5
Examples of surfaces

In this chapter we describe some of the simplest classes of surfaces. Others will be introduced later in the book.

5.1 Level surfaces

As we have already seen (Examples 4.1.3–5 and Exercise 4.1.3), surfaces are often given to us as *level surfaces*

$$\{(x, y, z) \in \mathbb{R}^3 \mid f(x, y, z) = 0\},$$

where f is a smooth function. In those examples, we constructed atlases by fairly ad hoc methods. The following result gives general conditions under which a level surface is a smooth surface. In fact, it deals with a slightly more general situation in which different regions of a surface may be defined by different functions.

Theorem 5.1.1

Let $\mathcal{S}$ be a subset of $\mathbb{R}^3$ with the following property: for each point $\mathbf{p} \in \mathcal{S}$, there is an open subset W of $\mathbb{R}^3$ containing $\mathbf{p}$ and a smooth function $f : W \to \mathbb{R}$ such that

(i) $\mathcal{S} \cap W = \{(x, y, z) \in W \mid f(x, y, z) = 0\}$;

Andrew Pressley, *Elementary Differential Geometry: Second Edition*,
Springer Undergraduate Mathematics Series, DOI 10.1007/978-1-84882-891-9_5,

(ii) The gradient $\nabla f = (f_x, f_y, f_z)$ of f does not vanish at $\mathbf{p}$.

Then, $\mathcal{S}$ is a smooth surface.

We postpone the proof to Section 5.6.

Example 5.1.2

For the unit sphere S^2, we can take $W = \mathbb{R}^3$ and use the single function $f(x, y, z) = x^2+y^2+z^2-1$. Then, $\nabla f = (2x, 2y, 2z)$ so $\| \nabla f \| = 2$ at all points of S^2. In particular, ∇f is non-zero everywhere on S^2. Hence, Theorem 5.1.1 tells us that S^2 is a smooth surface.

Example 5.1.3

For the circular cone of Example 4.1.5, $f(x, y, z) = x^2 + y^2 - z^2$. Hence, $\nabla f = (2x, 2y, -2z)$, and this vanishes only at the vertex $(0, 0, 0)$. Theorem 5.1.1 applies with $W = \{(x, y, z) \in \mathbb{R}^3 \mid z \neq 0\}$, so the circular cone with the vertex removed is a smooth surface, as we have already seen.

A large class of level surfaces is studied in the next section.

EXERCISES

5.1.1 Show that the following are smooth surfaces:

(i) $x^2 + y^2 + z^4 = 1$.

(ii) $(x^2 + y^2 + z^2 + a^2 - b^2)^2 = 4a^2(x^2 + y^2)$, where $a > b > 0$ are constants.

Show that the surface in (ii) is, in fact, the torus of Exercise 4.2.5.

5.1.2 Consider the surface $\mathcal{S}$ defined by $f(x, y, z) = 0$, where f is a smooth function such that ∇f does not vanish at any point of $\mathcal{S}$. Show that ∇f is perpendicular to the tangent plane at every point of $\mathcal{S}$, and deduce that $\mathcal{S}$ is orientable. Suppose now that $F : \mathbb{R}^3 \to \mathbb{R}$ is a smooth function. Show that, if the restriction of F to $\mathcal{S}$ has a local maximum or a local minimum at $\mathbf{p}$ then, at $\mathbf{p}$, $\nabla F = \lambda \nabla f$ for some scalar λ. (This is called *Lagrange's Method of Undetermined Multipliers*.)

5.1.3 Show that the smallest value of $x^2 + y^2 + z^2$ subject to the condition $xyz = 1$ is 3, and that the points (x, y, z) that give this minimum value lie at the vertices of a regular tetrahedron in $\mathbb{R}^3$.

5.2 Quadric surfaces

The simplest level surfaces, namely planes, have Cartesian equations of the form $ax + by + cz = d$, where a, b, c, d are constants. From this point of view, the next simplest surfaces should be those whose Cartesian equations are given by quadratic expressions in x, y and z.

In this section, we identify any vector $\mathbf{v} = (x, y, z) \in \mathbb{R}^3$ with the column matrix $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, which we also denote by $\mathbf{v}$. Note that, if $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$, $\mathbf{v}^t\mathbf{w}$ is a 1×1 matrix, i.e., a number, namely the dot product $\mathbf{v} \cdot \mathbf{w}$.

Definition 5.2.1

A *quadric* is the subset of $\mathbb{R}^3$ defined by an equation of the form

$$\mathbf{v}^t A\mathbf{v} + \mathbf{b}^t\mathbf{v} + c = 0,$$

where $\mathbf{v} = (x, y, z)$, A is a constant symmetric 3×3 matrix, $\mathbf{b} \in \mathbb{R}^3$ is a constant vector, and $c \in \mathbb{R}$ is a constant scalar.

To see this more explicitly, let

$$A = \begin{pmatrix} a_1 & a_4 & a_6 \\ a_4 & a_2 & a_5 \\ a_6 & a_5 & a_3 \end{pmatrix}, \quad \mathbf{b} = (b_1, b_2, b_3).$$

Then, the equation of the quadric is

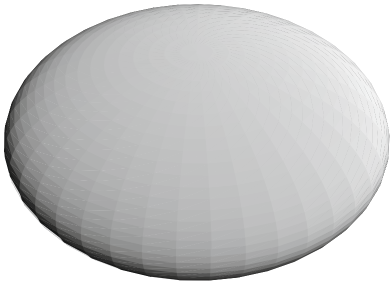
$$a_1x^2 + a_2y^2 + a_3z^2 + 2a_4xy + 2a_5yz + 2a_6xz + b_1x + b_2y + b_3z + c = 0. \quad (5.1)$$

A quadric is not necessarily a surface. For example, the quadric with equation $x^2+y^2+z^2 = 0$ is a single point, and that with equation $x^2+y^2 = 0$ is a straight line. A more interesting example is the quadric $xy = 0$, which is the union of the two intersecting planes $x = 0$ and $y = 0$, and is also not a surface. (Intuitively, it has a 'corner' along the line of intersection of the planes.) The following proposition shows that to understand all quadrics it is sufficient to consider quadrics whose equations take on a particularly simple form.

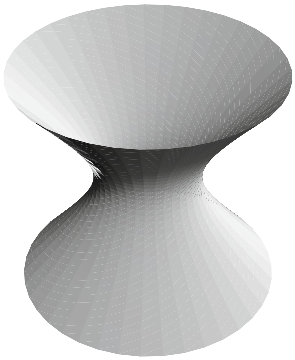
Theorem 5.2.2

By applying a direct isometry of $\mathbb{R}^3$, every non-empty quadric (5.1) in which the coefficients are not all zero can be transformed into one whose Cartesian equation is one of the following:

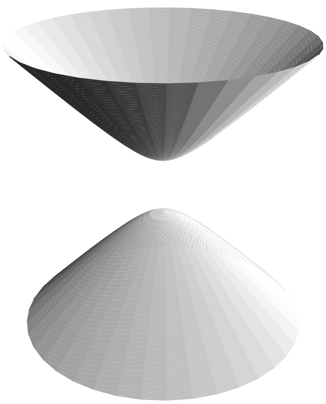
(i) Ellipsoid: $\frac{x^2}{p^2} + \frac{y^2}{q^2} + \frac{z^2}{r^2} = 1$.



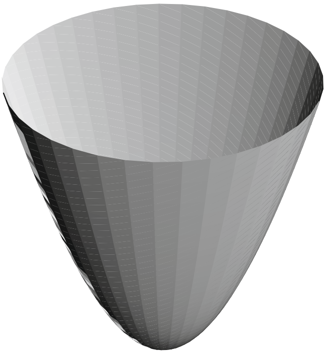
(ii) Hyperboloid of one sheet: $\frac{x^2}{p^2} + \frac{y^2}{q^2} - \frac{z^2}{r^2} = 1$.



