}(u) + v\boldsymbol{\delta}(u).$$

We found there that $\boldsymbol{\sigma}_u = \dot{\boldsymbol{\gamma}} + v\dot{\boldsymbol{\delta}}$, $\boldsymbol{\sigma}_v = \boldsymbol{\delta}$, the dot denoting d/du, and that the Gaussian curvature of $\boldsymbol{\sigma}$ is zero if and only if

$$\dot{\boldsymbol{\delta}} \cdot (\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v) = 0.$$

Since

$$\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v = \dot{\boldsymbol{\gamma}} \times \boldsymbol{\delta} + v\dot{\boldsymbol{\delta}} \times \boldsymbol{\delta},$$

and $\dot{\boldsymbol{\delta}} \cdot (\dot{\boldsymbol{\delta}} \times \boldsymbol{\delta}) = 0$,

$$K = 0 \quad \text{if and only if} \quad \dot{\boldsymbol{\delta}} \cdot (\dot{\boldsymbol{\gamma}} \times \boldsymbol{\delta}) = 0. \tag{8.10}$$

Thus, $K = 0$ *if and only if* $\dot{\boldsymbol{\gamma}}$, $\boldsymbol{\delta}$ *and* $\dot{\boldsymbol{\delta}}$ *are everywhere linearly dependent.*

To proceed further, let us assume, as we may, that $\boldsymbol{\delta}(u)$ is a unit vector for all values of u. Then, $\boldsymbol{\delta} \cdot \dot{\boldsymbol{\delta}} = 0$. Suppose first that $\dot{\boldsymbol{\delta}}(u) = \mathbf{0}$ for all values of u. Then, $\boldsymbol{\delta}$ is a constant vector and $\boldsymbol{\sigma}$ is a *generalized cylinder.*

Suppose now that $\dot{\boldsymbol{\delta}}$ is never zero. Then, $\boldsymbol{\delta}$ and $\dot{\boldsymbol{\delta}}$ are linearly independent as they are non-zero and perpendicular, so if $\dot{\boldsymbol{\gamma}}, \boldsymbol{\delta}$ and $\dot{\boldsymbol{\delta}}$ are linearly dependent, then

$$\dot{\boldsymbol{\gamma}}(u) = f(u)\boldsymbol{\delta}(u) + g(u)\dot{\boldsymbol{\delta}}(u)$$

for some smooth functions f and g. Assume first that $f = \dot{g}$ everywhere. Then, $\dot{\boldsymbol{\gamma}} = (g\boldsymbol{\delta})^{\cdot}$ and so $\boldsymbol{\gamma} = g\boldsymbol{\delta} + \mathbf{a}$, where $\mathbf{a}$ is a constant vector; hence,

$$\boldsymbol{\sigma}(u, v) = \mathbf{a} + (v + g(u))\boldsymbol{\delta}(u).$$

Putting $\tilde{u} = u$, $\tilde{v} = v + g(u)$, we see that this is a reparametrization of a *generalized cone.*

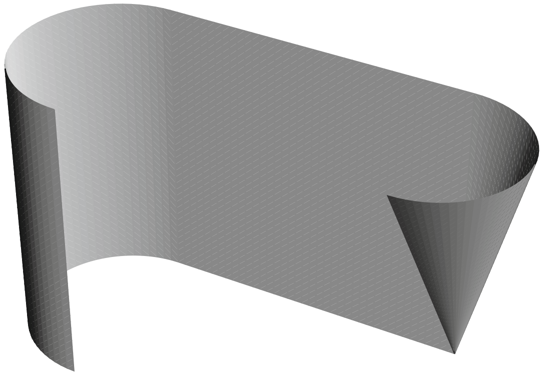
Suppose finally that $\dot{\boldsymbol{\delta}}$ and $f - \dot{g}$ are both nowhere zero. If we define

$$\tilde{\boldsymbol{\gamma}}(u) = \boldsymbol{\gamma}(u) - g(u)\boldsymbol{\delta}(u), \quad \tilde{v} = \frac{v + g(u)}{f(u) - \dot{g}(u)},$$

a short calculation gives

$$\boldsymbol{\sigma}(u,v) = \tilde{\boldsymbol{\gamma}}(u) + \tilde{v}\dot{\tilde{\boldsymbol{\gamma}}}(u),$$

so $\boldsymbol{\sigma}$ is a reparametrization of an open subset of the *tangent developable* of $\tilde{\boldsymbol{\gamma}}$.



Of course, it could be that none of the conditions on $\boldsymbol{\delta}$, f and g considered above are satisfied. In fact, we have only shown that certain open subsets of the surface are parts of generalized cylinders, generalized cones or tangent developables. It is not true that the whole surface must be one of these three types, since flat surfaces of different types can be joined together to make a smooth surface, as shown in the diagram above. It can be shown that the most general flat surface is a patchwork consisting of pieces of generalized cylinders, generalized cones and tangent developables, joined together along segments of straight lines.

