Chapter 7
Backstepping for Time-Varying Systems

Abstract Backstepping is one of the most popular frameworks for designing controllers for nonlinear systems. Its multiple advantages are well-known. It leads to a wide family of globally asymptotically stabilizing control laws, and it makes it possible to address robustness issues and solve adaptive control problems. This chapter begins with a review of classical backstepping for time-invariant systems. We then give several extensions that lead to time-varying strict Lyapunov functions and stabilizing feedbacks for time-varying systems. We first consider a general class of linear time-varying systems. Then we provide stronger results for linear systems in feedback form. Finally, we study nonlinear systems in feedback form and give conditions ensuring globally uniform stabilizability by bounded control laws.

7.1 Motivation: PVTOL

To motivate our results, we first consider the plane with vertical take off and landing (PVTOL) model; see, e.g., [149, Chap. 6] or Sect. 7.9 for the literature on the model. In the absence of disturbances, the equations of the PVTOL model are

$$\begin{cases} \dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 = -u_1 \sin(\theta) \\ \dot{z}_1 = z_2 \\ \dot{z}_2 = u_1 \cos(\theta) - 1 \\ \dot{\theta} = \omega \\ \dot{\omega} = u_2 \ , \end{cases} \tag{7.1}$$

where ξ_1 and z_1 are the horizontal and vertical positions of the aircraft center of mass, respectively; and θ is the roll angle that the aircraft makes with the horizon. The control inputs u_1 and u_2 are the thrust (directed out from the bottom of the aircraft) and the angular acceleration (a.k.a. rolling moment), respectively. The coefficient -1 in the z_2-dynamics is the normalized gravitational acceleration.

Assume that we wish to track the following admissible trajectory for (7.1):

$$\big(\xi_{1,r}, \xi_{2,r}, z_{1,r}, z_{2,r}, \theta_r, \omega_r\big)(t) \;=\; \big(0, 0, 2\cos(3t), -6\sin(3t), 0, 0\big) \; . \tag{7.2}$$

The inputs corresponding to (7.2) are

$$u_{1,r}(t) = 1 - 18\cos(3t) \;\text{ and }\; u_{2,r}(t) = 0. \tag{7.3}$$

Using the variables $\tilde{\xi}_i = \xi_i - \xi_{i,r}(t)$ and $\tilde{z}_i = z_i - z_{i,r}(t)$ for $i = 1, 2$, $\tilde{\theta} = \theta - \theta_r(t)$, and $\tilde{\omega} = \omega - \omega_r(t)$, and the change of feedback

$$\tilde{u}_1 = u_1 - u_{1,r}(t) \; , \; \tilde{u}_2 = u_2 - u_{2,r}(t) \tag{7.4}$$

gives the error dynamics

$$\begin{cases} \dot{\tilde{\xi}}_1 = \tilde{\xi}_2 \\ \dot{\tilde{\xi}}_2 = -\big[\tilde{u}_1 + 1 - 18\cos(3t)\big]\sin(\tilde{\theta}) \\ \dot{\tilde{z}}_1 = \tilde{z}_2 \\ \dot{\tilde{z}}_2 = \big[\tilde{u}_1 + 1 - 18\cos(3t)\big]\cos(\tilde{\theta}) - 1 + 18\cos(3t) \\ \dot{\tilde{\theta}} = \tilde{\omega} \\ \dot{\tilde{\omega}} = \tilde{u}_2 . \end{cases} \tag{7.5}$$

We wish to find feedback stabilizers that render (7.5) UGAS to the origin.

To this end, we first consider the auxiliary system

$$\begin{cases} \dot{\tilde{\xi}}_1 = \tilde{\xi}_2 \\ \dot{\tilde{\xi}}_2 = -\big[\tilde{u}_1 + 1 - 18\cos(3t)\big]\sin(v_2) \\ \dot{\tilde{z}}_1 = \tilde{z}_2 \\ \dot{\tilde{z}}_2 = \big[\tilde{u}_1 + 1 - 18\cos(3t)\big]\cos(v_2) - 1 + 18\cos(3t) \end{cases} \tag{7.6}$$

with $\tilde{u}_1$ and v_2 as inputs. Assume for the moment that we have constructed two control laws

