Chapter 4
Control Properties and Previous Work on Control of Ocean Vessels

In this chapter, first control properties of ocean vessels are presented. Then, the existing literature on the control of underactuated ocean vessels is reviewed. Through the review of the previous work in the areas of stabilization, trajectory-tracking, path-following, and output feedback control of underactuated ocean vessels, challenging questions are raised. Illustration of the background and process of solutions of those questions, as well as an explanation of the solutions in terms of their physical insights and practical applications are then presented in subsequent chapters.

4.1 Controllability Properties

4.1.1 Acceleration Constraints

The number (m_c) of independent control inputs (the number of nonzero elements of the propulsion force and moment vector $\boldsymbol{\tau}$) is smaller than the number (n_c) of degrees of freedom to be controlled for a standard model of the ocean vessels. As such, we remove all zero elements of $\boldsymbol{\tau}$ and denote the resulting vector by $\boldsymbol{\tau_a}$. Thus, if $\boldsymbol{\tau_a} \in \mathbb{R}^{m_c}$ and $\boldsymbol{\eta} \in \mathbb{R}^{n_c}$, then $m_c < n_c$. For example, for the case of the vessels with six degrees of freedom to be controlled we have $m_c < 6$, for the case of the vessels with five degrees of freedom to be controlled $m_c < 5$, and $m_c < 3$ for the case of the vessels with three degrees of freedom to be controlled. For clarity, we ignore the environmental disturbance forces and moments to investigate acceleration constraints on the aforementioned ocean vessels. Let $\boldsymbol{M_u}$, $\boldsymbol{C_u}(\boldsymbol{v})$, $\boldsymbol{D_u}(\boldsymbol{v})$, and $\boldsymbol{g_u}(\boldsymbol{\eta})$ denote the rows of $\boldsymbol{M}$, $\boldsymbol{C}(\boldsymbol{v})$, $\boldsymbol{D}(\boldsymbol{v})$, and $\boldsymbol{g}(\boldsymbol{\eta})$ that correspond to those rows without propulsion forces or moments, i.e.,

$$\boldsymbol{M_u}\dot{\boldsymbol{v}} + \boldsymbol{C_u}(\boldsymbol{v})\boldsymbol{v} + \boldsymbol{D_u}(\boldsymbol{v})\boldsymbol{v} + \boldsymbol{g_u}(\boldsymbol{\eta}) = \mathbf{0}. \tag{4.1}$$

The above equation describes the acceleration constraints, i.e., second-order constraints. The following results give the conditions whether the constraints given in (4.1) are partially integrable or totally integrable.

Lemma 4.1. *The constraints (4.1) are partially integrable if and only if the following conditions hold:*

1. $\boldsymbol{g_u}(\boldsymbol{\eta})$ *is a constant vector.*
2. $(\boldsymbol{C_u}(\boldsymbol{v}) + \boldsymbol{D_u}(\boldsymbol{v}))$ *is a constant matrix.*
3. *The distribution* $\Omega^{\perp}(\boldsymbol{\eta}) = \ker\left((\boldsymbol{C_u}(\boldsymbol{v}) + \boldsymbol{D_u}(\boldsymbol{v}))\boldsymbol{J}^{-\mathbf{1}}(\boldsymbol{\eta})\right)$ *is completely integrable.*

Proof. See [29]. □

Lemma 4.2. *The constraints (4.1) are totally integrable if and only if the following conditions hold:*

1. *The constraints are partially integrable.*
2. $(\boldsymbol{C_u}(\boldsymbol{v}) + \boldsymbol{D_u}(\boldsymbol{v}))=0$.
3. *The distribution* $\Delta(\boldsymbol{\eta}) = \ker\left(\boldsymbol{M_u}\boldsymbol{J}^{-\mathbf{1}}(\boldsymbol{\eta})\right)$ *is completely integrable.*

Proof. See [29]. □

The following lemma gives a result on the stabilizability of an underactuated ocean vessel.

Lemma 4.3. *Consider the system (3.31) with* $\boldsymbol{\tau_E} = \mathbf{0}^{n_c \times 1}$. *Assume that the elements of the restoring force and moment vector* $\boldsymbol{g}(\boldsymbol{\eta})$ *corresponding to the unactuated dynamics are zero, i.e., the vector* $\boldsymbol{g}(\boldsymbol{\eta})$ *can be written in the form of*

$$\boldsymbol{g}(\boldsymbol{\eta}) = \begin{bmatrix} \boldsymbol{g_a}(\boldsymbol{\eta}) \\ \mathbf{0}^{(n_c - m_c) \times 1} \end{bmatrix}, \tag{4.2}$$

where $\boldsymbol{g_a}(\boldsymbol{\eta}) \in \mathbb{R}^{m_c}$ *is the restoring force and moment vector corresponding to the actuated dynamics. Let* $(\boldsymbol{\eta}, \boldsymbol{v}) = (\boldsymbol{\eta^e}, \mathbf{0}^{n_c - m_c})$ *be an equilibrium. There is no* C^1 *state feedback law* $\boldsymbol{\alpha}(\boldsymbol{\eta}, \boldsymbol{v}) : \mathbb{R}^{n_c} \times \mathbb{R}^{n_c} \rightarrow \mathbb{R}^{m_c}$ *that makes the equilibrium* $(\boldsymbol{\eta^e}, \mathbf{0}^{n_c - m_c})$ *asymptotically stable.*

Proof. See [29]. □

4.1.2 Kinematic Constraints

In this section, we address controllability properties of ocean vessels. Since the vessel under consideration has a number of degrees of freedom to be controlled greater than control inputs (e.g., underwater vehicles do not have independent actuators in the heave and sway axes, see Section 3.4.2 and surface ships do not have an independent actuator in the sway axis, see Section 3.4.1), we can address the controllability issue of the vessel kinematic by considering the kinematics with the linear velocity vector

$$\boldsymbol{v_1} = \begin{bmatrix} u \\ 0 \\ 0 \end{bmatrix}. \tag{4.3}$$

We analyze controllability properties of six degrees of freedom vessels. The case of three degrees of freedom vessels can be obtained directly from the results for the six degrees of freedom vessels. With (4.3), we now write the kinematics of the vessel as follows:

$$\begin{gathered} \dot{\eta} = \boldsymbol{\gamma_1}(\eta)u + \boldsymbol{\gamma_2}(\eta)p + \boldsymbol{\gamma_3}(\eta)q + \boldsymbol{\gamma_4}(\eta)r \\ \Updownarrow \\ \dot{\eta} = \boldsymbol{\Upsilon}(\eta)\boldsymbol{u}, \end{gathered} \tag{4.4}$$

where

$$\boldsymbol{\gamma_1}(\eta) = \begin{bmatrix} \cos(\theta)\cos(\psi) \\ \cos(\theta)\sin(\psi) \\ -\sin(\theta) \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{\gamma_2}(\eta) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

$$\boldsymbol{\gamma_3}(\eta) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \sin(\phi)\tan(\theta) \\ \cos(\phi) \\ \sin(\phi)\sec(\theta) \end{bmatrix}, \quad \boldsymbol{\gamma_4}(\eta) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cos(\phi)\tan(\theta) \\ -\sin(\phi) \\ \cos(\phi)\sec(\theta) \end{bmatrix}, \tag{4.5}$$

and

$$\begin{aligned} \boldsymbol{\Upsilon}(\eta) &= \left[\boldsymbol{\gamma_1}(\eta)\ \boldsymbol{\gamma_2}(\eta)\ \boldsymbol{\gamma_3}(\eta)\ \boldsymbol{\gamma_4}(\eta)\right], \\ \boldsymbol{u} &= [u\ p\ q\ r]^T. \end{aligned} \tag{4.6}$$

From (4.4) and (4.5), a calculation shows that the following nonhonolomic (non-integrable) constraints are satisfied:

$$\begin{aligned} &\big(\cos(\psi)\sin(\theta)\sin(\phi) - \sin(\psi)\cos(\phi)\big)\dot{x} + \\ &\qquad \big(\sin(\psi)\sin(\theta)\sin(\phi) + \cos(\psi)\cos(\phi)\big)\dot{y} + \cos(\theta)\sin(\phi)\dot{z} = 0, \\ &\big(\sin(\psi)\sin(\theta)\cos(\phi) - \sin(\psi)\sin(\phi)\big)\dot{x} + \\ &\qquad \big(\sin(\psi)\sin(\theta)\cos(\phi) - \cos(\psi)\sin(\phi)\big)\dot{y} + \cos(\theta)\cos(\phi)\dot{z} = 0. \end{aligned} \tag{4.7}$$

Based on (4.4), we will address the following controllability issues: Controllability about a point (i.e., stabilization) and controllability about a trajectory (i.e., trajectory-tracking).

4.1.3 Controllability at a Point

We will first consider a linear approximation of the system (4.4) at an equilibrium point $\boldsymbol{\eta_e}$. Let the error associated with the equilibrium point $\boldsymbol{\eta_e}$ be as follows:

$$\tilde{\boldsymbol{\eta}} = \boldsymbol{\eta} - \boldsymbol{\eta_e}. \tag{4.8}$$

With (4.8), we can write the tangent linearization of (4.4) at the equilibrium point $\boldsymbol{\eta_e}$ as

$$\dot{\tilde{\boldsymbol{\eta}}} = \boldsymbol{\Upsilon}(\boldsymbol{\eta_e})\boldsymbol{u}, \tag{4.9}$$

which is not controllable because the rank of the matrix $\boldsymbol{\Upsilon}(\boldsymbol{\eta_e})$ is 4. This implies that a linear controller will never achieve posture stabilization, not even in a local sense. In order to study the controllability of the vessel in question, we need to use some tools (the Lie algebra rank condition and nilpotent concepts) from nonlinear control theory [4].

