Chapter 9
Way-point Tracking Control of Underactuated Ships

This chapter presents state feedback and output feedback controllers that force underactuated ships to globally ultimately track a straight line under environmental disturbances induced by waves, wind, and ocean currents. When there are no environmental disturbances, the controllers are able to drive the heading angle and cross-tracking error to zero asymptotically. Based on the backstepping technique and several technical lemmas introduced for a nonlinear system with nonvanishing disturbances, a full state feedback controller is first designed. An output feedback controller is then developed by using a nonlinear observer, which globally exponentially estimates the unmeasured sway and yaw velocities from the measured sway displacement and the measured yaw angle.

9.1 Control Objective

In addition to the assumptions made in Section 3.4.1.1, we assume that the surge velocity is controlled by the main propulsion control system. As such, the resulting mathematical model of the underactuated ship moving in sway and yaw is rewritten as

$$
\begin{aligned}
\dot{y} &= u\sin(\psi) + \cos(\psi)v, \\
\dot{\psi} &= r, \\
\dot{v} &= -\frac{m_{11}u}{m_{22}}r - \frac{d_{22}}{m_{22}}v - \sum_{i\geq 2}\frac{d_{vi}}{m_{22}}|v|^{i-1}v + \frac{1}{m_{22}}\tau_{wv}(t), \\
\dot{r} &= \frac{(m_{11}-m_{22})u}{m_{33}}v - \frac{d_{33}}{m_{33}}r - \sum_{i\geq 2}\frac{d_{ri}}{m_{33}}|r|^{i-1}r + \frac{1}{m_{33}}\tau_r + \frac{1}{m_{33}}\tau_{wr}(t),
\end{aligned}
\tag{9.1}
$$

where y, v, ψ, r, and u are sway displacement, sway velocity, yaw angle, yaw velocity, and forward speed controlled by the main thruster control system, respectively. Without loss of generality, we assume that the forward speed u is positive and if time-varying, has a bounded derivative $\dot{u}(t)$, i.e., $0 < u_{\min} \leq u(t) \leq u_{\max} < \infty$ and $|\dot{u}(t)| \leq M < \infty$, $\forall t \geq 0$. The positive constant terms m_{jj}, $1 \leq j \leq 3$ denote the ship's inertia including added mass. The positive constant terms d_{22}, d_{33}, d_{vi} and d_{ri}, $i \geq 2$ represent the hydrodynamic damping in sway and yaw. The bounded time-varying terms, $\tau_{wv}(t)$ and $\tau_{wr}(t)$, are the environmental disturbance moments induced by wave, wind, and ocean current with an assumption that $|\tau_{wv}(t)| \leq \tau_{wv\max} < \infty$ and $|\tau_{wr}(t)| \leq \tau_{wr\max} < \infty$. In this chapter, we study two control objectives. The first is full state feedback. In this case, we assume that all states y, v, ψ, and r are available for feedback. In the design of an output feedback controller, only sway and yaw displacements are measurable. For both full state and output feedback cases, we design a control law, τ_r, that forces the ship to track a linear course with ultimate boundedness, i.e. the tracking errors are globally ultimately bounded. When there are no environmental disturbances, the sway displacement and velocity, y and v, yaw angle and velocity, ψ and r, asymptotically converge to zero.

9.2 Full-state Feedback

9.2.1 Control Design

We define the following coordinate transformation

$$z_1 = \psi + \arcsin\left(\frac{ky}{\sqrt{1+(ky)^2}}\right), \tag{9.2}$$

where k is a positive constant to be selected later. Note that the convergence of z_1 and y to zero implies that of ψ. Upon application of the coordinate transformation (9.2), the ship dynamics (9.1) are rewritten as

$$\begin{aligned}
\dot{y} &= -\frac{kuy}{\sqrt{1+(ky)^2}} + \frac{v}{\sqrt{1+(ky)^2}} + \frac{u\left(\sin(z_1)-(\cos(z_1)-1)ky\right)}{\sqrt{1+(ky)^2}} + \\
&\quad \frac{v\left((\cos(z_1)-1)+ky\sin(z_1)\right)}{\sqrt{1+(ky)^2}}, \\
\dot{v} &= -\frac{m_{11}u}{m_{22}}r - \frac{d_{22}}{m_{22}}v - \sum_{i\geq 2}\frac{d_{vi}}{m_{22}}|v|^{i-1}v + \frac{1}{m_{22}}\tau_{wv}(t), \\
\dot{z}_1 &= r - \frac{k^2uy}{(1+(ky)^2)^{3/2}} + \frac{kv}{(1+(ky)^2)^{3/2}} + \frac{ku\left(\sin(z_1)-(\cos(z_1)-1)ky\right)}{(1+(ky)^2)^{3/2}} + \\
&\quad \frac{kv\left((\cos(z_1)-1)+ky\sin(z_1)\right)}{(1+(ky)^2)^{3/2}}, \\
\dot{r} &= \frac{(m_{11}-m_{22})u}{m_{33}}v - \frac{d_{33}}{m_{33}}r - \sum_{i\geq 2}\frac{d_{ri}}{m_{33}}|r|^{i-1}r + \frac{1}{m_{33}}\tau_r + \frac{1}{m_{33}}\tau_{wr}(t).
\end{aligned} \tag{9.3}$$

Therefore the problem of stabilizing (9.1) at the origin becomes that of stabilizing (9.3) at the origin. The structure of the model (9.3) suggests that we design the control τ_r in two stages by applying the popular backstepping technique. At the first step, we design an intermediate control r_d for r and at the second step the actual control τ_r will be designed to eliminate the error between r_d and r.

Step 1

Define

$$z_2 = r - r_d, \tag{9.4}$$

where r_d is an intermediate control designed as

$$\begin{aligned} r_d = &-k_1 z_1 + \frac{k^2 u y}{(1+(ky)^2)^{3/2}} - \frac{kv}{(1+(ky)^2)^{3/2}} - \\ &\frac{ku\,(\sin(z_1)-(\cos(z_1)-1)ky)}{(1+(ky)^2)^{3/2}} - \frac{kv\,((\cos(z_1)-1)+ky\sin(z_1))}{(1+(ky)^2)^{3/2}}, \end{aligned} \tag{9.5}$$

where k_1 is a positive constant to be selected later.

Step 2

With (9.5), the time derivative of (9.4) along the solutions of the last equation of (9.3) is

$$\begin{aligned} \dot{z}_2 = &\frac{(m_{11}-m_{22})u}{m_{33}} v - \frac{d_{33}}{m_{33}} r - \sum_{i\geq 2} \frac{d_{ri}}{m_{33}} |r|^{i-1} r + \frac{1}{m_{33}} \tau_r + \frac{1}{m_{33}} \tau_{wr}(t) - \\ &\frac{\partial r_d}{\partial u} \dot{u} - \frac{\partial r_d}{\partial z_1}(-k_1 z_1 + z_2) - \frac{\partial r_d}{\partial y}\Bigg(-\frac{kuy}{\sqrt{1+(ky)^2}} + \frac{v}{\sqrt{1+(ky)^2}} + \\ &\frac{u\,(\sin(z_1)-(\cos(z_1)-1)ky)}{\sqrt{1+(ky)^2}} + \frac{v\,((\cos(z_1)-1)+ky\sin(z_1))}{\sqrt{1+(ky)^2}} \Bigg) - \\ &\frac{\partial r_d}{\partial v}\left(-\frac{m_{11}u}{m_{22}} r - \frac{d_{22}}{m_{22}} v - \sum_{i\geq 2} \frac{d_{vi}}{m_{22}} |v|^{i-1} v + \frac{1}{m_{22}} \tau_{wv}(t) \right), \end{aligned} \tag{9.6}$$

where

$$\begin{aligned} \frac{\partial r_d}{\partial u} &= \frac{k^2 y}{(1+(ky)^2)^{3/2}} - \frac{k}{(1+(ky)^2)^{3/2}}(\sin(z_1)-(\cos(z_1)-1)ky), \\ \frac{\partial r_d}{\partial z_1} &= -k_1 - \frac{ku\,(\cos(z_1)+ky\sin(z_1))}{(1+(ky)^2)^{3/2}} - \frac{kv\,(-\sin(z_1)+ky\cos(z_1))}{(1+(ky)^2)^{3/2}}, \end{aligned}$$

$$\frac{\partial r_d}{\partial y} = \frac{-3k^3 y\,(kuy - u\,(\sin(z_1) - y(\cos(z_1)-1)))}{(1+(ky)^2)^{5/2}} -$$
$$\frac{v\,((\cos(z_1)-1)+y\sin(z_1))}{(1+(ky)^2)^{5/2}} + \frac{k^2\,(u\cos(z_1) - v\sin(z_1))}{(1+(ky)^2)^{3/2}},$$
$$\frac{\partial r_d}{\partial v} = -\frac{k}{(1+(ky)^2)^{3/2}}\,(\cos(z_1)+\sin(z_1)ky)\,. \tag{9.7}$$

We now choose the actual control without canceling the useful damping terms as

$$\tau_r = m_{33}\Bigg[-z_1 - k_2 z_2 - \frac{(m_{11}-m_{22})u}{m_{33}}v + \frac{d_{33}}{m_{33}}r_d + \sum_{i\geq 2}\frac{d_{ri}}{m_{33}}|r|^{i-1}r_d +$$
$$\frac{\partial r_d}{\partial u}\dot{u} + \frac{\partial r_d}{\partial z_1}(-k_1 z_1 + z_2) + \frac{\partial r_d}{\partial y}\left(-\frac{kuy}{\sqrt{1+(ky)^2}} + \frac{v}{\sqrt{1+(ky)^2}} +\right.$$
$$\left.\frac{u\,(\sin(z_1)-(\cos(z_1)-1)ky)}{\sqrt{1+(ky)^2}} + \frac{v\,((\cos(z_1)-1)+ky\sin(z_1))}{\sqrt{1+(ky)^2}}\right) +$$
$$\frac{\partial r_d}{\partial v}\left(-\frac{m_{11}u}{m_{22}}r - \frac{d_{22}}{m_{22}}v - \sum_{i\geq 2}\frac{d_{vi}}{m_{22}}|v|^{i-1}v\right) - \tag{9.8}$$
$$\frac{1}{m_{33}}\tau_{wr\max}\tanh\left(\frac{z_2}{\rho_1}\right) - \frac{1}{m_{22}}\tau_{wv\max}\frac{\partial r_d}{\partial v}\tanh\left(\frac{\partial r_d}{\partial v}\frac{z_2}{\rho_2}\right)\Bigg],$$

where k_2, ρ_1, and ρ_2 are positive constants to be chosen later. Substituting (9.4), (9.5), and (9.8) into (9.3) results in the following closed loop system:

$$\dot{y} = -\frac{kuy}{\sqrt{1+(ky)^2}} + \frac{v}{\sqrt{1+(ky)^2}} + \frac{u\,(\sin(z_1)-(\cos(z_1)-1)ky)}{\sqrt{1+(ky)^2}} +$$
$$\frac{v}{\sqrt{1+(ky)^2}}\,((\cos(z_1)-1)+ky\sin(z_1))\,,$$
$$\dot{v} = -\frac{d_{22}}{m_{22}}v - \sum_{i\geq 2}\frac{d_{vi}}{m_{22}}|v|^{i-1}v - \frac{m_{11}u}{m_{22}}\frac{k^2uy - kv}{(1+(ky)^2)^{3/2}} -$$
$$\frac{m_{11}u}{m_{22}}\left(-k_1 z_1 + z_2 - \frac{ku}{(1+(ky)^2)^{3/2}}\,(\sin(z_1)-(\cos(z_1)-1)ky) -\right.$$
$$\left.\frac{kv}{(1+(ky)^2)^{3/2}}\,((\cos(z_1)-1)+ky\sin(z_1))\right) + \frac{1}{m_{22}}\tau_{wv}(t),$$
$$\dot{z}_1 = -k_1 z_1 + z_2,$$
$$\dot{z}_2 = -z_1 - k_2 z_2 - \frac{d_{33}}{m_{33}}z_2 - \sum_{i\geq 2}\frac{d_{ri}}{m_{33}}|r|^{i-1}z_2 + \frac{1}{m_{33}}\,(\tau_{wr}(t) - \tau_{wr\max}\times$$
$$\tanh\left(\frac{z_2}{\rho_1}\right)\Bigg) + \frac{1}{m_{22}}\left(-\frac{\partial r_d}{\partial v}\tau_{wv}(t) - \frac{\partial r_d}{\partial v}\tau_{wv\max}\tanh\left(\frac{\partial r_d}{\partial v}\frac{z_2}{\rho_2}\right)\right). \tag{9.9}$$

9.2.2 Stability Analysis

The following two lemmas will be used extensively in stability analysis.