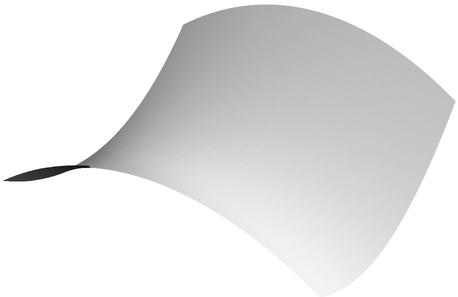
(iii) Hyperboloid of two sheets: $\frac{z^2}{r^2} - \frac{x^2}{p^2} - \frac{y^2}{q^2} = 1$.



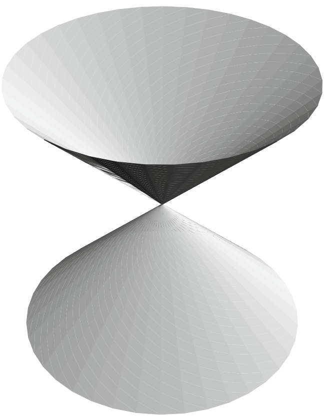
(iv) Elliptic paraboloid: $\frac{x^2}{p^2} + \frac{y^2}{q^2} = z$.



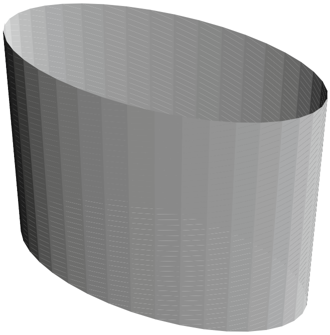
(v) Hyperbolic paraboloid: $\frac{x^2}{p^2} - \frac{y^2}{q^2} = z$.



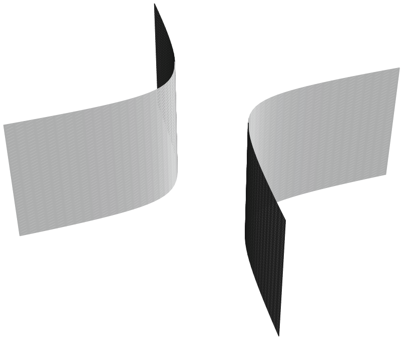
(vi) Quadric cone: $\frac{x^2}{p^2} + \frac{y^2}{q^2} - \frac{z^2}{r^2} = 0$.



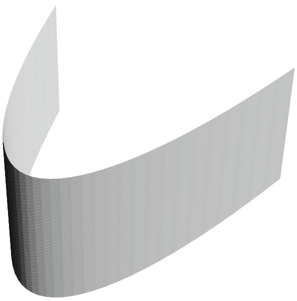
(vii) Elliptic cylinder: $\frac{x^2}{p^2} + \frac{y^2}{q^2} = 1$.



(viii) Hyperbolic cylinder: $\frac{x^2}{p^2} - \frac{y^2}{q^2} = 1$.



(ix) Parabolic cylinder: $\frac{x^2}{p^2} = y$.



(x) Plane: $x = 0$.

(xi) Two parallel planes: $x^2 = p^2$.

(xii) Two intersecting planes: $\frac{x^2}{p^2} - \frac{y^2}{q^2} = 0$.

(xiii) Straight line: $\frac{x^2}{p^2} + \frac{y^2}{q^2} = 0$.

(xiv) Single point: $\frac{x^2}{p^2} + \frac{y^2}{q^2} + \frac{z^2}{r^2} = 0$.

In each case, p, q and r are non-zero constants.

Proof

By Theorem A.0.4, there is an orthogonal matrix P such that PAP^t is a *diagonal* matrix, say

$$A' = \begin{pmatrix} a'_1 & 0 & 0 \\ 0 & a'_2 & 0 \\ 0 & 0 & a'_3 \end{pmatrix}$$

(P^t denotes the transpose of P and I denotes the identity matrix). Then, $\det(P) = \pm 1$, and by replacing P by $-P$ if necessary, we can assume that $\det(P) = 1$. The diagonal entries of A' are the eigenvalues of A, and the rows of P are the corresponding eigenvectors. Define $\mathbf{v}' = (x', y', z')$, $\mathbf{b}' = (b'_1, b'_2, b'_3)$, where

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = P \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \begin{pmatrix} b'_1 \\ b'_2 \\ b'_3 \end{pmatrix} = P \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

Noting that

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = P^t \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}, \quad \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = P^t \begin{pmatrix} b'_1 \\ b'_2 \\ b'_3 \end{pmatrix},$$

the quadric in Definition 5.2.1 becomes

$$\begin{aligned} & (x'\,y'\,z')A'(x'\,y'\,z')^t + (b'_1\,b'_2\,b'_3)(x'\,y'\,z')^t + c = 0, \\ \text{i.e.,} \quad & a'_1 x'^2 + a'_2 y'^2 + a'_3 z'^2 + b'_1 x' + b'_2 y' + b'_3 z' + c = 0. \end{aligned}$$

This new quadric is obtained from the given one by applying the direct isometry $\mathbf{v} \mapsto P\mathbf{v}$ (see Appendix 1), so we might as well consider the quadric in (5.1), but assume that $a_4 = a_5 = a_6 = 0$, i.e.,

$$a_1 x^2 + a_2 y^2 + a_3 z^2 + b_1 x + b_2 y + b_3 z + c = 0. \tag{5.2}$$

Suppose now that, in Eq. 5.2, $a_1 \neq 0$. If we define $x' = x + b_1/2a_1$, corresponding to a translation of $\mathbb{R}^3$, the equation becomes

$$a_1 x'^2 + a_2 y^2 + a_3 z^2 + b_2 y + b_3 z + c' = 0,$$

where c' is a constant. In other words, if $a_1 \neq 0$, we can assume that $b_1 = 0$, and similarly for a_2 and a_3, of course.

If a_1, a_2 and a_3 in Eq. 5.2 are all non-zero, we may therefore reduce to the form

$$a_1x^2 + a_2y^2 + a_3z^2 + c = 0.$$

If $c \neq 0$, we get cases (i), (ii) and (iii), depending on the signs of a_1, a_2, a_3 and c, and if $c = 0$ we get cases (vi) and (xiv).

If exactly one of a_1, a_2 and a_3 is zero, say $a_3 = 0$, we are reduced to the form

$$a_1x^2 + a_2y^2 + b_3z + c = 0. \tag{5.3}$$

If $b_3 \neq 0$, we may define $z' = z + c/b_3$. Thus, by a translation (and by dividing by b_3), we are reduced to the case

$$a_1x^2 + a_2y^2 + z = 0.$$

This gives cases (iv) and (v).

If $b_3 = 0$ in Eq. 5.3, we have

$$a_1x^2 + a_2y^2 + c = 0.$$

If $c = 0$ we get cases (xii) and (xiii). If $c \neq 0$, dividing through by it leads to cases (vii) and (viii).

Suppose now that $a_2 = a_3 = 0$, but $a_1 \neq 0$. Then we have

$$a_1x^2 + b_2y + b_3z + c = 0. \tag{5.4}$$

If b_2 and b_3 are not both zero, by applying a rotation in the xz-plane that takes the y-axis to a line parallel to the vector (b_2, b_3), we can arrive at the situation $b_2 \neq 0, b_3 = 0$, and then by a translation along the y-axis we can arrange that $c = 0$. This leads to the equation

$$a_1x^2 + y = 0,$$

which gives case (ix). If $b_2 = b_3 = 0$ in Eq. 5.4, then $c = 0$ gives case (x) and $c \neq 0$ gives case (xi).

Finally, if $a_1 = a_2 = a_3 = 0$, (5.6) is the equation of a plane, so after applying a Suitable composite of rotations and translations we are in case (x) again. $\square$

Corollary 5.2.3

Every non-empty quadric of types (i)–(x) is a surface (for type (vi) one must remove the vertex of the cone).

