Chapter 7
Partial-state and Output Feedback Trajectory-tracking Control of Underactuated Ships

Global partial-state feedback and output feedback control schemes are discussed in this chapter for tracking control of an underactuated surface ship without sway force. For the case of partial-state feedback, we do not require measurements of the ship sway and surge velocities, while for the case of output feedback, none of the ship velocities are required for feedback. The reference trajectory to be tracked can be a curve including a straight line. Global nonlinear coordinate changes are introduced to transform the ship dynamics to a system affine in the ship velocities to design observers to globally exponentially estimate unmeasured velocities. These observers plus the techniques in the previous chapter facilitate the development of controllers in the following sections.

7.1 Control Objective

For the convenience of the reader, the mathematical model of the underactuated ship moving in surge, sway and yaw, see Section 3.4.1.2 (i.e., (3.45) and (3.46)), is once again presented:

$$\begin{aligned} \dot{\boldsymbol{\eta}} &= \boldsymbol{J}(\boldsymbol{\eta})\boldsymbol{v}, \\ \boldsymbol{M}\dot{\boldsymbol{v}} &= -\boldsymbol{C}(\boldsymbol{v})\boldsymbol{v} - \boldsymbol{D}\boldsymbol{v} + \boldsymbol{\tau}, \end{aligned} \tag{7.1}$$

where the matrices $\boldsymbol{J}(\boldsymbol{\eta})$, $\boldsymbol{M}$, $\boldsymbol{C}(\boldsymbol{v})$, and $\boldsymbol{D}$ are given by

$$\begin{aligned} \boldsymbol{J}(\boldsymbol{\eta}) &= \begin{bmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ \boldsymbol{M} = \begin{bmatrix} m_{11} & 0 & 0 \\ 0 & m_{22} & 0 \\ 0 & 0 & m_{33} \end{bmatrix}, \\ \boldsymbol{C}(\boldsymbol{v}) &= \begin{bmatrix} 0 & 0 & -m_{22}v \\ 0 & 0 & m_{11}u \\ m_{22}v & -m_{11}u & 0 \end{bmatrix}, \ \boldsymbol{D} = \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix}, \end{aligned} \tag{7.2}$$

with

$$m_{11} = m - X_{\dot{u}},\ m_{22} = m - Y_{\dot{v}},\ m_{33} = I_z - N_{\dot{r}},$$
$$d_{11} = -X_u,\ d_{22} = -Y_v,\ d_{33} = -N_r. \tag{7.3}$$

The propulsion force and moment vector $\boldsymbol{\tau}$ is still given by (3.43), i.e.,

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_u \\ 0 \\ \tau_r \end{bmatrix}. \tag{7.4}$$

We assume that the reference trajectory is generated by a virtual ship as follows:

$$\begin{aligned} \dot{\boldsymbol{\eta}}_d &= \boldsymbol{J}(\boldsymbol{\eta}_d)\boldsymbol{v}_d, \\ \dot{v}_d &= -\frac{m_{11}}{m_{22}} u_d r_d - \frac{d_{22}}{m_{22}} v_d, \end{aligned} \tag{7.5}$$

where all the variables have similar meanings as in system (7.1). It is noted that we do not require the reference surge and yaw velocities to be generated by the virtual ship. In this chapter we impose the following assumptions on the reference model (7.5):

Assumption 7.1. *The reference signals x_d, y_d, u_d, r_d, $\dot{u}_d$, $\ddot{u}_d$ and $\dot{r}_d$ are bounded. There exists a strictly positive constant $u_{d\min}$, such that $|u_d(t)| \geq u_{d\min}$, $\forall t \geq 0$. The reference sway velocity satisfies $|v_d(t)| < |u_d(t)|$, $\forall t \geq 0$.*

Assumption 7.2. *One of the following conditions holds:*

C1. The surge and sway displacements (x, y), yaw angle, ψ, and yaw velocity, r, are measurable but the surge and sway velocities, u and v, are not.
C2. The surge and sway displacements (x, y) and yaw angle ψ are measurable but none of the velocities u, v, and r are measurable.

Remark 7.1. Condition $|u_d(t)| \geq u_{d\min}$, $\forall t \geq 0$ implies that the reference surge velocity is always nonzero but can be either positive or negative. This means that we consider both forward and backward tracking. From a practical control viewpoint of surface ships, the condition $|u_d(t)| \geq u_{d\min}$, $\forall t \geq 0$ is much less restrictive than a persistently exciting condition on the yaw reference velocity in the references [19, 71, 106] in the sense that tracking of a straight line is included. Surface ships are often equipped with a rudder or a pair of propellers or water jets. The yaw moment to steer the ship is generated by changing the rudder angle or the speed of each propeller or water jet. These facts imply that the tracking control is carried out only when the surge speed is nonzero. The condition $|v_d(t)| < |u_d(t)|$, $\forall t \geq 0$ implies that the ship cannot track a circle with arbitrarily small radius due to the ship's high inertia and underactuation in the sway direction.

Remark 7.2. Assumption 7.2.C1 means that we need to solve a partial-state feedback control problem. Although the yaw velocity is measurable, there is still a cross term uv in the yaw velocity dynamics, see the last equation of (7.1). Assumption 7.2.C2 implies that we need to solve an output feedback control problem. Indeed, Assumption 7.2.C2 covers Assumption 7.2.C1. We will, however, show later that design of an output feedback tracking controller is much more involved than that of the partial-state feedback controller.

7.2 Partial-state Feedback

7.2.1 Observer Design

As discussed above, since the term $\boldsymbol{C}(\boldsymbol{v})\boldsymbol{v}$ in (7.1) causes difficulties in observer design, we first remove this term by proposing the following coordinate transformation:

$$\boldsymbol{X} = e^{\boldsymbol{Q}(t,\boldsymbol{\eta})\boldsymbol{v}}, \tag{7.6}$$

where $\boldsymbol{Q}(t,\boldsymbol{\eta})$ is a matrix to be determined. Differentiating both sides of (7.6) along the solutions of the second equation of (7.1) yields

$$\dot{\boldsymbol{X}} = e^{\boldsymbol{Q}(t,\boldsymbol{\eta})}[\dot{\boldsymbol{Q}}(t,\boldsymbol{\eta}) - \boldsymbol{M}^{-1}\boldsymbol{C}(\boldsymbol{v})]\boldsymbol{v} + e^{\boldsymbol{Q}(t,\boldsymbol{\eta})}(-\boldsymbol{D}\boldsymbol{v} + \boldsymbol{\tau}). \tag{7.7}$$

It can be seen that the square bracket on the right-hand side of (7.7) is zero, if the matrix $\boldsymbol{Q}(t,\boldsymbol{\eta})$ is chosen such that

$$\dot{\boldsymbol{Q}}(t,\boldsymbol{\eta}) - \boldsymbol{M}^{-1}\boldsymbol{C}(\boldsymbol{v}) = 0. \tag{7.8}$$

By using the first equation of (7.1), a particular solution of (7.8) is

$$\boldsymbol{Q}(t,\boldsymbol{\eta}) = \boldsymbol{M}^{-1}\begin{bmatrix} 0 & 0 & -m_{22}q_{13} \\ 0 & 0 & m_{11}q_{23} \\ m_{22}q_{13} & -m_{11}q_{23} & 0 \end{bmatrix}, \tag{7.9}$$

with

$$\begin{aligned} q_{13} &= y\cos(\psi) - x\sin(\psi) + p_{13}(t), \\ q_{23} &= y\sin(\psi) + x\cos(\psi) + p_{23}(t), \\ \dot{p}_{13} &= y\sin(\psi)r + x\cos(\psi)r, \\ \dot{p}_{23} &= -y\cos(\psi)r + x\sin(\psi)r. \end{aligned} \tag{7.10}$$

Note that the matrix $\boldsymbol{Q}(t,\boldsymbol{\eta})$ contains only the available signals since we assume that x, y, ψ, and r are measurable. Using the Taylor expansion the matrix $e^{\boldsymbol{Q}(t,\boldsymbol{\eta})}$ can be expanded as

$$e^{\mathbf{Q}(t,\boldsymbol{\eta})} = \begin{bmatrix} \frac{1}{2a_5}\left(2a_2a_4 + a_1a_3\left(e^{-\sqrt{a_5}} + e^{\sqrt{a_5}}\right)\right) & \frac{1}{2a_5}\left(a_1a_4\left(e^{-\sqrt{a_5}} + e^{\sqrt{a_5}} - 2\right)\right) & \frac{1}{2\sqrt{a_5}}\left(a_1\left(-e^{-\sqrt{a_5}} + e^{\sqrt{a_5}}\right)\right) \\ \frac{1}{2a_5}\left(a_2a_3\left(e^{-\sqrt{a_5}} + e^{\sqrt{a_5}} - 2\right)\right) & \frac{1}{2a_5}\left(2a_1a_3 + a_2a_4\left(e^{-\sqrt{a_5}} + e^{\sqrt{a_5}}\right)\right) & \frac{1}{2\sqrt{a_5}}\left(a_2\left(-e^{-\sqrt{a_5}} + e^{\sqrt{a_5}}\right)\right) \\ \frac{1}{2\sqrt{a_5}}\left(a_3\left(-e^{-\sqrt{a_5}} + e^{\sqrt{a_5}}\right)\right) & \frac{1}{2\sqrt{a_5}}\left(a_4\left(-e^{-\sqrt{a_5}} + e^{\sqrt{a_5}}\right)\right) & \frac{1}{2}\left(e^{-\sqrt{a_5}} + e^{\sqrt{a_5}}\right) \end{bmatrix}, \tag{7.11}$$

where

$$\begin{aligned} a_1 &= -\frac{m_{22}}{m_{11}}q_{13}, \\ a_2 &= \frac{m_{11}}{m_{22}}q_{23}, \\ a_3 &= \frac{m_{22}}{m_{33}}q_{13}, \\ a_4 &= -\frac{m_{11}}{m_{33}}q_{23}, \\ a_5 &= a_1a_3 + a_2a_4. \end{aligned} \tag{7.12}$$

Similarly, $e^{-\boldsymbol{Q}(t,\boldsymbol{\eta})}$ has the same form as (7.11), but all of the terms a_i, $1 \le i \le 4$ have an opposite sign to those defined in (7.12). From (7.11) and noticing that $a_5 \le 0$, $\forall (q_{13}, q_{23}) \in \mathbb{R}^2$, it is easily seen that all elements of $e^{\boldsymbol{Q}(t,\boldsymbol{\eta})}$ or $e^{-\boldsymbol{Q}(t,\boldsymbol{\eta})}$ are bounded by some constants, which depend only on the ship parameters, m_{11}, m_{22}, and m_{33}. Using the coordinate change (7.6), the ship system (7.1) is written in $(\boldsymbol{\eta}, \boldsymbol{X})$ coordinates as

$$\begin{aligned} \dot{\boldsymbol{\eta}} &= \boldsymbol{J}(\boldsymbol{\eta})e^{-\boldsymbol{Q}(t,\boldsymbol{\eta})}\boldsymbol{X}, \\ \dot{\boldsymbol{X}} &= -e^{\boldsymbol{Q}(t,\boldsymbol{\eta})}\boldsymbol{M}^{-1}\boldsymbol{D}e^{-\boldsymbol{Q}(t,\boldsymbol{\eta})}\boldsymbol{X} + e^{\boldsymbol{Q}(t,\boldsymbol{\eta})}\boldsymbol{M}^{-1}\boldsymbol{\tau}. \end{aligned} \tag{7.13}$$

The system (7.13) has a very nice structure, namely linear in the unmeasured states. Of course, a reduced-order observer can be designed but it is often noise-sensitive. Here we use the following nonlinear observer to construct the unmeasured surge and sway velocities:

$$\begin{aligned} \dot{\hat{\boldsymbol{\eta}}} &= \boldsymbol{J}(\boldsymbol{\eta})e^{-\boldsymbol{Q}(t,\boldsymbol{\eta})}\hat{\boldsymbol{X}} + \boldsymbol{K_0}(\boldsymbol{\eta} - \hat{\boldsymbol{\eta}}), \\ \dot{\hat{\boldsymbol{X}}} &= -e^{\boldsymbol{Q}(t,\boldsymbol{\eta})}\boldsymbol{M}^{-1}\boldsymbol{D}e^{-\boldsymbol{Q}(t,\boldsymbol{\eta})}\hat{\boldsymbol{X}} + e^{\boldsymbol{Q}(t,\boldsymbol{\eta})}\boldsymbol{M}^{-1}\boldsymbol{\tau} + \\ &\quad (\boldsymbol{J}(\boldsymbol{\eta})e^{-\boldsymbol{Q}(t,\boldsymbol{\eta})})^T(\boldsymbol{\eta} - \hat{\boldsymbol{\eta}}), \end{aligned} \tag{7.14}$$

where $\hat{\boldsymbol{\eta}}$ and $\hat{\boldsymbol{X}}$ are the estimates of $\boldsymbol{\eta}$ and $\boldsymbol{X}$, respectively; $\boldsymbol{K_0} = \boldsymbol{K_0^T}$ is the positive diagonal observer gain matrix. From (7.13) and (7.14), we have

$$\begin{aligned}\dot{\tilde{\boldsymbol{\eta}}} &= \boldsymbol{J}(\eta)e^{-\boldsymbol{Q}(t,\boldsymbol{\eta})}\tilde{\boldsymbol{X}} - \boldsymbol{K_0}\tilde{\boldsymbol{\eta}},\\ \dot{\tilde{\boldsymbol{X}}} &= -e^{\boldsymbol{Q}(t,\boldsymbol{\eta})}\boldsymbol{M^{-1}D}e^{-\boldsymbol{Q}(t,\boldsymbol{\eta})}\tilde{\boldsymbol{X}} - (\boldsymbol{J}(\eta)e^{-\boldsymbol{Q}(t,\boldsymbol{\eta})})^T\tilde{\boldsymbol{\eta}},\end{aligned} \tag{7.15}$$

where $\tilde{\boldsymbol{\eta}} = \boldsymbol{\eta} - \hat{\boldsymbol{\eta}}$ and $\tilde{\boldsymbol{X}} = \boldsymbol{X} - \hat{\boldsymbol{X}}$. From (7.15), one can show that

$$\left\|(\tilde{\boldsymbol{\eta}}(t), \tilde{\boldsymbol{X}}(t))\right\| \leq \left\|(\tilde{\boldsymbol{\eta}}(t_0), \tilde{\boldsymbol{X}}(t_0))\right\| e^{-\sigma_0(t-t_0)},\ \forall\ 0 \leq t_0 \leq t < \infty, \tag{7.16}$$

with $\sigma_0 = \min\left(\lambda_{\min}(\boldsymbol{K_0}), \lambda_{\min}(\boldsymbol{M^{-1}D})\right)$, which in turn implies that (7.14) is a global exponential observer of (7.13). We define $\hat{\boldsymbol{v}} = [\hat{u},\ \hat{v},\ \hat{r}]^T$ being an estimate of the velocity vector $\boldsymbol{v}$ as

$$\hat{\boldsymbol{v}} = e^{-\boldsymbol{Q}(t,\boldsymbol{\eta})}\hat{\boldsymbol{X}}. \tag{7.17}$$