Given a set of generators or basis vector fields $\boldsymbol{\gamma_1}$, $\boldsymbol{\gamma_2}$,..., $\boldsymbol{\gamma_{m_c}}$, the length of a Lie product recursively defined as

$$\begin{aligned} &\ell\{\gamma_i\} = 1, \quad i = 1, 2, ..., m_c \\ &\ell([A, B]) = \ell[A] + \ell[B], \end{aligned} \tag{4.10}$$

where A and B are themselves Lie products. Alternatively, $\ell[A]$ is the number of generators in the expansion for A. A Lie algebra or basis is nilpotent if there exists an integer k such that all Lie products of length greater than k are zero. The integer k is called the order of nilpotency. The use of the nilpotent basis eliminates the need for cumbersome computations as we see that all higher order Lie brackets above some particular order are zero.

The above concepts and conditions imply that Lie algebra $L\{\boldsymbol{\gamma_1}, \boldsymbol{\gamma_2}, \boldsymbol{\gamma_3}, \boldsymbol{\gamma_4}\}$ is nilpotent algebra of order $k = 2$, i.e., the vector fields $\boldsymbol{\gamma_1}$, $\boldsymbol{\gamma_2}$, $\boldsymbol{\gamma_3}$, and $\boldsymbol{\gamma_4}$ are the nilpotent basis. Thus all Lie brackets of order greater than two are zero. The only independent Lie brackets computed from the four basis vector fields are $[\boldsymbol{\gamma_1}, \boldsymbol{\gamma_3}]$ and $[\boldsymbol{\gamma_1}, \boldsymbol{\gamma_4}]$. Therefore, for our system the Lie algebra rank condition becomes

$$\text{rank}[\boldsymbol{C_c}] = 6 \Leftrightarrow \text{rank}[\boldsymbol{\gamma_1}, \boldsymbol{\gamma_2}, \boldsymbol{\gamma_3}, \boldsymbol{\gamma_4}, [\boldsymbol{\gamma_1}, \boldsymbol{\gamma_3}], [\boldsymbol{\gamma_1}, \boldsymbol{\gamma_4}]] = 6, \tag{4.11}$$

where $[\boldsymbol{\gamma_1}, \boldsymbol{\gamma_3}]$ and $[\boldsymbol{\gamma_2}, \boldsymbol{\gamma_4}]$ are the two independent Lie brackets computed from the four vector fields $(\boldsymbol{\gamma_1}, \boldsymbol{\gamma_2}, \boldsymbol{\gamma_3}, \boldsymbol{\gamma_4})$ and C_c is called the controllability matrix. For two vector fields $g(x)$ and $h(x)$, a Lie bracket is computed based on the following formula:

$$[g,h](x) = \frac{\partial h}{\partial x} g - \frac{\partial g}{\partial x} h. \tag{4.12}$$

Using the definition (4.12), Lie brackets $[\boldsymbol{\gamma_1}, \boldsymbol{\gamma_3}]$ and $[\boldsymbol{\gamma_1}, \boldsymbol{\gamma_4}]$ are given by

$$[\boldsymbol{\gamma_1}, \boldsymbol{\gamma_3}] = \frac{\partial \boldsymbol{\gamma_3}}{\partial \boldsymbol{\eta}} \boldsymbol{\gamma_1} - \frac{\partial \boldsymbol{\gamma_1}}{\partial \boldsymbol{\eta}} \boldsymbol{\gamma_3} = \begin{bmatrix} \cos(\psi)\sin(\theta)\cos(\phi) + \sin(\psi)\sin(\phi) \\ \sin(\psi)\sin(\theta)\cos(\phi) - \cos(\psi)\sin(\phi) \\ \cos(\theta)\cos(\phi) \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$[\boldsymbol{\gamma_1}, \boldsymbol{\gamma_4}] = \frac{\partial \boldsymbol{\gamma_4}}{\partial \boldsymbol{\eta}} \boldsymbol{\gamma_1} - \frac{\partial \boldsymbol{\gamma_1}}{\partial \boldsymbol{\eta}} \boldsymbol{\gamma_4} = \begin{bmatrix} -\cos(\psi)\sin(\theta)\sin(\phi) + \sin(\psi)\cos(\phi) \\ -\sin(\psi)\sin(\theta)\sin(\phi) - \cos(\psi)\cos(\phi) \\ -\cos(\theta)\sin(\phi) \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{4.13}$$

Therefore, the controllability matrix $\boldsymbol{C_c}$ is given by

$$\boldsymbol{C_c} = \left[\begin{matrix} \cos(\psi)\cos(\theta) & 0 & 0 & 0 \\ \sin(\psi)\cos(\theta) & 0 & 0 & 0 \\ -\sin(\theta) & 0 & 0 & 0 \\ 0 & 1 & \sin(\phi)\tan(\theta) & \cos(\phi)\tan(\theta) \\ 0 & 0 & \cos(\phi) & -\sin(\phi) \\ 0 & 0 & \sin(\phi)\sec(\theta) & \cos(\phi)\sec(\theta) \end{matrix}\right. \begin{matrix} \cos(\psi)\sin(\theta)\cos(\phi) + \sin(\psi)\sin(\phi) \\ \sin(\psi)\sin(\theta)\cos(\phi) - \cos(\psi)\sin(\phi) \\ \cos(\theta)\cos(\phi) \\ 0 \\ 0 \\ 0 \end{matrix} \left.\begin{matrix} -\cos(\psi)\sin(\theta)\sin(\phi) + \sin(\psi)\cos(\phi) \\ -\sin(\psi)\sin(\theta)\sin(\phi) - \cos(\psi)\cos(\phi) \\ -\cos(\theta)\sin(\phi) \\ 0 \\ 0 \\ 0 \end{matrix}\right]. \tag{4.14}$$

It can be seen that the above matrix $\boldsymbol{C_c}$ has one nonzero minor of order 6. Therefore, this matrix is full rank provided that $\theta \neq \frac{\pi}{2}$. This implies that the vessel is locally controllable and also globally controllable as long as the singular condition $\theta \neq \frac{\pi}{2}$ is avoided. As for the stabilizability of system (4.4) to a point, the failure of the previous linear analysis indicates that exponential stability cannot be achieved by smooth feedback [21]. Things turn out to be even worse: If smooth (in fact, even continuous) time-invariant feedback laws are used, Lyapunov stability cannot be used directly. This negative result is established on the basis of a necessary condition due to Brockett [21], see Section 2.7.4: Smooth stabilizability of a driftless regular system (i.e., such that the input vector fields are well defined and linearly independent at $\boldsymbol{\eta_e}$) requires that the number of inputs be equal to the number of states. The above difficulty has a deep impact on the control design. In fact, to obtain a posture stabilizing controller it is either necessary to give up the continuity requirement and/or to resort to time-varying control laws.

4.1.4 Controllability About a Trajectory

For the system (4.4), let the reference trajectory $\boldsymbol{\eta_d}$ and the reference trajectory input $\boldsymbol{u_d}$ be

$$\boldsymbol{\eta_d} = \begin{bmatrix} x_d(t) \\ y_d(t) \\ z_d(t) \\ \phi_d(t) \\ \theta_d(t) \\ \psi_d(t) \end{bmatrix}, \ \boldsymbol{u_d} = \begin{bmatrix} u_d(t) \\ p_d(t) \\ q_d(t) \\ r_d(t) \end{bmatrix}. \tag{4.15}$$

Indeed, the reference trajectory $\boldsymbol{\eta_d}$ and the reference trajectory input $\boldsymbol{u_d}$ should satisfy the nonholonomic constraints (4.7), i.e.,

$$\begin{aligned}
&\big(\cos(\psi_d)\sin(\theta_d)\sin(\phi_d) - \sin(\psi_d)\cos(\phi_d)\big)\dot{x}_d + \\
&\quad \big(\sin(\psi_d)\sin(\theta_d)\sin(\phi_d) + \cos(\psi_d)\cos(\phi_d)\big)\dot{y}_d + \cos(\theta_d)\sin(\phi_d)\dot{z}_d = 0, \\
&\big(\sin(\psi_d)\sin(\theta_d)\cos(\phi_d) - \sin(\psi_d)\sin(\phi_d)\big)\dot{x}_d + \\
&\quad \big(\sin(\psi_d)\sin(\theta_d)\cos(\phi_d) - \cos(\psi_d)\sin(\phi_d)\big)\dot{y}_d + \cos(\theta_d)\cos(\phi_d)\dot{z}_d = 0.
\end{aligned} \tag{4.16}$$

Let the errors associated with the reference trajectory and the reference input trajectory be

$$\begin{aligned}
\boldsymbol{\eta_e} &= \boldsymbol{\eta} - \boldsymbol{\eta_d}, \\
\boldsymbol{u_e} &= \boldsymbol{u} - \boldsymbol{u_d}.
\end{aligned} \tag{4.17}$$

Using (4.17), we can write (4.4) as

$$\dot{\boldsymbol{\eta}} = \boldsymbol{\Upsilon}(\boldsymbol{\eta_d} + \boldsymbol{\eta_e})\big(\boldsymbol{u_d} + \boldsymbol{u_e}\big). \tag{4.18}$$

The Taylor series expansion of $\boldsymbol{\Upsilon}(\boldsymbol{\eta_d} + \boldsymbol{\eta_e})$ about the nominal solution $\boldsymbol{\eta_d}$ is given by

$$\dot{\boldsymbol{\eta}} = \left(\boldsymbol{\Upsilon}(\boldsymbol{\eta_d},t) + \left.\frac{\partial \boldsymbol{\Upsilon}(\boldsymbol{\eta})}{\partial \boldsymbol{\eta}}\right|_{\boldsymbol{\eta}=\boldsymbol{\eta_d}} \boldsymbol{\eta_e}(t) + \text{HOT}\right)\left(\boldsymbol{u_d}(t) + \boldsymbol{u_e}(t)\right). \tag{4.19}$$