$$\tilde{u}_{1s}\big(t, \tilde{\xi}_1, \tilde{\xi}_2, \tilde{z}_1, \tilde{z}_2\big) \;\text{ and }\; v_{2s}\big(t, \tilde{\xi}_1, \tilde{\xi}_2, \tilde{z}_1, \tilde{z}_2\big)$$

that have period 2π in t and that render the origin of the system (7.6) UGAS. Then a variant of classical backstepping (which we review in Sect. 7.2.3) gives a control law $\mu_s(t, \tilde{\xi}_1, \tilde{\xi}_2, \tilde{z}_1, \tilde{z}_2, \tilde{\theta})$ that also has period 2π in t such that

$$\begin{cases} \dot{\tilde{\xi}}_1 = \tilde{\xi}_2 \\ \dot{\tilde{\xi}}_2 = -\left[\tilde{u}_{1s}(t, \tilde{\xi}_1, \tilde{\xi}_2, \tilde{z}_1, \tilde{z}_2) + 1 - 18\cos(3t)\right]\sin(\tilde{\theta}) \\ \dot{\tilde{z}}_1 = \tilde{z}_2 \\ \dot{\tilde{z}}_2 = \left[\tilde{u}_{1s}(t, \tilde{\xi}_1, \tilde{\xi}_2, \tilde{z}_1, \tilde{z}_2) + 1 - 18\cos(3t)\right]\cos(\tilde{\theta}) - 1 + 18\cos(3t) \\ \dot{\tilde{\theta}} = \mu_s(t, \tilde{\xi}_1, \tilde{\xi}_2, \tilde{z}_1, \tilde{z}_2, \tilde{\theta}) \end{cases} \tag{7.7}$$

is also UGAS to the origin. Repeating this argument gives a control law

$$\tilde{u}_{2s}\big(t, \tilde{\xi}_1, \tilde{\xi}_2, \tilde{z}_1, \tilde{z}_2, \tilde{\theta}, \tilde{\omega}\big),$$

also having period 2π in t, such that the origin of (7.5) in closed-loop with $\tilde{u}_{1s}(t, \tilde{\xi}_1, \tilde{\xi}_2, \tilde{z}_1, \tilde{z}_2)$ and $\tilde{u}_{2s}(t, \tilde{\xi}_1, \tilde{\xi}_2, \tilde{z}_1, \tilde{z}_2, \tilde{\theta}, \tilde{\omega})$ is UGAS. However, it is by no means clear how to construct the necessary control laws $\tilde{u}_{1s}(t, \tilde{\xi}_1, \tilde{\xi}_2, \tilde{z}_1, \tilde{z}_2)$ and $v_{2s}(t, \tilde{\xi}_1, \tilde{\xi}_2, \tilde{z}_1, \tilde{z}_2)$ to stabilize (7.6). We will return to this example in Sect. 7.8, where we construct $\tilde{u}_{1s}$ and v_{2s} as a special case of a general backstepping theory for time-varying systems.

7.2 Classical Backstepping

Backstepping involves constructing stabilizing controllers for nonlinear systems having a lower triangular structure called *feedback form.* The backstepping approach is not a single technique, but rather is a collection of techniques sharing some key ideas. There is a backstepping technique based on cancelation of nonlinearities, and another involving domination of nonlinearities. We review these two methods next. Throughout the chapter, all inequalities and equalities should be understood to hold globally unless otherwise indicated, and we omit the arguments of our functions when they are clear from the context. Also, we assume that all of the functions encountered are sufficiently smooth.

7.2.1 Backstepping with Cancelation

We first recall the most important steps of backstepping by applying a basic version of backstepping with cancelation (which is also called *exact backstepping*) to the following family of time-invariant systems:

$$\begin{cases} \dot{x}_i = x_{i+1} + f_i(x_1, x_2, ..., x_i), \quad 1 \leq i \leq n-1 \\ \dot{x}_n = u + f_n(x_1, x_2, ..., x_n) \end{cases} \tag{7.8}$$

where each $x_i \in \mathbb{R}$, $u \in \mathbb{R}$ is the input, and each function f_i is assumed to be zero at the origin and C^1. Systems of the form (7.8) are said to be in *strict feedback form.*

