Chapter 3
Modeling of Ocean Vessels

In this chapter, we classify the basic motion tasks for ocean vessels and their mathematical models, which will be used for the design of various control systems in the subsequent chapters.

3.1 Introduction

In automatic control, feedback improves system performance by allowing the successful completion of a task even in the presence of external disturbances and initial errors, and inaccuracy of the system parameters. To this end, real-time sensor measurements are used to reconstruct the vehicle state. Throughout this study, the latter is assumed to be available at every instant, as provided by local/global position and orientation measurement sensors. In some cases, we also assume that the vehicle velocities are measurable or constructible from position measurements.

We will concentrate on the case of a vessel workspace free of obstacles. In fact, we implicitly consider the vessel controller to be embedded in a hierarchical architecture in which a higher-level planner solves the obstacle avoidance problem and provides a series of motion goals to the lower control layer. In this perspective, the controller deals with the basic issue of converting ideal plans into actual motion execution. The nonholonomic nature of the ocean vessels is related to the fact that the vessel does not usually have independent actuators in the sway and heave axes. This implies the presence of a nonintegrable set of second-order differential constraints on the configuration variables. While these nonholonomic constraints reduce the instantaneous motions that the vessel can perform, they still allow almost global controllability in the configuration space. This feature leads to some challenging problems in the synthesis of feedback controllers, which parallel the new research issues arising in nonholonomic motion planning. Indeed, the ocean vessel application has triggered the search for innovative types of feedback controllers that can be used also for more general nonlinear systems that describe the motion of more complicated vessel systems such as ocean vessels and air vehicles working in a group.

3.2 Basic Motion Tasks

In order to derive the most suitable feedback controllers for each case, it is convenient to classify the possible motion tasks as follows:

- Point-to-point motion: The vessel must reach a desired goal configuration starting from a given initial configuration, see Figure 3.1a.
- Path-following: The vessel must reach and follow a geometric reference path in the Cartesian space starting from a given initial configuration (on or off the path), see Figure 3.1b.
- Trajectory-tracking and path-tracking: The vessel must reach and follow a reference trajectory/path in the Cartesian space (i.e., a geometric path with an associated timing law) starting from a given initial configuration (on or off the trajectory/path), see Figure 3.1c. Trajectory-tracking is referred to as the case where the reference trajectory is generated by a suitable virtual vessel whereas the reference path is not required to be generated by a virtual vessel for the path-tracking.

The above tasks for an ocean vessel are sketched in Figure 3.1. Execution of these tasks can be achieved using either feedforward commands, or feedback control, or a combination of the two. Indeed, feedback solutions exhibit an intrinsic degree of robustness.

Using a more control-oriented terminology, the point-to-point motion task is a stabilization problem for a (equilibrium) point in the vessel state space. When using a feedback strategy, the point-to-point motion task leads to a state regulation control problem for a point in the vessel state space. Posture stabilization is another frequently used term. Without loss of generality, the goal can be taken as the origin of the n-dimensional vessel configuration space. Contrary to the usual situation, trajectory-tracking, path-tracking, and path-following are easier than regulation for a nonholonomic vessel. An intuitive explanation of this can be given in terms of a comparison between the number of controlled variables (outputs) and the number of control inputs. For the ship or underwater vehicle moving in a horizontal plane, two input commands are available while three variables (position and orientation) are needed to determine its configuration. Thus, regulation of the surface ship or the underwater vehicle in a horizontal position to a desired configuration implies zeroing three independent configuration errors.

In the path-following task, the controller is given a geometric description of the assigned Cartesian path. This information is usually available in a parameterized form expressing the desired motion in terms of a path parameter, which may be in particular the arc length along the path. For this task, time dependence is not relevant because one is concerned only with the geometric displacement between the vessel

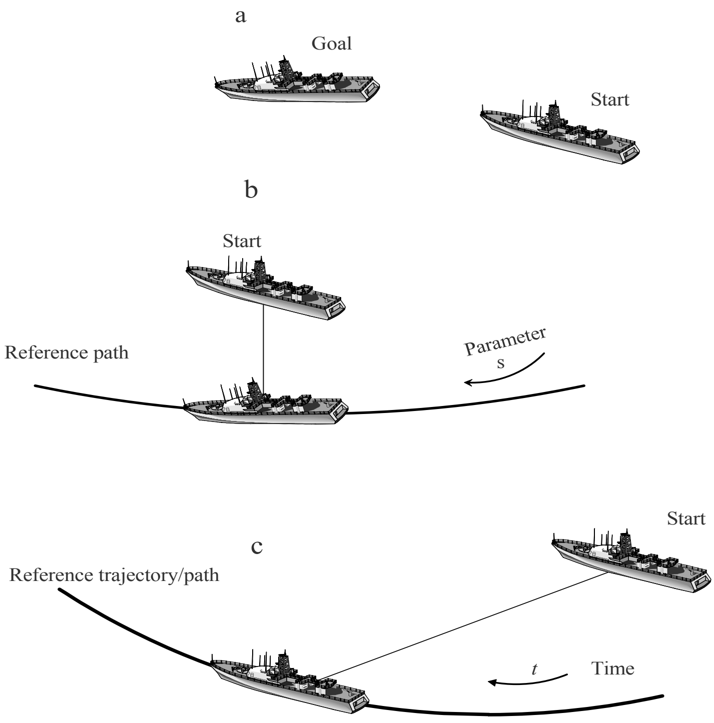

Figure 3.1 Basic motion tasks for an ocean vessel

and the path. In this context, the time evolution of the path parameter is usually free and, accordingly, the command inputs can be arbitrarily scaled with respect to time without changing the resulting vessel path. It is then customary to set the vessel forward velocity (one of the inputs) to an arbitrary constant or time-varying value, leaving the other input variables for control. The path-following problem is thus rephrased as the stabilization to zero of a suitable scalar path error function using only the rest of the control inputs.

In the trajectory-tracking and path-tracking tasks, the vessel must follow the desired Cartesian path with a specified timing law. Although the reference trajectory/path can be split into a parameterized geometric path and a timing law for the parameter, such separation is not strictly necessary. Often, it is simpler to specify the workspace trajectory as the desired time evolution for the position of some representative point of the vessel. The trajectory-tracking and path-tracking problems consist then in the stabilization to zero of the Cartesian errors using all the available control inputs.

The point stabilization problem can be formulated in a local or in a global sense, the latter meaning that we allow for initial configurations that are arbitrarily far from the destination. The same is true also for path-following, trajectory-tracking, and path-tracking, although locality has two different meanings in these tasks. For path-following, a local solution means that the controller works properly, provided that we start sufficiently close to the path; for trajectory-tracking and path-tracking, closeness should be evaluated with respect to the current position of the reference vessel and a current reference point, which moves on the reference path with a specified time law, on the reference path, respectively. The amount of information that should be provided by a high-level motion planner varies for each control task. In point-to-point motion, information is reduced to a minimum (i.e., the goal configuration only) when a globally stabilizing feedback control solution is available. However, if the initial error is large, such a control may produce erratic behavior and/or large control effort, which are unacceptable in practice. On the other hand, a local feedback solution requires the definition of intermediate subgoals at the task planning level in order to get closer to the final desired configuration. For the other motion tasks, the planner should provide a path that is kinematically feasible (namely, that complies with the nonholonomic constraints of the specific vessel), so as to allow its perfect execution in nominal conditions. While for a fully or overactuated vessel in which any path is feasible, some degree of geometric smoothness is in general required for nonholonomic vessels. Nevertheless, the intrinsic feedback structure of the driving commands enables it to recover transient errors due to isolated path discontinuities. Note also that the infeasibility arising from a lack of continuity in some higher-order derivative of the path may be overcome by appropriate motion timing. For example, paths with discontinuous curvature (like the Reeds and Shepp optimal paths under maximum curvature constraint) can be executed by choosing an appropriate point on the vessel provided that the vessel is allowed to stop, whereas paths with discontinuous tangent are not feasible. In this analysis, the selection of the vessel representative point for path/trajectory planning is critical. The timing profile is the additional item needed in trajectory-tracking and path-tracking control tasks. This information is seldom provided by current motion planners, also because the actual dynamics of the specific vessel are typically neglected at this level. The above example suggests that it may be reasonable at the planning stage to enforce requirements such as “move slower where the path curvature is higher”.