Since the reference trajectory $\boldsymbol{\eta}$ and the reference input trajectory $\boldsymbol{u_d}$ satisfy the nonholonomic constraints (4.16), we have

$$\dot{\boldsymbol{\eta}}_{\boldsymbol{d}} = \boldsymbol{\Upsilon}(\boldsymbol{\eta_d},t)\boldsymbol{u_d}(t). \tag{4.20}$$

Subtracting (4.19) by (4.20) and ignoring the high-order terms (HOT) gives

$$\begin{aligned}\dot{\boldsymbol{\eta}}_{\boldsymbol{e}} &= \left(\left.\frac{\partial \boldsymbol{\Upsilon}(\boldsymbol{\eta})}{\partial \boldsymbol{\eta}}\right|_{\boldsymbol{\eta}=\boldsymbol{\eta_d}} \boldsymbol{\eta_e}(t)\right)\boldsymbol{u_d}(t) + \boldsymbol{\Upsilon}(\boldsymbol{\eta_d},t)\boldsymbol{u_e}(t) \\ &:= \boldsymbol{A}(t)\boldsymbol{\eta_e}(t) + \boldsymbol{B}(t)\boldsymbol{u_e}(t),\end{aligned} \tag{4.21}$$

where

$$\boldsymbol{A}(t) = \begin{bmatrix} 0_{3\times 3} & \boldsymbol{A_1}(t) \\ 0_{3\times 3} & \boldsymbol{A_2}(t) \end{bmatrix}, \; \boldsymbol{B}(t) = \begin{bmatrix} \boldsymbol{J_{d1}}(t) & 0_{3\times 3} \\ 0_{3\times 1} & \boldsymbol{J_{d2}}(t) \end{bmatrix}, \tag{4.22}$$

with $\boldsymbol{A_1}$ and $\boldsymbol{A_2}$ given by

$$\boldsymbol{A_1}(t) = \begin{bmatrix} 0 & -\cos(\psi_d)\sin(\theta_d)u_d & -\sin(\psi_d)\cos(\theta_d)u_d \\ 0 & -\sin(\psi_d)\sin(\theta_d)u_d & \cos(\psi_d)\cos(\theta_d)u_d \\ 0 & -\cos(\theta_d)u_d & 0 \end{bmatrix},$$

$$\boldsymbol{A_2}(t) = \begin{bmatrix} \cos(\phi_d)\tan(\phi_d)q_d - \sin(\phi_d)\tan(\phi_d)r_d & \sin(\phi_d)\sec^2(\theta_d)q_d + \cos(\phi_d)\sec^2(\theta_d)r_d & 0 \\ -\sin(\phi_d)q_d - \cos(\phi_d)r_d & 0 & 0 \\ \cos(\phi_d)\sec(\theta_d)q_d - \sin(\phi_d)\sec(\theta_d)r_d & \sin(\phi_d)\sec(\theta_d)\tan(\theta_d)q_d + \cos(\phi_d)\sec(\theta_d)\tan(\theta_d)r_d & 0 \end{bmatrix}, \tag{4.23}$$

and $\boldsymbol{J_{d1}}(t)$ and $\boldsymbol{J_{d2}}(t)$ given by

$$\begin{aligned}\boldsymbol{J_{d1}}(t) &= \begin{bmatrix} \cos(\theta_d)\cos(\psi_d) \\ \cos(\theta_d)\sin(\psi_d) \\ -\sin(\theta_d) \end{bmatrix}, \\ \boldsymbol{J_{d2}}(t) &= \begin{bmatrix} 1 & \sin(\phi_d)\tan(\theta_d) & \cos(\phi_d)\tan(\theta_d) \\ 0 & \cos(\phi_d) & -\sin(\phi_d) \\ 0 & \sin(\phi_d)\sec(\theta_d) & \cos(\phi_d)\sec(\theta_d) \end{bmatrix}.\end{aligned} \tag{4.24}$$

In (4.23) and (4.24), the argument t of ϕ_d, θ_d, ψ_d, u_d, p_d, and r_d is omitted for simplicity.

The system (4.21) is linear time-varying. The controllability condition becomes

$$\text{rank}\{\boldsymbol{B}, \boldsymbol{A}\boldsymbol{B}, \boldsymbol{A}^2\boldsymbol{B}, \boldsymbol{A}^3\boldsymbol{B}, \boldsymbol{A}^4\boldsymbol{B}, \boldsymbol{A}^5\boldsymbol{B}\} = 6. \tag{4.25}$$

A calculation shows that the above matrix has a nonzero minor of order 6 provided that ($u_d \neq 0$, $p_d \neq 0$, $q_d \neq 0$, $r_d \neq 0$) and ($\theta_d \neq \frac{\pi}{2}$). Therefore, we conclude that the kinematic system (4.21) can be locally stabilized by linear feedback about trajectories consisting of linear or circular or helix paths, which do not collapse to a point.

4.2 Previous Work on Control of Underactuated Ocean Vessels

This section starts with a brief review on the control of nonholonomic systems, due to their relevance to the control of underactuated ocean vessels. Next, the existing methods on control of underactuated ocean vessels are reviewed. Limitations of the existing methods are then pointed out and hence motivate the contributions of the book.

4.2.1 Control of Nonholonomic Systems

The term "nonholonomic system" originates from classical mechanics and has its widely accepted meaning as a "Lagrange system with linear constraints being nonintegrable". A mechanical system is said to be nonholonomic if its generalized velocity satisfies an equality condition that cannot be written as an equivalent condition on the generalized position, see [30]. Control of nonholonomic dynamic systems has formed an active area in the control community – see surveys by Kolmanovsky and McClamroch in [31], Canudas de Wit et al. in [15], Murray and Sastry in [32], and references therein for an overview and interesting introductory examples in this expanding area.

Nonholonomic systems have inherent difficulties in feedback stabilization at the origin or at a given equilibrium point since the tangent linearization of these systems is uncontrollable. In fact, a direct application of Brockett's necessary condition, see Section 2.7.4 for more details, for feedback stabilization implies that nonholonomic systems cannot be stabilized by any stationary continuous state feedback although they are open loop controllable. As a consequence, the classical smooth control theory cannot be applied. This motivates researchers to seek novel approaches. These approaches can be roughly classified into discontinuous feedback, see for example [33–45] and time-varying feedback, see for example, [15, 32, 46–48]. The discontinuous feedback approach often uses the state scaling originated from the σ-process [49] and a switching control strategy to overcome the difficulty due to the loss of controllability. This approach results in a fast transient response and usually an exponential convergence can be achieved. The drawback is discontinuity in the control input. On the other hand, the time-varying feedback approach provides a smooth/continuous controller, i.e., no switching is required, however the price is slow convergence. The stability analysis is often based on linear time-varying system theory and Barbalat's lemma. The backstepping technique [3] is usually used for high-order chained form systems in both discontinuous and time-varying approaches. Those aforementioned systems are either driftless or have weak nonlinear drifts. When nonholonomic systems are perturbed by drifts with uncertainties, robust and adaptive control approaches are often applied. The robust control design schemes are based on the size domination concept [50]. The control is conservative when a priori knowledge of uncertainties is poor. A class of nonholonomic systems with strong nonlinear uncertainties was recently considered in [51]. Discontinuous state feedback and output feedback controllers were designed to achieve global exponential stability. However the x_0-subsystem is required to be Lipschitz since a constant control input u_0 is used to get around the difficulty due to the loss of controllability. The adaptive approach [38, 40, 46] provides less conservative control input but increases the dynamics of the closed loop system. The systems studied in these papers do not allow drifts in the x_0-subsystem. A difficulty in designing adaptive stabilization controllers for chained systems with drifts is that the state of the x_0-subsystem can have several zero crossings due to transient behavior of the unknown parameter estimate. This phenomenon causes difficulties in applying the state scaling. For a solution of the stabilization of nonholonomic systems in a chained form with strong nonlinear drifts and unknown parameters, the reader is referred to [52].

4.2.2 Control of Underactuated Ships and Underwater Vehicles

Control of underactuated ships and autonomous underwater vehicles (AUVs) is an active field due to its important applications such as passenger and goods transportation, environmental surveying, undersea cable inspection, and offshore oil installations.

Based on its practical requirement, motion control of underactuated ocean vessels has been divided into three areas: Stabilization, trajectory-tracking, and path-following. These control problems are challenging due to the fact that the motion of underactuated surface ships and AUVs possesses more degrees of freedom to be controlled than the number of the independent controls under some nonintegrable second-order nonholonomic constraints [29, 53, 54]. In particular, underactuated ships do not usually have an actuator in the sway axis while in the case of AUVs there are no actuators in the sway and heave directions. This configuration is by far the most common among marine vessels. Therefore, Brockett's condition indicates that any continuous time-invariant feedback control law does not make a null solution of the underactuated surface ship and AUV dynamics asymptotically stable in the sense of Lyapunov. Furthermore as observed in [22, 54], the underactuated ship and AUV system is not transformable into a standard chain system. Consequently, existing control schemes [15, 32–48] developed for chained systems cannot be applied directly. Nevertheless, in the past decade, stabilization, trajectory-tracking control, and path-following of underactuated ocean vessels have been studied separately from different viewpoints.