The remainder of this section is devoted to the proof of Proposition 8.4.1 and can safely be omitted by readers who are uncomfortable with the use of the inverse function theorem. In fact, we can prove a more general result with no additional effort:

Proposition 8.4.3

Let $\tilde{\boldsymbol{\sigma}} : \tilde{U} \to \mathbb{R}^3$ be a surface patch, and suppose that for all $(\tilde{u},\tilde{v}) \in \tilde{U}$ we are given tangent vectors

$$\boldsymbol{e}_1(\tilde{u},\tilde{v}) = a(\tilde{u},\tilde{v})\tilde{\boldsymbol{\sigma}}_{\tilde{u}} + b(\tilde{u},\tilde{v})\tilde{\boldsymbol{\sigma}}_{\tilde{v}}, \quad \boldsymbol{e}_2(\tilde{u},\tilde{v}) = c(\tilde{u},\tilde{v})\tilde{\boldsymbol{\sigma}}_{\tilde{u}} + d(\tilde{u},\tilde{v})\tilde{\boldsymbol{\sigma}}_{\tilde{v}},$$

whose components a, b, c, d are smooth functions of $(\tilde{u},\tilde{v})$. Assume that, at some point $(\tilde{u}_0,\tilde{v}_0) \in \tilde{U}$, the vectors $\boldsymbol{e}_1(\tilde{u}_0,\tilde{v}_0)$ and $\boldsymbol{e}_2(\tilde{u}_0,\tilde{v}_0)$ are linearly independent. Then, there is an open subset $\tilde{V}$ of $\tilde{U}$ containing $(\tilde{u}_0,\tilde{v}_0)$ and a reparametrization $\boldsymbol{\sigma}(u,v)$ of $\tilde{\boldsymbol{\sigma}}(\tilde{u},\tilde{v})$, for $(\tilde{u},\tilde{v}) \in \tilde{V}$, such that $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$ are parallel to $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$, respectively.

Proposition 8.4.1 is a special case of Proposition 8.4.3. In fact, let $\tilde{\boldsymbol{\sigma}}$ be any surface patch of $\mathcal{S}$ containing $\mathbf{p}$, and let $\mathbf{p} = \tilde{\boldsymbol{\sigma}}(\tilde{u}_0, \tilde{v}_0)$. Since the principal curvatures κ_1 and κ_2 of $\tilde{\boldsymbol{\sigma}}$ are distinct at $\mathbf{p}$, and are continuous functions by Exercise 8.2.8, they remain distinct for $(\tilde{u}, \tilde{v})$ in some open set $\tilde{U}$ containing $(\tilde{u}_0, \tilde{v}_0)$ on which $\tilde{\boldsymbol{\sigma}}$ is defined. Let

$$\boldsymbol{e}_1 = \xi_1 \tilde{\boldsymbol{\sigma}}_{\tilde{u}} + \eta_1 \tilde{\boldsymbol{\sigma}}_{\tilde{v}}, \quad \boldsymbol{e}_2 = \xi_2 \tilde{\boldsymbol{\sigma}}_{\tilde{u}} + \eta_2 \tilde{\boldsymbol{\sigma}}_{\tilde{v}}$$

be unit principal vectors corresponding to κ_1 and κ_2; they are perpendicular by Proposition 8.2.1. Let $\boldsymbol{\sigma}(u, v)$ be a reparametrization of $\tilde{\boldsymbol{\sigma}}$ as in Proposition 8.4.3. Then, $\boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_v = 0$ because $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are perpendicular, so the first fundamental form of $\boldsymbol{\sigma}$ is of the form $Edu^2 + Gdv^2$. Also, $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$ are principal vectors corresponding to κ_1 and κ_2, so we have

$$(\mathcal{F}_{II} - \kappa_1 \mathcal{F}_I) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (\mathcal{F}_{II} - \kappa_2 \mathcal{F}_I) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where $\mathcal{F}_I$ and $\mathcal{F}_{II}$ are the matrices associated to the first and second fundamental forms of $\boldsymbol{\sigma}$. Since $\mathcal{F}_I = \begin{pmatrix} E & 0 \\ 0 & G \end{pmatrix}$, these equations imply that $\mathcal{F}_{II} = \begin{pmatrix} \kappa_1 E & 0 \\ 0 & \kappa_2 G \end{pmatrix}$, so the second fundamental form of $\boldsymbol{\sigma}$ is $Ldu^2 + Ndv^2$, where $L = \kappa_1 E$ and $N = \kappa_2 G$.

