12 Minimal surfaces

In Section 9.4 we considered the problem of finding the shortest paths between two points on a surface. We now consider the analogous problem in one higher dimension, that of finding a surface of minimal area with a fixed curve as its boundary. This is called *Plateau's Problem*. The solutions to Plateau's problem turn out to be surfaces whose mean curvature vanishes everywhere. The study of these so-called minimal surfaces was initiated by Euler and Lagrange in the mid-eighteenth century, but new examples of minimal surfaces are still being discovered.

12.1 Plateau's problem

In Section 9.4, we found the condition for a curve on a surface to minimize distance between its endpoints by embedding the given curve in a family of curves passing through the same two points, and studying how the length of the curve varies as the curve varies through the family. Accordingly, we shall now study a family of surface patches $\boldsymbol{\sigma}^\tau : U \to \mathbb{R}^3$, where U is an open subset of $\mathbb{R}^2$ independent of τ, and τ lies in some open interval $(-\delta, \delta)$, for some $\delta > 0$. Let $\boldsymbol{\sigma} = \boldsymbol{\sigma}^0$. The family is required to be *smooth*, in the sense that the map $(u, v, \tau) \mapsto \boldsymbol{\sigma}^\tau(u, v)$ from the open subset $\{(u, v, \tau) \mid (u, v) \in U,\ \tau \in (-\delta, \delta)\}$ of $\mathbb{R}^3$ to $\mathbb{R}^3$ is smooth. The *surface variation* of the family is the function $\boldsymbol{\varphi} : U \to \mathbb{R}^3$ given by

$$\boldsymbol{\varphi} = \dot{\boldsymbol{\sigma}}^\tau|_{\tau=0}\,,$$

where here and elsewhere in this section, a dot denotes $d/d\tau$.

Andrew Pressley, *Elementary Differential Geometry: Second Edition*,
Springer Undergraduate Mathematics Series, DOI 10.1007/978-1-84882-891-9_12,

Let $\boldsymbol{\pi}$ be a simple closed curve that is contained, along with its interior $\mathrm{int}(\boldsymbol{\pi})$, in U (see Section 3.1). Then $\boldsymbol{\pi}$ corresponds to a closed curve $\boldsymbol{\gamma}^\tau = \boldsymbol{\sigma}^\tau \circ \boldsymbol{\pi}$ in the surface patch $\boldsymbol{\sigma}^\tau$, and we define the area function $\mathcal{A}(\tau)$ to be the area of the part of $\boldsymbol{\sigma}^\tau$ inside $\boldsymbol{\gamma}^\tau$:

$$\mathcal{A}(\tau) = \int_{\mathrm{int}(\boldsymbol{\pi})} d\mathcal{A}_{\boldsymbol{\sigma}^\tau}.$$

Note that, if we are considering a family of surfaces with a *fixed* boundary curve $\boldsymbol{\gamma}$, then $\boldsymbol{\gamma}^\tau = \boldsymbol{\gamma}$ for all τ, and hence $\boldsymbol{\varphi}^\tau(u,v) = \mathbf{0}$ when (u,v) is a point on the curve $\boldsymbol{\pi}$.

Theorem 12.1.1

With the above notation, assume that the surface variation $\boldsymbol{\varphi}^\tau$ vanishes along the boundary curve $\boldsymbol{\pi}$. Then,

$$\dot{\mathcal{A}}(0) = -2\int_{\mathrm{int}(\boldsymbol{\pi})} H(EG - F^2)^{1/2}\alpha\, dudv, \qquad (12.1)$$

where H is the mean curvature of $\boldsymbol{\sigma}$, E, F and G are the coefficients of its first fundamental form, and $\alpha = \boldsymbol{\varphi} \cdot \mathbf{N}$ where $\mathbf{N}$ is the standard unit normal of $\boldsymbol{\sigma}$.

We defer the proof of this theorem to the end of this section.

If $\boldsymbol{\sigma}$ has the smallest area among all surfaces with the given boundary curve $\boldsymbol{\gamma}$, then $\mathcal{A}$ must have an absolute minimum at $\tau = 0$, so $\dot{\mathcal{A}}(0) = 0$ for *all* smooth families of surfaces as above. This means that the integral in (12.1) must vanish for all smooth functions $\alpha : U \to \mathbb{R}$. As in the proof of Theorem 9.4.1, this can happen only if the term that multiplies α in the integrand vanishes, in other words only if $H = 0$. This suggests the following definition.

Definition 12.1.2

A *minimal surface* is a surface whose mean curvature is zero everywhere.

Theorem 12.1.1 and the preceding discussion then give

Corollary 12.1.3

If a surface $\mathcal{S}$ has least area among all surfaces with the same boundary curve, then $\mathcal{S}$ is a minimal surface.

Minimal surfaces have an interesting physical interpretation as the shapes taken up by soap films. A soap film has energy by virtue of surface tension, and this energy is proportional to its area. A soap film spanning a wire in the shape of a curve $\mathcal{C}$ should therefore adopt the shape of a surface of least area with boundary $\mathcal{C}$. By Corollary 12.1.3, this will be a minimal surface.

More generally, if the soap film separates two regions of different pressure, the film will adopt the shape of a surface of *constant* mean curvature. This is the case for a soap bubble, for example, for which the air pressure inside the bubble is greater than the pressure outside. To see this, we apply the principle of 'virtual work'. This tells us that, if the soap film is in equilibrium, and we imagine a ('virtual') change in the surface, the change in the energy of the film must be the same as the work done by the film against the air pressure. If p is the pressure difference, the force exerted by the air on a small piece of the surface of area $\Delta\mathcal{A}$ is $p\Delta\mathcal{A}$, and so the work done when it moves a small distance α perpendicular to itself is $\alpha p\Delta\mathcal{A}$. On the other hand, the formula in Theorem 12.1.1 shows that the change in area of the surface is proportional to $\alpha H\Delta\mathcal{A}$ (note that α is the component of the variation $\boldsymbol{\varphi}$ perpendicular to the surface). So p is proportional to H. Since the pressure difference must be the same across the whole surface, so must the mean curvature H. Surfaces of constant non-zero mean curvature were discussed in Section 8.5.

For the moment, we give only one example of a minimal surface; others will be given in the next section. This example already shows, however, that the converse of Corollary 12.1.3 is false.

Example 12.1.4

The surface obtained by revolving the curve $x = \cosh z$ in the xz-plane around the z-axis is called a *catenoid* (a picture of a catenoid can be found in Exercise 5.3.1). The catenoid can be parametrized by

$$\boldsymbol{\sigma}(u, v) = (\cosh u \cos v, \cosh u \sin v, u).$$

Then,

$$\begin{gathered}
\boldsymbol{\sigma}_u = (\sinh u \cos v, \sinh u \sin v, 1), \boldsymbol{\sigma}_v = (-\cosh u \sin v, \cosh u \cos v, 0), \\
\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v = (-\cosh u \cos v, -\cosh u \sin v, \sinh u \cosh u), \\
\mathbf{N} = (-\operatorname{sech} u \cos v, -\operatorname{sech} u \sin v, \operatorname{tanh} u), \\
\boldsymbol{\sigma}_{uu} = (\cosh u \cos v, \cosh u \sin v, 0), \\
\boldsymbol{\sigma}_{uv} = (-\sinh u \sin v, \sinh u \cos v, 0), \\
\boldsymbol{\sigma}_{vv} = (-\cosh u \cos v, -\cosh u \sin v, 0).
\end{gathered}$$

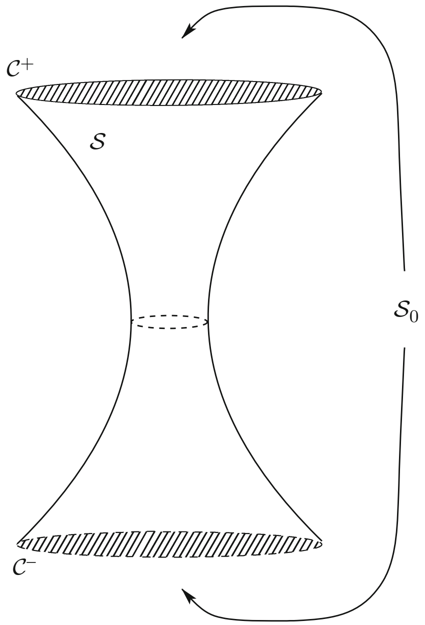
This gives the coefficients of the first and second fundamental forms of $\boldsymbol{\sigma}$ as

$$E = G = \cosh^2 u, \ F = 0, \quad L = -1, \ M = 0, \ N = 1.$$

The first three of these equations show that the parametrization $\boldsymbol{\sigma}$ is conformal, and Corollary 8.1.3 gives

$$H = \frac{LG - 2MF + NE}{2(EG - F^2)} = \frac{-\cosh^2 u + \cosh^2 u}{2\cosh^4 u} = 0,$$

showing that the catenoid is a minimal surface.

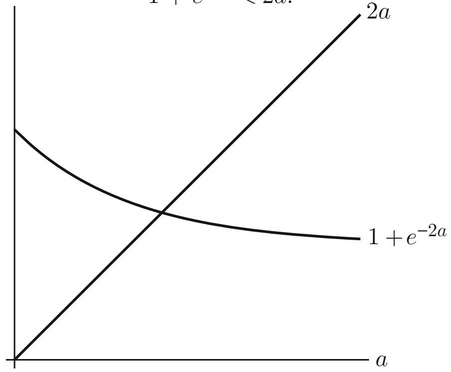


Fix $a > 0$, and let $b = \cosh a$. The surface $\mathcal{S}$ consisting of the part of the catenoid with $|z| < a$ has the two circles $\mathcal{C}^\pm$ of radius b in the planes $z = \pm a$ with centres on the z-axis as boundary. Another surface spanning the same two circles is, of course, the surface $\mathcal{S}_0$ consisting of the two discs $x^2 + y^2 \leq b^2$ in the planes $z = \pm a$. The area of $\mathcal{S}$ is, by Proposition 6.4.2,

$$\int_0^{2\pi} \int_{-a}^{a} (EG - F^2)^{1/2} du dv = \int_0^{2\pi} \int_{-a}^{a} \cosh^2 u \, du dv = 2\pi(a + \sinh a \cosh a).$$

The area of $\mathcal{S}_0$ is, of course, $2\pi b^2 = 2\pi \cosh^2 a$. So the minimal surface $\mathcal{S}$ will *not* minimize the area among all surfaces with boundary the two circles $\mathcal{C}^\pm$ if $\cosh^2 a < a + \sinh a \cosh a$, i.e., if

$$1 + e^{-2a} < 2a. \tag{12.2}$$



The graphs of $1 + e^{-2a}$ and $2a$ as functions of a clearly intersect in exactly one point $a = a_0$, say, and the inequality (12.2) holds if $a > a_0$. If this condition is satisfied, the catenoid is not area minimizing.

