Chapter 2
Positivity Gap

This chapter investigates the gap between positive forms and SOS forms. Conservatism of the LMI relaxations described in Chapter 1 is related to the existence of positive forms which are not SOS, called PNS forms. *a priori* conditions for non-conservatism of these relaxations are presented for some classes of forms. The class of SMR-tight forms is introduced in order to derive *a posteriori* tightness conditions. A further contribution of this chapter consists of providing a parametrization of the set of PNS forms. It is shown that the set of PNS forms is dense in the space of forms, that each PNS form is the vertex of a cone of PNS forms, and how PNS forms can be constructed via the SMR.

2.1 Hilbert's 17th Problem

Is it true that any positive semidefinite form is an SOS form? This question is closely related to Hilbert's 17th problem [122], which concerns the possibility of representing nonnegative polynomials as a sum of squares of rational functions.

The answer to the former question is negative. This fact was discovered by Hilbert himself in 1888 via a non-constructive proof [118]. In 1967, Motzkin provided an example of form which is positive semidefinite but not SOS. This form has degree 6 in 3 scalar variables, and is given by [122]

$$h_{Mot}(x) = x_1^4x_2^2 + x_1^2x_2^4 + x_3^6 - 3x_1^2x_2^2x_3^2. \tag{2.1}$$

Indeed, it can be verified that $h_{Mot}(x)$ is positive semidefinite and $h_{Mot}(x)$ is not SOS. In particular, one has that

$$\mu(h_{Mot}) = 0, \quad \lambda(h_{Mot}) = -0.0070.$$

Hence, there are forms that are positive semidefinite but not SOS. The following result, found by Artin in 1927, states that any positive semidefinite form is the ratio of two SOS forms [70].

G. Chesi et al.: Homogeneous Polynomial Forms, LNCIS 390, pp. 39–61.
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Theorem 2.1. *A form $h \in \Xi_{n,2m}$ is positive semidefinite if and only if there exist $h_1 \in \Sigma_{n,2(a+m)}$ and $h_2 \in \Sigma_{n,2a}$ for some integer $a \geq 0$, such that*

$$h(x) = \frac{h_1(x)}{h_2(x)}. \tag{2.2}$$

The following result, found by Polya in 1928, characterizes the forms that are positive on the simplex [70].

Theorem 2.2. *A form $h \in \Xi_{n,d}$ is positive on the simplex Υ_n in* (1.68) *if and only if there exists an integer $k \geq 0$ such that the coefficients of*

$$h(x) \left(\sum_{i=1}^{n} x_i \right)^k \tag{2.3}$$

are positive.

In the sequel we will investigate forms that are positive semidefinite but not SOS. First of all, let us introduce the following definition.

Definition 2.1 (PNS). A form $h \in \Xi_{n,2m}$ is *PNS* if it is positive semidefinite but not SOS.

We will indicate the set of PNS forms of degree $2m$ in n scalar variables as follows:

$$\Delta_{n,2m} = \left\{ h \in \Xi_{n,2m} : \ h(x) \text{ is PNS} \right\}. \tag{2.4}$$

Therefore, the set $\Omega_{n,2m}$ in (1.51) can be expressed as

$$\Omega_{n,2m} = \Sigma_{n,2m} \cup \Delta_{n,2m}.$$

An interesting fact is that the set $\Delta_{n,2m}$ is empty for some values of n,m. Indeed, let us define the set

$$\mathscr{E} = \left\{ (n,2), \ n \in \mathbb{N} \right\} \cup \left\{ (2,2m), \ m \in \mathbb{N} \right\} \cup \left\{ (3,4) \right\}. \tag{2.5}$$

The following result states an important property of $\Delta_{n,2m}$ for any pair $(n,2m)$ in $\mathscr{E}$. A formal proof can be found in [70].

Theorem 2.3. *Let $(n,2m) \in \mathscr{E}$. Then, $\Delta_{n,2m} = \emptyset$, i.e. for all $h \in \Xi_{n,2m}$ one has*

$$h(x) \textit{ is positive semidefinite} \iff \lambda(h) \geq 0. \tag{2.6}$$

The following result provides a further property of the forms in $\Xi_{n,2m}$ with $(n,2m)$ in $\mathscr{E}$, in particular stating that these forms are positive definite if and only if they admit a positive definite SMR matrix.

Theorem 2.4. *Let $(n,2m) \in \mathscr{E}$. Then, for all $h \in \Xi_{n,2m}$ one has that*

$$h(x) \text{ is positive definite } \iff \lambda(h) > 0. \tag{2.7}$$

Proof. (Necessity) Let us suppose that $h(x)$ is positive definite. From Theorem 1.7 this means that $\mu(h) > 0$. Let us define the form

$$h_1(x) = h(x) - \mu(h)\|x^{\{m\}}\|^{2m}. \tag{2.8}$$

We have that

$$\begin{aligned} \mu(h_1) &= \min_{x \in \mathscr{C}_{n,m}} h_1(x) \\ &= \min_{x \in \mathscr{C}_{n,m}} \left(h(x) - \mu(h)\|x^{\{m\}}\|^{2m}\right) \\ &= \mu(h) - \mu(h) \\ &= 0. \end{aligned}$$

From Theorem 1.7 this implies that $h_1(x)$ is positive semidefinite. Moreover, $(n,2m) \in \mathscr{E}$, and hence from (2.6) it follows that $h_1(x)$ is SOS. Therefore, we have that

$$0 \le \lambda(h_1) \le \mu(h_1) = 0$$

which implies that $\lambda(h_1) = 0$. From Theorem 1.4, $h_1(x)$ can be written as

$$h_1(x) = x^{\{m\}'} H_1 x^{\{m\}}$$

where $H_1 \in \mathbb{S}^{\sigma(n,m)}$ is positive semidefinite. Now, let us express $h(x)$ as $h(x) = x^{\{m\}'} H x^{\{m\}}$. It follows from (2.8) that

$$H = H_1 + \mu(h) I_{\sigma(n,m)}$$

which implies that

$$\lambda(h) = \lambda(h_1) + \mu(h) = \mu(h). \tag{2.9}$$

Since $\mu(h) > 0$, it follows that $\lambda(h) > 0$.

(Sufficiency) Let us suppose that $\lambda(h) > 0$. From Theorem 1.9 it follows that $\mu(h) \ge \lambda(h) > 0$. From Theorem 1.7 this implies that $h(x)$ is positive definite. □

A direct consequence of Theorem 2.4 is that for forms with $(n,2m) \in \mathscr{E}$, the SOS index coincides with the positivity index.

Corollary 2.1. *Let $(n,2m) \in \mathscr{E}$. Then,*

$$\lambda(h) = \mu(h) \quad \forall h \in \Xi_{n,2m}. \tag{2.10}$$

Proof. It follows from (2.9) in the proof of Theorem 2.4. □

Example 2.1. Let us consider the form

$$h(x) = x_1^4 + x_2^4 + x_3^4 - 3x_1^2 x_2 x_3. \tag{2.11}$$

One has that $h \in \Xi_{3,4}$. By solving the EVP (1.29), one gets $\lambda(h) = -0.0310$. By Theorem 1.4, this implies that $h(x)$ is not SOS. Moreover, since $(3,4) \in \mathscr{E}$, we can conclude from Theorem 2.3 that $h(x)$ is not positive semidefinite, i.e.

$$\exists x \in \mathbb{R}^3 : \; h(x) < 0.$$

Indeed, for $x_1 = 4/3$, $x_2 = 1$, $x_3 = 1$, one has $h(x) = -14/81$.

Example 2.2. Let us consider

$$h(x) = x_1^{2m} + x_2^{2m}, \quad m \in \mathbb{N},\; m > 0. \tag{2.12}$$

We have that $h \in \Xi_{2,2m}$. Moreover, it is straightforward to verify that $h(x)$ is positive definite. Then, since $(2,2m) \in \mathscr{E}$ for any considered m, one can conclude from Theorem 2.4 that $\lambda(h) > 0$, or in other words, $h(x)$ admits a positive definite SMR matrix according to Lemma 1.2.