The key feature of (7.8) is that each $\dot{x}_i$ depends only on $x_1, x_2, ..., x_{i+1}$ and is affine in x_{i+1}. The idea behind backstepping is to consider x_2 as a "pseudo-control" (which is also frequently called a "virtual input") for the x_1-subsystem. Thus, if it were possible to simply replace x_2 with $-x_1 - f_1(x_1)$, then the x_1-subsystem would become

$$\dot{x}_1 \; = \; -x_1 \tag{7.9}$$

which has the Lyapunov function $V_1(x_1) = \frac{1}{2}x_1^2$. Since x_2 cannot be replaced with $-x_1 - f_1(x_1)$, we instead use the change of coordinates

$$\begin{array}{l} z_1 = x_1 \\ z_2 = x_2 - \alpha_1(x_1) \end{array} \tag{7.10}$$

where $\alpha_1(x_1) = -x_1 - f_1(x_1)$. This change of coordinates transforms the (x_1, x_2)-subsystem of (7.8) into

$$\begin{cases} \dot{z}_1 = -z_1 + z_2 \\ \dot{z}_2 = x_3 + \overline{f}_2(z_1, z_2) \; , \end{cases} \tag{7.11}$$

where

$$\overline{f}_2(z_1, z_2) \; = \; f_2(x_1, x_2) - \alpha_1'(x_1)[x_2 + f_1(x_1)].$$

The time derivative of $V_1(z_1)$ along the trajectories of (7.11) satisfies

$$\dot{V}_1 = -z_1^2 + z_1 z_2 \; . \tag{7.12}$$

Assume $n \geq 4$. The backstepping now proceeds recursively. We view x_3 in (7.11) as a virtual input, and we use the new coordinate $z_3 = x_3 - \alpha_2(z_1, z_2)$, where $\alpha_2(z_1, z_2) = -z_1 - z_2 - \overline{f}_2(z_1, z_2)$. This gives the system

$$\begin{cases} \dot{z}_1 = -z_1 + z_2 \\ \dot{z}_2 = z_3 + \alpha_2(z_1, z_2) + \overline{f}_2(z_1, z_2) = z_3 - z_1 - z_2 \\ \dot{z}_3 = x_4 + \overline{f}_3(z_1, z_2, z_3), \end{cases} \tag{7.13}$$

where

$$\overline{f}_3(z_1, z_2, z_3) \; = \; f_3\big(z_1, z_2 + \alpha_1(z_1), z_3 + \alpha_2(z_1, z_2)\big) - \dot{\alpha}_2(z_1, z_2).$$

The time derivative of

$$V_2(z_1, z_2) = V_1(z_1) + \frac{1}{2}z_2^2 \tag{7.14}$$

along the solutions of (7.13) satisfies

$$\dot{V}_2 = -z_1^2 + z_1 z_2 + z_2(z_3 - z_1 - z_2) = -z_1^2 - z_2^2 + z_2 z_3. \tag{7.15}$$

At the i-th step, the last component of the dynamics is

$$\dot{z}_i = x_{i+1} + \overline{f}_i(z_1, ..., z_i) \tag{7.16}$$

for a suitable function $\overline{f}_i$, and we introduce the variable

$$z_{i+1} = x_{i+1} - \alpha_i(z_1, \ldots, z_i)\ , \tag{7.17}$$

where $\alpha_i(z_1, \ldots, z_i) = -z_{i-1} - z_i - \overline{f}_i(z_1, ..., z_i)$ and

$$V_i(z_1, ..., z_i) = \frac{1}{2}\sum_{r=1}^{i} z_r^2. \tag{7.18}$$

Then

$$\dot{z}_i = z_{i+1} + \alpha_i(z_1, \ldots, z_i) + \overline{f}_i(z_1, ..., z_i) = z_{i+1} - z_{i-1} - z_i \tag{7.19}$$

and the time derivative of V_i along trajectories of the $(z_1, ..., z_i)$-subsystem satisfies

$$\dot{V}_i = -\sum_{r=1}^{i} z_r^2 + z_i z_{i+1}. \tag{7.20}$$

At the last step, we have

$$\dot{z}_n = u + \overline{f}_n(z_1, ..., z_n)\ . \tag{7.21}$$

Choosing

$$u = \alpha_n(z_1, ..., z_n) \doteq -z_{n-1} - z_n - \overline{f}_n(z_1, ..., z_n) \tag{7.22}$$