Lemma 9.1. *Consider the following nonlinear system:*

$$\dot{x} = f(t,x) + g(t,x,\xi(t)), \tag{9.10}$$

where $x \in \mathbb{R}^n$, $\xi(t) \in \mathbb{R}^m$, $f(t,x)$ *is piecewise continuous in* t *and locally Lipschitz in* x. *If there exist positive constants* c_i, $1 \le i \le 4$, λ_j, $1 \le j \le 2$, σ_0, ε_0, μ_0, c_0, *and a class-K function* α_0 *such that the following conditions are satisfied:*

C1. There exists a proper function $V(t,x)$ *satisfying:*

$$c_1 \|x\|^2 \le V(t,x) \le c_2 \|x\|^2,$$
$$\left\| \frac{\partial V}{\partial x}(t,x) \right\| \le c_3 \|x\|,$$
$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \le -c_4 \|x\|^2 + c_0.$$

C2. The vector function $g(t,x,\xi(t))$ *satisfies:*

$$\|g(t,x,\xi(t))\| \le (\lambda_1 + \lambda_2 \|x\|) \|\xi(t)\|.$$

C3. $\xi(t)$ *globally exponentially converges to a ball centered at the origin:*

$$\|\xi(t)\| \le \alpha_0 (\|\xi(t_0)\|) e^{-\sigma_0(t-t_0)} + \varepsilon_0, \ \forall t \ge t_0 \ge 0.$$

C4. The following gain condition is satisfied:

$$c_4 - \lambda_2 c_3 \varepsilon_0 - \frac{\lambda_1 c_3 \varepsilon_0}{4\mu_0} > 0.$$

Then the solution $x(t)$ *of (9.10) globally exponentially converges to a ball centered at the origin, i.e.,*

$$\|x(t)\| \le \alpha (\|(x(t_0), \xi(t_0))\|) e^{-\sigma(t-t_0)} + \varepsilon, \ \forall t \ge t_0 \ge 0, \tag{9.11}$$

where $\varepsilon = \sqrt{a_4/c_1 a_1}$ *and*

if $a_1 = \sigma_0$ *then*

$$\alpha(s) = \sqrt{\frac{e^{\frac{a_2(s)}{\sigma_0}}}{c_1} \left(c_2 s^2 + \left(a_3(s) + a_1^{-1} a_2(s) a_4\right) \phi\right)}$$
$$\sigma = 0.5(a_1 - d);$$

$$\text{if } a_1 \neq \sigma_0 \text{ then}$$
$$\alpha(s) = \sqrt{\frac{e^{\frac{a_2(s)}{\sigma_0}}}{c_1}\left(c_2 s^2 + \frac{a_1 a_3(s) + a_2(s) a_4}{a_1 |a_1 - \sigma_0|}\right)}$$
$$\sigma = 0.5 \min(a_1, |a_1 - \sigma_0|);$$

with

$$\begin{aligned}
a_1 &= \frac{1}{c_2}\left(c_4 - \lambda_2 c_3 \varepsilon_0 - \frac{\lambda_1 c_3 \varepsilon_0}{4\mu_0}\right),\\
a_2(s) &= \frac{c_3}{c_1}(\lambda_1 + \lambda_2)\alpha_0(s),\\
a_3(s) &= \frac{\lambda_1 c_3}{4}\alpha_0(s),\\
a_4 &= c_0 + \lambda_1 c_3 \varepsilon_0 \mu_0,\\
0 < d &< a_1, \quad \phi \geq (t - t_0)e^{-d(t-t_0)}, \quad \forall t \geq t_0 \geq 0, s \geq 0.
\end{aligned}$$

When $c_0 = 0$ and $\varepsilon_0 = 0$, we have $\varepsilon = 0$ and the system (9.10) is globally K-exponentially stable. Note that a finite value of the constant ϕ exists for an arbitrarily small positive d.

Proof. From conditions C1, C2, and C3, we have

$$\begin{aligned}
\dot{V} &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) + \frac{\partial V}{\partial x} g(t,x,\xi(t))\\
&\leq -(c_4 - \lambda_2 c_3 \varepsilon_0 - \lambda_1 c_3 \varepsilon_0 / 4\mu_0)\|x\|^2 +\\
&\quad c_3 \|x\| (\lambda_1 + \lambda_2 \|x\|)\alpha_0(\|\xi(t_0)\|) e^{-\sigma_0(t-t_0)} +\\
&\quad \lambda_1 c_3 \varepsilon_0 \mu_0 + c_0.
\end{aligned} \tag{9.12}$$

Upon application of the completing square, (9.12) can be rewritten as

$$\dot{V} \leq -\left(a_1 - a_2 e^{-\sigma_0(t-t_0)}\right) V + a_3 e^{-\sigma_0(t-t_0)} + a_4. \tag{9.13}$$

Solving the above differential inequality results in

$$V(t) \leq V(t_0) e^{\frac{a_2}{\sigma_0}} e^{-a_1(t-t_0)} + \left(a_3 + \frac{a_2 a_4}{a_1}\right) e^{\frac{a_2}{\sigma_0}} e^{-a_1 t + \sigma_0 t_0} \int_{t_0}^{t} e^{(a_1 - \sigma_0)\tau} d\tau + \frac{a_4}{a_1}, \tag{9.14}$$

which yields (9.11) readily. □

Lemma 9.2. *Consider the following nonlinear system:*

$$\dot{x} = f(t,x) + g(t,x,\xi(t)), \tag{9.15}$$

where $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$, $x = [x_1\ x_2]^T \in \mathbb{R}^{n_1+n_2}$, $\xi(t) \in \mathbb{R}^m$, $f(t,x)$ *is piecewise continuous in* t *and locally Lipschitz in* x. *If there exist positive constants* c_0, c_1, c_2, c_{31}, c_{32}, λ_i, $0 \le i \le 2$, σ_0, ε_0, c_0, *and a class-K function* α_0 *such that the following conditions are satisfied.*

C1. There exists a proper function $V(t,x)$ *such that:*

$$c_1 \|x\|^2 \le V(t,x) \le c_2 \|x\|^2,$$
$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \le -c_{31}\|x_1\|^2 - \frac{c_{32}\|x_2\|^2}{\sqrt{1+c_4\|x_2\|^2}} + c_0,$$
$$\left\| \frac{\partial V}{\partial x} g(t,x,\xi(t)) \right\| \le \left(\lambda_0 + \lambda_1 \|x_1\|^2 + \frac{\lambda_2 \|x_2\|^2}{\sqrt{1+c_4\|x_2\|^2}} \right) \|\xi(t)\|.$$

C2. $\xi(t)$ *globally exponentially converges to a ball centered at the origin:*

$$\|\xi(t)\| \le \alpha_0(\|\xi(t_0)\|)\, e^{-\sigma_0(t-t_0)} + \varepsilon_0, \ \forall t \ge t_0 \ge 0.$$

C3. The following gain conditions are satisfied:

$$c_{31} - \lambda_1 \varepsilon_0 > 0 \quad \text{and} \quad c_{32} - \lambda_2 \varepsilon_0 > 0.$$

C4. $x_2(t)$ *is bounded:*

$$\|x_2(t)\| \le \varpi,$$

where ϖ *is a nondecreasing function of* $\|(x(t_0), \xi(t_0))\|$,

then the solution $x(t)$ *of (9.15) globally asymptotically converges to a ball centered at the origin, i.e.,*

$$\|x(t)\| \le \alpha(\|(x(t_0),\xi(t_0))\|)\, e^{-\sigma(\|(x(t_0),\xi(t_0))\|)(t-t_0)} + \varepsilon(s), \ \forall t \ge t_0 \ge 0, \quad (9.16)$$

where $\varepsilon(s) = \sqrt{\frac{a_4}{c_1 a_1(s)}}$ *and*

if $a_1(s) = \sigma_0$ *then*

$$\alpha(s) = \sqrt{c_1^{-1} e^{a_2(s)/\sigma_0} \left(c_2 s^2 + \left(a_3(s) + a_1^{-1}(s) a_2(s) a_4 \right) \phi \right)}$$
$$\sigma(s) = 0.5(a_1(s) - d);$$

if $a_1(s) \ne \sigma_0$ *then*

$$\alpha(s) = \sqrt{c_1^{-1} e^{a_2(s)/\sigma_0} \left(c_2 s^2 + \frac{a_1(s)a_3(s) + a_2(s)a_4}{a_1(s)\,|a_1(s) - \sigma_0|} \right)}$$
$$\sigma(s) = 0.5 \min(a_1(s), |a_1(s) - \sigma_0|);$$

with

$$
\begin{aligned}
a_1(s) &= \frac{1}{c_2}\min\left(c_{31}-\lambda_1\varepsilon_0, \frac{c_{32}-\lambda_2\varepsilon_0}{\sqrt{1+c_4\varpi^2(s)}}\right),\\
a_2(s) &= \frac{1}{c_1}\max(\lambda_1,\lambda_2)\alpha_0(s),\\
a_3(s) &= \lambda_0\alpha_0(s),\\
a_4 &= c_0+\lambda_0\varepsilon_0,\\
0 &< d < a_1(s),\ \phi \geq (t-t_0)e^{-d(t-t_0)}, \quad \forall t \geq t_0 \geq 0, s \geq 0.
\end{aligned}
$$

When $c_0 = 0$ and $\varepsilon_0 = 0$, we have $\varepsilon = 0$ and the system (9.15) is GAS. Note that a finite value of the constant ϕ exists for an arbitrarily small positive d.

Proof. The proof of this lemma is similar to that of Lemma 9.1. □

Remark 9.1. It is important to note that the rate $\sigma > 0$ in (9.16) and a_1 depend on the initial conditions. In addition, around the origin, both σ and a_1 are bounded below from zero.

We first need to show that the closed loop system (9.9) is forward complete. It is straightforward to show that the derivative of the function $V_0 = z_1^2 + z_2^2 + v^2 + y^2$ along the solutions of the closed loop system (9.9) satisfies $\dot{V}_0 \leq a_0 V_0 + b_0$ where a_0 and b_0 are nonnegative constants. The inequality $\dot{V}_0 \leq a_0 V_0 + b_0$ implies that the closed loop system (9.9) is forward complete. We now apply Lemmas 9.1 and 9.2 to analyze the closed loop system (9.9). We view (z_1, z_2) as $\xi(t)$, v as x in Lemma 9.1, and (v, y) as x in Lemma 9.2. Hence it is necessary to verify all the conditions of Lemmas 9.1 and 9.2.