Throughout the book, the results in Theorems 2.3 and 2.4 will be exploited to formulate *a priori* conditions, which guarantee that results based on SOS relaxations of problems involving positivity of forms are not conservative.

As an example, let us consider the problem of checking positivity of a polynomial over an ellipsoid, addressed in Section 1.7. It can be observed that the LMI conditions in Theorem 1.16 are not only sufficient but also necessary, for some values of n,m. The next result is a direct consequence of Theorem 2.4.

Theorem 2.5. *Let $(n,2m) \in \mathscr{E}$. Then, the conditions in Theorem 1.16 are not only sufficient but also necessary for* (1.59) *to hold.*

Example 2.3. Let us consider the problem to establish whether (1.59) holds with

$$f(x) = 0.5 + x_1 + x_2^2, \quad Q = I_2, \quad c = 1.$$

The SOS index of the resulting $w(x;c)$ is negative, in particular $\lambda(w(\cdot;c)) = -0.7500$. Since $n = 2$ and $m = 2$ we have that $(n,2m) \in \mathscr{E}$. Therefore, from Theorems 1.16 and 2.5 we have that (1.59) does not hold, i.e. there exists some $x \in \mathscr{B}(Q,c)$ such that $f(x) \leq 0$.

2.2 Maximal SMR Matrices

In Chapter 1 it has been shown that a form may be represented by different SMR matrices. This section investigates the SMR matrices whose minimum eigenvalue coincides with the SOS index of the form. Roughly speaking, such matrices can be considered the "most positive definite" SMR matrices of the form. As it will be explained in the next section, these matrices are useful in order to study the gap between positive forms and SOS forms.

Definition 2.2 (Maximal SMR Matrix). Let $H^* \in \mathbb{S}^{\sigma(n,m)}$ be an SMR matrix of $h \in \Xi_{n,2m}$. Then, H^* is called a *maximal SMR matrix* of $h(x)$ if

$$\lambda_{min}(H^*) = \lambda(h). \tag{2.13}$$

Given a form $h(x)$, its maximal SMR matrices can be obtained as

$$H^* = H + L(\alpha^*) \tag{2.14}$$

where α^* is a value of α for which the maximum in (1.29) is achieved, and $H + L(\cdot)$ is the complete SMR matrix of $h(x)$ in (1.29).

2.2.1 Minimum Eigenvalue Decomposition

The following definition introduces a key decomposition of symmetric matrices which will be exploited in the sequel. For ease of presentation, the decomposition is formulated for a matrix of size $\sigma(n,m)$, though it can be defined for matrices of any size.

Definition 2.3 (Minimum Eigenvalue Decomposition). For a matrix $H \in \mathbb{S}^{\sigma(n,m)}$ we say that the quadruplet $\langle \lambda_{min}(H), \beta, V_0, V_p \rangle$ is a *minimum eigenvalue decomposition* of H if

$$H = VDV' \tag{2.15}$$

where $D \in \mathbb{S}^{\sigma(n,m)}$ is the diagonal matrix

$$D = \lambda_{min}(H) I_{\sigma(n,m)} + \mathrm{diag}\begin{pmatrix} 0_{\sigma(n,m)-r} \\ \beta \end{pmatrix} \tag{2.16}$$

with

$$\begin{cases} \beta \in \mathbb{R}^r \\ \beta > 0 \end{cases} \tag{2.17}$$

and $V \in \mathbb{R}^{\sigma(n,m)\times\sigma(n,m)}$ is an orthogonal matrix such that

$$\begin{cases} V = \begin{pmatrix} V_0 & V_p \end{pmatrix} \\ V_0 \in \mathbb{R}^{\sigma(n,m)\times(\sigma(n,m)-r)}, \quad V_p \in \mathbb{R}^{\sigma(n,m)\times r} \\ VV' = V'V = I_{\sigma(n,m)}. \end{cases} \tag{2.18}$$

It follows that the diagonal of D contains the eigenvalues of H, V is a matrix of eigenvectors, and r is an integer satisfying $1 \leq r \leq \sigma(n,m)$ which represents the number of eigenvalues of H distinct from $\lambda_{min}(H)$ (including their multiplicity).

It is useful to observe that $\langle \lambda_{min}(H), \beta, V_0, V_p \rangle$ is a minimum eigenvalue decomposition of H if and only if $\langle \lambda_{min}(H), T_1\beta, V_0T_2, V_pT_1^{-1} \rangle$ is, for all matrices $T_1 \in \mathbb{R}^{r\times r}$ and $T_2 \in \mathbb{R}^{(\sigma(n,m)-r)\times(\sigma(n,m)-r)}$ such that T_1 is a permutation matrix and T_2 is a nonsingular matrix.

2.2.2 Structure of Maximal SMR Matrices

The following result provides a fundamental property of maximal SMR matrices.

Theorem 2.6. *Let $h \in \Xi_{n,2m}$, $H + L(\alpha)$ be a complete SMR matrix of $h(x)$, and $\langle \lambda_{min}(H), \beta, V_0, V_p \rangle$ be a minimum eigenvalue decomposition of H. Let us define*

$$\eta^*(V_0) = \max_{\alpha:\ \|\alpha\|=1} \lambda_{min}\left(V_0' L(\alpha) V_0\right). \tag{2.19}$$

Then, H is a maximal SMR matrix of h if and only if

$$\eta^*(V_0) \leq 0. \tag{2.20}$$

Proof. From (2.13) it follows that H is a maximal SMR matrix if and only if

$$\lambda_{min}\left(H + L(\alpha)\right) \leq \lambda_{min}(H) \quad \forall \alpha$$

and, hence, if and only if

$$\forall \alpha\ \exists y, \|y\| = 1 :\ y'\left(H + L(\alpha)\right) y \leq \lambda_{min}(H). \tag{2.21}$$

Let $\langle \lambda_{min}(H), \beta, V_0, V_p \rangle$ be a minimum eigenvalue decomposition of H. Then, (2.21) can be rewritten as

$$\forall \alpha\ \exists y, \|y\| = 1 :\ y' V_p \operatorname{diag}(\beta) V_p' y \leq -y' L(\alpha) y. \tag{2.22}$$

Let us observe that $L(\alpha)$ depends linearly on α. This means that $V_p'y$ tends to zero as α tends to zero because

$$\operatorname{diag}(\beta) > 0.$$

Moreover, if (2.22) holds for the pair (y, α), it also holds for the pair $(y, c\alpha)$ for all $c \geq 1$. Therefore, it turns out that H is a maximal SMR matrix if and only if

$$\forall \alpha\ \forall \varepsilon > 0\ \exists y, \|y\| = 1 :\ \|V_p' y\| < \varepsilon \text{ and } y' V_p \operatorname{diag}(\beta) V_p' y \leq -y' L(\alpha) y$$

or, equivalently, if and only if

$$\forall\alpha\ \exists y, \|y\| = 1 :\ V_p' y = 0_r \text{ and } y'V_p \operatorname{diag}(\beta)V_p' y \le -y'L(\alpha)y. \tag{2.23}$$

Let us observe that

$$\ker(V_p') = \operatorname{img}(V_0) \tag{2.24}$$

and hence

$$V_p' y = 0 \iff y \in \operatorname{img}(V_0).$$

Therefore, (2.23) can be rewritten as

$$\forall\alpha\ \exists y \in \operatorname{img}(V_0), \|y\| = 1 :\ y'L(\alpha)y \le 0. \tag{2.25}$$

Let us observe that

$$y \in \operatorname{img}(V_0) \quad \iff \quad y = V_0 p, \quad p \in \mathbb{R}^{\sigma(n,m)-r}.$$

Since $y'L(\alpha)y$ depends linearly on α, the condition (2.25) can be rewritten as

$$\forall\alpha, \|\alpha\| = 1,\ \exists p, \|p\| = 1 :\ p'V_0'L(\alpha)V_0 p \le 0$$

which is equivalent to (2.20). □

Theorem 2.6 provides a necessary and sufficient condition to establish if a given SMR matrix H is a maximal SMR matrix. This condition is important because it states that the property of being a maximal SMR matrix is related only to the matrix V_0 in the minimum eigenvalue decomposition of H, which represents the eigenspace of the minimum eigenvalue of H. In particular, this eigenspace is given by $\operatorname{img}(V_0)$. Hence, Theorem 2.6 provides a way to construct maximal SMR matrices.