It can be shown that if $a < a_0$ the catenoid does have least area among all surfaces spanning the circles $\mathcal{C}^+$ and $\mathcal{C}^-$.

It is time to prove Theorem 12.1.1.

Proof

Let $\boldsymbol{\varphi}^\tau = \dot{\boldsymbol{\sigma}}^\tau$, so that $\boldsymbol{\varphi}^0 = \boldsymbol{\varphi}$, and let $\mathbf{N}^\tau$ be the standard unit normal of $\boldsymbol{\sigma}^\tau$. There are smooth functions α^τ, β^τ and γ^τ of (u, v, τ) such that

$$\boldsymbol{\varphi}^\tau = \alpha^\tau \mathbf{N}^\tau + \beta^\tau \boldsymbol{\sigma}_u^\tau + \gamma^\tau \boldsymbol{\sigma}_v^\tau,$$

so that $\alpha = \alpha^0$. To simplify the notation, we drop the superscript τ for the rest of the proof; at the end of the proof we put $\tau = 0$.

We have

$$\mathcal{A}(\tau) = \int_{\mathrm{int}(\boldsymbol{\pi})} \| \boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v \| \, dudv = \int_{\mathrm{int}(\boldsymbol{\pi})} \mathbf{N} \cdot (\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v) \, dudv,$$

so

$$\dot{\mathcal{A}} = \int_{\mathrm{int}(\boldsymbol{\pi})} \frac{\partial}{\partial \tau} \left(\mathbf{N} \cdot (\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v) \right) dudv. \tag{12.3}$$

Now,

$$\frac{\partial}{\partial \tau}(\mathbf{N} \cdot (\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v)) = \dot{\mathbf{N}} \cdot (\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v) + \mathbf{N} \cdot (\dot{\boldsymbol{\sigma}}_u \times \boldsymbol{\sigma}_v) + \mathbf{N} \cdot (\boldsymbol{\sigma}_u \times \dot{\boldsymbol{\sigma}}_v). \tag{12.4}$$

Since $\mathbf{N}$ is a unit vector,

$$\dot{\mathbf{N}} \cdot (\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v) = \dot{\mathbf{N}} \cdot \mathbf{N} \, \| \boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v \| = 0.$$

On the other hand,

$$\begin{aligned}\mathbf{N}\cdot(\dot{\boldsymbol{\sigma}}_u\times\boldsymbol{\sigma}_v) &= \frac{(\boldsymbol{\sigma}_u\times\boldsymbol{\sigma}_v)\cdot(\dot{\boldsymbol{\sigma}}_u\times\boldsymbol{\sigma}_v)}{\|\boldsymbol{\sigma}_u\times\boldsymbol{\sigma}_v\|}\\ &= \frac{(\boldsymbol{\sigma}_u\cdot\dot{\boldsymbol{\sigma}}_u)(\boldsymbol{\sigma}_v\cdot\boldsymbol{\sigma}_v)-(\boldsymbol{\sigma}_u\cdot\boldsymbol{\sigma}_v)(\boldsymbol{\sigma}_v\cdot\dot{\boldsymbol{\sigma}}_u)}{\|\boldsymbol{\sigma}_u\times\boldsymbol{\sigma}_v\|}\\ &= \frac{G(\boldsymbol{\sigma}_u\cdot\dot{\boldsymbol{\sigma}}_u)-F(\boldsymbol{\sigma}_v\cdot\dot{\boldsymbol{\sigma}}_u)}{(EG-F^2)^{1/2}},\end{aligned}$$

using Proposition 6.4.2. Similarly,

$$\mathbf{N}\cdot(\boldsymbol{\sigma}_u\times\dot{\boldsymbol{\sigma}}_v)=\frac{E(\boldsymbol{\sigma}_v\cdot\dot{\boldsymbol{\sigma}}_v)-F(\boldsymbol{\sigma}_u\cdot\dot{\boldsymbol{\sigma}}_v)}{(EG-F^2)^{1/2}}.$$

Substituting these results into Eq. 12.4, we get

$$\frac{\partial}{\partial\tau}(\mathbf{N}\cdot(\boldsymbol{\sigma}_u\times\boldsymbol{\sigma}_v))=\frac{E(\boldsymbol{\sigma}_v\cdot\dot{\boldsymbol{\sigma}}_v)-F(\dot{\boldsymbol{\sigma}}_u\cdot\boldsymbol{\sigma}_v+\boldsymbol{\sigma}_u\cdot\dot{\boldsymbol{\sigma}}_v)+G(\boldsymbol{\sigma}_u\cdot\dot{\boldsymbol{\sigma}}_u)}{(EG-F^2)^{1/2}}. \quad (12.5)$$

Now

$$\begin{aligned}\dot{\boldsymbol{\sigma}}_u &= \boldsymbol{\varphi}_u=\alpha_u\mathbf{N}+\beta_u\boldsymbol{\sigma}_u+\gamma_u\boldsymbol{\sigma}_v+\alpha\mathbf{N}_u+\beta\boldsymbol{\sigma}_{uu}+\gamma\boldsymbol{\sigma}_{uv},\\ \therefore\quad \boldsymbol{\sigma}_u\cdot\dot{\boldsymbol{\sigma}}_u &= E\beta_u+F\gamma_u+(\boldsymbol{\sigma}_u\cdot\mathbf{N}_u)\alpha+(\boldsymbol{\sigma}_u\cdot\boldsymbol{\sigma}_{uu})\beta+(\boldsymbol{\sigma}_u\cdot\boldsymbol{\sigma}_{uv})\gamma.\end{aligned}$$

Since $\boldsymbol{\sigma}_u\cdot\mathbf{N}_u=-\boldsymbol{\sigma}_{uu}\cdot\mathbf{N}=-L$, $\boldsymbol{\sigma}_u\cdot\boldsymbol{\sigma}_{uu}=\frac{1}{2}E_u$ and $\boldsymbol{\sigma}_u\cdot\boldsymbol{\sigma}_{uv}=\frac{1}{2}E_v$, we get

$$\boldsymbol{\sigma}_u\cdot\dot{\boldsymbol{\sigma}}_u=E\beta_u+F\gamma_u-L\alpha+\frac{1}{2}E_u\beta+\frac{1}{2}E_v\gamma.$$

Similarly,

$$\begin{aligned}\boldsymbol{\sigma}_v\cdot\dot{\boldsymbol{\sigma}}_u &= F\beta_u+G\gamma_u-M\alpha+(F_u-\frac{1}{2}E_v)\beta+\frac{1}{2}G_u\gamma,\\ \boldsymbol{\sigma}_u\cdot\dot{\boldsymbol{\sigma}}_v &= E\beta_v+F\gamma_v-M\alpha+\frac{1}{2}E_v\beta+(F_v-\frac{1}{2}G_u)\gamma,\\ \boldsymbol{\sigma}_v\cdot\dot{\boldsymbol{\sigma}}_v &= F\beta_v+G\gamma_v-N\alpha+\frac{1}{2}G_u\beta+\frac{1}{2}G_v\gamma.\end{aligned}$$

Substituting these last four equations into the right-hand side of Eq. 12.5, simplifying, and using the formula for H in Corollary 8.1.3, we find that

$$\begin{aligned}\frac{\partial}{\partial\tau}(\mathbf{N}\cdot(\boldsymbol{\sigma}_u\times\boldsymbol{\sigma}_v)) = \Big(\beta(EG-F^2)^{1/2}\Big)_u &+ \Big(\gamma(EG-F^2)^{1/2}\Big)_v\\ &-2\alpha H(EG-F^2)^{1/2}.\end{aligned} \quad (12.6)$$

Comparing with Eq. 12.3, and reinstating the superscripts, we see that we must prove that

$$\int_{\mathrm{int}(\boldsymbol{\pi})}\left\{\Big(\beta^0(EG-F^2)^{1/2}\Big)_u+\Big(\gamma^0(EG-F^2)^{1/2}\Big)_v\right\}dudv=0. \quad (12.7)$$

But by Green's theorem (see Section 3.2), this integral is equal to

$$\int_{\boldsymbol{\pi}} (EG - F^2)^{1/2}(\beta^0 dv - \gamma^0 du),$$

and this obviously vanishes because $\beta^0 = \gamma^0 = 0$ along the boundary curve $\boldsymbol{\pi}$. This completes the proof of Theorem 12.1.1. □

Note that we did not quite use the full force of the assumptions in Theorem 12.1.1, since they imply that α^0 $(= \alpha)$ vanishes along the boundary curve, and this was not used in the proof. So Eq. 12.1 holds provided the surface variation $\boldsymbol{\varphi}$ is normal to the surface along the boundary curve.

Note also that Theorem 12.1.1 is intuitively obvious for variations $\boldsymbol{\varphi}$ that are parallel to the surface, i.e., those for which $\alpha = 0$ everywhere on the surface, since such a parallel variation causes the surface to slide along itself and will not change the shape, and in particular the area, of the surface. Thus, the main point is to prove Theorem 12.1.1 for normal variations, i.e., those for which $\beta = \gamma = 0$ everywhere on the surface. Making this restriction simplifies the above proof considerably.

EXERCISES

12.1.1 Show that the Gaussian curvature of a minimal surface is ≤ 0 everywhere, and that it is zero everywhere if and only if the surface is an open subset of a plane. We shall obtain a much more precise result in Corollary 12.5.6.