3.3 Modeling of Ocean Vessels

Modeling of the ocean vessels is usually based on mechanics, principles of statics and dynamics. Statics is concerned with the equilibrium of bodies at rest or moving with a constant velocity. Dynamics deals with bodies having accelerated motion resulting from disturbances or/and control forces. Since we are interested in a mathematical model of the ocean vessels for the purpose of designing the control systems, this section focuses on dynamics of the vessels rather than statics. The following briefly presents the ocean vessel equations of motion based on the results in [11]. The resulting nonlinear model presented in this section is mainly intended for designing control systems in the next chapters. For a detailed and comprehensive derivation of the model, the reader is referred to [11,12,25]. The physical and control properties of the model are also presented for control design and stability analysis. In this section, we use the notation, see Table 3.1 and Figure 3.2, that complies with the Society of Naval Architects and Marine Engineers (SNAME) [26].

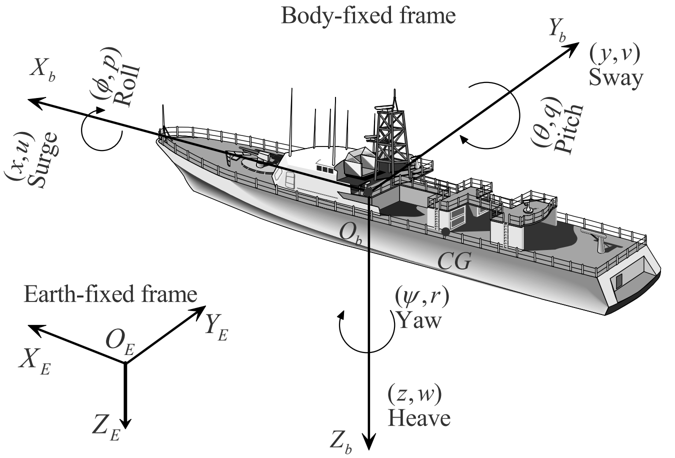

Figure 3.2 Motion variables for an ocean vessel

For an ocean vessel moving in six degrees of freedom, six independent coordinates are required to determine its position and orientation. The first three coordinates (x, y, z) and their first time derivatives correspond to the position and translational motion along the x-, y- and z-axes, while the last three coordinates (ϕ, θ, ψ) and their first time derivatives describe orientation and rotational motion.

Table 3.1 SNAME Notation for ocean vessels

Degree of freedom		Force and moment	Linear and angular velocity	Position and Euler angles
1	Surge	X	u	x
2	Sway	Y	v	y
3	Heave	Z	w	z
4	Roll	K	p	ϕ
5	Pitch	M	q	θ
6	Yaw	N	r	ψ

According to SNAME, the six different motion components are defined as *surge, sway, heave, roll, pitch*, and *yaw*. To determine the equations of motion, two reference frames are considered: the inertial or fixed to earth frame $O_E X_E Y_E Z_E$ that may be taken to coincide with the vessel fixed coordinates in some initial condition and the body-fixed frame $O_b X_b Y_b Z_b$ see Figure 3.2. Since the motion of the Earth hardly affects ocean vessels (different from air vehicles), the earth-fixed frame $O_E X_E Y_E Z_E$ can be considered to be inertial. For ocean vessels in general, the most commonly adopted position for the body-fixed frame is such that it gives hull symmetry about the $O_bX_bZ_b$-plane and approximate symmetry about the $O_bY_bZ_b$-plane. In this sense, the body axes O_bX_b, O_bY_b, and O_bZ_b coincide with the principal axes of inertia and are usually defined as follows: O_bX_b is the longitudinal axis (directed from aft to fore); O_bY_b is the transverse axis (directed to starboard); and O_bZ_b is normal axis (directed from top to bottom. Based on the notion in Table 3.1, the general motion of an ocean vessel can be described by the following vectors:

$$\begin{aligned} \boldsymbol{\eta} &= [\boldsymbol{\eta_1}\ \boldsymbol{\eta_2}]^T, & \boldsymbol{\eta_1} &= [x\ y\ z]^T, & \boldsymbol{\eta_2} &= [\phi\ \theta\ \psi]^T, \\ \boldsymbol{v} &= [\boldsymbol{v_1}\ \boldsymbol{v_2}]^T, & \boldsymbol{v_1} &= [u\ v\ w]^T, & \boldsymbol{v_2} &= [p\ q\ r]^T, \\ \boldsymbol{\tau} &= [\boldsymbol{\tau_1}\ \boldsymbol{\tau_2}]^T, & \boldsymbol{\tau_1} &= [X\ Y\ Z]^T, & \boldsymbol{\tau_2} &= [K\ M\ N]^T, \end{aligned}$$

where $\boldsymbol{\eta}$ denotes the position and orientation vector with coordinates in the earth-fixed frame, $\boldsymbol{v}$ denotes the linear and angular velocity vector with coordinates in the body-fixed frame, and $\boldsymbol{\tau}$ denotes the forces and moments acting on the vessel in the body-fixed frame.

In deriving equations of motion of the ocean vessels, we divide the study of vessel dynamics into two parts *kinematics*, which treats only geometrical aspects of motion, and *kinetics*, which is the analysis of the forces resulting in the motion.

3.3.1 Kinematics

The first time derivative of the position vector $\boldsymbol{\eta_1}$ is related to the linear velocity vector $\boldsymbol{v_1}$ via the following transformation:

$$\dot{\boldsymbol{\eta}}_{\boldsymbol{1}} = \boldsymbol{J_1}(\boldsymbol{\eta_2})\boldsymbol{v_1}, \tag{3.1}$$

where $\boldsymbol{J_1}(\boldsymbol{\eta_2})$ is a transformation matrix, which is related through the functions of the Euler angles: roll (ϕ), pitch (θ), and yaw (ψ). This matrix is given by

$$\boldsymbol{J_1}(\boldsymbol{\eta_2}) = \begin{bmatrix} \cos(\psi)\cos(\theta) & -\sin(\psi)\cos(\phi)+\sin(\phi)\sin(\theta)\cos(\psi) & \sin(\psi)\sin(\phi)+\sin(\theta)\cos(\psi)\cos(\phi) \\ \sin(\psi)\cos(\theta) & \cos(\psi)\cos(\phi)+\sin(\phi)\sin(\theta)\sin(\psi) & -\cos(\psi)\sin(\phi)+\sin(\theta)\sin(\psi)\cos(\phi) \\ -\sin(\theta) & \sin(\phi)\cos(\theta) & \cos(\phi)\cos(\theta) \end{bmatrix}. \tag{3.2}$$

It is noted that the matrix $\boldsymbol{J_1}(\boldsymbol{\eta_2})$ is globally invertible since $\boldsymbol{J_1^{-1}}(\boldsymbol{\eta_2}) = \boldsymbol{J_1^T}(\boldsymbol{\eta_2})$.

On the other hand, the first time derivative of the Euler angle vector $\boldsymbol{\eta_2}$ is related to the body-fixed velocity vector $\boldsymbol{v_2}$ through the following transformation:

$$\dot{\boldsymbol{\eta}}_{\boldsymbol{2}} = \boldsymbol{J_2}(\boldsymbol{\eta_2})\boldsymbol{v_2}, \tag{3.3}$$

where the transformation matrix $\boldsymbol{J_2}(\boldsymbol{\eta_2})$ is given by

$$\boldsymbol{J_2}(\boldsymbol{\eta_2}) = \begin{bmatrix} 1 & \sin(\phi)\tan(\theta) & \cos(\phi)\tan(\theta) \\ 0 & \cos(\phi) & -\sin(\phi) \\ 0 & \sin(\phi)/\cos(\theta) & \cos(\phi)/\cos(\theta) \end{bmatrix}. \tag{3.4}$$

Note that the transformation matrix $\boldsymbol{J_2}(\boldsymbol{\eta_2})$ is singular at $\theta = \pm\frac{\pi}{2}$. However, during practical operations ocean vessels are not likely to enter the neighborhood of $\theta = \pm\frac{\pi}{2}$ because of the metacentric restoring forces. For the case where it is essential to consider a region containing $\theta = \pm\frac{\pi}{2}$, a four-parameter description based on Euler parameters can be used instead. The interested reader is referred to [12] for more details. Combining (3.1) and (3.3) results in the kinematics of the ocean vessels:

$$\begin{bmatrix} \dot{\boldsymbol{\eta}}_{\mathbf{1}} \\ \dot{\boldsymbol{\eta}}_{\mathbf{2}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{J_1}(\boldsymbol{\eta_2}) & 0_{3\times 3} \\ 0_{3\times 3} & \boldsymbol{J_2}(\boldsymbol{\eta_2}) \end{bmatrix} \begin{bmatrix} \boldsymbol{v_1} \\ \boldsymbol{v_2} \end{bmatrix} \quad \Leftrightarrow \quad \dot{\boldsymbol{\eta}} = \boldsymbol{J}(\boldsymbol{\eta})\boldsymbol{v}. \tag{3.5}$$

3.3.2 Kinetics

3.3.2.1 Rigid Body Equations of Motion

Let us define the following vectors:

- $\boldsymbol{f}_{Ob} = [X\ Y\ Z]^T$: force decomposed in the body-fixed frame.
- $\boldsymbol{m}_{Ob} = [K\ M\ N]^T$: moment decomposed in the body-fixed frame.
- $\boldsymbol{v}_{Ob} = [u\ v\ w]^T$: linear velocity decomposed in the body-fixed frame.
- $\boldsymbol{\omega}_{Ob}^{E} = [p\ q\ r]^T$: angular velocity of the body-fixed frame relative to the earth-fixed frame.
- $\boldsymbol{r}_{Ob} = [x_g\ y_g\ z_g]^T$: vector from O_b to CG (center of gravity of the vessel) decomposed in the body-fixed frame.

