2
How much does a curve curve?

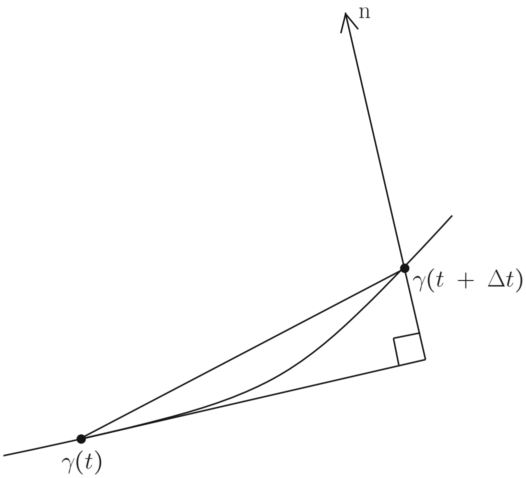
In this chapter, we associate two scalar functions, its curvature and torsion, to any curve in $\mathbb{R}^3$. The curvature measures the extent to which a curve is not contained in a straight line (so that straight lines have zero curvature), and the torsion measures the extent to which a curve is not contained in a plane (so that plane curves have zero torsion). It turns out that the curvature and torsion together determine the shape of a curve.

2.1 Curvature

We are going to try to find a measure of how 'curved' a curve is, and to simplify matters we shall work with plane curves initially. Since a straight line should certainly have zero curvature, a measure of the curvature of a plane curve at a point $\mathbf{p}$ of the curve should be its deviation from the tangent line at $\mathbf{p}$.

Suppose then that $\boldsymbol{\gamma}$ is a unit-speed curve in $\mathbb{R}^2$. As the parameter t of $\boldsymbol{\gamma}$ changes to $t + \Delta t$, the curve moves away from its tangent line at $\boldsymbol{\gamma}(t)$ by a distance $(\boldsymbol{\gamma}(t + \Delta t) - \boldsymbol{\gamma}(t)) \cdot \mathbf{n}$, where $\mathbf{n}$ is a unit vector perpendicular to the tangent vector $\dot{\boldsymbol{\gamma}}(t)$ of $\boldsymbol{\gamma}$ at the point $\boldsymbol{\gamma}(t)$.

Andrew Pressley, *Elementary Differential Geometry: Second Edition*,
Springer Undergraduate Mathematics Series, DOI 10.1007/978-1-84882-891-9_2,



By Taylor's theorem,

$$\boldsymbol{\gamma}(t+\Delta t) = \boldsymbol{\gamma}(t) + \dot{\boldsymbol{\gamma}}(t)\Delta t + \frac{1}{2}\ddot{\boldsymbol{\gamma}}(t)(\Delta t)^2 + \text{ remainder}, \tag{2.1}$$

where (remainder)/$(\Delta t)^2$ tends to zero as Δt tends to zero. Since $\dot{\boldsymbol{\gamma}} \cdot \mathbf{n} = 0$, the deviation of $\boldsymbol{\gamma}$ from its tangent line at $\boldsymbol{\gamma}(t)$ is

$$\frac{1}{2}\ddot{\boldsymbol{\gamma}}(t) \cdot \mathbf{n}(\Delta t)^2 + \text{ remainder}.$$

Since $\boldsymbol{\gamma}$ is unit-speed, $\ddot{\boldsymbol{\gamma}}$ is perpendicular to $\dot{\boldsymbol{\gamma}}$ and therefore parallel to $\mathbf{n}$. Hence, neglecting the remainder terms, the magnitude of the deviation of $\boldsymbol{\gamma}$ from its tangent line is

$$\frac{1}{2} \parallel \ddot{\boldsymbol{\gamma}}(t) \parallel (\Delta t)^2.$$

This suggests the following definition:

Definition 2.1.1

If $\boldsymbol{\gamma}$ is a unit-speed curve with parameter t, its *curvature* $\kappa(t)$ at the point $\boldsymbol{\gamma}(t)$ is defined to be $\parallel \ddot{\boldsymbol{\gamma}}(t) \parallel$.

Note that we make this definition for unit-speed curves in $\mathbb{R}^n$ for all $n \geq 2$. Note also that this definition is consistent with Proposition 1.1.6, which tells us that if $\ddot{\boldsymbol{\gamma}} = \mathbf{0}$ everywhere then $\boldsymbol{\gamma}$ is part of a straight line, and so should certainly have zero curvature.

Let us see if Definition 2.1.1 is consistent with what we expect for the curvature of circles. Consider the circle in $\mathbb{R}^2$ centred at (x_0, y_0) and of radius R. This has a unit-speed parametrization

$$\boldsymbol{\gamma}(t) = \left(x_0 + R\cos\frac{t}{R}, y_0 + R\sin\frac{t}{R}\right).$$

We have $\dot{\boldsymbol{\gamma}}(t) = \left(-\sin\frac{t}{R}, \cos\frac{t}{R}\right)$, and so

$$\parallel \dot{\boldsymbol{\gamma}}(t) \parallel = \sqrt{\left(-\sin\frac{t}{R}\right)^2 + \left(\cos\frac{t}{R}\right)^2} = 1,$$

confirming that $\boldsymbol{\gamma}$ is unit-speed, and hence $\ddot{\boldsymbol{\gamma}}(t) = \left(-\frac{1}{R}\cos\frac{t}{R}, -\frac{1}{R}\sin\frac{t}{R}\right)$, so the curvature

$$\parallel \ddot{\boldsymbol{\gamma}}(t) \parallel = \sqrt{\left(-\frac{1}{R}\cos\frac{t}{R}\right)^2 + \left(-\frac{1}{R}\sin\frac{t}{R}\right)^2} = \frac{1}{R}$$

is the reciprocal of the radius of the circle. This is in accordance with our expectation that small circles should have large curvature and large circles small curvature.

So far we have only considered unit-speed curves. If $\boldsymbol{\gamma}$ is any *regular* curve, then by Proposition 1.3.6, $\boldsymbol{\gamma}$ has a unit-speed parametrization $\tilde{\boldsymbol{\gamma}}$, say, and we can define the curvature of $\boldsymbol{\gamma}$ to be that of $\tilde{\boldsymbol{\gamma}}$. For this to make sense, we need to know that if $\hat{\boldsymbol{\gamma}}$ is another unit-speed parametrization of $\boldsymbol{\gamma}$, the curvatures of $\tilde{\boldsymbol{\gamma}}$ and $\hat{\boldsymbol{\gamma}}$ are the same. To see this, note that $\hat{\boldsymbol{\gamma}}$ will be a reparametrization of $\tilde{\boldsymbol{\gamma}}$ (Exercise 1.3.4), so by Corollary 1.3.7,

$$\tilde{\boldsymbol{\gamma}}(t) = \hat{\boldsymbol{\gamma}}(u),$$

where $u = \pm t + c$ and c is a constant. Then, by the chain rule, $\frac{d\tilde{\boldsymbol{\gamma}}}{dt} = \pm\frac{d\hat{\boldsymbol{\gamma}}}{du}$, so

$$\frac{d^2\tilde{\boldsymbol{\gamma}}}{dt}^2 = \pm\frac{d}{du}\left(\pm\frac{d\hat{\boldsymbol{\gamma}}}{du}\right) = \frac{d^2\hat{\boldsymbol{\gamma}}}{du^2},$$

which shows that $\tilde{\boldsymbol{\gamma}}$ and $\hat{\boldsymbol{\gamma}}$ do indeed have the same curvature.

Although every regular curve $\boldsymbol{\gamma}$ has a unit-speed reparametrization, it may be complicated or impossible to write it down *explicitly* (see Examples 1.3.8 and 1.3.9), and so it is desirable to have a formula for the curvature of $\boldsymbol{\gamma}$ in terms of $\boldsymbol{\gamma}$ itself rather than a reparametrization of it.

Proposition 2.1.2

Let $\boldsymbol{\gamma}(t)$ be a regular curve in $\mathbb{R}^3$. Then, its curvature is

$$\kappa = \frac{\parallel \ddot{\boldsymbol{\gamma}} \times \dot{\boldsymbol{\gamma}} \parallel}{\parallel \dot{\boldsymbol{\gamma}} \parallel^3}, \tag{2.2}$$

where the $\times$ indicates the vector (or cross) product and the dot denotes d/dt.

Of course, since a curve in $\mathbb{R}^2$ can be viewed as a curve in the xy-plane (say) in $\mathbb{R}^3$, Eq. 2.2 can also be used to calculate the curvature of plane curves.

Proof

Let s be a unit-speed parameter for $\boldsymbol{\gamma}$. Then, by the chain rule,

$$\dot{\boldsymbol{\gamma}} = \frac{d\boldsymbol{\gamma}}{dt} = \frac{d\boldsymbol{\gamma}}{ds}\frac{ds}{dt},$$

so

$$\kappa = \left\| \frac{d^2\boldsymbol{\gamma}}{ds^2} \right\| = \left\| \frac{d}{ds}\left(\frac{d\boldsymbol{\gamma}/dt}{ds/dt}\right) \right\| = \left\| \frac{\frac{d}{dt}\left(\frac{d\boldsymbol{\gamma}/dt}{ds/dt}\right)}{ds/dt} \right\| = \left\| \frac{\frac{ds}{dt}\frac{d^2\boldsymbol{\gamma}}{dt^2} - \frac{d^2s}{dt^2}\frac{d\boldsymbol{\gamma}}{dt}}{(ds/dt)^3} \right\|. \quad (2.3)$$

Now,

$$\left(\frac{ds}{dt}\right)^2 = \| \dot{\boldsymbol{\gamma}} \|^2 = \dot{\boldsymbol{\gamma}} \cdot \dot{\boldsymbol{\gamma}},$$

and differentiating with respect to t gives

$$\frac{ds}{dt}\frac{d^{2s}}{dt^2} = \dot{\boldsymbol{\gamma}} \cdot \ddot{\boldsymbol{\gamma}}.$$

Using this and Eq. 2.3, we get

$$\kappa = \left\| \frac{\left(\frac{ds}{dt}\right)^2 \ddot{\boldsymbol{\gamma}} - \frac{d^2s}{dt^2}\frac{ds}{dt}\dot{\boldsymbol{\gamma}}}{(ds/dt)^4} \right\| = \frac{\| (\dot{\boldsymbol{\gamma}} \cdot \dot{\boldsymbol{\gamma}})\ddot{\boldsymbol{\gamma}} - (\dot{\boldsymbol{\gamma}} \cdot \ddot{\boldsymbol{\gamma}})\dot{\boldsymbol{\gamma}} \|}{\| \dot{\boldsymbol{\gamma}} \|^4}.$$

