3
Global properties of curves

All the properties of curves that we have discussed so far are 'local': they depend only on the behaviour of a curve near a given point and not on the 'global' shape of the curve. Proving global results about curves often requires concepts from *topology*, in addition to the calculus techniques we have used in the first two chapters of this book. Since we are not assuming that readers of this book have extensive familiarity with topological ideas, we will not be able to give complete proofs of some of the global results about curves that we discuss in this chapter.

3.1 Simple closed curves

In this chapter, we shall consider plane curves of the following type.

Definition 3.1.1

A *simple closed curve* in $\mathbb{R}^2$ is a closed curve in $\mathbb{R}^2$ that has no self-intersections.

It is a standard, but highly non-trivial, result of the topology of $\mathbb{R}^2$, called the *Jordan Curve Theorem*, that any simple closed curve in the plane has an 'interior' and an 'exterior': more precisely, the *complement* of the image of $\boldsymbol{\gamma}$

Andrew Pressley, *Elementary Differential Geometry: Second Edition*,
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(i.e., the set of points of $\mathbb{R}^2$ that are *not* in the image of $\boldsymbol{\gamma}$) is the disjoint union of two subsets of $\mathbb{R}^2$, denoted by $\operatorname{int}(\boldsymbol{\gamma})$ and $\operatorname{ext}(\boldsymbol{\gamma})$, with the following properties:

(i) $\operatorname{int}(\boldsymbol{\gamma})$ is *bounded*, i.e., it is contained inside a circle of sufficiently large radius.

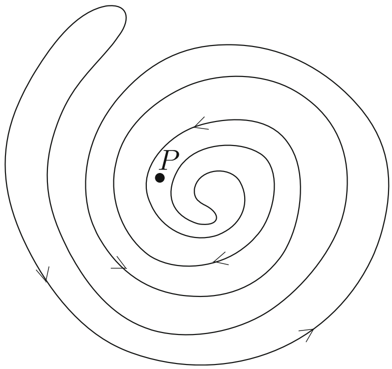
(ii) $\operatorname{ext}(\boldsymbol{\gamma})$ is unbounded.

(iii) Both of the regions $\operatorname{int}(\boldsymbol{\gamma})$ and $\operatorname{ext}(\boldsymbol{\gamma})$ are *connected*, i.e., they have the property that any two points in the same region can be joined by a curve contained entirely in the region (but any curve joining a point of $\operatorname{int}(\boldsymbol{\gamma})$ to a point of $\operatorname{ext}(\boldsymbol{\gamma})$ must cross the curve $\boldsymbol{\gamma}$).

Example 3.1.2

The ellipse $\boldsymbol{\gamma}(t) = (p\cos t, q\sin t)$, where p and q are non-zero constants, is a simple closed curve with period 2π. The interior and exterior of $\boldsymbol{\gamma}$ are, of course, given by $\left\{(x,y) \in \mathbb{R}^2 \mid \frac{x^2}{p^2} + \frac{y^2}{q^2} < 1\right\}$ and $\left\{(x,y) \in \mathbb{R}^2 \mid \frac{x^2}{p^2} + \frac{y^2}{q^2} > 1\right\}$, respectively.

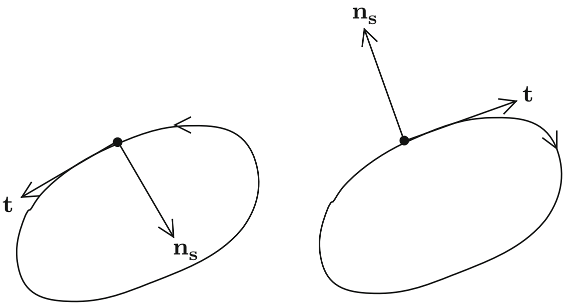
Not all examples of simple closed curves have such an obvious interior and exterior, however. Is the point $\mathbf{p}$ in the interior or the exterior of the simple closed curve shown below?



Example 3.1.3

The limaçon in Example 1.1.7 is closed but is not a simple closed curve as it has a self-intersection – see Exercise 3.1.1.