4.2.2.1 Stabilization

An underactuated ocean vessel belongs to a class of underactuated mechanical systems subject to some nonintegrable second-order nonholonomic constraints, see [29,53,54]. Therefore, design of a feedback stabilizer using linear and classical nonlinear control theories is not possible. There are two main approaches to deal with stabilization of an underactuated ocean vessel. They are (time-invariant and time-varying) discontinuous feedback and time-varying continuous/smooth feedback. We here mention some typical results of both approaches.

Time-invariant and Time-varying Discontinuous Approach

A discontinuous state feedback control law was proposed in [55] using the σ-process to exponentially stabilize an underactuated ship at the origin where the ship model is discontinuously transformed to an extended chained form system. The dynamics of an underactuated ship is considered in [55], see also Section 3.4.1:

$$\begin{aligned}
&\dot{\boldsymbol{\eta}} = \boldsymbol{J}(\boldsymbol{\eta})\boldsymbol{v},\\
&\boldsymbol{M}\dot{\boldsymbol{v}} = -\boldsymbol{C}(\boldsymbol{v})\boldsymbol{v} - \boldsymbol{D}\boldsymbol{v} + \boldsymbol{\tau},\\
&\boldsymbol{\eta} = [x,\ y,\ \psi]^T,\ \boldsymbol{v} = [u,\ v,\ r]^T,\ \boldsymbol{\tau} = [\tau_u,\ 0,\ \tau_r]^T,
\end{aligned} \tag{4.26}$$

where (x, y) denotes the earth-fixed position of the center of mass of the ship, ψ denotes the orientation angle, (u, v) and r are the linear and angular velocities in the body-fixed frame, and (τ_u, τ_r) are the surge force and yaw moment. The matrices $\boldsymbol{J}(\boldsymbol{\eta})$, $\boldsymbol{M}$, $\boldsymbol{C}(\boldsymbol{v})$, and $\boldsymbol{D}$ are given by

$$\boldsymbol{J}(\boldsymbol{\eta}) = \begin{bmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \boldsymbol{M} = \begin{bmatrix} m_{11} & 0 & 0 \\ 0 & m_{22} & 0 \\ 0 & 0 & m_{33} \end{bmatrix},$$

$$\boldsymbol{C}(\boldsymbol{v}) = \begin{bmatrix} 0 & 0 & -m_{22}v \\ 0 & 0 & m_{11}u \\ m_{22}v & -m_{11}u & 0 \end{bmatrix},\ \boldsymbol{D} = \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix}, \tag{4.27}$$

where m_{11}, m_{22}, and m_{33} denote the ship inertia including added mass, and d_{11}, d_{22}, and d_{33} are hydrodynamic damping constants, see Chapter 3 for more details. The control objective is to design the control inputs τ_u and τ_r to stabilize (4.26) asymptotically at the origin. In [55], the coordinate transformation

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} \psi \\ x\cos(\psi) + y\sin(\psi) \\ -x\sin(\psi) + y\cos(\psi) \\ v \\ r \\ u \end{bmatrix} \tag{4.28}$$

is used to transform the ship model (4.26) to the following system

$$\begin{aligned} \dot{x}_1 &= x_5, \\ \dot{x}_2 &= x_6 + x_3 x_5, \\ \dot{x}_3 &= x_4 - x_2 x_5, \\ \dot{x}_4 &= -\alpha x_4 - \beta x_5 x_6, \\ \dot{x}_5 &= \Omega_1, \\ \dot{x}_6 &= \Omega_2, \end{aligned} \tag{4.29}$$

where $\alpha = d_{22}/m_{22}$, $\beta = m_{11}/m_{22}$, and

$$\Omega_1 = \frac{\tau_r - d_{33} r + (m_{11} - m_{22}) uv}{m_{33}}, \; \Omega_2 = \frac{\tau_u + m_{22} vr - d_{11} u}{m_{11}}. \tag{4.30}$$

It can be seen that the system (4.29) consists of two subsystems, namely (x_1, x_2, x_3, x_4) and (x_5, x_6), connected to each other in a strict feedback form [3]. The control design can be simply carried out in two steps as follows.

Step 1

In this step, the author of [55] considers the first four equations of (4.29), and (x_5, x_6) as controls (v_1, v_2). With the assumption of $x_1 \neq 0$, the coordinate transformation (σ-process)

$$y = x_1, \; z_1 = x_2, \; z_2 = \frac{x_3}{x_1}, \; z_3 = \frac{x_4}{x_1} \tag{4.31}$$

results in

$$\begin{aligned} \dot{y} &= v_1, \\ \dot{z}_1 &= v_2 + y z_2 v_1, \\ \dot{z}_2 &= z_3 - \frac{z_1 + z_2}{y} v_1, \\ \dot{z}_3 &= -\alpha z_3 - \frac{z_3 + \beta v_2}{y} v_1. \end{aligned} \tag{4.32}$$

The feedback control law is designed as

$$\begin{aligned} v_1 &= -k_1 y, \\ v_2 &= -k_{21} z_1 - k_{22} z_2 - k_{23} z_3, \end{aligned} \tag{4.33}$$

where k_1, k_{21}, k_{22}, and k_{23} are the control gains chosen such that the matrix

$$A_1 = \begin{bmatrix} -k_{21} & -k_{22} & -k_{23} \\ k_1 & k_1 & 1 \\ -k_1 \beta k_{21} & -k_1 \beta k_{22} & k_1 - \alpha - k_1 \beta k_{23} \end{bmatrix} \tag{4.34}$$

is Hurwitz.

Step 2

At this step, the last two equations of (4.29) are considered. Using the standard backstepping technique results in the following control law

$$\begin{aligned} \Omega_1 &= -k_3(x_5 - v_1) - k_1 x_5, \\ \Omega_2 &= -k_4(x_6 - v_2) - k_{21}(x_6 + x_3 x_5) \\ &\quad - k_{22}\frac{x_4 - x_2 x_5}{x_1} + k_{23}\frac{\alpha x_4 + \beta x_5 x_6}{x_1} + k_{22}\frac{x_3 x_5}{x_1^2} + k_{23}\frac{x_4 x_5}{x_1^2}, \end{aligned} \tag{4.35}$$

where $k_3 > k_1$ and k_4 are positive constants. The actual controls τ_u and τ_r can be found from (4.35) and (4.30). In [55] it is shown that if the initial conditions $x_1(t_0) \neq 0$ and $x_1(t_0)(x_5(t_0) + k_1 x_1(t_0)) \geq 0$ then $(x_1(t), x_2(t), x_3(t), x_4(t), x_5(t), x_6(t))$ is bounded for all $t \geq t_0 \geq 0$, and exponentially converges to zero. If the above conditions do not hold, the controls

$$\begin{aligned} \Omega_1 &= -|x_1 - \epsilon|^a \mathrm{sign}(x_1 - \epsilon) - |x_5|^b \mathrm{sign}(x_5), \\ \Omega_2 &= 0, \end{aligned} \tag{4.36}$$

with $\epsilon \neq 0$, $b \in (0,1)$, and $a > b/(2-b)$ being constants, can be used to make the above conditions hold in finite time. For more details, the reader is referred to [55].

Remark 4.1. The aforementioned discontinuous stabilizer provides a fast convergence of the stabilizing errors to zero. However, the control inputs τ_u and τ_r are discontinuous. Moreover, under arbitrarily small nonvanishing environmental disturbances induced by waves, wind, and ocean currents, the closed loop system consisting of (4.35) and (4.29) can be unstable in the sense that the states $(x_1(t), x_2(t), x_3(t), x_4(t), x_5(t), x_6(t))$ can go to infinity exponentially fast.

The work mentioned in [22,53,54,56–58] can also be grouped in the discontinuous approach. The authors of [22] developed a discontinuous time-varying feedback stabilizer for a nonholonomic system and applied it to underactuated ships. Some local exponential stabilization results were reported in [53,54] based on the time-varying homogeneous control approach. An application of averaging and backstepping techniques was proposed in [59] to design a global practical controller for stabilization and tracking control of surface ships. Experimental results on dynamic positioning of underactuated ships were reported in [56]. By transforming the underactuated ship kinematics and dynamics into the so-called skew form, some dynamic feedback results on stabilization were given in [57]. In [58], the authors proposed a discontinuous solution to the problem of steering an underactuated AUV to a point with desired orientation using the polar coordinate transformation motivated from the work in [60].

Time-varying Continuous/Smooth Approach

A typical result on stabilization of the underactuated ship (4.26) in the time-varying continuous/smooth approach is given in [61]. In [61], the coordinate transformations (similar to the ones given in (4.30) and (4.28))

$$\begin{aligned} z_1 &= \cos(\psi)x + \sin(\psi)y, \\ z_2 &= -\sin(\psi)x + \cos(\psi)y, \\ z_3 &= \psi, \\ \Omega_1 &= \frac{\tau_r - d_{33}r + (m_{11} - m_{22})uv}{m_{33}}, \\ \Omega_2 &= \frac{\tau_u + m_{22}vr - d_{11}u}{m_{11}} \end{aligned} \tag{4.37}$$

are first used to transform the ship system (4.26) to

$$\begin{aligned} \dot{z}_1 &= u + z_2 r, \\ \dot{z}_2 &= v - z_1 r, \\ \dot{z}_3 &= r, \\ \dot{u} &= \Omega_2, \\ \dot{v} &= -cur - dv, \\ \dot{r} &= \Omega_1, \end{aligned} \tag{4.38}$$

where $c = m_{11}/m_{22}$ and $d = d_{22}/m_{22}$. Then the following nontrivial coordinate transformations

$$\begin{aligned} Z_2 &= z_2 + \frac{v}{d}, \\ u &= -\frac{d}{c}z_1 - \frac{d}{c}\mu, \\ \Omega_{2\mu} &= \frac{d}{c}z_1 + \frac{d}{c}\mu - Z_2 r + \frac{v}{d}r - \frac{c}{d}\Omega_2 \end{aligned} \tag{4.39}$$

are applied to (4.38) to obtain the system

$$
\begin{aligned}
\dot{z}_1 &= -\frac{d}{c}z_1 - \frac{d}{c}\mu + Z_2 r - \frac{v}{d}r,\\
\dot{Z}_2 &= \mu r,\\
\dot{z}_3 &= r,\\
\dot{v} &= -dv + d(z_1 + \mu)r,\\
\dot{\mu} &= \Omega_{2\mu},\\
\dot{r} &= \Omega_1.
\end{aligned} \tag{4.40}
$$