By the Newton–Euler formulation for a rigid body with a mass of m, we have the following balancing forces and moments:

$$\begin{aligned} &m[\dot{\boldsymbol{v}}_{Ob} + \dot{\boldsymbol{\omega}}_{Ob}^{E} \times \boldsymbol{r}_{Ob} + \boldsymbol{\omega}_{Ob}^{E} \times \boldsymbol{v}_{Ob} + \boldsymbol{\omega}_{Ob}^{E} \times (\boldsymbol{\omega}_{Ob}^{E} \times \boldsymbol{r}_{Ob})] = \boldsymbol{f}_{Ob}, \\ &\boldsymbol{I}_o\dot{\boldsymbol{\omega}}_{Ob}^{E} + \boldsymbol{\omega}_{Ob}^{E} \times \boldsymbol{I}_o\boldsymbol{\omega}_{Ob}^{E} + m\boldsymbol{r}_{Ob} \times (\dot{\boldsymbol{v}}_{Ob} + \boldsymbol{\omega}_{Ob}^{E} \times \boldsymbol{v}_{Ob}) = \boldsymbol{m}_{Ob}, \end{aligned} \tag{3.6}$$

where $\boldsymbol{I}_o$ is the inertia matrix about O_b defined by

$$\boldsymbol{I_o} = \begin{bmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{yx} & I_y & -I_{yz} \\ -I_{zx} & -I_{zy} & I_z \end{bmatrix}. \tag{3.7}$$

Here I_x, I_y, and I_z are the moments of inertia about the O_bX_b, O_bY_b, and O_bZ_b axes, and $I_{xy} = I_{yx}$, $I_{xz} = I_{zx}$, and $I_{yz} = I_{zy}$ are the products of inertia. These quantities are defined as

$$
\begin{aligned}
I_x &= \int_V (y^2+z^2)\rho_m \mathrm{d}V, \; I_{xy} = \int_V xy\rho_m \mathrm{d}V, \\
I_y &= \int_V (x^2+z^2)\rho_m \mathrm{d}V, \; I_{xz} = \int_V xz\rho_m \mathrm{d}V, \\
I_z &= \int_V (x^2+y^2)\rho_m \mathrm{d}V, \; I_{zy} = \int_V zy\rho_m \mathrm{d}V,
\end{aligned} \tag{3.8}
$$

where ρ_m and V are, respectively, the mass density and the volume of the rigid body. Substituting the definitions of $\boldsymbol{f}_{Ob}$, $\boldsymbol{m}_{Ob}$, $\boldsymbol{v}_{Ob}$, $\boldsymbol{\omega}_{Ob}^E$, and $\boldsymbol{r}_{Ob}$ into (3.6) results in the following equations of motion of a rigid body:

$$
\boldsymbol{M_{RB}}\dot{\boldsymbol{v}} + \boldsymbol{C_{RB}}(\boldsymbol{v})\boldsymbol{v} = \boldsymbol{\tau_{RB}}, \tag{3.9}
$$

where $\boldsymbol{v} = [u \; v \; w \; p \; q \; r]^T$ is the generalized velocity vector decomposed in the body-fixed frame, $\boldsymbol{\tau_{RB}} = [X \; Y \; Z \; K \; M \; N]^T$ is the generalized vector of external forces and moments, the rigid body system inertia matrix $\boldsymbol{M_{RB}}$ is given by

$$
\boldsymbol{M_{RB}} = \begin{bmatrix}
m & 0 & 0 & 0 & mz_g & -my_g \\
0 & m & 0 & -mz_g & 0 & mx_g \\
0 & 0 & m & my_g & -mx_g & 0 \\
0 & -mz_g & my_g & I_x & -I_{xy} & -I_{xz} \\
mz_g & 0 & -mx_g & -I_{yx} & I_y & -I_{yz} \\
-my_g & mx_g & 0 & -I_{zx} & -I_{zy} & I_z
\end{bmatrix}, \tag{3.10}
$$

and the rigid body Coriolis and centripetal matrix $\boldsymbol{C_{RB}}(\boldsymbol{v})$ is given by

$$
\boldsymbol{C_{RB}}(\boldsymbol{v}) = \left[\begin{matrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
-m(y_g q + z_g r) & m(y_g p + w) & m(z_g p - v) \\
m(x_g q - w) & -m(z_g r + x_g p) & m(z_g q + u) \\
m(x_g r + v) & m(y_g r - u) & -m(x_g p + y_g q)
\end{matrix}\right.
$$

$$
\left.\begin{matrix}
m(y_g q + z_g r) & -m(x_g q - w) & -m(x_g r + v) \\
-m(y_g p + w) & m(z_g r + x_g p) & -m(y_g r - u) \\
-m(z_g p - v) & -m(z_g q + u) & m(x_g p + y_g q) \\
0 & -I_{yz} q - I_{xz} p + I_z r & I_{yz} r + I_{xy} p - I_y q \\
I_{yz} q + I_{xz} p - I_z r & 0 & -I_{xz} r - I_{xy} q + I_x p \\
-I_{yz} r - I_{xy} p + I_y q & I_{xz} r + I_{xy} q - I_x p & 0
\end{matrix}\right]. \tag{3.11}
$$

The generalized external force and moment vector, $\boldsymbol{\tau_{RB}}$, is a sum of hydrodynamic force and moment vector $\boldsymbol{\tau_H}$, external disturbance force and moment vector $\boldsymbol{\tau_E}$, and propulsion force and moment vector $\boldsymbol{\tau}$. Each of these vectors is detailed in the following sections.

3.3.2.2 Hydrodynamic Forces and Moments

In hydrodynamics, it is usually assumed that the hydrodynamic forces and moments on a rigid body can be linearly superimposed, see [27]. The hydrodynamic forces and moments are forces and moments on the body when the body is forced to oscillate with the wave excitation frequency and there are no incident waves. These forces and moments can be identified as the sum of three components: (1) added mass due to the inertia of the surrounding fluid, (2) radiation-induced potential damping due to the energy carried away by generated surface waves, and (3) restoring forces due to Archimedian forces (weight and buoyancy). The hydrodynamic force and moment vector $\boldsymbol{\tau_H}$ is given by

$$\boldsymbol{\tau_H} = -\boldsymbol{M_A}\dot{\boldsymbol{v}} - \boldsymbol{C_A}(\boldsymbol{v})\boldsymbol{v} - \boldsymbol{D}(\boldsymbol{v})\boldsymbol{v} - \boldsymbol{g}(\boldsymbol{\eta}), \tag{3.12}$$

where $\boldsymbol{M_A}$ is the added mass matrix, $\boldsymbol{C_A}(\boldsymbol{v})$ is the hydrodynamic Coriolis and centripetal matrix, $\boldsymbol{D}(\boldsymbol{v})$ is the damping matrix, and $\boldsymbol{g}(\boldsymbol{\eta})$ is the position and orientation depending vector of restoring forces and moments.