The fact that a simple closed curve has an interior and an exterior enables us to distinguish between the two possible orientations of $\boldsymbol{\gamma}$. We shall say that $\boldsymbol{\gamma}$ is *positively-oriented* if the signed unit normal $\mathbf{n}_s$ of $\boldsymbol{\gamma}$ (see Section 2.2) points into int($\boldsymbol{\gamma}$) at every point of $\boldsymbol{\gamma}$. This can always be achieved by replacing the parameter t of $\boldsymbol{\gamma}$ by $-t$, if necessary. In the diagrams below, the arrow indicates the direction of increasing parameter. Is the simple closed curve shown above positively-oriented?



Positively-oriented Not positively-oriented

We conclude this section by stating the following important result.

Theorem 3.1.4 (Hopf's Umlaufsatz)

The total signed curvature of a simple closed curve in $\mathbb{R}^2$ is $\pm 2\pi$.

The proof of Theorem 3.1.4 would take us a little further into the realm of topology than is appropriate for this book. A heuristic proof (of a slightly more general result) is given in Section 13.1.

Note that Corollary 2.2.5 shows that the total signed curvature of any closed curve in $\mathbb{R}^2$ is an integer multiple of 2π. The point of Hopf's theorem is that if the curve is *simple* closed, this integer must be ± 1. The German word 'Umlaufsatz' means 'rotation theorem': from the proof of Corollary 2.2.5 we see that Hopf's theorem says that any turning angle φ of a simple closed curve changes by $\pm 2\pi$ on going once round the curve, which means that the tangent vector rotates by $\pm 2\pi$. The reader might like to check that this property holds for the maze-like simple closed curve preceding Example 3.1.3.

EXERCISES

3.1.1 Show that

$$\boldsymbol{\gamma}(t) = ((1 + a\cos t)\cos t, (1 + a\cos t)\sin t),$$

where a is a constant, is a simple closed curve if $|a| < 1$, but that if $|a| > 1$ its complement is the disjoint union of three connected subsets of $\mathbb{R}^2$, two of which are bounded and one is unbounded. What happens if $a = \pm 1$?

3.2 The isoperimetric inequality

The *area* contained by a simple closed curve $\boldsymbol{\gamma}$ is

$$\mathcal{A}(\boldsymbol{\gamma}) = \int_{\mathrm{int}(\boldsymbol{\gamma})} dxdy. \tag{3.1}$$

This can be computed by using the following theorem.

Green's Theorem *Let $f(x,y)$ and $g(x,y)$ be smooth functions (i.e., functions with continuous partial derivatives of all orders), and let $\boldsymbol{\gamma}$ be a positively-oriented simple closed curve. Then,*

$$\int_{\mathrm{int}(\boldsymbol{\gamma})} \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dxdy = \int_{\boldsymbol{\gamma}} f(x,y)dx + g(x,y)dy.$$

A proof can be found in standard books on multivariable calculus.

Proposition 3.2.1

If $\boldsymbol{\gamma}(t) = (x(t), y(t))$ is a positively-oriented simple closed curve in $\mathbb{R}^2$ with period T, then

$$\mathcal{A}(\boldsymbol{\gamma}) = \frac{1}{2} \int_0^T (x\dot{y} - y\dot{x})dt. \tag{3.2}$$

Proof

Taking $f = -\frac{1}{2}y$, $g = \frac{1}{2}x$ in Green's theorem, we get

$$\mathcal{A}(\boldsymbol{\gamma}) = \frac{1}{2} \int_{\boldsymbol{\gamma}} xdy - ydx,$$

which gives Eq. 3.2 immediately. □

Note that, although the formula in Eq. 3.2 involves the parameter t of $\boldsymbol{\gamma}$, it is clear from the Definition 3.1.1 that $\mathcal{A}(\boldsymbol{\gamma})$ is unchanged if $\boldsymbol{\gamma}$ is reparametrized.

One of the most famous global results about plane curves is the following theorem.

Theorem 3.2.2 (Isoperimetric Inequality)

Let $\boldsymbol{\gamma}$ be a simple closed curve, let $\ell(\boldsymbol{\gamma})$ be its length and let $\mathcal{A}(\boldsymbol{\gamma})$ be the area contained by it. Then,

$$\mathcal{A}(\boldsymbol{\gamma}) \leq \frac{1}{4\pi}\ell(\boldsymbol{\gamma})^2,$$

and equality holds if and only if $\boldsymbol{\gamma}$ is a circle.