Using the vector triple product identity

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

(where $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$), we get

$$\dot{\boldsymbol{\gamma}} \times (\ddot{\boldsymbol{\gamma}} \times \dot{\boldsymbol{\gamma}}) = (\dot{\boldsymbol{\gamma}} \cdot \dot{\boldsymbol{\gamma}})\ddot{\boldsymbol{\gamma}} - (\dot{\boldsymbol{\gamma}} \cdot \ddot{\boldsymbol{\gamma}})\dot{\boldsymbol{\gamma}}.$$

Further, $\dot{\boldsymbol{\gamma}}$ and $\ddot{\boldsymbol{\gamma}} \times \dot{\boldsymbol{\gamma}}$ are perpendicular vectors, so

$$\| \dot{\boldsymbol{\gamma}} \times (\ddot{\boldsymbol{\gamma}} \times \dot{\boldsymbol{\gamma}}) \| = \| \dot{\boldsymbol{\gamma}} \| \| \ddot{\boldsymbol{\gamma}} \times \dot{\boldsymbol{\gamma}} \|.$$

Hence,

$$\frac{\| (\dot{\boldsymbol{\gamma}} \cdot \dot{\boldsymbol{\gamma}})\ddot{\boldsymbol{\gamma}} - (\dot{\boldsymbol{\gamma}} \cdot \ddot{\boldsymbol{\gamma}})\dot{\boldsymbol{\gamma}} \|}{\| \dot{\boldsymbol{\gamma}} \|^4} = \frac{\| \dot{\boldsymbol{\gamma}} \times (\ddot{\boldsymbol{\gamma}} \times \dot{\boldsymbol{\gamma}}) \|}{\| \dot{\boldsymbol{\gamma}} \|^4} = \frac{\| \dot{\boldsymbol{\gamma}} \| \| \ddot{\boldsymbol{\gamma}} \times \dot{\boldsymbol{\gamma}} \|}{\| \dot{\boldsymbol{\gamma}} \|^4} = \frac{\| \ddot{\boldsymbol{\gamma}} \times \dot{\boldsymbol{\gamma}} \|}{\| \dot{\boldsymbol{\gamma}} \|^3}. \qquad \square$$

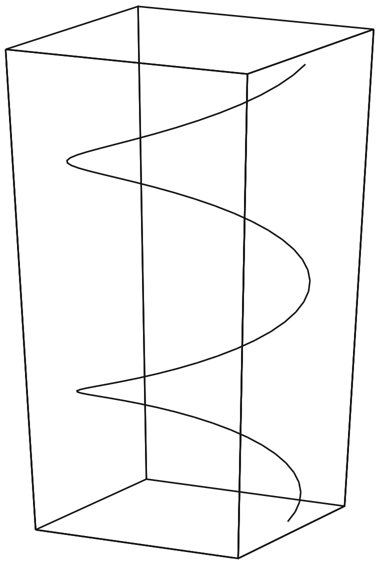
Note that formula (2.2) makes sense provided that $\dot{\boldsymbol{\gamma}} \neq \mathbf{0}$. Thus, the curvature is defined at all regular points of the curve.

Example 2.1.3

A *circular helix* with axis the z-axis is a curve of the form

$$\boldsymbol{\gamma}(\theta) = (a\cos\theta, a\sin\theta, b\theta), \quad \theta \in \mathbb{R},$$

where a and b are constants.



If (x, y, z) is a point on the helix, so that

$$x = a\cos\theta, \ y = a\sin\theta, \ z = b\theta,$$

for some value of θ, then $x^2+y^2 = a^2$, showing that the helix lies on the cylinder with axis the z-axis and radius $|a|$; the positive number $|a|$ is called the *radius* of the helix. As θ increases by 2π, the point $(a\cos\theta, a\sin\theta, b\theta)$ rotates once round the z-axis and moves parallel to the z-axis by $2\pi b$; the positive number $2\pi|b|$ is called the *pitch* of the helix.

Let us compute the curvature of the helix using the formula in Proposition 2.1.2. Denoting $d/d\theta$ by a dot, we have $\dot{\boldsymbol{\gamma}}(\theta) = (-a\sin\theta, a\cos\theta, b)$ so

$$\| \dot{\boldsymbol{\gamma}}(\theta) \| = \sqrt{a^2 + b^2}.$$

This shows that $\dot{\boldsymbol{\gamma}}(\theta)$ is never zero, so $\boldsymbol{\gamma}$ is regular (unless $a = b = 0$, in which case the image of the helix is a single point). Hence, the formula in

Proposition 2.1.2 applies, and we have $\ddot{\gamma} = (-a\cos\theta, -a\sin\theta, 0)$ so $\ddot{\gamma} \times \dot{\gamma} = (-ab\sin\theta, ab\cos\theta, -a^2)$ and hence

$$\kappa = \frac{\| (-ab\sin\theta, ab\cos\theta, -a^2) \|}{\| (-a\sin\theta, a\cos\theta, b) \|^3} = \frac{(a^2b^2 + a^4)^{1/2}}{(a^2+b^2)^{3/2}} = \frac{|a|}{a^2+b^2}. \tag{2.4}$$

Thus, the curvature of the helix is constant.

Let us examine some limiting cases to see if this result agrees with what we already know. First, suppose that $b = 0$ (but $a \neq 0$). Then, the helix is simply a circle in the xy-plane of radius $|a|$, so by the calculation following Definition 2.1.1 its curvature is $1/|a|$. On the other hand, the formula (2.4) gives the curvature as

$$\frac{|a|}{a^2+0^2} = \frac{|a|}{a^2} = \frac{|a|}{|a|^2} = \frac{1}{|a|}.$$

Next, suppose that $a = 0$ (but $b \neq 0$). Then, the image of the helix is just the z-axis, a straight line, so the curvature is zero. And formula (2.4) gives zero when $a = 0$ too.

EXERCISES

2.1.1 Compute the curvature of the following curves:

(i) $\gamma(t) = \left(\frac{1}{3}(1+t)^{3/2} \frac{1}{3}(1-t)^{3/2}, \frac{t}{\sqrt{2}}\right)$.

(ii) $\gamma(t) = \left(\frac{4}{5}\cos t, 1-\sin t, -\frac{3}{5}\cos t\right)$.

(iii) $\gamma(t) = (t, \cosh t)$.

(iv) $\gamma(t) = (\cos^3 t, \sin^3 t)$.

For the astroid in (iv), show that the curvature tends to ∞ as we approach one of the points $(\pm 1, 0)$, $(0, \pm 1)$. Compare with the sketch found in Exercise 1.1.5.

2.1.2 Show that, if the curvature $\kappa(t)$ of a regular curve $\gamma(t)$ is > 0 everywhere, then $\kappa(t)$ is a smooth function of t. Give an example to show that this may not be the case without the assumption that $\kappa > 0$.

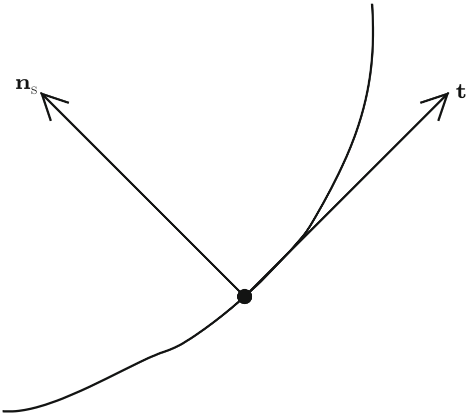
2.2 Plane curves

For plane curves, it is possible to refine the definition of curvature slightly and give it an appealing geometric interpretation.

Suppose that $\boldsymbol{\gamma}(s)$ is a unit-speed curve in $\mathbb{R}^2$. Denoting d/ds by a dot, let

$$\mathbf{t} = \dot{\boldsymbol{\gamma}}$$

be the tangent vector of $\boldsymbol{\gamma}$; note that $\mathbf{t}$ is a unit vector. There are two unit vectors perpendicular to $\mathbf{t}$; we make a choice by defining $\mathbf{n}_s$, the *signed unit normal* of $\boldsymbol{\gamma}$, to be the unit vector obtained by rotating $\mathbf{t}$ anticlockwise by $\pi/2$.



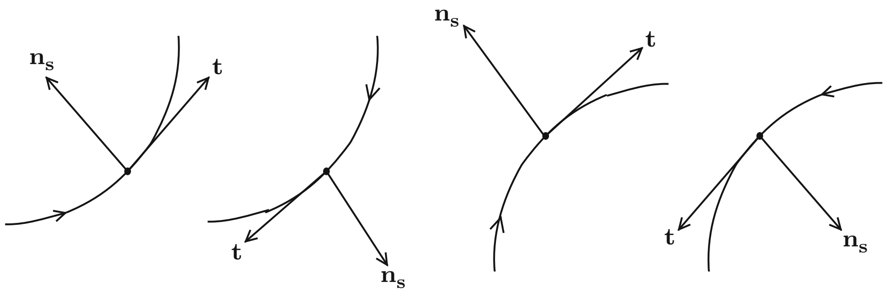
By Proposition 1.2.4, $\dot{\mathbf{t}} = \ddot{\boldsymbol{\gamma}}$ is perpendicular to $\mathbf{t}$, and hence parallel to $\mathbf{n}_s$. Thus, there is a scalar κ_s such that

$$\ddot{\boldsymbol{\gamma}} = \kappa_s \mathbf{n}_s;$$

κ_s is called the *signed curvature* of $\boldsymbol{\gamma}$ (it can be positive, negative or zero). Note that, since $\| \mathbf{n}_s \| = 1$, we have

$$\kappa = \| \ddot{\boldsymbol{\gamma}} \| = \| \kappa_s \mathbf{n}_s \| = |\kappa_s|, \tag{2.5}$$

so the curvature of $\boldsymbol{\gamma}$ is the absolute value of its signed curvature. The following diagrams show how the sign of the signed curvature is determined (in each case, the arrow on the curve indicates the direction of increasing s); κ_s is negative for the two middle diagrams and positive for the other two.