(z_1, z_2)-subsystem

We take the following quadratic function:

$$
V_1 = \frac{1}{2}(z_1^2 + z_2^2), \tag{9.17}
$$

whose time derivative along the solutions of the last two equations of (9.9) satisfies

$$
\begin{aligned}
\dot{V}_1 = -k_1 z_1^2 - k_2 z_2^2 - \frac{d_{33}}{m_{33}} z_2^2 - \sum_{i\geq 2} \frac{d_{ri}}{m_{33}} |r|^{i-1} z_2^2 + \frac{z_2}{m_{33}} (\tau_{wr}(t) - \tau_{wr\max} \times \\
\tanh\left(\frac{z_2}{\rho_1}\right)\right) + \frac{z_2}{m_{22}}\left(-\frac{\partial r_d}{\partial v}\tau_{wv}(t) - \frac{\partial r_d}{\partial v}\tau_{wv\max}\tanh\left(\frac{\partial r_d}{\partial v}\frac{z_2}{\rho_2}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&\leq -k_1 z_1^2 - k_2 z_2^2 + \frac{\tau_{wr\max}}{m_{33}} \times \\
&\quad \left(|z_2| - z_2 \tanh\left(\frac{z_2}{\rho_1}\right)\right) + \frac{\tau_{wv\max}}{m_{22}}\left(\left|\frac{\partial r_d}{\partial v} z_2\right| - \frac{\partial r_d}{\partial v} z_2 \tanh\left(\frac{\partial r_d}{\partial v}\frac{z_2}{\rho_2}\right)\right) \\
&\leq -k_1 z_1^2 - k_2 z_2^2 + 0.2785\left(\frac{1}{m_{33}}\tau_{wr\max}\rho_1 + \frac{1}{m_{22}}\tau_{wv\max}\rho_2\right),
\end{aligned} \tag{9.18}
$$

where we have used $|x| - x\tanh(x/\lambda) \leq 0.2785\lambda,\ \forall x \in \mathbb{R}$ and $\lambda > 0$. From (9.17) and (9.18), it can be shown that

$$
\|z(t)\| \leq \|z(t_0)\| e^{-\sigma_0(t-t_0)} + \varepsilon_0 \forall t \geq t_0 \geq 0\,, \tag{9.19}
$$

where $z = [z_1\ z_2]^T$ and

$$
\begin{aligned}
\sigma_0 &= \min(k_1, k_2), \\
\varepsilon_0 &= \sqrt{\frac{0.2785\,(\tau_{wr\max}\rho_1/m_{33} + \tau_{wv\max}\rho_2/m_{22})}{\sigma_0}}.
\end{aligned} \tag{9.20}
$$

Therefore the (z_1, z_2)-subsystem is globally ultimately stable at the origin. Furthermore, (9.19) implies that $\xi(t) := (z_1, z_2)^T$ globally exponentially converges to a ball centered at the origin. The radius of this ball can be made arbitrarily small by increasing k_1 and k_2 and/or reducing ρ_1 and ρ_2.

Boundedness of v

To prove that v is bounded, we consider the second equation of (9.9). In order to apply Lemma 9.1, define $x = v$, $\xi(t) = [z_1 z_2]^T$ and consider y as a function of time t,

$$
\begin{aligned}
f(\bullet) &= -\frac{d_{22}}{m_{22}} v - \sum_{i \geq 2} \frac{d_{vi}}{m_{22}} |v|^{i-1} v - \\
&\quad \frac{m_{11} u}{m_{22}}\left(\frac{k^2 u y}{(1+(ky)^2)^{3/2}} - \frac{kv}{(1+(ky)^2)^{3/2}}\right) + \frac{1}{m_{22}}\tau_{wv}(t), \\
g(\bullet) &= -\frac{m_{11} u}{m_{22}}(-k_1 z_1 + z_2 - \\
&\quad \left.\frac{k\,(u\,(\sin(z_1) - (\cos(z_1) - 1)ky) + v\,((\cos(z_1) - 1) + ky\sin(z_1)))}{(1+(ky)^2)^{3/2}}\right).
\end{aligned} \tag{9.21}
$$

This abuse of notation is introduced for simplicity and is possible because:

$$
0 \leq \frac{1}{(1+(ky)^2)^{3/2}} \leq 1,\ 0 \leq \left|\frac{ky}{(1+(ky)^2)^{3/2}}\right| < 1,\ \forall\ y \in \mathbb{R},
$$

and we have shown that the closed loop system is forward complete. We now verify all of the conditions of Lemma 9.1.

Verifying Condition C1. We take the function $V_2 = 0.5v^2$ whose time derivative along the solutions of the differential equation $\dot{v} = f(t,v)$, see (9.21), satisfies

$$\begin{aligned}\dot{V}_2 &= -\frac{d_{22}}{m_{22}}v^2 - \sum_{i\geq 2}\frac{d_{vi}}{m_{22}}|v|^{i-1}v^2 - \frac{m_{11}uv}{m_{22}}\frac{k^2uy-kv}{(1+(ky)^2)^{3/2}} + \frac{v}{m_{22}}\tau_{wv}(t)\\ &\leq -\left(\frac{d_{22}}{m_{22}} - \frac{m_{11}ku_{\max}}{m_{22}} - \frac{m_{11}ku_{\max}^2\mu_1}{m_{22}} - \frac{\mu_1}{m_{22}}\right)v^2 + \frac{m_{11}ku_{\max}^2 + \tau_{wv\max}^2}{4\mu_1 m_{22}}.\end{aligned} \tag{9.22}$$

Hence, the condition C1 is satisfied with

$$\begin{aligned}c_0 &= \frac{1}{4\mu_1 m_{22}}\left(m_{11}ku_{\max}^2 + \tau_{wv\max}^2\right),\ c_1 = c_2 = 0.5,\ c_3 = 1,\\ c_4 &= \frac{d_{22}}{m_{22}} - \frac{m_{11}ku_{\max}^2}{m_{22}}\mu_1 - \frac{m_{11}ku_{\max}}{m_{22}} - \frac{\mu_1}{m_{22}},\end{aligned} \tag{9.23}$$

where $\mu_1 > 0$ and $k > 0$ are chosen such that $c_4 > 0$.

Verifying Condition C2. It is directly shown from (9.21) that

$$|g(t,v,z(t))| \leq (\lambda_1 + \lambda_2|v|)\,\|z(t)\|, \tag{9.24}$$

where

$$\lambda_1 = \frac{m_{11}u_{\max}}{m_{22}}(1 + k_1 + 2ku_{\max}),\ \lambda_2 = \frac{2km_{11}u_{\max}}{m_{22}}. \tag{9.25}$$

Verifying Condition C3. This condition follows directly from (9.19).

Verifying Condition C4. It can be shown from (9.20), (9.23), and (9.25) that we can find positive constant k such that the condition C4 is satisfied, i.e.,

$$c_4 - \lambda_2 c_3\varepsilon_0 - \frac{\lambda_1 c_3\varepsilon_0}{4\mu_0} > 0. \tag{9.26}$$

All of the conditions of Lemma 9.1 have been verified, hence the sway velocity is bounded and satisfies

$$|v(t)| \leq \alpha_1(\|(v(t_0), z(t_0))\|)e^{\sigma_1(t-t_0)} + \varepsilon_1, \tag{9.27}$$

where ε_1, σ_1, and α_1 are calculated as in Lemma 9.1, and the constants c_i, $1 \leq i \leq 4$, λ_j, $1 \leq j \leq 2$, σ_0, ε_0, μ_0, and c_0 are given in (9.20), (9.23), and (9.26).

(v, y)-subsystem

In this section, we will apply Lemma 9.2 to prove global ultimate boundedness of the (v, y)-subsystem. It can be seen that the first two equations of (9.9) are in the form of the system in Lemma 9.2 with $x_1 = v$, $x_2 = y$, $\xi(t) = z(t)$, and

$$f(\bullet) = \begin{bmatrix} -\frac{d_{22}}{m_{22}}v - \sum_{i\geq 2}\frac{d_{vi}}{m_{22}}|v|^{i-1}v - \frac{m_{11}u}{m_{22}}\frac{k^2uy - kv}{(1+(ky)^2)^{3/2}} + \frac{1}{m_{22}}\tau_{wv}(t) \\ -\frac{kuy}{\sqrt{1+(ky)^2}} + \frac{v}{\sqrt{1+(ky)^2}} \end{bmatrix},$$

$$g(\bullet) = \begin{bmatrix} \frac{m_{11}u}{m_{22}}\left(k\frac{\sin(z_1)(u+kvy) + (\cos(z_1)-1)(v-kuy)}{(1+(ky)^2)^{3/2}} + k_1z_1 - z_2\right) \\ \frac{u\left(\sin(z_1) - (\cos(z_1)-1)ky\right)}{\sqrt{1+(ky)^2}} + \frac{v\left((\cos(z_1)-1) + \sin(z_1)ky\right)}{\sqrt{1+(ky)^2}} \end{bmatrix}. \tag{9.28}$$

We now need to verify all of the conditions of Lemma 9.2.

Verifying Condition C1. To verify this condition, we take the function $V_3 = 0.5(v^2 + y^2)$. It can be directly shown that this function satisfies condition C1 with $|v(t)| \leq \alpha_1\left(\|(v(t_0), z(t_0))\|\right)e^{-\sigma_1(t-t_0)} + \varepsilon_1$ and

$$\begin{aligned} c_0 &= \frac{\tau^2_{wv\max}}{4\mu_3 m_{22}}, c_1 = c_2 = 0.5, \\ c_{31} &= \frac{d_{22}}{m_{22}} - \frac{m_{11}ku_{\max}}{m_{22}} - \mu_2 - \frac{m_{11}k^2u^2_{\max}}{m_{22}}\mu_2 - \frac{\mu_3}{m_{22}}, \\ c_{32} &= ku_{\min} - \frac{1}{4\mu_2}\left(1 + \frac{m_{11}k^2u^2_{\max}}{m_{22}}\right), \\ \lambda_0 &= \frac{m_{11}u_{\max}}{4\mu_4 m_{22}}(1 + k_1 + 2ku_{\max}) + \frac{u_{\max}}{4\mu_4}, \\ \lambda_1 &= \frac{m_{11}u_{\max}}{m_{22}}(2ku_{\max} + k_1 + 1)\mu_4 + \frac{2km_{11}u_{\max}}{m_{22}} + \mu_4, \\ \lambda_2 &= \frac{1}{4\mu_4} + u_{\max}\mu_4 + ku_{\max} + k(\alpha_1 + \varepsilon_1), \end{aligned} \tag{9.29}$$

where $k > 0$ and $\mu_2 > 0$ are chosen such that $c_{31} > 0$ and $c_{32} > 0$.

Verifying Condition C2. This condition follows directly from (9.19).

Verifying Condition C3. It can be shown that there exists a positive constant k such that the condition C3 satisfies, i.e.,

$$c_{31} - \lambda_1 \varepsilon_0 > 0,\ c_{32} - \lambda_2 \varepsilon_0 > 0. \tag{9.30}$$

Verifying Condition C4. From the boundedness of the sway velocity $v(t)$ proven in the previous subsection and noting that ε_0 in (9.19) can be made arbitrarily small, it is shown that there exists a nondecreasing function ϖ of $\|((v(t_0), y(t_0)), z(t_0))\|$ such that $|y(t)| \leq \varpi$ by applying Lemma 9.1 to the first equation of (9.9) with the Lyapunov function $V_y = 0.5y^2$.