Of course, it is obvious that equality holds when γ is a circle, since in that case $\ell(\gamma) = 2\pi R$ and $\mathcal{A}(\gamma) = \pi R^2$, where R is the radius of the circle.

To prove this theorem, we need the following result from analysis:

Proposition 3.2.3 (Wirtinger's Inequality)

Let $F : [0, \pi] \to \mathbb{R}$ be a smooth function such that $F(0) = F(\pi) = 0$. Then,

$$\int_0^\pi \left(\frac{dF}{dt}\right)^2 dt \geq \int_0^\pi F(t)^2 \, dt,$$

and equality holds if and only if $F(t) = D \sin t$ for all $t \in [0, \pi]$, where D is a constant.

Assuming this result for the moment, we show how to deduce the isoperimetric inequality from it.

Proof

We start by making some assumptions about γ that will simplify the proof. First, we can, if we wish, assume that γ is parametrized by arc-length s. However, because of the π that appears in Theorem 3.2.2, it turns out to be more convenient to assume that the period of γ is π. If we change the parameter of γ from s to

$$t = \frac{\pi s}{\ell(\gamma)}, \tag{3.3}$$

the resulting curve is still simple closed, and has period π because when s increases by $\ell(\gamma)$, t increases by π. We shall therefore assume that γ is parametrized using the parameter t in Eq. 3.3 from now on.

For the second simplification, we note that both $\ell(\gamma)$ and $\mathcal{A}(\gamma)$ are unchanged if γ is subjected to a translation $\gamma(t) \mapsto \gamma(t) + \mathbf{b}$, where $\mathbf{b}$ is any constant vector (see Exercise 3.2.1). Taking $\mathbf{b} = -\gamma(0)$, we might as well assume that $\gamma(0) = \mathbf{0}$ to begin with, i.e., we assume that γ begins and ends at the origin.

To prove Theorem 3.2.2, we shall calculate $\ell(\gamma)$ and $\mathcal{A}(\gamma)$ by using polar coordinates

$$x = r\cos\theta, \quad y = r\sin\theta.$$

Using the chain rule, it is easy to show that

$$\dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\theta}^2, \qquad x\dot{y} - y\dot{x} = r^2\dot{\theta},$$

with d/dt denoted by a dot. Then, using Eq. 3.3,

$$\dot{r}^2 + r^2\dot{\theta}^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2\right)\left(\frac{ds}{dt}\right)^2 = \frac{\ell(\boldsymbol{\gamma})^2}{\pi^2}, \quad (3.4)$$

since $(dx/ds)^2 + (dy/ds)^2 = 1$. Further, by Eq. 3.2, we have

$$\mathcal{A}(\boldsymbol{\gamma}) = \frac{1}{2}\int_0^\pi (x\dot{y} - y\dot{x})\,dt = \frac{1}{2}\int_0^\pi r^2\dot{\theta}\,dt. \quad (3.5)$$

To prove Theorem 3.2.2, we have to show that

$$\frac{\ell(\boldsymbol{\gamma})^2}{4\pi} - \mathcal{A}(\boldsymbol{\gamma}) \geq 0,$$

with equality holding if and only if $\boldsymbol{\gamma}$ is a circle. By Eq. 3.4,

$$\int_0^\pi (\dot{r}^2 + r^2\dot{\theta}^2)\,dt = \frac{\ell(\boldsymbol{\gamma})^2}{\pi}.$$

Hence, using Eq. 3.5,

$$\frac{\ell(\boldsymbol{\gamma})^2}{4\pi} - \mathcal{A}(\boldsymbol{\gamma}) = \frac{1}{4}\int_0^\pi (\dot{r}^2 + r^2\dot{\theta}^2)dt - \frac{1}{2}\int_0^\pi r^2\dot{\theta}\,dt = \frac{1}{4}\mathcal{I},$$

where

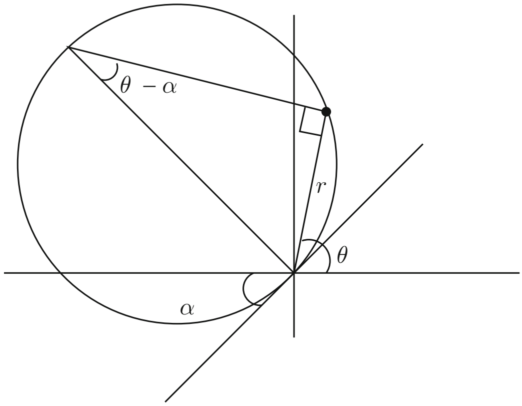
$$\mathcal{I} = \int_0^\pi (\dot{r}^2 + r^2\dot{\theta}^2 - 2r^2\dot{\theta})\,dt. \quad (3.6)$$