We are thus left with the proof of Proposition 8.4.3. To begin, we observe that, if

$$\boldsymbol{e} = A \tilde{\boldsymbol{\sigma}}_{\tilde{u}} + B \tilde{\boldsymbol{\sigma}}_{\tilde{v}},$$

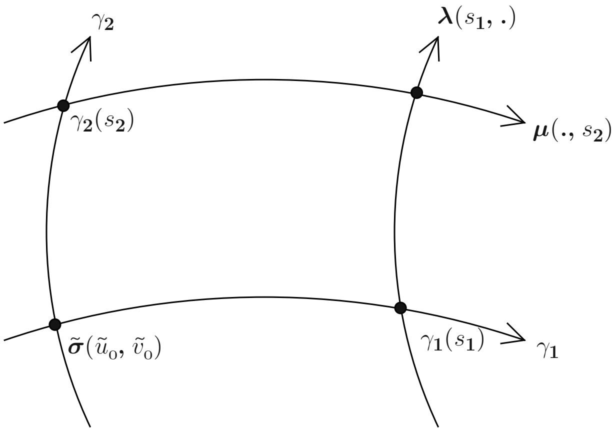
where A and B are any given smooth functions of $(\tilde{u}, \tilde{v}) \in \tilde{U}$, we can find a curve $\boldsymbol{\gamma}$ in $\tilde{\boldsymbol{\sigma}}$ with $\dot{\boldsymbol{\gamma}} = \boldsymbol{e}$ and with any given point $\mathbf{q} = \tilde{\boldsymbol{\sigma}}(\alpha, \beta)$ as starting point $\boldsymbol{\gamma}(0)$. For, finding such a curve $\boldsymbol{\gamma}(t) = \tilde{\boldsymbol{\sigma}}(\tilde{u}(t), \tilde{v}(t))$ is equivalent to solving the pair of ordinary differential equations

$$\dot{\tilde{u}} = A(\tilde{u}, \tilde{v}), \quad \dot{\tilde{v}} = B(\tilde{u}, \tilde{v})$$

with initial conditions $\tilde{u}(0) = \alpha$, $\tilde{v}(0) = \beta$. It is proved in the theory of ordinary differential equations that this problem has a unique solution $\tilde{u}(t), \tilde{v}(t)$ defined on some open interval containing $t = 0$. Moreover, $\tilde{u}$ and $\tilde{v}$ are smooth functions of the three variables t, α and β.

Applying this observation to $\boldsymbol{e} = \boldsymbol{e}_1$, we can find a curve $\boldsymbol{\gamma}_1(s_1)$ in $\tilde{\boldsymbol{\sigma}}$ with $\boldsymbol{\gamma}_1(0) = \tilde{\boldsymbol{\sigma}}(\tilde{u}_0, \tilde{v}_0)$ and $d\boldsymbol{\gamma}_1/ds_1 = \boldsymbol{e}_1$. Now applying the same observation to $\boldsymbol{e} = \boldsymbol{e}_2$, we can find, for each value of s_1 close to 0, a curve $s_2 \mapsto \boldsymbol{\lambda}(s_1, s_2)$ in $\tilde{\boldsymbol{\sigma}}$ with $\partial \boldsymbol{\lambda}/\partial s_2 = \boldsymbol{e}_2$ and $\boldsymbol{\lambda}(s_1, 0) = \boldsymbol{\gamma}_1(s_1)$. Define $(\tilde{u}, \tilde{v})$ as functions of (s_1, s_2) by

$$\tilde{\boldsymbol{\sigma}}(\tilde{u}, \tilde{v}) = \boldsymbol{\lambda}(s_1, s_2). \tag{8.11}$$



Differentiating with respect to s_1 and s_2 gives

$$\tilde{\boldsymbol{\sigma}}_{\tilde{u}}\frac{\partial \tilde{u}}{\partial s_1} + \tilde{\boldsymbol{\sigma}}_{\tilde{v}}\frac{\partial \tilde{v}}{\partial s_1} = \boldsymbol{\lambda}_{s_1}, \quad \tilde{\boldsymbol{\sigma}}_{\tilde{u}}\frac{\partial \tilde{u}}{\partial s_2} + \tilde{\boldsymbol{\sigma}}_{\tilde{v}}\frac{\partial \tilde{v}}{\partial s_2} = \boldsymbol{\lambda}_{s_2}.$$

We have

$$\boldsymbol{\lambda}_{s_1}|_{s_2=0} = \frac{d}{ds_1}\boldsymbol{\lambda}(s_1, 0) = \frac{d\boldsymbol{\gamma}_1}{ds_1} = \boldsymbol{e}_1, \quad \boldsymbol{\lambda}_{s_2} = \frac{\partial \lambda}{\partial s_2} = \boldsymbol{e}_2. \tag{8.12}$$

Equating coefficients of $\tilde{\boldsymbol{\sigma}}_{\tilde{u}}$ and $\tilde{\boldsymbol{\sigma}}_{\tilde{v}}$, we see from the last two sets of equations that, at the point $\tilde{\boldsymbol{\sigma}}(\tilde{u}_0, \tilde{v}_0)$, where $s_1 = s_2 = 0$, the Jacobian matrix

$$\begin{pmatrix} \frac{\partial \tilde{u}}{\partial s_1} & \frac{\partial \tilde{u}}{\partial s_2} \\ \frac{\partial \tilde{v}}{\partial s_1} & \frac{\partial \tilde{v}}{\partial s_2} \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}. \tag{8.13}$$