Let k_2, k_3, k_μ, and k_r be strictly positive constants such that $1 \geq k_2 \geq k_3$. The controls Ω_1 and $\Omega_{2\mu}$ are designed in [61] as

$$
\begin{aligned}
\Omega_1 &= -k_r(r - r_f) + \dot{r}_f - \lambda\big(Z_2\mu_f + 2Z_3 + 2Z_2k_2\cos(t)\mu_f\big),\\
\Omega_{2\mu} &= -k_\mu(\mu - \mu_f) + \dot{\mu}_f - \lambda\big(Z_2 + 2Z_3k_2\cos(t)\big)r,
\end{aligned} \tag{4.41}
$$

where

$$
\begin{aligned}
\lambda &= 2 + \frac{k_3}{3} - \frac{k_3\sin(2t)}{6}\frac{2V_1 + V_1^2}{(1+V_1)^2},\\
Z_3 &= z_3 + k_2\cos(t)Z_2,\\
V_1 &= Z_2^2 + 2Z_3^2,\\
\mu_f &= -\frac{\sin(t)Z_2^2}{2(0.001 + Z_2^2)},\\
r_f &= \frac{-k_3Z_3 + k_2\sin(t)Z_2}{1 + k_2\cos(t)\mu_f}.
\end{aligned} \tag{4.42}
$$

In [61], it is proven that the closed loop system consisting of (4.41), (4.39), (4.37), and (4.26) is GAS at the origin.

Remark 4.2. The design of the feedback given in (4.41) is nontrivial. Overall, convergence of the stabilizing errors to zero is slow. This is a well-known phenomenon of the continuous/smooth time-varying approach applying not to only underactuated ships but also to mobile robots. Moreover, since the stabilizer design mentioned above is nontrivial, it is difficult to extend the control design scheme to solve a trajectory-tracking problem, see the next section. In addition, the physical meaning of the feedback is not clear.

In addition to the aforementioned results on stabilization of underactuated vessels, the following results are also related to the topic under discussion. In [62], several control configurations were considered, and a technique for synthesizing open loop controls was given. The first control scheme with the dynamic AUV model taken into account was proposed in [63]. A kinematic drift free model of the underwater vehicles with four control inputs was used to design a regulation controller in [64]. The authors of [65] proposed a controller that is able to stabilize an AUV to some equilibria based on the interconnection and damping assignment passivity-based control approach, which has been successfully applied to many other mechanical systems [66]. See also [67] for stabilization results of underactuated mechanical systems on Riemannian manifolds.

4.2.2.2 Trajectory-tracking

Trajectory-tracking is here defined as a control problem of forcing an underactuated surface ship or AUV to track a reference trajectory generated by a suitable vessel model, i.e., the vessel model that has the same parameters as the real one. There are two main approaches to solve the trajectory-tracking control problems. The first approach is based on linear time-varying control system theory while the second approach relies on the Lyapunov direct method. We here briefly describe typical results of the two approaches.

Linear Time-varying Approach

A typical work belonging to this approach is given in [68] on a global K-exponential tracking result for the underactuated ship (4.26). In [68], the authors consider a problem of designing the control τ_u and τ_r to force the position (x, y) and orientation ψ of the ship (4.26) to track the reference position (x_d, y_d) and orientation ψ_d generated by the reference ship model

$$\begin{aligned}
\dot{\boldsymbol{\eta}}_{\boldsymbol{d}} &= \boldsymbol{J}(\boldsymbol{\eta}_{\boldsymbol{d}})\boldsymbol{v}_{\boldsymbol{d}},\\
\boldsymbol{M}\dot{\boldsymbol{v}}_{\boldsymbol{d}} &= -\boldsymbol{C}(\boldsymbol{v}_{\boldsymbol{d}})\boldsymbol{v}_{\boldsymbol{d}} - \boldsymbol{D}\boldsymbol{v}_{\boldsymbol{d}} + \boldsymbol{\tau}_{\boldsymbol{d}},\\
\boldsymbol{\eta}_{\boldsymbol{d}} &= \begin{bmatrix} x_d \\ y_d \\ \psi_d \end{bmatrix},\ \boldsymbol{v}_{\boldsymbol{d}} = \begin{bmatrix} u_d \\ v_d \\ r_d \end{bmatrix},\ \boldsymbol{\tau}_{\boldsymbol{d}} = \begin{bmatrix} \tau_{ud} \\ 0 \\ \tau_{rd} \end{bmatrix}.
\end{aligned} \tag{4.43}$$

In [68], the coordinate transformations

$$\begin{cases} z_1 = \cos(\psi)x + \sin(\psi)y, \\ z_2 = -\sin(\psi)x + \cos(\psi)y, \\ z_3 = \psi, \end{cases} \qquad \begin{cases} z_{1d} = \cos(\psi_d)x_d + \sin(\psi_d)y_d, \\ z_{2d} = -\sin(\psi_d)x_d + \cos(\psi_d)y_d, \\ z_{3d} = \psi_d, \end{cases} \tag{4.44}$$

and the tracking errors

$$\begin{aligned}
&u_e = u - u_d,\ v_e = v - v_d,\ r_e = r - r_d,\\
&z_{1e} = z_1 - z_{1d},\ z_{2e} = z_2 - z_{2d},\ z_{3e} = z_3 - z_{3d}
\end{aligned} \tag{4.45}$$

are used to obtain the tracking error dynamics of a chained form

$$\dot{u}_e = \frac{m_{22}}{m_{11}}\big(v_e r_e + v_e r_d + v_d r_e\big) - \frac{d_{11}}{m_{11}}u_e + \frac{1}{m_{11}}(\tau_u - \tau_{ud}),$$

$$\begin{aligned}
\dot{v}_e &= -\frac{m_{11}}{m_{22}}\big(u_e r_e + u_e r_d + u_d r_e\big) - \frac{d_{22}}{m_{22}} v_e,\\
\dot{r}_e &= \frac{m_{11}-m_{22}}{m_{33}}\big(u_e v_e + u_e v_d + u_d v_e\big) - \frac{d_{33}}{m_{33}} r_e + \frac{1}{m_{33}}(\tau_r - \tau_{rd}),\\
\dot{z}_{1e} &= u_e + z_{2e} r_e + z_{2e} r_d + z_{2d} r_e,\\
\dot{z}_{2e} &= v_e - z_{1e} r_e - z_{1e} r_d - z_{1d} r_e,\\
\dot{z}_{3e} &= r_e.
\end{aligned} \tag{4.46}$$

Assuming that u_d, v_d, z_{1d}, and z_{2d} are bounded, and that $r_d(t)$ is persistently exciting, the controls

$$\begin{aligned}
\tau_u &= \tau_{ud} - k_1 u_e + k_2 r_d v_e - k_3 z_{1e} + k_4 r_d z_{2e},\\
\tau_r &= \tau_{rd} - (m_{11}-m_{22})(u_e v_e + v_d u_e + u_d v_e) -\\
&\quad k_5 r_e - k_6 z_{3e},
\end{aligned} \tag{4.47}$$

where the control gains k_i, $i = 1,\ldots,6$ satisfy

$$\begin{aligned}
&k_1 > d_{22} - d_{11},\\
&k_2 = \frac{m_{22}k_4(k_4 + k_1 + d_{11} - d_{22})}{d_{22}k_4 + m_{11}k_3},\\
&0 < k_3 < (k_1 + d_{11} - d_{22})\frac{d_{22}}{m_{11}},\\
&k_4 > 0,\\
&k_5 > -d_{33},\\
&k_6 > 0,
\end{aligned} \tag{4.48}$$

make the closed loop system consisting of (4.47), (4.43), and (4.26), that is,

$$\begin{bmatrix} \dot{u}_e \\ \dot{v}_e \\ \dot{z}_{1e} \\ \dot{z}_{2e} \end{bmatrix} = \begin{bmatrix} -\frac{k_1+d_{11}}{m_{11}} & \frac{k_2+m_{22}}{m_{11}} r_d(t) & -\frac{k_3}{m_{11}} & \frac{k_4}{m_{11}} r_d(t) \\ -\frac{m_{11}}{m_{22}} r_d(t) & -\frac{d_{22}}{m_{22}} & 0 & 0 \\ 1 & 0 & 0 & r_d(t) \\ 0 & 1 & -r_d(t) & 0 \end{bmatrix} \begin{bmatrix} u_e \\ v_e \\ z_{1e} \\ z_{2e} \end{bmatrix} +$$

$$\begin{bmatrix} \frac{m_{22}}{m_{11}}(v_e + v_d) & 0 \\ -\frac{m_{11}}{m_{22}}(u_e + u_d) & 0 \\ z_{2e} + z_{2d} & 0 \\ -(z_{1e} + z_{1d}) & 0 \end{bmatrix} \begin{bmatrix} r_e \\ z_{3e} \end{bmatrix},$$

$$\begin{bmatrix} \dot{r}_e \\ \dot{z}_{3e} \end{bmatrix} = \begin{bmatrix} -\frac{d_{33}+k_5}{m_{33}} & -\frac{k_6}{m_{33}} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} r_e \\ z_{3e} \end{bmatrix}, \tag{4.49}$$

globally K-exponentially stable at the origin. Proof of K-exponential stability of the above closed loop system is straightforward using the results on stability of cascade systems in [17] and [69], and those of linear time-varying system theory in [6], see [68] for details. It should be noted that the persistently exciting condition on the yaw reference velocity, r_d, is required to prove K-exponential stability of the closed loop system (4.49).