12.1.2 Let $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ be a minimal surface patch, and assume that $\mathcal{A}_{\boldsymbol{\sigma}}(\mathcal{U}) < \infty$ (see Definition 6.4.1). Let $\lambda \neq 0$ and assume that the principal curvatures κ of $\boldsymbol{\sigma}$ satisfy $|\lambda\kappa| < 1$ everywhere, so that the parallel surface $\boldsymbol{\sigma}^\lambda$ of $\boldsymbol{\sigma}$ (Definition 8.5.1) is a regular surface patch. Prove that

$$\mathcal{A}_{\boldsymbol{\sigma}^\lambda}(\mathcal{U}) \leq \mathcal{A}_{\boldsymbol{\sigma}}(\mathcal{U})$$

and that equality holds for some $\lambda \neq 0$ if and only if $\boldsymbol{\sigma}(U)$ is an open subset of a plane. (Thus, any minimal surface is area-minimizing among its family of parallel surfaces.)

12.1.3 Show that there is no compact minimal surface.

12.1.4 Show that applying a dilation or an isometry of $\mathbb{R}^3$ to a minimal surface gives another minimal surface. Can there be a *local* isometry between a minimal surface and a non-minimal surface?

12.2 Examples of minimal surfaces

The simplest minimal surface is, of course, the plane, for which both principal curvatures are zero everywhere. Apart from this, the first minimal surfaces to be discovered were those in the following two examples.

Example 12.2.1

If a is a non-zero constant, the surface $\mathcal{S}_a$ obtained by rotating the curve $x = \frac{1}{a}\cosh az$ in the xz-plane around the z-axis is called a *catenoid*. More generally, a catenoid is a surface obtained by applying an isometry of $\mathbb{R}^3$ to a surface $\mathcal{S}_a$. Any catenoid $\mathcal{S}$ is a minimal surface, since $\mathcal{S}$ can be obtained from the special catenoid $\mathcal{S}_1$ in Example 12.1.4 by applying an isometry and a dilation (Exercise 12.1.4). A picture of a catenoid can be found in Exercise 5.3.1.

Catenoids are surfaces of revolution. In fact, apart from the plane, they are the only minimal surfaces of revolution:

Proposition 12.2.2

Any minimal surface of revolution $\mathcal{S}$ is an open subset of a plane or a catenoid.

Proof

By applying an isometry of $\mathbb{R}^3$, we can assume that the axis of the surface $\mathcal{S}$ is the z-axis and the profile curve lies in the xz-plane. We parametrize $\mathcal{S}$ in the usual way (see Example 5.3.2):

$$\boldsymbol{\sigma}(u, v) = (f(u)\cos v, f(u)\sin v, g(u)),$$

where the profile curve $u \mapsto (f(u), 0, g(u))$ is assumed to be unit-speed and $f > 0$. From Examples 6.1.3 and 7.1.2, the first and second fundamental forms are

$$du^2 + f(u)^2 dv^2 \quad \text{and} \quad (\dot{f}\ddot{g} - \ddot{f}\dot{g})du^2 + f\dot{g}dv^2,$$

respectively, a dot denoting d/du. By Corollary 8.1.3, the mean curvature is

$$H = \frac{1}{2}\left(\dot{f}\ddot{g} - \ddot{f}\dot{g} + \frac{\dot{g}}{f}\right).$$

We suppose now that, for some value of u, say $u = u_0$, we have $\dot{g}(u_0) \neq 0$. Since $\dot{g}$ is continuous (because g is smooth), we shall then have $\dot{g}(u) \neq 0$ for u

in some open interval containing u_0. Let (α, β) be the largest such interval. Supposing now that $u \in (\alpha, \beta)$, the unit-speed condition $\dot{f}^2 + \dot{g}^2 = 1$ gives (as in Example 8.1.4)

$$\dot{f}\ddot{g} - \ddot{f}\dot{g} = -\frac{\ddot{f}}{\dot{g}},$$

and so we get

$$H = \frac{1}{2}\left(\frac{\dot{g}}{f} - \frac{\ddot{f}}{\dot{g}}\right).$$

Since $\dot{g}^2 = 1 - \dot{f}^2$, $\mathcal{S}$ is minimal if and only if

$$f\ddot{f} = 1 - \dot{f}^2. \tag{12.8}$$

To solve the differential equation (12.8), put $h = \dot{f}$, and note that

$$\ddot{f} = \frac{dh}{dt} = \frac{dh}{df}\frac{df}{dt} = h\frac{dh}{df}.$$

Hence, Eq. 12.8 becomes

$$fh\frac{dh}{df} = 1 - h^2.$$

Note that, since $\dot{g} \neq 0$, we have $h^2 \neq 1$, and so we can integrate this equation as follows:

$$\int \frac{hdh}{1-h^2} = \int \frac{df}{f},$$

$$\therefore \quad h = \frac{\sqrt{a^2f^2 - 1}}{af},$$

where a is a non-zero constant. (We have omitted a $\pm$, but the sign can be changed by replacing u by $-u$ if necessary.) Writing $h = df/du$ and integrating again,

$$\int \frac{af\,df}{\sqrt{a^2f^2-1}} = \int du,$$

$$\therefore \quad f = \frac{1}{a}\sqrt{1 + a^2(u+b)^2},$$

where b is a constant. By a change of parameter $u \mapsto u + b$, we can assume that $b = 0$. So,

$$f = \frac{1}{a}\sqrt{1 + a^2u^2}.$$

To compute g, we have

$$\begin{aligned}
\dot{g}^2 &= 1 - \dot{f}^2 = 1 - h^2 = \frac{1}{a^2 f^2},\\
\therefore \quad \frac{dg}{du} &= \pm\frac{1}{\sqrt{1+a^2u^2}},\\
\therefore \quad g &= \pm\frac{1}{a}\sinh^{-1}(au) + c \qquad \text{(where } c \text{ is a constant)},\\
\therefore \quad au &= \pm\sinh(a(g-c)),\\
\therefore \quad f &= \frac{1}{a}\cosh(a(g-c)).
\end{aligned}$$

Thus, the profile curve of $\mathcal{S}$ is

$$x = \frac{1}{a}\cosh(a(z-c)).$$

This surface is obtained by applying to the catenoid $\mathcal{S}_a$ a translation along the z-axis.

We are not quite finished, however. So far, we have only shown that the open subset of $\mathcal{S}$ corresponding to $u \in (\alpha, \beta)$ is part of the catenoid, for in the proof we used in an essential way that $\dot{g} \neq 0$. This is why the proof has so far excluded the possibility that $\mathcal{S}$ is a plane. To complete the proof, we argue as follows. Suppose that $\beta < \infty$. Then, if the profile curve is defined for values of $u \geq \beta$, we must have $\dot{g}(\beta) = 0$, for otherwise $\dot{g}$ would be non-zero on an open interval containing β, which would contradict our assumption that (α, β) is the *largest* open interval containing u_0 on which $\dot{g} \neq 0$. But the formulas above show that

$$\dot{g}^2 = \frac{1}{1+a^2u^2} \quad \text{if } u \in (\alpha, \beta),$$

and so, since $\dot{g}$ is a continuous function of u, $\dot{g}(\beta) = \pm(1 + a^2\beta^2)^{-1/2} \neq 0$. This contradiction shows that the profile curve is not defined for values of $u \geq \beta$. Of course, this also holds trivially if $\beta = \infty$. A similar argument applies to α, and shows that (α, β) is the entire domain of definition of the profile curve. Hence, the whole of $\mathcal{S}$ is an open subset of a catenoid.

The only remaining case to consider is that in which $\dot{g}(u) = 0$ for all values of u for which the profile curve is defined. But then $g(u)$ is a constant, say d, and $\mathcal{S}$ is an open subset of the plane $z = d$. □

Example 12.2.3

A *helicoid* is a ruled surface swept out by a straight line that rotates at constant speed about an axis perpendicular to the line while simultaneously moving at constant speed along the axis. By applying an isometry of $\mathbb{R}^3$ we can take the axis to be the z-axis. Let ω be the angular velocity of the rotating line and α its speed along the z-axis. If the line starts along the x-axis, at time v the centre of the line is at $(0, 0, \alpha v)$ and it has rotated by an angle ωv. Hence, the point of the line initially at $(u, 0, 0)$ is now at the point

$$\boldsymbol{\sigma}(u, v) = (u \cos \omega v, u \sin \omega v, \alpha v).$$

We leave it to Exercise 12.2.1 to check that this is a minimal surface. (A picture of a helicoid can be found in Exercise 4.2.6.)

We have the following analogue of Proposition 12.2.2.

Proposition 12.2.4

Any ruled minimal surface is an open subset of a plane or a helicoid.

Proof

We take the usual parametrization

$$\boldsymbol{\sigma}(u, v) = \boldsymbol{\gamma}(u) + v\boldsymbol{\delta}(u)$$

(see Example 5.3.3), where $\boldsymbol{\gamma}$ is a curve that meets each of the rulings and $\boldsymbol{\delta}(u)$ is a vector parallel to the ruling through $\boldsymbol{\gamma}(u)$. We begin the proof by making some simplifications to the parametrization.

First, we can certainly assume that $\| \boldsymbol{\delta}(u) \| = 1$ for all values of u. We assume also that $\dot{\boldsymbol{\delta}}$ *is never zero*, where the dot denotes d/du. (We shall consider later what happens if $\dot{\boldsymbol{\delta}}(u) = \mathbf{0}$ for some values of u.) We can then assume that $\dot{\boldsymbol{\gamma}} \cdot \dot{\boldsymbol{\delta}} = 0$ (see Exercise 5.3.4).