Using (7.17) and (7.14), we rewrite (7.1) in $(\boldsymbol{\eta}, \hat{\boldsymbol{v}})$ coordinates as

$$\begin{bmatrix}\dot{\boldsymbol{\eta}}\\ \dot{\hat{\boldsymbol{v}}}\end{bmatrix} = \begin{bmatrix}\boldsymbol{J}(\eta)\hat{\boldsymbol{v}}\\ -\boldsymbol{M^{-1}C}(\hat{\boldsymbol{v}})\hat{\boldsymbol{v}} - \boldsymbol{M^{-1}D}\hat{\boldsymbol{v}} + \boldsymbol{M^{-1}\tau}\end{bmatrix} + \begin{bmatrix}0_{3\times 3} & H_{12}\\ H_{21} & H_{22}\end{bmatrix}\begin{bmatrix}\tilde{\boldsymbol{\eta}}\\ \tilde{\boldsymbol{X}}\end{bmatrix}, \tag{7.18}$$

where

$$\begin{aligned}H_{12} &= \boldsymbol{J}(\eta)e^{-\boldsymbol{Q}(t,\boldsymbol{\eta})},\\ H_{21} &= e^{-\boldsymbol{Q}(t,\boldsymbol{\eta})}(\boldsymbol{J}(\eta)e^{-\boldsymbol{Q}(t,\boldsymbol{\eta})})^T,\\ H_{22} &= \boldsymbol{M^{-1}C}(\hat{\boldsymbol{v}} + e^{-\boldsymbol{Q}(t,\boldsymbol{\eta})}\tilde{\boldsymbol{X}})e^{-\boldsymbol{Q}(t,\boldsymbol{\eta})} - \boldsymbol{M^{-1}C}^*(\hat{\boldsymbol{v}})e^{-\boldsymbol{Q}(t,\boldsymbol{\eta})} +\\ &\quad e^{-\boldsymbol{Q}(t,\boldsymbol{\eta})}\boldsymbol{M^{-1}C}(\hat{\boldsymbol{v}} + e^{-\boldsymbol{Q}(t,\boldsymbol{\eta})}\tilde{\boldsymbol{X}}),\end{aligned} \tag{7.19}$$

with $\boldsymbol{C}^*(\hat{\boldsymbol{v}})$ being defined such that $\boldsymbol{C}^*(\hat{\boldsymbol{v}})\tilde{\boldsymbol{v}} = \boldsymbol{C}(\tilde{\boldsymbol{v}})\hat{\boldsymbol{v}}$. It is now observed that the systems (7.15) and (7.18) are in a cascaded structure. It is also observed that the system $(\tilde{\boldsymbol{\eta}}, \tilde{\boldsymbol{X}})$ is GES at the origin and that the connected terms H_{12}, H_{21}, and H_{22} are Lipschitz in $\boldsymbol{\eta}$ and $\hat{\boldsymbol{v}}$. Furthermore from (7.17) and (7.6), the velocity estimate error vector, $\tilde{\boldsymbol{v}} = [\tilde{u},\ \tilde{v},\ \tilde{r}]^T = \boldsymbol{v} - \hat{\boldsymbol{v}}$, satisfies

$$\tilde{\boldsymbol{v}} = e^{-\boldsymbol{Q}(t,\boldsymbol{\eta})}\tilde{\boldsymbol{X}}. \tag{7.20}$$

Since all elements of $e^{-\boldsymbol{Q}(t,\boldsymbol{\eta})}$ are bounded, (7.16) and (7.20) imply that there exists a positive constant γ_0 such that

$$\|(\tilde{\boldsymbol{\eta}}(t),\ \tilde{\boldsymbol{v}}(t))\| \leq \gamma_0 \|(\tilde{\boldsymbol{\eta}}(t_0),\ \tilde{\boldsymbol{v}}(t_0))\| e^{-\sigma_0(t-t_0)},\ \forall 0 \leq t_0 \leq t < \infty, \tag{7.21}$$

which means that the estimation errors $\tilde{\boldsymbol{\eta}}(t)$ and $\tilde{\boldsymbol{v}}(t)$ globally exponentially converge to the origin.

7.2.2 Coordinate Transformations

We now interpret the position and orientation errors $x-x_d$, $y-y_d$, and $\psi-\psi_d$ in a frame attached to the ship body. That is, we consider the error coordinates

$$\begin{bmatrix} x_e \\ y_e \\ \psi_e \end{bmatrix} = \boldsymbol{J}^{-1}(\eta) \begin{bmatrix} x-x_d \\ y-y_d \\ \psi-\psi_d \end{bmatrix},$$
$$\begin{bmatrix} u_e \\ v_e \\ r_e \end{bmatrix} = \begin{bmatrix} \hat{u}-u_d \\ \hat{v}-v_d \\ \hat{r}-r_d \end{bmatrix}. \tag{7.22}$$

Differentiating both sides of (7.22) along the solutions of (7.18) and (7.5) yields the error dynamics of the "kinematic part" in the transformed coordinates:

$$\begin{aligned} \dot{x}_e &= u_e - u_d(\cos(\psi_e)-1) - v_d\sin(\psi_e) + r_e y_e + r_d y_e + h_x, \\ \dot{y}_e &= v_e - v_d(\cos(\psi_e)-1) + u_d\sin(\psi_e) - r_e x_e - r_d x_e + h_y, \\ \dot{\psi}_e &= r_e + h_\psi \end{aligned} \tag{7.23}$$

where h_x, h_y, and h_ψ are the first, second, and third rows of $\boldsymbol{J}^{-1}(\eta)H_{12}\tilde{\boldsymbol{X}} + \tilde{r}\begin{bmatrix} y_e \\ -x_e \\ 0 \end{bmatrix}$, respectively.

By looking at (7.23), we see that x_e and ψ_e can be stabilized by u_e and r_e. There are several options to stabilize y_e, namely r_e, v_e, or ψ_e. If r_e is used, the control design will be extremely complicated since r_e enters all of the three equations of (7.23). On the other hand, the use of v_e to stabilize y_e will result in an undesired feature of ship control practice, namely the ship will slide in the sway direction. Hence we will use ψ_e to stabilize the sway error y_e. As such, we define the following coordinate transformation

$$z_e = \psi_e + \arcsin\left(\frac{k y_e}{\sqrt{c^2 + x_e^2 + y_e^2 + v_e^2}}\right), \tag{7.24}$$

where the constants k and c are such that $|k| < 1$ and $c \geq 1$ and will be specified later. It is seen that (7.24) is well defined and that convergence of z_e and y_e implies that of ψ_e. By using the nonlinear coordinate transformation (7.24) instead of a linear one like $z_e = \psi_e + k y_e$, we avoid the ship whirling around when y_e is large. The coordinate (7.24) is slightly different from the one in the preceding chapter. This will result in bounded virtual velocity controls. Using the nonlinear coordinate (7.24) together with (7.23), the ship error dynamics are rewritten as

$$\begin{aligned} \dot{x}_e =&\ u_e + u_d\varpi_2^{-1}(\varpi_2 - \varpi_1) + k v_d \varpi_2^{-1} y_e + r_e y_e + r_d y_e + p_x + h_x, \\ \dot{y}_e =&\ v_e + v_d\varpi_2^{-1}(\varpi_2 - \varpi_1) - k u_d \varpi_2^{-1} y_e - r_e x_e - r_d x_e + p_y + h_y, \end{aligned}$$

$$
\begin{aligned}
\dot{z}_e = & \left(1-k\varpi_1^{-1}x_e+k\beta\varpi_1^{-1}\varpi_2^{-2}y_e v_e(u_e+u_d)\right) r_e + \\
& k\varpi_1^{-1}\left(v_e+v_d\varpi_2^{-1}(\varpi_2-\varpi_1)-ku_d\varpi_2^{-1}y_e-r_d x_e- y_e\varpi_2^{-2}\times\right. \\
& x_e u_e+y_e v_e+(x_e u_d+y_e v_d)\varpi_2^{-1}(\varpi_2-\varpi_1)+(x_e v_d-y_e u_d)\big) \\
& \left(k\varpi_2^{-1}y_e-v_e(\alpha v_e+\beta u_e r_d)\right)\big)+p_z+h_z, \\
\dot{u}_e = & \frac{m_{22}}{m_{11}}\hat{v}\hat{r}-\frac{d_{11}}{m_{11}}\hat{u}+\frac{1}{m_{11}}\tau_u-\dot{u}_d+h_u, \\
\dot{v}_e = & -\frac{m_{11}}{m_{22}}u_e r_d-\frac{m_{11}}{m_{22}}(u_e+u_d)r_e-\frac{d_{22}}{m_{22}}v_e+h_v, \\
\dot{r}_e = & \frac{(m_{11}-m_{22})}{m_{33}}\hat{u}\hat{v}-\frac{d_{33}}{m_{33}}\hat{r}+\frac{1}{m_{33}}\tau_r-\dot{r}_d+h_r,
\end{aligned}
\tag{7.25}
$$

where h_u, h_v, and h_r are the first, second, and third rows of $H_{21}\tilde{\eta}+H_{22}\tilde{X}$, respectively. Also, for notational simplicity, we have defined

$$
\begin{aligned}
\varpi_1 = & \sqrt{c^2+x_e^2+(1-k^2)y_e^2+v_e^2}, \\
\varpi_2 = & \sqrt{c^2+x_e^2+y_e^2+v_e^2}, \\
\alpha = & \frac{d_{22}}{m_{22}}, \quad \beta=\frac{m_{11}}{m_{22}}, \\
p_x = & -u_d\left((\cos(z_e)-1)\varpi_1\varpi_2^{-1}+\sin(z_e)k\varpi_2^{-1}y_e\right)- \\
& v_d\left(\sin(z_e)\varpi_1\varpi_2^{-1}-(\cos(z_e)-1)k\varpi_2^{-1}y_e\right), \\
p_y = & -v_d\left((\cos(z_e)-1)\varpi_1\varpi_2^{-1}+\sin(z_e)k\varpi_2^{-1}y_e\right)+ \\
& u_d\left(\sin(z_e)\varpi_1\varpi_2^{-1}-(\cos(z_e)-1)k\varpi_2^{-1}y_e\right), \\
p_z = & k\varpi_1^{-1}\left(p_y-y_e\varpi_2^{-2}\left(x_e p_x+y_e p_y\right)\right), \\
h_z = & k\varpi_1^{-1}\left(h_y-y_e\varpi_2^{-1}\left(x_e h_x+y_e h_y+v_e h_v\right)\right)+h_\psi.
\end{aligned}
\tag{7.26}
$$

It is now clear that the problem of forcing the underactuated ship (7.1) to track the virtual ship (7.5) becomes one of stabilizing the system (7.25) at the origin.

7.2.3 Control Design

The triangular structure of (7.25) suggests that we design the actual controls τ_u and τ_r in two stages. First, we design the virtual velocity controls for u_e and r_e to globally asymptotically stabilize x_e, y_e, z_e and v_e at the origin. Based on the backstepping technique, the controls τ_u and τ_r will be then designed. It is noted that the term $\left(1-kx_e/\varpi_1+k\beta y_e v_e(u_e+u_d)/(\varpi_1\varpi_2^2)\right)$ in the z_e-dynamics may vanish and therefore might prevent a global design. This problem can be fixed by decomposing u_e and r_e as

$$\begin{aligned} u_e &= u_e^d + \tilde{u}_e, \\ r_e &= r_e^d + \tilde{r}_e, \end{aligned} \tag{7.27}$$

where u_e^d and r_e^d are the virtual velocity controls of u_e and r_e; $\tilde{u}_e$ and $\tilde{r}_e$ are the virtual control errors.