Thus, to prove Theorem 3.2.2, we have to show that $\mathcal{I} \geq 0$, and that $\mathcal{I} = 0$ if and only if $\boldsymbol{\gamma}$ is a circle.

By simple algebra,

$$\mathcal{I} = \int_0^\pi r^2(\dot{\theta} - 1)^2\,dt + \int_0^\pi (\dot{r}^2 - r^2)\,dt. \quad (3.7)$$

The first integral on the right-hand side of Eq. 3.7 is obviously ≥ 0, and the second integral is ≥ 0 by Wirtinger's inequality (we are taking $F = r$: note that $r(0) = r(\pi) = 0$ since $\boldsymbol{\gamma}(0) = \boldsymbol{\gamma}(\pi) = \mathbf{0}$). Hence, $\mathcal{I} \geq 0$. Further, since both integrals on the right-hand side of Eq. 3.7 are ≥ 0, their sum $\mathcal{I}$ is zero if and only if both of these integrals are zero. But the first integral is zero only if $\dot{\theta} = 1$ for all t, and the second is zero only if $r = D\sin t$ for some constant D (by Wirtinger again). So $\theta = t + \alpha$, where α is a constant, and hence $r = D\sin(\theta - \alpha)$. It is easy to see that this is the polar equation of a circle of diameter D, thus completing the proof of Theorem 3.2.2 (see the diagram below). □



We now prove Wirtinger's inequality.

Let $G(t) = F(t)/\sin t$. Then, denoting d/dt by a dot as usual,

$$\begin{aligned}\int_0^\pi \dot{F}^2\,dt &= \int_0^\pi (\dot{G}\sin t + G\cos t)^2\,dt \\ &= \int_0^\pi \dot{G}^2\sin^2 t\,dt + 2\int_0^\pi G\dot{G}\sin t\cos t\,dt + \int_0^\pi G^2\cos^2 t\,dt.\end{aligned}$$

Integrating by parts[1]:

$$\begin{aligned}2\int_0^\pi G\dot{G}\sin t\cos t\,dt &= G^2\sin t\cos t\Big|_0^\pi - \int_0^\pi G^2(\cos^2 t - \sin^2 t)\,dt \\ &= \int_0^\pi G^2(\sin^2 t - \cos^2 t)\,dt.\end{aligned}$$

Hence,

$$\begin{aligned}\int_0^\pi \dot{F}^2 dt &= \int_0^\pi \dot{G}^2\sin^2 t\,dt + \int_0^\pi G^2(\sin^2 t - \cos^2 t)\,dt + \int_0^\pi G^2\cos^2 t\,dt \\ &= \int_0^\pi (G^2 + \dot{G}^2)\sin^2 t\,dt = \int_0^\pi F^2 dt + \int_0^\pi \dot{G}^2\sin^2 t\,dt,\end{aligned}$$

and so

$$\int_0^\pi \dot{F}^2\,dt - \int_0^\pi F^2\,dt = \int_0^\pi \dot{G}^2\sin^2 t\,dt.$$

The integral on the right-hand side is obviously ≥ 0, and it is zero if and only if $\dot{G} = 0$ for all t, i.e., if and only if $G(t)$ is equal to a constant, say D, for all t, which means that $F(t) = D\sin t$. □

[1] In performing the integration by parts, we assume that G is continuously differentiable (for we assume that the function $G(t)^2\sin t\cos t$ is equal to the integral of its derivative). Unfortunately, $G(t)$ is not even defined when $t = 0$ or π, as the ratio $F(t)/\sin t$ is $0/0$ there. So we must show that G can be defined at these points so as to become continuously differentiable everywhere. This can be done by using l'Hospital's rule.