Since $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are linearly independent at $(\tilde{u}_0, \tilde{v}_0)$, this matrix is invertible. By the Inverse Function Theorem 5.6.1, Eq. 8.11 can be solved for (s_1, s_2) as smooth functions of $(\tilde{u}, \tilde{v})$ when $(\tilde{u}, \tilde{v})$ is in some open set $\tilde{W}$ of $\tilde{U}$ containing $(\tilde{u}_0, \tilde{v}_0)$. Thus, $\boldsymbol{\lambda}$ is an allowable surface patch; by Eq. 8.12, it has the property that $\boldsymbol{\lambda}_{s_1} = \boldsymbol{e}_1$ when $s_2 = 0$, and $\boldsymbol{\lambda}_{s_2} = \boldsymbol{e}_2$ everywhere.

We now repeat the procedure, this time starting with a curve $\boldsymbol{\gamma}_2(t_2)$ with $d\boldsymbol{\gamma}_2/dt_2 = \boldsymbol{e}_2$ and $\boldsymbol{\gamma}_2(0) = \tilde{\boldsymbol{\sigma}}(\tilde{u}_0, \tilde{v}_0)$, and then taking a curve $t_1 \mapsto \boldsymbol{\mu}(t_1, t_2)$ with $\partial\boldsymbol{\mu}/\partial t_1 = \boldsymbol{e}_1$ and $\boldsymbol{\mu}(0, t_2) = \boldsymbol{\gamma}_2(t_2)$. This gives an allowable patch $\boldsymbol{\mu}(t_1, t_2)$ such that

$$\boldsymbol{\mu}(t_1, t_2) = \tilde{\boldsymbol{\sigma}}(\tilde{u}, \tilde{v})$$

for $(\tilde{u}, \tilde{v})$ in some open subset $\tilde{Z}$ of $\tilde{U}$ containing $(\tilde{u}_0, \tilde{v}_0)$. This patch has the property that $\boldsymbol{\mu}_{t_1} = \boldsymbol{e}_1$ everywhere and $\boldsymbol{\mu}_{t_2} = \boldsymbol{e}_2$ when $t_1 = 0$.

The parametrization we want is $\boldsymbol{\sigma}(u,v)$, where $\boldsymbol{\sigma}(u,v)$ is the intersection of the curve $s_2 \mapsto \boldsymbol{\lambda}(u,s_2)$ with the curve $t_1 \mapsto \boldsymbol{\mu}(t_1,v)$. Thus, we consider the equations

$$\tilde{\boldsymbol{\sigma}}(\tilde{u},\tilde{v}) = \boldsymbol{\lambda}(u,s_2) = \boldsymbol{\mu}(t_1,v).$$

From Eq. 8.13,

$$\frac{\partial \tilde{u}}{\partial u} = a, \quad \frac{\partial \tilde{v}}{\partial u} = b,$$

and similarly

$$\frac{\partial \tilde{u}}{\partial v} = c, \quad \frac{\partial \tilde{v}}{\partial v} = d.$$

Hence, the Jacobian matrix

$$\begin{pmatrix} \frac{\partial \tilde{u}}{\partial u} & \frac{\partial \tilde{u}}{\partial v} \\ \frac{\partial \tilde{v}}{\partial u} & \frac{\partial \tilde{v}}{\partial v} \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

As usual, the fact that this matrix is invertible means that (u,v) can be expressed as smooth functions of $(\tilde{u},\tilde{v})$, for $(\tilde{u},\tilde{v})$ in some open subset $\tilde{V}$ of $\tilde{W} \cap \tilde{Z}$ containing $(\tilde{u}_0,\tilde{v}_0)$, and we get a reparametrization $\boldsymbol{\sigma}(u,v)$ of $\tilde{\boldsymbol{\sigma}}(\tilde{u},\tilde{v})$. Finally, the equation $\boldsymbol{\sigma}(u,v) = \boldsymbol{\mu}(t_1,v)$ implies that

$$\boldsymbol{\sigma}_u = \frac{\partial t_1}{\partial u}\boldsymbol{\mu}_{t_1} = \frac{\partial t_1}{\partial u}\boldsymbol{e}_1,$$

and similarly

$$\boldsymbol{\sigma}_v = \frac{\partial s_2}{\partial v}\boldsymbol{e}_2,$$

so $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$ are parallel to $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ everywhere. □

EXERCISES

8.4.1 Let $\mathbf{p}$ be a hyperbolic point of a surface $\mathcal{S}$ (see Section 8.2). Show that there is a patch of $\mathcal{S}$ containing $\mathbf{p}$ whose parameter curves are asymptotic curves (see Exercise 7.3.6). Show that the second fundamental form of such a patch is of the form $2Mdudv$.