We have $\boldsymbol{\sigma}_u = \dot{\boldsymbol{\gamma}} + v\dot{\boldsymbol{\delta}}$, $\boldsymbol{\sigma}_v = \boldsymbol{\delta}$, so

$$E = \| \dot{\boldsymbol{\gamma}} + v\dot{\boldsymbol{\delta}} \|^2, \quad F = (\dot{\boldsymbol{\gamma}} + v\dot{\boldsymbol{\delta}}) \cdot \boldsymbol{\delta} = \dot{\boldsymbol{\gamma}} \cdot \boldsymbol{\delta}, \quad G = 1.$$

Let $A = \sqrt{EG - F^2}$. Then,

$$\mathbf{N} = A^{-1}(\dot{\boldsymbol{\gamma}} + v\dot{\boldsymbol{\delta}}) \times \boldsymbol{\delta}.$$

Next, we have $\boldsymbol{\sigma}_{uu} = \ddot{\boldsymbol{\gamma}} + v\ddot{\boldsymbol{\delta}}$, $\boldsymbol{\sigma}_{uv} = \dot{\boldsymbol{\delta}}$, $\boldsymbol{\sigma}_{vv} = \mathbf{0}$, so

$$\begin{aligned} L &= A^{-1}(\ddot{\boldsymbol{\gamma}} + v\ddot{\boldsymbol{\delta}}) \cdot ((\dot{\boldsymbol{\gamma}} + v\dot{\boldsymbol{\delta}}) \times \boldsymbol{\delta}), \\ M &= A^{-1}\dot{\boldsymbol{\delta}} \cdot ((\dot{\boldsymbol{\gamma}} + v\dot{\boldsymbol{\delta}}) \times \boldsymbol{\delta}) = A^{-1}\dot{\boldsymbol{\delta}}.(\dot{\boldsymbol{\gamma}} \times \boldsymbol{\delta}), \\ N &= 0. \end{aligned}$$

Hence, the minimal surface condition

$$H = \frac{LG - 2MF + NE}{2A^2} = 0$$

gives

$$(\ddot{\boldsymbol{\gamma}} + v\ddot{\boldsymbol{\delta}}) \cdot ((\dot{\boldsymbol{\gamma}} + v\dot{\boldsymbol{\delta}}) \times \boldsymbol{\delta}) = 2(\boldsymbol{\delta} \cdot \dot{\boldsymbol{\gamma}})(\dot{\boldsymbol{\delta}} \cdot (\dot{\boldsymbol{\gamma}} \times \boldsymbol{\delta})).$$

This equation must hold for all values of (u, v). Equating coefficients of powers of v gives

$$\ddot{\boldsymbol{\gamma}} \cdot (\dot{\boldsymbol{\gamma}} \times \boldsymbol{\delta}) = 2(\boldsymbol{\delta} \cdot \dot{\boldsymbol{\gamma}})(\dot{\boldsymbol{\delta}} \cdot (\dot{\boldsymbol{\gamma}} \times \boldsymbol{\delta})), \quad (12.9)$$

$$\ddot{\boldsymbol{\gamma}} \cdot (\dot{\boldsymbol{\delta}} \times \boldsymbol{\delta}) + \ddot{\boldsymbol{\delta}} \cdot (\dot{\boldsymbol{\gamma}} \times \boldsymbol{\delta}) = 0, \quad (12.10)$$

$$\ddot{\boldsymbol{\delta}} \cdot (\dot{\boldsymbol{\delta}} \times \boldsymbol{\delta}) = 0. \quad (12.11)$$

Equation 12.11 shows that $\boldsymbol{\delta}, \dot{\boldsymbol{\delta}}$ and $\ddot{\boldsymbol{\delta}}$ are linearly dependent. Since $\boldsymbol{\delta}$ and $\dot{\boldsymbol{\delta}}$ are perpendicular unit vectors, there are smooth functions $\alpha(u)$ and $\beta(u)$ such that

$$\ddot{\boldsymbol{\delta}} = \alpha\boldsymbol{\delta} + \beta\dot{\boldsymbol{\delta}}.$$

But, since $\boldsymbol{\delta}$ is unit-speed, $\dot{\boldsymbol{\delta}} \cdot \ddot{\boldsymbol{\delta}} = 0$. Also, differentiating $\boldsymbol{\delta} \cdot \dot{\boldsymbol{\delta}} = 0$ gives $\boldsymbol{\delta} \cdot \ddot{\boldsymbol{\delta}} = -\dot{\boldsymbol{\delta}} \cdot \dot{\boldsymbol{\delta}} = -1$. Hence, $\alpha = -1$ and $\beta = 0$, so

$$\ddot{\boldsymbol{\delta}} = -\boldsymbol{\delta}. \quad (12.12)$$

Equation 12.12 shows that the curvature of the curve $\boldsymbol{\delta}$ is 1, and that its principal normal is $-\boldsymbol{\delta}$. Hence, its binormal is $\dot{\boldsymbol{\delta}} \times (-\boldsymbol{\delta})$, and since

$$\frac{d}{du}(\dot{\boldsymbol{\delta}} \times \boldsymbol{\delta}) = \ddot{\boldsymbol{\delta}} \times \boldsymbol{\delta} + \dot{\boldsymbol{\delta}} \times \dot{\boldsymbol{\delta}} = -\boldsymbol{\delta} \times \boldsymbol{\delta} = \mathbf{0},$$

it follows that the torsion of $\boldsymbol{\delta}$ is zero. Hence, $\boldsymbol{\delta}$ parametrizes a circle of radius 1 (see Proposition 2.3.5). By applying an isometry of $\mathbb{R}^3$, we can assume that $\boldsymbol{\delta}$ is the circle with radius 1 and centre the origin in the xy-plane, so that

$$\boldsymbol{\delta}(u) = (\cos u, \sin u, 0).$$

From Eq. 12.12, we get $\ddot{\boldsymbol{\delta}} \cdot (\dot{\boldsymbol{\gamma}} \times \boldsymbol{\delta}) = -\boldsymbol{\delta} \cdot (\dot{\boldsymbol{\gamma}} \times \boldsymbol{\delta}) = 0$, so by Eq. 12.10,

$$\ddot{\boldsymbol{\gamma}} \cdot (\dot{\boldsymbol{\delta}} \times \boldsymbol{\delta}) = 0.$$

It follows that $\ddot{\boldsymbol{\gamma}}$ is parallel to the xy-plane, and hence that

$$\boldsymbol{\gamma}(u) = (f(u), g(u), au + b),$$

where f and g are smooth functions and a and b are constants. If $a = 0$, the surface is an open subset of the plane $z = b$. Otherwise, Eq. 12.9 gives

$$\ddot{g}\cos u - \ddot{f}\sin u = 2(\dot{f}\cos u + \dot{g}\sin u). \quad (12.13)$$

We finally make use of the condition $\dot{\boldsymbol{\gamma}} \cdot \dot{\boldsymbol{\delta}} = 0$, which gives

$$\dot{f} \sin u = \dot{g} \cos u. \tag{12.14}$$

Differentiating this gives

$$\ddot{f} \sin u + \dot{f} \cos u = \ddot{g} \cos u - \dot{g} \sin u. \tag{12.15}$$

Equations 12.13 and 12.15 together give

$$\dot{f} \cos u + \dot{g} \sin u = 0$$

and using Eq. 12.14 we get $\dot{f} = \dot{g} = 0$. Thus, f and g are constants. By a translation of the surface, we can assume that the constants f, g and b are zero, so that $\boldsymbol{\gamma}(u) = (0, 0, au)$ and

$$\boldsymbol{\sigma}(u, v) = (v \cos u, v \sin u, au),$$

which is a helicoid.

We assumed at the beginning that $\dot{\boldsymbol{\delta}}$ is never zero. If $\dot{\boldsymbol{\delta}}$ is *always* zero, then $\boldsymbol{\delta}$ is a constant vector and the surface is a generalized cylinder. But in fact a generalized cylinder is a minimal surface only if the cylinder is an open subset of a plane (Exercise 12.2.3). The proof is now completed by an argument similar to that used at the end of the proof of Proposition 12.2.2, which shows that the whole surface is an open subset of a plane or a helicoid. □

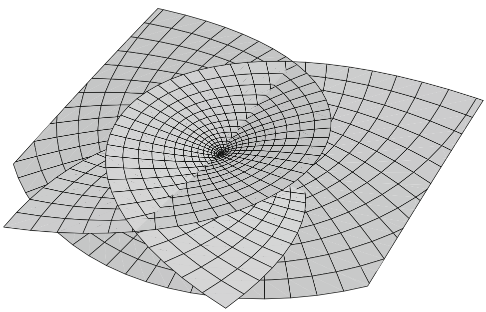
After the catenoid and helicoid, the next minimal surfaces to be discovered were the following two.

Example 12.2.5

Enneper's surface is

$$\boldsymbol{\sigma}(u, v) = \left(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + u^2 v, u^2 - v^2\right).$$

It was shown in Exercise 8.5.1 that this is a minimal surface.



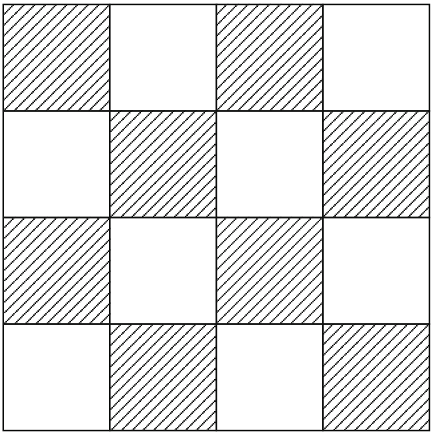
Strictly speaking, this is not a surface patch as it is not injective. The self-intersections are clearly visible in the picture above. However, if we restrict (u, v) to lie in sufficiently small open sets, $\boldsymbol{\sigma}$ will be injective (see Exercise 5.6.3).