$\kappa_s > 0$ $\kappa_s < 0$ $\kappa_s < 0$ $\kappa_s > 0$

If $\boldsymbol{\gamma}(t)$ is a regular, but not necessarily unit-speed, curve we define the unit tangent vector $\mathbf{t}$, signed unit normal $\mathbf{n}_s$ and signed curvature κ_s of $\boldsymbol{\gamma}$ to be those of its unit-speed parametrization $\tilde{\boldsymbol{\gamma}}(s)$, where s is the arc-length of $\boldsymbol{\gamma}$. Thus,

$$\mathbf{t} = \frac{d\boldsymbol{\gamma}/dt}{ds/dt} = \frac{d\boldsymbol{\gamma}/dt}{\| d\boldsymbol{\gamma}/dt \|},$$

$\mathbf{n}_s$ is obtained by rotating $\mathbf{t}$ anticlockwise by $\pi/2$, and

$$\frac{d\mathbf{t}}{dt} = \frac{d\mathbf{t}}{ds}\frac{ds}{dt} = \kappa_s \frac{ds}{dt}\mathbf{n}_s = \kappa_s \left\| \frac{d\boldsymbol{\gamma}}{dt} \right\| \mathbf{n}_s.$$

The signed curvature has a simple geometric interpretation in terms of the rate at which the tangent vector rotates. If $\boldsymbol{\gamma}$ is a unit-speed curve, the direction of the tangent vector $\dot{\boldsymbol{\gamma}}(s)$ is measured by the angle $\varphi(s)$ such that

$$\dot{\boldsymbol{\gamma}}(s) = (\cos\varphi(s), \sin\varphi(s)). \tag{2.6}$$

The angle $\varphi(s)$ is not unique, however, as we can add to any particular choice any integer multiple of 2π. The following result guarantees that there is always a *smooth* choice:

Proposition 2.2.1

Let $\boldsymbol{\gamma} : (\alpha, \beta) \to \mathbb{R}^2$ be a unit-speed curve, let $s_0 \in (\alpha, \beta)$ and let φ_0 be such that

$$\dot{\boldsymbol{\gamma}}(s_0) = (\cos\varphi_0, \sin\varphi_0).$$

Then there is a unique smooth function $\varphi : (\alpha, \beta) \to \mathbb{R}$ such that $\varphi(s_0) = \varphi_0$ and that Eq. 2.6 holds for all $s \in (\alpha, \beta)$.

Proof

Let

$$\dot{\boldsymbol{\gamma}}(s) = (f(s), g(s));$$

note that

$$f(s)^2 + g(s)^2 = 1 \quad \text{for all } s \tag{2.7}$$

since $\boldsymbol{\gamma}$ is unit-speed. Define

$$\varphi(s) = \varphi_0 + \int_{s_0}^{s} (f\dot{g} - g\dot{f})\, dt.$$

Obviously $\varphi(s_0) = \varphi_0$. Moreover, since the functions f and g are smooth, so is $\dot{\varphi} = f\dot{g} - g\dot{f}$, and hence so is φ.

Let

$$F = f\cos\varphi + g\sin\varphi, \qquad G = f\sin\varphi - g\cos\varphi.$$

Then,

$$\dot{F} = (\dot{f} + g\dot{\varphi})\cos\varphi + (\dot{g} - f\dot{\varphi})\sin\varphi.$$

But

$$\dot{f} + g\dot{\varphi} = \dot{f}(1-g^2) + fg\dot{g} = f(f\dot{f} + g\dot{g}) = 0,$$

where the second equality used Eq. 2.7 and the last equality used its consequence

$$f\dot{f} + g\dot{g} = 0.$$

Similarly, $\dot{g} - f\dot{\varphi} = 0$. Hence, $\dot{F} = 0$ and F is constant. A similar argument shows that G is constant. But

$$F(s_0) = f(s_0)\cos\varphi_0 + g(s_0)\sin\varphi_0 = \cos^2\varphi_0 + \sin^2\varphi_0 = 1,$$

and similarly $G(s_0) = 0$. It follows that

$$f\cos\varphi + g\sin\varphi = 1, \qquad f\sin\varphi - g\cos\varphi = 0$$

for all s. These equations imply that $f = \cos\varphi$, $g = \sin\varphi$, and hence that the smooth function φ satisfies Eq. 2.6.

As to the uniqueness, if ψ is another smooth function such that $\psi(s_0) = \varphi_0$ and $\dot{\boldsymbol{\gamma}}(s) = (\cos\psi(s), \sin\psi(s))$ for $s \in (\alpha, \beta)$, there is an integer $n(s)$ such that

$$\psi(s) - \varphi(s) = 2\pi n(s) \qquad \text{for all } s \in (\alpha, \beta).$$

Because φ and ψ are smooth, n is a smooth, hence continuous, function of s. This implies that n is a constant: otherwise we would have $n(s_0) \neq n(s_1)$ for some $s_1 \in (\alpha, \beta)$, and then by the intermediate value theorem the continuous function $n(s)$ would have to take all values between $n(s_0)$ and $n(s_1)$ when s is between s_0 and s_1. But most real numbers between $n(s_0)$ and $n(s_1)$ are not integers! Thus, n is actually independent of s, and since $\psi(s_0) = \varphi(s_0) = \varphi_0$, we must have $n = 0$ and hence $\psi(s) = \varphi(s)$ for all $s \in (\alpha, \beta)$. □

Definition 2.2.2

The smooth function φ in Proposition 2.2.1 is called the *turning angle* of $\boldsymbol{\gamma}$ determined by the condition $\varphi(s_0) = \varphi_0$.

We are now in a position to give the geometric interpretation of the signed curvature that we promised earlier.

Proposition 2.2.3

Let $\boldsymbol{\gamma}(s)$ be a unit-speed plane curve, and let $\varphi(s)$ be a turning angle for $\boldsymbol{\gamma}$. Then,

$$\kappa_s = \frac{d\varphi}{ds}.$$

Thus, *the signed curvature is the rate at which the tangent vector of the curve rotates*. As the diagrams following Eq. 2.5 show, the signed curvature is positive or negative accordingly as $\mathbf{t}$ rotates anticlockwise or clockwise as one moves along the curve in the direction of increasing s.

Proof

By Eq. 2.6 the tangent vector $\mathbf{t} = (\cos\varphi, \sin\varphi)$, so

$$\dot{\mathbf{t}} = \dot{\varphi}(-\sin\varphi, \cos\varphi).$$

Since $\mathbf{n}_s = (-\sin\varphi, \cos\varphi)$, the equation $\dot{\mathbf{t}} = \kappa_s \mathbf{n}_s$ gives the stated result. □

Example 2.2.4

Let us find the signed curvature of the catenary (Exercise 1.2.1). Using the parametrization $\boldsymbol{\gamma}(t) = (t, \cosh t)$ we get $\dot{\boldsymbol{\gamma}} = (1, \sinh t)$ and hence

$$s = \int_0^t \sqrt{1 + \sinh^2 t}\, dt = \sinh t,$$

so if φ is the angle between $\dot{\boldsymbol{\gamma}}$ and the x-axis,

$$\begin{aligned} \tan\varphi = \sinh t &= s, \\ \therefore \quad \sec^2\varphi \frac{d\varphi}{ds} &= 1, \\ \therefore \quad \kappa_s = \frac{d\varphi}{ds} = \frac{1}{\sec^2\varphi} &= \frac{1}{1+\tan^2\varphi} = \frac{1}{1+s^2}. \end{aligned}$$

Proposition 2.2.3 has an interesting consequence in terms of the *total signed curvature* of a unit-speed *closed* curve $\boldsymbol{\gamma}$ of length ℓ, namely

$$\int_0^\ell \kappa_s(s)\, ds. \tag{2.8}$$

Corollary 2.2.5

The total signed curvature of a closed plane curve is an integer multiple of 2π.

Proof

Let $\boldsymbol{\gamma}$ be a unit-speed closed plane curve and let ℓ be its length. By Proposition 2.2.3, the total signed curvature of $\boldsymbol{\gamma}$ is

$$\int_0^\ell \frac{d\varphi}{ds}\, ds = \varphi(\ell) - \varphi(0),$$

where φ is a turning angle for $\boldsymbol{\gamma}$. Now, $\boldsymbol{\gamma}$ is l-periodic (see Section 1.4):

$$\boldsymbol{\gamma}(s + \ell) = \boldsymbol{\gamma}(s).$$

Differentiating both sides gives

$$\dot{\boldsymbol{\gamma}}(s + \ell) = \dot{\boldsymbol{\gamma}}(s),$$

and in particular $\dot{\boldsymbol{\gamma}}(\ell) = \dot{\boldsymbol{\gamma}}(0)$. Hence, by Eq. 2.6,

$$(\cos\varphi(\ell), \sin\varphi(\ell)) = (\cos\varphi(0), \sin\varphi(0)),$$

which implies that $\varphi(\ell) - \varphi(0)$ is an integer multiple of 2π. □

The next result shows that a unit-speed plane curve is essentially determined once we know its signed curvature at each point of the curve. The meaning of 'essentially' here is 'up to a direct isometry of $\mathbb{R}^2$', i.e., a map $M : \mathbb{R}^2 \to \mathbb{R}^2$ of the form

$$M = T_{\mathbf{a}} \circ \rho_\theta,$$

where ρ_θ is an anticlockwise rotation by an angle θ about the origin,

$$\rho_\theta(x, y) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta),$$

and $T_{\mathbf{a}}$ is the translation by the vector $\mathbf{a}$,

$$T_{\mathbf{a}}(\mathbf{v}) = \mathbf{v} + \mathbf{a},$$

for any vectors (x, y) and $\mathbf{v}$ in $\mathbb{R}^2$ (see Appendix 1).

Theorem 2.2.6

Let $k : (\alpha, \beta) \to \mathbb{R}$ be any smooth function. Then, there is a unit-speed curve $\boldsymbol{\gamma} : (\alpha, \beta) \to \mathbb{R}^2$ whose signed curvature is k.