Let us observe that the feasible set for α in (2.19) is nonconvex, which makes the computation of the index $\eta^*(V_0)$ difficult. The following result provides an alternative way for characterizing maximal SMR matrices.

Theorem 2.7. *Let V_0 and $L(\alpha)$ be defined as in Theorem 2.6, and define*

$$\eta(V_0) = \max\ \{\eta(V_0, 1), \eta(V_0, -1)\} \tag{2.26}$$

where

$$\eta(V_0, z) = \sup_{\alpha:\ y'\alpha = z} \lambda_{min}\left(V_0'L(\alpha)V_0\right) \tag{2.27}$$

and $y \in \mathbb{R}_0^{\omega(n,m)}$. Then, for all $y \in \mathbb{R}_0^{\omega(n,m)}$, one has

$$\eta^*(V_0) \le 0 \quad \iff \quad \eta(V_0) \le 0. \tag{2.28}$$

Proof. (Necessity) Let us assume that $\eta^*(V_0) \le 0$ and let us suppose by contradiction that $\eta(V_0) > 0$. Then, there exists $\tilde{\alpha} \in \mathbb{R}^{\omega(n,m)}$ such that $|y'\tilde{\alpha}| = 1$ and

$$\lambda_{min}\left(V_0'L(\tilde{\alpha})V_0\right) > 0. \tag{2.29}$$

Let us define

$$\bar{\alpha} = \|\tilde{\alpha}\|^{-1}\tilde{\alpha}.$$

We have that $\|\bar{\alpha}\| = 1$ and

$$\lambda_{min}\left(V_0'L(\bar{\alpha})V_0\right) = \|\tilde{\alpha}\|^{-1}\lambda_{min}\left(V_0'L(\tilde{\alpha})V_0\right) > 0.$$

But this is impossible since we have assumed that $\eta^*(V_0) \leq 0$.

(Sufficiency) Let us assume that $\eta(V_0) \leq 0$ and let us suppose by contradiction that $\eta^*(V_0) > 0$. Then, there exists $\tilde{\alpha} \in \mathbb{R}^{\omega(n,m)}$ such that $\|\tilde{\alpha}\| = 1$ and (2.29) holds. First, let us suppose that

$$y'\tilde{\alpha} \neq 0 \tag{2.30}$$

and let us define

$$\bar{\alpha} = |y'\tilde{\alpha}|_2^{-1}\tilde{\alpha}.$$

We have that $\|y'\bar{\alpha}\| = 1$ and

$$\lambda_{min}\left(V_0'L(\bar{\alpha})V_0\right) = |w'\tilde{\alpha}|_2^{-1}\lambda_{min}\left(V_0'L(\tilde{\alpha})V_0\right) > 0.$$

But this is impossible since we have assumed that $\eta(V_0) \leq 0$.
Now, let us suppose that

$$y'\tilde{\alpha} = 0.$$

Then, for all $\varepsilon > 0$ there exists $\hat{\alpha} \in \mathbb{R}^{\omega(n,m)}$ such that $\|\hat{\alpha}\| = 1$ and

$$\|\hat{\alpha} - \tilde{\alpha}\| < \varepsilon \text{ and } y'\hat{\alpha} \neq 0.$$

Since the function $\lambda_{min}(V_0'L(\alpha)V_0)$ is continuous with respect to α and since $\hat{\alpha}$ is arbitrarily close to $\tilde{\alpha}$ which satisfies (2.29), it follows that $\hat{\alpha}$ can be chosen to satisfy also the condition $\lambda_{min}(V_0'L(\hat{\alpha})V_0) > 0$. By repeating the proof from (2.30) by using $\hat{\alpha}$ instead of $\tilde{\alpha}$, we finally conclude that (2.28) holds. □

Theorem 2.7 provides an alternative way to establish whether an SMR matrix is a maximal SMR matrix or not. This is achieved via the index $\eta(V_0)$, which can be computed through two convex optimizations. In fact, it turns out that $\eta(V_0, z)$ is the solution of the EVP

$$\begin{aligned} \eta(V_0, z) = \ & \sup_{t,\alpha} t \\ & \text{s.t. } \begin{cases} y'\alpha - z = 0 \\ V_0'L(\alpha)V_0 - tI_{\sigma(n,m)-r} \geq 0. \end{cases} \end{aligned} \tag{2.31}$$

Let us observe that the free vector y defines two hyperplanes on which the function $\lambda_{min}(V_0'L(\alpha)V_0)$ is evaluated.

Example 2.4. Let us consider the form $h(x)$ in (1.20) and its SMR in (1.21). It can be verified that a minimum eigenvalue decomposition $\langle \lambda_{min}(H), \beta, V_0, V_p \rangle$ of the SMR matrix H in (1.21) is given by

$$\begin{aligned}
\lambda_{min}(H) &= -0.6180 \\
\beta &= (2.2361, 2.6180)' \\
V_0 &= (0.5257, -0.8507, 0)' \\
V_p &= \begin{pmatrix} -0.8507 & -0.5257 & 0 \\ 0 & 0 & 1.0000 \end{pmatrix}'.
\end{aligned}$$

By applying (2.26)-(2.27), we find that $\eta(V_0) > 0$, which implies from Theorem 2.7 that H is not a maximal SMR matrix. This is confirmed by the fact that there exists another SMR matrix of $h(x)$ whose minimum eigenvalue is larger than the minimum eigenvalue of H. This SMR matrix is given by H^* in (2.14) with $\alpha^* = 0.8008$, which is an optimal value of α in the EVP (1.29). Indeed we have:

$$H^* = \begin{pmatrix} 1.0000 & 1.0000 & -0.8008 \\ \star & 1.6016 & 0.0000 \\ \star & \star & 2.0000 \end{pmatrix} \tag{2.32}$$

and

$$\lambda_{min}(H^*) = 0.0352, \quad \lambda_{min}(H) = -0.6180.$$

Lastly, we test Theorem 2.7 on the SMR matrix H^*. To this end, consider the minimum eigenvalue decomposition of H^* given by

$$\begin{aligned}
\lambda_{min}(H^*) &= 0.0352 \\
\beta^* &= (1.7895, 2.7065)' \\
V_0^* &= (0.7972, -0.5089, 0.3249)' \\
V_p^* &= \begin{pmatrix} 0.1544 & 0.6920 & 0.7052 \\ -0.5837 & -0.5120 & 0.6302 \end{pmatrix}'.
\end{aligned} \tag{2.33}$$

From (2.26) we find $\eta(V_0^*) = 0.0000$ by solving (2.31) with $y = 1$, which verifies by Theorems 2.6 and 2.7 that H^* is a maximal SMR matrix.

2.3 SMR-tight Forms

This section introduces and characterizes a special class of forms, specifically the forms whose positivity index coincides with their SOS index.

Definition 2.4 (SMR-tight Form). Let us suppose $h \in \Xi_{n,2m}$ satisfies

$$\lambda(h) = \mu(h). \tag{2.34}$$

Then, $h(x)$ is said to be *SMR-tight*.

Before proceeding with the characterization of SMR-tight forms, let us make the following observations:

1. PNS forms are not SMR-tight. In fact, if $h(x)$ is PNS then $\mu(h) \geq 0$ and $\lambda(h) < 0$.
2. A form can be SMR-tight even if it is not SOS. This is shown by the following example.