8.5 Surfaces of constant mean curvature

We now consider surfaces whose mean curvature H is constant. Such surfaces have an interesting physical interpretation: we shall show in Section 12.1 that soap bubbles always adopt the form of a surface of constant mean curvature. In this section we give two simple constructions of surfaces of constant non-zero mean curvature; the case in which $H = 0$ is treated in much more detail in Chapter 12.

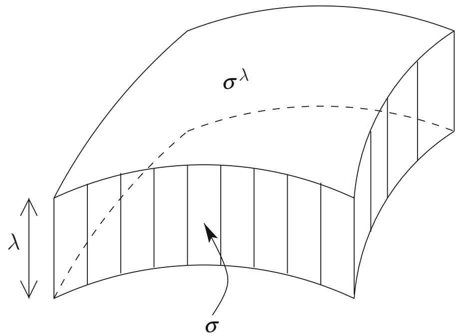
The first of these gives a correspondence between surfaces of constant non-zero mean curvature and surfaces of constant positive Gaussian curvature.

Definition 8.5.1

Let $\mathcal{S}$ be an oriented surface and let $\lambda \in \mathbb{R}$. The *parallel surface* $\mathcal{S}^\lambda$ of $\mathcal{S}$ is

$$\mathcal{S}^\lambda = \{\mathbf{p} + \lambda \mathbf{N_p} \,|\, \mathbf{p} \in \mathcal{S}\},$$

where $\mathbf{N_p}$ is the unit normal of $\mathcal{S}$ at the point $\mathbf{p}$.



Roughly speaking, $\mathcal{S}^\lambda$ is obtained by translating the surface $\mathcal{S}$ at a distance λ perpendicular to itself (but this will not be a genuine translation since $\mathbf{N_p}$ will in general depend on $\mathbf{p}$).

Proposition 8.5.2

Let κ_1 and κ_2 be the principal curvatures of an oriented surface $\mathcal{S}$, let $\lambda \in \mathbb{R}$ and let $\mathcal{S}^\lambda$ be the corresponding parallel surface of $\mathcal{S}$. Assume that neither κ_1 nor κ_2 is equal to $1/\lambda$ at any point of $\mathcal{S}$. Then,

(i) $\mathcal{S}^\lambda$ is a (smooth) oriented surface, the unit normal of $\mathcal{S}^\lambda$ at $\mathbf{p} + \lambda \mathbf{N_p}$ being equal to $\epsilon \mathbf{N_p}$, where ϵ is the sign of $(1 - \lambda\kappa_1)(1 - \lambda\kappa_2)$.

(ii) The principal curvatures of $\mathcal{S}^\lambda$ are $\epsilon\kappa_1/(1 - \lambda\kappa_1)$ and $\epsilon\kappa_2/(1 - \lambda\kappa_2)$, and the corresponding principal vectors are the same as those of $\mathcal{S}$ for the principal curvatures κ_1 and κ_2, respectively.

(iii) The Gaussian and mean curvatures of $\mathcal{S}^\lambda$ are

$$\frac{K}{1 - 2\lambda H + \lambda^2 K} \quad \text{and} \quad \frac{\epsilon(H - \lambda K)}{1 - 2\lambda H + \lambda^2 K},$$

respectively, where K and H are the Gaussian and mean curvatures of $\mathcal{S}$.

Proof

Let $\boldsymbol{\sigma}(u, v)$ be a surface patch of $\mathcal{S}$ with standard unit normal $\mathbf{N}(u, v)$. Define

$$\boldsymbol{\sigma}^\lambda(u, v) = \boldsymbol{\sigma}(u, v) + \lambda \mathbf{N}(u, v).$$

By Proposition 8.1.2,

$$\begin{aligned} \boldsymbol{\sigma}^\lambda_u &= \boldsymbol{\sigma}_u + \lambda \mathbf{N}_u = (1 - \lambda a)\, \boldsymbol{\sigma}_u - \lambda b\, \boldsymbol{\sigma}_v, \\ \boldsymbol{\sigma}^\lambda_v &= \boldsymbol{\sigma}_v + \lambda \mathbf{N}_v = -\lambda c\, \boldsymbol{\sigma}_u + (1 - \lambda d)\, \boldsymbol{\sigma}_v, \end{aligned} \tag{8.14}$$

where

$$\mathcal{W}_{\boldsymbol{\sigma}} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

is the matrix of the Weingarten map of $\mathcal{S}$ with respect to the basis $\{\boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v\}$ of the tangent plane. Hence,

$$\boldsymbol{\sigma}^\lambda_u \times \boldsymbol{\sigma}^\lambda_v = (1 - \lambda(a + d) + \lambda^2(ad - bc))\, \boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v.$$

Since κ_1 and κ_2 are the eigenvalues of $\mathcal{W}_{\boldsymbol{\sigma}}$ (see Section 8.2), and since the sum and product of the eigenvalues of a matrix are equal to the trace and the determinant of the matrix, respectively,

$$\kappa_1 + \kappa_2 = a + d, \quad \kappa_1 \kappa_2 = ad - bc.$$

Hence,

$$\boldsymbol{\sigma}^\lambda_u \times \boldsymbol{\sigma}^\lambda_v = (1 - \lambda\kappa_1)(1 - \lambda\kappa_2)\, \boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v. \tag{8.15}$$

The assertions in part (i) follow from this equation.