Further, if $\tilde{\boldsymbol{\gamma}} : (\alpha, \beta) \to \mathbb{R}^2$ is any other unit-speed curve whose signed curvature is k, there is a direct isometry M of $\mathbb{R}^2$ such that

$$\tilde{\boldsymbol{\gamma}}(s) = M(\boldsymbol{\gamma}(s)) \quad \text{for all } s \in (\alpha, \beta).$$

Proof

For the first part, fix $s_0 \in (\alpha, \beta)$ and define, for any $s \in (\alpha, \beta)$,

$$\begin{aligned}\varphi(s) &= \int_{s_0}^{s} k(u)du, \quad \text{(cf. Proposition 2.2.3)},\\ \boldsymbol{\gamma}(s) &= \left(\int_{s_0}^{s} \cos\varphi(t)dt, \int_{s_0}^{s} \sin\varphi(t)dt\right).\end{aligned}$$

Then, the tangent vector of $\boldsymbol{\gamma}$ is

$$\dot{\boldsymbol{\gamma}}(s) = (\cos\varphi(s), \sin\varphi(s)),$$

which is a unit vector making an angle $\varphi(s)$ with the x-axis. Thus, $\boldsymbol{\gamma}$ is unit-speed and, by Proposition 2.2.3, its signed curvature is

$$\frac{d\varphi}{ds} = \frac{d}{ds}\int_{s_0}^{s} k(u)du = k(s).$$

For the second part, let $\tilde{\varphi}(s)$ be a smooth turning angle for $\tilde{\boldsymbol{\gamma}}$. Thus,

$$\begin{aligned}\dot{\tilde{\boldsymbol{\gamma}}}(s) &= (\cos\tilde{\varphi}(s), \sin\tilde{\varphi}(s)),\\ \therefore \quad \tilde{\boldsymbol{\gamma}}(s) &= \left(\int_{s_0}^{s} \cos\tilde{\varphi}(t)dt, \int_{s_0}^{s} \sin\tilde{\varphi}(t)dt\right) + \tilde{\boldsymbol{\gamma}}(s_0). \qquad (2.9)\end{aligned}$$

By Proposition 2.2.3, $k(s) = d\tilde{\varphi}/ds$ so

$$\tilde{\varphi}(s) = \int_{s_0}^{s} k(u)du + \tilde{\varphi}(s_0).$$

Inserting this into Eq. 2.9, and writing $\mathbf{a}$ for the constant vector $\tilde{\boldsymbol{\gamma}}(s_0)$ and θ for the constant scalar $\tilde{\varphi}(s_0)$, we get

$$\begin{aligned}\tilde{\boldsymbol{\gamma}}(s) &= T_{\mathbf{a}}\left(\int_{s_0}^{s} \cos(\varphi(t)+\theta)dt, \int_{s_0}^{s} \sin(\varphi(t)+\theta)dt\right)\\ &= T_{\mathbf{a}}\left(\cos\theta\int_{s_0}^{s} \cos\varphi(t)dt - \sin\theta\int_{s_0}^{s} \sin\varphi(t)dt,\right.\\ &\qquad\qquad \left.\sin\theta\int_{s_0}^{s} \cos\varphi(t)dt + \cos\theta\int_{s_0}^{s} \sin\varphi(t)dt\right)\\ &= T_{\mathbf{a}}\rho_\theta\left(\int_{s_0}^{s} \cos\varphi(t)dt, \int_{s_0}^{s} \sin\varphi(t)dt\right)\\ &= T_{\mathbf{a}}\rho_\theta(\boldsymbol{\gamma}(s)).\end{aligned}$$

□

Example 2.2.7

Any regular plane curve $\boldsymbol{\gamma}$ whose curvature is a positive *constant* is part of a circle. To see this, let κ be the curvature of $\boldsymbol{\gamma}$, and let κ_s be its signed curvature. Then, by Eq. 2.5,

$$\kappa_s = \pm\kappa.$$

A priori, we could have $\kappa_s = \kappa$ at some points of the curve and $\kappa_s = -\kappa$ at others, but in fact this cannot happen since κ_s is a continuous function of s (see Exercise 2.2.2), so the intermediate value theorem tells us that, if κ_s takes both the value κ and the value $-\kappa$, it must take all values between. Thus, either $\kappa_s = \kappa$ at all points of the curve, or $\kappa_s = -\kappa$ at all points of the curve. In particular, κ_s is constant.

The idea now is to show that, whatever the value of κ_s, we can find a parametrized circle whose signed curvature is κ_s. The theorem then tells us that *every* curve whose signed curvature is κ_s can be obtained by applying a direct isometry to this circle. Since rotations and translations obviously take circles to circles, it follows that *every* curve whose signed curvature is constant is (part of) a circle.

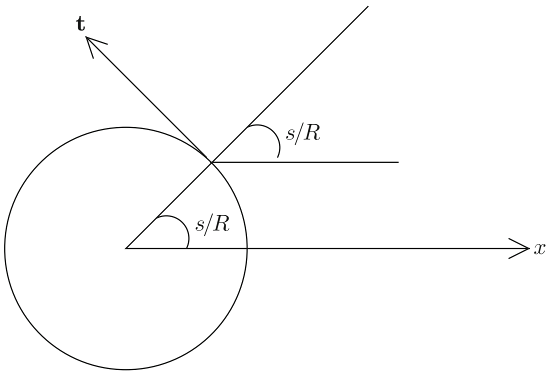
A unit-speed parametrization of the circle with centre the origin and radius R is

$$\boldsymbol{\gamma}(s) = \left(R\cos\frac{s}{R}, R\sin\frac{s}{R}\right).$$

Its tangent vector

$$\mathbf{t} = \dot{\boldsymbol{\gamma}} = \left(-\sin\frac{s}{R}, \cos\frac{s}{R}\right)$$

is the unit vector making an angle $\pi/2 + s/R$ with the positive x-axis:



Hence, the signed curvature of $\boldsymbol{\gamma}$ is

$$\frac{d}{ds}\left(\frac{\pi}{2} + \frac{s}{R}\right) = \frac{1}{R}.$$

Thus, if $\kappa_s > 0$, the circle of radius $1/\kappa_s$ has signed curvature κ_s.

If $\kappa_s < 0$, it is easy to check that the curve

$$\tilde{\gamma}(s) = \left(R\cos\frac{s}{R}, -R\sin\frac{s}{R}\right)$$

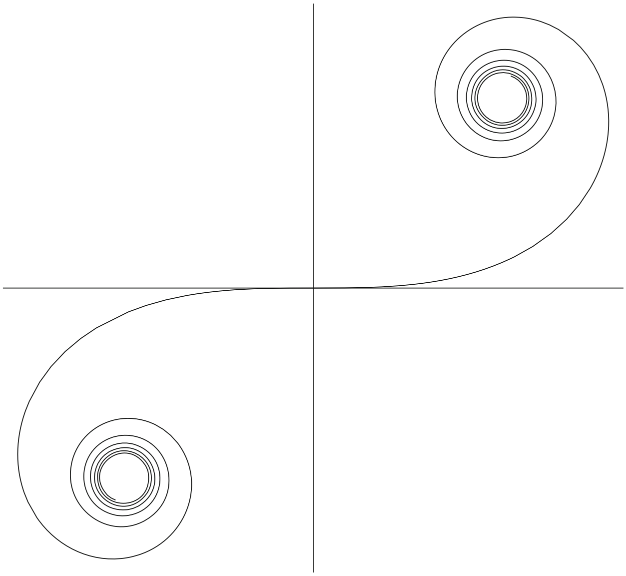
(which is just another parametrization of the circle with centre the origin and radius R) has signed curvature $-1/R$. Thus, if $R = -1/\kappa_s$ we again get a circle with signed curvature κ_s.

These calculations should be compared to the analogous ones for curvature (as opposed to signed curvature) following Definition 2.1.1.

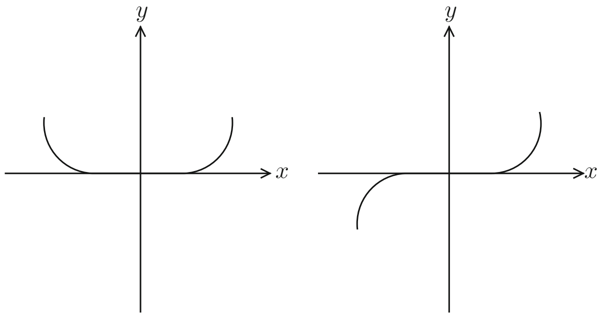
Example 2.2.8

Theorem 2.2.6 shows that we can find a plane curve with any given smooth function as its signed curvature. But simple curvatures can lead to complicated curves. For example, let the signed curvature be $\kappa_s(s) = s$. Following the proof of Theorem 2.2.6, and taking $s_0 = 0$, we get $\varphi(s) = \int_0^s u du = \frac{s^2}{2}$ so

$$\boldsymbol{\gamma}(s) = \left(\int_0^s \cos\left(\frac{t^2}{2}\right) dt, \int_0^s \sin\left(\frac{t^2}{2}\right) dt\right).$$



These integrals cannot be evaluated in terms of 'elementary' functions. (They arise in the theory of diffraction of light, where they are called *Fresnel's integrals*, and the curve $\boldsymbol{\gamma}$ is called *Cornu's Spiral*, although it was first considered by Euler.) The picture of $\boldsymbol{\gamma}$ above is obtained by computing the integrals numerically.



It is natural to ask whether Theorem 2.2.6 remains true if we replace 'signed curvature' by 'curvature'. The first part holds if (and only if) we assume that $k \geq 0$, for then $\boldsymbol{\gamma}$ can be chosen to have signed curvature k and so will have curvature k as well. The second part of Theorem 2.2.6, however, no longer holds. For, we can take a (smooth) curve $\boldsymbol{\gamma}$ that coincides with the x-axis for $-1 \leq x \leq 1$ (say), and is otherwise above the x-axis. (The reader who wishes to write down such a curve explicitly will find the solution of Exercise 9.4.3 helpful.) We now reflect the part of the curve with $x \leq 0$ in the x-axis. The new curve has the same curvature as $\boldsymbol{\gamma}$, but obviously cannot be obtained by applying an isometry to $\boldsymbol{\gamma}$. See Exercise 2.2.3 for a version of Theorem 2.2.6 that *is* valid for curvature instead of signed curvature.

EXERCISES

2.2.1 Show that, if $\boldsymbol{\gamma}$ is a unit-speed plane curve,

$$\dot{\mathbf{n}}_s = -\kappa_s \mathbf{t}.$$

2.2.2 Show that the signed curvature of any regular plane curve $\boldsymbol{\gamma}(t)$ is a smooth function of t. (Compare with Exercise 2.1.2.)