Proof

This is easily verified using Exercise 4.2.8, Theorem 5.1.1 and the special form of the equations of quadrics in Theorem 5.2.2. □

Example 5.2.4

Consider the quadric

$$x^2 + 2y^2 + 6x - 4y + 3z = 7.$$

Setting $x' = x + 3, y' = y - 1$ (a translation), we get

$$x'^2 + 2y'^2 + 3z = 18.$$

Setting $z' = z - 6$ (another translation) gives

$$x'^2 + 2y'^2 + 3z' = 0.$$

Finally, setting $x'' = x', y'' = -y', z'' = -z'$ (a rotation by π about the x-axis) gives

$$\frac{1}{3}x''^2 + \frac{2}{3}y''^2 = z'',$$

which is an elliptic paraboloid. It can be parametrized by setting $x'' = u, y'' = v, z'' = \frac{1}{3}u^2 + \frac{2}{3}v^2$. This corresponds to $x = u - 3, y = 1 - v, z = 6 - \frac{1}{3}u^2 - \frac{2}{3}v^2$, and shows that the given quadric is a smooth surface with an atlas consisting of the single surface patch

$$\boldsymbol{\sigma}(u, v) = \left(u - 3, 1 - v, 6 - \frac{1}{3}u^2 - \frac{2}{3}v^2\right).$$

EXERCISES

5.2.1 Write down parametrizations of each of the quadrics in parts (i)–(xi) of Theorem 5.2.2 (in case (vi) one must remove the origin).

5.2.2 Show that the quadric

$$x^2 + y^2 - 2z^2 - \frac{2}{3}xy + 4z = c$$

is a hyperboloid of one sheet if $c > 2$, and a hyperboloid of two sheets if $c < 2$. What if $c = 2$? (This exercise requires a knowledge of eigenvalues and eigenvectors.)

5.2.3 Show that, if a quadric contains three points on a straight line, it contains the whole line. Deduce that, if L_1, L_2 and L_3 are non-intersecting straight lines in $\mathbb{R}^3$, there is a quadric containing all three lines.

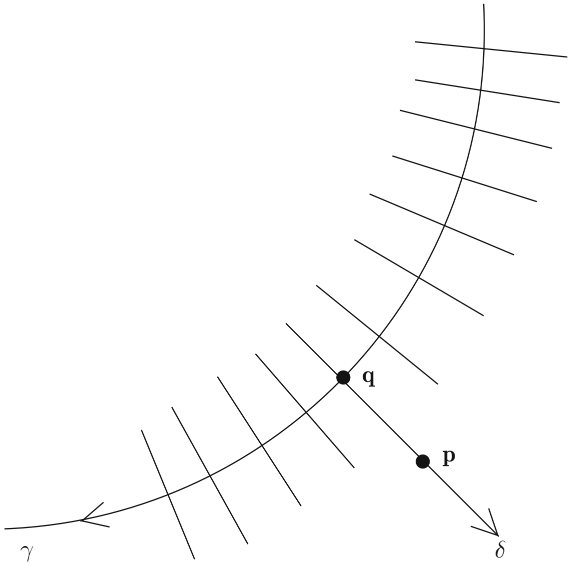
5.2.4 Use the preceding exercise to show that any doubly ruled surface is (part of) a quadric surface. (A surface is doubly ruled if it is the union of each of two families of straight lines such that no two lines of the same family intersect, but every line of the first family intersects every line of the second family, with at most a finite number of exceptions.) Which quadric surfaces are doubly ruled?

5.3 Ruled surfaces and surfaces of revolution

Level surfaces have an 'algebraic' origin, in that they arise from a function $f(x, y, z)$. On the other hand, the two classes of surfaces considered in this section arise from geometric constructions.

Example 5.3.1

A *ruled surface* is a surface that is a union of straight lines, called the *rulings* (or sometimes the *generators*) of the surface.



Suppose that $\mathcal{C}$ is a curve in $\mathbb{R}^3$ that meets each of these lines. Any point $\mathbf{p}$ of the surface lies on one of the given straight lines, which intersects $\mathcal{C}$ at $\mathbf{q}$, say. If $\boldsymbol{\gamma} : (\alpha, \beta) \to \mathbb{R}^3$ is a parametrization of $\mathcal{C}$ with $\boldsymbol{\gamma}(u) = \mathbf{q}$, and if $\boldsymbol{\delta}(u)$ is a non-zero vector in the direction of the line passing through $\boldsymbol{\gamma}(u)$, then

$$\mathbf{p} = \boldsymbol{\gamma}(u) + v\boldsymbol{\delta}(u),$$

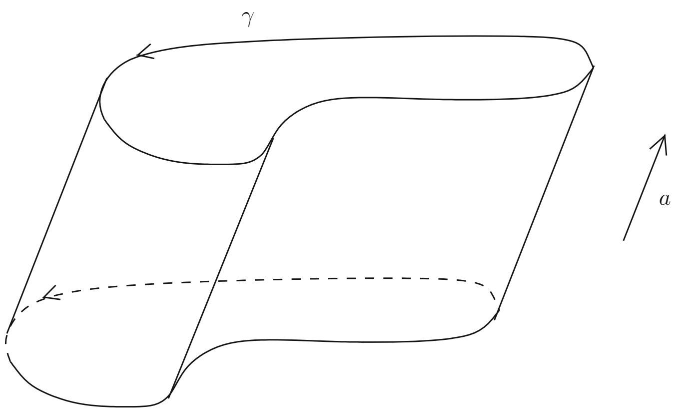
for some scalar v. Denoting the right-hand side by $\boldsymbol{\sigma}(u,v)$, it is clear that $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ is a smooth map, where $U = \{(u,v) \in \mathbb{R}^2 \mid \alpha < u < \beta\}$. Moreover, denoting d/du by a dot,

$$\boldsymbol{\sigma}_u = \dot{\boldsymbol{\gamma}} + v\dot{\boldsymbol{\delta}}, \quad \boldsymbol{\sigma}_v = \boldsymbol{\delta}.$$

Thus, $\boldsymbol{\sigma}$ is regular if $\dot{\boldsymbol{\gamma}} + v\dot{\boldsymbol{\delta}}$ and $\boldsymbol{\delta}$ are linearly independent. This will be true, for example, if $\dot{\boldsymbol{\gamma}}$ and $\boldsymbol{\delta}$ are linearly independent and v is sufficiently small. Thus, to get a surface, the curve $\mathcal{C}$ must never be tangent to the rulings.

An important special case is that in which the rulings are all parallel to each other; the ruled surface $\mathcal{S}$ is then called a *generalized cylinder*. In the above notation, we can take $\boldsymbol{\delta}$ to be a constant unit vector, say $\mathbf{a}$, parallel to the rulings, and the parametrization becomes

$$\boldsymbol{\sigma}(u,v) = \boldsymbol{\gamma}(u) + v\mathbf{a}.$$



Since

$$\boldsymbol{\sigma}(u,v) = \boldsymbol{\sigma}(u',v') \Longleftrightarrow \boldsymbol{\gamma}(u) - \boldsymbol{\gamma}(u') = (v'-v)\mathbf{a},$$

for $\boldsymbol{\sigma}$ to be a injective (and hence a surface patch), no straight line parallel to $\mathbf{a}$ should meet $\boldsymbol{\gamma}$ in more than one point. Finally, $\boldsymbol{\sigma}_u = \dot{\boldsymbol{\gamma}}$, $\boldsymbol{\sigma}_v = \mathbf{a}$, so $\boldsymbol{\sigma}$ is regular if and only if $\boldsymbol{\gamma}$ is never tangent to the rulings.