Example 2.5. Let us consider the form

$$h(x) = x_1^2 + 4x_1x_2 + x_2^2.$$

We have that a complete SMR of $h(x)$ is given by

$$x^{\{m\}} = \begin{pmatrix} x_1 & x_2 \end{pmatrix}', \quad H = \begin{pmatrix} 1 & 2 \\ \star & 1 \end{pmatrix}, \quad L(\alpha) = 0_{2\times 2}$$

which implies that the SOS index of $h(x)$ is

$$\lambda(h) = \lambda_{min}(H) = -1.$$

Then, it can be verified that the positivity index of $h(x)$ is

$$\begin{aligned} \mu(h) &= \min_{x \in \mathscr{C}_{n,m}} h(x) \\ &= \min_{x:\ x_1^2 + x_2^2 = 1} h(x) \\ &= -1. \end{aligned}$$

Therefore, $h(x)$ is SMR-tight because $\lambda(h) = \mu(h)$. However, $h(x)$ is not SOS: indeed, $\mu(h)$ is negative, which means that $h(x)$ can take negative values.

2.3.1 Minimal Point Set

A necessary and sufficient condition for establishing whether a form is SMR-tight can be obtained by searching for power vectors in a linear space. To this end, let us introduce the following definition.

Definition 2.5 (Minimal Point Set). Let $h \in \Xi_{n,2m}$, $H \in \mathbb{S}^{\sigma(n,m)}$ be a maximal SMR matrix of $h(x)$, and define the linear space

$$\mathscr{N}(H) = \ker\left(H - \lambda_{min}(H) I_{\sigma(n,m)}\right). \tag{2.35}$$

Then, the set

$$\mathrm{mps}(h) = \left\{ x \in \mathbb{R}^n : \ \|x\| = 1, \ x^{\{m\}} \in \mathcal{N}(H) \right\} \tag{2.36}$$

is called *minimal point set* of $h(x)$.

The following lemma clarifies the relationship between $\mathcal{N}(H)$ and the minimal eigenvalue decompositions of H.

Lemma 2.1. *Let $h \in \Xi_{n,2m}$, and $H \in \mathbb{S}^{\sigma(n,m)}$ be a maximal SMR matrix of $h(x)$. Let $\langle \lambda_{min}(H), \beta, V_0, V_p \rangle$ be a minimum eigenvalue decomposition of H. Then,*

$$\mathcal{N}(H) = \mathrm{img}(V_0). \tag{2.37}$$

Proof. From Definition 2.3 we have that the columns of V_0 are a base of the eigenspace of the minimum eigenvalue of H, which is $\mathcal{N}(H)$ according to (2.35). Therefore, (2.37) holds. □

It is worthwhile to observe that the minimal point set of $h(x)$ does not depend on the chosen maximal SMR matrix H. This is explained in the following result.

Theorem 2.8. *Let $h \in \Xi_{n,2m}$, and for $i = 1,2$ define*

$$\mathscr{A}_i = \left\{ x \in \mathbb{R}^n : \ \|x\| = 1, \ x^{\{m\}} \in \mathcal{N}(H_i) \right\}$$

where $H_1, H_2 \in \mathbb{S}^{\sigma(n,m)}$ are any pair of maximal SMR matrices of $h(x)$. Then,

$$\mathscr{A}_1 = \mathscr{A}_2$$

i.e. $\mathrm{mps}(h)$ *is independent on the chosen maximal SMR matrix H of $h(x)$.*

Proof. Let us suppose by contradiction that there exists $\bar{x} \in \mathscr{A}_1$ such that $\bar{x} \notin \mathscr{A}_2$. Since $\bar{x} \in \mathscr{A}_1$ we have that

$$\begin{aligned} 0 &= \bar{x}^{\{m\}'} \left(H_1 - \lambda_{min}(H_1) I_{\sigma(n,m)} \right) \bar{x}^{\{m\}} \\ &= h(\bar{x}) - \lambda_{min}(H_1) \left\| \bar{x}^{\{m\}} \right\|^2. \end{aligned} \tag{2.38}$$

Since H_2 is a maximal SMR matrix of $h(x)$ we have that

$$\begin{aligned} h(\bar{x}) &= \bar{x}^{\{m\}'} H_2 \bar{x}^{\{m\}} \\ \lambda_{min}(H_1) &= \lambda_{min}(H_2) \end{aligned}$$

which, from (2.38), provides

$$0 = \bar{x}^{\{m\}'} \left(H_2 - \lambda_{min}(H_2) I_{\sigma(n,m)} \right) \bar{x}^{\{m\}}.$$

Moreover,

$$H_2 - \lambda_{min}(H_2) I_{\sigma(n,m)} \geq 0$$

which implies that

$$\|\bar{x}\| = 1 \text{ and } \bar{x}^{\{m\}} \in \mathcal{N}(H_2)$$

hence contradicting the assumption $\bar{x} \notin \mathcal{A}_2$. □

The following result states that a necessary and sufficient condition for a form to be SMR-tight is that the minimal point set of the form is not empty.

Theorem 2.9. *Let $h \in \Xi_{n,2m}$. Then, $h(x)$ is SMR-tight if and only if*

$$\text{mps}(h) \neq \emptyset. \tag{2.39}$$

Proof. (Sufficiency) Let us suppose that $\text{mps}(h) \neq \emptyset$. Let $\bar{x}$ be any vector in $\text{mps}(h)$ and define

$$\hat{x} = \frac{\bar{x}}{\|\bar{x}^{\{m\}}\|}.$$

By letting $H \in \mathbb{S}^{\sigma(n,m)}$ be a maximal SMR matrix of $h(x)$, we have that

$$\begin{aligned} 0 &= \hat{x}^{\{m\}'} \left(H - \lambda_{min}(H) I_{\sigma(n,m)}\right) \hat{x}^{\{m\}} \\ &= h(\hat{x}) - \lambda_{min}(H) \left\|\hat{x}^{\{m\}}\right\|^2 \\ &= h(\hat{x}) - \lambda(h) \end{aligned}$$

which implies that

$$\exists \hat{x} \in \mathcal{C}_{n,m} : \ h(\hat{x}) = \lambda(h). \tag{2.40}$$

By Definition 1.14, $\mu(h)$ is the minimum of $h(x)$ over the set $\mathcal{C}_{n,m}$; moreover, by Theorem 1.9, $\lambda(h)$ is a lower bound of $\mu(h)$. Therefore, from (2.40) we conclude that $\mu(h) = \lambda(h)$, i.e. $h(x)$ is SMR-tight.

(Necessity) Let us suppose that $h(x)$ is SMR-tight, i.e. $\mu(h) = \lambda(h)$. Then, (2.40) is satisfied. Let $H \in \mathbb{S}^{\sigma(n,m)}$ be a maximal SMR matrix of $h(x)$. We have that:

$$\begin{aligned} 0 &= h(\hat{x}) - \lambda(h) \\ &= h(\hat{x}) - \lambda_{min}(H) \left\|\hat{x}^{\{m\}}\right\|^2 \\ &= \hat{x}^{\{m\}'} \left(H - \lambda_{min}(H) I_{\sigma(n,m)}\right) \hat{x}^{\{m\}}. \end{aligned}$$

Since $H - \lambda_{min}(H) I_{\sigma(n,m)} \geq 0$ it follows that there exists $\hat{x} \in \mathbb{R}_0^n$ such that $x^{\{m\}}$ belongs to $\mathcal{N}(H)$. Therefore, let us define

$$\bar{x} = \frac{\hat{x}}{\|\hat{x}\|}.$$

We have that $\bar{x} \in \text{mps}(h)$, and hence (2.39) holds. □

2.3.2 Rank Conditions

Clearly, it is possible to establish whether $\mathrm{mps}(h)$ is empty or not by computing vectors in $\mathrm{mps}(h)$ through the extraction procedure described in Section 1.9. In the sequel, we aim to provide alternative conditions for establishing whether $\mathrm{mps}(h)$ is empty, which do not require the actual computation of the set $\mathrm{mps}(h)$ itself.