2.2.3 Let $\boldsymbol{\gamma}$ and $\tilde{\boldsymbol{\gamma}}$ be two plane curves. Show that, if $\tilde{\boldsymbol{\gamma}}$ is obtained from $\boldsymbol{\gamma}$ by applying an isometry M of $\mathbb{R}^2$, the signed curvatures κ_s and $\tilde{\kappa}_s$ of $\boldsymbol{\gamma}$ and $\tilde{\boldsymbol{\gamma}}$ are equal if M is direct but that $\tilde{\kappa}_s = -\kappa_s$ if M is opposite (in particular, $\boldsymbol{\gamma}$ and $\tilde{\boldsymbol{\gamma}}$ have the same curvature). Show, conversely, that if $\boldsymbol{\gamma}$ and $\tilde{\boldsymbol{\gamma}}$ have the same nowhere-vanishing curvature, then $\tilde{\boldsymbol{\gamma}}$ can be obtained from $\boldsymbol{\gamma}$ by applying an isometry of $\mathbb{R}^2$.

2.2.4 Let k be the signed curvature of a plane curve $\mathcal{C}$ expressed in terms of its arc-length. Show that, if $\mathcal{C}_a$ is the image of $\mathcal{C}$ under the dilation

$\mathbf{v} \mapsto a\mathbf{v}$ of the plane (where a is a non-zero constant), the signed curvature of $\mathcal{C}_a$ in terms of *its* arc-length s is $\frac{1}{a}k(\frac{s}{a})$.

A heavy chain suspended at its ends hanging loosely takes the form of a plane curve $\mathcal{C}$. Show that, if s is the arc-length of $\mathcal{C}$ measured from its lowest point, φ the angle between the tangent of $\mathcal{C}$ and the horizontal, and T the tension in the chain, then

$$T\cos\varphi = \lambda, \quad T\sin\varphi = \mu s,$$

where λ, μ are non-zero constants (we assume that the chain has constant mass per unit length). Show that the signed curvature of $\mathcal{C}$ is

$$\kappa_s = \frac{1}{a}\left(1 + \frac{s^2}{a^2}\right)^{-1},$$

where $a = \lambda/\mu$, and deduce that $\mathcal{C}$ can be obtained from the catenary in Example 2.2.4 by applying a dilation and an isometry of the plane.

2.2.5 Let $\boldsymbol{\gamma}(t)$ be a regular plane curve and let λ be a constant. The *parallel curve* $\boldsymbol{\gamma}^\lambda$ of $\boldsymbol{\gamma}$ is defined by

$$\boldsymbol{\gamma}^\lambda(t) = \boldsymbol{\gamma}(t) + \lambda\mathbf{n}_s(t).$$

Show that, if $\lambda\kappa_s(t) \neq 1$ for all values of t, then $\boldsymbol{\gamma}^\lambda$ is a regular curve and that its signed curvature is $\kappa_s/|1 - \lambda\kappa_s|$.

2.2.6 Another approach to the curvature of a unit-speed plane curve $\boldsymbol{\gamma}$ at a point $\boldsymbol{\gamma}(s_0)$ is to look for the 'best approximating circle' at this point. We can then *define* the curvature of $\boldsymbol{\gamma}$ to be the reciprocal of the radius of this circle.

Carry out this programme by showing that the centre of the circle which passes through three nearby points $\boldsymbol{\gamma}(s_0)$ and $\boldsymbol{\gamma}(s_0 \pm \delta s)$ on $\boldsymbol{\gamma}$ approaches the point

$$\boldsymbol{\epsilon}(s_0) = \boldsymbol{\gamma}(s_0) + \frac{1}{\kappa_s(s_0)}\mathbf{n}_s(s_0)$$

as δs tends to zero. The circle $\mathcal{C}$ with centre $\boldsymbol{\epsilon}(s_0)$ passing through $\boldsymbol{\gamma}(s_0)$ is called the *osculating circle* to $\boldsymbol{\gamma}$ at the point $\boldsymbol{\gamma}(s_0)$, and $\boldsymbol{\epsilon}(s_0)$ is called the *centre of curvature* of $\boldsymbol{\gamma}$ at $\boldsymbol{\gamma}(s_0)$. The radius of $\mathcal{C}$ is $1/|\kappa_s(s_0)| = 1/\kappa(s_0)$, where κ is the curvature of $\boldsymbol{\gamma}$ – this is called the *radius of curvature* of $\boldsymbol{\gamma}$ at $\boldsymbol{\gamma}(s_0)$.

2.2.7 With the notation in the preceding exercise, we regard $\boldsymbol{\epsilon}$ as the parametrization of a new curve, called the *evolute* of $\boldsymbol{\gamma}$ (if $\boldsymbol{\gamma}$ is any regular plane curve, its evolute is defined to be that of a unit-speed reparametrization of $\boldsymbol{\gamma}$). Assume that $\dot{\kappa}_s(s) \neq 0$ for all values of s (a dot denoting d/ds), say $\dot{\kappa}_s > 0$ for all s (this can be achieved by replacing s by $-s$ if necessary). Show that the arc-length of $\boldsymbol{\epsilon}$ is $-\frac{1}{\kappa_s(s)}$ (up to adding a constant), and calculate the signed curvature of $\boldsymbol{\epsilon}$. Show also that all the normal lines to $\boldsymbol{\gamma}$ are tangent to $\boldsymbol{\epsilon}$ (for this reason, the evolute of $\boldsymbol{\gamma}$ is sometimes described as the 'envelope' of the normal lines to $\boldsymbol{\gamma}$). Show that the evolute of the cycloid

$$\boldsymbol{\gamma}(t) = a(t - \sin t, 1 - \cos t), \qquad 0 < t < 2\pi,$$

where $a > 0$ is a constant, is

$$\boldsymbol{\epsilon}(t) = a(t + \sin t, -1 + \cos t)$$

(see Exercise 1.1.7) and that, after a suitable reparametrization, $\boldsymbol{\epsilon}$ can be obtained from $\boldsymbol{\gamma}$ by a translation of the plane.

2.2.8 A string of length ℓ is attached to the point $\boldsymbol{\gamma}(0)$ of a unit-speed plane curve $\boldsymbol{\gamma}(s)$. Show that when the string is wound onto the curve while being kept taught, its endpoint traces out the curve

$$\boldsymbol{\iota}(s) = \boldsymbol{\gamma}(s) + (\ell - s)\dot{\boldsymbol{\gamma}}(s),$$

where $0 < s < \ell$ and a dot denotes d/ds. The curve $\boldsymbol{\iota}$ is called the *involute* of $\boldsymbol{\gamma}$ (if $\boldsymbol{\gamma}$ is any regular plane curve, we define its involute to be that of a unit-speed reparametrization of $\boldsymbol{\gamma}$). Suppose that the signed curvature κ_s of $\boldsymbol{\gamma}$ is never zero, say $\kappa_s(s) > 0$ for all s. Show that the signed curvature of $\boldsymbol{\iota}$ is $1/(\ell - s)$.

2.2.9 Show that the involute of the catenary

$$\boldsymbol{\gamma}(t) = (t, \cosh t)$$

with $l = 0$ (see the preceding exercise) is the *tractrix*

$$x = \cosh^{-1}\left(\frac{1}{y}\right) - \sqrt{1 - y^2}.$$

See Section 8.3 for a simple geometric characterization of this curve.

2.2.10 A unit-speed plane curve $\boldsymbol{\gamma}(s)$ rolls without slipping along a straight line ℓ parallel to a unit vector $\mathbf{a}$, and initially touches ℓ at a point $\mathbf{p} = \boldsymbol{\gamma}(0)$. Let $\mathbf{q}$ be a point fixed *relative to* $\boldsymbol{\gamma}$. Let $\boldsymbol{\Gamma}(s)$ be the point to which $\mathbf{q}$ has moved when $\boldsymbol{\gamma}$ has rolled a distance s

along ℓ (note that $\mathbf{\Gamma}$ will not usually be unit-speed). Let $\theta(s)$ be the angle between $\mathbf{a}$ and the tangent vector $\dot{\boldsymbol{\gamma}}$. Show that

$$\mathbf{\Gamma}(s) = \mathbf{p} + s\mathbf{a} + \rho_{-\theta(s)}(\mathbf{q} - \boldsymbol{\gamma}(s)),$$

where ρ_φ is the rotation about the origin through an angle φ. Show further that

$$\dot{\mathbf{\Gamma}}(s) \cdot \rho_{-\theta(s)}(\mathbf{q} - \boldsymbol{\gamma}(s)) = 0.$$

Geometrically, this means that a point on $\mathbf{\Gamma}$ moves as if it is rotating about the instantaneous point of contact of the rolling curve with ℓ. See Exercise 1.1.7 for a special case.

2.3 Space curves

Our main interest in this book is in curves (and surfaces) in $\mathbb{R}^3$, i.e., space curves. While a plane curve is essentially determined by its curvature (see Theorem 2.2.6), this is no longer true for space curves. For example, a circle of radius 1 in the xy-plane and a circular helix with $a = b = 1/2$ (see Example 2.1.3) both have curvature 1 everywhere, but it is obviously impossible to change one curve into the other by any isometry of $\mathbb{R}^3$. We shall define another type of curvature for space curves, called the *torsion*, and we shall prove that the curvature and torsion of a curve together determine the curve up to a direct isometry of $\mathbb{R}^3$.