The added mass matrix $\boldsymbol{M_A}$ is given by

$$\boldsymbol{M_A} = -\begin{bmatrix} X_{\dot{u}} & X_{\dot{v}} & X_{\dot{w}} & X_{\dot{p}} & X_{\dot{q}} & X_{\dot{r}} \\ Y_{\dot{u}} & Y_{\dot{v}} & Y_{\dot{w}} & Y_{\dot{p}} & Y_{\dot{q}} & Y_{\dot{r}} \\ Z_{\dot{u}} & Z_{\dot{v}} & Z_{\dot{w}} & Z_{\dot{p}} & Z_{\dot{q}} & Z_{\dot{r}} \\ K_{\dot{u}} & K_{\dot{v}} & K_{\dot{w}} & K_{\dot{p}} & K_{\dot{q}} & K_{\dot{r}} \\ M_{\dot{u}} & M_{\dot{v}} & M_{\dot{w}} & M_{\dot{p}} & M_{\dot{q}} & M_{\dot{r}} \\ N_{\dot{u}} & N_{\dot{v}} & N_{\dot{w}} & N_{\dot{p}} & N_{\dot{q}} & N_{\dot{r}} \end{bmatrix}, \tag{3.13}$$

where the SNAME notation has been used. For example, the hydrodynamic added mass force Y along the y-axis due to an acceleration $\dot{u}$ in the x-direction is written as

$$Y = -Y_{\dot{u}}\dot{u},\ Y_{\dot{u}} := \frac{\partial Y}{\partial \dot{u}}. \tag{3.14}$$

The hydrodynamic Coriolis and centripetal matrix is given by

$$\boldsymbol{C_A}(\boldsymbol{v}) = \begin{bmatrix} 0 & 0 & 0 & 0 & -a_3 & a_2 \\ 0 & 0 & 0 & a_3 & 0 & -a_1 \\ 0 & 0 & 0 & -a_2 & a_1 & 0 \\ 0 & -a_3 & a_2 & 0 & -b_3 & b_2 \\ a_3 & 0 & -a_1 & b_3 & 0 & -b_1 \\ -a_2 & a_1 & 0 & -b_2 & b_1 & 0 \end{bmatrix}, \tag{3.15}$$

where

$$\begin{aligned} a_1 &= X_{\dot{u}}u + X_{\dot{v}}v + X_{\dot{w}}w + X_{\dot{p}}p + X_{\dot{q}}q + X_{\dot{r}}r, \\ a_2 &= Y_{\dot{u}}u + Y_{\dot{v}}v + Y_{\dot{w}}w + Y_{\dot{p}}p + Y_{\dot{q}}q + Y_{\dot{r}}r, \\ a_3 &= Z_{\dot{u}}u + Z_{\dot{v}}v + Z_{\dot{w}}w + Z_{\dot{p}}p + Z_{\dot{q}}q + Z_{\dot{r}}r, \\ b_1 &= K_{\dot{u}}u + K_{\dot{v}}v + K_{\dot{w}}w + K_{\dot{p}}p + K_{\dot{q}}q + K_{\dot{r}}r, \end{aligned}$$

$$
\begin{aligned}
b_2 &= M_{\dot{u}}u + M_{\dot{v}}v + M_{\dot{w}}w + M_{\dot{p}}p + M_{\dot{q}}q + M_{\dot{r}}r, \\
b_3 &= N_{\dot{u}}u + N_{\dot{v}}v + N_{\dot{w}}w + N_{\dot{p}}p + N_{\dot{q}}q + N_{\dot{r}}r. \qquad (3.16)
\end{aligned}
$$

In general, hydrodynamic damping for ocean vessels is mainly caused by potential damping, skin friction, wave drift damping, and damping due to vortex shedding. It is difficult to give a general expression of the hydrodynamic damping matrix $\boldsymbol{D}(\boldsymbol{v})$. However, it is common to write the hydrodynamic damping matrix $\boldsymbol{D}(\boldsymbol{v})$ as

$$\boldsymbol{D}(\boldsymbol{v}) = \boldsymbol{D} + \boldsymbol{D_n}(\boldsymbol{v}). \qquad (3.17)$$

Here the linear damping matrix $\boldsymbol{D}$ is given by

$$\boldsymbol{D} = -\begin{bmatrix} X_u & X_v & X_w & X_p & X_q & X_r \\ Y_u & Y_v & Y_w & Y_p & Y_q & Y_r \\ Z_u & Z_v & Z_w & Z_p & Z_q & Z_r \\ K_u & K_v & K_w & K_p & K_q & K_r \\ M_u & M_v & M_w & M_p & M_q & M_r \\ N_u & N_v & N_w & N_p & N_q & N_r \end{bmatrix}. \qquad (3.18)$$

The nonlinear damping matrix $\boldsymbol{D_n}(\boldsymbol{v})$ is usually modeled by using a third-order Taylor series expansion or modulus functions (quadratic drag). If the xz-plane is a plane of symmetry (starboard/port symmetry) an odd Taylor series expansion containing first-order and third-order terms in velocity can be sufficient to describe most manoeuvres. An approximate expression of each of this matrices will be given in the next section when specific vessels are considered.

3.3.2.3 Restoring Forces and Moments

In this section, a model for $\boldsymbol{g}(\boldsymbol{\eta})$ is described. Let ∇ be the volume of fluid displaced by the vessel, g the acceleration of gravity (positive downwards), and ρ the water density. The submerged weight of the body and buoyancy force are defined as

$$
\begin{aligned}
W &= mg, \\
B &= \rho g \nabla. \qquad (3.19)
\end{aligned}
$$

With the above definition, the restoring force and moment vector $\boldsymbol{g}(\boldsymbol{\eta})$ is due to gravity and buoyancy forces, and is given by

$$\boldsymbol{g}(\boldsymbol{\eta}) = \begin{bmatrix} (W-B)\sin(\theta) \\ -(W-B)\cos(\theta)\sin(\phi) \\ -(W-B)\cos(\theta)\cos(\phi) \\ -(y_g W - y_b B)\cos(\theta)\cos(\phi) + (z_g W - z_b B)\cos(\theta)\sin(\phi) \\ (z_g W - z_b B)\sin(\theta) + (x_g W - x_b B)\cos(\theta)\cos(\phi) \\ -(x_g W - x_b B)\cos(\theta)\sin(\phi) - (y_g W - y_b B)\sin(\theta) \end{bmatrix}, \qquad (3.20)$$

where (x_b, y_b, z_b) denote coordinates of the center of buoyancy.

3.3.2.4 Environmental Disturbances

In this section, we detail the vector, $\boldsymbol{\tau_E}$, of forces and moments induced by environmental disturbances including ocean currents, waves (wind generated) and wind, i.e., we can write

$$\boldsymbol{\tau_E} = \boldsymbol{\tau_E^{cu}} + \boldsymbol{\tau_E^{wa}} + \boldsymbol{\tau_E^{wi}}, \tag{3.21}$$

where $\boldsymbol{\tau_E^{cu}}$, $\boldsymbol{\tau_E^{wa}}$, and $\boldsymbol{\tau_E^{wi}}$ are vectors of forces and moments induced by ocean currents, waves and wind, respectively.

Current-induced Forces and Moments

The vector $\boldsymbol{\tau_E^{cu}}$ of the current-induced forces and moments is given by

$$\boldsymbol{\tau_E^{cu}} = (\boldsymbol{M_{RB}} + \boldsymbol{M_A})\dot{\boldsymbol{v}}_{\boldsymbol{c}} + \boldsymbol{C}(\boldsymbol{v_r})\boldsymbol{v_r} - \boldsymbol{C}(\boldsymbol{v})\boldsymbol{v} + \boldsymbol{D}(\boldsymbol{v_r})\boldsymbol{v_r} - \boldsymbol{D}(\boldsymbol{v})\boldsymbol{v}, \tag{3.22}$$

where $\boldsymbol{v_r} = \boldsymbol{v} - \boldsymbol{v_c}$ and $\boldsymbol{v_c} = [u_c, v_c, w_c, 0, 0, 0]^T$ is a vector of irrotational body-fixed current velocities. Let the earth-fixed current velocity vector be denoted by $[u_c^E, v_c^E, w_c^E]^T$. Then, the body-fixed components $[u_c, v_c, w_c]^T$ can be computed as

$$\begin{bmatrix} u_c \\ v_c \\ w_c \end{bmatrix} = \boldsymbol{J_1^T}(\boldsymbol{\eta_2}) \begin{bmatrix} u_c^E \\ v_c^E \\ w_c^E \end{bmatrix}. \tag{3.23}$$

Wave-induced Forces and Moments

The vector $\boldsymbol{\tau_E^{wa}}$ of the wave-induced forces and moments is given by

$$\boldsymbol{\tau_E^{wi}} = \begin{bmatrix} \sum_{i=1}^{N} \rho g B L T \cos(\beta) s_i(t) \\ \sum_{i=1}^{N} -\rho g B L T \sin(\beta) s_i(t) \\ 0 \\ 0 \\ 0 \\ \sum_{i=1}^{N} \frac{1}{24} \rho g B L (L^2 - B^2) \sin(2\beta) s_i^2(t) \end{bmatrix}, \tag{3.24}$$

where β is the vessel's heading (encounter) angle, see Figure 3.3, ρ is the water density, L is the length of the vessel, B is the breadth of the vessel, and T is the draft of the vessel. Ignoring the higher-order terms of the wave amplitude, the wave slope $s_i(t)$ for the wave component i is defined as:

$$s_i(t) = A_i \frac{2\pi}{\lambda_i} \sin(\omega_{ei} t + \phi_i), \tag{3.25}$$

where A_i is the wave amplitude, λ_i is the wave length, ω_{ei} is the encounter frequency, and ϕ_i is a random phase uniformly distributed and constant with time in $[0\ 2\pi)$ corresponding to the wave component i.