Step 1

In this step, the virtual surge and yaw velocity controls are chosen as

$$u_e^d = -\frac{k_1 x_e}{\varpi_2},\ r_e^d = r_{1e}^d + r_{2e}^d, \tag{7.28}$$

where

$$\begin{aligned} r_{1e}^d &= -\frac{1}{1 - kx_e/\varpi_1 + k\beta y_e v_e(u_e^d + u_d)/(\varpi_1 \varpi_2^2)} \left(\frac{k}{\varpi_1} \left(v_e + \frac{v_d}{\varpi_2} \times \right.\right. \\ &\quad (\varpi_2 - \varpi_1)) - \frac{k u_d y_e}{\varpi_2} - r_d x_e - \left(\frac{y_e}{\varpi_2} \left(x_e u_e^d + y_e v_e + (x_e u_d + \right.\right. \\ &\quad \left.\left.\left.\left. y_e v_d) \frac{(\varpi_2 - \varpi_1)}{\varpi_2^{-1}} + \ (x_e v_d - y_e u_d) \frac{k y_e}{\varpi_2} - v_e(\alpha v_e + \beta u_e^d r_d) \right) \right) \right), \\ r_{2e}^d &= -\frac{1}{1 - kx_e/\varpi_1 + k\beta y_e v_e(u_e^d + u_d)/(\varpi_1 \varpi_2^2)} \left(\frac{k_2 z_e}{\sqrt{1 + z_e^2}} + p_z \right), \end{aligned} \tag{7.29}$$

and k_i, $i = 1,\ 2$, are positive constants to be selected later. We have written r_e^d as a sum of r_{1e}^d and r_{2e}^d to simplify notation in the stability analysis later. Notice that

$$1 - \frac{kx_e}{\varpi_1} + \frac{k\beta y_e v_e(u_e^d + u_d)}{\varpi_1 \varpi_2^2} \geq 1 - |k| \left(1 + \frac{0.5\beta(k_1 + |u_d|)}{c} \right), \tag{7.30}$$

therefore r_{1e}^d and r_{2e}^d are well defined if the design constants k, c and k_1 are chosen such that

$$1 - |k|\,(1 + 0.5\beta(k_1 + |u_d|)/c) \geq k^* > 0. \tag{7.31}$$

By noting

$$\begin{aligned} |p_x| &\leq |u_d|\,(2 + |k|) + |v_d|\,(1 + 2\,|k|), \\ |p_y| &\leq |v_d|\,(2 + |k|) + |u_d|\,(1 + 2\,|k|), \\ |p_z| &\leq |k|\,(2\,|p_y| + |p_x|), \end{aligned} \tag{7.32}$$

we can show from (7.28) and (7.29) that u_e^d and r_e^d are bounded by some constants.

Remark 7.3. Unlike the standard application of the backstepping technique, in order to reduce complexity of the controller expressions, we have chosen a simple virtual control law u_e^d without canceling the known terms. From (7.29) and (7.28), we observe that r_{1e}^d is Lipschitz in (x_e, y_e, v_e) and r_{2e}^d vanishes when z_e does. This observation plays a crucial role in the stability analysis of the closed loop system.

Step 2

By differentiating (7.27) along the solutions of (7.25) and (7.29), the actual controls τ_u and τ_r without canceling the useful damping terms are chosen as

$$\begin{aligned}
\tau_u =& -m_{11}\Big(c_1\tilde{u}_e + m_{22}m_{11}^{-1}\hat{v}\hat{r} - d_{11}m_{11}^{-1}(u_e^d + u_d) - \dot{u}_d - \\
&\frac{\partial u_e^d}{\partial x_e}\big(u_e + u_d\,\varpi_2^{-1}(\varpi_2 - \varpi_1) + kv_d\varpi_2^{-1}y_e + r_e y_e + r_d y_e\big) - \\
&\frac{\partial u_e^d}{\partial y_e}\big(v_e + v_d\,\varpi_2^{-1}(\varpi_2 - \varpi_1) - ku_d\varpi_2^{-1}y_e - r_e x_e - r_d x_e\big) - \\
&\frac{\partial u_e^d}{\partial v_e}m_{22}^{-1}(-m_{11}u_e r_d - m_{11}(u_e + u_d)r_e - d_{22}v_e) + \\
&\varpi_1^{-1}\varpi_2^{-2}(k\beta y_e v_e r - ky_e(x_e - \beta r_d v_e))\,z_e\Big), \\
\tau_r =& -m_{33}\Big(c_2\tilde{r}_e + (m_{11} - m_{22})m_{33}^{-1}\hat{u}\hat{v} - d_{33}m_{33}^{-1}(r_e^d + r_d) - \dot{r}_d - \\
&\frac{\partial r_e^d}{\partial u_d}\dot{u}_d - \frac{\partial r_e^d}{\partial v_d}\dot{v}_d - \frac{\partial r_e^d}{\partial r_d}\dot{r}_d - \frac{\partial r_e^d}{\partial x_e}\big(u_e + u_d\,\varpi_2^{-1}(\varpi_2 - \varpi_1) + \\
&kv_d\varpi_2^{-1}y_e + r_e y_e + r_d y_e\big) - \frac{\partial r_e^d}{\partial y_e}\big(v_e + v_d\,\varpi_2^{-1}(\varpi_2 - \varpi_1) - \\
&ku_d\varpi_2^{-1}y_e - r_e x_e - r_d x_e\big) - \frac{\partial r_e^d}{\partial v_e}m_{22}^{-1}(-m_{11}u_e r_d - m_{11}(u_e + \\
&u_d)r_e - d_{22}v_e) - \frac{\partial r_e^d}{\partial \psi_e}r_e + \big(1 - k\varpi_1^{-1}x_e + k\beta\varpi_1^{-1}\varpi_2^{-2} \times \\
&y_e v_e(u_e^d + u_d)\big)\,z_e\Big),
\end{aligned} \tag{7.33}$$

where c_i, $i = 1, 2$, are positive constants. We now state the first main result of this chapter, the proof of which is given in the next section.

Theorem 7.1. *Under Assumption 7.1 assume the following:*

1. *There are no environmental disturbances*
2. *The ship parameters are known*
3. *The reference signals are generated by the virtual ship model (7.5) and the reference velocities satisfy Assumption 7.1.*

If the partial state feedback control law (7.33) together with the observer (7.18) are applied to the ship system (7.1), then the tracking errors $x(t) - x_d(t)$, $y(t) - y_d(t)$, $\psi(t) - \psi_d(t)$, *and* $v(t) - v_d(t)$ *globally asymptotically and locally exponentially* *converge to zero with an appropriate choice of the design constants c, k and k_i, $i = 1,2$. Furthermore, the virtual velocity controls, u_e^d and r_e^d, are bounded by some computable positive constants.*

7.2.4 Stability Analysis

Substituting (7.33) and (7.28) into (7.25) results in the following closed loop system:

$$\begin{aligned}\dot{\boldsymbol{X}}_{\boldsymbol{1e}} &= \boldsymbol{f_1}(t,\boldsymbol{X_e}) + \boldsymbol{g_1}(t,\boldsymbol{X_e}) + \boldsymbol{\phi_1}(t,\hat{\boldsymbol{v}},\boldsymbol{X_e},\boldsymbol{\eta},\tilde{\boldsymbol{\eta}},\tilde{\boldsymbol{X}}),\\ \dot{\boldsymbol{X}}_{\boldsymbol{2e}} &= \boldsymbol{f_2}(t,\boldsymbol{X_e}) + \boldsymbol{\phi_2}(t,\hat{\boldsymbol{v}},\boldsymbol{X_e},\boldsymbol{\eta},\tilde{\boldsymbol{\eta}},\tilde{\boldsymbol{X}}),\end{aligned} \tag{7.34}$$

where

$$\boldsymbol{X}_{\boldsymbol{1e}} = \begin{bmatrix} x_e \\ y_e \\ v_e \end{bmatrix},\ \boldsymbol{X}_{\boldsymbol{2e}} = \begin{bmatrix} z_e \\ \tilde{u}_e \\ \tilde{r}_e \end{bmatrix},\ \boldsymbol{X_e} = \begin{bmatrix} \boldsymbol{X}_{\boldsymbol{1e}} \\ \boldsymbol{X}_{\boldsymbol{2e}} \end{bmatrix},$$

$$\boldsymbol{f_1}(t,\boldsymbol{X_e}) = \begin{bmatrix} f_{11} \\ f_{12} \\ f_{13} \end{bmatrix},\ \boldsymbol{f_2}(t,\boldsymbol{X_e}) = \begin{bmatrix} f_{21} \\ f_{22} \\ f_{23} \end{bmatrix},\ \boldsymbol{g_1}(t,\boldsymbol{X_e}) = \begin{bmatrix} p_x + \tilde{u}_e \\ p_y \\ g_{13} \end{bmatrix},$$

$$\boldsymbol{\phi_1}(t,\hat{\boldsymbol{v}},\boldsymbol{X_e},\boldsymbol{\eta},\tilde{\boldsymbol{\eta}},\tilde{\boldsymbol{X}}) = \begin{bmatrix} h_x^* \\ h_y^* \\ h_v \end{bmatrix},\ \boldsymbol{\phi_2}(t,\hat{\boldsymbol{v}},\boldsymbol{X_e},\boldsymbol{\eta},\tilde{\boldsymbol{\eta}},\tilde{\boldsymbol{X}}) = \begin{bmatrix} h_z \\ \varphi_{22} \\ \varphi_{23} \end{bmatrix},$$

$$\begin{aligned}
f_{11} &= -\frac{k_1 x_e}{\varpi_2} + u_d\,\frac{\varpi_2 - \varpi_1}{\varpi_2} + \frac{k v_d\, y_e}{\varpi_2} + y_e(r_e + r_d + \tilde{r}),\\
f_{12} &= v_e + v_d\,\frac{\varpi_2 - \varpi_1}{\varpi_2} - \frac{k u_d\, y_e}{\varpi_2} - x_e(r_e + r_d + \tilde{r}),\\
f_{13} &= -\alpha\, v_e - \beta\, u_e^d r_d - \beta\,(u_e^d + u_d) r_{1e}^d,\\
g_{13} &= -\beta\left(\tilde{u}_e(r_e^d + \tilde{r}_e + r_d) + (u_e^d + u_d)(r_{2e}^d + \tilde{r}_e)\right),\\
f_{21} &= -\frac{k_2 z_e}{\sqrt{1+z_e^2}} + \left(1 - \frac{k x_e}{\varpi_1} + \frac{k\beta y_e v_e (u_e^d + u_d)}{\varpi_1 \varpi_2^2}\right)\tilde{r}_e + \\
&\left(\frac{k\beta y_e v_e r_e}{\varpi_1 \varpi_2^2} - \frac{k y_e}{\varpi_1 \varpi_2^2}(x_e - \beta r_d v_e)\right)\tilde{u}_e,\\
f_{22} &= -\left(c_1 + \frac{d_{11}}{m_{11}}\right)\tilde{u}_e - \left(\frac{k\beta y_e v_e r_e}{\varpi_1 \varpi_2^2} - \frac{k y_e}{\varpi_1 \varpi_2^2}(x_e - \beta r_d v_e)\right) z_e,\\
f_{23} &= -\left(c_2 + \frac{d_{33}}{m_{33}}\right)\tilde{r}_e - \left(1 - \frac{k x_e}{\varpi_1} + \frac{k\beta y_e v_e (u_e^d + u_d)}{\varpi_1 \varpi_2^2}\right) z_e,
\end{aligned}$$

$$\varphi_{22} = h_u - \frac{\partial u_e^d}{\partial x_e}(p_x + h_x) - \frac{\partial u_e^d}{\partial y_e}\left(p_y + h_y\right) - \frac{\partial u_e^d}{\partial v_e} h_v,$$
$$\varphi_{23} = h_r - \frac{\partial r_e^d}{\partial x_e}(p_x + h_x) - \frac{\partial r_e^d}{\partial y_e}\left(p_y + h_y\right) - \frac{\partial r_e^d}{\partial v_e} h_v - \frac{\partial r_e^d}{\partial \psi_e} h_\psi, \quad (7.35)$$

with h_x^* and h_y^* being the first and second rows of $\boldsymbol{J}^{-1}(\boldsymbol{\eta})H_{12}\tilde{\boldsymbol{X}}$. The time dependence of $\boldsymbol{f_i}(t, \boldsymbol{X_e})$, $\boldsymbol{g_1}(t, \boldsymbol{X_e})$, and $\boldsymbol{\phi_i}(t, \hat{\boldsymbol{v}}, \boldsymbol{X_e}, \boldsymbol{\eta}, \tilde{\boldsymbol{\eta}}, \tilde{\boldsymbol{X}})$, $i = 1,2$, is due to the time-varying reference velocities. Observe that the closed loop system (7.34) consists of the $(\boldsymbol{X_{1e}}, \boldsymbol{X_{2e}})$-subsystem and $(\tilde{\boldsymbol{\eta}}, \tilde{\boldsymbol{X}})$-subsystem (see (7.15)) in a cascaded structure. From (7.35) it can be readily shown that the connected terms $\boldsymbol{\phi_i}(t, \hat{\boldsymbol{v}}, \boldsymbol{X_e}, \boldsymbol{\eta}, \tilde{\boldsymbol{\eta}}, \tilde{\boldsymbol{X}})$, $i = 1,2$, satisfy

$$\left\|\boldsymbol{\phi_i}(t, \hat{\boldsymbol{v}}, \boldsymbol{X_e}, \boldsymbol{\eta}, \tilde{\boldsymbol{\eta}}, \tilde{\boldsymbol{X}})\right\| \le \phi_i(\hat{\boldsymbol{v}}, \boldsymbol{\eta})\left\|(\tilde{\boldsymbol{\eta}}, \tilde{\boldsymbol{X}})\right\|, \quad (7.36)$$

where the functions $\phi_i(\hat{\boldsymbol{v}}, \boldsymbol{\eta})$ are Lipschitz in $\hat{\boldsymbol{v}}$ and are bounded with respect to any $\boldsymbol{\eta}$. Also we note that by definition $\hat{\mathbf{v}} = [\hat{u}, \hat{v}, \hat{r}]^T = [u_e^d + u_d + \tilde{u}_e, v_e + v_d, r_e^d + r_d + \tilde{r}_e]^T$, the $(\tilde{\boldsymbol{\eta}}, \tilde{\boldsymbol{X}})$-subsystem is GES at the origin, and the reference velocities u_d, v_d and r_d are bounded. On the other hand, the virtual velocity controls, u_e^d and r_e^d are bounded. Hence, using the recent stability results for cascade systems given in [17,69], we need to show that there exist the design constants c, k, k_1, and k_2 such that the $(\boldsymbol{X_{1e}}, \boldsymbol{X_{2e}})$-subsystem without the connected terms $\boldsymbol{\phi_i}(t, \hat{\boldsymbol{v}}, \boldsymbol{\eta}, \tilde{\boldsymbol{\eta}}, \tilde{\boldsymbol{X}})$, $i = 1,2$, is GAS at the origin. That is why we did not include some nonlinear damping terms in the control law (7.33). From the above discussions, we will study the system given by

$$\begin{aligned} \dot{\boldsymbol{X}}_{\boldsymbol{1e}} &= \boldsymbol{f_1}(t, \boldsymbol{X_e}) + \boldsymbol{g_1}(t, \boldsymbol{X_e}), \\ \dot{\boldsymbol{X}}_{\boldsymbol{2e}} &= \boldsymbol{f_2}(t, \boldsymbol{X_e}). \end{aligned} \quad (7.37)$$

To further simplify the investigation of stability of the system (7.37), we note that this system also consists of the $\boldsymbol{X_{1e}}$-subsystem and the $\boldsymbol{X_{2e}}$-subsystem in a cascaded structure. From (7.35), it is not hard to show that the connected term $\boldsymbol{g_1}(t, \boldsymbol{X_e})$ satisfies $\|\boldsymbol{g_1}(t, \boldsymbol{X_e})\| \le \kappa_1 \|\boldsymbol{X_{2e}}\|$ with κ_1 being some positive constant. Therefore global stability of $\dot{\boldsymbol{X}}_{\boldsymbol{1e}} = \boldsymbol{f_1}(t, \boldsymbol{X_e})$ and $\dot{\boldsymbol{X}}_{\boldsymbol{2e}} = \boldsymbol{f_2}(t, \boldsymbol{X_e})$ implies that of (7.37). We will first study stability of the subsystem $\dot{\boldsymbol{X}}_{\boldsymbol{2e}} = \boldsymbol{f_2}(t, \boldsymbol{X_e})$ then move to the subsystem $\dot{\boldsymbol{X}}_{\boldsymbol{1e}} = \boldsymbol{f_1}(t, \boldsymbol{X_e})$.