The parametrization is simplest when $\boldsymbol{\gamma}$ lies in a plane perpendicular to $\mathbf{a}$ (in fact, this can always be achieved by replacing $\boldsymbol{\gamma}$ by its perpendicular projection onto such a plane – see Exercise 5.3.3). The regularity condition is then clearly satisfied provided $\dot{\boldsymbol{\gamma}}$ is never zero, i.e., provided $\boldsymbol{\gamma}$ is regular. We might as well take the plane to be the xy-plane and $\mathbf{a} = (0, 0, 1)$ to be parallel to the z-axis. Then, $\boldsymbol{\gamma}(u) = (f(u), g(u), 0)$ for some smooth functions f and g, and the parametrization becomes

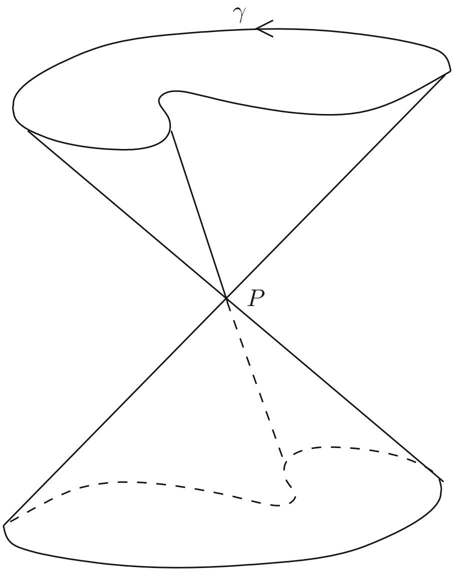
$$\boldsymbol{\sigma}(u, v) = (f(u), g(u), v).$$

For example, starting with a circle, we get a circular cylinder. Taking the circle to have centre the origin, radius 1 and to lie in the xy-plane, it can be parametrized by

$$\boldsymbol{\gamma}(u) = (\cos u, \sin u, 0),$$

defined for $0 < u < 2\pi$ and $-\pi < u < \pi$, say. This gives the atlas for the unit cylinder found in Example 4.1.3.

The second special case we shall consider is that in which the rulings all pass through a certain fixed point, say $\mathbf{v}$; then $\mathcal{S}$ is called a *generalized cone with vertex* $\mathbf{v}$.



We can take $\boldsymbol{\delta}(u) = \boldsymbol{\gamma}(u) - \mathbf{v}$, giving

$$\boldsymbol{\sigma}(u, v) = (1 + v)\boldsymbol{\gamma}(u) - v\mathbf{v}.$$

Now,

$$\boldsymbol{\sigma}(u, v) = \boldsymbol{\sigma}(u', v') \iff (1 + v)\boldsymbol{\gamma}(u) - (1 + v')\boldsymbol{\gamma}(u') + (v' - v)\mathbf{v} = \mathbf{0};$$

since $(1+v)-(1+v')+(v'-v) = 0$, the equation on the right-hand side means that the points $\mathbf{v}$, $\boldsymbol{\gamma}(u)$ and $\boldsymbol{\gamma}(u')$ are collinear. So, for $\boldsymbol{\sigma}$ to be a surface patch, no straight line passing through $\mathbf{v}$ should pass through more than one point of $\boldsymbol{\gamma}$ (in particular, $\boldsymbol{\gamma}$ should not pass through $\mathbf{v}$). Finally, we have $\boldsymbol{\sigma}_u = (1+v)\dot{\boldsymbol{\gamma}}$, $\boldsymbol{\sigma}_v = \boldsymbol{\gamma} - \mathbf{v}$, so $\boldsymbol{\sigma}$ is regular provided $v \neq -1$, i.e., the vertex of the cone is omitted (cf. Example 4.1.5), and none of the straight lines forming the cone is tangent to $\boldsymbol{\gamma}$.

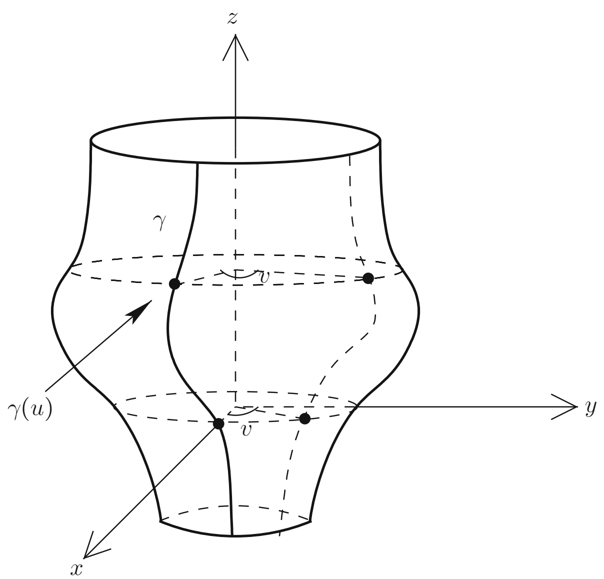
The parametrization is simplest when $\boldsymbol{\gamma}$ lies in a plane. If this plane contains $\mathbf{v}$, the cone is simply part of that plane. Otherwise, we can take $\mathbf{v}$ to be the origin and the plane to be $z = 1$. Then, $\boldsymbol{\gamma}(u) = (f(u), g(u), 1)$ for some smooth functions f and g, and the parametrization takes the form

$$\boldsymbol{\sigma}(u, v) = v(f(u), g(u), 1),$$

after making the reparametrization $v \mapsto v - 1$.

Example 5.3.2

A *surface of revolution* is the surface obtained by rotating a plane curve, called the *profile curve*, around a straight line in the plane. The circles obtained by rotating a fixed point on the profile curve around the axis of rotation are called the *parallels* of the surface, and the curves on the surface obtained by rotating the profile curve through a fixed angle are called its *meridians*. (This agrees with the use of these terms in geography, if we think of the earth as the surface obtained by rotating a circle passing through the poles about the polar axis and we take u and v to be latitude and longitude, respectively.)



Let us take the axis of rotation to be the z-axis and the plane to be the xz-plane. Any point $\mathbf{p}$ of the surface is obtained by rotating some point $\mathbf{q}$ of the profile curve through an angle v (say) around the z-axis. If

$$\gamma(u) = (f(u), 0, g(u))$$

is a parametrization of the profile curve containing $\mathbf{q}$, $\mathbf{p}$ is of the form

$$\boldsymbol{\sigma}(u, v) = (f(u)\cos v, f(u)\sin v, g(u)).$$

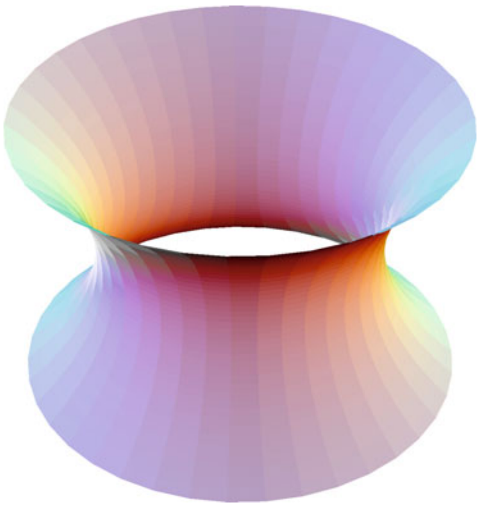
To check regularity, we compute (with a dot denoting d/du):

$$\begin{aligned} \boldsymbol{\sigma}_u &= (\dot{f}\cos v, \dot{f}\sin v, \dot{g}), \quad \boldsymbol{\sigma}_v = (-f\sin v, f\cos v, 0), \\ \therefore \quad \boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v &= (f\dot{g}\cos v, -f\dot{g}\sin v, f\dot{f}), \\ \therefore \quad \| \boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v \|^2 &= f^2(\dot{f}^2 + \dot{g}^2). \end{aligned}$$

Thus, $\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v$ will be non-vanishing if $f(u)$ is never zero, i.e., if $\boldsymbol{\gamma}$ does not intersect the z-axis, and if $\dot{f}$ and $\dot{g}$ are never zero simultaneously, i.e., if $\boldsymbol{\gamma}$ is regular. In this case, we might as well assume that $f(u) > 0$, so that $f(u)$ is the distance of $\boldsymbol{\sigma}(u, v)$ from the axis of rotation. Then, $\boldsymbol{\sigma}$ is injective provided that $\boldsymbol{\gamma}$ does not self-intersect and the angle of rotation v is restricted to lie in an open interval of length $\leq 2\pi$. Under these conditions, surface patches of the form $\boldsymbol{\sigma}$ give the surface of revolution the structure of a surface.