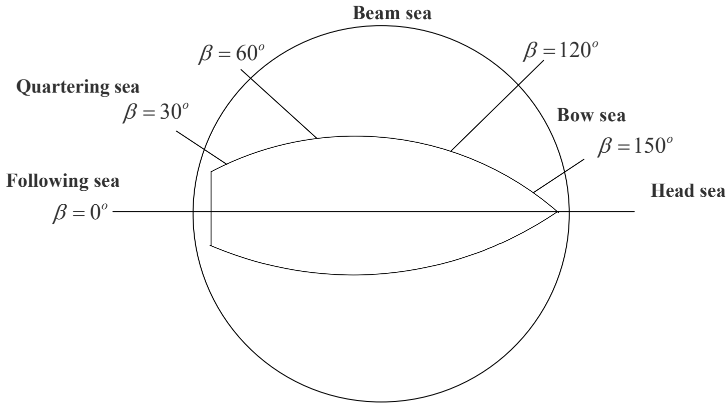

Figure 3.3 Definition of a vessel's heading (encounter) angle

Wind-induced Forces and Moments

For the case where the vessel is at rest (zero speed), the vector $\boldsymbol{\tau}_{\boldsymbol{E}}^{\boldsymbol{wi}}$ of the wind-induced forces and moments is given by

$$\boldsymbol{\tau}_{\boldsymbol{E}}^{\boldsymbol{wi}} = \frac{1}{2}\rho_a V_w^2 \begin{bmatrix} C_X(\gamma_w) A_{Fw} \\ C_Y(\gamma_w) A_{Lw} \\ C_Z(\gamma_w) A_{Fw} \\ C_K(\gamma_w) A_{Lw} H_{Lw} \\ C_M(\gamma_w) A_{Fw} H_{Fw} \\ C_N(\gamma_w) A_{Lw} L_{oa} \end{bmatrix}, \tag{3.26}$$

where V_w is the wind speed, ρ_a is the air density, A_{Fw} is the frontal projected area, A_{Lw} is the lateral projected area, H_{Fw} is the centroid of A_{Fw} above the water line, H_{Lw} is the centroid of A_{Lw} above the water line, L_{oa} is the over all length of the vessel, γ_w is the angle of relative wind of the vessel bow, see Figure 3.4, and is given by

$$\gamma_w = \psi - \beta_w - \pi, \tag{3.27}$$

with β_w being the wind direction. All the wind coefficients (look-up tables) $C_X(\gamma_w)$, $C_Y(\gamma_w)$, $C_Z(\gamma_w)$, $C_K(\gamma_w)$, $C_M(\gamma_w)$, and $C_N(\gamma_w)$ are computed numerically or by experiments in a wind tunnel, see [28].

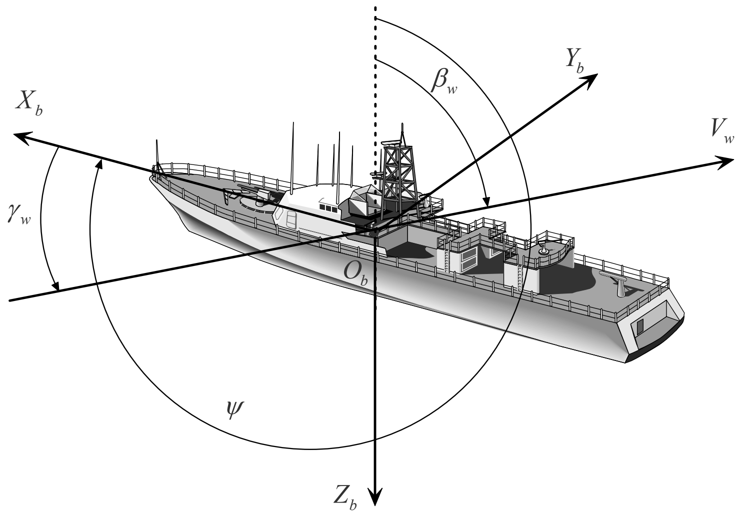

Figure 3.4 Definition of wind speed and direction

For the case where the vessel is moving, the vector $\boldsymbol{\tau}_{\boldsymbol{E}}^{\boldsymbol{wi}}$ is given by

$$\boldsymbol{\tau}_{\boldsymbol{E}}^{\boldsymbol{wi}} = \frac{1}{2}\rho_a V_{rw}^2 \begin{bmatrix} C_X(\gamma_{rw})A_{Fw} \\ C_Y(\gamma_{rw})A_{Lw} \\ C_Z(\gamma_{rw})A_{Fw} \\ C_K(\gamma_{rw})A_{Lw}H_{Lw} \\ C_M(\gamma_{rw})A_{Fw}H_{Fw} \\ C_N(\gamma_{rw})A_{Lw}L_{oa} \end{bmatrix} \tag{3.28}$$

where

$$\begin{aligned} V_{rw} &= \sqrt{u_{rw}^2 + v_{rw}^2}, \\ \gamma_{rw} &= -\text{arctan2}(v_{rw}, u_{rw}), \end{aligned} \tag{3.29}$$

with

$$\begin{aligned} u_{rw} &= u - V_w \cos(\beta_w - \psi), \\ v_{rw} &= v - V_w \sin(\beta_w - \psi). \end{aligned} \tag{3.30}$$

3.3.2.5 Propulsion Forces and Moments

The vector, $\boldsymbol{\tau}$, of propulsion forces and moments depends on a specific configuration of actuators such as propellers, rudders, and water jets on a particular vessel. In the next section where $\boldsymbol{\tau}$ is specified, we consider some classes of the ocean vessels that are common in practice. In this book, we neglect the dynamics of the actuators that provide the propulsion forces and moments since the response of the actuators such as hydraulic systems and electrical motors is much faster than the response of the vessel.

3.3.2.6 Model Summary and its Properties

Body-fixed Representation

Now substituting $\boldsymbol{\tau}_{\boldsymbol{RB}} = \boldsymbol{\tau}_{\boldsymbol{H}} + \boldsymbol{\tau}_{\boldsymbol{E}} + \boldsymbol{\tau}$ into (3.9) and combining it with (3.5) results in the equations of motion of an ocean vessel in six degrees of freedom as follows:

$$\begin{aligned} \dot{\eta} &= \boldsymbol{J}(\eta)\boldsymbol{v}, \\ \boldsymbol{M}\dot{\boldsymbol{v}} &= -\boldsymbol{C}(\boldsymbol{v})\boldsymbol{v} - \boldsymbol{D}(\boldsymbol{v})\boldsymbol{v} - \boldsymbol{g}(\eta) + \boldsymbol{\tau} + \boldsymbol{\tau}_{\boldsymbol{E}}, \end{aligned} \tag{3.31}$$

where

$$\begin{aligned} \boldsymbol{M} &= \boldsymbol{M}_{\boldsymbol{RB}} + \boldsymbol{M}_{\boldsymbol{A}}, \\ \boldsymbol{C}(\boldsymbol{v}) &= \boldsymbol{C}_{\boldsymbol{RB}}(\boldsymbol{v}) + \boldsymbol{C}_{\boldsymbol{A}}(\boldsymbol{v}). \end{aligned} \tag{3.32}$$

Under the assumption that the body is at rest (or at most is moving at low speed) in ideal fluid, the matrix $\boldsymbol{M}$ is always symmetric positive definite, i.e.,

$$\boldsymbol{M} = \boldsymbol{M}^{\boldsymbol{T}} > \boldsymbol{0}. \tag{3.33}$$

For a rigid body moving in fluid, the Coriolis and centripetal matrix $\boldsymbol{C}(\boldsymbol{v})$ can always be parameterized such that it is skew-symmetric, i.e.,

$$\boldsymbol{C}(\boldsymbol{v}) = -\boldsymbol{C}^{\boldsymbol{T}}(\boldsymbol{v}),\ \forall\, \boldsymbol{v} \in \mathbb{R}^{\boldsymbol{6}}. \tag{3.34}$$

For a rigid body moving in an ideal fluid, the hydrodynamic damping matrix $\boldsymbol{D}(\boldsymbol{v})$ is real, non-symmetric and strictly positive, i.e.,

$$\boldsymbol{D}(\boldsymbol{v}) > \boldsymbol{0},\ \forall\, \boldsymbol{v} \in \mathbb{R}^{\boldsymbol{6}}. \tag{3.35}$$