Subsystem $\dot{\boldsymbol{X}}_{\boldsymbol{2e}} = \boldsymbol{f_2}(t, \boldsymbol{X_e})$. By differentiating $V_1 = 0.5\left(z_e^2 + \tilde{u}_e^2 + \tilde{r}_e^2\right)$ along the solutions of $\dot{\boldsymbol{X}}_{\boldsymbol{2e}} = \boldsymbol{f_2}(t, \boldsymbol{X_e})$, one can show that this system is globally asymptotically and locally exponentially stable at the origin for any constants $k_2 > 0$, $c_1 \ge 0$, and $c_2 \ge 0$.

Subsystem $\dot{\boldsymbol{X}}_{\boldsymbol{1e}} = \boldsymbol{f_1}(t, \boldsymbol{X_e})$. To investigate the stability of this subsystem, we take the Lyapunov function

$$V_2 = \sqrt{c^2 + x_e^2 + y_e^2 + v_e^2} + \frac{1}{2}k_3 v_e^2 - c, \tag{7.38}$$

where k_3 is a positive constant to be selected later. After some lengthy but simple calculation using the completing squares, the time derivative of (7.38) along the solutions of $\dot{\boldsymbol{X}}_{\mathbf{1e}} = \boldsymbol{f}_{\mathbf{1}}(t, \boldsymbol{X}_{\boldsymbol{e}})$ satisfies

$$\dot{V}_2 \leq -p_1(t)x_e^2\varpi_2^{-2} - p_2(t)y_e^2\varpi_2^{-2} - p_3(t)v_e^2, \tag{7.39}$$

with

$$\begin{aligned}
p_1(t) &= k_1 - \beta(k_3 + 1/c)k_1 |r_d|/(4\varepsilon_3) - [|kv_d|/(4\varepsilon_1) + \beta(k_3 + 1/c) \times \\
&\quad (k_1 + |u_d|)|k|(|r_d|/(4\varepsilon_3(1-k^2)) + k_1/(2c\sqrt{1-k^2}))/k^*\Big], \\
p_2(t) &= ku_d - 1/(4\varepsilon_2) - \big[k^2(0.5|u_d| + |v_d|)/c + \varepsilon_1 |kv_d| + \beta(k_3 + 1/c)\times \\
&\quad (k_1 + |u_d|)|k|\big(k^2|v_d| + (k^2 + |k|)(|u_d| + |v_d|) + |ku_d|/(c4\varepsilon_3)\big)/k^*\big], \\
p_3(t) &= \alpha k_3 - \varepsilon_2 - \beta(k_3 + 1/c)k_1 |r_d|\varepsilon_3 - [\beta(k_3 + 1/c)(k_1 + |u_d|)|k| \times \\
&\quad (2/c + |ku_d|\varepsilon_3/c + \varepsilon_3|r_d| + 0.5\alpha/c + \beta k_1|r_d|/c^2)/k^*\big],
\end{aligned} \tag{7.40}$$

where ε_i, $1 \leq i \leq 3$ are positive constants. Hence the subsystem $\dot{\boldsymbol{X}}_{\mathbf{1e}} = \boldsymbol{f}_{\mathbf{1}}(t, \boldsymbol{X}_{\boldsymbol{e}})$ is GAS at the origin if the design constants are chosen such that

$$p_i(t) \geq p_i^* \tag{7.41}$$

for some positive constants $p_i^*, i = 1,2,3$. In summary, we need to choose the constants c, k, k_1, and k_3 such that they satisfy (7.31) and (7.41). Note that the condition (7.31) automatically implies that $|k| < 1$ is required in (7.24). In the next section, we will show that under Assumption 7.1, there always exist the constants c, k, k_1, and k_3 such that (7.31) and (7.41) hold.

7.2.5 Selection of Design Constants

To choose the design constants c, k, k_1, and k_3, we observe the following: First, it is noted from the expression of $p_2(t)$ that the sign of constant k must have the same sign as that of the reference surge velocity, u_d (this sign does not change under Assumption 7.1). Second, it is observed that the condition (7.31) can be rewritten in the form of

$$1 - |k|\big(1 + 0.5\beta(k_1 + u_d^{\max})/c\big) \geq k^* > 0, \tag{7.42}$$

which implies that for each fixed $k^* < 1$, $k_1 > 0$, $u_d^{\max}$, $|k| < k^*$, we can always pick a large enough constant c such that (7.42) holds. Third, under Assumption 7.1, the magnitude of the reference sway velocity is always less than that of the reference surge velocity. Fourth, the mass including added mass in the sway dynamics, m_{22}, is always larger than that in the surge dynamics, m_{11}, for surface ships, i.e., $\beta < 1$.

Finally, all of the terms in the square brackets in $p_i, i = 1,2,3$ have the constant k as a factor. These terms also decrease when the constant c increases. Looking closely at p_i with the above observations, if the constant k is chosen small enough and the constant c is selected large enough, then we can pick a positive constant k_1 such that (7.31) and (7.41) hold with some large enough k_3. It is noted that $|k|$ should be decreased and c should be increased if u_d is large. This physically means that the distance from the ship to the point it aims to track should be increased if the velocities and surge acceleration are large, otherwise the ship will miss that point. Furthermore when α is small, $|k|$ and k_1 should be decreased, and c should be increased. This can be physically interpreted as follows: If the damping in the sway dynamics is small, the control gain in the surge dynamics should also be small otherwise the ship will slide in the sway direction. Due to complicated expressions of $p_i(t)$, we provide some general guidelines to choose the design constants rather than present their extremely complex explicit expressions: Pick $k^* < 1$, small values for $|k|$ and k_1, larger values for c and k_3. Then increase c and k_3 until (7.31) and (7.41) hold.

7.3 Output Feedback

7.3.1 Observer Design

We now introduce a more general coordinate change than (7.6) to cancel the term $\boldsymbol{C}(\boldsymbol{v})\boldsymbol{v}$ in (7.1) as follows:

$$\boldsymbol{X} = \boldsymbol{P}(\boldsymbol{\eta})\boldsymbol{v}, \tag{7.43}$$

where $\boldsymbol{P}(\boldsymbol{\eta}) \in \mathbb{R}^{3\times3}$ is a global invertible matrix to be determined. With (7.43), the second equation of (7.1) is written as

$$\dot{\boldsymbol{X}} = \left[\dot{\boldsymbol{P}}(\boldsymbol{\eta})\boldsymbol{v} - \boldsymbol{P}(\boldsymbol{\eta})\boldsymbol{M}^{-1}\boldsymbol{C}(\boldsymbol{v})\boldsymbol{v}\right] - \boldsymbol{P}(\boldsymbol{\eta})\boldsymbol{M}^{-1}\boldsymbol{D}\boldsymbol{P}^{-1}(\boldsymbol{\eta})\boldsymbol{X} + \boldsymbol{P}(\boldsymbol{\eta})\boldsymbol{M}^{-1}\boldsymbol{\tau}. \tag{7.44}$$

Our goal is to cancel the terms in the square bracket on the right-hand side of (7.44). Assuming that the elements of $\boldsymbol{P}(\boldsymbol{\eta})$ are $p_{ij}(\boldsymbol{\eta}), i = 1,2,3, j = 1,2,3$, the first bracket in the right-hand side of (7.44) is zero if

$$\begin{aligned}&\dot{p}_{i1}u + \dot{p}_{i2}v + \dot{p}_{i3}r + \frac{m_{22}}{m_{11}}p_{i1}vr - \frac{m_{11}}{m_{22}}p_{i2}ur + \frac{m_{11}-m_{22}}{m_{33}}p_{i3}uv = 0,\\ &i = 1,2,3,\ \forall(\boldsymbol{\eta}, u, v, r) \in \mathbb{R}^6,\end{aligned} \tag{7.45}$$

where for brevity, we omit the argument $\boldsymbol{\eta}$ of $p_{ij}(\boldsymbol{\eta})$. With the first equation of (7.1), we expand (7.45) as

$$\left(\frac{\partial p_{i1}}{\partial x}\cos(\psi)+\frac{\partial p_{i1}}{\partial y}\sin(\psi)\right)u^2+\left(-\frac{\partial p_{i2}}{\partial x}\sin(\psi)+\frac{\partial p_{i2}}{\partial y}\cos(\psi)\right)v^2+$$
$$\frac{\partial p_{i3}}{\partial \psi}r^2+\left(-\frac{\partial p_{i1}}{\partial x}\sin(\psi)+\frac{\partial p_{i1}}{\partial y}\cos(\psi)+\frac{\partial p_{i2}}{\partial x}\cos(\psi)+\frac{\partial p_{i2}}{\partial y}\sin(\psi)+\right.$$
$$\left.\frac{m_{11}-m_{22}}{m_{33}}p_{i3}\right)uv+\left(\frac{\partial p_{i1}}{\partial \psi}+\frac{\partial p_{i3}}{\partial x}\cos(\psi)+\frac{\partial p_{i3}}{\partial y}\sin(\psi)-\frac{m_{11}}{m_{22}}p_{i2}\right)ur+$$
$$\left(\frac{\partial p_{i2}}{\partial \psi}-\frac{\partial p_{i3}}{\partial x}\sin(\psi)+\frac{\partial p_{i3}}{\partial y}\cos(\psi)+\frac{m_{22}}{m_{11}}p_{i1}\right)vr=0. \tag{7.46}$$

Therefore (7.46) holds for all $(\boldsymbol{\eta},u,v,r)\in\mathbb{R}^6$ if

$$\begin{aligned}
&\frac{\partial p_{i1}}{\partial x}\cos(\psi)+\frac{\partial p_{i1}}{\partial y}\sin(\psi)=0,\\
&-\frac{\partial p_{i2}}{\partial x}\sin(\psi)+\frac{\partial p_{i2}}{\partial y}\cos(\psi)=0,\\
&\frac{\partial p_{i3}}{\partial \psi}=0,\\
&\left(\frac{\partial p_{i2}}{\partial y}-\frac{\partial p_{i1}}{\partial x}\right)\sin(\psi)+\left(\frac{\partial p_{i1}}{\partial y}+\frac{\partial p_{i2}}{\partial x}\right)\cos(\psi)+\frac{m_{11}-m_{22}}{m_{33}}p_{i3}=0,\\
&\frac{\partial p_{i1}}{\partial \psi}+\frac{\partial p_{i3}}{\partial x}\cos(\psi)+\frac{\partial p_{i3}}{\partial y}\sin(\psi)-\frac{m_{11}}{m_{22}}p_{i2}=0,\\
&\frac{\partial p_{i2}}{\partial \psi}-\frac{\partial p_{i3}}{\partial x}\sin(\psi)+\frac{\partial p_{i3}}{\partial y}\cos(\psi)+\frac{m_{22}}{m_{11}}p_{i1}=0.
\end{aligned} \tag{7.47}$$