Let $\boldsymbol{\gamma}(s)$ be a unit-speed curve in $\mathbb{R}^3$, and let $\mathbf{t} = \dot{\boldsymbol{\gamma}}$ be its unit tangent vector. *If the curvature* $\kappa(s)$ *is non-zero*, we define the *principal normal* of $\boldsymbol{\gamma}$ at the point $\boldsymbol{\gamma}(s)$ to be the vector

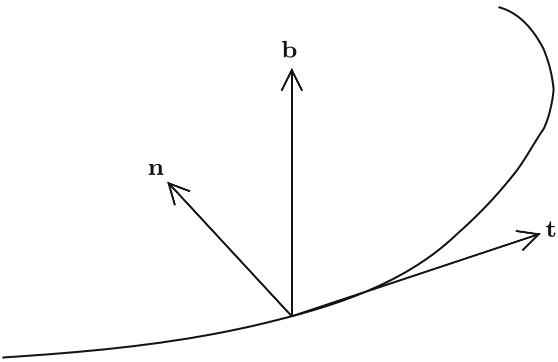
$$\mathbf{n}(s) = \frac{1}{\kappa(s)}\dot{\mathbf{t}}(s). \tag{2.10}$$

Since $\| \dot{\mathbf{t}} \| = \kappa$, $\mathbf{n}$ is a unit vector. Further, by Proposition 1.2.4, $\mathbf{t} \cdot \dot{\mathbf{t}} = 0$, so $\mathbf{t}$ and $\mathbf{n}$ are actually perpendicular unit vectors. It follows that

$$\mathbf{b} = \mathbf{t} \times \mathbf{n} \tag{2.11}$$

is a unit vector perpendicular to both $\mathbf{t}$ and $\mathbf{n}$. The vector $\mathbf{b}(s)$ is called the *binormal* vector of $\boldsymbol{\gamma}$ at the point $\boldsymbol{\gamma}(s)$. Thus, $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ is an orthonormal basis of $\mathbb{R}^3$, and is *right-handed*, i.e.,

$$\mathbf{b} = \mathbf{t} \times \mathbf{n}, \quad \mathbf{n} = \mathbf{b} \times \mathbf{t}, \quad \mathbf{t} = \mathbf{n} \times \mathbf{b}.$$



Since $\mathbf{b}(s)$ is a unit vector for all s, $\dot{\mathbf{b}}$ is perpendicular to $\mathbf{b}$. Now we use the 'product rule' for differentiating the vector product of vector-valued functions $\mathbf{u}$ and $\mathbf{v}$ of a parameter s:

$$\frac{d}{ds}(\mathbf{u} \times \mathbf{v}) = \frac{d\mathbf{u}}{ds} \times \mathbf{v} + \mathbf{u} \times \frac{d\mathbf{v}}{ds}.$$

(This is easily proved by writing out both sides in component form and using the usual product rule for differentiating scalar functions.) Applying this to $\mathbf{b} = \mathbf{t} \times \mathbf{n}$ gives

$$\dot{\mathbf{b}} = \dot{\mathbf{t}} \times \mathbf{n} + \mathbf{t} \times \dot{\mathbf{n}} = \mathbf{t} \times \dot{\mathbf{n}}, \tag{2.12}$$

since by the definition (2.10) of $\mathbf{n}$, $\dot{\mathbf{t}} \times \mathbf{n} = \kappa \mathbf{n} \times \mathbf{n} = \mathbf{0}$. Equation 2.12 shows that $\dot{\mathbf{b}}$ is perpendicular to $\mathbf{t}$. Being perpendicular to both $\mathbf{t}$ and $\mathbf{b}$, $\dot{\mathbf{b}}$ must be parallel to $\mathbf{n}$, so

$$\dot{\mathbf{b}} = -\tau \mathbf{n}, \tag{2.13}$$

for some scalar τ, which is called the *torsion* of $\boldsymbol{\gamma}$ (inserting the minus sign here will reduce the total number of minus signs later). Note that the torsion is only defined if the curvature is non-zero.

Of course, we define the torsion of an arbitrary regular curve $\boldsymbol{\gamma}$ to be that of a unit-speed reparametrization of $\boldsymbol{\gamma}$. As in the case of the curvature, to see that this makes sense, we have to investigate how the torsion is affected by a change in the unit-speed parameter of $\boldsymbol{\gamma}$ of the form

$$u = \pm s + c,$$

where c is a constant. But this change of parameter clearly has the following effect on the vectors introduced above:

$$\mathbf{t} \mapsto \pm \mathbf{t}, \ \dot{\mathbf{t}} \mapsto \dot{\mathbf{t}}, \ \mathbf{n} \mapsto \mathbf{n}, \ \mathbf{b} \mapsto \pm \mathbf{b}, \ \dot{\mathbf{b}} \mapsto \dot{\mathbf{b}};$$

it follows from Eq. 2.13 that $\tau \mapsto \tau$. Thus, *the curvature and torsion are well-defined for any regular curve.*

Just as we did for the curvature in Proposition 2.1.2, it is possible to give a formula for the torsion of a regular space curve $\boldsymbol{\gamma}$ in terms of $\boldsymbol{\gamma}$ itself, rather than in terms of a unit-speed reparametrization:

Proposition 2.3.1

Let $\boldsymbol{\gamma}(t)$ be a regular curve in $\mathbb{R}^3$ with nowhere-vanishing curvature. Then, denoting d/dt by a dot, its torsion is given by

$$\tau = \frac{(\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}}) \cdot \dddot{\boldsymbol{\gamma}}}{\| \dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}} \|^2}. \tag{2.14}$$

Note that this formula shows that $\tau(t)$ is defined at all points $\boldsymbol{\gamma}(t)$ of the curve at which its curvature $\kappa(t)$ is non-zero, since by Proposition 2.1.2 this is the condition for the denominator on the right-hand side to be non-zero.

Proof

We could 'derive' Eq. 2.14 by imitating the proof of Proposition 2.1.2. But it is easier and clearer to proceed as follows, even though this method has the disadvantage that one must know the formula (2.14) for τ in advance.

We first treat the case in which $\boldsymbol{\gamma}$ is unit-speed. Using Eqs. 2.11 and 2.13,

$$\tau = -\mathbf{n} \cdot \dot{\mathbf{b}} = -\mathbf{n} \cdot (\mathbf{t} \times \mathbf{n})^{\cdot} = -\mathbf{n} \cdot (\dot{\mathbf{t}} \times \mathbf{n} + \mathbf{t} \times \dot{\mathbf{n}}) = -\mathbf{n} \cdot (\mathbf{t} \times \dot{\mathbf{n}}).$$

Now, $\mathbf{n} = \frac{1}{\kappa}\dot{\mathbf{t}} = \frac{1}{\kappa}\ddot{\boldsymbol{\gamma}}$, so

$$\tau = -\frac{1}{\kappa}\ddot{\boldsymbol{\gamma}} \cdot \left(\dot{\boldsymbol{\gamma}} \times \frac{d}{dt}\left(\frac{1}{\kappa}\ddot{\boldsymbol{\gamma}}\right)\right) = -\frac{1}{\kappa}\ddot{\boldsymbol{\gamma}} \cdot \left(\dot{\boldsymbol{\gamma}} \times \left(\frac{1}{\kappa}\dddot{\boldsymbol{\gamma}} - \frac{\dot{\kappa}}{\kappa^2}\ddot{\boldsymbol{\gamma}}\right)\right) = \frac{1}{\kappa^2}\dddot{\boldsymbol{\gamma}} \cdot (\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}}),$$

since $\ddot{\boldsymbol{\gamma}} \cdot (\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}}) = 0$ and $\ddot{\boldsymbol{\gamma}} \cdot (\dot{\boldsymbol{\gamma}} \times \dddot{\boldsymbol{\gamma}}) = -\dddot{\boldsymbol{\gamma}} \cdot (\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}})$. This agrees with Eq. 2.14, for, since $\boldsymbol{\gamma}$ is unit-speed, $\dot{\boldsymbol{\gamma}}$ and $\ddot{\boldsymbol{\gamma}}$ are perpendicular, so

$$\| \dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}} \| = \| \dot{\boldsymbol{\gamma}} \| \| \ddot{\boldsymbol{\gamma}} \| = \| \ddot{\boldsymbol{\gamma}} \| = \kappa.$$

In the general case, let s be arc-length along $\boldsymbol{\gamma}$. Then,

$$\frac{d\boldsymbol{\gamma}}{dt} = \frac{ds}{dt}\frac{d\boldsymbol{\gamma}}{ds}, \quad \frac{d^2\boldsymbol{\gamma}}{dt^2} = \left(\frac{ds}{dt}\right)^2 \frac{d^2\boldsymbol{\gamma}}{ds^2} + \frac{d^2 s}{dt^2}\frac{d\boldsymbol{\gamma}}{ds},$$

$$\frac{d^3\boldsymbol{\gamma}}{dt^3} = \left(\frac{ds}{dt}\right)^3 \frac{d^3\boldsymbol{\gamma}}{ds^3} + 3\frac{ds}{dt}\frac{d^2 s}{dt^2}\frac{d^2\boldsymbol{\gamma}}{ds^2} + \frac{d^3 s}{dt^3}\frac{d\boldsymbol{\gamma}}{ds}.$$

Hence,

$$\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}} = \left(\frac{ds}{dt}\right)^3 \left(\frac{d\boldsymbol{\gamma}}{ds} \times \frac{d^2\boldsymbol{\gamma}}{ds^2}\right),$$

$$\dddot{\boldsymbol{\gamma}} \cdot (\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}}) = \left(\frac{ds}{dt}\right)^6 \left(\frac{d^3\boldsymbol{\gamma}}{ds^3} \cdot \left(\frac{d\boldsymbol{\gamma}}{ds} \times \frac{d^2\boldsymbol{\gamma}}{ds^2}\right)\right).$$

So the torsion of $\boldsymbol{\gamma}$ is

$$\tau = \frac{\left(\frac{d^3\boldsymbol{\gamma}}{ds^3} \cdot \left(\frac{d\boldsymbol{\gamma}}{ds} \times \frac{d^2\boldsymbol{\gamma}}{ds^2}\right)\right)}{\left\| \frac{d\boldsymbol{\gamma}}{ds} \times \frac{d^2\boldsymbol{\gamma}}{ds^2} \right\|^2} = \frac{\dddot{\boldsymbol{\gamma}} \cdot (\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}})}{\| \dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}} \|^2}. \qquad \square$$

Example 2.3.2

We compute the torsion of the circular helix $\boldsymbol{\gamma}(\theta) = (a\cos\theta, a\sin\theta, b\theta)$ studied in Example 2.1.3. We have

$$\begin{aligned} \dot{\boldsymbol{\gamma}}(\theta) &= (-a\sin\theta, a\cos\theta, b), \quad \ddot{\boldsymbol{\gamma}}(\theta) = (-a\cos\theta, -a\sin\theta, 0), \\ \dddot{\boldsymbol{\gamma}}(\theta) &= (a\sin\theta, -a\cos\theta, 0), \quad \dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}} = (ab\sin\theta, -ab\cos\theta, a^2), \\ \| \dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}} \|^2 &= a^2(a^2+b^2), \quad (\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}}) \cdot \dddot{\boldsymbol{\gamma}} = a^2 b, \end{aligned}$$

so the torsion

$$\tau = \frac{(\dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}}) \cdot \dddot{\boldsymbol{\gamma}}}{\| \dot{\boldsymbol{\gamma}} \times \ddot{\boldsymbol{\gamma}} \|^2} = \frac{a^2 b}{a^2(a^2+b^2)} = \frac{b}{a^2+b^2}.$$

Note that the torsion of the circular helix in Example 2.3.2 becomes zero when $b = 0$, in which case the helix is just a circle in the xy-plane. This gives us a clue to the geometrical interpretation of torsion, contained in the next proposition.