Earth-fixed Representation

The mathematical model (3.31) can also be written using a representation of the earth-fixed coordinates by applying the following kinematic transformations (with the assumption that $\boldsymbol{J}^{-1}(\boldsymbol{\eta})$ exists, i.e., $\theta \neq \pm\frac{\pi}{2}$):

$$\begin{aligned} \boldsymbol{v} &= \boldsymbol{J}^{-1}(\boldsymbol{\eta})\dot{\boldsymbol{\eta}}, \\ \dot{\boldsymbol{v}} &= \boldsymbol{J}^{-1}(\boldsymbol{\eta})\Big[\ddot{\boldsymbol{\eta}} - \dot{\boldsymbol{J}}(\boldsymbol{\eta})\boldsymbol{J}^{-1}(\boldsymbol{\eta})\dot{\boldsymbol{\eta}}\Big]. \end{aligned} \tag{3.36}$$

Now substituting (3.36) into the second equation of (3.31) results in

$$\boldsymbol{M}^{\star}(\boldsymbol{\eta})\ddot{\boldsymbol{\eta}} = -\boldsymbol{C}^{\star}(\boldsymbol{v},\boldsymbol{\eta})\dot{\boldsymbol{\eta}} - \boldsymbol{D}^{\star}(\boldsymbol{v},\boldsymbol{\eta})\dot{\boldsymbol{\eta}} - \boldsymbol{g}^{\star}(\boldsymbol{\eta}) + \boldsymbol{J}^{T}(\boldsymbol{\eta})(\boldsymbol{\tau} + \boldsymbol{\tau}_{\boldsymbol{E}}), \tag{3.37}$$

where

$$\begin{aligned} \boldsymbol{M}^{\star}(\boldsymbol{\eta}) &= \boldsymbol{J}^{-T}(\boldsymbol{\eta})\boldsymbol{M}\boldsymbol{J}^{-1}(\boldsymbol{\eta}), \\ \boldsymbol{C}^{\star}(\boldsymbol{v},\boldsymbol{\eta}) &= \boldsymbol{J}^{-T}(\boldsymbol{\eta})\big[\boldsymbol{C}(\boldsymbol{v}) - \boldsymbol{M}\boldsymbol{J}^{-1}(\boldsymbol{\eta})\dot{\boldsymbol{J}}(\boldsymbol{\eta})\big]\boldsymbol{J}^{-1}(\boldsymbol{\eta}), \\ \boldsymbol{D}^{\star}(\boldsymbol{v},\boldsymbol{\eta}) &= \boldsymbol{J}^{-T}(\boldsymbol{\eta})\boldsymbol{D}(\boldsymbol{v})\boldsymbol{J}^{-1}(\boldsymbol{\eta}), \\ \boldsymbol{g}^{\star}(\boldsymbol{\eta}) &= \boldsymbol{J}^{-T}(\boldsymbol{\eta})\boldsymbol{g}(\boldsymbol{\eta}). \end{aligned} \tag{3.38}$$

Under the same assumptions used in the body-fixed representation, the model (3.37) using the earth-fixed representation has the following properties:

$$\begin{aligned} &\boldsymbol{M}^{\star}(\boldsymbol{\eta}) = \boldsymbol{M}^{\star}(\boldsymbol{\eta})^{T}, \ \forall\, \boldsymbol{\eta} \in \mathbb{R}^{\boldsymbol{6}}, \\ &\boldsymbol{s}^{T}\big[\dot{\boldsymbol{M}}^{\star}(\boldsymbol{\eta}) - \boldsymbol{2}\boldsymbol{C}^{\star}(\boldsymbol{v},\boldsymbol{\eta})\big]\boldsymbol{s} = \boldsymbol{0}, \ \forall\, \boldsymbol{\eta} \in \mathbb{R}^{\boldsymbol{6}}, \boldsymbol{v} \in \mathbb{R}^{\boldsymbol{6}}, \ \boldsymbol{s} \in \mathbb{R}^{6}, \\ &\boldsymbol{D}^{\star}(\boldsymbol{v},\boldsymbol{\eta}) > \boldsymbol{0}, \quad \forall\, \boldsymbol{\eta} \in \mathbb{R}^{\boldsymbol{6}}, \boldsymbol{v} \in \mathbb{R}^{\boldsymbol{6}}. \end{aligned} \tag{3.39}$$

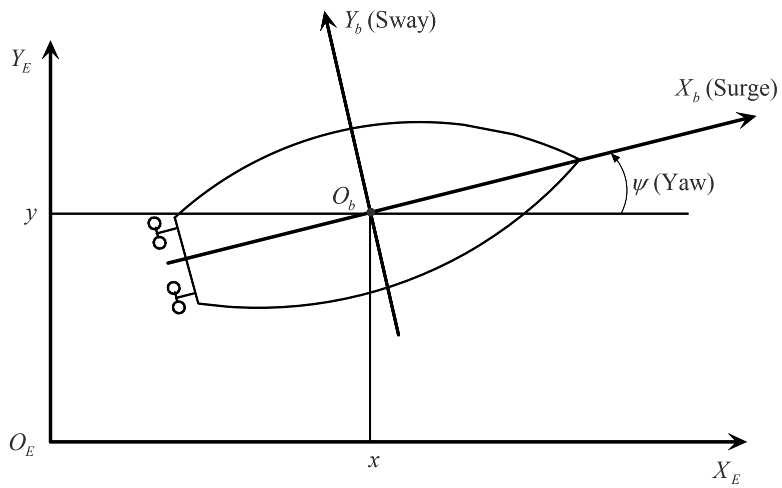

Figure 3.5 Motion variables for an ocean vessel moving in a horizontal plane

3.4 Standard Models for Ocean Vessels

In this section, the main results of the previous section are simplified to give a set of standard models for surface ships and underwater vehicles. These standard models will be extensively used for the control design in the coming chapters.

3.4.1 Three Degrees of Freedom Horizontal Model

3.4.1.1 Standard Three Degrees of Freedom Horizontal Model

The horizontal motion of a surface ship or an underwater vehicle moving in a horizontal plane is often described by the motion components in surge, sway, and yaw. Figure 3.5 illustrates the motion variables in this case. Therefore, we choose $\boldsymbol{\eta} = [x\ y\ \psi]^T$ and $\boldsymbol{v} = [u\ v\ r]^T$. This model is obtained from the general model (3.31) under the following assumption.

Assumption 3.1.

1. *The motion in roll, pitch, and heave is ignored. This means that we ignore the dynamics associated with the motion in heave, roll, and pitch, i.e.,* $z = 0$, $w = 0$, $\phi = 0$, $p = 0$, $\theta = 0$, *and* $q = 0$.
2. *The vessel has homogeneous mass distribution and* xz*-plane of symmetry so that*

$$I_{xy} = I_{yz} = 0. \tag{3.40}$$

3. *The center of gravity* CG *and the center of buoyancy,* CB*, are located vertically on the* z*-axis.*

With Assumption 3.1, the dynamics of a surface ship or an underwater vehicle moving in a horizontal plane is simplified from the general model (3.31) as follows:

$$\begin{aligned} \dot{\boldsymbol{\eta}} &= \boldsymbol{J}(\boldsymbol{\eta})\boldsymbol{v}, \\ \boldsymbol{M}\dot{\boldsymbol{v}} &= -\boldsymbol{C}(\boldsymbol{v})\boldsymbol{v} - (\boldsymbol{D} + \boldsymbol{D_n}(\boldsymbol{v}))\boldsymbol{v} + \boldsymbol{\tau} + \boldsymbol{\tau_E}, \end{aligned} \tag{3.41}$$

where the matrices $\boldsymbol{J}(\boldsymbol{\eta})$, $\boldsymbol{M}$, $\boldsymbol{C}(\boldsymbol{v})$, $\boldsymbol{D}$, and $\boldsymbol{D_n}(\boldsymbol{v})$ are given by

$$\boldsymbol{J}(\boldsymbol{\eta}) = \begin{bmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ \boldsymbol{M} = \begin{bmatrix} m - X_{\dot{u}} & 0 & 0 \\ 0 & m - Y_{\dot{v}} & mx_g - Y_{\dot{r}} \\ 0 & mx_g - Y_{\dot{r}} & I_z - N_{\dot{r}} \end{bmatrix},$$

$$\boldsymbol{C}(\boldsymbol{v}) = \begin{bmatrix} 0 & 0 & -m(x_g r + v) + Y_{\dot{v}} v + Y_{\dot{r}} r \\ 0 & 0 & mu - X_{\dot{u}} u \\ m(x_g r + v) - Y_{\dot{v}} v - Y_{\dot{r}} r & -mu + X_{\dot{u}} u & 0 \end{bmatrix},$$

$$\boldsymbol{D} = -\begin{bmatrix} X_u & 0 & 0 \\ 0 & Y_v & Y_r \\ 0 & N_v & N_r \end{bmatrix},$$

$$\boldsymbol{D_n}(\boldsymbol{v}) = -\begin{bmatrix} X_{|u|u}|u| & 0 & 0 \\ 0 & Y_{|v|v}|v| + Y_{|r|v}|r| & Y_{|v|r}|v| \\ 0 & N_{|v|v}|v| + N_{|r|v}|r| & N_{|v|r}|v| + N_{|r|r}|r| \end{bmatrix}. \tag{3.42}$$

The propulsion force and moment vector $\boldsymbol{\tau}$ is given by

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_u \\ 0 \\ \tau_r \end{bmatrix}. \tag{3.43}$$

The above propulsion force and moment vector $\boldsymbol{\tau}$ implies that we are considering a surface vessel, which does not have an independent actuator in the sway, i.e., an underactuated vessel is under consideration. Such a vessel can be one equipped with a pair of water jets or a pair of propellers.