A family of solutions of the above set of six partial differential equations is

$$\begin{aligned}
p_{i1}&=\frac{((m_{11}C_{i3}x+m_{33}C_{i1})\sin(\psi)-(m_{11}C_{i3}y-m_{33}C_{i2})\cos(\psi))}{m_{33}},\\
p_{i2}&=\frac{m_{22}\left((m_{11}C_{i3}x+m_{33}C_{i1})\cos(\psi)+(m_{11}C_{i3}y-m_{33}C_{i2})\sin(\psi)\right)}{m_{11}m_{33}},\\
p_{i3}&=C_{i3},
\end{aligned} \tag{7.48}$$

where C_{i1}, C_{i2} and C_{i3} are arbitrary constants. It is noted that the above solutions can be obtained by the following Maple™ code:

```
>PDE1:=diff(p11(x,y,\psi),x)*cos(\psi)+
   diff(p11(x,y,\psi),y)*sin(\psi)=0,
   -diff(p12(x,y,\psi),x)*sin(\psi)+
   diff(p12(x,y,\psi),y)*cos(\psi)=0,
   diff(p13(x,y,\psi),\psi)=0,
   (diff(p12(x,y,\psi),y)-diff(p11(x,y,\psi),x))*
   sin(\psi)+(diff(p11(x,y,\psi),y)+diff(p12(x,y,
   \psi),x))*cos(\psi)+(m11-m22)/m33*p13(x,y,\psi)=0,
   diff(p11(x,y,\psi),\psi)+diff(p13(x,y,\psi),x)*
```

```
    cos(\psi)+diff(p13(x,y,\psi),y)*sin(\psi)-
    m11/m22*p12(x,y,\psi)=0,
    diff(p12(x,y,\psi),\psi)-diff(p13(x,y,\psi),x)*
    sin(\psi)+diff(p13(x,y,\psi),y)*cos(\psi)+
    m22/m11*p11(x,y,\psi)=0;


>PDE2:=diff(p21(x,y,\psi),x)*cos(\psi)+
    diff(p21(x,y,\psi),y)*sin(\psi)=0,
    -diff(p22(x,y,\psi),x)*sin(\psi)+
    diff(p22(x,y,\psi),y)*cos(\psi)=0,
    diff(p23(x,y,\psi),\psi)=0,
    (diff(p22(x,y,\psi),y)-diff(p21(x,y,\psi),x))*
    sin(\psi)+(diff(p21(x,y,\psi),y)+diff(p22(x,y,
    \psi),x))*cos(\psi)+(m11-m22)/m33*p23(x,y,\psi)=0,
    diff(p21(x,y,\psi),\psi)+diff(p23(x,y,\psi),x)*
    cos(\psi)+diff(p23(x,y,\psi),y)*sin(\psi)-
    m11/m22*p22(x,y,\psi)=0,
    diff(p22(x,y,\psi),\psi)-diff(p23(x,y,\psi),x)*
    sin(\psi)+diff(p23(x,y,\psi),y)*cos(\psi)+
    m22/m11*p21(x,y,\psi)=0;


>PDE3:=diff(p31(x,y,\psi),x)*cos(\psi)+
    diff(p31(x,y,\psi),y)*sin(\psi)=0,
    -diff(p32(x,y,\psi),x)*sin(\psi)+
    diff(p32(x,y,\psi),y)*cos(\psi)=0,
    diff(p33(x,y,\psi),\psi)=0,
    (diff(p32(x,y,\psi),y)-diff(p31(x,y,\psi),x))*
    sin(\psi)+(diff(p31(x,y,\psi),y)+diff(p32(x,y,
    \psi),x))*cos(\psi)+(m11-m22)/m33*p33(x,y,\psi)=0,
    diff(p31(x,y,\psi),\psi)+diff(p33(x,y,\psi),x)*
    cos(\psi)+diff(p33(x,y,\psi),y)*sin(\psi)-
    m11/m22*p32(x,y,\psi)=0,
    diff(p32(x,y,\psi),\psi)-diff(p33(x,y,\psi),x)*
    sin(\psi)+diff(p33(x,y,\psi),y)*cos(\psi)+
    m22/m11*p31(x,y,\psi)=0;

>solutions:=pdsolve([PDE1, PDE2, PDE3]);
```

We now choose the constants C_{i1}, C_{i2}, and C_{i3} such that the matrix $\boldsymbol{P}(\boldsymbol{\eta})$ is invertible. A choice of $C_{13} = C_{11} = 0$, $C_{12} = 1$, $C_{23} = C_{22} = 0$, $C_{21} = 1$, $C_{31} = C_{32} = 0$, and $C_{33} = 1$ results in

$$\boldsymbol{P}(\eta) = \begin{bmatrix} \cos(\psi) & -\frac{m_{22}\sin(\psi)}{m_{11}} & 0 \\ \sin(\psi) & \frac{m_{22}\cos(\psi)}{m_{11}} & 0 \\ \frac{m_{11}(\sin(\psi)x - \cos(\psi)y)}{m_{33}} & \frac{m_{22}(\cos(\psi)x + \sin(\psi)y)}{m_{33}} & 1 \end{bmatrix}. \tag{7.49}$$

Substituting (7.49) into (7.44) and using (7.43) and the first equation of (7.1), we have

$$\begin{aligned} \dot{\boldsymbol{\eta}} &= \boldsymbol{J}(\eta)\boldsymbol{P}^{-1}(\eta)\boldsymbol{X}, \\ \dot{\boldsymbol{X}} &= -\boldsymbol{D}_{\eta}(\eta)\boldsymbol{X} + \boldsymbol{P}(\eta)\boldsymbol{M}^{-1}\boldsymbol{\tau}, \end{aligned} \tag{7.50}$$

with $\boldsymbol{D}_{\eta}(\eta) = \boldsymbol{P}(\eta)\boldsymbol{M}^{-1}\boldsymbol{D}\boldsymbol{P}^{-1}(\eta)$. It can be seen that the matrix $\boldsymbol{P}(\eta)$ given in (7.49) does not use any ship velocities. This feature results in the main difference between the partial-state feedback design in Section 7.2 and the output feedback design in this section. From (7.50), we use the following full-order nonlinear observer to construct the unmeasured ship velocities:

$$\begin{aligned} \dot{\hat{\boldsymbol{\eta}}} &= \boldsymbol{J}(\eta)\boldsymbol{P}^{-1}(\eta)\hat{\boldsymbol{X}} + \boldsymbol{K}_{01}(\eta - \hat{\eta}), \\ \dot{\hat{\boldsymbol{X}}} &= -\boldsymbol{D}_{\eta}(\eta)\hat{\boldsymbol{X}} + \boldsymbol{P}(\eta)\boldsymbol{M}^{-1}\boldsymbol{\tau} + \boldsymbol{K}_{02}(\eta - \hat{\eta}), \end{aligned} \tag{7.51}$$

where $\hat{\boldsymbol{\eta}}$ and $\hat{\boldsymbol{X}}$ are the estimates of $\boldsymbol{\eta}$ and $\boldsymbol{X}$, respectively. The observer gain matrices $\boldsymbol{K}_{01}$ and $\boldsymbol{K}_{02}$ are chosen such that

$$\begin{aligned} \boldsymbol{Q}_{01} &= \boldsymbol{K}_{01}^{T}\boldsymbol{P}_{01} + \boldsymbol{P}_{01}\boldsymbol{K}_{01}, \\ \boldsymbol{Q}_{02} &= \boldsymbol{D}_{\eta}^{T}(\eta)\boldsymbol{P}_{02} + \boldsymbol{P}_{02}\boldsymbol{D}_{\eta}(\eta) \end{aligned}$$

are positive definite, and that

$$(\boldsymbol{J}(\eta)\boldsymbol{P}^{-1}(\eta))^{T}\boldsymbol{P}_{01} - \boldsymbol{P}_{02}\boldsymbol{K}_{02} = 0, \tag{7.52}$$

with $\boldsymbol{P}_{01}$ and $\boldsymbol{P}_{02}$ being positive definite matrices. It is straightforward to show that $\boldsymbol{K}_{01}$ and $\boldsymbol{K}_{02}$ always exist since $\boldsymbol{D}_{\eta}(\eta)$ is positive definite. From (7.50) and (7.51), we have

$$\begin{aligned} \dot{\tilde{\boldsymbol{\eta}}} &= \boldsymbol{J}(\eta)\boldsymbol{P}^{-1}(\eta)\tilde{\boldsymbol{X}} - \boldsymbol{K}_{01}\tilde{\boldsymbol{\eta}}, \\ \dot{\tilde{\boldsymbol{X}}} &= -\boldsymbol{D}_{\eta}(\eta)\tilde{\boldsymbol{X}} - \boldsymbol{K}_{02}\tilde{\boldsymbol{\eta}}, \end{aligned} \tag{7.53}$$

where $\tilde{\boldsymbol{\eta}} := (\tilde{x}, \tilde{y}, \tilde{\psi})^{T} = \boldsymbol{\eta} - \hat{\boldsymbol{\eta}}$ and $\tilde{\boldsymbol{X}} := (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)^{T} = \boldsymbol{X} - \hat{\boldsymbol{X}}$. From (7.53) and (7.52), we can show that there exist strictly positive constants γ_0 and σ_0 such that

$$\left\| (\tilde{\boldsymbol{\eta}}(t), \tilde{\boldsymbol{X}}(t)) \right\| \le \gamma_0 \left\| (\tilde{\boldsymbol{\eta}}(t_0), \tilde{\boldsymbol{X}}(t_0)) \right\| e^{-\sigma_0(t - t_0)}, \ \forall\, t \ge t_0 \ge 0. \tag{7.54}$$

Define $\hat{\boldsymbol{v}} = [\hat{u}, \hat{v}, \hat{r}]^T$ being an estimate of the velocity vector $\boldsymbol{v}$ as

$$\hat{\boldsymbol{v}} = \boldsymbol{P}^{-1}(\boldsymbol{\eta})\hat{\boldsymbol{X}}. \tag{7.55}$$

Then the velocity estimate error vector, $\tilde{\boldsymbol{v}} := [\tilde{u}, \tilde{v}, \tilde{r}]^T = \boldsymbol{v} - \hat{\boldsymbol{v}}$, satisfies

$$\tilde{\boldsymbol{v}} = \boldsymbol{P}^{-1}(\boldsymbol{\eta})\tilde{\boldsymbol{X}}. \tag{7.56}$$

Based on (7.56), we cannot conclude anything about the convergence of the velocity estimate errors since some elements of the matrix $\boldsymbol{P}(\boldsymbol{\eta})$, see (7.49), depend linearly on x and y. However, our controller will guarantee that (x, y) are globally bounded. Then (7.56) implies that the velocity estimate errors globally exponentially converge to zero. Indeed, the linear dependence of $\boldsymbol{P}(\boldsymbol{\eta})$ on x and y will result in a challenging problem, which the control design will have to take care of. To prepare for the control design, using (7.55), we rewrite (7.51) as

$$\begin{aligned}
\dot{x} &= \hat{u}\cos(\psi) - \hat{v}\sin(\psi) + \left(\cos^2(\psi) + m_{11}m_{22}^{-1}\sin^2(\psi)\right)\tilde{x}_1 + \\
&\quad 0.5(m_{22} - m_{11})m_{22}^{-1}\sin(2\psi)\tilde{x}_2, \\
\dot{y} &= \hat{u}\sin(\psi) + \hat{v}\cos(\psi) + 0.5(m_{22} - m_{11})m_{22}^{-1}\sin(2\psi)\tilde{x}_1 + \\
&\quad \left(\sin^2(\psi) + m_{11}m_{22}^{-1}\cos^2(\psi)\right)\tilde{x}_2, \\
\dot{\psi} &= \hat{r} + m_{11}m_{33}^{-1}(y\tilde{x}_1 - x\tilde{x}_2) + \tilde{x}_3, \\
\dot{\hat{u}} &= m_{22}m_{11}^{-1}\hat{v}\hat{r} - d_{11}m_{11}^{-1}\hat{u} + m_{11}^{-1}\tau_u + \hat{v}\left(m_{22}m_{33}^{-1}(y\tilde{x}_1 - x\tilde{x}_2) + \right. \\
&\quad \left. m_{22}m_{11}^{-1}\tilde{x}_3\right) + \cos(\psi)\tilde{x} + \sin(\psi)\tilde{y} - m_{11}m_{33}^{-1}(\sin(\psi)x - \cos(\psi)y)\tilde{\psi}, \\
\dot{\hat{v}} &= -m_{11}m_{22}^{-1}\hat{u}\hat{r} - d_{22}m_{22}^{-1}\hat{v} - \hat{u}m_{11}m_{22}^{-1}\left(m_{11}m_{33}^{-1}(y\tilde{x}_1 - x\tilde{x}_2) + \tilde{x}_3\right) - \\
&\quad m_{11}^2m_{22}^{-2}(\sin(\psi)\tilde{x} - \cos(\psi)\tilde{y}) - m_{11}^2m_{22}^{-1}m_{33}^{-1}(\cos(\psi)x + \sin(\psi)y)\tilde{\psi}, \\
\dot{\hat{r}} &= (m_{11} - m_{22})m_{33}^{-1}\hat{u}\hat{v} - d_{33}m_{33}^{-1}\hat{r} + m_{33}^{-1}\tau_r + m_{11}^2m_{22}^{-1}m_{33}^{-1}\hat{u} \times \\
&\quad (-\sin(\psi)\tilde{x}_1 + \cos(\psi)\tilde{x}_2) - m_{22}m_{33}^{-1}\hat{v} \times (\cos(\psi)\tilde{x}_1 + \\
&\quad \sin(\psi)\tilde{x}_2) + (m_{11}^2 + m_{33}^2)m_{33}^{-2}\left(x^2 + y^2\right)\tilde{\psi} + m_{22}^{-1}m_{33}^{-1} \times \\
&\quad \left(\left(0.5(m_{11}^2 - m_{11}m_{22})\sin(2\psi)x + (m_{11}m_{22}\cos^2(\psi) + \right.\right. \\
&\quad \left. m_{11}^2\sin^2(\psi))y\right)\tilde{x} + \left(0.5(m_{11}^2 - m_{11}m_{22})\sin(2\psi)y - \right. \\
&\quad \left.\left. (m_{11}m_{22}\sin^2(\psi) + m_{11}^2\cos^2(\psi))x\right)\tilde{y}\right),
\end{aligned} \tag{7.57}$$

where for simplicity, we have taken $\boldsymbol{K_{02}} = (\boldsymbol{J}(\boldsymbol{\eta})\boldsymbol{P}^{-1}(\boldsymbol{\eta}))^{\boldsymbol{T}}$.