EXERCISES

5.3.1 The surface obtained by rotating the curve $x = \cosh z$ in the xz-plane around the z-axis is called a *catenoid*. Describe an atlas for this surface.



5.3.2 Show that

$$\boldsymbol{\sigma}(u,v) = (\operatorname{sech} u \cos v, \operatorname{sech} u \sin v, \tanh u)$$

is a regular surface patch for S^2 (it is called *Mercator's projection*). Show that meridians and parallels on S^2 correspond under $\boldsymbol{\sigma}$ to perpendicular straight lines in the plane. (This patch is 'derived' in Exercise 6.3.3.)

5.3.3 Show that, if $\boldsymbol{\sigma}(u,v)$ is the (generalized) cylinder in Example 5.3.1:

(i) The curve $\tilde{\boldsymbol{\gamma}}(u) = \boldsymbol{\gamma}(u) - (\boldsymbol{\gamma}(u)\cdot\mathbf{a})\mathbf{a}$ is contained in a plane perpendicular to $\mathbf{a}$.

(ii) $\boldsymbol{\sigma}(u,v) = \tilde{\boldsymbol{\gamma}}(u) + \tilde{v}\mathbf{a}$, where $\tilde{v} = v + \boldsymbol{\gamma}(u)\cdot\mathbf{a}$.

(iii) $\tilde{\boldsymbol{\sigma}}(u,\tilde{v}) = \tilde{\boldsymbol{\gamma}}(u) + \tilde{v}\mathbf{a}$ is a reparametrization of $\boldsymbol{\sigma}(u,v)$.

This exercise shows that, when considering a generalized cylinder $\boldsymbol{\sigma}(u,v) = \boldsymbol{\gamma}(u) + v\mathbf{a}$, we can always assume that the curve $\boldsymbol{\gamma}$ is contained in a plane perpendicular to the vector $\mathbf{a}$.

5.3.4 Consider the ruled surface

$$\boldsymbol{\sigma}(u,v) = \boldsymbol{\gamma}(u) + v\boldsymbol{\delta}(u), \tag{5.5}$$

where $\| \boldsymbol{\delta}(u) \| = 1$ and $\dot{\boldsymbol{\delta}}(u) \neq \mathbf{0}$ for all values of u (a dot denotes d/du). Show that there is a unique point $\boldsymbol{\Gamma}(u)$, say, on the ruling through $\boldsymbol{\gamma}(u)$ at which $\dot{\boldsymbol{\delta}}(u)$ is perpendicular to the surface. The curve $\boldsymbol{\Gamma}$ is called the *line of striction* of the ruled surface $\boldsymbol{\sigma}$ (of course, it need not be a straight line). Show that $\dot{\boldsymbol{\Gamma}}\cdot\dot{\boldsymbol{\delta}} = 0$. Let $\tilde{v} = v + \frac{\dot{\boldsymbol{\gamma}}\cdot\dot{\boldsymbol{\delta}}}{\|\dot{\boldsymbol{\delta}}\|^2}$, and let $\tilde{\boldsymbol{\sigma}}(u,\tilde{v})$ be the corresponding reparametrization of $\boldsymbol{\sigma}$. Then, $\tilde{\boldsymbol{\sigma}}(u,\tilde{v}) = \boldsymbol{\Gamma}(u) + \tilde{v}\boldsymbol{\delta}(u)$. This means that, when considering ruled surfaces as in (5.5), *we can always assume that* $\dot{\boldsymbol{\gamma}}\cdot\dot{\boldsymbol{\delta}} = 0$. We shall make use of this in Chapter 12.

5.4 Compact surfaces

A subset X of $\mathbb{R}^3$ is called *compact* if it is *closed* (i.e., the set of points in $\mathbb{R}^3$ that are *not* in X is open) and *bounded* (i.e., X is contained in some open ball). On several occasions later in the book we shall be particularly interested in compact surfaces.

Example 5.4.1

Any sphere is compact. Let us consider the unit sphere S^2 for simplicity. Obviously S^2 is bounded as it is contained in the open ball $D_2(\mathbf{0})$ (for example).

To show that S^2 is closed, let $\mathbf{p}$ be a point not in S^2, so that $\parallel \mathbf{p} \parallel \neq 1$. Suppose, for example, that $\parallel \mathbf{p} \parallel > 1$ (a similar argument applies if $\parallel \mathbf{p} \parallel < 1$). Let $\epsilon = \parallel \mathbf{p} \parallel -1$. Then the open ball $D_\epsilon(\mathbf{p})$ does not intersect S^2, for if $\mathbf{q} \in D_\epsilon(\mathbf{p})$ the triangle inequality $\parallel \mathbf{p} \parallel = \parallel (\mathbf{p} - \mathbf{q}) + \mathbf{q} \parallel \leq \parallel \mathbf{p} - \mathbf{q} \parallel + \parallel \mathbf{q} \parallel$ gives

$$\parallel \mathbf{q} \parallel \geq \parallel \mathbf{p} \parallel - \parallel \mathbf{p} - \mathbf{q} \parallel > \parallel \mathbf{p} \parallel -\epsilon = 1,$$

so $\parallel \mathbf{q} \parallel > 1$ It follows that the set of points of $\mathbb{R}^3$ that are not in S^2 is open.

Example 5.4.2

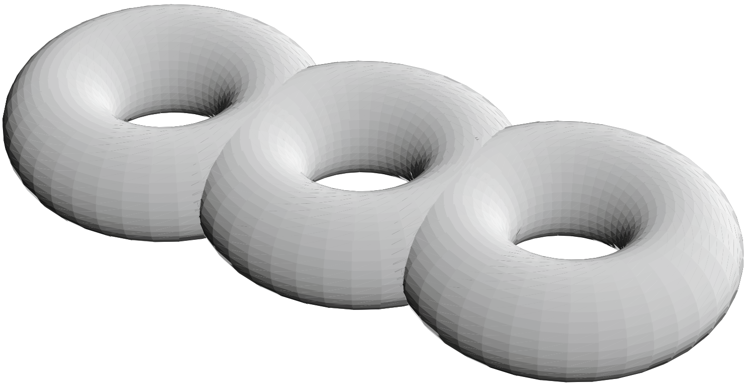
A plane is not compact since it is obviously unbounded.

Example 5.4.3

The open disc

$$\mathcal{D} = \{(x, y, z) \in \mathbb{R}^2 \,|\, x^2 + y^2 < 1, \ z = 0\}$$

is a non-compact surface. It is obviously bounded (it is contained in $D_1(\mathbf{0})$); it is not closed, however, since the point $\mathbf{p} = (1, 0, 0)$ is not in $\mathcal{D}$ and for any $\epsilon > 0$ the open ball $D_\epsilon(\mathbf{p})$ contains the point $(1 - \frac{1}{2}\epsilon, 0, 0)$ which is in $\mathcal{D}$.



It is a surprising result that there are very few compact surfaces in $\mathbb{R}^3$ up to diffeomorphism, and they can all be described explicitly. We have already seen the simplest example, the sphere. The next simplest is the *torus* considered in Exercise 4.2.5. More generally, one can join such tori together (see above). This surface is denoted by T_g, where g is the number of holes, called the *genus* of the surface (we take $g = 0$ for the sphere). We accept the following theorem without proof:

Theorem 5.4.4

For any integer $g \geq 0$, T_g has an atlas making it a smooth surface. Moreover, every compact surface is diffeomorphic to one of the T_g.

Corollary 5.4.5

Every compact surface is orientable.

Proof

Each of the surfaces T_g obviously has an 'interior', which is bounded, and an 'exterior' which is unbounded. Hence, we can choose the unit normal at each point of the surface to point into the exterior region. This provides a smooth choice of unit normal at every point of the surface T_g, so T_g is orientable. Since every compact surface is diffeomorphic to one of the surfaces T_g, the corollary follows from Exercise 4.5.2. □

EXERCISES

5.4.1 One of the following surfaces is compact and one is not:

(i) $x^2 - y^2 + z^4 = 1$.

(ii) $x^2 + y^2 + z^4 = 1$.

Which is which, and why? Sketch the compact surface.