EXERCISES

3.2.1 Show that the length $\ell(\boldsymbol{\gamma})$ and the area $\mathcal{A}(\boldsymbol{\gamma})$ are unchanged by applying an isometry to $\boldsymbol{\gamma}$.

3.2.2 By applying the isoperimetric inequality to the ellipse

$$\frac{x^2}{p^2} + \frac{y^2}{q^2} = 1$$

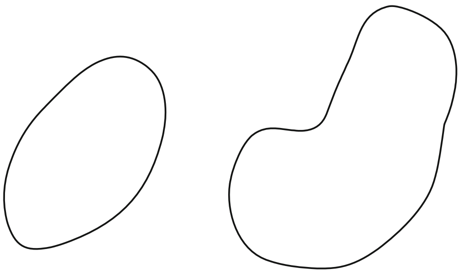
(where p and q are positive constants), prove that

$$\int_0^{2\pi} \sqrt{p^2 \sin^2 t + q^2 \cos^2 t}\, dt \geq 2\pi\sqrt{pq},$$

with equality holding if and only if $p = q$.

3.3 The four vertex theorem

We conclude this chapter with a famous result about convex curves in the plane. A simple closed curve $\boldsymbol{\gamma}$ is called *convex* if its interior $\mathrm{int}(\boldsymbol{\gamma})$ is convex, in the sense that the straight line segment joining any two points of $\mathrm{int}(\boldsymbol{\gamma})$ is contained entirely in $\mathrm{int}(\boldsymbol{\gamma})$.



convex not convex

Definition 3.3.1

A *vertex* of a curve $\boldsymbol{\gamma}(t)$ in $\mathbb{R}^2$ is a point where its signed curvature κ_s has a stationary point, i.e., where $d\kappa_s/dt = 0$.

It is easy to see that this definition is independent of the parametrization of $\boldsymbol{\gamma}$.

Example 3.3.2

The signed curvature of the ellipse $\boldsymbol{\gamma}(t) = (p\cos t, q\sin t)$, where p and q are positive constants, is easily found to be

$$\kappa_s(t) = \frac{pq}{(p^2\sin^2 t + q^2\cos^2 t)^{3/2}}.$$

Then,

$$\frac{d\kappa_s}{dt} = \frac{3pq(q^2 - p^2)\sin t\cos t}{(p^2\sin^2 t + q^2\cos^2 t)^{5/2}}$$

vanishes at exactly four points of the ellipse, namely the points with $t = 0, \pi/2, \pi$ and $3\pi/2$, which are the ends of the two axes of the ellipse.

The following theorem says that this is the smallest number of vertices a convex simple closed curve can have.

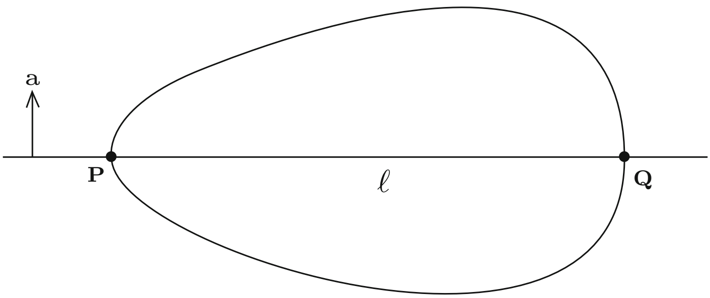
Theorem 3.3.3 (Four Vertex Theorem)

Every convex simple closed curve in $\mathbb{R}^2$ has at least four vertices.

The conclusion of this theorem actually remains true without the assumption of convexity, but the proof is then more difficult than the one we are about to give.