The principal curvatures of $\mathcal{S}^\lambda$ are the eigenvalues of the matrix $\mathcal{W}_{\boldsymbol{\sigma}^\lambda}$ of the Weingarten map of $\mathcal{S}^\lambda$ with respect to the basis $\{\boldsymbol{\sigma}^\lambda_u, \boldsymbol{\sigma}^\lambda_v\}$. By the proof of Proposition 8.1.2, this is the negative of the matrix expressing $\mathbf{N}^\lambda_u$ and $\mathbf{N}^\lambda_v$ in terms of $\boldsymbol{\sigma}^\lambda_u$ and $\boldsymbol{\sigma}^\lambda_v$, where $\mathbf{N}^\lambda$ is the standard unit normal of $\boldsymbol{\sigma}^\lambda$. Equation 8.14 says that the matrix expressing $\boldsymbol{\sigma}^\lambda_u$ and $\boldsymbol{\sigma}^\lambda_v$ in terms of $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$ is $I - \lambda \mathcal{W}_{\boldsymbol{\sigma}}$, and the fact that $\mathbf{N}^\lambda = \epsilon \mathbf{N}$ implies that $-\epsilon \mathcal{W}_{\boldsymbol{\sigma}}$ is the matrix expressing $\mathbf{N}^\lambda_u$ and $\mathbf{N}^\lambda_v$ in terms of $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$. Combining these two observations we get

$$\mathcal{W}_{\boldsymbol{\sigma}^\lambda} = \epsilon (I - \lambda \mathcal{W}_{\boldsymbol{\sigma}})^{-1} \mathcal{W}_{\boldsymbol{\sigma}}.$$

If T is an eigenvector of $\mathcal{W}_{\boldsymbol{\sigma}}$ with eigenvalue κ, then T is also an eigenvector of $\mathcal{W}_{\boldsymbol{\sigma}^\lambda}$ with eigenvalue $\epsilon\kappa/(1 - \lambda\kappa)$. The assertions in part (ii) follows from this.

Part (iii) follows from part (ii) by straightforward algebra. $\square$

Corollary 8.5.3

If $\mathcal{S}$ has constant Gaussian curvature $1/R^2$, the parallel surfaces $\mathcal{S}^{\pm R}$ have constant mean curvature $1/2R$. Conversely, if $\mathcal{S}$ has constant mean curvature $1/2R$, the parallel surface $\mathcal{S}^R$ has constant Gaussian curvature $1/R^2$.

Proof

This follows from part (iii) of the proposition by straightforward algebra. For example, if $H = 1/2R$ the Gaussian curvature of $\mathcal{S}^R$ is

$$\frac{K}{1 - 2RH + R^2K} = \frac{K}{R^2K} = \frac{1}{R^2}. \qquad \square$$

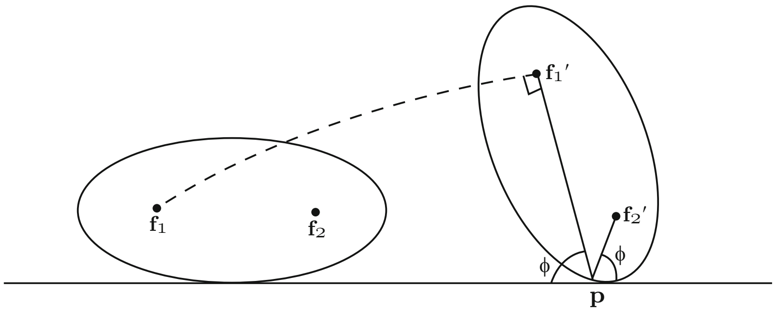
The next construction gives a beautiful geometric description of the surfaces of revolution which have constant non-zero mean curvature in terms of the curve traced out by the focus of an ellipse that rolls without slipping along a straight line (cf. Exercise 2.2.10). Take the ellipse to be

$$\frac{x^2}{p^2} + \frac{(y-q)^2}{q^2} = 1,$$

where $p > q > 0$ are constants. Thus, the ellipse is tangent to the x-axis at the origin. The foci of the ellipse are the points $\mathbf{f}_1 = (-\epsilon p, q)$ and $\mathbf{f}_2 = (\epsilon p, q)$, where the eccentricity $\epsilon = \sqrt{1 - \frac{q^2}{p^2}}$.