Example 12.2.6

Scherk's surface is the surface with Cartesian equation

$$z = \ln\left(\frac{\cos y}{\cos x}\right).$$

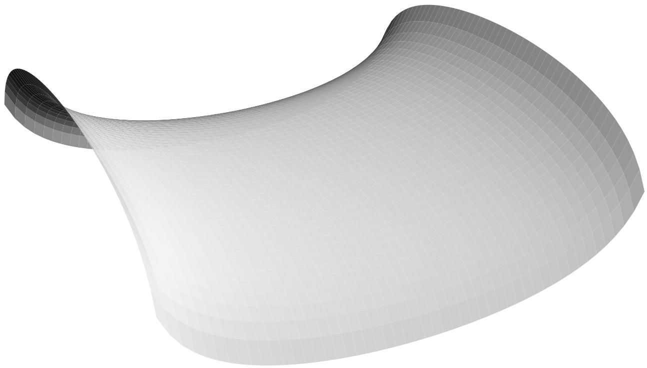
It was shown in Exercise 8.5.2 that this is a minimal surface. Note that the surface exists only when $\cos x$ and $\cos y$ are both > 0 or both < 0, in other words in the interiors of the white squares of the following chess board pattern, in which the squares have vertices at the points $(\pi/2 + m\pi, \pi/2 + n\pi)$, where m and n are integers, no two squares with a common edge have the same colour, and the square containing the origin is white:



The white squares have centres of the form $(m\pi, n\pi)$, where m and n are integers with $m + n$ even. Since, for such m, n,

$$\frac{\cos(y + n\pi)}{\cos(x + m\pi)} = \frac{\cos y}{\cos x},$$

it follows that the part of the surface over the square with centre $(m\pi, n\pi)$ is obtained from the part over the square with centre $(0, 0)$ by the translation $(x, y, z) \mapsto (x + m\pi, y + n\pi, z)$. So it suffices to exhibit the part of the surface over a single square (see below).



EXERCISES

12.2.1 Show that every helicoid is a minimal surface.

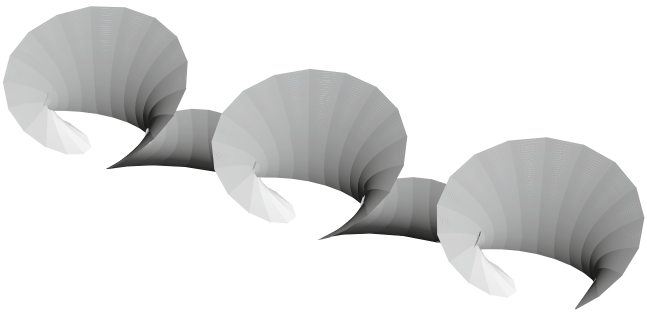
12.2.2 Show that the surfaces $\boldsymbol{\sigma}^t$ in the isometric deformation of a helicoid into a catenoid given in Exercise 6.2.2 are minimal surfaces. (This is 'explained' in Exercise 12.5.4.)

12.2.3 Show that a generalized cylinder is a minimal surface only when the cylinder is an open subset of a plane.

12.2.4 Verify that *Catalan's surface*

$$\boldsymbol{\sigma}(u,v) = \left(u - \sin u \cosh v, 1 - \cos u \cosh v, -4 \sin \frac{u}{2} \sinh \frac{v}{2}\right)$$

is a conformally parametrized minimal surface. (As in the case of Enneper's surface, Catalan's surface has self-intersections, so it is only a surface if we restrict (u, v) to sufficiently small open sets.)



Show that:

(i) The parameter curve on the surface given by $u = 0$ is a straight line.

(ii) The parameter curve $u = \pi$ is a parabola.

(iii) The parameter curve $v = 0$ is a cycloid (see Exercise 1.1.7).

Show also that each of these curves, when suitably parametrized, is a geodesic on Catalan's surface. (There is a sense in which Catalan's surface is 'designed' to have a cycloidal geodesic – see Exercise 12.5.5.)

12.3 Gauss map of a minimal surface

Recall from Section 7.2 that the Gauss map $\mathcal{G}$ of an oriented surface $\mathcal{S}$ associates to each point $\mathbf{p} \in \mathcal{S}$ the unit normal $\mathbf{N_p}$ of $\mathcal{S}$ at $\mathbf{p}$ regarded as a point of the unit sphere S^2. We begin with the following 'local' result:

Proposition 12.3.1

With the above notation, suppose that the Gaussian curvature of $\mathcal{S}$ is non-zero at the point $\mathbf{p}$. Then, there is an open subset V of $\mathcal{S}$ containing $\mathbf{p}$ such that the restriction of $\mathcal{G}$ to V is injective.

This result (and its proof) implies that, if the Gaussian curvature of $\mathcal{S}$ is nowhere zero, the Gauss map of $\mathcal{S}$ is a *local diffeomorphism*.

Proof

Let $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ be a surface patch of $\mathcal{S}$ containing $\mathbf{p}$, say $\mathbf{p} = \boldsymbol{\sigma}(u_0, v_0)$, and let $\mathbf{N} : U \to \mathbb{R}^3$ be the standard unit normal of $\boldsymbol{\sigma}$. By Eq. 8.2,

$$\mathbf{N}_u \times \mathbf{N}_v = K\, \boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v,$$

where K is the Gaussian curvature of $\mathcal{S}$, so by Exercise 5.6.3 there is an open subset W of U containing (u_0, v_0) such that the restriction of the map $\mathbf{N}$ to W is injective. Then, $\boldsymbol{\sigma}(W)$ is an open subset of $\mathcal{S}$ containing $\mathbf{p}$ and the restriction of $\mathcal{G}$ to $\boldsymbol{\sigma}(W)$ is injective. □

Theorem 12.3.2

Let $\mathcal{S}$ be a minimal surface with nowhere vanishing Gaussian curvature. Then, the Gauss map is a conformal map from $\mathcal{S}$ to S^2.

Proof

By Theorem 6.3.3, we have to show that the bilinear forms $\langle\ ,\ \rangle$ and $\mathcal{G}^*\langle\ ,\ \rangle$ are proportional. Now, if $\mathbf{p} \in \mathcal{S}$ and $\mathbf{v}, \mathbf{w} \in T_{\mathbf{p}}\mathcal{S}$,

$$\mathcal{G}^*\langle \mathbf{v}, \mathbf{w}\rangle = \langle \mathcal{D}_{\mathbf{p}}\mathcal{G}(\mathbf{v}), \mathcal{D}_{\mathbf{p}}\mathcal{G}(\mathbf{w})\rangle = \langle -\mathcal{W}(\mathbf{v}), -\mathcal{W}(\mathbf{w})\rangle = \langle \mathcal{W}^2(\mathbf{v}), \mathbf{w}\rangle,$$

where $\mathcal{W}$ is the Weingarten map; the last equation follows from the fact that $\mathcal{W}$ is self-adjoint (Corollary 7.2.4). But, by Exercise 8.1.6 and the fact that the mean curvature H is zero, we have

$$\mathcal{W}^2 = -K,$$

the Gaussian curvature of $\mathcal{S}$. It follows that

$$\mathcal{G}^*\langle\ ,\ \rangle = -K\langle\ ,\ \rangle,$$

as we want. □

We saw in Exercise 6.3.4 that a conformal parametrization of the plane is necessarily holomorphic or anti-holomorphic, so this proposition strongly suggests a connection between minimal surfaces and holomorphic functions. This connection turns out to be very extensive, and we shall give an introduction to it in Section 12.5.

EXERCISES

12.3.1 Let $\mathcal{S}$ be a connected surface whose Gauss map is conformal.

(i) Show that, if $\mathbf{p} \in \mathcal{S}$ and if the mean curvature H of $\mathcal{S}$ at $\mathbf{p}$ is non-zero, there is an open subset of $\mathcal{S}$ containing $\mathbf{p}$ that is part of a sphere.

(ii) Deduce that, if H is non-zero at $\mathbf{p}$, there is an open subset of $\mathcal{S}$ containing $\mathbf{p}$ on which H is constant.

(iii) Deduce that $\mathcal{S}$ is either a minimal surface or an open subset of a sphere.

12.3.2 Show that:

(i) The Gauss map of a catenoid is injective and its image is the whole of S^2 except for the north and south poles.

(ii) The image of the Gauss map of a helicoid is the same as that of a catenoid, but that infinitely many points on the helicoid are sent by the Gauss map to any given point in its image.

(The fact that the Gauss maps of a catenoid and a helicoid have the same image is 'explained' in Exercise 12.5.3 (ii).)

12.4 Conformal parametrization of minimal surfaces

Our goal in this section is to prove the following theorem.

Theorem 12.4.1

Let $\mathcal{S}$ be a minimal surface and let $\mathbf{p} \in \mathcal{S}$. Then, there is a surface patch $\boldsymbol{\sigma}$ of $\mathcal{S}$ containing $\mathbf{p}$ that is conformal.

Recall from Section 6.3 that this means that the first fundamental form of $\boldsymbol{\sigma}(u, v)$ is of the form $E(du^2 + dv^2)$ for some smooth function $E(u, v)$.

Proof

Let $\mathbf{p} = (x_0, y_0, z_0)$. By Exercise 5.6.4, if the tangent plane of $\mathcal{S}$ at $\mathbf{p}$ does not contain the z-axis, there is an open set U in $\mathbb{R}^2$ containing (x_0, y_0) and a smooth function $f : V \to \mathbb{R}$ such that an open subset of $\mathcal{S}$ consisting of the points (x, y, z) with $(x, y) \in V$ coincides with the graph of the function f. (If the tangent plane at $\mathbf{p}$ does contain the z-axis, then $\mathcal{S}$ will be a graph of the form $x = f(y, z)$ or $y = f(x, z)$ near $\mathbf{p}$.) We can also assume that V is an open disc

$$D = \{(x, y) \mid (x - x_0)^2 + (y - y_0)^2 < r^2\},$$

for some $r > 0$, since any open set in $\mathbb{R}^2$ containing (x_0, y_0) contains such a disc. We must therefore show that the surface patch

$$\tilde{\boldsymbol{\sigma}}(x, y) = (x, y, f(x, y)), \quad (x, y) \in D,$$

has a conformal reparametrization.