5.4.2 Explain, without giving a detailed proof, why the tube (Exercise 4.2.7) around a closed curve in $\mathbb{R}^3$ with no self-intersections is a compact surface diffeomorphic to a torus (provided the tube has sufficiently small radius).

5.5 Triply orthogonal systems

A *triply orthogonal system of surfaces* consists of three families of surfaces such that

(i) Exactly one surface of each family passes through each point of $\mathbb{R}^3$ (or of some open subset of $\mathbb{R}^3$).

(ii) Any two surfaces belonging to different families intersect orthogonally.

The simplest example of such a system is given by the families of planes parallel to the coordinates planes, namely

$$x = u, \; y = v, \; z = w.$$

Fixing the value of u (say) determines a particular plane in the first family, and similarly for the other families. If $\mathbf{p} = (a, b, c) \in \mathbb{R}^3$, there is a unique plane from each family passing through $\mathbf{p}$, namely those corresponding to $u = a$, $v = b$ and $w = c$. The orthogonality property (ii) is obviously satisfied.

More generally, suppose that the three families are of the form

$$U(x, y, z) = u, \; V(x, y, z) = v, \; W(x, y, z) = w, \tag{5.6}$$

where U, V and W are smooth functions of (x, y, z). By Theorem 5.1.1, these equations determine three families of smooth surfaces provided the vectors ∇U, ∇V and ∇W are non-zero everywhere. Assuming that this condition holds, by Exercise 5.1.2 the non-zero vector ∇U is then perpendicular to the tangent plane of the surface $U(x, y, z) = u$ (and similarly for V, W), so condition (ii) in the definition of a triply orthogonal system becomes

$$\nabla U \cdot \nabla V = \nabla V \cdot \nabla W = \nabla W \cdot \nabla U = 0. \tag{5.7}$$

Now consider the smooth function

$$F(x, y, z) = (U(x, y, z), V(x, y, z), W(x, y, z)).$$

The Jacobian matrix of F is

$$J(F) = \begin{pmatrix} U_x & U_y & U_z \\ V_x & V_y & V_z \\ W_x & W_y & W_z \end{pmatrix}$$

so the rows of $J(F)$ are the components of the non-zero vectors ∇U, ∇V and ∇W. By Eq. 5.7, these vectors are orthogonal, and hence linearly independent, so the matrix $J(F)$ is invertible. By the inverse function theorem (see Section 5.6), Eq. 5.6 can be solved for (x, y, z) in terms of (u, v, w) (at least if (u, v, w) is restricted to lie in a suitable open subset of $\mathbb{R}^3$), say

$$(x, y, z) = \boldsymbol{\Sigma}(u, v, w). \tag{5.8}$$

Then, setting u equal to a constant u_0 (say) gives a parametrization $(v, w) \mapsto \boldsymbol{\Sigma}(u_0, v, w)$ of the surface $U(x, y, z) = u_0$ (and similarly for the other two families of surfaces).

Regarding x, y, z as functions of u, v, w via Eq. 5.8, we can differentiate both sides of the equation

$$U(x, y, z) = u$$

with respect to u, v and w. This gives

$$\begin{aligned} U_x x_u + U_y y_u + U_z z_u &= 1 \\ U_x x_v + U_y y_v + U_z z_v &= 0 \\ U_x x_w + U_y y_w + U_z z_w &= 0. \end{aligned}$$

These three equations, together with the corresponding equations for V and W, can be written in vector form as follows:

$$\begin{aligned} \nabla U \cdot \boldsymbol{\Sigma}_u = 1,\ \nabla U \cdot \boldsymbol{\Sigma}_v = 0,\ \nabla U \cdot \boldsymbol{\Sigma}_w &= 0, \\ \nabla V \cdot \boldsymbol{\Sigma}_u = 0,\ \nabla V \cdot \boldsymbol{\Sigma}_v = 1,\ \nabla V \cdot \boldsymbol{\Sigma}_w &= 0, \\ \nabla W \cdot \boldsymbol{\Sigma}_u = 0,\ \nabla W \cdot \boldsymbol{\Sigma}_v = 0,\ \nabla W \cdot \boldsymbol{\Sigma}_w &= 1. \end{aligned} \tag{5.9}$$

By Eqs. 5.8 and 5.9, ∇U and $\boldsymbol{\Sigma}_u$ are both perpendicular to ∇V and ∇W, so they are parallel to each other. Thus, $\boldsymbol{\Sigma}_u$ is normal to the surface $U(x, y, z) = u$, and $\boldsymbol{\Sigma}_v$ and $\boldsymbol{\Sigma}_w$ are tangent to it (the last statement is also obvious from the statement at the end of the preceding paragraph).

We shall have more to say about triply orthogonal systems later, but now we shall describe one of the most beautiful examples of such systems, which is provided by the theory of quadric surfaces. Let p, q and r be constants such that $0 < p^2 < q^2 < r^2$. For $(x, y, z) \in \mathbb{R}^3$, $t \neq p^2, q^2$ or r^2, let

$$F_t(x, y, z) = \frac{x^2}{p^2 - t} + \frac{y^2}{q^2 - t} + \frac{z^2}{r^2 - t}.$$

Fix a point $(a, b, c) \in \mathbb{R}^3$ with a, b and c all non-zero. The following properties are clear:

(i) $F_t(a, b, c)$ is a continuous function of t in each of the open intervals $(-\infty, p^2)$, (p^2, q^2), (q^2, r^2) and (r^2, ∞).

(ii) $F_t(a, b, c) \to 0$ as $t \to \pm\infty$.

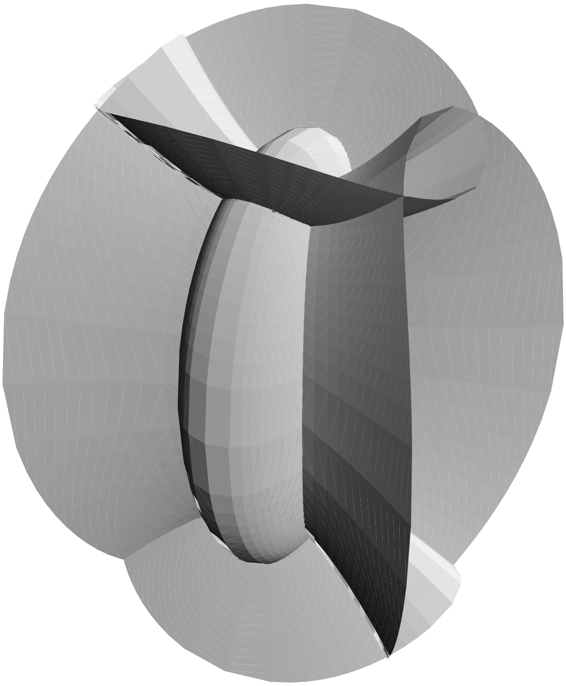
(iii) $F_t(a, b, c) \to \infty$ as t approaches p^2, q^2 or r^2 from the left, and $F_t(a, b, c) \to -\infty$ as t approaches p^2, q^2 or r^2 from the right.

It follows from these properties and the intermediate value theorem that there is at least one value of t in each open interval $(-\infty, p^2)$, (p^2, q^2) and (q^2, r^2) such that $F_t(a, b, c) = 1$. On the other hand, the equation $F_t(a, b, c) = 1$ is equivalent to the cubic equation $G_t(a, b, c) = 0$, where

$$\begin{aligned} G_t(a, b, c) = a^2(q^2 - t)(r^2 - t) + b^2(p^2 - t)(r^2 - t) + c^2(p^2 - t)(q^2 - t) \\ -(p^2 - t)(q^2 - t)(r^2 - t), \end{aligned} \tag{5.10}$$

which has at most three real roots. It follows that there are unique numbers $u \in (-\infty, p^2)$, $v \in (p^2, q^2)$ and $w \in (q^2, r^2)$ (depending on (a, b, c), of course) such that

$$F_u(a, b, c) = 1, \quad F_v(a, b, c) = 1, \quad F_w(a, b, c) = 1. \tag{5.11}$$



The three quadrics $F_u(x, y, z) = 1$, $F_v(x, y, z) = 1$ and $F_w(x, y, z) = 1$ are ellipsoids, hyperboloids of one sheet and hyperboloids of two sheets, respectively, and we have shown that there is one of each passing through each point $(a, b, c) \in \mathbb{R}^3$ that does not lie on any of the coordinate planes. We show that they form a triply orthogonal system.