Proposition 8.5.4

With the above notation, let $\mathcal{C}$ be the curve traced out by one of the foci of the ellipse as it rolls without slipping along the x-axis. Let $\mathcal{S}$ be the surface obtained by rotating $\mathcal{C}$ around the x-axis. Then, $\mathcal{S}$ has constant non-zero mean curvature.



Proof

We consider a situation in which the ellipse has rolled along the x-axis so that its point of contact with the x-axis is at a point $\mathbf{p}$, the focus $\mathbf{f}_1$ has moved to a point $\mathbf{f}_1{}' = (x, y)$ on $\mathcal{C}$, and the focus $\mathbf{f}_2$ has moved to $\mathbf{f}_2{}' = (X, Y)$, say. Let φ be the angle between $\mathbf{p} - \mathbf{f}_1{}'$ and the x-axis; then φ is also the angle between $\mathbf{p} - \mathbf{f}_2{}'$ and the x-axis by Exercise 1.1.6(iii). Hence,

$$y = \| \mathbf{p} - \mathbf{f}_1{}' \| \sin\varphi, \quad Y = \| \mathbf{p} - \mathbf{f}_2{}' \| \sin\varphi$$

and so

$$y + Y = 2p \sin\varphi$$

by Exercise 1.1.6(i). But Exercise 1.1.6(ii) gives $yY = q^2$ so

$$y + \frac{q^2}{y} = 2p \sin\varphi. \tag{8.16}$$

Now, since the ellipse rolls without slipping, the point of contact of the ellipse with the x-axis is stationary. This implies that the point $\mathbf{f}_1{}'$ moves as if rotating instantaneously about $\mathbf{p}$, so that the tangent vector to $\mathcal{C}$ at $\mathbf{f}_1{}'$ is perpendicular to $\mathbf{p} - \mathbf{f}_1{}'$. (If this heuristic argument is unconvincing, an analytical proof can be found in Exercise 2.2.10.) It follows that

$$\frac{dy}{dx} = \cot\varphi. \tag{8.17}$$

Eliminating φ between Eqs. 8.16 and 8.17 gives

$$y^2 + q^2 = \frac{2py}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}. \tag{8.18}$$

The surface $\mathcal{S}$ obtained by rotating $\mathcal{C}$ around the x-axis can be parametrized by

$$\boldsymbol{\sigma}(x, \theta) = (x, y\cos\theta, y\sin\theta)$$

where θ is the angle of rotation. The first and second fundamental forms of $\boldsymbol{\sigma}$ are

$$\left(1 + \left(\frac{dy}{dx}\right)^2\right) dx^2 + y^2 d\theta^2 \quad \text{and} \quad -\frac{d^2y}{dx^2}dx^2 + \frac{y}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}d\theta^2,$$

respectively. Using the formula in Corollary 8.1.3, the mean curvature is found to be

$$H = \frac{1}{2y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} - \frac{\frac{d^2y}{dx^2}}{2\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{3/2}}. \tag{8.19}$$

Differentiating both sides of Eq. 8.18 we get

$$2y\frac{dy}{dx} = \frac{2p\frac{dy}{dx}}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}} - \frac{2py\frac{dy}{dx}\frac{d^2y}{dx^2}}{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{3/2}}.$$

Dividing both sides by $4py\frac{dy}{dx}$ and comparing with Eq. 8.19 shows that the surface $\mathcal{S}$ has mean curvature $1/2p$. □

EXERCISES

8.5.1 Suppose that the first fundamental form of a surface patch $\boldsymbol{\sigma}(u, v)$ is of the form $E(du^2 + dv^2)$. Prove that $\boldsymbol{\sigma}_{uu} + \boldsymbol{\sigma}_{vv}$ is perpendicular to $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$. Deduce that the mean curvature $H = 0$ everywhere if and only if the Laplacian

$$\boldsymbol{\sigma}_{uu} + \boldsymbol{\sigma}_{vv} = \mathbf{0}.$$

Show that the surface patch

$$\boldsymbol{\sigma}(u, v) = \left(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + u^2v, u^2 - v^2\right)$$

has $H = 0$ everywhere. (A picture of this surface can be found in Section 12.2.)

8.5.2 Prove that $H = 0$ for the surface

$$z = \ln\left(\frac{\cos y}{\cos x}\right).$$

(A picture of this surface can also be found in Section 12.2.)

8.5.3 Let $\boldsymbol{\sigma}(u, v)$ be a surface with first and second fundamental forms $Edu^2 + Gdv^2$ and $Ldu^2 + Ndv^2$, respectively (cf. Proposition 8.4.1). Define

$$\boldsymbol{\Sigma}(u, v, w) = \boldsymbol{\sigma}(u, v) + w\mathbf{N}(u, v),$$

where $\mathbf{N}$ is the standard unit normal of $\boldsymbol{\sigma}$. Show that the three families of surfaces obtained by fixing the values of u, v or w in $\boldsymbol{\Sigma}$ form a triply orthogonal system (see Section 5.5). The surfaces $w =$ constant are parallel surfaces of $\boldsymbol{\sigma}$. Show that the surfaces $u =$ constant and $v =$ constant are flat ruled surfaces.