Proof

Let $\boldsymbol{\gamma}$ be a parametrization of a convex simple closed curve in $\mathbb{R}^2$, and let ℓ be its length. Assume for a contradiction that $\boldsymbol{\gamma}$ has fewer than four vertices. We show first that there is a straight line L that divides $\boldsymbol{\gamma}$ into two segments, in one of which $\dot{\kappa}_s > 0$ and in the other $\dot{\kappa}_s \leq 0$ (or possibly $\dot{\kappa}_s \geq 0$ on one and $\dot{\kappa}_s < 0$ on the other). Indeed, κ_s attains all of its values on the closed interval $[0, \ell]$, so κ_s must attain its maximum and minimum values at some points $\mathbf{p}$ and $\mathbf{q}$ of $\boldsymbol{\gamma}$. We can assume that $\mathbf{p} \neq \mathbf{q}$, since otherwise κ_s would be constant, $\boldsymbol{\gamma}$ would be a circle (by Example 2.2.7), and every point of $\boldsymbol{\gamma}$ would be a vertex. If $\mathbf{p}$ and $\mathbf{q}$ were the only vertices of $\boldsymbol{\gamma}$, we would have $\dot{\kappa}_s > 0$ on one of the segments into which the line through $\mathbf{p}$ and $\mathbf{q}$ divides $\boldsymbol{\gamma}$ and $\dot{\kappa}_s < 0$ on the other. Suppose now that there is just one more vertex, say $\mathbf{r}$. Then, $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ divide $\boldsymbol{\gamma}$ into three segments, on each of which either $\dot{\kappa}_s > 0$ or $\dot{\kappa}_s < 0$. It follows that there are two adjacent segments on which $\dot{\kappa}_s > 0$ or two on which $\dot{\kappa}_s < 0$ (except at the point at which the two segments meet). This proves our assertion. □



Let $\mathbf{a}$ be a unit vector perpendicular to L, so that $\boldsymbol{\gamma}\cdot\mathbf{a} > 0$ on one side of L and $\boldsymbol{\gamma}\cdot\mathbf{a} < 0$ on the other. Then, the quantity $\dot{\kappa}_s(\boldsymbol{\gamma}\cdot\mathbf{a})$ is either always > 0 or always < 0, except at the two points in which L intersects the curve. It follows that

$$\int_0^\ell \dot{\kappa}_s(\boldsymbol{\gamma}\cdot\mathbf{a})\,dt \neq 0, \tag{3.8}$$

as this integral is definitely > 0 in the first case and < 0 in the second. But, using the equation $\dot{\mathbf{n}}_s = -\kappa_s\mathbf{t}$ (see Exercise 2.2.1), we get

$$\dot{\kappa}_s\boldsymbol{\gamma} = (\kappa_s\boldsymbol{\gamma})^{\cdot} - \kappa_s\dot{\boldsymbol{\gamma}} = (\kappa_s\boldsymbol{\gamma} + \mathbf{n}_s)^{\cdot},$$

so the integrand on the left-hand side of (3.8) is the derivative of $(\kappa_s\boldsymbol{\gamma}+\mathbf{n}_s)\cdot\mathbf{a} = \lambda$, say. Since $\boldsymbol{\gamma}$ is ℓ-periodic,

$$\boldsymbol{\gamma}(t+\ell) = \boldsymbol{\gamma}(t) \quad \text{for all} \quad t,$$

differentiating with respect to t shows that the tangent vector $\mathbf{t}$ of $\boldsymbol{\gamma}$ is also ℓ-periodic:

$$\mathbf{t}(t+\ell) = \dot{\boldsymbol{\gamma}}(t+\ell) = \dot{\boldsymbol{\gamma}}(t) = \mathbf{t}(t).$$

Rotating by $\pi/2$ gives

$$\mathbf{n}_s(t+\ell) = \mathbf{n}_s(t),$$

and hence $\kappa_s(t+\ell) = \kappa_s(t)$. It follows that $\lambda(t+\ell) = \lambda(t)$ for all t, so the integral in (3.8) is equal to

$$\int_0^\ell \dot{\lambda}(t)\,dt = \lambda(\ell) - \lambda(0) = 0.$$

This contradiction proves that $\boldsymbol{\gamma}$ must have at least four vertices. □

EXERCISES

3.3.1 Show that the ellipse in Example 3.1.2 is convex.

3.3.2 Show that the limacọn in Example 1.1.7 has only two vertices (cf. Example 3.1.3).

3.3.3 Show that a plane curve $\boldsymbol{\gamma}$ has a vertex at $t = t_0$ if and only if the evolute $\boldsymbol{\epsilon}$ of $\boldsymbol{\gamma}$ (Exercise 2.2.7) has a singular point at $t = t_0$.