Indeed, the vector

$$\left(\frac{x}{p^2 - t}, \frac{y}{q^2 - t}, \frac{z}{r^2 - t}\right)$$

is perpendicular to the tangent plane of the surface $F_t(x, y, z) = 1$ at (x, y, z). Thus, to show that the first two surfaces in (5.11) are perpendicular at (a, b, c), for example, we have to show that

$$\frac{a^2}{(p^2 - u)(p^2 - v)} + \frac{b^2}{(q^2 - u)(q^2 - v)} + \frac{c^2}{(r^2 - u)(r^2 - v)} = 0.$$

But the left-hand side of this equation is

$$\frac{F_u(a, b, c) - F_v(a, b, c)}{u - v} = \frac{1 - 1}{u - v} = 0.$$

We can also construct a simultaneous parametrization of the three families. Note that the cubic $G_t(a,b,c)$ in (5.10) is equal to $(t-u)(t-v)(t-w)$, since it is divisible by this product and the coefficients of t^3 agree. Putting $t = p^2, q^2$ and r^2 and solving the resulting equations for a^2, b^2 and c^2, we find that

$$\begin{aligned} a &= \pm\sqrt{\frac{(p^2-u)(p^2-v)(p^2-w)}{(r^2-p^2)(q^2-p^2)}}, \\ b &= \pm\sqrt{\frac{(q^2-u)(q^2-v)(q^2-w)}{(p^2-q^2)(r^2-q^2)}}, \\ c &= \pm\sqrt{\frac{(r^2-u)(r^2-v)(r^2-w)}{(p^2-r^2)(q^2-r^2)}}. \end{aligned} \tag{5.12}$$

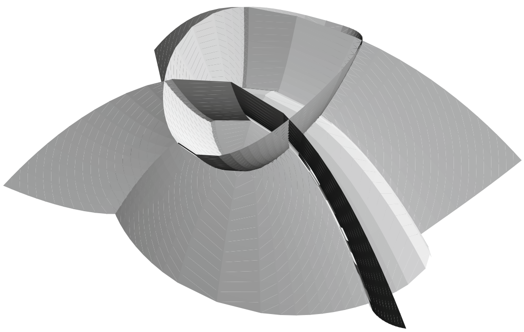
Define $\boldsymbol{\sigma}(u,v,w) = (x,y,z)$, where x, y and z are the right-hand sides of the three equations in (5.12), respectively, with any combination of signs. For fixed u (resp. fixed v, fixed w), this gives eight surface patches for the corresponding ellipsoid $F_u(x,y,z) = 1$ (resp. hyperboloid of one sheet $F_v(x,y,z) = 1$, hyperboloid of two sheets $F_w(x,y,z) = 1$).

EXERCISES

5.5.1 Show that the following are triply orthogonal systems:

(i) The spheres with centre the origin, the planes containing the z-axis and the circular cones with axis the z-axis.

(ii) The planes parallel to the xy-plane, the planes containing the z-axis and the circular cylinders with axis the z-axis.



5.5.2 By considering the quadric surface $F_t(x,y,z) = 0$, where

$$F_t(x,y,z) = \frac{x^2}{p^2-t} + \frac{y^2}{q^2-t} - 2z + t,$$

construct a triply orthogonal system (illustrated above) consisting of two families of elliptic paraboloids and one family of hyperbolic paraboloids. Find a parametrization of these surfaces analogous to (5.12).

5.6 Applications of the inverse function theorem

In this section we give the proofs of Propositions 4.2.6 and 4.4.6 and Theorem 5.1.1.

Suppose first that $f : U \to \mathbb{R}^n$ is a smooth map, where U is an open subset of $\mathbb{R}^m$. If we write $(\tilde{u}_1, \ldots, \tilde{u}_n) = f(u_1, \ldots, u_m)$, the *Jacobian matrix* of f is

$$J(f) = \begin{pmatrix} \frac{\partial \tilde{u}_1}{\partial u_1} & \frac{\partial \tilde{u}_1}{\partial u_2} & \cdots & \frac{\partial \tilde{u}_1}{\partial u_m} \\ \frac{\partial \tilde{u}_2}{\partial u_1} & \frac{\partial \tilde{u}_2}{\partial u_2} & \cdots & \frac{\partial \tilde{u}_2}{\partial u_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \tilde{u}_n}{\partial u_1} & \frac{\partial \tilde{u}_n}{\partial u_2} & \cdots & \frac{\partial \tilde{u}_n}{\partial u_m} \end{pmatrix}.$$

This has already been used in the case $m = n = 2$ in Section 4.2, but now we shall need it in other cases too.

The main tool that we use is the following theorem.

Theorem 5.6.1 (Inverse Function Theorem)

Let $f : U \to \mathbb{R}^n$ be a smooth map defined on an open subset U of $\mathbb{R}^n$ $(n \geq 1)$. Assume that, at some point $x_0 \in U$, the Jacobian matrix $J(f)$ is invertible. Then, there is an open subset V of $\mathbb{R}^n$ and a smooth map $g : V \to \mathbb{R}^n$ such that

(i) $y_0 = f(x_0) \in V$

(ii) $g(y_0) = x_0$

(iii) $g(V) \subseteq U$

(iv) $g(V)$ is an open subset of $\mathbb{R}^n$

(v) $f(g(y)) = y$ for all $y \in V$

In particular, $g : V \to g(V)$ and $f : g(V) \to V$ are inverse bijections.

Thus, the inverse function theorem says that, if $J(f)$ is invertible at some point, then f is bijective near that point and its inverse map is smooth. A proof of this theorem can be found in books on multivariable calculus.

As our first application of Theorem 5.6.1, we complete the proof of Proposition 4.4.6. Suppose then that $f : \mathcal{S} \to \tilde{\mathcal{S}}$ is a smooth map between surfaces $\mathcal{S}$ and $\tilde{\mathcal{S}}$, let $\mathbf{p} \in \mathcal{S}$ and assume that the linear map $D_{\mathbf{p}}f : T_{\mathbf{p}}\mathcal{S} \to T_{f(\mathbf{p})}\tilde{\mathcal{S}}$ is invertible. Let $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ be a surface patch of $\mathcal{S}$ containing $\mathbf{p}$, say $\boldsymbol{\sigma}(u_0, v_0) = \mathbf{p}$, and let $\tilde{\boldsymbol{\sigma}} : \tilde{U} \to \mathbb{R}^3$ be a surface patch of $\tilde{\mathcal{S}}$ containing $f(\mathbf{p})$. By shrinking U if necessary, we can assume that f maps $\boldsymbol{\sigma}(U)$ into $\tilde{\boldsymbol{\sigma}}(\tilde{U})$. Since f is smooth, there are smooth functions $\alpha : U \to \mathbb{R}$ and $\beta : U \to \mathbb{R}$ such that

$$f(\boldsymbol{\sigma}(u, v)) = \tilde{\boldsymbol{\sigma}}(\alpha(u, v), \beta(u, v)).$$

From the remarks following Proposition 4.4.4, the matrix of $D_{\mathbf{p}}f$ with respect to the bases $\{\boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v\}$ of $T_{\mathbf{p}}\mathcal{S}$ and $\{\tilde{\boldsymbol{\sigma}}_{\tilde{u}}, \tilde{\boldsymbol{\sigma}}_{\tilde{v}}\}$ of $T_{f(\mathbf{p})}\tilde{\mathcal{S}}$ is the Jacobian matrix

$$\begin{pmatrix} \alpha_u & \alpha_v \\ \beta_u & \beta_v \end{pmatrix}.$$

Since $D_{\mathbf{p}}f$ is invertible, so is this matrix. By the inverse function theorem, the smooth map $U \to \mathbb{R}^2$ given by $(u, v) \mapsto (\alpha(u, v), \beta(u, v))$ is a diffeomorphism from an open subset V (say) of U containing (u_0, v_0) to an open subset $\tilde{V}$ (say) of $\tilde{U}$. Then $\mathcal{O} = \boldsymbol{\sigma}(V)$ and $\tilde{\mathcal{O}} = \tilde{\boldsymbol{\sigma}}(\tilde{V})$ are open subsets of $\mathcal{S}$ and $\tilde{\mathcal{S}}$, respectively, and f is a diffeomorphism from $\mathcal{O}$ to $\tilde{\mathcal{O}}$. This proves that f is a local diffeomorphism.