Theorem 2.10. *Let $h \in \Xi_{n,2m}$, $H \in \mathbb{S}^{\sigma(n,m)}$ be a maximal SMR matrix of $h(x)$, and $\mathcal{N}(H)$ be the linear space in* (2.35). *Let us suppose that one of the following conditions holds:*

1. *m is odd and* $\dim(\mathcal{N}(H)) > \sigma(n,m) - n$;
2. *m is even and* $\dim(\mathcal{N}(H)) = \sigma(n,m)$.

Then, $h(x)$ is SMR-tight.

Proof. Let us suppose that item 1 holds, and let $\langle \lambda_{min}(H), \beta, V_0, V_p \rangle$ be a minimum eigenvalue decomposition of H. Let us consider the equation

$$V_p' x^{\{m\}} = 0. \tag{2.41}$$

We have that (2.41) defines a system of $\sigma(n,m) - \dim(\mathcal{N}(H))$ homogeneous equations of degree m in n scalar variables. In particular, the degree of these homogeneous equations is odd, and their number is smaller than the number of scalar variables because

$$\sigma(n,m) - \dim(\mathcal{N}(H)) < n.$$

This implies that

$$\exists x \in \mathbb{R}^n : \ \|x\| = 1, \ V_p' x^{\{m\}} = 0.$$

From Definition 2.3, $\mathrm{img}(V_0) = \ker(V_p')$, and hence it immediately follows that

$$\exists x \in \mathbb{R}^n : \ \|x\| = 1, \ x^{\{m\}} \in \mathrm{img}(V_0).$$

Moreover, by Lemma 2.1, $\mathrm{img}(V_0) = \mathcal{N}(H)$, which implies that there exists $x \in \mathbb{R}^n$ such that $\|x\| = 1$ and $x^{\{m\}} \in \mathcal{N}(H)$. This means that $\mathrm{mps}(h) \neq \emptyset$ from Definition 2.5, and hence $h(x)$ is SMR-tight by Theorem 2.9.

Lastly, let us suppose that item 2 holds. It immediately follows that

$$\mathcal{N}(H) = \mathbb{R}^{\sigma(n,m)}$$

and hence

$$\mathrm{mps}(h) = \{x \in \mathbb{R}^n : \ \|x\| = 1\}$$

i.e. $\mathrm{mps}(h) \neq \emptyset$ and hence $h(x)$ is SMR-tight by Definition 2.5 and Theorem 2.9. □

Theorem 2.10 provides a simple condition to establish whether a form is SMR-tight, which consists only in checking whether the dimension of the linear space $\mathcal{N}(H)$ lies in a given range.

Example 2.6. Let us consider the form $h(x)$ in Example 1.2. A maximal SMR matrix of $h(x)$ has been found in Example 2.4 and is given by H^* in (2.32). From Lemma 2.1 and (2.33) we have that the linear space $\mathcal{N}(H^*)$ is given by

$$\mathcal{N}(H^*) = \text{img}\begin{pmatrix} 0.7972 \\ -0.5089 \\ 0.3249 \end{pmatrix}.$$

Since H^* is constructed with respect to the power vector $x^{\{m\}} = (x_1^2, x_1x_2, x_2^2)'$, it can be verified from Definition 2.5 that

$$\text{mps}(h) = \left\{ \pm \begin{pmatrix} 0.8429 \\ -0.5381 \end{pmatrix} \right\}.$$

Therefore, $\text{mps}(h)$ is not empty, and hence $h(x)$ is SMR-tight according to Theorem 2.9.

Example 2.7. Let us consider the form

$$\begin{aligned} h(x) = \; & 27x_1^4x_2^2 - 36\sqrt{3}x_1^2x_2^4 + 72x_1^2x_2^2x_3^2 + 36x_2^6 - 24\sqrt{3}x_2^4x_3^2 + 12x_2^2x_3^4 \\ & +12x_1^4x_3^2 - 24\sqrt{3}x_1^2x_3^4 + 27x_2^4x_3^2 - 36\sqrt{3}x_2^2x_3^4 + 36x_3^6. \end{aligned}$$

We have $n = 3$, $m = 3$ and $\sigma(n,m) = 10$. After computing a maximal SMR matrix H of $h(x)$, we find that $\dim(\mathcal{N}(H)) = 8$. Let us observe that $\text{mps}(h)$ cannot be computed via the extraction procedure described in Section 1.9 because (1.89) does not hold, indeed $8 = u \not\leq (n-1)(m-1)+2 = 6$. Then, let us consider Theorem 2.10. We have that m is odd, and the first condition of the theorem holds since $8 = \dim(\mathcal{N}(H)) > \sigma(n,m) - n = 7$. This implies that $h(x)$ is SMR-tight and hence the positivity index $\mu(h)$ is equal to the SOS index $\lambda(h)$, which in this case is equal to 0.

Example 2.8. Let us consider Motzkin's form in (2.1). We have $n = 3$, $m = 3$ and $\sigma(n,m) = 10$. After computing a maximal SMR matrix H of $h_{Mot}(x)$, we find that $\dim(\mathcal{N}(H)) = 7$. Let us observe that $\text{mps}(h)$ cannot be computed via the extraction procedure described in Section 1.9 because (1.89) does not hold, indeed $7 = u \not\leq (n-1)(m-1)+2 = 6$. Then, let us consider Theorem 2.10. We have that m is odd, however the first condition of the theorem does not hold since $7 = \dim(\mathcal{N}(H)) \not> \sigma(n,m) - n = 7$. This means that we cannot conclude that $h_{Mot}(x)$ is SMR-tight. This is in accordance with the fact that $h_{Mot}(x)$ cannot be SMR-tight since it is PNS, which implies $\lambda(h_{Mot}) < 0$ and $\mu(h_{Mot}) = 0$.

2.4 Characterizing PNS Forms via the SMR

This section investigates the structure of PNS forms through the SMR. In particular, it is shown that each PNS form is the vertex of a cone of PNS forms. Moreover, a parametrization of PNS forms is proposed.

2.4.1 Basic Properties of PNS Forms

First of all, let us observe that, while the sets $\Omega_{n,2m}$ and $\Sigma_{n,2m}$ are convex, the set $\Delta_{n,2m}$ is nonconvex. This is shown by the following example.

Example 2.9. Let us consider Motzkin's form in (2.1) and Stengle's form [122]

$$h_{Ste}(x) = x_1^3 x_3^3 + (x_2^2 x_3 - x_1^3 - x_1 x_3^2)^2, \tag{2.42}$$

which are both in $\Delta_{3,6}$. Let us define the form

$$h(x) = \frac{1}{2}\left(h_{Mot}(x) + h_{Ste}(x)\right).$$

It can be verified that

$$\lambda(h) = 0$$

which means that $h(x)$ is SOS. Therefore, $h \notin \Delta_{3,6}$, which implies that $\Delta_{3,6}$ is not convex.

For any $h \in \Xi_{n,m}$ let us define the ball in $\Xi_{n,m}$ with radius $\delta \in \mathbb{R}$ centered in $h(x)$ as

$$\mathscr{B}_\delta(h) = \{h_1 \in \Xi_{n,m} : \ d(h_1, h) \leq \delta\} \tag{2.43}$$

where $d : \Xi_{n,m} \times \Xi_{n,m} \to \mathbb{R}$ is the distance in $\Xi_{n,m}$ defined as

$$d(h_1, h) = \|g_1 - g\| \tag{2.44}$$

being $g_1, g \in \mathbb{R}^{\sigma(n,m)}$ vectors representing respectively h_1, h according to the power vector representation (1.6).

The following result introduces some key properties of $\Delta_{n,2m}$.