Remark 4.3. The persistent exciting condition on the yaw reference velocity r_d excludes a straight-line reference trajectory. In comparison with the direct Lyapunov approach summarized below, it is difficult to deal with any external disturbances and/or actuator dynamics using the linear time-varying approach.

Direct Lyapunov Approach

The direct Lyapunov method has been widely used in designing controllers for underactuated ocean vessels. However, the use of the Lyapunov direct method for designing control systems for underactuated ocean vessels is not straightforward due to the underactuated nature of ocean vessels. We here describe typical results of trajectory-tracking control based on the Lyapunov direct method.

Local Trajectory-tracking Results

An application of the recursive technique proposed in [70] for the standard chain-form systems yields a high-gain based local tracking result in [59] for surface ships. The experimental results of this proposed controller were reported in [71]. The control design in [59,71] starts from (4.46) as follows. First of all, the following restrictive assumption is made on the yaw reference velocity:

$$0 < r_{d\,\min} < |r_d(t)| < r_{d\,\max}, \tag{4.50}$$

where $r_{d\,\min}$ and $r_{d\,\max}$ are strictly positive constants. Motivated by the work in [70], the authors define new error variables as

$$\begin{aligned} \omega_1 &= z_{1e} - z_2 z_{3e}, \\ \omega_2 &= z_{2e} + z_1 z_{3e}, \\ y_1 &= v_e + cuz_{3e} + k_2\omega_2, \\ y_2 &= u_e + k_1\omega_1 - \frac{k_2(d-k_2)}{cr_d}\omega_2, \\ y_3 &= z_{3e}, \end{aligned} \tag{4.51}$$

where $c = m_{11}/m_{22}$, $d = d_{22}/m_{22}$, and k_2 is a parameter to be determined later. It should be stressed that the condition (4.50) on the yaw reference velocity, r_d is required so that the error transformations (4.51) are valid. With (4.51), the tracking error system (4.46) is written in a triangular-like structure as follows:

$$\begin{aligned}
\dot{\omega}_1 &= y_2 - k_1\omega_1 + \frac{k_2(d-k_2)}{cr_d}\omega_2 + \omega_2 r_d - (v - z_1 r_e)y_3,\\
\dot{\omega}_2 &= y_1 - k_2\omega_2 - \omega_1 r_d + ((1-c)u + z_2 r_e)y_3,\\
\dot{y}_1 &= -cy_2 r_d + (ck_1 - k_2)r_d\omega_1 - (d-k_2)y_1 + (c\Omega_2 + cdu +\\
&\quad k_2((1-c)u + z_2 r_e))y_3,\\
\dot{y}_2 &= \Omega_2 - \Omega_{2d} + k_1 y_2 - k_1^2\omega_1 + \frac{1}{cr_d}k_1k_2(d-k_2)\omega_2 + k_1 r_d\omega_2 +\\
&\quad \frac{\dot{r}_d}{cr_d^2}k_2(d-k_2)\omega_2 - \frac{1}{cr_d}k_2(d-k_2)(y_1 - k_2\omega_2 - r_d\omega_1) -\\
&\quad (k_1(v - z_1 r_e) + \frac{1}{cr_d}k_2(d-k_2)((1-c)u + z_2 r_e))y_3,\\
\dot{y}_3 &= r_e,\\
\dot{r}_e &= \Omega_1 - \Omega_{1d},
\end{aligned} \tag{4.52}$$

where Ω_1 and Ω_2 are given in (4.30), and

$$\begin{aligned}
\Omega_{1d} &= \frac{\tau_{rd} - d_{33}r_d + (m_{11} - m_{22})u_d v_d}{m_{33}},\\
\Omega_{2d} &= \frac{\tau_{ud} + m_{22}v_d r_d - d_{11}u_d}{m_{11}}.
\end{aligned} \tag{4.53}$$

The triangular structure (4.52) allows us to use the backstepping technique [3] to design the controls Ω_1 and Ω_2. In [71], the the controls Ω_1 and Ω_2 are designed as

$$\begin{aligned}
\Omega_1 &= -a_3(r - \alpha_r) + \dot{\alpha}_r - \kappa y_3,\\
\Omega_2 &= -a_1 y_2 - \gamma\omega_1 + car_d y_1 - \Big(-\Omega_{2d} + k_1 y_2 - k_1^2\omega_1 +\\
&\quad \frac{1}{cr_d}k_1k_2(d-k_2)\omega_2 + k_1 r_d\omega_2 + \frac{\dot{r}_d}{cr_d^2}k_2(d-k_2)\omega_2 - \frac{1}{cr_d}\times\\
&\quad k_2(d-k_2)(y_1 - k_2\omega_2 - r_d\omega_1)\Big),
\end{aligned} \tag{4.54}$$

where

$$\begin{aligned}
\alpha_r = &\left(\lambda + \gamma(\omega_1 z_1 + \omega_2 z_2) + ak_2 y_1 z_2 + k_1 y_2 z_1 - \frac{1}{cr_d}k_2(d-k_2)y_2 z_2\right)^{-1}\times\\
&\Big(-a_2 y_3 + \gamma\omega_1 v - \gamma\omega_2(1-c)u - ay_1(c(\Omega_2 + du) + k_2(1-c)u) +
\end{aligned}$$

$$k_1 y_2 v + \frac{1}{c r_d} k_2 (d - k_2)(1 - c) u y_2 \Bigg) + r_d,$$

$$\kappa = \lambda + \gamma \omega_1 z_1 + \gamma \omega_2 z_2 + a k_2 y_1 z_2 + k_1 y_2 z_1 - \frac{1}{c r_d} k_2 (d - k_2) y_2 z_2. \quad (4.55)$$

In (4.54) and (4.55), the control parameters k_1, k_2, a, a_1, a_2, a_3, and λ are positive constants, and are chosen such that

$$k_2 < d, \; k_1 > \frac{k_2 (d - k_2)^2}{c^2 r_{d\,\min}^2},$$
$$\frac{1}{k_2 (d - k_2)} < \frac{a}{\gamma} < \frac{k_1 (d - k_2)}{(c k_1 - k_2)^2 r_{d\,\max}^2}. \quad (4.56)$$

It is noted that the virtual control α_r given in (4.55) is solvable if and only if

$$\lambda > -\left(\gamma \omega_1 z_1 + \gamma \omega_2 z_2 + a k_2 y_1 z_2 + k_1 y_2 z_1 - \frac{1}{c r_d} k_2 (d - k_2) y_2 z_2 \right). \quad (4.57)$$

Proof of local exponential stability of the closed loop system consisting of (4.54), (4.46), and (4.30) can be carried out by using the Lyapunov function

$$V = \frac{1}{2} \gamma \omega_1^2 + \frac{1}{2} \gamma \omega_2^2 + \frac{1}{2} a y_1^2 + \frac{1}{2} y_2^2 + \frac{\lambda}{2} y_3^2 + \frac{1}{2} (r - \alpha_r)^2. \quad (4.58)$$

Remark 4.4. There are two limitations of the aforementioned tracking controllers. These limitations are described in conditions (4.50) and (4.57). The condition (4.50) implies that the reference yaw velocity r_d cannot be zero at any time. This restrictive condition excludes a straight-line reference trajectory. The condition (4.57) implies that the aforementioned trajectory-tracking result is inherently local. One can argue that by the control parameters k_1, k_2, a, a_1, a_2, a_3, and λ may increase the size of the attraction region. However, it is very hard to ensure this property since the control parameters must satisfy various conditions specified in (4.56). In fact, this is true, as said in [71].

Global Trajectory-tracking Results

Based on Lyapunov's direct method and the passivity approach [72], two restricted tracking solutions of an underactuated surface ship were proposed in [19]. We here briefly summarize the result based on the passivity approach in [19]. The result based on the standard backstepping technique [3] is discussed later. In [19], the starting point is the tracking error system (4.46). The passivity based method consists of two steps as follows.

Step 1

Design of the surge force τ_u: This force is designed based on the Lyapunov function

$$V_1 = \frac{1}{2}\left(z_{1e} - \lambda_1 z_{2e} r_d\right)^2 + \frac{1}{2} z_{2e}^2 + \frac{\lambda_0}{2} v_e^2 + \frac{1}{2}\bar{u}_e^2, \tag{4.59}$$

where $\bar{u}_e = u_e - \alpha_0$ with

$$\alpha_0 = -\lambda_2 (z_{1e} - \lambda_1 z_{2e} r_d). \tag{4.60}$$

In (4.59) and (4.60), the control parameters λ_0, λ_1 and λ_2 are chosen such that

$$c(t) = \min\left(2\epsilon(\lambda_2 - \lambda_1 r_d^2(t)),\, 2\left(\lambda_1 r_d^2(t) - \frac{m_{22}}{(1-\epsilon)\lambda_0 d_{22}}\right),\, 2\epsilon\frac{d_{22}}{m_{22}},\, 2c_1\right) \geq c^*, \tag{4.61}$$

where c_1 is a positive constant, $0 < \epsilon < 1$, and c^* is strictly positive. In (4.59) and (4.60), α_0 is understood as a virtual control of u_e. From the first time derivative of the Lyapunov function V_1 given in (4.59) along the solutions of (4.46), a choice of the surge force τ_u

$$\begin{aligned} \tau_u = \; & \tau_{ud} + m_{11}\Bigg[-\frac{m_{22}}{m_{11}}(vr - v_d r_d) + \frac{d_{11}}{m_{11}} u_e - c_1(u_e + \lambda_2(z_{1e} - \lambda_1 z_{2e} r_d)) - \\ & \left((z_{1e} - \lambda_1 z_{2e} r_d) - \frac{\lambda_0 m_{11}}{m_{22}} r_d v_e\right) - \lambda_2(u_e + z_{2e} r_d + z_2 r_e) + \\ & \lambda_1 \lambda_2 \dot{r}_d z_{2e} + \lambda_1 \lambda_2 r_d (v_e - z_{1e} r_d - z_1 r_e)\Bigg] \end{aligned} \tag{4.62}$$

gives

$$\dot{V}_1 \leq -c(t) V_1 + \left[(z_{1e} - \lambda_1 z_{2e} r_d)(z_2 + \lambda_1 r_d z_1) - z_{2e} z_1 - \frac{\lambda_0 m_{11}}{m_{22}} v_e u\right] r_e, \tag{4.63}$$

where $c(t)$ is given in (4.61).