The coefficients of the first fundamental form of $\tilde{\boldsymbol{\sigma}}$ are

$$E = 1 + f_x^2, \quad F = f_x f_y, \quad G = 1 + f_y^2.$$

We show first that

$$\left(\frac{F}{A}\right)_x = \left(\frac{E}{A}\right)_y, \quad \left(\frac{G}{A}\right)_x = \left(\frac{F}{A}\right)_y, \tag{12.16}$$

where $A = \sqrt{EG - F^2}$. Indeed,

$$\begin{aligned}
\left(\frac{F}{A}\right)_x - \left(\frac{E}{A}\right)_y &= \frac{(1 + f_x^2 + f_y^2)(f_{xx} f_y + f_x f_{xy}) - f_x f_y (f_x f_{xx} + f_y f_{xy})}{(1 + f_x^2 + f_y^2)^{3/2}} \\
&\quad - \frac{2(1 + f_x^2 + f_y^2) f_x f_{xy} - (1 + f_x^2)(f_x f_{xy} + f_y f_{yy})}{(1 + f_x^2 + f_y^2)^{3/2}} \\
&= \frac{f_y((1 + f_y^2) f_{xx} - 2 f_x f_y f_{xy} + (1 + f_x^2) f_{yy})}{(1 + f_x^2 + f_y^2)^{3/2}} \\
&= 0,
\end{aligned}$$

by Exercise 8.1.1. The second equation in (12.16) is proved similarly.

From advanced calculus, we know that Eqs. 12.16 imply the existence of smooth functions $\varphi, \psi : D \to \mathbb{R}$ such that

$$\varphi_x = \frac{E}{A}, \quad \varphi_y = \frac{F}{A}, \quad \psi_x = \frac{F}{A}, \quad \psi_y = \frac{G}{A}.$$

In fact, we can just define

$$\varphi(x, y) = \int_0^1 \frac{xE((1-t)\mathbf{r}_0 + t\mathbf{r}) + yF((1-t)\mathbf{r}_0 + t\mathbf{r})}{A((1-t)\mathbf{r}_0 + t\mathbf{r})}\, dt,$$

where $\mathbf{r} = (x, y)$, $\mathbf{r}_0 = (x_0, y_0)$; and similarly for ψ.

The reparametrization map we want is

$$u(x, y) = x + \varphi(x, y), \quad v(x, y) = y + \psi(x, y).$$

Note that

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} 1 + \varphi_x & \varphi_y \\ \psi_x & 1 + \psi_y \end{pmatrix} = \begin{pmatrix} 1 + \frac{E}{A} & \frac{F}{A} \\ \frac{F}{A} & 1 + \frac{G}{A} \end{pmatrix} \tag{12.17}$$

so

$$\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \left(1 + \frac{E}{A}\right)\left(1 + \frac{G}{A}\right) - \frac{F^2}{A^2} = 2 + \frac{E + G}{A} > 0.$$

By the Inverse Function Theorem 5.6.1, the function $F : D \to \mathbb{R}^2$ given by $F(x, y) = (u(x, y), v(x, y))$ has a smooth inverse function F^{-1} (we may have to replace D by a smaller open disc with centre (x_0, y_0)). Let

$$F^{-1}(u, v) = (x(u, v), y(u, v)).$$

We shall show that the reparametrization

$$\boldsymbol{\sigma}(u,v) = (x(u,v), y(u,v), f(x(u,v), y(u,v)))$$

of $\tilde{\boldsymbol{\sigma}}$ is conformal.

By the chain rule,

$$\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = I,$$

so

$$\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}^{-1} = \frac{1}{E+G+2A} \begin{pmatrix} G+A & -F \\ -F & E+A \end{pmatrix}$$

by Eq. 12.17. Letting $z(u,v) = f(x(u,v), y(u,v))$, we get (again using the chain rule)

$$\begin{aligned} z_u &= f_x x_u + f_y y_u = \frac{f_x(G+A) - f_y F}{E+G+2A}, \\ z_v &= f_x x_v + f_y y_v = \frac{f_y(E+A) - f_x F}{E+G+2A}. \end{aligned}$$

Hence,

$$\begin{aligned} \boldsymbol{\sigma}_u . \boldsymbol{\sigma}_u &= x_u^2 + y_u^2 + z_u^2 \\ &= \frac{(G+A)^2 + F^2 + (f_x(G+A) - f_y F)^2}{(E+G+2A)^2} \\ &= \frac{(G+A)^2 + F^2 + (E-1)(G+A)^2 + (G-1)F^2 - 2(G+A)F^2}{(E+G+2A)^2} \\ &= \frac{E(G+A)^2 + GF^2 - 2(G+A)F^2}{(E+G+2A)^2} \\ &= \frac{EA^2 + 2A(EG-F^2) + G(EG-F^2)}{(E+G+2A)^2} \\ &= \frac{A^2}{E+G+2A}, \end{aligned}$$

using $f_x^2 = E-1$, $f_y^2 = G-1$ to pass from the second line to the third and $A^2 = EG - F^2$ to pass from the fifth line to the sixth. Similar calculations show that

$$\boldsymbol{\sigma}_v . \boldsymbol{\sigma}_v = \frac{A^2}{E+G+2A}, \quad \boldsymbol{\sigma}_u . \boldsymbol{\sigma}_v = 0. \qquad \square$$

EXERCISES

12.4.1 Use Proposition 12.3.2 to give another proof of Theorem 12.4.1 for surfaces $\mathcal{S}$ with nowhere-vanishing Gaussian curvature.

12.5 Minimal surfaces and holomorphic functions

In this section, we shall make use of certain elementary properties of holomorphic functions. Readers without the necessary background in complex analysis may safely omit this section, the results of which are not used anywhere else in the book.

Let $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ be a conformal surface patch. We introduce complex coordinates in the plane of which U is an open subset by setting

$$\zeta = u + iv, \qquad (u, v) \in U,$$

and we define

$$\boldsymbol{\varphi}(\zeta) = \boldsymbol{\sigma}_u - i\boldsymbol{\sigma}_v. \tag{12.18}$$

Thus, $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \varphi_3)$ has three components, each of which is a complex-valued function of (u, v), i.e., of ζ. The basic result which establishes the connection between minimal surfaces and holomorphic functions is the following proposition.

Proposition 12.5.1

Let $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ be a conformal surface patch. Then $\boldsymbol{\sigma}$ is minimal if and only if the function $\boldsymbol{\varphi}$ defined in Eq. 12.18 is holomorphic on U.

To say that $\boldsymbol{\varphi}$ is holomorphic means that each of its components φ_1, φ_2 and φ_3 is holomorphic.

Proof

Let $\varphi(u, v)$ be a complex-valued smooth function, and let α and β be its real and imaginary parts, so that $\varphi = \alpha + i\beta$. The Cauchy–Riemann equations

$$\alpha_u = \beta_v \quad \text{and} \quad \alpha_v = -\beta_u$$

are the necessary and sufficient conditions for φ to be holomorphic. Applying this to each of the components of $\boldsymbol{\varphi}$, we see that $\boldsymbol{\varphi}$ is holomorphic if and only if

$$(\boldsymbol{\sigma}_u)_u = (-\boldsymbol{\sigma}_v)_v \quad \text{and} \quad (\boldsymbol{\sigma}_u)_v = -(-\boldsymbol{\sigma}_v)_u. \qquad (12.19)$$

The second equation imposes no condition on $\boldsymbol{\sigma}$, and the first is equivalent to $\boldsymbol{\sigma}_{uu} + \boldsymbol{\sigma}_{vv} = \mathbf{0}$. But it was shown in Exercise 8.5.1 that a conformal surface patch $\boldsymbol{\sigma}$ is minimal if and only if $\boldsymbol{\sigma}_{uu} + \boldsymbol{\sigma}_{vv}$ is zero. □

The holomorphic function $\boldsymbol{\varphi}$ associated to a minimal surface $\boldsymbol{\sigma}$ is not arbitrary:

Theorem 12.5.2

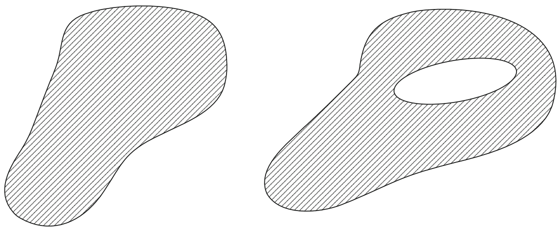
If $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ is a conformally parametrized minimal surface, the vector-valued holomorphic function $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \varphi_3)$ defined in Eq. 12.18 satisfies the following conditions:

(i) $\varphi_1^2 + \varphi_2^2 + \varphi_3^2 = 0$.

(ii) $\boldsymbol{\varphi}$ is nowhere zero.

Conversely, if U is simply-connected, and if φ_1, φ_2 and φ_3 are holomorphic functions on U satisfying conditions (i) and (ii) above, there is a conformally parametrized minimal surface $\boldsymbol{\sigma} : U \to \mathbb{R}^3$ such that $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \varphi_3)$ satisfies Eq. 12.18. Moreover, $\boldsymbol{\sigma}$ is uniquely determined by φ_1, φ_2 and φ_3 up to a translation.

An open subset U of $\mathbb{R}^2$ is said to be *simply-connected* if every simple closed curve in U can be shrunk to a point staying inside U. Intuitively, this means that U has no 'holes'.



Simply-connected Not simply-connected

In the course of the following proof, and in the proof of Proposition 12.5.5 below, we shall need to recall that, if F is a holomorphic function of $\zeta = u + iv$, then

$$F_u = F', \quad F_v = iF', \quad (\overline{F})_u = \overline{F'}, \quad (\overline{F})_v = -i\overline{F'},$$

where $F' = dF/d\zeta$ is the complex derivative of F, and the bar denotes complex-conjugate.

Proof

Suppose first that $\boldsymbol{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ is minimal, where $\sigma^k : U \to \mathbb{R}$ for $k = 1, 2, 3$. We have to show that $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \varphi_3)$ satisfies conditions (i) and (ii). Since $\varphi_k = \sigma^k_u - i\sigma^k_v$ for $k = 1, 2, 3$,

$$\sum_{k=1}^{3} \varphi_k^2 = \sum_{k=1}^{3} \left((\sigma^k_u)^2 - (\sigma^k_v)^2 - 2i\sigma^k_u\sigma^k_v\right) = \| \boldsymbol{\sigma}_u \|^2 - \| \boldsymbol{\sigma}_v \|^2 - 2i\boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_v, \tag{12.20}$$

which vanishes since $\boldsymbol{\sigma}$ is conformal. Finally, $\boldsymbol{\varphi} = \mathbf{0}$ if and only if $\boldsymbol{\sigma}_u = \boldsymbol{\sigma}_v = \mathbf{0}$, and this is impossible since $\boldsymbol{\sigma}$ is regular.