We now give the proof of Proposition 4.2.6. We want to show that, if $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ and $\tilde{\boldsymbol{\sigma}} : \tilde{U} \to \mathbb{R}^3$ are two regular patches in the atlas of a surface $\mathcal{S}$, the transition map from $\boldsymbol{\sigma}$ to $\tilde{\boldsymbol{\sigma}}$ is smooth where it is defined.

Suppose that a point $\mathbf{p}$ lies in both patches, say $\boldsymbol{\sigma}(u_0, v_0) = \tilde{\boldsymbol{\sigma}}(\tilde{u}_0, \tilde{v}_0) = \mathbf{p}$. Write

$$\boldsymbol{\sigma}(u, v) = (f(u, v), g(u, v), h(u, v)).$$

Since $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$ are linearly independent, the Jacobian matrix

$$\begin{pmatrix} f_u & f_v \\ g_u & g_v \\ h_u & h_v \end{pmatrix}$$

of $\boldsymbol{\sigma}$ has rank 2 everywhere. Hence, at least one of its three 2×2 submatrices is invertible at each point. Suppose that the submatrix

$$\begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix}$$

is invertible at $\mathbf{p}$. (The proof is similar in the other two cases.) By the inverse function theorem applied to the map $F : U \to \mathbb{R}^2$ given by

$$F(u, v) = (f(u, v), g(u, v)),$$

there is an open subset V of $\mathbb{R}^2$ containing $F(u_0, v_0)$ and an open subset W of U containing (u_0, v_0) such that $F : W \to V$ is bijective with a smooth inverse $F^{-1} : V \to W$. Since $\boldsymbol{\sigma} : W \to \boldsymbol{\sigma}(W)$ is bijective, the projection $\pi : \boldsymbol{\sigma}(W) \to V$ given by $\pi(x, y, z) = (x, y)$ is also bijective, since $\pi = F \circ \boldsymbol{\sigma}^{-1}$ on $\boldsymbol{\sigma}(W)$. It follows that $\tilde{W} = \tilde{\boldsymbol{\sigma}}^{-1}(\boldsymbol{\sigma}(W))$ is an open subset of $\tilde{U}$ and that

$$\boldsymbol{\sigma}^{-1} \circ \tilde{\boldsymbol{\sigma}} = F^{-1} \circ \tilde{F}$$

on $\tilde{W}$, where $\tilde{F} = \pi \circ \tilde{\boldsymbol{\sigma}}$. Since F^{-1} and $\tilde{F}$ are smooth on $\tilde{W}$, so is the transition map $\boldsymbol{\sigma}^{-1} \circ \tilde{\boldsymbol{\sigma}}$. Since $\boldsymbol{\sigma}^{-1} \circ \tilde{\boldsymbol{\sigma}}$ is smooth on an open set containing any point (u_0, v_0) where it is defined, it is smooth.

Finally, we give the proof of Theorem 5.1.1. Let $\mathbf{p}$, W and f be as in the statement of the theorem, and suppose that $\mathbf{p} = (x_0, y_0, z_0)$ and that $f_z \neq 0$ at $\mathbf{p}$. (The proof is similar in the other two cases.) Consider the map $F : W \to \mathbb{R}^3$ defined by

$$F(x, y, z) = (x, y, f(x, y, z)).$$

The Jacobian matrix of F is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ f_x & f_y & f_z \end{pmatrix},$$

and is clearly invertible at $\mathbf{p}$ since $f_z \neq 0$. By the inverse function theorem, there is an open subset V of $\mathbb{R}^3$ containing $F(x_0, y_0, z_0) = (x_0, y_0, 0)$ and a smooth map $G : V \to W$ such that $\tilde{W} = G(V)$ is open and $F : \tilde{W} \to V$ and $G : V \to \tilde{W}$ are inverse bijections.

Since V is open, there are open subsets U_1 of $\mathbb{R}^2$ containing (x_0, y_0) and U_2 of $\mathbb{R}$ containing 0 such that V contains the open set $U_1 \times U_2$ of all points (x, y, w) with $(x, y) \in U_1$ and $w \in U_2$. Hence, we might as well assume that $V = U_1 \times U_2$. The fact that F and G are inverse bijections means that

$$G(x, y, w) = (x, y, g(x, y, w))$$

for some smooth map $g : U_1 \times U_2 \to \mathbb{R}$, and

$$f(x, y, g(x, y, w)) = w$$

for all $(x, y) \in U_1$, $w \in U_2$.

Define $\boldsymbol{\sigma} : U_1 \to \mathbb{R}^3$ by

$$\boldsymbol{\sigma}(x, y) = (x, y, g(x, y, 0)).$$

Then $\boldsymbol{\sigma}$ is a homeomorphism from U_1 to $\mathcal{S} \cap \tilde{W}$ (whose inverse is the restriction to $\mathcal{S} \cap \tilde{W}$ of the projection $\pi(x, y, z) = (x, y)$). It is obvious that $\boldsymbol{\sigma}$ is smooth, and it is regular because

$$\boldsymbol{\sigma}_x \times \boldsymbol{\sigma}_y = (-g_x, -g_y, 1)$$

is nowhere zero. So $\boldsymbol{\sigma}$ is a regular surface patch on $\mathcal{S}$ containing the given point $\mathbf{p}$. Since $\mathbf{p}$ was an arbitrary point of $\mathcal{S}$, we have constructed an atlas for $\mathcal{S}$ making it into a (smooth) surface.

EXERCISES

5.6.1 Show that, if $\boldsymbol{\gamma} : (\alpha, \beta) \to \mathbb{R}^3$ is a curve whose image is contained in a surface patch $\boldsymbol{\sigma} : U \to \mathbb{R}^3$, then $\boldsymbol{\gamma}(t) = \boldsymbol{\sigma}(u(t), v(t))$ for some smooth map $(\alpha, \beta) \to U$, $t \mapsto (u(t), v(t))$.

5.6.2 Prove Theorem 1.5.1 and its analogue for level curves in $\mathbb{R}^3$ (Exercise 1.5.1).

5.6.3 Let $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ be a smooth map such that $\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v \neq \mathbf{0}$ at some point $(u_0, v_0) \in U$. Show that there is an open subset W of U containing (u_0, v_0) such that the restriction of $\boldsymbol{\sigma}$ to W is injective. Note that, in the text, surface patches are injective by definition, but this exercise shows that injectivity near a given point is a consequence of regularity.

5.6.4 Let $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ be a regular surface patch, let $(u_0, v_0) \in U$ and let $\boldsymbol{\sigma}(u_0, v_0) = (x_0, y_0, z_0)$. Suppose that the unit normal $\mathbf{N}(u_0, v_0)$ is not parallel to the xy-plane. Show that there is an open set V in $\mathbb{R}^2$ containing (x_0, y_0), an open subset W of U containing (u_0, v_0) and a smooth function $\varphi : V \to \mathbb{R}$ such that $\tilde{\boldsymbol{\sigma}}(x, y) = (x, y, \varphi(x, y))$ is a reparametrization of $\boldsymbol{\sigma} : W \to \mathbb{R}^3$. Thus, 'near' $\mathbf{p}$, the surface is part of the graph $z = \varphi(x, y)$.
What happens if $\mathbf{N}(u_0, v_0)$ is parallel to the xy-plane?