8.6 Gaussian curvature of compact surfaces

We have seen in Section 8.2 how the relative signs of the principal curvatures at a point $\mathbf{p}$ of a surface $\mathcal{S}$ determine the shape of $\mathcal{S}$ near $\mathbf{p}$. In fact, since the Gaussian curvature K of $\mathcal{S}$ is the product of its principal curvatures, the discussion there shows that

(i) If $K > 0$ at $\mathbf{p}$, then $\mathbf{p}$ is an elliptic point.

(ii) If $K < 0$ at $\mathbf{p}$, then $\mathbf{p}$ is a hyperbolic point.

(iii) If $K = 0$ at $\mathbf{p}$, then $\mathbf{p}$ is either a parabolic point or a planar point.

In this section, we give a result which shows how the Gaussian curvature influences the *global* shape of a surface. We shall give another result of a similar nature in Section 13.4.

Proposition 8.6.1

If $\mathcal{S}$ is a compact surface, there is a point of $\mathcal{S}$ at which its Gaussian curvature K is > 0.

In the proof, we shall make use of the following fact about compact sets: if X is a compact subset of $\mathbb{R}^3$ and $f : \mathbb{R}^3 \to \mathbb{R}$ is a continuous function, then there are points $\mathbf{p}, \mathbf{q} \in X$ such that $f(\mathbf{q}) \leq f(\mathbf{r}) \leq f(\mathbf{p})$ for all points $\mathbf{r} \in X$, so that f attains its maximum value on X at $\mathbf{p}$ and its minimum at $\mathbf{q}$.

Proof

Define $f : \mathbb{R}^3 \to \mathbb{R}$ by $f(\mathbf{v}) = \parallel \mathbf{v} \parallel^2$. Then, f is continuous, so the fact that $\mathcal{S}$ is compact implies that there is a point $\mathbf{p} \in \mathcal{S}$ where f attains its maximum value. Then $\mathcal{S}$ is contained inside the closed ball of radius $\parallel \mathbf{p} \parallel$ and centre the origin, and $\mathcal{S}$ intersects its boundary sphere at $\mathbf{p}$. The idea is that $\mathcal{S}$ is at least as curved as the sphere at $\mathbf{p}$, so its Gaussian curvature should be at least that of the sphere at $\mathbf{p}$, i.e., at least $1/ \parallel \mathbf{p} \parallel^2$.

To make this argument precise, let $\boldsymbol{\gamma}(t)$ be any unit-speed curve in $\mathcal{S}$ passing through $\mathbf{p}$ when $t = 0$. Then, $f(\boldsymbol{\gamma}(t))$ has a local maximum at $t = 0$, so

$$\frac{d}{dt} f(\boldsymbol{\gamma}(t)) = 0, \quad \frac{d^2}{dt^2} f(\boldsymbol{\gamma}(t)) \leq 0$$

at $t = 0$, i.e.,

$$\boldsymbol{\gamma}(0) \cdot \dot{\boldsymbol{\gamma}}(0) = 0, \quad \boldsymbol{\gamma}(0) \cdot \ddot{\boldsymbol{\gamma}}(0) + 1 \leq 0. \tag{8.20}$$

The equation in (8.20) shows that $\mathbf{p} = \boldsymbol{\gamma}(0)$ is perpendicular to every unit tangent vector to $\mathcal{S}$ at $\mathbf{p}$, and hence is perpendicular to the tangent plane $T_{\mathbf{p}}\mathcal{S}$.

Choose a surface patch $\boldsymbol{\sigma}$ of $\mathcal{S}$ containing $\mathbf{p}$, and let $\mathbf{N}$ be its standard unit normal. By the preceding remark,

$$\mathbf{N} = \pm \frac{\mathbf{p}}{\| \mathbf{p} \|}. \tag{8.21}$$

The inequality in (8.20) implies that the normal curvature $\kappa_n = \ddot{\boldsymbol{\gamma}}(0) \cdot \mathbf{N}$ of $\boldsymbol{\gamma}$ at $\mathbf{p}$ (computed in the patch $\boldsymbol{\sigma}$) is $\leq -1/ \| \mathbf{p} \|$ or $\geq 1/ \| \mathbf{p} \|$, according to whether the sign in Eq. 8.21 is $+$ or $-$, respectively. By Corollary 8.2.5, the principal curvatures of $\boldsymbol{\sigma}$ at $\mathbf{p}$ are either both $\leq -1/ \| \mathbf{p} \|$ or both $\geq 1/ \| \mathbf{p} \|$. In each case, $K \geq 1/ \| \mathbf{p} \|^2 > 0$ at $\mathbf{p}$. $\square$