7.3.2 Coordinate Transformations

If one applies the coordinate change (7.22) to (7.57), it will result in a very complicated system, namely some quadratic terms of (x_e, y_e, ψ_e) multiplied by the observer errors appearing in the kinematic part of the transformed system due to linear dependence of x and y on some elements of the matrix $\boldsymbol{P}(\boldsymbol{\eta})$. This makes the control design extremely difficult and might result in a finite escape. To avoid the said difficulty, we propose the following coordinate transformation:

$$\begin{bmatrix} x_e \\ y_e \\ \psi_e \end{bmatrix} = \boldsymbol{J}^{-1}(\boldsymbol{\eta_d}) \begin{bmatrix} x - x_d \\ y - y_d \\ \psi - \psi_d \end{bmatrix}, \begin{cases} u_e = \hat{u} - u_d, \\ v_e = \hat{v} - v_d, \\ r_e = \hat{r} - r_d. \end{cases} \tag{7.58}$$

Indeed, convergence to zero of (x_e, y_e, ψ_e) implies that of $(x - x_d, y - y_d, \psi - \psi_d)$. Differentiating both sides of (7.58) along the solutions of (7.57) and (7.5) yields the error dynamics of the "kinematic part":

$$\begin{aligned} \dot{x}_e &= u_e + (u_e + u_d)(\cos(\psi_e) - 1) - (v_e + v_d)\sin(\psi_e) + r_d y_e + h_x, \\ \dot{y}_e &= v_e + (v_e + v_d)(\cos(\psi_e) - 1) + (u_e + u_d)\sin(\psi_e) - r_d x_e + h_y, \\ \dot{\psi}_e &= r_e + \Omega_\psi + h_\psi, \end{aligned} \tag{7.59}$$

where, for notational simplicity, we have defined

$$\begin{aligned} h_x &= \cos(\psi_d)\Delta_x + \sin(\psi_d)\Delta_y, \\ h_y &= -\sin(\psi_d)\Delta_x + \cos(\psi_d)\Delta_y, \\ h_\psi &= m_{11}m_{33}^{-1}(y_d\tilde{x}_1 - x_d\tilde{x}_2) + \tilde{x}_3, \\ \Omega_\psi &= m_{11}m_{33}^{-1}((\sin(\psi_d)x_e + \cos(\psi_d)y_e)\tilde{x}_1 - (\cos(\psi_d)x_e - \mathrm{sins}(\psi_d)y_e)\tilde{x}_2), \\ \Delta_x &= (\cos^2(\psi) + \sin^2(\psi)m_{11}m_{22}^{-1})\tilde{x}_1 + 0.5\sin(2\psi)\tilde{x}_2(m_{11} - m_{22})m_{22}^{-1}, \\ \Delta_y &= 0.5\sin(2\psi)\tilde{x}_1(m_{11} - m_{22})m_{22}^{-1} + (\sin^2(\psi) + \cos^2(\psi)m_{11}m_{22}^{-1})\tilde{x}_2. \end{aligned} \tag{7.60}$$

We define the following coordinate transformation, which is slightly different from (7.24):

$$z_e = \psi_e + \arcsin\left(\frac{k u_d y_e}{\sqrt{1 + x_e^2 + y_e^2}}\right), \tag{7.61}$$

where the constant k is such that $|k u_d(t)| < 1,\ \forall t \geq 0$. This constant will be specified later. Using the nonlinear coordinate transformation (7.61) together with (7.59), the ship error dynamics are rewritten as

$$\begin{aligned} \dot{x}_e &= u_e + (u_e + u_d)p_x - (v_e + v_d)p_y + (v_e + v_d)k u_d \varpi_2^{-1} y_e + \\ &\quad r_d y_e + h_x, \\ \dot{y}_e &= v_e + (v_e + v_d)p_x + (u_e + u_d)p_y - k u_d^2 \varpi_2^{-1} y_e - \\ &\quad k u_d \varpi_2^{-1} u_e y_e - r_d x_e + h_y, \\ \dot{z}_e &= r_e + f_z + g_z u_e + \Omega_z + h_z, \\ \dot{u}_e &= m_{22}m_{11}^{-1}\hat{v}\hat{r} - d_{11}m_{11}^{-1}\hat{u} + m_{11}^{-1}\tau_u - \dot{u}_d + \Omega_u + h_u, \\ \dot{v}_e &= -m_{11}m_{22}^{-1}(u_e r_e + u_d r_e + u_e r_d) - d_{22}m_{22}^{-1}v_e + \Omega_v + h_v, \\ \dot{r}_e &= (m_{11} - m_{22})m_{33}^{-1}\hat{u}\hat{v} - d_{33}m_{33}^{-1}\hat{r} + m_{33}^{-1}\tau_r - \dot{r}_d + \Omega_r + h_r, \end{aligned} \tag{7.62}$$

where, for notational simplicity and convenience of the control design, we have defined the following:

The terms $\omega_1, \omega_2, p_x, p_y, f_z$, and g_z are defined as

$$\begin{aligned}
\varpi_1 &= \sqrt{1+x_e^2+(1-k^2u_d^2)y_e^2},\ \varpi_2 = \sqrt{1+x_e^2+y_e^2},\\
p_x &= \varpi_2^{-1}\left((\cos(z_e)-1)\varpi_1+(\varpi_1-\varpi_2)+\sin(z_e)ku_d y_e\right),\\
p_y &= \varpi_2^{-1}\left(\sin(z_e)\varpi_1-(\cos(z_e)-1)ku_d y_e\right),\\
f_z &= k\varpi_1^{-1}\left(\dot{u}_d+u_d\left((1+x_e^2)\varpi_2^{-2}\left(v_e-ku_d^2\varpi_2^{-1}y_e+(v_e+v_d)\times\right.\right.\right.\\
&\quad \left.p_x+u_d p_y\right)-r_d x_e-\varpi_2^{-2}x_e y_e\left(u_d p_x-(v_e+v_d)(p_y-\right.\\
&\quad ku_d\varpi_2^{-1}y_e)))),\\
g_z &= ku_d\varpi_1^{-1}((1+x_e^2)\varpi_2^{-2}(-ku_d\varpi_2^{-1}y_e+p_y)-\varpi_2^{-2}y_e x_e(1+p_x)).
\end{aligned} \tag{7.63}$$

The terms Ω_z, Ω_u, Ω_v, and Ω_r containing states multiplied by the observer errors are defined as

$$\begin{aligned}
\Omega_z =&\ \Omega_\psi,\\
\Omega_u =&\ \hat{v}(m_{22}m_{33}^{-1}(y\tilde{x}_1-x\tilde{x}_2)+m_{22}m_{11}^{-1}\tilde{x}_3)-\\
&\ m_{11}m_{33}^{-1}(\sin(\psi_e)x_e-\cos(\psi_e)y_e)\tilde{\psi},\\
\Omega_v =&\ -(u_e+u_d)m_{11}m_{22}^{-1}(m_{11}m_{33}^{-1}(y\tilde{x}_1-x\tilde{x}_2)+\tilde{x}_3)-\\
&\ m_{11}^2m_{22}^{-1}m_{33}^{-1}(\cos(\psi_e)x_e+\sin(\psi_e)y_e)\tilde{\psi},\\
\Omega_r =&\ m_{11}^2m_{22}^{-1}m_{33}^{-1}\hat{u}\,(-\sin(\psi)\tilde{x}_1+\cos(\psi)\tilde{x}_2)-\\
&\ m_{22}m_{33}^{-1}\hat{v}\,(\cos(\psi)\tilde{x}_1+\sin(\psi)\tilde{x}_2)+\\
&\ m_{22}^{-1}m_{33}^{-1}(0.5(m_{11}^2-m_{11}m_{22})\sin(2\psi)\Delta_{xd}+\\
&\ (m_{11}m_{22}\cos^2(\psi)+m_{11}^2\sin^2(\psi))\Delta_{yd})\tilde{x}+\\
&\ m_{22}^{-1}m_{33}^{-1}(0.5(m_{11}^2-m_{11}m_{22})\sin(2\psi)\Delta_{yd}-\\
&\ (m_{11}m_{22}\sin^2(\psi)+m_{11}^2\cos^2(\psi))\Delta_{xd})\tilde{y}+\\
&\ (m_{11}^2m_{33}^{-2}+1)(x_e^2+y_e^2+2\sin(\psi_d)(x_e y_d-\\
&\ y_e x_d)+2\cos(\psi_d)(x_e x_d+y_e y_d))\tilde{\psi},
\end{aligned} \tag{7.64}$$

with $\Delta_{xd}=\cos(\psi_d)x_e-\sin(\psi_d)y_e$, and $\Delta_{yd}=\sin(\psi_d)x_e+\cos(\psi_d)y_e$.

The terms h_z, h_u, h_v, and h_r containing the observer errors multiplied by bounded terms are defined as

$$\begin{aligned}
h_z &= ku_d\varpi_1^{-1}((1+x_e^2)\varpi_2^{-2}h_y-\varpi_2^{-2}y_e x_e h_x)+h_\psi,\\
h_u &= \cos(\psi)\tilde{x}+\sin(\psi)\tilde{y}-m_{11}m_{33}^{-1}\,(\sin(\psi)x_d-\cos(\psi)y_d)\,\tilde{\psi},\\
h_v &= -m_{11}^2m_{22}^{-1}\,(\sin(\psi)\tilde{x}-\cos(\psi)\tilde{y})-m_{11}^2m_{22}^{-1}m_{33}^{-1}\,(\cos(\psi)x_d+\sin(\psi)y_d)\,\tilde{\psi},\\
h_r &= m_{22}^{-1}m_{33}^{-1}(0.5(m_{11}^2-m_{11}m_{22})\sin(2\psi)(x_d\tilde{x}-y_d\tilde{y})+(m_{11}m_{22}\cos^2(\psi)+\\
&\quad m_{11}^2\sin^2(\psi))y_d\tilde{x}-(m_{11}m_{22}\sin^2(\psi)+m_{11}^2\cos^2(\psi))x_d\tilde{y})+m_{33}^{-2}(m_{11}^2+\\
&\quad m_{33}^2)(x_d^2+y_d^2)\tilde{\psi}.
\end{aligned} \tag{7.65}$$

The problem of forcing the underactuated ship (7.1) to track the virtual ship (7.5) becomes one of stabilizing the system (7.62) at the origin. The effort, we have made so far is to obtain the stabilizing term $-ku_d^2 y_e/\varpi_2$ in the y_e-dynamics.

7.3.3 Control Design

Before designing the control inputs, it is important to note that the terms Ω_u and Ω_r can be dominated by adding some nonlinear damping terms in the control inputs τ_u and τ_r. However the term Ω_v cannot be dominated by any nonlinear damping terms in τ_u and τ_r. Since Ω_v contains $u_e x_e$ and $u_e y_e$ multiplied by the observer errors, with x and y being substituted in from (7.58), if one designs a virtual control of u_e, which is linear in x_e and y_e, the sway velocity dynamics might have a finite escape time due to the fact that separation principle does not hold for the nonlinear system in question. The coordinate change (7.58) allows us to design a virtual control of u_e such that it is bounded for all x_e and y_e and stabilizes the x_e-dynamics at the origin. We design the controls τ_u and τ_r in two steps.

Step 1

Define the virtual control errors as

$$\begin{aligned}\tilde{u}_e &= u_e - u_e^d,\\ \tilde{r}_e &= r_e - r_e^d,\end{aligned} \tag{7.66}$$

where u_e^d and r_e^d are the virtual velocity controls of u_e and r_e, respectively. The virtual controls u_e^d and r_e^d are chosen as follows:

$$\begin{aligned}u_e^d &= -\frac{k_1 x_e}{\varpi_2},\\ r_e^d &= -k_2 z_e - f_z - g_z u_e^d,\end{aligned} \tag{7.67}$$

where k_1 and k_2 are positive design constants to be specified later.

Step 2

By differentiating (7.66) along the solutions of (7.62) and (7.67), the actual controls τ_u and τ_r with some nonlinear damping terms to overcome the effect of observer errors, and without canceling the useful damping terms, are chosen as

$$\tau_u = m_{11}\Bigg(-m_{22}m_{11}^{-1}\hat{v}\hat{r} + d_{11}m_{11}^{-1}(u_e^d + u_d) + \dot{u}_d + \frac{\partial u_e^d}{\partial x_e}(u_e + (u_e + u_d)\times$$

$$p_x + (v_e + v_d)\left(-p_y + k u_d \varpi_2^{-1} y_e\right) + r_d y_e\big) \frac{\partial u_e^d}{\partial y_e} (v_e + (v_e + v_d) p_x +$$
$$(u_e + u_d) p_y - k u_d y_e \varpi_2^{-1} (u_e + u_d) - r_d x_e\big) - c_1 \tilde{u}_e - g_z z_e +$$
$$k_3 k_4^{-1} m_{11} m_{22}^{-1} \hat{r} v_e - \delta_1 \tilde{u}_e \tau_{u\mathrm{dam}}\big),$$

$$\begin{aligned}\tau_r = m_{33} \Big(&-(m_{11} - m_{22}) m_{33}^{-1} \hat{u}\hat{v} + d_{33} m_{33}^{-1} (r_e^d + r_d) + \dot{r}_d + \\ &\frac{\partial r_e^d}{\partial x_e} \left(u_e + (u_e + u_d) p_x + (v_e + v_d)\left(-p_y + k u_d \varpi_2^{-1} y_e\right) + r_d y_e\right) + \\ &\frac{\partial r_e^d}{\partial y_e} \big(v_e + (v_e + v_d) p_x + (u_e + u_d) p_y - k u_d y_e \varpi_2^{-1} (u_e + u_d) - \\ &r_d x_e\big) + \frac{\partial r_e^d}{\partial z_e} (r_e + f_z + g_z u_e) - \frac{\partial r_e^d}{\partial v_e} \big(d_{22} m_{22}^{-1} v_e + m_{11} m_{22}^{-1} (u_e r_e + \\ &u_d r_e + u_e r_d)\big) + \frac{\partial r_e^d}{\partial u_d} \dot{u}_d + \frac{\partial r_e^d}{\partial \dot{u}_d} \ddot{u}_d + \frac{\partial r_e^d}{\partial v_d} \dot{v}_d + \frac{\partial r_e^d}{\partial r_d} \dot{r}_d - c_2 \tilde{r}_e - z_e + \\ &k_3 k_4^{-1} m_{11} m_{22}^{-1} (u_e^d + u_d) v_e - \delta_2 \tilde{r}_e \tau_{r\mathrm{dam}}\Big),\end{aligned} \tag{7.68}$$

where c_1, c_2, k_3, and k_4 are positive constants to be specified later, δ_1 and δ_2 are arbitrarily positive constants. We introduced the ratio k_3/k_4 to enhance the feasibility of design constants. The nonlinear damping terms $\tau_{u\mathrm{dam}}$ and $\tau_{r\mathrm{dam}}$ are defined as:

$$\begin{aligned}\tau_{u\mathrm{dam}} &= (\hat{v}^2 + 1)(x_e^2 + y_e^2) + \hat{v}^2, \\ \tau_{r\mathrm{dam}} &= \hat{u}^2 + \hat{v}^2 + (x_e^2 + y_e^2)^2 + (x_e^2 + y_e^2)(\hat{u}^2 + 1).\end{aligned} \tag{7.69}$$

We now state the second main result of this chapter, the proof of which is given in the next section.