Proposition 2.3.3

Let $\boldsymbol{\gamma}$ be a regular curve in $\mathbb{R}^3$ with nowhere vanishing curvature (so that the torsion τ of $\boldsymbol{\gamma}$ is defined). Then, the image of $\boldsymbol{\gamma}$ is contained in a plane if and only if τ is zero at every point of the curve.

Proof

We can assume that $\boldsymbol{\gamma}$ is unit-speed (for this can be achieved by reparametrizing $\boldsymbol{\gamma}$, and reparametrizing changes neither the torsion nor the fact that $\boldsymbol{\gamma}$ is, or is not, contained in a plane). We denote the parameter of $\boldsymbol{\gamma}$ by s and d/ds by a dot as usual.

Suppose first that the image of $\boldsymbol{\gamma}$ is contained in the plane $\mathbf{v} \cdot \mathbf{N} = d$, where $\mathbf{N}$ is a constant vector and d is a constant scalar and $\mathbf{v} \in \mathbb{R}^3$. We can assume that $\mathbf{N}$ is a unit vector. Differentiating $\boldsymbol{\gamma} \cdot \mathbf{N} = d$ with respect to s, we get

$$\mathbf{t} \cdot \mathbf{N} = 0, \tag{2.15}$$

$$\therefore \quad \dot{\mathbf{t}} \cdot \mathbf{N} = 0 \qquad (\text{since } \dot{\mathbf{N}} = \mathbf{0}),$$

$$\therefore \quad \kappa\mathbf{n}\cdot\mathbf{N} = 0 \qquad (\text{since } \dot{\mathbf{t}} = \kappa\mathbf{n}),$$
$$\therefore \quad \mathbf{n}\cdot\mathbf{N} = 0 \qquad (\text{since } \kappa \neq 0). \tag{2.16}$$

Equations 2.15 and 2.16 show that $\mathbf{t}$ and $\mathbf{n}$ are perpendicular to $\mathbf{N}$. It follows that $\mathbf{b} = \mathbf{t}\times\mathbf{n}$ is parallel to $\mathbf{N}$. Since $\mathbf{N}$ and $\mathbf{b}$ are both unit vectors, and $\mathbf{b}(s)$ is a smooth (hence continuous) function of s, we must have $\mathbf{b}(s) = \mathbf{N}$ for all s or $\mathbf{b}(s) = -\mathbf{N}$ for all s. In both cases, $\mathbf{b}$ is a constant vector. But then $\dot{\mathbf{b}} = \mathbf{0}$, so $\tau = 0$.

Conversely, suppose that $\tau = 0$ everywhere. By Eq. 2.13, $\dot{\mathbf{b}} = \mathbf{0}$, so $\mathbf{b}$ is a constant vector. The first part of the proof suggests that $\boldsymbol{\gamma}$ should be contained in a plane $\mathbf{v}\cdot\mathbf{b} = \text{constant}$. We therefore consider

$$\frac{d}{ds}(\boldsymbol{\gamma}\cdot\mathbf{b}) = \dot{\boldsymbol{\gamma}}\cdot\mathbf{b} = \mathbf{t}\cdot\mathbf{b} = 0,$$

so $\boldsymbol{\gamma}\cdot\mathbf{b}$ is a constant (scalar), say d. This means that $\boldsymbol{\gamma}$ is indeed contained in the plane $\mathbf{v}\cdot\mathbf{b} = d$. □

There is a gap in our calculations which we would like to fill. Namely, we know that, for a unit-speed curve, we have

$$\dot{\mathbf{t}} = \kappa\mathbf{n} \quad \text{and} \quad \dot{\mathbf{b}} = -\tau\mathbf{n}$$

(these were our definitions of $\mathbf{n}$ and τ, respectively), but we have not computed $\dot{\mathbf{n}}$. This is not difficult. Since $\{\mathbf{t},\mathbf{n},\mathbf{b}\}$ is a right-handed orthonormal basis of $\mathbb{R}^3$,

$$\mathbf{t}\times\mathbf{n} = \mathbf{b}, \quad \mathbf{n}\times\mathbf{b} = \mathbf{t}, \quad \mathbf{b}\times\mathbf{t} = \mathbf{n}.$$

Hence,

$$\dot{\mathbf{n}} = \dot{\mathbf{b}}\times\mathbf{t} + \mathbf{b}\times\dot{\mathbf{t}} = -\tau\mathbf{n}\times\mathbf{t} + \kappa\mathbf{b}\times\mathbf{n} = -\kappa\mathbf{t} + \tau\mathbf{b}.$$

Putting all these together, we get the following theorem.

Theorem 2.3.4

Let $\boldsymbol{\gamma}$ be a unit-speed curve in $\mathbb{R}^3$ with nowhere vanishing curvature. Then,

$$\begin{aligned}\dot{\mathbf{t}} &= \kappa\mathbf{n}\\ \dot{\mathbf{n}} &= -\kappa\mathbf{t} + \tau\mathbf{b}\\ \dot{\mathbf{b}} &= -\tau\mathbf{n}.\end{aligned} \tag{2.17}$$

Equations 2.17 are called the *Frenet–Serret equations*. Notice that the matrix

$$\begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}$$

which expresses $\dot{\mathbf{t}}$, $\dot{\mathbf{n}}$ and $\dot{\mathbf{b}}$ in terms of $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b}$ is *skew-symmetric*, i.e., it is equal to the negative of its transpose. This helps when trying to remember the equations. (The 'reason' for this skew-symmetry can be seen in Exercise 2.3.6.)

Here is a simple application of Frenet–Serret:

Proposition 2.3.5

Let $\boldsymbol{\gamma}$ be a unit-speed curve in $\mathbb{R}^3$ with constant curvature and zero torsion. Then, $\boldsymbol{\gamma}$ is a parametrization of (part of) a circle.

Proof

This result is actually an immediate consequence of Example 2.2.7 and Proposition 2.3.3, but the following proof is instructive and gives more information, namely the centre and radius of the circle and the plane in which it lies.

By the proof of Proposition 2.3.3, the binormal $\mathbf{b}$ is a constant vector and $\boldsymbol{\gamma}$ is contained in a plane Π, say, perpendicular to $\mathbf{b}$. Now

$$\frac{d}{ds}\left(\boldsymbol{\gamma} + \frac{1}{\kappa}\mathbf{n}\right) = \mathbf{t} + \frac{1}{\kappa}\dot{\mathbf{n}} = \mathbf{0},$$

using the fact that the curvature κ is constant and the Frenet–Serret equation

$$\dot{\mathbf{n}} = -\kappa\mathbf{t} + \tau\mathbf{b} = -\kappa\mathbf{t} \quad (\text{since } \tau = 0)$$

(the reason for considering $\boldsymbol{\gamma} + \frac{1}{\kappa}\mathbf{n}$ can be found in Exercise 2.2.6). Hence, $\boldsymbol{\gamma} + \frac{1}{\kappa}\mathbf{n}$ is a constant vector, say $\mathbf{a}$, and we have

$$\| \boldsymbol{\gamma} - \mathbf{a} \| = \| -\frac{1}{\kappa}\mathbf{n} \| = \frac{1}{\kappa}.$$

This shows that $\boldsymbol{\gamma}$ lies on the sphere $\mathcal{S}$, say, with centre $\mathbf{a}$ and radius $1/\kappa$. The intersection of Π and $\mathcal{S}$ is a circle, say $\mathcal{C}$, and we have shown that $\boldsymbol{\gamma}$ is a parametrization of part of $\mathcal{C}$. If r is the radius of $\mathcal{C}$, we have $\kappa = 1/r$ so $r = 1/\kappa$ is also the radius of $\mathcal{S}$. It follows that $\mathcal{C}$ is a *great circle* on $\mathcal{S}$, i.e., that Π passes through the centre $\mathbf{a}$ of $\mathcal{S}$. Thus, $\mathbf{a}$ is the centre of $\mathcal{C}$ and the equation of Π is $\mathbf{v} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{b}$. □

We conclude this chapter with the analogue of Theorem 2.2.6 for space curves.

Theorem 2.3.6

Let $\boldsymbol{\gamma}(s)$ and $\tilde{\boldsymbol{\gamma}}(s)$ be two unit-speed curves in $\mathbb{R}^3$ with the same curvature $\kappa(s) > 0$ and the same torsion $\tau(s)$ for all s. Then, there is a direct isometry M of $\mathbb{R}^3$ such that

$$\tilde{\boldsymbol{\gamma}}(s) = M(\boldsymbol{\gamma}(s)) \quad \text{for all } s.$$

Further, if k and t are smooth functions with $k > 0$ everywhere, there is a unit-speed curve in $\mathbb{R}^3$ whose curvature is k and whose torsion is t.

Proof

Let $\mathbf{t}, \mathbf{n}$ and $\mathbf{b}$ be the tangent vector, principal normal and binormal of $\boldsymbol{\gamma}$, and let $\tilde{\mathbf{t}}, \tilde{\mathbf{n}}$ and $\tilde{\mathbf{b}}$ be those of $\tilde{\boldsymbol{\gamma}}$. Let s_0 be a fixed value of the parameter s, let θ be the angle between $\mathbf{t}(s_0)$ and $\tilde{\mathbf{t}}(s_0)$ and let ρ be the rotation through an angle θ around the axis passing through the origin and perpendicular to both of these vectors. Then, $\rho(\mathbf{t}(s_0)) = \tilde{\mathbf{t}}(s_0)$; let $\hat{\mathbf{n}} = \rho(\mathbf{n}(s_0))$, $\hat{\mathbf{b}} = \rho(\mathbf{b}(s_0))$. If φ is the angle between $\hat{\mathbf{n}}$ and $\tilde{\mathbf{n}}(s_0)$, let ρ' be the rotation through an angle φ around the axis passing through the origin parallel to $\tilde{\mathbf{t}}(s_0)$. Then, ρ' fixes $\tilde{\mathbf{t}}(s_0)$ and takes $\hat{\mathbf{n}}$ to $\tilde{\mathbf{n}}(s_0)$. Since $\{\mathbf{t}(s_0), \mathbf{n}(s_0), \mathbf{b}(s_0)\}$ and $\{\tilde{\mathbf{t}}(s_0), \tilde{\mathbf{n}}(s_0), \tilde{\mathbf{b}}(s_0)\}$ are both right-handed orthonormal bases of $\mathbb{R}^3$, $\rho' \circ \rho$ takes the vectors $\mathbf{t}(s_0), \mathbf{n}(s_0), \mathbf{b}(s_0)$ to the vectors $\tilde{\mathbf{t}}(s_0), \tilde{\mathbf{n}}(s_0), \tilde{\mathbf{b}}(s_0)$, respectively. Now let M be the direct isometry $M = T_{\tilde{\boldsymbol{\gamma}}(s_0) - \boldsymbol{\gamma}(s_0)} \circ \rho' \circ \rho$. By Exercise 2.3.5, the curve $\boldsymbol{\Gamma} = M(\boldsymbol{\gamma})$ is unit-speed, and if $\mathbf{T}$, $\mathbf{N}$ and $\mathbf{B}$ denote its unit tangent vector, principal normal and binormal, we have

$$\boldsymbol{\Gamma}(s_0) = \tilde{\boldsymbol{\gamma}}(s_0), \;\; \mathbf{T}(s_0) = \tilde{\mathbf{t}}(s_0), \;\; \mathbf{N}(s_0) = \tilde{\mathbf{n}}(s_0), \;\; \mathbf{B}(s_0) = \tilde{\mathbf{b}}(s_0). \tag{2.18}$$