Theorem 2.11. *Suppose that $\Delta_{n,2m} \neq \emptyset$. Then:*

1. *there exists $h \in \Delta_{n,2m}$ such that $\mu(h) > 0$;*
2. *for any $h \in \Delta_{n,2m}$ such that $\mu(h) > 0$, it follows that*

$$\exists \delta > 0 : \ \mathscr{B}_\delta(h) \subset \Delta_{n,2m}; \tag{2.45}$$

3. *for any $h \in \Delta_{n,2m}$ there exists $\delta > 0$ such that*

$$\mathscr{B}_\delta(h) \cap \Omega_{n,2m} \subset \Delta_{n,2m}. \tag{2.46}$$

Proof. Consider item 1, and let $\Delta_{n,2m} \neq \emptyset$. Then, there exists $h \in \Delta_{n,2m}$ such that $\mu(h) \geq 0$. Let us suppose that $\mu(h) = 0$ and let us define

$$h_1(x) = h(x) + \varepsilon x^{\{m\}'} x^{\{m\}}.$$

It follows that

$$\mu(h_1) = \mu(h) + \varepsilon = \varepsilon.$$

Moreover, let H be an SMR matrix of $h(x)$. We have that

$$H_1 = H + \varepsilon I_{\sigma(n,m)}$$

is an SMR matrix of $h_1(x)$. Hence, it follows that

$$\lambda(h_1) = \lambda(h) + \varepsilon.$$

Since $\lambda(h) < 0$, by choosing $\varepsilon \in (0, -\lambda(h))$, one gets $h_1 \in \Delta_{n,2m}$ and $\mu(h_1) > 0$. Hence, item 1 holds.

Consider item 2, and let $h \in \Delta_{n,2m}$ with $\mu(h) > 0$. We have also $\lambda(h) < 0$. For continuity of $\mu(h)$ and $\lambda(h)$ with respect to the coefficients of $h(x)$, it follows that

$$\exists \delta > 0 : \ \mu(h_1) > 0 \text{ and } \lambda(h_1) < 0 \quad \forall h_1 \in \mathscr{B}_\delta(h)$$

i.e. (2.45) holds.

Lastly, consider item 3, and let $h \in \Delta_{n,2m}$. If $\mu(h) > 0$, then (2.45) holds, which directly implies (2.46) since $\Delta_{n,2m} \subset \Omega_{n,2m}$. Hence, let us suppose $\mu(h) = 0$. Similarly to the proof of (2.45) it follows that

$$\exists \delta > 0 : \ \lambda(h_1) < 0 \quad \forall h_1 \in \mathscr{B}_\delta(h)$$

i.e. $\mathscr{B}_\delta(h) \cap \Sigma_{n,2m} = \emptyset$. Therefore, (2.46) holds. □

Theorem 2.11 states three properties for the set of PNS forms $\Delta_{n,2m}$. The first says that, if this set is not empty, then it contains positive definite forms. The second property says that positive definite forms in $\Delta_{n,2m}$ are interior points of $\Delta_{n,2m}$. The third property establishes that every PNS form owns a neighborhood with shape defined by (2.43)–(2.44) where all positive semidefinite forms are PNS. This means that arbitrarily small changes of the coefficients of a PNS form cannot turn this form into an SOS form.

As it has been explained in the previous sections, to establish whether a form $h(x)$ is PNS amounts to establishing whether $\mu(h) \geq 0$ and $\lambda(h) < 0$. The following result provides a further characterization of PNS forms which turns out to be useful for their construction.

Theorem 2.12. *Let $h \in \Delta_{n,2m}$, and $H \in \mathbb{S}^{\sigma(n,m)}$ be a maximal SMR matrix of $h(x)$. Let $\langle \lambda_{min}(H), \beta, V_0, V_p \rangle$ be a minimum eigenvalue decomposition of H. Then,*

$$\nexists x \in \mathbb{R}_0^n : \ V_p' x^{\{m\}} = 0. \tag{2.47}$$

Proof. Let us suppose by contradiction that there exists $\tilde{x} \in \mathbb{R}_0^n$ such that $\tilde{x}^{\{m\}} \in \ker(V_p')$. Let r be the length of β. Then, we have

$$\begin{aligned} h(\tilde{x}) &= \tilde{x}^{\{m\}'} \begin{pmatrix} V_0 & V_p \end{pmatrix} \left(\lambda_{min}(H) I_{\sigma(n,m)} + \operatorname{diag}\begin{pmatrix} 0_{\sigma(n,m)-r} \\ \beta \end{pmatrix} \right) \begin{pmatrix} V_0' \\ V_p' \end{pmatrix} \tilde{x}^{\{m\}} \\ &= \lambda_{min}(H) \|V_0' \tilde{x}^{\{m\}}\|^2. \end{aligned}$$

Let us observe that $\lambda_{min}(H) < 0$, because H is a maximal SMR matrix of a PNS form. Moreover, $\|V_0'\tilde{x}^{\{m\}}\| > 0$, since $\operatorname{img}(V_0) = \ker(V_p')$. This implies that $h(\tilde{x}) < 0$, which is impossible since $h(x)$ is PNS. □

Theorem 2.12 provides a necessary condition for a form to be PNS: the absence of solutions $x \in \mathbb{R}_0^n$ in the homogeneous polynomial system $V_p' x^{\{m\}} = 0$. By Definition 2.3, this condition is equivalent to

$$\nexists x \in \mathbb{R}_0^n : \; x^{\{m\}} \in \operatorname{img}(V_0).$$

2.4.2 Cones of PNS Forms

The following result provides a way to generate a set of PNS forms from a given PNS form.

Theorem 2.13. *Let $h \in \Delta_{n,2m}$, $H \in \mathbb{S}^{\sigma(n,m)}$ be a maximal SMR matrix of $h(x)$, and $\langle \lambda_{min}(H), \beta, V_0, V_p \rangle$ be a minimum eigenvalue decomposition of H. Let us define the parametrized form*

$$s(x;\gamma) = x^{\{m\}'} V_p \operatorname{diag}(\gamma) V_p' x^{\{m\}} \tag{2.48}$$

for some $\gamma \in \mathbb{R}^r$, where r is the length of β. Moreover, let us define the set

$$\operatorname{cone}(h) = \left\{ h_1 \in \Xi_{n,2m} : \; h_1(x) = h(x) + s(x;\gamma), \; \gamma > 0 \right\}. \tag{2.49}$$

Then,

$$\operatorname{cone}(h) \subset \Delta_{n,2m}. \tag{2.50}$$

Moreover,

$$\exists \delta > 0 : \; \mu\left(h + s(\cdot,\gamma)\right) \geq \mu(h) + \delta \min_{1 \leq i \leq r} \gamma_i. \tag{2.51}$$

Proof. First of all, let us observe that $s(x;\gamma)$ is SOS for all $\gamma \geq 0$ because a positive semidefinite SMR matrix of $s(x;\gamma)$ for all $\gamma \geq 0$ is given by

$$S(\gamma) = V_p \operatorname{diag}(\gamma) V_p'.$$

In order to prove (2.50), let us observe that

$$H_1 = H + S(\gamma)$$

is a maximal SMR matrix of

$$h_1(x) = h(x) + s(x;\gamma).$$

In fact, we have that

$$\begin{aligned} H_1 &= \begin{pmatrix} V_0 & V_p \end{pmatrix} \left(\lambda_{min}(H) I_{\sigma(n,m)} + \mathrm{diag}\begin{pmatrix} 0_{\sigma(n,m)-r} \\ \gamma \end{pmatrix} \right) \begin{pmatrix} V_0' \\ V_p' \end{pmatrix} + V_p \,\mathrm{diag}(\beta) V_p' \\ &= \begin{pmatrix} V_0 & V_p \end{pmatrix} \left(\lambda_{min}(H) I_{\sigma(n,m)} + \mathrm{diag}\begin{pmatrix} 0_{\sigma(n,m)-r} \\ \beta+\gamma \end{pmatrix} \right) \begin{pmatrix} V_0' \\ V_p' \end{pmatrix} \end{aligned}$$

which clearly implies that

$$\langle \lambda_{min}(H), \beta+\gamma, V_0, V_p \rangle \tag{2.52}$$

is a minimum eigenvalue decomposition of H_1. Since H is a maximal SMR matrix of $h(x)$, we have from Theorem 2.6 that $\eta^*(V_0) \le 0$, which implies that also H_1 is a maximal SMR matrix.