Step 2

Design of the yaw moment τ_r: This moment is designed based on the Lyapunov function

$$V_2 = V_1 + \frac{1}{2} z_{3e}^2 + \frac{1}{2}\bar{r}_e^2, \tag{4.64}$$

where $\bar{r}_e = r_e - \alpha_1$ and

$$\alpha_1 = -c_2\left[(z_{1e} - \lambda_1 z_{2e} r_d)(z_2 + \lambda_1 r_d z_1) - z_{2e} z_1 - \frac{\lambda_0 m_{11}}{m_{22}} v_e u + z_{3e}\right], \tag{4.65}$$

with $c_2 > 0$. From the first time derivative of the Lyapunov function V_2 given in (4.64) along the solutions of (4.46), a choice of the yaw moment

$$\tau_r = \tau_{rd} + m_{33}\Bigg[-\frac{m_{11}-m_{22}}{m_{33}}(uv - u_d v_d) + \frac{d_{33}}{m_{33}} r_e - c_3 \bar{r}_e + \dot{\alpha}_1 - \Bigg((z_{1e} - \lambda_1 z_{2e} r_d)(z_2 + \lambda_1 r_d z_1) - z_{2e} z_1 - \frac{\lambda_0 m_{11}}{m_{22}} v_e u + z_{3e} \Bigg) \Bigg] \quad (4.66)$$

where $c_3 > 0$ results in

$$\dot{V}_2 \leq -c(t)V_1 - c_2\Bigg((z_{1e} - \lambda_1 z_{2e} r_d)(z_2 + \lambda_1 r_d z_1) - z_{2e} z_1 - \frac{\lambda_0 m_{11}}{m_{22}} v_e u + z_{3e} \Bigg)^2 - c_3 \bar{r}_e^2. \quad (4.67)$$

This implies global asymptotic stability of the closed loop system at the origin as long as the control parameters λ_0, λ_1, and λ_2 are chosen such that (4.61) holds. Note that this condition is feasible only when the reference yaw velocity r_d satisfies the following restrictive condition

$$0 < r_\star \leq |r_d(t)| \leq r^\star, \quad (4.68)$$

where $r_\star$ and $r^\star$ are positive constants. In [19], the result based on the standard backstepping technique also consists of two steps. The first step is to design the surge force τ_u. This step is the same as Step 1 mentioned above. The second step is to design the yaw moment τ_r. This step is slightly different from Step 2. In this step, a simple controller to stabilize the (z_{3e}, r_e)-subsystem, that is the third and last equations of (4.46), is designed as

$$\tau_r = \tau_{rd} + m_{33}\Bigg[-\frac{m_{11}-m_{22}}{m_{33}}(uv - u_d v_d) + \frac{d_{33}}{m_{33}} r_e - k_1 z_{3e} - k_2 r_e \Bigg]. \quad (4.69)$$

With the surge force τ_u and the yaw moment τ_r designed as in (4.62) and (4.70), it is proven in [19] that the tracking errors (z_{1e}, z_{2e}, z_{3e}, u_e, v_e, r_e) exponentially converge (*not exponential stability of the closed loop system*) to zero as long as the following restrictive condition on the reference yaw velocity r_d holds

$$\int_{t_0}^{t} r_d^2(\tau)\mathrm{d}\tau \geq \sigma_r(t - t_0), \quad \forall 0 \leq t_0 \leq t < \infty, \quad (4.70)$$

where σ_r is a strictly positive constant.

Remark 4.5. In comparison with the trajectory-tracking results in [71], we see that the control design in [19] is much simpler and gives global solutions. However, the restrictive conditions on the yaw reference velocity cannot be relaxed, see (4.68) and (4.70). Moreover, it is only possible to find the control parameters such that $c(t)$ given in (4.61) is strictly positive for vessels with a small ratio $\frac{m_{22}}{d_{22}}$, i.e., the vessels with large damping in the sway axis.

Remark 4.6. A common restriction of the above results on trajectory-tracking control of underactuated ships is that the reference yaw velocity has to satisfy various kinds of persistently exciting conditions. This implies that the reference trajectory must be curved, and indeed excludes a straight-line reference trajectory, hence, it substantially limits the practical use of the aforementioned control systems. A curious question is why all the above controllers suffer from the *must-be-curved* reference trajectory restriction. An answer is that the design of the above controllers starts from the chained form (4.46). The reader will find that this book provides various solutions for trajectory-tracking control of underactuated ships without imposing a persistent exciting condition on the yaw reference velocity. As such, we will not use the chained form (4.46) but will project the tracking errors, $x - x_d$, $y - y_d$, and $\psi - \psi_d$, on the body-fixed frame.

Apart from the aforementioned results on trajectory-tracking control of underactuated ships there are a few more results that are worth reviewing. Using sliding mode control, output redefinition and results on tracking of nonlinear nonminimum phase system [73], a path controller for surface ships was proposed in [74]. However, the convergence of the combined output does not guarantee convergence of its components. A continuous time-invariant state feedback controller was developed in [75] to achieve global exponential position tracking under the assumption that the reference surge velocity is always positive. Unfortunately, the orientation of the ship was not controlled. In [76, 77], (see also [78]), the authors developed a high-gain dynamic feedback control law to achieve global ultimate regulation and tracking of underactuated ships. The dynamics of the closed loop system is increased due to the controller designed to make the state of the transformed system track the auxiliary signals generated by some oscillator. The same approach was extended to the case of adaptive tracking control in [77]. It is worth mentioning that in [47], a time-varying velocity feedback controller was proposed to achieve both stabilization and tracking of unicycle mobile robots at the kinematics level motivated by the work in [18]. However this controller cannot be extended directly to the case of underactuated ships or AUVs due to the nonintegrable second-order constraint. Some related independent work includes [79, 80] on local H_∞ tracking control and output redefinition, and the trajectory planning approach, see [81, 82].

4.2.2.3 Path-following

Path-following is here defined as a control problem of forcing an underactuated ship or AUV to follow a specified path at a desired forward speed. Due to the high dependence on the reference model and complicated control laws of the trajectory-tracking approach, several researchers have studied the path-following problem, which is more suitable for practical implementation. The problem of path-following for air and underwater vehicles was introduced in [83] where some local results were obtained using linearization techniques. In [84], a feedforward cancelation of simplified vessel dynamics scheme followed by a linear quadratic regulator design was proposed to obtain local results on "track-keeping". A fourth-order ship model in Serret–Frenet frame was used in [85] to develop a control strategy to track both a straight line and a circumference under constant ocean current disturbance. The ocean-current direction was assumed to be known. A path-following controller was proposed in [86] by using a kinematic model written in polar coordinates, which is inspired by the solution for mobile robots in [60]. However, the controller was designed at the kinematic level with an assumption of constant ocean current and its direction known to be to achieve an adjustable boundedness of the path-following error. Since ocean vessels do not have direct control over velocities, a static mapping implementation might result in an unstable closed loop system due to nonvanishing environmental disturbances. Recently, a path-following controller based on a transformation of the ship kinematics to the Serret–Frenet frame, which was used for mobile robot control [44], on the path was proposed in [87], where an acceleration feedback and linearization of ship dynamics were used. It is worth mentioning that in [88, 89], a simple control scheme was proposed to make mobile robots follow a specified path using a polar coordinate transformation. Since underactuated surface ships have fewer numbers of actuators than the to-be-controlled degrees of freedom and are subject to nonintegrable acceleration constraints, their dynamic models are not transformable into a system without drifts. Therefore, the above control scheme is not directly applicable. In [54], a continuous, periodic time-varying feedback control law was proposed to locally exponentially stabilize an underactuated underwater vehicle at the origin. When the hydrodynamic restoring force in roll is large enough, this controller can be used without a roll control torque. However the closed loop system exhibits undesired oscillatory motions.

In [90], a linearization technique with an assumption of reference trajectories of underwater vehicles, which are helices parameterized by the vehicles' linear speed, yaw rate, and path angle, was introduced to develop the so-called time-invariant generalized vehicle error dynamics and kinematics. Various controllers were then designed based on the gain-scheduling technique to yield some local stability result about the trimming trajectories.

4.2.2.4 Output Feedback

Output feedback control of an underactuated ocean vessel is here defined as a control problem of forcing the vessel to achieve the aforementioned tasks (stabilization, trajectory-tracking, and path-following) without using measurements of the vessel's velocities for feedback. For ocean vessels, output feedback control usually consists of two stages. The first stage is to design an observer to reconstruct unmeasured states. Using the reconstructed states, a controller is designed to achieve control objectives in the second stage. In the literature, there are two main approaches to designing an observer for ocean vessels.