All of the conditions of Lemma 9.2 have been verified. Therefore we have

$$\|(v(t), y(t))\| \leq \alpha_2 (\|((v(t_0), y(t_0)), z(t_0))\|) e^{-\sigma_2 (t - t_0)} + \varepsilon_2,\ \ \forall t \geq t_0 \geq 0, \tag{9.31}$$

where ε_2, σ_2, and α_2 are calculated as in Lemma 9.2, and all other constants given in (9.29). It can be seen that when there are no environmental disturbances, since $\varepsilon_2 = 0$, $(v(t), y(t))$ globally asymptotically converges to zero. We have thus proven the first main result of this chapter.

Theorem 9.1. *The full-state feedback control problem stated in Section 9.1 is solved by the control law (9.8) as long as the design constants k, k_1, and k_2 are chosen such that (9.26) and (9.30) hold.*

9.3 Output Feedback

This section is devoted to the development of an output feedback controller to fulfill the output feedback control objective. A nonlinear observer is first designed so that it globally exponentially drives the observer error dynamics to a ball centered at the origin. When there are no environmental disturbances, the observer error dynamics are GES at the origin. A controller is then designed based on the approach in the preceding section and the proposed observer. Before designing an observer and output feedback controller, we impose the following assumption, see [12].

Assumption 9.1. *For the ship model (9.1), the matrix*

$$\boldsymbol{K}_2 = \begin{bmatrix} -\dfrac{d_{22}}{m_{22}} & -\dfrac{m_{11} u}{m_{22}} \\ \dfrac{(m_{11} - m_{22}) u}{m_{33}} & -\dfrac{d_{33}}{m_{33}} \end{bmatrix} \tag{9.32}$$

is Hurwitz.

The above assumption implies that the ship (when the nonlinear damping terms $\sum\limits_{i \geq 2} \frac{d_{vi}}{m_{33}} |v|^{i-1} v$ and $\sum\limits_{i \geq 2} \frac{d_{ri}}{m_{33}} |r|^{i-1} r$ are ignored) is dynamic stable in straight-line motion. Straight-line stability physically implies that a new path of the ship will be a straight line after an action in yaw. The direction of the new path will usually be different from that of the initial path, as mentioned in [12]. On the other hand, unstable ships will go into a starboard or port turn without any rudder deflection. We impose Assumption 9.1 to make our observer design possible. Note that this assumption does not hold for several types of surface ships such as large tankers and high-speed crafts with sufficiently small ratios d_{22}/m_{22} and d_{33}/m_{33}, and the added mass in the sway axis sufficiently larger than the added mass in the surge axis. Consequently, for these ships the real part of at least one of the eigenvalues of the matrix $\boldsymbol{K_2}$ is positive.

9.3.1 Observer Design

The ship dynamics (9.1) represent some difficulties for output feedback control design. These difficulties are mainly due to the nonlinear terms $\sum_{i\geq 2} \frac{d_{vi}}{m_{22}} |v|^{i-1} v$ and $\sum_{i\geq 2} \frac{d_{ri}}{m_{33}} |r|^{i-1} r$, the nonlinear kinematic term $\cos(\psi)$, and the underactuated situation. However we first observe that the nonlinear terms are monotonic, i.e., they satisfy

$$
\begin{aligned}
&(v_1 - v_2)\left(\sum_{i\geq 2} \frac{d_{vi}}{m_{22}} |v_1|^{i-1} v_1 - \sum_{i\geq 2} \frac{d_{vi}}{m_{22}} |v_2|^{i-1} v_2\right) \geq 0, \forall\ v_1 \in \mathbb{R}, v_2 \in \mathbb{R},\\
&(r_1 - r_2)\left(\sum_{i\geq 2} \frac{d_{ri}}{m_{33}} |r_1|^{i-1} r_1 - \sum_{i\geq 2} \frac{d_{ri}}{m_{33}} |r_2|^{i-1} r_2\right) \geq 0, \forall r_1 \in \mathbb{R}, r_2 \in \mathbb{R}.
\end{aligned}
\tag{9.33}
$$

Based on the structure of the underatuated ship dynamics (9.1) and property (9.33), we propose the following nonlinear observer:

$$
\begin{aligned}
\dot{\hat{y}} &= u\sin(\psi) + \cos(\psi)\hat{v} + k_{11}(y - \hat{y}) + k_{12}(\psi - \hat{\psi}),\\
\dot{\hat{\psi}} &= \hat{r} + k_{21}(y - \hat{y}) + k_{22}(\psi - \hat{\psi}),\\
\dot{\hat{v}} &= -\frac{m_{11}u}{m_{22}}\hat{r} - \frac{d_{22}}{m_{22}}\hat{v} - \sum_{i\geq 2} \frac{d_{vi}}{m_{22}} |\hat{v}|^{i-1}\hat{v} + k_{31}(y - \hat{y}) +\\
&\quad (k_{13} + \cos(\psi))(y - \hat{y}),\\
\dot{\hat{r}} &= \frac{(m_{11} - m_{22})u}{m_{33}}\hat{v} - \frac{d_{33}}{m_{33}}\hat{r} - \sum_{i\geq 2} \frac{d_{ri}}{m_{33}} |\hat{r}|^{i-1}\hat{r} + \frac{1}{m_{33}}\tau_r +\\
&\quad k_{42}(\psi - \hat{\psi}) + (k_{24} + 1)(\psi - \hat{\psi}),
\end{aligned}
\tag{9.34}
$$

where $\hat{y}, \hat{\psi}, \hat{v}$, and $\hat{r}$ are the estimate of y, ψ, v and r respectively. All the constants $k_{11}, k_{12}, k_{21}, k_{22}, k_{13}, k_{31}$, and k_{42} will be chosen later.

By defining the observer errors as $\tilde{y} = y - \hat{y}$, $\tilde{\psi} = \psi - \hat{\psi}$, $\tilde{v} = v - \hat{v}$, and $\tilde{r} = r - \hat{r}$, the observer error dynamics can be rewritten as

$$\begin{aligned}
\dot{\tilde{y}} &= -k_{11}\tilde{y} - k_{12}\tilde{\psi} - k_{13}\tilde{v} + (k_{13} + \cos(\psi))\tilde{v},\\
\dot{\tilde{\psi}} &= -k_{21}\tilde{y} - k_{22}\tilde{\psi} - k_{24}\tilde{r} + (k_{24} + 1)\tilde{r},\\
\dot{\tilde{v}} &= -k_{31}\tilde{y} - \frac{d_{22}}{m_{22}}\tilde{v} - \sum_{i\geq 2}\frac{d_{vi}}{m_{22}}\left(|v|^{i-1}v - |\hat{v}|^{i-1}\hat{v}\right) - \frac{m_{11}u}{m_{22}}\tilde{r} -\\
&\quad (k_{13} + \cos(\psi))\tilde{y} + \frac{1}{m_{22}}\tau_{wv}(t),\\
\dot{\tilde{r}} &= -k_{42}\tilde{\psi} - (k_{24}+1)\tilde{\psi} + \frac{(m_{11} - m_{22})u}{m_{33}}\tilde{v} - \frac{d_{33}}{m_{33}}\tilde{r} -\\
&\quad \sum_{i\geq 2}\frac{d_{ri}}{m_{33}}\left(|r|^{i-1}r - |\hat{r}|^{i-1}\hat{r}\right) + \frac{1}{m_{33}}\tau_{wr}(t).
\end{aligned} \tag{9.35}$$

We now show that there exist suitable observer gains k_{11}, k_{12}, k_{13}, k_{21}, k_{22}, k_{24}, k_{31}, and k_{42} such that the observer error dynamics (9.35) is globally ultimately stable. Consider the Lyapunov function

$$V_{\text{obs}} = \frac{1}{2}\tilde{x}^T\tilde{x} \tag{9.36}$$

where $\tilde{x} = \left[\,\tilde{y}\ \tilde{\psi}\ \tilde{v}\ \tilde{r}\,\right]^T$. The time derivative of (9.36) along the solutions of (9.35) and property (9.33) results in

$$\dot{V}_{\text{obs}} \leq -p_0\|\tilde{x}\|^2 + q_0, \tag{9.37}$$

where

$$\begin{aligned}
p_0 &= -\lambda_{\max}(A) - \max\left(\frac{\tau_{wv\max}}{m_{22}}, \frac{\tau_{wr\max}}{m_{33}}\right)\frac{1}{4\mu_0},\\
q_0 &= \max\left(\frac{\tau_{wv\max}}{m_{22}}, \frac{\tau_{wr\max}}{m_{33}}\right)\mu_0,\ \mu_0 > 0,
\end{aligned}$$

$$A = \begin{bmatrix} -k_{11} & -k_{12} & -k_{13} & 0\\ -k_{21} & -k_{22} & 0 & -k_{24}\\ -k_{31} & 0 & -\frac{d_{22}}{m_{22}} & -\frac{m_{11}u}{m_{22}}\\ 0 & -k_{42} & \frac{(m_{11}-m_{22})u}{m_{33}} & -\frac{d_{33}}{m_{33}} \end{bmatrix}.$$

The above matrix A is made negative definite by choosing

$$\begin{aligned} &k_{13} = k_{24} = k_{31} = k_{42}, \\ &K_1 := \begin{bmatrix} -k_{11} & -k_{12} \\ -k_{21} & -k_{22} \end{bmatrix} < 0, \\ &K_2 - K_{12} K_1^{-1} K_{12} < 0, \end{aligned} \tag{9.38}$$

where

$$K_{12} = \begin{bmatrix} -k_{13} & 0 \\ 0 & -k_{24} \end{bmatrix}, \tag{9.39}$$

and K_2 is defined in (9.32). Here are details of choosing the observer gains such that (9.38) holds. The condition (9.38) is expanded as

$$\begin{aligned} &\begin{bmatrix} -k_{11} & -k_{12} \\ -k_{21} & -k_{22} \end{bmatrix} < 0, \\ &\begin{bmatrix} -\frac{d_{22}}{m_{22}} + \frac{k_{13}^2 k_{22}}{k_{11}k_{22} - k_{12}k_{21}} & -\frac{m_{11}u}{m_{22}} - \frac{k_{13}k_{12}k_{24}}{k_{11}k_{22} - k_{12}k_{21}} \\ \frac{(m_{11} - m_{22})u}{m_{33}} - \frac{k_{24}k_{21}k_{13}}{k_{11}k_{22} - k_{12}k_{21}} & -\frac{d_{33}}{m_{33}} + \frac{k_{24}^2 k_{11}}{k_{11}k_{22} - k_{12}k_{21}} \end{bmatrix} < 0. \end{aligned} \tag{9.40}$$

From (9.40), it suffices that

$$\begin{aligned} &\frac{d_{22}}{m_{22}} - \frac{k_{13}^2 k_{22}}{k_{11}k_{22} - k_{12}k_{21}} > 0, \\ &\frac{d_{33}}{m_{33}} - \frac{k_{24}^2 k_{11}}{k_{11}k_{22} - k_{12}k_{21}} > 0, \\ &\frac{m_{11}u}{m_{22}} = -\frac{k_{13}k_{12}k_{24}}{k_{11}k_{22} - k_{12}k_{21}}, \\ &\frac{(m_{11} - m_{22})u}{m_{33}} = \frac{k_{13}k_{21}k_{24}}{k_{11}k_{22} - k_{12}k_{21}}, \\ &k_{11} > 0,\ k_{22} > 0, \\ &k_{11}k_{22} - k_{12}k_{21} > 0. \end{aligned} \tag{9.41}$$

For simplicity, we choose

$$\begin{aligned} &k_{13} = k_{24} = \rho\sqrt{u}, \\ &k_{11}k_{22} - k_{12}k_{21} = \rho, \end{aligned} \tag{9.42}$$

where $\rho > 0$ is to be selected later. Substituting (9.42) into (9.41) yields

$$0 < k_{22} < \frac{d_{22}}{\rho u_{\max} m_{22}},\ 0 < k_{11} < \frac{d_{33}}{\rho u_{\max} m_{33}},$$

$$k_{12} = -\frac{m_{11}}{\rho m_{22}},\ k_{21} = \frac{(m_{11}-m_{22})}{\rho m_{33}},$$
$$k_{11}k_{22} > -\frac{m_{11}}{\rho^2 m_{22}}\frac{(m_{11}-m_{22})}{m_{33}}. \tag{9.43}$$

Hence, under Assumption 9.1, we can always pick a suitable constant $\rho > 0$ such that (9.43) holds. In summary, the observer gains k_{11}, k_{12}, k_{13}, k_{21}, k_{22}, k_{24}, k_{31}, and k_{42} are chosen such that (9.42) and (9.43) hold.