For the converse, take $\boldsymbol{\varphi}$ satisfying conditions (i) and (ii). We must show that $\boldsymbol{\varphi}$ arises from a minimal surface as above, and that this minimal surface is unique up to a translation of $\mathbb{R}^3$. Fix $(u_0, v_0) \in U$ and define $\boldsymbol{\sigma}$ as the real part of a complex line integral:

$$\boldsymbol{\sigma}(u, v) = \mathfrak{Re} \int_{\boldsymbol{\pi}} \boldsymbol{\varphi}(\xi) d\xi,$$

where $\boldsymbol{\pi}$ is any curve in U from (u_0, v_0) to $(u, v) \in U$. The fact that U is simply-connected implies, by virtue of Cauchy's Theorem, that $\int_{\boldsymbol{\pi}} \boldsymbol{\varphi}(\xi)d\xi$ is independent of the path $\boldsymbol{\pi}$ chosen, and hence so is $\boldsymbol{\sigma}(u, v)$. Now, $\boldsymbol{\Phi}(\zeta) = \int_{\boldsymbol{\pi}} \boldsymbol{\varphi}(\xi)d\xi$ is a holomorphic function of $\zeta = u + iv$, and $\boldsymbol{\Phi}'(\zeta) = \boldsymbol{\varphi}(\zeta)$. Hence, by the facts stated just before the beginning of the proof,

$$\begin{aligned} \boldsymbol{\sigma}_u &= \mathfrak{Re}(\boldsymbol{\Phi}_u) = \mathfrak{Re}(\boldsymbol{\Phi}') = \mathfrak{Re}(\boldsymbol{\varphi}), \\ \boldsymbol{\sigma}_v &= \mathfrak{Re}(\boldsymbol{\Phi}_v) = \mathfrak{Re}(i\boldsymbol{\Phi}') = -\mathfrak{Im}(\boldsymbol{\varphi}), \end{aligned} \tag{12.21}$$

so $\boldsymbol{\varphi} = \boldsymbol{\sigma}_u - i\boldsymbol{\sigma}_v$.

To complete the proof, we have to show that $\boldsymbol{\sigma}$ is a conformal surface patch. But, condition (ii) and Eqs. 12.21 show that $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$ are not both zero. By condition (i) and Eq. 12.20, $\| \boldsymbol{\sigma}_u \| = \| \boldsymbol{\sigma}_v \|$ and $\boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_v = 0$. Since $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$ are not both zero, this proves that $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$ are both non-zero and perpendicular, hence linearly independent, so that $\boldsymbol{\sigma}$ is a regular surface patch; it also proves that $\boldsymbol{\sigma}$ is conformal.

If another conformal minimal surface $\tilde{\boldsymbol{\sigma}}$ corresponds to the same holomorphic function $\boldsymbol{\varphi}$ as $\boldsymbol{\sigma}$, then $\tilde{\boldsymbol{\sigma}}_u = \boldsymbol{\sigma}_u$ and $\tilde{\boldsymbol{\sigma}}_v = \boldsymbol{\sigma}_v$ everywhere on U, which implies that $\tilde{\boldsymbol{\sigma}} - \boldsymbol{\sigma}$ is a constant, say $\mathbf{a}$, so that $\tilde{\boldsymbol{\sigma}}$ is obtained from $\boldsymbol{\sigma}$ by translating by the vector $\mathbf{a}$. □

Before giving some examples, we observe that, if a holomorphic function $\boldsymbol{\varphi}$ satisfies the conditions in Theorem 12.5.2, so does $i\boldsymbol{\varphi}$. If $\boldsymbol{\varphi}$ is the holomorphic function corresponding to a minimal surface $\mathcal{S}$, the minimal surface to which $i\boldsymbol{\varphi}$ corresponds is called the *conjugate* of $\mathcal{S}$. It is well defined by $\mathcal{S}$ up to a translation.

Example 12.5.3

The parametrization

$$\boldsymbol{\sigma}(u,v) = (\cosh u \cos v, \cosh u \sin v, u)$$

of the catenoid is conformal (see the solution of Exercise 6.2.3). The associated holomorphic function is

$$\begin{aligned}\boldsymbol{\varphi}(\zeta) &= \boldsymbol{\sigma}_u - i\boldsymbol{\sigma}_v \\ &= (\sinh u \cos v + i\cosh u \sin v, \sinh u \sin v - i\cosh u \cos v, 1) \\ &= (\sinh(u+iv), -i\cosh(u+iv), 1) \\ &= (\sinh\zeta, -i\cosh\zeta, 1).\end{aligned}$$

Note that conditions (i) and (ii) in Theorem 12.5.2 are satisfied, since $\boldsymbol{\varphi}$ is clearly never zero and the sum of the squares of its components is

$$\sinh^2\zeta - \cosh^2\zeta + 1 = 0.$$

Let us determine the conjugate minimal surface $\tilde{\boldsymbol{\sigma}}$ of the catenoid. From the proof of Theorem 12.5.2,

$$\begin{aligned}\tilde{\boldsymbol{\sigma}}(u,v) &= \mathfrak{Re}\int_{\boldsymbol{\pi}} (i\sinh\xi, \cosh\xi, i)\, d\xi \\ &= \mathfrak{Re}(i\cosh\zeta, \sinh\zeta, i\zeta) \\ &= (-\sinh u \sin v, \sinh u \cos v, -v),\end{aligned}$$

up to a translation. If we reparametrize by defining $\tilde{u} = \sinh u, \tilde{v} = v + \pi/2$, we get the surface

$$(\tilde{u},\tilde{v}) \mapsto (\tilde{u}\cos\tilde{v}, \tilde{u}\sin\tilde{v}, -\tilde{v}),$$

after translating by $(0,0,-\pi/2)$, which is obtained from the helicoid in Exercise 4.2.6 by reflecting in the z-axis. Note that the parametrization of the helicoid given in Exercise 4.2.6 is not conformal, so the constructions in this section cannot be applied to it.

It is actually possible to 'solve' the conditions on $\boldsymbol{\varphi}$ in Theorem 12.5.2.

Proposition 12.5.4

Let $f(\zeta)$ be a holomorphic function on an open set U in the complex plane, not identically zero, and let $g(\zeta)$ be a meromorphic function on U such that, if $\zeta_0 \in U$ is a pole of g of order $m \geq 1$, say, then ζ_0 is also a zero of f of order $\geq 2m$. Then,

$$\boldsymbol{\varphi} = \left(\frac{1}{2}f(1-g^2), \frac{i}{2}f(1+g^2), fg\right) \tag{12.22}$$

satisfies conditions (i) and (ii) in Theorem 12.5.2, and conversely every holomorphic function $\boldsymbol{\varphi}$ satisfying these conditions arises in this way.

The correspondence given by Theorem 12.5.2 and Proposition 12.5.4 between pairs of functions f and g and minimal surfaces is called *Weierstrass' representation.*

Proof

Suppose that f and g are as in the statement of the proposition. If g has a pole of order $m \geq 1$ at $\zeta_0 \in U$, and f has a zero of order $n \geq 2m$ at ζ_0, then the Laurent expansions of f and g about ζ_0 are of the form

$$f(\zeta) = a(\zeta - \zeta_0)^n + \cdots \quad \text{and} \quad g(\zeta) = \frac{b}{(\zeta - \zeta_0)^m} + \cdots ,$$

where a and b are non-zero complex numbers and the $\cdots$ indicates terms involving higher powers of $\zeta - \zeta_0$. Then,

$$f(1 \pm g^2) = \pm ab^2(\zeta - \zeta_0)^{n-2m} + \cdots \quad \text{and} \quad fg = ab(\zeta - \zeta_0)^{n-m} + \cdots$$

involve only non-negative powers of $\zeta - \zeta_0$, so $\boldsymbol{\varphi}$ is holomorphic near ζ_0. Since it is clear that $\boldsymbol{\varphi}$ is holomorphic wherever g is holomorphic, it follows that the function $\boldsymbol{\varphi}$ defined by Eq. 12.22 is holomorphic everywhere on U. It is clear that $\boldsymbol{\varphi}$ is identically zero only if f is identically zero, and simple algebra shows that $\boldsymbol{\varphi}$ satisfies condition (i) in Theorem 12.5.2.

Conversely, suppose that $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \varphi_3)$ is a holomorphic function satisfying conditions (i) and (ii) in Theorem 12.5.2. If $\varphi_1 - i\varphi_2$ is not identically zero, define

$$f = \varphi_1 - i\varphi_2, \quad g = \frac{\varphi_3}{\varphi_1 - i\varphi_2}. \tag{12.23}$$

Since $\boldsymbol{\varphi}$ is holomorphic, f is holomorphic and g is meromorphic. Condition (i) implies that $(\varphi_1 + i\varphi_2)(\varphi_1 - i\varphi_2) = -\varphi_3^2$, and hence that

$$\varphi_1 + i\varphi_2 = -fg^2. \tag{12.24}$$

Simple algebra shows that Eqs. 12.23 and 12.24 imply Eq. 12.22. Equation 12.24 implies that fg^2 is holomorphic, and the argument with Laurent expansions in the first part of the proof now gives the condition on the zeros and poles of f and g. Finally, if $\varphi_1 - i\varphi_2 = 0$, we repeat the above argument replacing $\varphi_1 \pm i\varphi_2$ by $\varphi_1 \mp i\varphi_2$ (note that $\varphi_1 - i\varphi_2$ and $\varphi_1 + i\varphi_2$ cannot both be zero, for if they were we would have $\varphi_1 = \varphi_2 = 0$, hence $\varphi_3 = 0$ by condition (i), and this would violate condition (ii)). □

We give only one application of Weierstrass' representation.