The environmental disturbance vector $\boldsymbol{\tau_E}$ is given by

$$\boldsymbol{\tau_E} = \begin{bmatrix} \tau_{uE} \\ \tau_{vE} \\ \tau_{rE} \end{bmatrix}, \tag{3.44}$$

where τ_{uE} and τ_{vE} are disturbance forces acting in surge and sway respectively, and τ_{rE} is the disturbance moment acting in yaw.

3.4.1.2 Simplified Three Degrees of Freedom Horizontal Model

In some cases, in addition to Assumption 3.1 we ignore the off-diagonal terms of the matrices $\boldsymbol{M}$ and $\boldsymbol{D}$, all elements of the nonlinear damping matrix $\boldsymbol{D_n}(\boldsymbol{v})$. These assumptions hold when the vessel has three planes of symmetry, for which the axes of the body-fixed reference frame are chosen to be parallel to the principal axis of the displaced fluid, which are equal to the principal axis of the vessel. Most ships have port/starboard symmetry, and moreover, bottom/top symmetry is not required for horizontal motion. Ship fore/aft nonsymmetry implies that the off-diagonal terms of the inertia and damping matrices are nonzero. However, these terms are small compared to the main diagonal terms. Furthermore, disturbances induced by waves, wind, and ocean currents are ignored. Under the just-mentioned assumptions, the dynamics of a surface ship or an underwater vehicle moving in a horizontal plane is simplified from the three degrees of freedom model (3.41) as follows:

$$\begin{aligned}\dot{\boldsymbol{\eta}} &= \boldsymbol{J}(\boldsymbol{\eta})\boldsymbol{v},\\ \boldsymbol{M}\dot{\boldsymbol{v}} &= -\boldsymbol{C}(\boldsymbol{v})\boldsymbol{v} - \boldsymbol{D}\boldsymbol{v} + \boldsymbol{\tau},\end{aligned} \tag{3.45}$$

where the matrices $\boldsymbol{J}(\boldsymbol{\eta})$, $\boldsymbol{M}$, $\boldsymbol{C}(\boldsymbol{v})$ and $\boldsymbol{D}$ are given by

$$\begin{aligned}\boldsymbol{J}(\boldsymbol{\eta}) &= \begin{bmatrix}\cos(\psi) & -\sin(\psi) & 0\\ \sin(\psi) & \cos(\psi) & 0\\ 0 & 0 & 1\end{bmatrix}, \boldsymbol{M} = \begin{bmatrix} m_{11} & 0 & 0\\ 0 & m_{22} & 0\\ 0 & 0 & m_{33}\end{bmatrix},\\ \boldsymbol{C}(\boldsymbol{v}) &= \begin{bmatrix}0 & 0 & -m_{22}v\\ 0 & 0 & m_{11}u\\ m_{22}v & -m_{11}u & 0\end{bmatrix}, \boldsymbol{D} = \begin{bmatrix} d_{11} & 0 & 0\\ 0 & d_{22} & 0\\ 0 & 0 & d_{33}\end{bmatrix},\end{aligned} \tag{3.46}$$

with

$$\begin{aligned}&m_{11} = m - X_{\dot{u}},\ m_{22} = m - Y_{\dot{v}},\ m_{33} = I_z - N_{\dot{r}},\\ &d_{11} = -X_u,\ d_{22} = -Y_v,\ d_{33} = -N_r.\end{aligned} \tag{3.47}$$

The propulsion force and moment vector $\boldsymbol{\tau}$ is still given by (3.43), i.e., $\boldsymbol{\tau} = [\tau_u\ 0\ \tau_r]^T$.

3.4.1.3 Spherical Three Degrees of Freedom Horizontal Model

In addition to the assumptions made in Subsection 3.4.1.2, we assume that the vessel has bottom/top symmetry. An example of this type of vessels is an ODIN moving in a horizontal plane, see Figure 3.6. In this case, the model is further simplified to

$$\begin{aligned}\dot{\boldsymbol{\eta}} &= \boldsymbol{J}(\boldsymbol{\eta})\boldsymbol{v},\\ \boldsymbol{M}\dot{\boldsymbol{v}} &= -\boldsymbol{C}(\boldsymbol{v})\boldsymbol{v} - \boldsymbol{D}\boldsymbol{v} + \boldsymbol{\tau}\end{aligned} \tag{3.48}$$

where the matrices $\boldsymbol{J}(\boldsymbol{\eta})$, $\boldsymbol{M}$, $\boldsymbol{C}(\boldsymbol{v})$ and $\boldsymbol{D}$ are given by

$$\boldsymbol{J}(\boldsymbol{\eta}) = \begin{bmatrix}\cos(\psi) & -\sin(\psi) & 0\\ \sin(\psi) & \cos(\psi) & 0\\ 0 & 0 & 1\end{bmatrix}, \boldsymbol{M} = \begin{bmatrix} m_{xy} & 0 & 0\\ 0 & m_{xy} & 0\\ 0 & 0 & m_{33}\end{bmatrix},$$

$$\boldsymbol{C}(\boldsymbol{v}) = \begin{bmatrix}0 & 0 & -m_{xy}v\\ 0 & 0 & m_{xy}u\\ m_{xy}v & -m_{xy}u & 0\end{bmatrix}, \boldsymbol{D} = \begin{bmatrix} d_{xy} & 0 & 0\\ 0 & d_{xy} & 0\\ 0 & 0 & d_{33}\end{bmatrix}, \tag{3.49}$$

with

$$m_{xy} = m - X_{\dot{u}} = m - Y_{\dot{v}},\ m_{33} = I_z - N_{\dot{r}},$$
$$d_{xy} = -X_u = -Y_v,\ d_{33} = -N_r. \tag{3.50}$$

The propulsion force and moment vector $\boldsymbol{\tau}$ is still given by (3.43), i.e. $\boldsymbol{\tau} = [\tau_u\ 0\ \tau_r]^T$.

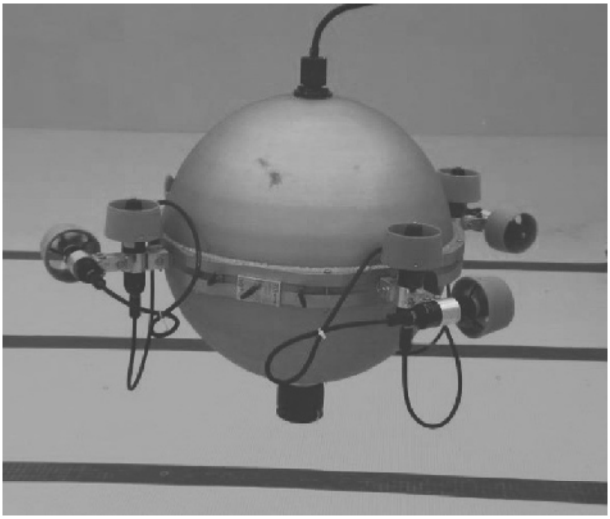

Figure 3.6 An omnidirectional intelligent navigator (ODIN).
Courtesy http://www.math.hawaii.edu/∼ryan/STOMP/Photos 20ODIN.html