We choose A and μ_0 such that $p_0 > 0$. Hence (9.36) and (9.37) yield

$$\|\tilde{x}(t)\| \leq \|\tilde{x}(t_0)\| e^{-\eta(t-t_0)} + \eta_0, \forall t \geq t_0 \geq 0, \tag{9.44}$$

with $\eta_0 = \sqrt{q_0/p_0}$ and $\eta = p_0$. When there are no environmental disturbances, we have $\eta_0 = 0$. The observer error dynamics (9.35) is thus GES at the origin.

9.3.2 Control Design

We use the coordinate transformation (9.2) to rewrite the ship dynamics (9.3) in conjunction with (9.34) as follows

$$\begin{aligned}
\dot{y} &= -\frac{kuy}{\sqrt{1+(ky)^2}} + \frac{\hat{v}}{\sqrt{1+(ky)^2}} + \frac{u\,(\sin(z_1) - ky(\cos(z_1)-1))}{\sqrt{1+(ky)^2}} + \\
&\quad \frac{\hat{v}\,((\cos(z_1)-1)+\sin(z_1)ky)}{\sqrt{1+(ky)^2}} + \frac{\tilde{v}}{\sqrt{1+(ky)^2}}(\cos(z_1)+\sin(z_1)ky),\\
\dot{z}_1 &= \hat{r} + \tilde{r} - \frac{k^2uy - k\hat{v}}{(1+(ky)^2)^{3/2}} + \frac{ku\,(\sin(z_1)-ky(\cos(z_1)-1))}{(1+(ky)^2)^{3/2}} + \\
&\quad \frac{k\hat{v}\,((\cos(z_1)-1)+\sin(z_1)ky)}{(1+(ky)^2)^{3/2}} + \frac{k\tilde{v}\,(\cos(z_1)+\sin(z_1)ky)}{(1+(ky)^2)^{3/2}},\\
\dot{\hat{v}} &= -\frac{m_{11}u}{m_{22}}\hat{r} - \frac{d_{22}}{m_{22}}\hat{v} - \sum_{i\geq 2}\frac{d_{vi}}{m_{22}}|\hat{v}|^{i-1}\hat{v} + (k_{31}+k_{13}+\cos(\psi))\tilde{y},\\
\dot{\hat{r}} &= \frac{(m_{11}-m_{22})u}{m_{33}}\hat{v} - \frac{d_{33}}{m_{33}}\hat{r} - \sum_{i\geq 2}\frac{d_{ri}}{m_{33}}|\hat{r}|^{i-1}\hat{r} + \frac{1}{m_{33}}\tau_r + (k_{42}+k_{24}+1)\tilde{\psi}.
\end{aligned} \tag{9.45}$$

Similarly to the full state feedback case, we design the control law τ_r in two steps.

Step 1

Define

$$z_2 = \hat{r} - \hat{r}_d, \tag{9.46}$$

where $\hat{r}_d$ is an intermediate control designed as

$$\hat{r}_d = -k_1 z_1 - \frac{ku\,(\sin(z_1) - ky\cos(z_1))}{(1+(ky)^2)^{3/2}} - \frac{k\hat{v}\,(\cos(z_1) + \sin(z_1)ky)}{(1+(ky)^2)^{3/2}}, \tag{9.47}$$

with k_1 being a positive constant to be selected later.

Step 2

The first time derivative of (9.46) along the solutions of the last equation of (9.45) together with (9.47) is

$$\begin{aligned}
\dot{z}_2 = {} & \frac{(m_{11}-m_{22})u}{m_{33}}\hat{v} - \frac{d_{33}}{m_{33}}\hat{r} - \sum_{i\geq 2}\frac{d_{ri}}{m_{33}}|\hat{r}|^{i-1}\hat{r} + \frac{1}{m_{33}}\tau_r + \\
& (k_{42}+k_{24}+1)\tilde{\psi} - \frac{\partial \hat{r}_d}{\partial u}\dot{u} - \frac{\partial \hat{r}_d}{\partial z_1}(-k_1 z_1 + z_2) - \\
& \frac{\partial \hat{r}_d}{\partial y}\left(\frac{u\,(\sin(z_1) - ky\cos(z_1))}{\sqrt{1+(ky)^2}} + \frac{\hat{v}\,(\cos(z_1)+\sin(z_1)ky)}{\sqrt{1+(ky)^2}}\right) - \\
& \frac{\partial \hat{r}_d}{\partial \hat{v}}\left(-\frac{m_{11}u}{m_{22}}\hat{r} - \frac{d_{22}}{m_{22}}\hat{v} - \sum_{i\geq 2}\frac{d_{vi}}{m_{22}}|\hat{v}|^{i-1}\hat{v}\right) - \\
& \frac{\partial \hat{r}_d}{\partial z_1}\left(\frac{k}{(1+(ky)^2)^{3/2}}(\cos(z_1)+\sin(z_1)ky)\tilde{v} + \tilde{r}\right) - \\
& \frac{\partial \hat{r}_d}{\partial y}\frac{1}{\sqrt{1+(ky)^2}}(\cos(z_1)+\sin(z_1)ky)\,\tilde{v} - \frac{\partial \hat{r}_d}{\partial \hat{v}}(k_{31}+k_{13}+\cos(\psi))\,\tilde{y},
\end{aligned} \tag{9.48}$$

where

$$\begin{aligned}
\frac{\partial \hat{r}_d}{\partial u} &= -\frac{k\,(-\cos(z_1)ky + \sin(z_1))}{(1+(ky)^2)^{3/2}}, \quad \frac{\partial \hat{r}_d}{\partial \hat{v}} = -\frac{k\,(\cos(z_1)+\sin(z_1)ky)}{(1+(ky)^2)^{3/2}}, \\
\frac{\partial \hat{r}_d}{\partial z_1} &= -k_1 - \frac{ku\,(\cos(z_1)+ky\sin(z_1))}{(1+(ky)^2)^{3/2}} - \frac{k\hat{v}\,(-\sin(z_1)+ky\cos(z_1))}{(1+(ky)^2)^{3/2}}, \\
\frac{\partial \hat{r}_d}{\partial y} &= -\frac{3k^3 y\,(kuy - u\,(\sin(z_1)-(y+\hat{v})(\cos(z_1)-1)) - \hat{v}y\sin(z_1))}{(1+(ky)^2)^{5/2}} + \\
& \quad \frac{k^2\,(u\cos(z_1) - \hat{v}\sin(z_1))}{(1+(ky)^2)^{3/2}}.
\end{aligned} \tag{9.49}$$

We now choose the actual control without canceling the useful damping terms as

$$\begin{aligned}\tau_r = m_{33}\Bigg(&-z_1 - k_2 z_2 - \frac{(m_{11}-m_{22})u}{m_{33}}\hat{v} + \frac{d_{33}}{m_{33}}\hat{r}_d + \\ &\sum_{i\geq 2}\frac{d_{ri}}{m_{33}}|\hat{r}|^{i-1}\hat{r}_d + \frac{\partial \hat{r}_d}{\partial u}\dot{u} + \frac{\partial \hat{r}_d}{\partial z_1}(-k_1 z_1 + z_2) + \\ &\frac{\partial \hat{r}_d}{\partial y}\left(\frac{u(\sin(z_1) - ky\cos(z_1))}{\sqrt{1+(ky)^2}} + \frac{\hat{v}(\cos(z_1)+\sin(z_1)ky)}{\sqrt{1+(ky)^2}}\right) + \\ &\frac{\partial \hat{r}_d}{\partial \hat{v}}\left(-\frac{m_{11}u}{m_{22}}\hat{r} - \frac{d_{22}}{m_{22}}\hat{v} - \sum_{i\geq 2}\frac{d_{vi}}{m_{22}}|\hat{v}|^{i-1}\hat{v}\right)\Bigg),\end{aligned} \tag{9.50}$$

where k_2 is a positive constant to be chosen later. Substituting (9.46), (9.47), and (9.50) into (9.45) results in the following closed loop system:

$$\begin{aligned}\dot{y} &= -\frac{kuy}{\sqrt{1+(ky)^2}} + \frac{\hat{v}}{\sqrt{1+(ky)^2}} + \frac{u(\sin(z_1) - ky(\cos(z_1)-1))}{\sqrt{1+(ky)^2}} + \\ &\quad \frac{\hat{v}((\cos(z_1)-1)+\sin(z_1)ky)}{\sqrt{1+(ky)^2}} + \frac{\tilde{v}(\cos(z_1)+\sin(z_1)ky)}{\sqrt{1+(ky)^2}}, \\ \dot{\hat{v}} &= -\frac{d_{22}}{m_{22}}\hat{v} - \sum_{i\geq 2}\frac{d_{vi}}{m_{22}}|\hat{v}|^{i-1}\hat{v} - \frac{m_{11}u}{m_{22}}\Bigg(-k_1 z_1 + z_2 - \\ &\quad \frac{ku(\sin(z_1) - ky\cos(z_1))}{(1+(ky)^2)^{3/2}} - \frac{k\hat{v}(\cos(z_1)+\sin(z_1)ky)}{(1+(ky)^2)^{3/2}}\Bigg) + \\ &\quad (k_{31}+k_{13}+\cos(\psi))\tilde{y}, \\ \dot{z}_1 &= -k_1 z_1 + z_2 + \frac{k(\cos(z_1)+\sin(z_1)ky)\tilde{v}}{(1+(ky)^2)^{3/2}} + \tilde{r}, \\ \dot{z}_2 &= -z_1 - k_2 z_2 - \frac{d_{33}}{m_{33}}z_2 - \sum_{i\geq 2}\frac{d_{ri}}{m_{33}}|\hat{r}|^{i-1}z_2 - \\ &\quad \frac{\partial \hat{r}_d}{\partial z_1}\left(\frac{k(\cos(z_1)+\sin(z_1)ky)\tilde{v}}{(1+(ky)^2)^{3/2}} + \tilde{r}\right) + (k_{42}+k_{24}+1)\tilde{\psi} - \\ &\quad \frac{\partial \hat{r}_d}{\partial \hat{v}}(k_{31}+k_{13}+\cos(\psi))\tilde{y} - \frac{\partial \hat{r}_d}{\partial y}\frac{1(\cos(z_1)+\sin(z_1)ky)\tilde{v}}{\sqrt{1+(ky)^2}}.\end{aligned} \tag{9.51}$$

9.3.3 Stability Analysis

It is not difficult to show that the closed loop system (9.51) is forward complete. We now use Lemmas 9.1 and 9.2 to prove that the closed loop (9.51) is globally ultimately stable. From (9.49), it can be seen that the closed loop (9.51) is different from (9.9) since $\hat{v}$ enters the (z_1, z_2)-subsystem. To remove this obstacle, we first prove that $\hat{v}$ is bounded. We then prove the convergence of (z_1, z_2) and finally $\hat{v}$ and y.