The trick now is to consider the expression

$$A(s) = \tilde{\mathbf{t}} \cdot \mathbf{T} + \tilde{\mathbf{n}} \cdot \mathbf{N} + \tilde{\mathbf{b}} \cdot \mathbf{B}.$$

In view of Eq. 2.18, we have $A(s_0) = 3$. On the other hand, since $\tilde{\mathbf{t}}$ and $\mathbf{T}$ are unit vectors, $\tilde{\mathbf{t}} \cdot \mathbf{T} \leq 1$, with equality holding if and only if $\tilde{\mathbf{t}} = \mathbf{T}$; and similarly for $\tilde{\mathbf{n}} \cdot \mathbf{N}$ and $\tilde{\mathbf{b}} \cdot \mathbf{B}$. It follows that $A(s) \leq 3$, with equality holding if and only if $\tilde{\mathbf{t}} = \mathbf{T}, \tilde{\mathbf{n}} = \mathbf{N}$ and $\tilde{\mathbf{b}} = \mathbf{B}$. Thus, if we can prove that A is constant, it will follow in particular that $\tilde{\mathbf{t}} = \mathbf{T}$, i.e., that $\dot{\tilde{\boldsymbol{\gamma}}} = \dot{\boldsymbol{\Gamma}}$, and hence that $\tilde{\boldsymbol{\gamma}}(s) - \boldsymbol{\Gamma}(s)$ is a constant. But by Eq. 2.18 again, this constant vector must be zero, so $\tilde{\boldsymbol{\gamma}} = \boldsymbol{\Gamma}$.

For the first part of the theorem, we are therefore reduced to proving that A is constant. But, using the Frenet–Serret equations,

$$\begin{aligned} \dot{A} &= \dot{\tilde{\mathbf{t}}} \cdot \mathbf{T} + \dot{\tilde{\mathbf{n}}} \cdot \mathbf{N} + \dot{\tilde{\mathbf{b}}} \cdot \mathbf{B} + \tilde{\mathbf{t}} \cdot \dot{\mathbf{T}} + \tilde{\mathbf{n}} \cdot \dot{\mathbf{N}} + \tilde{\mathbf{b}} \cdot \dot{\mathbf{B}} \\ &= \kappa \tilde{\mathbf{n}} \cdot \mathbf{T} + (-\kappa \tilde{\mathbf{t}} + \tau \tilde{\mathbf{b}}) \cdot \mathbf{N} + (-\tau \tilde{\mathbf{n}}) \cdot \mathbf{B} + \tilde{\mathbf{t}} \cdot \kappa \mathbf{N} \\ &\quad + \tilde{\mathbf{n}} \cdot (-\kappa \mathbf{T} + \tau \mathbf{B}) + \tilde{\mathbf{b}} \cdot (-\tau \mathbf{N}), \end{aligned}$$

and this vanishes since the terms cancel in pairs.

For the second part of the theorem, we observe first that it follows from the theory of ordinary differential equations that the equations

$$\dot{\mathbf{T}} = k\mathbf{N}, \tag{2.19}$$

$$\dot{\mathbf{N}} = -\,k\mathbf{T} + t\mathbf{B}, \tag{2.20}$$

$$\dot{\mathbf{B}} = -\,t\mathbf{N} \tag{2.21}$$

have a unique solution $\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)$ such that $\mathbf{T}(s_0), \mathbf{N}(s_0), \mathbf{B}(s_0)$ are the standard orthonormal vectors $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, $\mathbf{k} = (0, 0, 1)$, respectively. Since the matrix

$$\begin{pmatrix} 0 & k & 0 \\ -k & 0 & t \\ 0 & -t & 0 \end{pmatrix}$$

expressing $\dot{\mathbf{T}}, \dot{\mathbf{N}}$ and $\dot{\mathbf{B}}$ in terms of $\mathbf{T}, \mathbf{N}$ and $\mathbf{B}$ is skew-symmetric, it follows that the vectors $\mathbf{T}$, $\mathbf{N}$ and $\mathbf{B}$ are orthonormal for all values of s (see Exercise 2.3.6).

Now define

$$\boldsymbol{\gamma}(s) = \int_{s_0}^{s} \mathbf{T}(u)du.$$

Then, $\dot{\boldsymbol{\gamma}} = \mathbf{T}$, so since $\mathbf{T}$ is a unit vector, $\boldsymbol{\gamma}$ is unit-speed. Next, $\dot{\mathbf{T}} = k\mathbf{N}$ by Eq. 2.19, so since $\mathbf{N}$ is a unit vector, k is the curvature of $\boldsymbol{\gamma}$ and $\mathbf{N}$ is its principal normal. Next, since $\mathbf{B}$ is a unit vector perpendicular to $\mathbf{T}$ and $\mathbf{N}$, $\mathbf{B} = \lambda\mathbf{T} \times \mathbf{N}$ where λ is a smooth function of s that is equal to ± 1 for all s. Since $\mathbf{k} = \mathbf{i} \times \mathbf{j}$, we have $\lambda(s_0) = 1$, so it follows that $\lambda(s) = 1$ for all s. Hence, $\mathbf{B}$ is the binormal of $\boldsymbol{\gamma}$ and by Eq. 2.21, t is its torsion. □

EXERCISES

2.3.1 Compute $\kappa, \tau, \mathbf{t}, \mathbf{n}$ and $\mathbf{b}$ for each of the following curves, and verify that the Frenet–Serret equations are satisfied:

(i) $\boldsymbol{\gamma}(t) = \left(\frac{1}{3}(1+t)^{3/2} \frac{1}{3}(1-t)^{3/2}, \frac{t}{\sqrt{2}}\right)$.

(ii) $\boldsymbol{\gamma}(t) = \left(\frac{4}{5}\cos t, 1 - \sin t, -\frac{3}{5}\cos t\right)$.

Show that the curve in (ii) is a circle, and find its centre, radius and the plane in which it lies.

2.3.2 Describe all curves in $\mathbb{R}^3$ which have *constant* curvature $\kappa > 0$ and *constant* torsion τ.

2.3.3 A regular curve $\boldsymbol{\gamma}$ in $\mathbb{R}^3$ with curvature > 0 is called a *generalized helix* if its tangent vector makes a fixed angle θ with a fixed unit vector $\mathbf{a}$. Show that the torsion τ and curvature κ of $\boldsymbol{\gamma}$ are related by $\tau = \pm\kappa \cot\theta$. Show conversely that, if the torsion and curvature of a regular curve are related by $\tau = \lambda\kappa$ where λ is a constant, then the curve is a generalized helix.
In view of this result, Examples 2.1.3 and 2.3.2 show that a circular helix is a generalized helix. Verify this directly.

2.3.4 Let $\boldsymbol{\gamma}(t)$ be a unit-speed curve with $\kappa(t) > 0$ and $\tau(t) \neq 0$ for all t. Show that, if $\boldsymbol{\gamma}$ is *spherical*, i.e., if it lies on the surface of a sphere, then

$$\frac{\tau}{\kappa} = \frac{d}{ds}\left(\frac{\dot{\kappa}}{\tau\kappa^2}\right). \tag{2.22}$$

Conversely, show that if Eq. 2.22 holds, then

$$\rho^2 + (\dot{\rho}\sigma)^2 = r^2$$

for some (positive) constant r, where $\rho = 1/\kappa$ and $\sigma = 1/\tau$, and deduce that $\boldsymbol{\gamma}$ lies on a sphere of radius r. Verify that Eq. 2.22 holds for Viviani's curve (Exercise 1.1.8).

2.3.5 Let P be an $n \times n$ orthogonal matrix and let $\mathbf{a} \in \mathbb{R}^n$, so that $M(\mathbf{v}) = P\mathbf{v} + \mathbf{a}$ is an isometry of $\mathbb{R}^3$ (see Appendix 1). Show that, if $\boldsymbol{\gamma}$ is a unit-speed curve in $\mathbb{R}^n$, the curve $\boldsymbol{\Gamma} = M(\boldsymbol{\gamma})$ is also unit-speed. Show also that, if $\mathbf{t}, \mathbf{n}, \mathbf{b}$ and $\mathbf{T}, \mathbf{N}, \mathbf{B}$ are the tangent vector, principal normal and binormal of $\boldsymbol{\gamma}$ and $\boldsymbol{\Gamma}$, respectively, then $\mathbf{T} = P\mathbf{t}$, $\mathbf{N} = P\mathbf{n}$ and $\mathbf{B} = P\mathbf{b}$.

2.3.6 Let (a_{ij}) be a skew-symmetric 3×3 matrix (i.e., $a_{ij} = -a_{ji}$ for all i, j). Let $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ be smooth functions of a parameter s satisfying the differential equations

$$\dot{\mathbf{v}}_i = \sum_{j=1}^{3} a_{ij}\mathbf{v}_j,$$

for $i = 1, 2$ and 3, and suppose that for some parameter value s_0 the vectors $\mathbf{v}_1(s_0), \mathbf{v}_2(s_0)$ and $\mathbf{v}_3(s_0)$ are orthonormal. Show that the vectors $\mathbf{v}_1(s), \mathbf{v}_2(s)$ and $\mathbf{v}_3(s)$ are orthonormal for all values of s.

For the remainder of this book,
all parametrized curves will be assumed to be regular.