3.4.2 Six Degrees of Freedom Model

3.4.2.1 Standard Model

In addition to the assumptions made in Section 3.3, we assume that the center of gravity and the center of buoyancy are located vertically on the $O_b Z_b$-axis, and that there are no couplings (off-diagonal terms) in the matrices $\boldsymbol{M}$, $\boldsymbol{D}$, and $\boldsymbol{D_n}(\boldsymbol{v})$. In this case, the model presented in Section 3.3 is simplified to

$$\begin{aligned}
\dot{\boldsymbol{\eta}}_1 &= \boldsymbol{J}_1(\boldsymbol{\eta}_2)\boldsymbol{v}_1,\\
\boldsymbol{M}_1\dot{\boldsymbol{v}}_1 &= -\boldsymbol{C}_1(\boldsymbol{v}_1)\boldsymbol{v}_2 - \boldsymbol{D}_1\boldsymbol{v}_1 - \boldsymbol{D}_{n1}(\boldsymbol{v}_1)\boldsymbol{v}_1 + \boldsymbol{\tau}_1 + \boldsymbol{\tau}_{1E},\\
\dot{\boldsymbol{\eta}}_2 &= \boldsymbol{J}_2(\boldsymbol{\eta}_2)\boldsymbol{v}_2,\\
\boldsymbol{M}_2\dot{\boldsymbol{v}}_2 &= -\boldsymbol{C}_1(\boldsymbol{v}_1)\boldsymbol{v}_1 - \boldsymbol{C}_2(\boldsymbol{v}_2)\boldsymbol{v}_2 - \boldsymbol{D}_2\boldsymbol{v}_2 - \boldsymbol{D}_{n2}(\boldsymbol{v}_2)\boldsymbol{v}_2 -\\
&\qquad \boldsymbol{g}_2(\boldsymbol{\eta}_2) + \boldsymbol{\tau}_2 + \boldsymbol{\tau}_{2E},
\end{aligned} \tag{3.51}$$

where $\boldsymbol{J_1}(\boldsymbol{\eta_2})$ and $\boldsymbol{J_2}(\boldsymbol{\eta_2})$ are given in (3.2) and (3.4). The matrices $\boldsymbol{M_1}$ and $\boldsymbol{M_2}$ are

$$\boldsymbol{M_1} = \begin{bmatrix} m_{11} & 0 & 0 \\ 0 & m_{22} & 0 \\ 0 & 0 & m_{33} \end{bmatrix},$$
$$\boldsymbol{M_2} = \begin{bmatrix} m_{44} & 0 & 0 \\ 0 & m_{55} & 0 \\ 0 & 0 & m_{66} \end{bmatrix}, \tag{3.52}$$

where

$$m_{11} = m - X_{\dot{u}},\ m_{22} = m - Y_{\dot{v}},$$
$$m_{33} = m - Z_{\dot{w}},\ m_{44} = I_x - K_{\dot{p}},$$
$$m_{55} = I_y - M_{\dot{q}},\ m_{66} = I_z - N_{\dot{r}}.$$

The matrices $\boldsymbol{C_1}(\boldsymbol{v_1})$ and $\boldsymbol{C_2}(\boldsymbol{v_2})$ are

$$\boldsymbol{C_1}(\boldsymbol{v_1}) = \begin{bmatrix} 0 & m_{33}w & -m_{22}v \\ -m_{33}w & 0 & m_{11}u \\ m_{22}v & -m_{11}u & 0 \end{bmatrix},$$
$$\boldsymbol{C_2}(\boldsymbol{v_2}) = \begin{bmatrix} 0 & m_{66}r & -m_{55}q \\ -m_{66}r & 0 & m_{44}p \\ m_{55}q & -m_{44}p & 0 \end{bmatrix}. \tag{3.53}$$

The linear damping matrices $\boldsymbol{D_1}$ and $\boldsymbol{D_2}$ are

$$\boldsymbol{D_1} = \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix},$$
$$\boldsymbol{D_2} = \begin{bmatrix} d_{44} & 0 & 0 \\ 0 & d_{55} & 0 \\ 0 & 0 & d_{66} \end{bmatrix}, \tag{3.54}$$

where

$$d_{11} = -X_u,$$
$$d_{22} = -Y_v,$$
$$d_{33} = -Z_w,$$
$$d_{44} = -K_p,$$
$$d_{55} = -M_q,$$
$$d_{66} = -N_r.$$

The nonlinear damping matrices $\boldsymbol{D_{n1}}(\boldsymbol{v_1})$ and $\boldsymbol{D_{n2}}(\boldsymbol{v_2})$ are

$$\boldsymbol{D_{n1}}(\boldsymbol{v_1}) = \begin{bmatrix} \sum_{i=2}^{3} d_{ui}|u|^{i-1} & 0 & 0 \\ 0 & \sum_{i=2}^{3} d_{vi}|v|^{i-1} & 0 \\ 0 & 0 & \sum_{i=2}^{3} d_{wi}|w|^{i-1} \end{bmatrix},$$

$$\boldsymbol{D_{n2}}(\boldsymbol{v_2}) = \begin{bmatrix} \sum_{i=2}^{3} d_{pi}|p|^{i-1} & 0 & 0 \\ 0 & \sum_{i=2}^{3} d_{qi}|q|^{i-1} & 0 \\ 0 & 0 & \sum_{i=2}^{3} d_{ri}|r|^{i-1} \end{bmatrix}, \quad (3.55)$$

where d_{ui}, d_{vi}, d_{wi}, d_{pi}, d_{qi}, and d_{ri} with $i = 2, 3$ are the nonlinear hydrodynamic damping coefficients.

The restoring force and moment vector $\boldsymbol{g_2}(\boldsymbol{\eta_2})$ is given by

$$\boldsymbol{g_2}(\boldsymbol{\eta_2}) = \begin{bmatrix} \rho g \nabla \overline{GM_T} \sin(\phi)\cos(\theta) \\ \rho g \nabla \overline{GM_L} \sin(\theta) \\ 0 \end{bmatrix}, \quad (3.56)$$

where ρ, g, ∇, $\overline{GM_T}$ and $\overline{GM_L}$ are the water density, gravity acceleration, displaced volume of water, transverse metacentric height and longitudinal metacentric height, respectively.

The propulsion force and moment vectors $\boldsymbol{\tau_1}$ and $\boldsymbol{\tau_2}$ are

$$\boldsymbol{\tau_1} = \begin{bmatrix} \tau_u \\ 0 \\ 0 \end{bmatrix}, \ \boldsymbol{\tau_2} = \begin{bmatrix} \tau_p \\ \tau_q \\ \tau_r \end{bmatrix}, \quad (3.57)$$

which imply that the vessel under consideration does not have independent actuators in the sway and heave.

The environmental disturbance vectors $\boldsymbol{\tau_{1E}}$ and $\boldsymbol{\tau_{2E}}$ are given by

$$\boldsymbol{\tau_{1E}} = \begin{bmatrix} \tau_{Eu} \\ \tau_{Ev} \\ \tau_{Ew} \end{bmatrix}, \ \boldsymbol{\tau_{2E}} = \begin{bmatrix} \tau_{Ep} \\ \tau_{Eq} \\ \tau_{Er} \end{bmatrix}, \quad (3.58)$$

where τ_{Eu}, τ_{Ev}, τ_{Ew}, τ_{Ep}, τ_{Eq}, and τ_{Er} are the environmental disturbance forces or moments acting on the surge, sway, heave, roll, pitch, and yaw axes, respectively.

3.4.2.2 Ignoring Nonlinear Damping Terms and Roll Model

In addition to the assumptions made in Section 3.4.2.1, it is sometimes reasonable to ignore nonlinear hydrodynamic damping terms and roll, and environmental disturbances. This holds when the vessel is operating at low speed and is equipped with independent internal/external roll actuators. As such, the model (3.51) is simplified to:

1. Kinematics

$$\begin{aligned}
\dot{x} &= \cos(\psi)\cos(\theta)u - \sin(\psi)v + \sin(\theta)\cos(\psi)w,\\
\dot{y} &= \sin(\psi)\cos(\theta)u + \cos(\psi)v + \sin(\theta)\sin(\psi)w,\\
\dot{z} &= -\sin(\theta)u + \cos(\theta)w,\\
\dot{\theta} &= q,\\
\dot{\psi} &= \frac{r}{\cos(\theta)}.
\end{aligned} \tag{3.59}$$

2. Kinetics

$$\begin{aligned}
\dot{u} &= \frac{m_{22}}{m_{11}}vr - \frac{m_{33}}{m_{11}}wq - \frac{d_{11}}{m_{11}}u + \frac{1}{m_{11}}\tau_u,\\
\dot{v} &= -\frac{m_{11}}{m_{22}}ur - \frac{d_{22}}{m_{22}}v,\\
\dot{w} &= \frac{m_{11}}{m_{33}}uq - \frac{d_{33}}{m_{33}}w,\\
\dot{q} &= \frac{m_{33}-m_{11}}{m_{55}}uw - \frac{d_{55}}{m_{55}}q - \frac{\rho g \nabla \overline{GM_L}\sin(\theta)}{m_{55}} + \frac{1}{m_{55}}\tau_q,\\
\dot{r} &= \frac{m_{11}-m_{22}}{m_{66}}uv - \frac{d_{66}}{m_{66}}r + \frac{1}{m_{66}}\tau_r.
\end{aligned} \tag{3.60}$$

3.5 Conclusions

This chapter sets out material about the basic motion tasks and mathematical models of the ocean vessels that will be used in the subsequent chapters. More details on deriving the mathematical models of the ocean vessels are given in [11, 12, 25]. It has also been pointed out that regulation/stabilization is much more difficult than trajectory-tracking, path-tracking, and path-following for underactuated ocean vessels.