Now, from the fact that H_1 is a maximal SMR matrix and taking into account its minimum eigenvalue decomposition in (2.52), it follows that

$$\lambda(h_1) = \lambda_{min}(H_1) = \lambda_{min}(H) = \lambda(h).$$

Moreover, we have that

$$\mu(h_1) \ge \mu(h)$$

because $s(x;\gamma)$ is SOS. Since $h \in \Delta_{n,2m}$ we conclude that $\lambda(h_1) = \lambda(h) < 0$ and $\mu(h_1) \ge \mu(h) \ge 0$, which imply that $h_1(x)$ is PNS. Therefore, (2.50) holds.

Finally, let us observe that

$$\mu\left(h + s(\cdot,\gamma)\right) \ge \mu(h) + \mu\left(s(\cdot,\gamma)\right)$$

and

$$s(x;\gamma) \ge \|V_p' x^{\{m\}}\|^2 \min_{1\le i\le r} \gamma_i \quad \forall x \; \forall \gamma.$$

According to Theorem 2.12, we have that $V_p' x^{\{m\}} \ne 0$ for all $x \in \mathbb{R}_0^n$. Hence, (2.51) holds with $\delta = \mu(h_2)$, where $h_2(x)$ is the form $h_2(x) = \|V_p' x^{\{m\}}\|^2$. □

Theorem 2.13 states that any PNS form $h(x)$ is the vertex of a cone of PNS forms given by $\mathrm{cone}(h)$. In particular, the directions of this cone correspond to the SOS forms given by $s(x;\gamma)$ for $\gamma > 0$. Let us also observe that, according to (2.51), there exist PNS forms in this cone whose positivity index is arbitrarily large.

2.4.3 Parametrization of PNS Forms

Maximal SMR matrices can be exploited to derive a parametrization of all PNS forms. Let us define the set

$$\begin{aligned}\Theta^P_{n,2m}(r) = \Big\{ & V_p \in \mathbb{R}^{\sigma(n,m)\times r} : \\ & V_p' V_p = I_r,\ \ \eta^*(\mathrm{cmp}(V_p)) \le 0, \text{ and (2.47) holds} \Big\}.\end{aligned} \tag{2.53}$$

The notation $\mathrm{cmp}(V_p)$ denotes any matrix in $\mathbb{R}^{\sigma(n,m)\times(\sigma(n,m)-r)}$ whose columns are an orthonormal base of $\ker(V_p')$. Hence $\mathrm{cmp}(V_p)$ satisfies the conditions

$$\begin{cases} \mathrm{cmp}(V_p)'\,\mathrm{cmp}(V_p) = I_{\sigma(n,m)-r} \\ \mathrm{img}(\mathrm{cmp}(V_p)) = \ker(V_p'). \end{cases}$$

Now, let us introduce the set

$$\Theta_{n,2m} = \bigcup_{1\le r\le \sigma(n,m)} \Theta_{n,2m}(r) \tag{2.54}$$

where

$$\Theta_{n,2m}(r) = \left\{ \langle \delta, \beta, V_p \rangle :\ \delta \in (0,1];\ \ \beta \in \mathbb{R}^r, \beta > 0;\ \ V_p \in \Theta^P_{n,2m}(r) \right\}. \tag{2.55}$$

For any $\theta = \langle \delta, \beta, V_p \rangle \in \Theta_{n,2m}(r)$, let $s(x;\beta) = x^{\{m\}'} V_p \,\mathrm{diag}(\beta) V_p' x^{\{m\}}$ and define the form

$$\pi(x;\theta) = s(x;\beta) - \delta\mu\left(s(\cdot,\beta)\right) x^{\{m\}'} x^{\{m\}}. \tag{2.56}$$

The following result provides a parametrization of the set of PNS forms $\Delta_{n,2m}$.

Theorem 2.14. *Let* $\Theta_{n,2m}$ *be defined by* (2.53)–(2.55)*, and* $\pi(x;\theta)$ *be given by* (2.56)*. Then,*

$$h \in \Delta_{n,2m} \quad \Longleftrightarrow \quad \exists \theta \in \Theta_{n,2m} :\ h(x) = \pi(x;\theta). \tag{2.57}$$

Proof. (Necessity) Let $h \in \Delta_{n,2m}$. Let H be a maximal SMR matrix of $h(x)$, and let $\langle \lambda_{min}(H), \beta, V_0, V_p \rangle$ be a minimum eigenvalue decomposition of H. Let r be the length of β. We have that

$$\begin{aligned} h(x) &= x^{\{m\}'} \begin{pmatrix} V_0 & V_p \end{pmatrix} \left(\lambda_{min}(H) I_{\sigma(n,m)} + \mathrm{diag}\begin{pmatrix} 0_{\sigma(n,m)-r} \\ \beta \end{pmatrix} \right) \begin{pmatrix} V_0' \\ V_p' \end{pmatrix} x^{\{m\}} \\ &= x^{\{m\}'} \left(\lambda_{min}(H) I_{\sigma(n,m)} + V_p \,\mathrm{diag}(\beta) V_p' \right) x^{\{m\}} \\ &= \lambda_{min}(H) x^{\{m\}'} x^{\{m\}} + s(x;\beta). \end{aligned}$$

Hence, $h(x) = \pi(x;\theta)$ where

$$\theta = \langle \delta, \beta, V_p \rangle$$
$$\delta = -\frac{\lambda_{min}(H)}{\mu\left(s(\cdot,\beta)\right)}.$$

Let us observe that $\delta \in (0,1]$ because $\lambda_{min}(H) = \lambda(h) < 0$ and $\lambda_{min}(H) + \mu\left(s(\cdot,\beta)\right) = \mu(g(x)) \geq 0$. Moreover, $\beta > 0$ because $\langle \lambda_{min}(H), \beta, V_0, V_p \rangle$ is a minimum eigenvalue decomposition of H. Then, by Theorem 2.6 and Theorem 2.12 it follows that $V_p \in \Theta^P_{n,2m}(r)$.

(Sufficiency) Let $\theta = \langle \delta, \beta, V_p \rangle \in \Theta_{n,2m}$. We have that an SMR matrix of $\pi(x;\theta)$ is given by

$$\begin{aligned} H &= V_p \operatorname{diag}(\beta) V_p' - \delta\mu\left(s(\cdot,\beta)\right) I_{\sigma(n,m)} \\ &= \begin{pmatrix} \operatorname{cmp}(V_p) & V_p \end{pmatrix} \left(\operatorname{diag}\begin{pmatrix} 0_{\sigma(n,m)-r} \\ \beta \end{pmatrix} - \delta\mu\left(s(\cdot,\beta)\right) I_{\sigma(n,m)} \right) \begin{pmatrix} \operatorname{cmp}(V_p)' \\ V_p' \end{pmatrix}. \end{aligned}$$

Since $V_p' V_p = I_r$ and $\beta > 0$, it follows that

$$\langle -\delta\mu\left(s(\cdot,\beta)\right), \beta, \operatorname{cmp}(V_p), V_p \rangle$$

is a minimum eigenvalue decomposition of H. Since $\eta^*(\operatorname{cmp}(V_p)) \leq 0$, this implies that H is a maximal SMR matrix from Theorem 2.6. Moreover, from Theorem 2.12 it follows that $\mu\left(s(\cdot,\beta)\right) > 0$. Hence,

$$\lambda\left(\pi(\cdot,\theta)\right) = -\delta\mu\left(s(\cdot,\beta)\right) < 0$$

and

$$\mu\left(\pi(\cdot,\theta)\right) = (1-\delta)\mu\left(s(\cdot,\beta)\right) \geq 0.$$

Therefore, (2.57) holds. □

Theorem 2.14 states that $\Delta_{n,2m}$ is the image of $\Theta_{n,2m}$ through the function $\pi(x;\theta)$. Hence, this result provides a technique to parametrize and construct all the PNS forms. This technique amounts to finding matrices V_p in $\Theta^P_{n,2m}(r)$ and calculating the positivity index $\mu\left(s(\cdot,\beta)\right)$. Unfortunately, it is difficult to find an explicit representation of the set $\Theta^P_{n,2m}(r)$. A method to find elements in this set consists of looking for matrices V_p with a fixed structure, for which the property (2.47) and the positivity index $\mu\left(s(\cdot,\beta)\right)$ can be easily checked, and using the remaining free parameters to satisfy the condition $\eta^*(\operatorname{cmp}(V_p)) \leq 0$.