Boundedness of $\hat{v}$

To prove that $\hat{v}$ is bounded, we view the last three equations of (9.51) as the system studied in Lemma 9.1 with $x_1 = \begin{bmatrix} \hat{v} & z_1 & z_2 \end{bmatrix}^T$ as x, $\tilde{x}$ as $\xi(t)$ and

$$\dot{x}_1 = f_1(t, x_1) + g_1(t, x_1, \tilde{x}), \tag{9.52}$$

where

$$f_1(t, x_1) = \begin{bmatrix} \Omega_1 \\ -k_1 z_1 + z_2 \\ -z_1 - \left(k_2 + \dfrac{d_{33}}{m_{33}} + \sum\limits_{i \geq 2} \dfrac{d_{ri}}{m_{33}} |\hat{r}|^{i-1} \right) z_2 \end{bmatrix},$$

$$g_1(t, x_1, \tilde{x}) = \begin{bmatrix} (k_{31} + k_{13} + \cos(\psi))\tilde{y} \\ \dfrac{k\tilde{v}\,(\cos(z_1) + \sin(z_1)ky)}{(1 + (ky)^2)^{3/2}} + \tilde{r} \\ \Omega_2 \end{bmatrix}, \tag{9.53}$$

with

$$\begin{aligned} \Omega_1 = {} & -\frac{d_{22}}{m_{22}}\hat{v} - \sum_{i \geq 2} \frac{d_{vi}}{m_{22}} |\hat{v}|^{i-1}\,\hat{v} - \frac{m_{11}u}{m_{22}} \times \\ & \left(-k_1 z_1 + z_2 - \frac{k\,(u\,(\sin(z_1) - ky\cos(z_1)) + \hat{v}\,(\cos(z_1) + \sin(z_1)ky))}{(1 + (ky)^2)^{3/2}} \right), \\ \Omega_2 = {} & (k_{42} + k_{24} + 1)\,\tilde{\psi} - \frac{\partial \hat{r}_d}{\partial z_1} \left(\frac{k\,(\cos(z_1) + \sin(z_1)ky)\,\tilde{v}}{(1 + (ky)^2)^{3/2}} + \tilde{r} \right) - \\ & \frac{\partial \hat{r}_d}{\partial \hat{v}} (k_{31} + k_{13} + \cos(\psi))\,\tilde{y} - \frac{\partial \hat{r}_d}{\partial y} \frac{(\cos(z_1) + \sin(z_1)ky)\,\tilde{v}}{\sqrt{1 + (ky)^2}}, \end{aligned}$$

where again with abuse of notation, $\hat{r}$ is considered as a function of time. We now need to verify all of the conditions of Lemma 9.1.

Verifying Condition C1. We take the following Lyapunov function:

$$V_1 = \frac{1}{2}\left(\hat{v}^2 + \delta_1(z_1^2 + z_2^2)\right), \tag{9.54}$$

where δ_1 is a positive constant. The first time derivative of (9.54) along the solutions of the differential equation $\dot{x}_1 = f_1(t, x_1)$, see (9.52) and (9.53), satisfies

$$\begin{aligned}
\dot{V}_1 \; &= -\frac{d_{22}}{m_{22}}\hat{v}^2 - \sum_{i\geq 2}\frac{d_{vi}}{m_{22}}|\hat{v}|^{i-1}\hat{v}^2 - \frac{m_{11}u\hat{v}}{m_{22}}\Big(-k_1 z_1 + z_2 - \\
&\quad \frac{k\left(u\left(\sin(z_1) - ky\cos(z_1)\right) + \hat{v}\left(\cos(z_1) + \sin(z_1)ky\right)\right)}{\left(1+(ky)^2\right)^{3/2}}\Big) - \\
&\quad \delta_1 k_1 z_1^2 - \delta_1 k_2 z_2^2 - \delta_1\frac{d_{33}}{m_{33}}z_2^2 - \delta_1\sum_{i\geq 2}\frac{d_{ri}}{m_{33}}|\hat{r}|^{i-1}z_2^2 \\
&\leq -\left(\frac{d_{22}}{m_{22}} - \frac{m_{11}u_{\max}}{m_{22}}(\mu_1 + k_1\mu_1 + 4k + 2k\mu_1 u_{mx})\right)\hat{v}^2 - \\
&\quad \left(\delta_1 k_1 - \frac{m_{11}k_1 u_{\max}}{4\mu_1 m_{22}}\right)z_1^2 - \left(\delta_1 k_2 - \frac{m_{11}u_{\max}}{4\mu_1 m_{22}}\right)z_2^2 + \frac{5m_{11}ku_{\max}^2}{2m_{22}}.
\end{aligned} \tag{9.55}$$

Hence the condition C1 of Lemma 9.1 is verified with

$$\begin{aligned}
c_0 &= \frac{5m_{11}ku_{\max}^2}{2m_{22}}, c_1 = \frac{1}{2}\min(1,\delta_1), \\
c_2 &= \frac{1}{2}\max(1,\delta_1), c_3 = \max(1,\delta_1), \\
c_4 &= \min\left(\left(\frac{d_{22}}{m_{22}} - \frac{m_{11}u_{\max}}{m_{22}}(\mu_1 + k_1\mu_1 + 4k + 2k\mu_1 u_{mx})\right),\right. \\
&\quad \left.\left(\delta_1 k_1 - \frac{m_{11}k_1 u_{\max}}{4\mu_1 m_{22}}\right), \left(\delta_1 k_2 - \frac{m_{11}u_{\max}}{4\mu_1 m_{22}}\right)\right),
\end{aligned} \tag{9.56}$$

where $\mu_1 > 0$ and $k > 0$ are chosen such that $c_4 > 0$.

Verifying Condition C2. To verify this condition of Lemma 9.1, we note from (9.49) that

$$\begin{aligned}
&\left|\frac{\partial \hat{r}_d}{\partial z_1}\right| \leq k_1 + 2k\left(u_{\max} + |\hat{v}|\right), \\
&\left|\frac{\partial \hat{r}_d}{\partial \hat{v}}\right| \leq 2k, \left|\frac{\partial \hat{r}_d}{\partial y}\right| \leq 3k^2\left(u_{\max} + 3ku_{\max} + 3k|\hat{v}|\right).
\end{aligned} \tag{9.57}$$

From (9.56) and (9.57), a simple calculation shows that the condition C2 of Lemma 9.1 is satisfied with

$$\lambda_1 = (2k+1)(k_{31}+k_{13}+1)+k_{42}+k_{24}+1+ \\ (2k+1)(k_1+2ku_{\max}+1)+6k^2u_{\max}(3k+1), \\ \lambda_2 = 2k(2k+1)+18k^3. \tag{9.58}$$

Verifying Condition C3. This condition follows directly from (9.44).

Verifying Condition C4. It can be shown that there exists a positive constant k such that the condition C4 of Lemma 9.1 satisfies

$$c_4 - \lambda_2 c_3 \varepsilon_0 - \frac{\lambda_1 c_3 \varepsilon_0}{4\mu_0} > 0, \tag{9.59}$$

where c_4, c_3, ε_0, λ_1, and λ_2 given in (9.44), (9.56) and (9.58).

All of the conditions of Lemma 9.1 have been verified, therefore we have

$$|\hat{v}| \leq \|x_1(t)\| \leq \alpha_1 (\|(x_1(t_0), \tilde{x}(t_0))\|) e^{-\sigma_1(t-t_0)} + \varepsilon_1, \ \forall t \geq t_0 \geq 0, \tag{9.60}$$

where α_1, σ_1, and ε_1 are in the form of α, σ, and ε in Lemma 9.1 with all constants given in (9.44), (9.56), and (9.58).

(z_1, z_2)-subsystem

Having proven that $\hat{v}$ is bounded in the previous section, we now apply Lemma 9.1 to the (z_1, z_2)-subsystem. It is clear that the last two equations of (9.51) are in the form of the system studied in Lemma 9.1 with $z = [z_1 \ z_2]^T$ as x, $\tilde{x}$ as $\xi(t)$ and

$$\dot{z} = f_z(t, z) + g_z(t, y, z, \tilde{x}), \tag{9.61}$$

where

$$f_z(t,z) = \begin{bmatrix} -k_1 z_1 + z_2 \\ -z_1 - \left(k_2 + \dfrac{d_{33}}{m_{33}} + \sum\limits_{i \geq 2} \dfrac{d_{ri}}{m_{33}} |\hat{r}|^{i-1} \right) z_2 \end{bmatrix}, \\ g_z(t,y,z,\tilde{x}) = \begin{bmatrix} \dfrac{k\,(\cos(z_1)+\sin(z_1)ky)\,\tilde{v}}{(1+(ky)^2)^{3/2}} + \tilde{r} \\ \Omega_2 \end{bmatrix}. \tag{9.62}$$

Proceeding with the same steps as in the previous section, it is shown that all the conditions of Lemma 9.1 hold with the Lyapunov function $V_2 = 0.5(z_1^2 + z_2^2)$ and

$$c_0 = 0, \ c_1 = c_2 = 0.5, \ c_3 = 1, \ c_4 = \min(k_1, k_2), \\ \lambda_1 = 2k\,(k_{13}+k_{31}+1)+(2k+1)(2+2ku_{\max})+k_{42}+$$

$$k_{24}+1+6k^2u_{\max}(1+3k)+(18k^3+2k(2k+1))(\alpha_1+\varepsilon_1),$$
$$\lambda_2=0, \tag{9.63}$$

where α_1 and ε_1 are given in (9.60). The condition C4 of Lemma 9.1 becomes

$$c_4-\frac{\lambda_1 c_3\varepsilon_0}{4\mu_0}>0, \tag{9.64}$$

where c_4, c_3, ε_0, and λ_2 are calculated from in (9.44) and (9.63).

All of the conditions of Lemma 9.1 have been verified, therefore we have

$$\|z(t)\|\le\alpha_2\left(\|(z(t_0),\tilde{x}(t_0))\|\right)e^{-\sigma_2(t-t_0)}+\varepsilon_2,\ \forall t\ge t_0\ge 0 \tag{9.65}$$

where α_2, σ_2, and ε_2 are in the form of α, σ, and ε in Lemma 9.1 with all constants given in (9.63).

$(y,\hat{\upsilon})$-subsystem

It can be seen that the first two equations of (9.51) are in the form of the system studied in Lemma 9.2, i.e., $x_3=\begin{bmatrix}\hat{\upsilon} & y\end{bmatrix}^T$, $\tilde{x}_3=\begin{bmatrix}z_1 & z_2 & \tilde{y} & \tilde{\upsilon}\end{bmatrix}^T$, and

$$\dot{x}_3=f_3(t,x_3)+g_3(t,x_3,\tilde{x}_3), \tag{9.66}$$

where

$$f_3(t,x_3)=\begin{bmatrix}-\frac{d_{22}}{m_{22}}\hat{\upsilon}-\sum_{i\ge 2}\frac{d_{\upsilon i}}{m_{22}}|\hat{\upsilon}|^{i-1}\hat{\upsilon}-\frac{m_{11}u}{m_{22}}\frac{k^2uy-k\hat{\upsilon}}{(1+(ky)^2)^{3/2}}\\ -\frac{kuy}{\sqrt{1+(ky)^2}}+\frac{\hat{\upsilon}}{\sqrt{1+(ky)^2}}\end{bmatrix}, \tag{9.67}$$

$$g_3(t,x_3,\tilde{x}_3)=\begin{bmatrix}\Omega_{31}\\ \Omega_{32}\end{bmatrix},$$

with

$$\Omega_{31}=-\frac{m_{11}u}{m_{22}}\left(-k_1z_1+z_2-\frac{ku\left(\sin(z_1)-ky(\cos(z_1)-1)\right)}{(1+(ky)^2)^{3/2}}-\right.$$
$$\left.\frac{k\hat{\upsilon}\left((\cos(z_1)-1)+\sin(z_1)ky\right)}{(1+(ky)^2)^{3/2}}\right)+(k_{31}+k_{13}+\cos(\psi))\tilde{y},$$
$$\Omega_{32}=\frac{u\left(\sin(z_1)-ky(\cos(z_1)-1)\right)}{\sqrt{1+(ky)^2}}+\frac{\hat{\upsilon}\left((\cos(z_1)-1)+\sin(z_1)ky\right)}{\sqrt{1+(ky)^2}}+$$
$$\frac{(\cos(z_1)-\sin(z_1)ky)\,\tilde{\upsilon}}{\sqrt{1+(ky)^2}}.$$

We now need to verify all conditions of Lemma 9.2 for the system (9.66).