Theorem 7.2. *Under Assumption 7.2, assume that (a) there are no environmental disturbances; (b) the ship parameters are known; (c) the reference signals* $(x_d,$ $y_d,$ $\psi_d,$ $v_d)$ *are generated by the virtual ship model (7.5), and Assumption 7.1 holds. If the output feedback control law (7.68) together with the observer (7.51) are applied to the ship system (7.1), then the tracking errors* $(x(t) - x_d(t), y(t) - y_d(t), \psi(t) - \psi_d(t), v(t) - v_d(t))$ *globally asymptotically and locally exponentially converge to zero with an appropriate choice of the design constants* k, c_1, c_2, *and* k_i, $i = 1, \ldots, 4$.

Remark 7.4. The main differences between the partial-state feedback and output feedback designs are the nonlinear coordinate transformations (7.6), (7.24), (7.43), (7.58), and (7.61). Furthermore, the partial-state feedback controller can allow the reference trajectory (x_d, y_d) to exponentially grow but the output feedback controller cannot. This is because the observer errors of the partial-state feedback design do not depend on x and y while those of the output feedback design depend linearly on x and y. Indeed, the output feedback control design can directly yield a controller for the partial-state feedback case but not vice versa. We have, however, presented both control designs for the sake of completeness.

7.3.4 Stability Analysis

Substituting (7.68), (7.67), and (7.66) into (7.62) results in the following closed loop system:

$$
\begin{aligned}
\dot{x}_e &= -k_1\varpi_2^{-1}x_e + (-k_1\varpi_2^{-1}x_e + u_d)p_x - (v_e+v_d)p_y + \\
&\quad (v_e+v_d)ku_d\varpi_2^{-1}y_e + r_d y_e + h_x + \tilde{u}_e(1+p_x), \\
\dot{y}_e &= v_e + (v_e+v_d)p_x + (-k_1\varpi_2^{-1}x_e + u_d)p_y - ku_d^2\varpi_2^{-1}y_e + \\
&\quad k_1ku_d\varpi_2^{-1}x_e y_e - r_d x_e + h_y + \tilde{u}_e(1-ku_d\varpi_2^{-1}y_e), \\
\dot{z}_e &= -k_2 z_e + \Omega_z + h_z + g_z\tilde{u}_e + \tilde{r}_e, \\
\dot{\tilde{u}}_e &= -\left(c_1 + d_{11}m_{11}^{-1}\right)\tilde{u}_e - g_z z_e + \frac{k_3}{k_4}\frac{m_{11}}{m_{22}}\hat{r}v_e - \delta_1\tilde{u}_e\tau_{u\text{dam}} + \\
&\quad \Omega_u + h_u - \frac{\partial u_e^d}{\partial x_e}h_x - \frac{\partial u_e^d}{\partial y_e}h_y, \\
\dot{v}_e &= -d_{22}m_{22}^{-1}v_e - m_{11}m_{22}^{-1}(u_e^d r_e^d + u_d r_e^d + u_e^d r_d) - m_{11}m_{22}^{-1}\times \\
&\quad (u_e^d + u_d)\tilde{r}_e - m_{11}m_{22}^{-1}\hat{r}\tilde{u}_e + \Omega_v + h_v, \\
\dot{\tilde{r}}_e &= -\left(c_2 + d_{33}m_{33}^{-1}\right)\tilde{r}_e - z_e - \frac{\partial r_e^d}{\partial x_e}h_x - \frac{\partial r_e^d}{\partial y_e}h_y - \frac{\partial r_e^d}{\partial z_e}(\Omega_z + h_z) - \\
&\quad \frac{\partial r_e^d}{\partial v_e}(\Omega_v + h_v) + k_3k_4^{-1}m_{11}m_{22}^{-1}(u_e^d + u_d)v_e - \delta_2\tilde{r}_e\tau_{r\text{dam}} + \Omega_r + h_r,
\end{aligned}
\tag{7.70}
$$

where, for brevity, we did not substitute the expressions of u_e^d and r_e^d into the sway dynamics. To prove Theorem 7.2, we just need to show that the closed loop system (7.70) is globally asymptotically and locally exponentially stable at the origin. It is noted that Ω_z contains x_e and y_e multiplied by the observer errors, see (7.64) and (7.60). On the other hand, the x_e and y_e-dynamics are stabilized by the terms $-k_1x_e/\varpi_2$ and $-ku_d^2y_e/\varpi_2$, respectively. This makes the stability analysis of (7.70) difficult, i.e., we cannot consider the (x_e, y_e, v_e) and $(z_e, \tilde{u}_e, \tilde{r}_e)$-subsystems separately as is often done in applying stability results for cascade systems. To illustrate our idea of proving asymptotic stability of (7.70), we first give a simple example. For any initial conditions $(\xi_1(t_0), \xi_2(t_0))$, the system

$$
\begin{aligned}
\dot{\xi}_1 &= -\frac{\xi_1}{\sqrt{1+\xi_1^2}} + \phi(\xi_2), \\
\dot{\xi}_2 &= -\xi_2 + \xi_1 e^{-\sigma_{\xi 2}(t-t_0)}
\end{aligned}
\tag{7.71}
$$

is GAS at the origin for $t \geq t_0 \geq 0$, any $\sigma_{\xi 2} > 0$, and $|\phi(\xi_2(t))| \leq A|\xi_2(t)|$ with A being any positive constant. In the first step, we show that there exists a positive constant $\sigma_{\xi 1} < \sigma_{\xi 2}$ such that $|\xi_1(t)| \leq \lambda_1(\cdot)e^{\sigma_{\xi 1}(t-t_0)}$, with $\lambda_1(\cdot)$ being a nondecreasing function of $\|(\xi_1(t_0), \xi_2(t_0))\|$. Take the Lyapunov function $W = 0.5\left(\xi_1^2 + K\xi_2^2\right)$, with K being a positive constant, whose derivative along the solution of (7.71) satisfies

$$\dot{W} \leq \sigma_{\xi 1}\xi_1^2 + \left(0.25A^2/\sigma_{\xi 1} - K\right)\xi_2^2 + K\xi_1\xi_2 e^{-\sigma_{\xi 2}(t-t_0)}. \tag{7.72}$$

Pick K such that $A^2/(4\sigma_{\xi 1}) - K < 0$, then (7.72) implies that

$$\dot{W} \leq 2\sigma_{\xi 1} W + K\max(1, K) W e^{-\sigma_{\xi 2}(t-t_0)}, \tag{7.73}$$

which in turn yields $|\xi_1(t)| \leq \lambda_1(\cdot)e^{\sigma_{\xi 1}(t-t_0)}$. Therefore the term $\xi_1 e^{-\sigma_{\xi 2}(t-t_0)}$ in (7.71) globally exponentially converges to zero. The second step of proving asymptotic stability can be carried out easily by taking the Lyapunov function $W_1 = \sqrt{1+\xi_1^2} - 1 + K_1\xi_2^2$, $K_1 > 0$. We now present the proof of asymptotic stability of the closed loop system (7.70) in two parts.

Part 1. In this part, we show that there exists a positive constant $\sigma_1 < \sigma_0$ such that

$$\|(x_e(t), y_e(t))\| \leq \gamma_{11}(\cdot)e^{\sigma_1(t-t_0)} + \gamma_{10}(\cdot), \tag{7.74}$$

where $\gamma_{11}(\cdot)$ and $\gamma_{10}(\cdot)$ are nondecreasing functions of $\|\boldsymbol{\Xi}(t_0)\|$ with

$$\begin{aligned} \boldsymbol{\Xi}(t_0) &:= [\tilde{\boldsymbol{\eta}}(t_0), \tilde{\boldsymbol{X}}(t_0), \boldsymbol{X_e}(t_0)]^T, \\ \boldsymbol{X_e} &:= [x_e, y_e, z_e, v_e, \tilde{u}_e, \tilde{r}_e]^T. \end{aligned}$$

Consider the following Lyapunov function:

$$V_1 = \frac{1}{2}\Big(x_e^2 + y_e^2 + k_3 v_e^2 + k_4(z_e^2 + \tilde{u}_e^2 + \tilde{r}_e^2)\Big), \tag{7.75}$$

with k_4 being a positive constant, whose time derivative along the solutions of (7.70), after some lengthy but simple calculation by completing squares satisfies

$$\begin{aligned} \dot{V}_1 \leq & -k_1\varpi_2^{-1}x_e^2 - ku_d^2\varpi_2^{-1}y_e^2 + 7\varepsilon_1(x_e^2 + y_e^2) - a_z z_e^2 - a_v v_e^2 - a_u\tilde{u}_e^2 - \\ & a_r\tilde{r}_e^2 + (\chi_{11}(\cdot)V_1 + \chi_{10}(\cdot))\, e^{-\sigma_0(t-t_0)} + a_0, \end{aligned} \tag{7.76}$$

where

$$\begin{aligned} a_z &= k_2k_4 - 0.25\varepsilon_1^{-1}(k_2k_3m_{11}m_{22}^{-1}(k_1 + |u_d|))^2 - \varepsilon_1^{-1}(k_1 + |u_d|)^2, \\ a_u &= k_4(c_1 + d_{11}m_{11}^{-1}) - 5\varepsilon_1^{-1}, \\ a_v &= k_3d_{22}m_{22}^{-1} - 6.75\varepsilon_1^{-1} - k_3m_{11}m_{22}^{-1}(k_1 + |u_d|)(8 + |ku_d|)\,|ku_d|, \\ a_r &= k_4(c_2 + d_{33}m_{33}^{-1}), \end{aligned} \tag{7.77}$$

with ε_1 and a_0 being positive constants. We choose the design constants k, c_1, c_2, and k_i, $i = 1,\ldots,4$ such that

$$\varepsilon_1 < \sigma_0/7,\ a_z > 0,\ a_v > 0,\ a_u > 0,\ a_r > 0. \tag{7.78}$$

It is not hard to show that there always exist k, c_1, c_2 and k_i, $i = 1,\ldots,4$ such that (7.78) holds for arbitrarily small ε_1. We will discuss more detail of (7.78) later. With the choice of (7.78), we can write (7.76) as

$$\begin{aligned}\dot V_1 \le\ & 14\varepsilon_1 (V_1 + (a_0 + \chi_{10}(\cdot))/(14\varepsilon_1)) + \\ & \chi_{11}(\cdot)(V_1 + (a_0 + \chi_{10}(\cdot))/(14\varepsilon_1))\, e^{-\sigma_0(t-t_0)},\end{aligned} \tag{7.79}$$

which, together with (7.75), yields (7.74).

Part 2. We now prove that the closed loop system (7.70) is GAS at the origin by taking the Lyapunov function

$$V_2 = \sqrt{1 + x_e^2 + y_e^2} - 1 + \frac{1}{2}\left(k_3 v_e^2 + k_4(z_e^2 + \tilde u_e^2 + \tilde r_e^2)\right) \tag{7.80}$$

whose time derivative along the solutions of (7.70), after some lengthy but simple calculation by completing squares, and using (7.74), satisfies

$$\begin{aligned}\dot V_2 \le\ & -\mu_x \varpi_2^{-2} x_e^2 - \mu_y(t)\varpi_2^{-2} y_e^2 - \mu_z(t) z_e^2 - \mu_v(t) v_e^2 - \mu_u \tilde u_e^2 - \mu_r \tilde r_e^2 + \\ & \chi_{21}(\cdot) V_2 e^{-\sigma_2(t-t_0)} + \chi_{20}(\cdot) e^{-\sigma_2(t-t_0)},\end{aligned} \tag{7.81}$$

where

$$\begin{aligned}
\mu_x =\ & k_1 - 6\rho_1, \\
\mu_y(t) =\ & k u_d^2 - (k u_d)^2(k_1 + |u_d| + |v_d| + 0.5) - |k u_d|(k_1 + \rho_1) - \\
& 0.25\rho_1^{-1}(k u_d v_d)^2 - 10\rho_1, \\
\mu_z(t) =\ & k_2 k_4 - \rho_1^{-1}(0.5(k_1 + |u_d|)^2 + 5(1 + v_d^2) + 0.25(k_3 m_{11} m_{22}^{-1} \times \\
& (k_1 + |u_d|))^2(k_2^2 + (k u_d)^2(k_1 + |u_d| + v_d^2) + 4 + 4v_d^2)), \\
\mu_v(t) =\ & k_3 d_{22} m_{22}^{-1} - 7\rho_1 - 0.25\rho_1^{-1}(|k u_d| + 1) - 0.5(k u_d)^2 - \\
& k_3 m_{11} m_{22}^{-1}(k_1 + |u_d|)\,|k u_d|\,(8 + |k u_d|) - 0.25\rho_1^{-1}(k_3 m_{11} m_{22}^{-1} \times \\
& (k_1 + |u_d|))^2((k \dot u_d)^2/(1 - (k u_d)^2) + (k u_d)^4(u_d^2 + k_1^2 + \\
& (k u_d v_d)^2 + r_d^2 + 2) + (k u_d)^2), \\
\mu_u =\ & k_4(c_1 + d_{11}/m_{11}) - 29/(4\rho_1), \\
\mu_r =\ & k_4(c_2 + d_{33}/m_{33}),
\end{aligned} \tag{7.82}$$

where $\sigma_2 = \sigma_0 - \sigma_1$, ρ_1 is a positive constant, and $\chi_{21}(\cdot)$ and $\chi_{20}(\cdot)$ are nondecreasing functions of $\|(\tilde{\boldsymbol\eta}(t_0), \tilde{\boldsymbol X}(t_0), \boldsymbol X_{\boldsymbol e}(t_0))\|$. We now choose the constants k, c_1, c_2, and k_i, $i = 1,\ldots,4$, such that

$$\begin{aligned} &\mu_x \geq \mu_x^*, \\ &\mu_y(t) \geq \mu_y^*, \\ &\mu_z(t) \geq \mu_z^*, \\ &\mu_v(t) \geq \mu_v^*, \\ &\mu_u \geq \mu_u^*, \\ &\mu_r \geq \mu_r^* \end{aligned} \tag{7.83}$$

for all $t \geq 0$, where μ_x^*, μ_y^*, μ_z^*, μ_v^*, μ_u^*, and μ_r^* are positive constants. Substituting (7.83) into (7.81) yields

$$\begin{aligned} \dot{V}_2 \leq & -\mu_x^* \varpi_2^{-2} x_e^2 - \mu_y^* \varpi_2^{-2} y_e^2 - \mu_z^* z_e^2 - \mu_v^* v_e^2 - \mu_u^* \tilde{u}_e^2 - \mu_r^* \tilde{r}_e^2 + \\ & \chi_{21}(\cdot) V_2 e^{-\sigma_2(t-t_0)} + \chi_{20}(\cdot) e^{-\sigma_2(t-t_0)}. \end{aligned} \tag{7.84}$$

From (7.84) we have $\dot{V}_2 \leq \chi_{21}(\cdot) V_2 e^{-\sigma_2(t-t_0)} + \chi_{20}(\cdot) e^{-\sigma_2(t-t_0)}$, which implies that $V_2 \leq \chi_{22}(\cdot)$, with $\chi_{22}(\cdot)$ being a nondecreasing function of $\|\boldsymbol{\Xi}(t_0)\|$. With $V_2 \leq \chi_{22}(\cdot)$ in mind, one can show from (7.84) that there exists $\sigma_3 > 0$ depending on $\|\boldsymbol{\Xi}(t_0)\|$ such that $\|\boldsymbol{X_e}(t)\| \leq \gamma_2(\cdot) e^{-\sigma_3(t-t_0)}$, where $\gamma_2(\cdot)$ is a nondecreasing function of $\|\boldsymbol{\Xi}(t_0)\|$, which in turn implies that the closed loop system (7.70) is asymptotically stable at the origin. However one can straightforwardly show that (7.70) is also locally exponentially stable at the origin. By carrying out a similar arguments in Section 7.2.5, one can show that there always exist the design constants k, c_1, and c_2 and k_i, $i = 1, 2, 3, 4$, such that $|k u_d(t)| < 1$ and that the conditions (7.78) and (7.83) hold.