Example 2.10. We show here the construction of a simple PNS by using Theorem 2.14, in the case with $n = 3$ and $m = 3$. Let us choose $x^{\{m\}}$ as

$$\begin{aligned} x^{\{m\}} = (&x_1^3, \sqrt{3}x_1^2x_2, \sqrt{3}x_1^2x_3, \sqrt{3}x_1x_2^2\sqrt{6}x_1x_2x_3, \sqrt{3}x_1x_3^2, x_2^3, \\ &\sqrt{3}x_2^2x_3, \sqrt{3}x_2x_3^2, x_3^3)'. \end{aligned} \tag{2.58}$$

This choice satisfies (1.9). Then, let us choose

$$V_p = \frac{1}{\sqrt{38}} \begin{pmatrix} 2\sqrt{3} & 0 & 0 & -5 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 2\sqrt{3} & 0 & -5 & 0 \\ 0 & 0 & -5 & 0 & 0 & 0 & 0 & 1 & 0 & 2\sqrt{3} \end{pmatrix}'.$$

The number of columns of V_p is $r = 3$. Observe that $V_p' V_p = I_3$. Moreover, by selecting $y = (1, 0, \ldots, 0)'$ in (2.27), we find that $\eta(\mathrm{cmp}(V_p)) = -0.0792$, which allows us to conclude that $\eta^*(\mathrm{cmp}(V_p)) \le 0$, by Theorem 2.7. Thanks to the structure of V_p, it is easy to verify the property (2.47) and to compute the positivity index $\mu(s(\cdot, \beta))$. In fact,

$$V_p' x^{\{m\}} = \sqrt{\frac{3}{38}} (w_1(x), w_2(x), w_3(x))'$$

where

$$\begin{aligned} w_1(x) &= x_1 \left(2x_1^2 - 5x_2^2 + x_3^2\right) \\ w_2(x) &= x_2 \left(x_1^2 + 2x_2^2 - 5x_3^2\right) \\ w_3(x) &= x_3 \left(-5x_1^2 + x_2^2 + 2x_3^2\right). \end{aligned}$$

It is straightforward to see that

$$w_i(x) = 0 \ \forall i = 1, 2, 3 \iff x = 0_3$$

and hence (2.47) holds. Therefore, $V_p \in \Theta^P_{3,6}(3)$ and

$$\theta = \langle \delta, \beta, V_p \rangle \in \Theta_{3,6} \quad \forall \delta \in (0, 1],\ \forall \beta \in \mathbb{R}^3, \beta > 0.$$

Moreover, let us select a vector β, for example $\beta = (38/3, 38/3, 38/3)'$. It follows

$$s(x; \beta) = \sum_{i=1}^{3} w_i(x)^2. \tag{2.59}$$

In order to compute $\mu(s(\cdot, \beta))$, we have to find the minimum of $s(x; \beta)$ over the set $\mathscr{C}_{n,m}$, which coincides with $\{x : \ \|x\| = 1\}$ due to the choice (2.58). Let us observe that, since $s(x; \beta)$ depends on x_1^2, x_2^2, x_3^2, one can first substitute $x_3^2 = 1 - x_1^2 - x_2^2$ in $s(x; \beta)$, and then find the minimum by computing the points where the derivatives of $s(x; \beta)$ with respect to x_1^2 and x_2^2 vanish. This operation amounts to solving a system of two quadratic equations in two variables, and can be done by finding the roots of a polynomial equation of degree four in one variable. We find

$$\mu(s(\cdot, \beta)) = 0.4360.$$

Let us define

$$\begin{aligned} h_0(x) &= \sum_{i=1}^{3} w_i(x)^2 \\ &= 4(x_1^6 + x_2^6 + x_3^6) - 19(x_1^4 x_2^2 + x_2^4 x_3^2 + x_3^4 x_1^2) + 29(x_1^4 x_3^2 + x_2^4 x_1^2 + x_3^4 x_2^2) \\ &\quad -30 x_1^2 x_2^2 x_3^2. \end{aligned}$$

Therefore, from Theorem 2.14 it follows that the form

$$h(x) = h_0(x) - 0.4360\|x\|^6\delta \tag{2.60}$$

is a PNS form for all $\delta \in (0,1]$. Figure 2.1 shows the plot of $h(x)$ on the upper semi-sphere for $\delta = 0.5$.

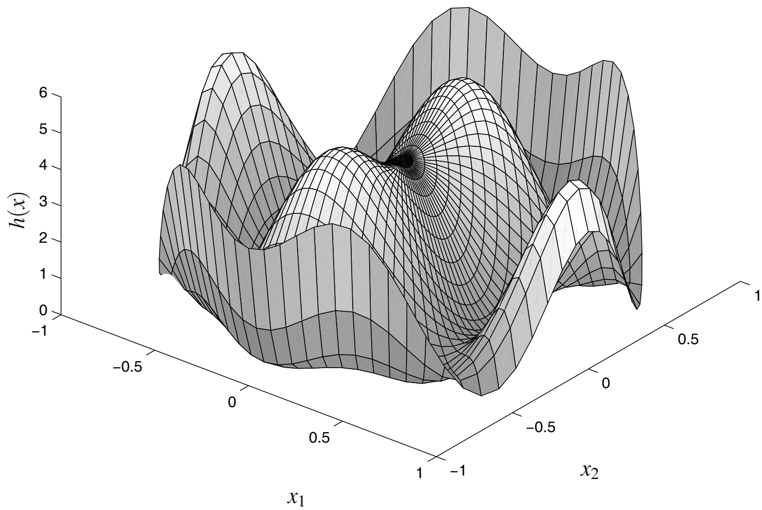

Fig. 2.1 Example 2.10: form $h(x)$ in (2.60) plotted with $\delta = 0.5$ for x such that $x_1^2 + x_2^2 \leq 1$ and $x_3 = \sqrt{1 - x_1^2 - x_2^2}$

Example 2.11. Let us consider the PNS form in (2.60), and let $\gamma \in \mathbb{R}^3$. From (2.59) we have that

$$h(x) + s(x;\gamma) = \sum_{i=1}^{3}(1+\gamma_i)w_i(x)^2 - 0.4360\|x\|^6\delta.$$

This implies that the cone (2.49) is given by

$$\text{cone}(h) = \left\{ h_1 \in \Xi_{3,6} : \; h_1(x) = \sum_{i=1}^{3}(1+\gamma_i)w_i(x)^2 - 0.4360\|x\|^6\delta, \; \gamma \geq 0 \right\}.$$

According to Theorem 2.13, such a cone contains only PNS forms.

2.5 Notes and References

There is an endless literature on Hilbert's 17th problem and related issues. The interested reader is referred to the classical book [70], and to more recent contributions such as [118, 122, 123] and references therein.

Theorem 2.4 was given in [34]. Maximal SMR matrices and related results in Section 2.2, as well as the characterization of PNS forms in Section 2.4, have been provided in [25]. SMR-tight forms have been introduced in [27].

The study of the gap between positive forms and SOS forms is a classical problem which has recently attracted much interest, see e.g. [9, 86, 25].