Verifying Condition C1. To verify this condition of Lemma 9.2, we take the following proper function

$$V_3 = \frac{1}{2}\left(\hat{v}^2 + y^2\right), \tag{9.68}$$

whose time derivative along (9.67) satisfies

$$\dot{V}_3 \leq -c_{31}\hat{v}^2 - c_{32}\frac{y^2}{\sqrt{1+c_4 y^2}}, \tag{9.69}$$

where

$$\begin{aligned}
c_{31} &= \frac{d_{22}}{m_{22}} - \frac{m_{11}ku_{\max}}{m_{22}} - \mu_2 - \frac{m_{11}k^2u_{\max}^2}{m_{22}}\mu_3, \\
c_{32} &= ku_{\min} - \frac{1}{4\mu_2} - \frac{1}{4\mu_3}\frac{m_{11}k^2u_{\max}^2}{m_{22}},
\end{aligned} \tag{9.70}$$

with $\mu_2 > 0$ and $\mu_3 > 0$ chosen such that $c_{31} > 0$ and $c_{32} > 0$.

From (9.67) and (9.68), it is easy to show that

$$\left|\frac{\partial V_3}{\partial x_3} g_3(t, x_3, \tilde{x}_3)\right| \leq \left(\lambda_0 + \lambda_1\hat{v}^2 + \lambda_2\frac{y^2}{\sqrt{1+c_4y^2}}\right)\|\tilde{x}_3\|, \tag{9.71}$$

with

$$\begin{aligned}
\lambda_0 &= \frac{1}{4\mu_4}\left(\frac{m_{11}u_{\max}}{m_{22}}(k_1 + 1 + 2ku_{\max}) + (k_{31} + k_{13} + 1) + \right. \\
&\quad \left. u_{\max} + 1 + (\alpha_1 + \varepsilon_1)^2\right), \\
\lambda_1 &= \frac{m_{11}u_{\max}\mu_4}{m_{22}}(k_1 + 1 + 2ku_{\max}) + \frac{2m_{11}ku_{\max}}{m_{22}} + \mu_4(k_{31} + k_{13} + 1), \\
\lambda_2 &= (k + \mu_4)u_{\max} + 2\mu_4 + k\left(\alpha_1 + \varepsilon_1 + \alpha_2 + \varepsilon_2\right),
\end{aligned} \tag{9.72}$$

where $\mu_4 > 0$, α_1, and ε_1 are given in (9.60), and α_2 and ε_2 are given in (9.65).

Verifying Condition C2. To verify this condition, we note that

$$\|\tilde{x}_3(t)\| \leq \left\|\begin{pmatrix} z_1(t) \\ z_2(t)\end{pmatrix}\right\| + \left\|\begin{pmatrix} \tilde{v}(t) \\ \tilde{y}(t)\end{pmatrix}\right\|. \tag{9.73}$$

Therefore we can write (9.73) from (9.44) and (9.65) as

$$\|\tilde{x}_3(t)\| \leq \alpha_3\left(\|(z(t_0), \tilde{x}(t_0))\|\right)e^{-\sigma_3(t-t_0)} + \varepsilon_3, \tag{9.74}$$

where

$$\alpha_3(\|(z(t_0),\tilde{x}(t_0))\|) = \|(z(t_0),\tilde{x}(t_0))\| + \alpha_2(\|(z(t_0),\tilde{x}(t_0))\|),$$
$$\sigma_3 = \min(\eta,\sigma_2),\ \varepsilon_3 = \eta_0 + \varepsilon_2, \tag{9.75}$$

with α_2 and ε_2 given in (9.65), η_0 and η given in (9.44).

Verifying Condition C3. This condition is satisfied if

$$c_{31} - \lambda_1\varepsilon_3 > 0 \text{ and } c_{32} - \lambda_2\varepsilon_3 > 0, \tag{9.76}$$

where c_{31}, c_{32}, λ_1, λ_2, and ε_3 are given in (9.72) and (9.75). After some lengthy but simple calculation, it can be shown that, under the assumption of small enough environmental disturbances, the condition (9.76) holds for a suitable choice of the observer gains k_{11}, k_{12}, k_{13}, k_{21}, k_{22}, k_{24}, k_{31}, and k_{42}, and the control gains k, k_1, and k_2.

Verifying Condition C4. From the boundedness of the sway velocity estimate, $\hat{v}(t)$, proven in the previous section and noting that ε_2 in (9.65) can be made arbitrarily small, it is directly shown that there exists a nondecreasing function ϖ of $\|((v(t_0),y(t_0)),z(t_0))\|$ such that $|y(t)| \le \varpi$ by applying Lemma 9.1 to the first equation of (9.51) with the Lyapunov function $V_y = 0.5y^2$.

All of the conditions of Lemma 9.2 have been verified, the closed loop (9.51) is globally ultimately stable, i.e.,

$$\|x_3(t)\| \le \alpha_4(\|(x(t_0),\xi(t_0))\|)e^{-\sigma_4(t-t_0)} + \varepsilon_4,\ \forall t \ge t_0 \ge 0, \tag{9.77}$$

where α_4, σ_4, and ε_4 are calculated as in Lemma 9.2.

It is noted that when there are no environmental disturbances, $\varepsilon = 0$. Therefore the closed loop (9.51) is GAS. We note that the convergence of z_1 and z_2 implies the convergence of $\hat{r}$ and ψ. The convergence of v and r results from that of $\hat{v}$ and $\hat{r}$ due to the global exponential property of the observer. We have thus proven the second main result of this chapter.

Theorem 9.2. *Under Assumption 9.1, the output feedback control problem stated in Section 9.1 is solved by the observer (9.34) and the control law (9.50) as long as the observer gains k_{11}, k_{12}, k_{13}, k_{21}, k_{22}, k_{24}, k_{13}, k_{31}, and k_{42}, and the control gains k, k_1, and k_2 are chosen such that (9.64), (9.70), (9.76), and (9.43) hold.*

Remark 9.2. Due to underactuaction and nonzero-mean environmental disturbances in the sway dynamic, our controller is only able to force the sway and its velocity to converge to a ball centered at the origin. The radius of this ball cannot be made arbitrarily small. This phenomenon should not be surprising since there is no control force in the sway direction. In addition, the yaw angle cannot be made arbitrarily small due to the effect of the sway. In fact, to guarantee the sway displacement bounded under nonzero-mean environmental disturbances acting on the sway dynamics, our controller forces the heading angle to a small value. This value together with the forward speed will prevent the sway from growing unbounded.

Remark 9.3. The choice of k depends on the ship parameters and forward speed, which coincides with the steering practice of a helmsman. The helmsman uses the ship's course angle to steer the ship toward the straight line rather than use the sway velocity, which will cause the ship to glide sideways. Furthermore, the design constant k is reduced when the ship forward speed is large, see (9.2), (9.23), (9.30), (9.56), and (9.59), otherwise the ship will miss the point on the straight line and slide in the sway direction.

Remark 9.4. By setting the value of k equal to zero, our proposed controller reduces to a course-keeping controller. In this case, the heading angle can be made arbitrarily small. However the sway will grow linearly unbounded under nonzero-mean environmental disturbances, see Figures 9.3 and 9.6.

9.4 Simulations

This section validates the control laws (9.8) and (9.50) for both cases of state and output feedback on a monohull ship with the parameters given in Section 5.4. The ship surge velocity is chosen as $u = 10 + 0.5\sin(3t)\,\mathrm{ms}^{-1}$. The environmental disturbances $\tau_{wv}(t)$ and $\tau_{wr}(t)$ are taken as $\tau_{wv}(t) = 10^5 \times 0.5 \times (1 + \mathrm{rand}(\cdot))$ and $\tau_{wr}(t) = 1.5 \times 10^7 \times \mathrm{rand}(\cdot)$, with $\mathrm{rand}(\cdot)$ being zero mean random noise with the uniform distribution on the interval $[-0.5\ 0.5]$. We run simulations for both state feedback and output feedback cases.

9.4.1 State Feedback Simulation Results

The control design parameters are chosen as $k = 0.05$, $k_1 = 0.2$, $k_2 = 0.5$, and $\rho_1 = \rho_2 = 0.05$. It can be directly verified that this choice satisfies all the conditions stated in Theorem 9.1. The initial values are

$$[y(0), v(0), \psi(0), r(0)] = \left[15\,\mathrm{m}, 0.2\ \mathrm{ms}^{-1}, -0.5\,\mathrm{rad}, 0.1\,\mathrm{rads}^{-1}\right].$$

Simulation results are plotted in Figure 9.1 for the case without disturbances. In this case, it can be seen that all sway displacement, sway velocity, and yaw angle converge to zero as desired. The large control effort is due to the fact that we simulate our controllers on a real surface ship but it is within the limit of the maximum yaw moment. For the case with disturbances, simulation results are plotted in Figure 9.2. In this case, all the states converge to a ball centered at the origin as proven in Theorem 9.1. To illustrate Remark 9.4, we simulate our controller with the design constant $k = 0$. The simulation results for this case are given in Figure 9.3. The sway displacement y grows linearly unbounded due to nonvanishing environmental disturbances. It should be noted that all of the course-keeping controllers, see for

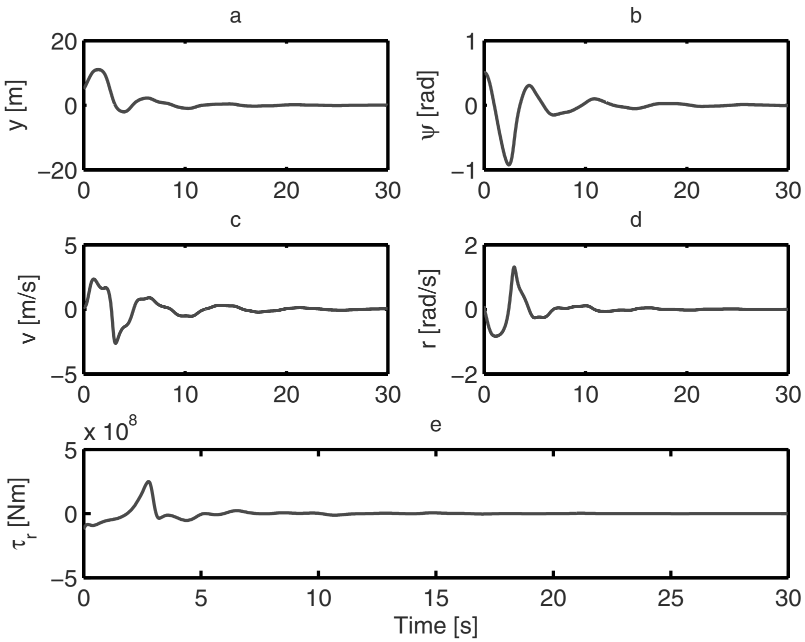

Figure 9.1 State feedback control results without disturbances: **a**. Sway displacement y; **b**. Heading angle ψ; **c**. Sway velocity v; **d**. Yaw velocity r; **e**. Yaw moment τ_r

example [12], which do not take the sway displacement into account, will result in similar unboundedness of the sway that was pointed out in Remark 9.4.