Proposition 12.5.5

The Gaussian curvature of the minimal surface corresponding to the functions f and g in Weierstrass' representation is

$$K = \frac{-16|dg/d\zeta|^2}{|f|^2(1+|g|^2)^4}.$$

Proof

This is a straightforward, if tedious, computation, and we shall omit many of the details. Define $\overline{\boldsymbol{\varphi}}$ by taking the complex-conjugate of each component of $\boldsymbol{\varphi}$. Then, $\boldsymbol{\sigma}_u = \frac{1}{2}(\boldsymbol{\varphi} + \overline{\boldsymbol{\varphi}})$, $\boldsymbol{\sigma}_v = \frac{1}{2i}(\overline{\boldsymbol{\varphi}} - \boldsymbol{\varphi})$. Since $\boldsymbol{\varphi} \cdot \boldsymbol{\varphi} = \overline{\boldsymbol{\varphi}} \cdot \overline{\boldsymbol{\varphi}} = 0$, the first fundamental form is $\frac{1}{2}\boldsymbol{\varphi} \cdot \overline{\boldsymbol{\varphi}}(du^2 + dv^2)$. Substituting the formula for $\boldsymbol{\varphi}$ from Eq. 12.22 and simplifying, we find that the first fundamental form is

$$\frac{1}{4}|f|^2(1+|g|^2)^2(du^2+dv^2). \tag{12.25}$$

Next,

$$\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v = \frac{1}{4i}(\boldsymbol{\varphi} + \overline{\boldsymbol{\varphi}}) \times (\overline{\boldsymbol{\varphi}} - \boldsymbol{\varphi}) = \frac{1}{2i}\boldsymbol{\varphi} \times \overline{\boldsymbol{\varphi}},$$

$$\therefore \quad \| \boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v \|^2 = -\frac{1}{4}(\boldsymbol{\varphi} \times \overline{\boldsymbol{\varphi}}) \cdot (\boldsymbol{\varphi} \times \overline{\boldsymbol{\varphi}}) = -\frac{1}{4}((\boldsymbol{\varphi} \cdot \boldsymbol{\varphi})(\overline{\boldsymbol{\varphi}} \cdot \overline{\boldsymbol{\varphi}}) - (\boldsymbol{\varphi} \cdot \overline{\boldsymbol{\varphi}})^2) = \frac{1}{4}(\boldsymbol{\varphi} \cdot \overline{\boldsymbol{\varphi}})^2,$$

$$\therefore \quad \mathbf{N} = i\frac{\overline{\boldsymbol{\varphi}} \times \boldsymbol{\varphi}}{\boldsymbol{\varphi} \cdot \overline{\boldsymbol{\varphi}}}.$$

In terms of f and g, this becomes

$$\mathbf{N} = \frac{1}{1+|g|^2}\left(g + \overline{g}, -i(g - \overline{g}), |g|^2 - 1\right). \tag{12.26}$$

Using the remarks preceding the proof of Theorem 12.5.2 and the formulas

$$L = -\boldsymbol{\sigma}_u \cdot \mathbf{N}_u, \quad M = -\boldsymbol{\sigma}_u \cdot \mathbf{N}_v, \quad N = -\boldsymbol{\sigma}_v \cdot \mathbf{N}_v$$

(which follow by differentiating $\boldsymbol{\sigma}_u \cdot \mathbf{N} = \boldsymbol{\sigma}_v \cdot \mathbf{N} = 0$), we find that the second fundamental form is

$$-\frac{1}{2}\left((fg' + \overline{fg'})(du^2 + dv^2) + 2i(fg' - \overline{fg'})du dv\right). \tag{12.27}$$

Combining Eqs. 12.25–12.27, and using the formula for the Gaussian curvature K in Corollary 8.1.3, we finally obtain the formula in the statement of the proposition. □

Corollary 12.5.6

Let $\mathcal{S}$ be a minimal surface that is not part of a plane. Then, the zeros of the Gaussian curvature of $\mathcal{S}$ are isolated.

This means that, if the Gaussian curvature K vanishes at a point $\mathbf{p} \in \mathcal{S}$, then K does not vanish at any other point of $\mathcal{S}$ sufficiently near to $\mathbf{p}$. More precisely, if $\mathbf{p}$ lies in a surface patch $\boldsymbol{\sigma}$ of $\mathcal{S}$, say $\mathbf{p} = \boldsymbol{\sigma}(u_0, v_0)$, there is a number $\epsilon > 0$ such that K does not vanish at the point $\boldsymbol{\sigma}(u, v) \in \mathcal{S}$ if $0 < (u - u_0)^2 + (v - v_0)^2 < \epsilon^2$.

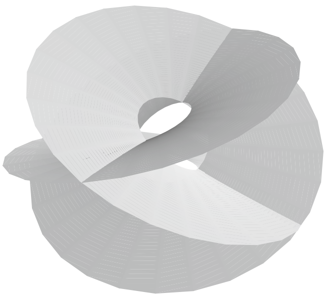
Proof

From the formula for K in Proposition 12.5.5, K vanishes exactly where the meromorphic function g' vanishes. If g' is zero everywhere, so is K and $\mathcal{S}$ is an open subset of a plane (this was shown in Proposition 8.2.9, but follows immediately from Eq. 12.26 which shows that $\mathbf{N}$ is constant if g is constant). But it is a standard result of complex analysis that the zeros of a non-zero meromorphic function are isolated, so if K is not identically zero its zeros must be isolated. □

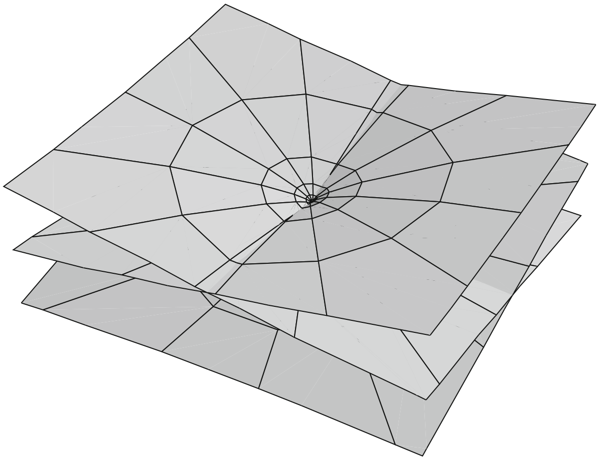
EXERCISES

12.5.1 Find the holomorphic function $\boldsymbol{\varphi}$ corresponding to Enneper's minimal surface given in Example 12.2.5. Show that its conjugate minimal surface coincides with a reparametrization of the same surface rotated by $\pi/4$ around the z-axis.

12.5.2 Find a parametrization of *Henneberg's surface*, the minimal surface corresponding to the functions $f(\zeta) = 1 - \zeta^{-4}$, $g(\zeta) = \zeta$ in Weierstrass' representation. The following are a 'close up' view and a 'large scale' view of this surface.



Henneberg: close up



Henneberg: Large scale

12.5.3 Show that, if $\boldsymbol{\varphi}$ satisfies the conditions in Theorem 12.5.2, so does $a\boldsymbol{\varphi}$ for any non-zero constant $a \in \mathbb{C}$; let $\boldsymbol{\sigma}^a$ be the minimal surface patch corresponding to $a\boldsymbol{\varphi}$, and let $\boldsymbol{\sigma}^1 = \boldsymbol{\sigma}$ be that corresponding to $\boldsymbol{\varphi}$. Show that:

(i) If $a \in \mathbb{R}$, then $\boldsymbol{\sigma}^a$ is obtained from $\boldsymbol{\sigma}$ by applying a dilation and a translation.

(ii) If $|a| = 1$, the map $\boldsymbol{\sigma}(u,v) \mapsto \boldsymbol{\sigma}^a(u,v)$ is an isometry, and the tangent planes of $\boldsymbol{\sigma}$ and $\tilde{\boldsymbol{\sigma}}$ at corresponding points are parallel (in particular, the images of the Gauss maps of $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}^a$ are the same).

12.5.4 Show that if the function $\boldsymbol{\varphi}$ in the preceding exercise is that corresponding to the catenoid (see Example 12.5.3), the surface $\boldsymbol{\sigma}^{e^{it}}$ coincides with the surface denoted by $\boldsymbol{\sigma}^t$ in Exercise 6.2.3.

12.5.5 Let $\boldsymbol{\gamma} : (\alpha, \beta) \to \mathbb{R}^3$ be a (regular) curve in the xy-plane, say

$$\boldsymbol{\gamma}(u) = (f(u), g(u), 0),$$

and assume that there are holomorphic functions F and G defined on a rectangle

$$\mathcal{U} = \{u + iv \in \mathbb{C} \,|\, \alpha < u < \beta,\ -\epsilon < v < \epsilon\},$$

for some $\epsilon > 0$, and such that $F(u) = f(u)$ and $G(u) = g(u)$ if u is real and $\alpha < u < \beta$. Note that (with a dash denoting d/dz as usual),

$$F'(z)^2 + G'(z)^2 \neq 0 \qquad \text{if } \mathfrak{Im}(z) = 0,$$

so by shrinking ϵ if necessary we can assume that $F'(z)^2+G'(z)^2 \neq 0$ for all $z \in \mathcal{U}$. Show that:

(i) The vector-valued holomorphic function

$$\boldsymbol{\varphi} = (F', G', i(F'^2 + G'^2)^{1/2})$$

satisfies the conditions of Theorem 12.5.2 and therefore defines a minimal surface $\boldsymbol{\sigma}(u, v)$.

(ii) Up to a translation, $\boldsymbol{\sigma}(u, 0) = \boldsymbol{\gamma}(u)$ for $\alpha < u < \beta$.

(iii) $\boldsymbol{\gamma}$ is a pre-geodesic on $\boldsymbol{\sigma}$ (see Exercise 9.1.2).

(iv) If we start with the cycloid

$$\boldsymbol{\gamma}(u) = (u - \sin u, 1 - \cos u, 0),$$

the resulting surface $\boldsymbol{\sigma}$ is, up to a translation, Catalan's surface and we have 'explained' why Catalan's surface has a cycloidal geodesic – see Exercise 12.2.4.