7.4 Robustness Discussion

In this section, we discuss robustness of the output feedback tracking controller. A discussion for the partial-state feedback can be carried out in a similar way. The control law (7.68) has been designed under the assumption that there are no environmental disturbances. Indeed, this assumption is unrealistic in practice. The aim of this section is to discuss the robustness property of our proposed controller in relation to environmental disturbances. Under additive environmental disturbances, it is not hard to show that the observer (7.51) guarantees that the observer errors $(\tilde{\boldsymbol{\eta}}(t), \tilde{\boldsymbol{X}}(t))$ globally exponentially converge to a ball centered at the origin. Moreover, one can prove that the tracking error vector $\boldsymbol{X_e}(t)$ also globally asymptotically converges to a ball centered at the origin. The radius of this ball can be adjusted by changing the control gains if the environmental disturbances are not too large. When the environmental disturbances are large enough, the observer (7.51) cannot provide a sufficiently accurate estimate of unmeasured velocities, and the control law (7.68) cannot compensate for considerable environmental disturbances acting on the sway axis. These will result in an unstable closed loop system, especially at a low forward speed. This phenomenon should not be surprising since the vessel in question is not actuated in the sway axis and does not have velocities available for feedback. One can see this phenomenon by observing the simple example system (7.71) with some additive disturbance in the first equation. It is easy to show that when this additive disturbance has magnitude larger than 1, then the system (7.71) will be unstable. The robustness issue is still a challenging problem in control of underactuated ocean vehicles and underactuated systems in general.

7.5 Simulations

This section illustrates the soundness of the control laws (7.68) by simulating them on the same monohull ship in the previous two chapters. Details of the ship parameters are listed in Section 5.4.

The initial conditions of the reference trajectories are chosen as $(x_d(0), y_d(0), \psi_d(0), v_d(0)) = (0\,\mathrm{m},\ 0\,\mathrm{m},\ 0\,\mathrm{rad},\ 0\,\mathrm{ms}^{-1})$. The reference velocities are $u_d = 4\mathrm{ms}^{-1}$ and $r_d = 0\mathrm{rads}^{-1}$ for the first 300 seconds, and $u_d = 4\,\mathrm{ms}^{-1}$ and $r_d = 0.02\,\mathrm{rads}^{-1}$ for the rest of the simulation time. This choice means that the reference trajectory is a straight line for the first 300 seconds and then followed by a circle with a radius of 200 m. Indeed the above choice of reference velocities satisfies Assumption 7.1. All of the initial conditions of $\hat{\boldsymbol{\eta}}$ and $\hat{\boldsymbol{v}}$ are chosen to be zero. We first choose $k_2 = 5$, $c_1 = 1$, $c_2 = 2$, $\delta_1 = \delta_2 = 0.1$, $\boldsymbol{K_{01}} = 10\,\mathrm{diag}(1,1,1)$, $\boldsymbol{P_{01}} = \boldsymbol{P_{02}} = 0.5\,\mathrm{diag}(1{,}1{,}1)$, and $\boldsymbol{K_{02}} = (\boldsymbol{J}(\eta)\boldsymbol{P}^{-1}(\eta))^T$. The other design constants are chosen as: $k = 0.05$, $k_1 = 1$, $k_3 = 5$, and $k_4 = 100$. Simulation results are plotted in Figures 7.1 and 7.2 with the initial conditions of the ship: $x = -10\,\mathrm{m}, y = 10\,\mathrm{m}, \psi = 0.1\,\mathrm{rad}, u = 0\,\mathrm{ms}^{-1}, v = 0\,\mathrm{ms}^{-1}$, and $r = 0\,\mathrm{rads}^{-1}$. It is seen from Figures 7.1a and 7.1b that all of the tracking and observer errors converge to zero. The control inputs, τ_u and τ_r, are within their limits. As always, the magnitude of τ_u and τ_r can be reduced by adjusting the control gains such as c_1 and c_2. However, the trade-off is a longer transient response time. For clarity, we only plot the tracking errors for the first 180 seconds, and the observer errors for the first 10 seconds. To illustrate robustness of our proposed controller, we also simulate with the same control gains and initial conditions chosen as above, and the environmental disturbance vector $\boldsymbol{\tau_w}(t) = 0.5\boldsymbol{M}([\sin(t), \cos(t), \sin(t)]^T + 1.5)$, i.e., the last equation of (7.1) is in the form of $\boldsymbol{M}\dot{\boldsymbol{v}} = -\boldsymbol{C}(\boldsymbol{v})\boldsymbol{v} - \boldsymbol{D}\boldsymbol{v} + \boldsymbol{\tau} + \boldsymbol{\tau_w}(t)$. Simulation results are plotted in Figures 7.3 and 7.4. For clarity, we only plot the tracking errors for the first 300 seconds and the observer errors for the first 10 seconds. It can be seen that the environmental disturbances deteriorate the performance of the controlled loop system in the sense that the tracking errors do not converge to zero but to a ball centered at the origin. This shows an important property of robustness of the controlled system with respect to the environmental disturbances.

7.6 Conclusions

The key to the control developments is an introduction of the global nonlinear coordinate transformations (7.6), (7.24), (7.43), (7.58), and (7.61) to obtain an exponential observer and to transform the tracking error dynamics to a suitable nonlinear system, to which Lyapunov's direct method and the backstepping technique can be applied. The work presented in this chapter is based on [94, 114, 116, 117].

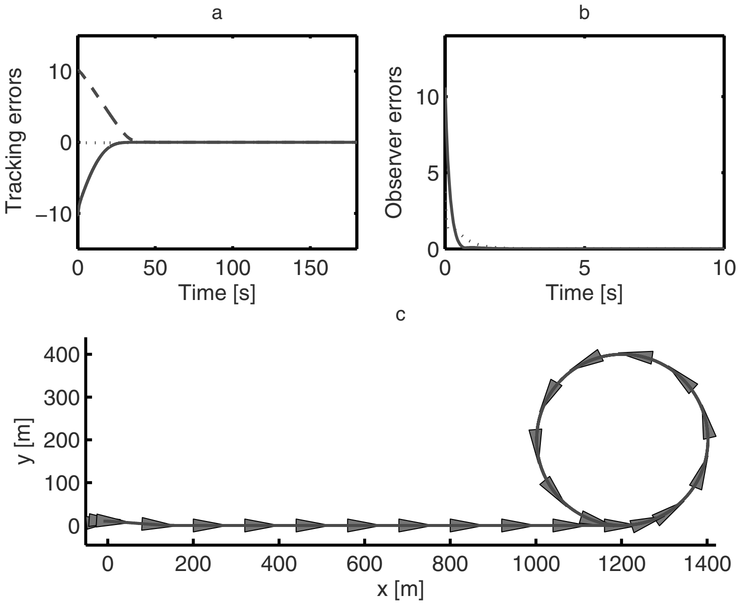

Figure 7.1 Simulation results without disturbances: **a**. Tracking errors ($x - x_d$ [m]: *solid line*, $y - y_d$ [m]: *dashed line*, $\psi - \psi_d$ [rad]]: *dotted line*); **b**. Observer errors ($\sqrt{(x-\hat{x})^2 + (y-\hat{y})^2 + (\psi-\hat{\psi})^2}$: *solid line*, $\sqrt{(u-\hat{u})^2 + (v-\hat{v})^2 + (r-\hat{r})^2}$): *dotted line*; **c**. Ship position and orientation in the (x, y)-plane

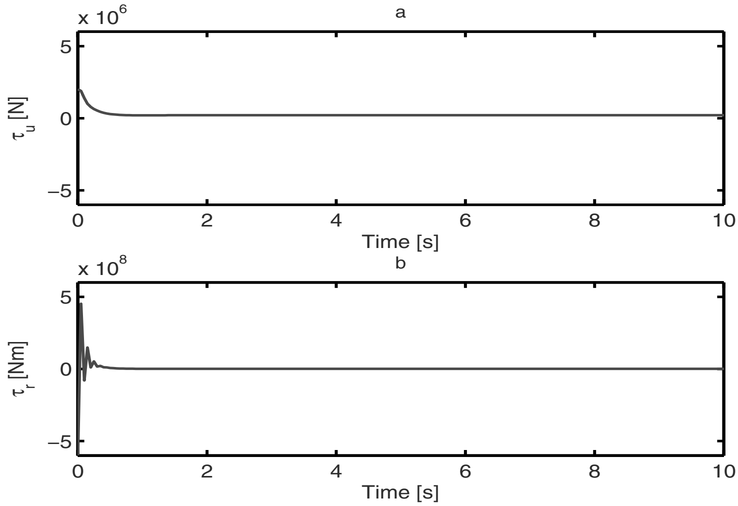

Figure 7.2 Simulation results without disturbances (control inputs): **a**. Surge force; **b**. Yaw moment

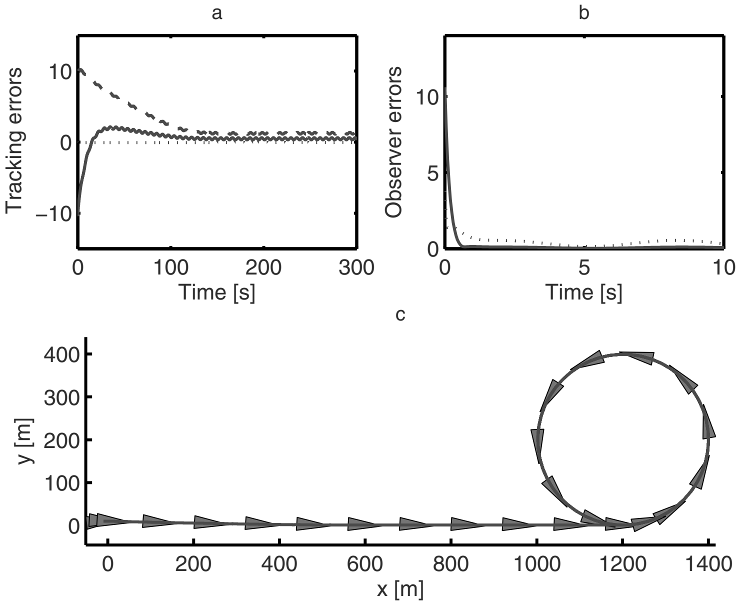

Figure 7.3 Simulation results with disturbances: **a**. Tracking errors ($x - x_d$[m]: *solid line*, $y - y_d$ [m]: *dashed line*, $\psi - \psi_d$ [rad]]: *dotted line*); **b**. Observer errors ($\sqrt{(x-\hat{x})^2 + (y-\hat{y})^2 + (\psi-\hat{\psi})^2}$: *solid line*, $\sqrt{(u-\hat{u})^2 + (v-\hat{v})^2 + (r-\hat{r})^2}$): *dotted line*; **c**. Ship position and orientation in the (x, y)-plane

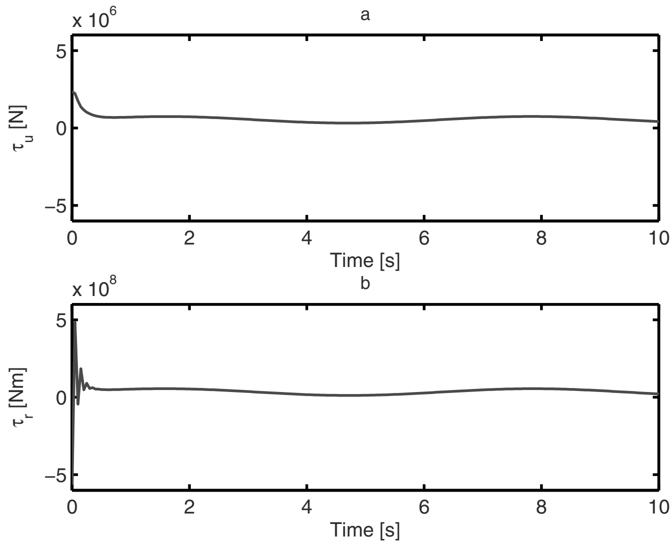

Figure 7.4 Simulation results with disturbances (control inputs): **a**. Surge force; **b**. Yaw moment