9.4.2 Output Feedback Simulation Results

The control design parameters are chosen as $k = 0.05$, $k_1 = 0.2$, $k_2 = 0.5$, and $\rho_1 = \rho_2 = 0.05$. The observer gains are selected as $k_{11} = k_{22} = 2$, $k_{12} = -\frac{m_{11}}{\rho m_{22}}$, $k_{21} = \frac{m_{11}-m_{22}}{\rho m_{33}}$, $k_{31} = k_{13} = k_{24} = k_{42} = \rho\sqrt{u}$, and $\rho = 0.015$. A calculation shows that this choice satisfies all the conditions stated in Theorem 9.2. The initial values are

$$[y(0), v(0), \psi(0), r(0)] = \left[15\,\text{m}, 0.2\ \text{ms}^{-1}, -0.5\,\text{rad}, 0.1\,\text{rads}^{-1}\right],$$
$$\left[\hat{y}(0), \hat{v}(0), \hat{\psi}(0), \hat{r}(0)\right] = \left[10\,\text{m}, 0\,\text{ms}^{-1}, -0.2\,\text{rad}, 0.2\,\text{rads}^{-1}\right].$$

Simulation results are plotted in Figure 9.4 for the case without disturbances and in Figure 9.5 for the case with disturbances. From Figure 9.4, it is seen that all sway displacement, sway velocity, and yaw angle converge to zero asymptotically. It is also observed that the observer states (dotted lines) exponentially converge to

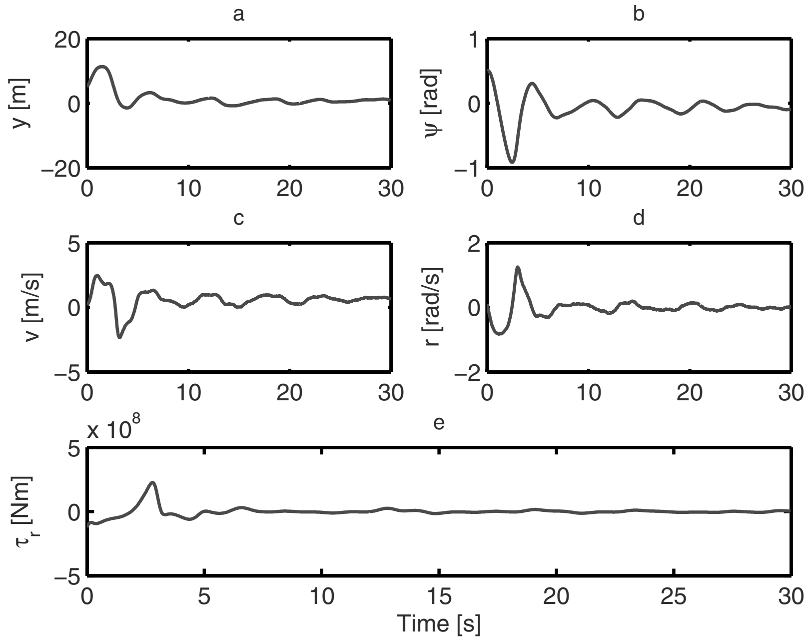

Figure 9.2 State feedback control results with disturbances: **a**. Sway displacement y; **b**. Heading angle ψ; **c**. Sway velocity v; **d**. Yaw velocity r; **e**. Yaw moment τ_r

their unknown estimated ones (solid lines). For the case with disturbances, all the states converge to a ball centered at the origin as proven in Theorem 9.2. The simulation results with the design constant $k = 0$ are plotted in Figure 9.6. Again, the sway displacement y grows linearly unbounded due to nonvanishing environmental disturbances as mentioned in Remark 9.4.

9.5 Conclusions

The control design was based on the idea of an interaction between the ship behavior and the action of a helmsman on a linear course. Although our proposed state feedback controller has been designed by using precise knowledge of the ship parameters, we can easily change them to an adaptive version to take inaccurate knowledge of the system parameters into account, see (9.6). However, for the case of output feedback, an adaptive observer will be required, see (9.34) and (9.45). This chapter is based on [116, 126, 127].

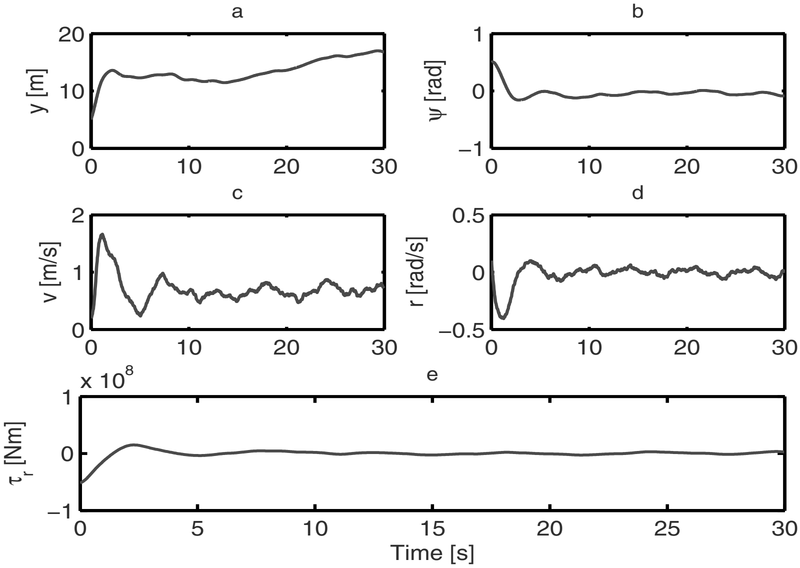

Figure 9.3 State feedback control results with disturbances and $k = 0$: **a**. Sway displacement y; **b**. Heading angle ψ; **c**. Sway velocity v; **d**. Yaw velocity r; **e**. Yaw moment τ_r

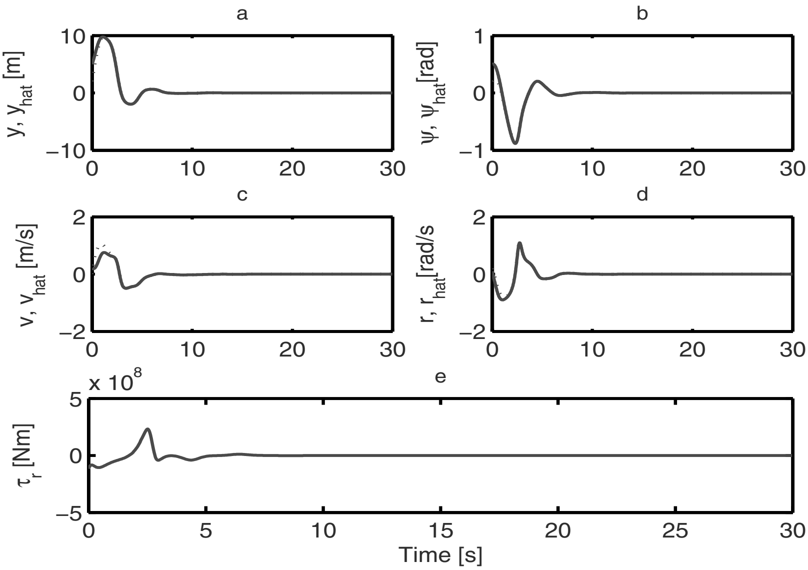

Figure 9.4 Output feedback control results without disturbances: **a**. Sway displacement y; **b**. Heading angle ψ; **c**. Sway velocity v (*solid line*) and its estimate $\hat{v}$ (*dotted line*); **d**. Yaw velocity r (*solid line*) and its estimate $\hat{r}$ (*dotted line*); **e**. Yaw moment τ_r

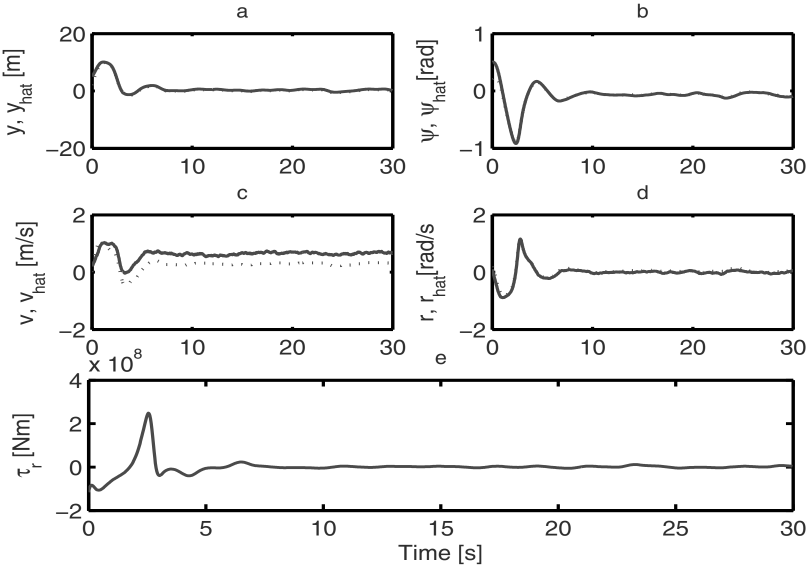

Figure 9.5 Output feedback control results with disturbances: **a**. Sway displacement y; **b**. Heading angle ψ; **c**. Sway velocity v (solid) and its estimate $\hat{v}$ (dot); **d**. Yaw velocity r (solid) and its estimate $\hat{r}$ (dot); **e**. Yaw moment τ_r

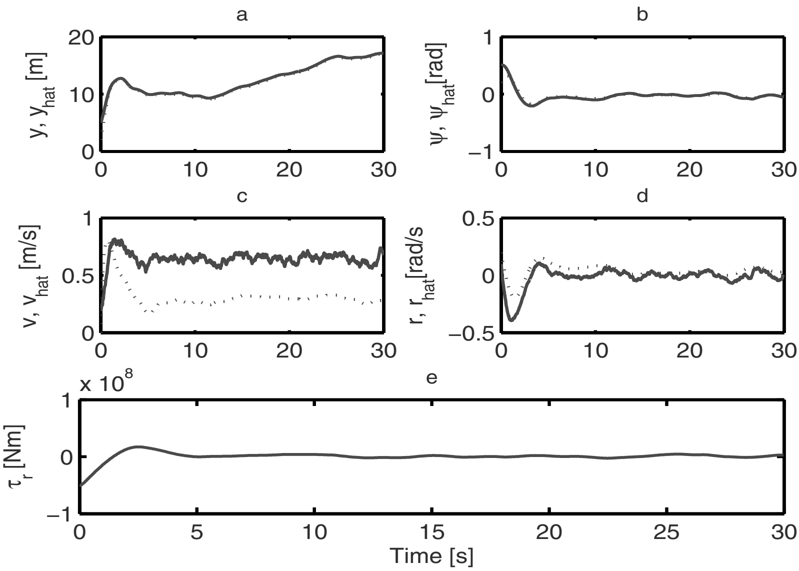

Figure 9.6 Output feedback control results with disturbances and $k = 0$: **a**. Sway displacement y; **b**. Heading angle ψ; **c**. Sway velocity v (*solid line*) and its estimate $\hat{v}$ (*dotted line*); **d**. Yaw velocity r (*solid line*) and its estimate $\hat{r}$ (*dotted line*); **e**. Yaw moment τ_r