The first approach is based on the output-injection method applied directly to the vessel's equations of motion. This approach is simple and usually results in a semiglobal observer due to the quadratic terms of the vessel's velocities. Belonging to this approach are the results presented in [11, 14, 91–94] on output feedback control of fully actuated ocean vessels or Lagrange systems. In addition, the reader is referred to [93] for an exponential observer and output feedback controller for a special class of multi-degree of freedom Lagrange systems without cross terms of quadratic velocities, and to [14, 95–98] for output feedback control of robot manipulators and rigid body without measurements of angular velocities. Another method to design an observer is the use of contraction theory, see for example, [99, 100]. This method has been applied to Lagrange systems with monotonic velocity terms but without any quadratic velocity terms. Below, we summarize the aforementioned results on an observer design. We will show that a standard observer design cannot be used to obtain a global exponential/asymptotical observer for the ocean vessel system (1.1). Assume that the vessel velocity vector $\boldsymbol{v}$ is not measurable for feedback. We would then design an output injection observer to estimate $\boldsymbol{v}$ as follows:

$$\begin{aligned}\dot{\hat{\boldsymbol{\eta}}} &= \boldsymbol{J}(\boldsymbol{\eta})\hat{\boldsymbol{v}} + \boldsymbol{K_{01}}(\boldsymbol{\eta} - \hat{\boldsymbol{\eta}}),\\ \boldsymbol{M}\dot{\hat{\boldsymbol{v}}} &= -\boldsymbol{C}(\hat{\boldsymbol{v}})\hat{\boldsymbol{v}} - \boldsymbol{D}(\hat{\boldsymbol{v}})\hat{\boldsymbol{v}} - \boldsymbol{g}(\boldsymbol{\eta}) + \boldsymbol{\tau} + \boldsymbol{K_{02}}(\boldsymbol{\eta} - \hat{\boldsymbol{\eta}}),\end{aligned} \tag{4.71}$$

where $\hat{\boldsymbol{\eta}}$ and $\hat{\boldsymbol{v}}$ are estimates of $\boldsymbol{\eta}$ and $\boldsymbol{v}$, respectively, and the positive definite symmetric matrices $\boldsymbol{K_{01}} \in \mathbb{R}^{6\times 6}$ and $\boldsymbol{K_{02}} \in \mathbb{R}^{6\times 6}$ are the observer gain matrices. It is noted that in some of the aforementioned work, the observer gain matrices $\boldsymbol{K_{01}}$ and $\boldsymbol{K_{02}}$ depend on the measurable state $\boldsymbol{\eta}$. Letting the observer errors be

$$\begin{aligned}\tilde{\boldsymbol{\eta}} &= \boldsymbol{\eta} - \hat{\boldsymbol{\eta}},\\ \tilde{\boldsymbol{v}} &= \boldsymbol{v} - \hat{\boldsymbol{v}}\end{aligned} \tag{4.72}$$

and differentiating (4.72) along the solutions of (1.1) and (4.71) results in

$$\begin{aligned}\dot{\tilde{\boldsymbol{\eta}}} &= -\boldsymbol{K_{01}}\tilde{\boldsymbol{\eta}} + \boldsymbol{J}(\boldsymbol{\eta})\tilde{\boldsymbol{v}},\\ \boldsymbol{M}\dot{\tilde{\boldsymbol{v}}} &= -\boldsymbol{K_{02}}\tilde{\boldsymbol{\eta}} - \Big(\boldsymbol{C}(\boldsymbol{v})\boldsymbol{v} - \boldsymbol{C}(\hat{\boldsymbol{v}})\hat{\boldsymbol{v}}\Big) - \Big(\boldsymbol{D}(\boldsymbol{v})\boldsymbol{v} - \boldsymbol{D}(\hat{\boldsymbol{v}})\hat{\boldsymbol{v}}\Big).\end{aligned} \tag{4.73}$$

The term $\big(\boldsymbol{D}(\boldsymbol{v})\boldsymbol{v} - \boldsymbol{D}(\hat{\boldsymbol{v}})\hat{\boldsymbol{v}}\big)$ does not cause a problem if the damping matrix $\boldsymbol{D}(\boldsymbol{v})$ is monotonic, i.e., $\big((\boldsymbol{v} - \hat{\boldsymbol{v}})^T \big(\boldsymbol{D}(\boldsymbol{v})\boldsymbol{v} - \boldsymbol{D}(\hat{\boldsymbol{v}})\hat{\boldsymbol{v}}\big)$ is nonnegative for all $\boldsymbol{v} \in \mathbb{R}^6$ and $\hat{\boldsymbol{v}} \in \mathbb{R}^6$. However, we can see a serious problem with (4.73) because of the Coriolis matrix, i.e., $(\boldsymbol{v} - \hat{\boldsymbol{v}})^{\boldsymbol{T}}\big(\boldsymbol{C}(\boldsymbol{v})\boldsymbol{v} - \boldsymbol{C}(\hat{\boldsymbol{v}})\hat{\boldsymbol{v}}\big)$ is not nonnegative for all $\boldsymbol{v} \in \mathbb{R}^6$ and $\hat{\boldsymbol{v}} \in \mathbb{R}^6$. Therefore, only a local or semiglobal observer can be obtained.

The second approach involves a nontrivial coordinate transformation to transform the vessel's equations of motion to a new set of differential equations that are linear in unmeasured states. Then the output-injection method is used to design an observer. This approach usually results in a global observer if the nontrivial coordinate transformation can be found. Unfortunately, this coordinate transformation depends heavily on a solution of a set of partial differential equations, which in general are hard to solve. The main idea of this approach is to find a coordinate transformation

$$\boldsymbol{X} = \boldsymbol{Q}(\boldsymbol{\eta})\boldsymbol{v}, \tag{4.74}$$

where $\boldsymbol{Q}(\eta)$ is an invertible. This matrix is to be determined later. Substituting (4.74) into (4.26) results in

$$\begin{aligned}\dot{\eta} &= \boldsymbol{J}(\eta)\boldsymbol{Q}^{-1}(\eta)\boldsymbol{X},\\ \dot{\boldsymbol{X}} &= \left[\dot{\boldsymbol{Q}}(\eta)\boldsymbol{v} - \boldsymbol{Q}(\eta)\boldsymbol{M}^{-1}\boldsymbol{C}(\boldsymbol{v})\boldsymbol{v}\right] - \boldsymbol{Q}(\eta)\boldsymbol{M}^{-1}\boldsymbol{D}\boldsymbol{Q}^{-1}\boldsymbol{X} + \boldsymbol{Q}(\eta)\boldsymbol{M}^{-1}\boldsymbol{\tau}.\end{aligned} \tag{4.75}$$

The goal is to determine the matrix $\boldsymbol{Q}(\eta)$ such that

$$\dot{\boldsymbol{Q}}(\eta)\boldsymbol{v} - \boldsymbol{Q}(\eta)\boldsymbol{M}^{-1}\boldsymbol{C}(\boldsymbol{v})\boldsymbol{v} = 0, \tag{4.76}$$

for all $\boldsymbol{\eta} \in \mathbb{R}^3$ and $\boldsymbol{v} \in \mathbb{R}^3$. With (4.76), we can write (4.75) as

$$\begin{aligned}\dot{\eta} &= \boldsymbol{J}(\eta)\boldsymbol{Q}^{-1}(\eta)\boldsymbol{X},\\ \dot{\boldsymbol{X}} &= -\boldsymbol{Q}(\eta)\boldsymbol{M}^{-1}\boldsymbol{D}\boldsymbol{Q}^{-1}\boldsymbol{X} + \boldsymbol{Q}(\eta)\boldsymbol{M}^{-1}\boldsymbol{\tau}.\end{aligned} \tag{4.77}$$

It is seen that the transformed system (4.77) is linear in the unmeasured state $\boldsymbol{X}$. This allows us to design an exponential/asymptotical observer to estimate $\boldsymbol{X}$. After that an estimate, $\hat{\boldsymbol{v}}$, of $\boldsymbol{v}$ can be found from (4.74), i.e.,

$$\hat{\boldsymbol{v}} = \boldsymbol{Q}^{-1}(\eta)\hat{\boldsymbol{X}}, \tag{4.78}$$

where $\hat{\boldsymbol{X}}$ denotes an estimate of $\boldsymbol{X}$. It is noted that combining the first equation of (4.26) and (4.76) results in a set of partial differential equations. Finding a solution to this set of partial differential equations is a hard task. A simple application of the above idea gives the results in [101–104] for some single degree of freedom Lagrange systems. It is noted that the method of solving the set of partial differential equations in [101–104] is not applicable for systems of more than one degree of freedom. For more complicated Lagrange systems, it is hard to find a result in this approach. However, the reader is referred to [105] where an output eedback control solution for simultaneous stabilization and tracking control of an underactuated ODIN is given.

Remark 4.7. The main difficulty in designing an observer-based output feedback for surface ships and Lagrange systems in general is because of the Coriolis matrix, which results in cross terms of unmeasured velocities. In addition, the underactuation of surface ships makes the output feedback problem much more challenging. For example, many solutions proposed for robot control, see [14] and references therein, cannot directly be applied. The reader will find that a set of special coordinate transformations is derived in this book to transform the ship dynamics to a system that is linear in unmeasured velocities, and another set of coordinate transformations that makes it possible to design global output feedback control controllers for underactuated ships.

4.3 Conclusions

This chapter presented the main control properties of ocean vessels. The literature on the control of underactuated ocean vessels including ships and underwater vehicles was then reviewed. Through this review, several challenging questions were raised. These questions motivate contributions of the coming chapters of this book.