and

$$V_n(z_1, ..., z_n) = \frac{1}{2}\sum_{r=1}^{n} z_r^2 \tag{7.23}$$

gives

$$\dot{z}_n = -z_{n-1} - z_n \tag{7.24}$$

and

$$\dot{V}_n = -\sum_{r=1}^{n} z_r^2. \tag{7.25}$$

Therefore, the system

$$\begin{cases} \dot{z}_1 = -z_1 + z_2 \\ \dot{z}_i = z_{i+1} - z_{i-1} - z_i, \quad i = 2, 3, \ldots, n-1 \\ \dot{z}_n = -z_{n-1} - z_n \end{cases} \tag{7.26}$$

is GAS. From the definition of the functions α_j, it follows that (7.8) in closed-loop with

$$u(x) = \alpha_n\big(\zeta_1(x), ..., \zeta_n(x)\big) \tag{7.27}$$

where $x = (x_1, .., x_n)$ and

$$\begin{aligned} \zeta_1(x) &= x_1 \\ \zeta_{i+1}(x) &= x_{i+1} - \alpha_i\big(\zeta_1(x), \dots, \zeta_i(x)\big), \quad i = 1, 2, \dots, n-1 \end{aligned} \tag{7.28}$$

is GAS.

7.2.2 Backstepping with Domination

The control law $u(x)$ in (7.27) depends explicitly on the nonlinear functions $f_i(x_1, x_2, ..., x_i)$ because

$$\alpha_i(z_1, \dots, z_i) = -z_{i-1} - z_i - \overline{f}_i(z_1, ..., z_i)$$

for each i. Consequently, when the functions f_i are unknown, the technique does not apply. In [75, pp. 84-85], it is explained how backstepping can be adapted to the case where the functions f_i are replaced by

$$f_i(t, x_1, x_2, ..., x_i, u) = \varphi_i(x_1, ..., x_i)^\top \Delta(t, x, u) \ , \tag{7.29}$$

where $\varphi_i(x_1, ..., x_i)$ is a $(p \times 1)$ vector of known smooth nonlinear functions, and $\Delta(t, x, u)$ is a globally bounded $(p \times 1)$ smooth vector of uncertain nonlinearities.

We next provide a variant of [75, pp. 84-85] that constructs a state feedback to prove UGAS of the uncertain system

$$\begin{cases} \dot{x}_1 = x_2 + \varphi_1(x_1)^\top \Delta_1(t, x, u) \\ \dot{x}_i = x_{i+1} + \varphi_i(x_1, ..., x_i)^\top \Delta_i(t, x, u), \quad i = 2, 3, \dots, n-1 \\ \dot{x}_n = u + \varphi_n(x_1, ..., x_n)^\top \Delta_n(t, x, u) \end{cases} \tag{7.30}$$

with state space $\mathbb{R}^n$ in feedback form. We do not require the functions Δ_i to be bounded. Rather, we assume that they satisfy

$$\big|\Delta_i(t, x, u)\big| \ \le \ \Delta_M \sqrt{\sum_{r=1}^{i} x_r^2} \quad \text{for} \quad i = 1, 2, \dots, n \tag{7.31}$$

for some known positive constant Δ_M. Let $\overline{\varphi}$ be an everywhere positive, increasing function such that

$$|\varphi_i(x_1, ..., x_i)| \ \leq \ \overline{\varphi}\left(\sqrt{\sum_{r=1}^{i} x_r^2}\right) \tag{7.32}$$

for each $i \in \{1, 2, \dots, n\}$ and $x \in \mathbb{R}^n$. Our backstepping involves a change of variables, followed by the construction of an appropriate set of dominating functions.

7.2.2.1 Change of Variables

We introduce the notation $\xi_i = (x_1, ..., x_i)$ for $i = 1, 2, \dots, n$. Given arbitrary positive constants c_i and everywhere positive functions $\kappa_i \in C^n$, we use the variables

$$\begin{aligned} z_1 &= x_1 \\ z_i &= x_i - \alpha_{i-1}(\xi_{i-1}) \qquad \forall i \geq 2 \ , \end{aligned} \tag{7.33}$$

where

$$\begin{aligned} \alpha_1(\xi_1) &= -\big[c_1 + \kappa_1(\xi_1)\big] z_1 \qquad \text{and} \\ \alpha_i(\xi_i) &= -\big[c_i + \kappa_i(\xi_i)\big] z_i - z_{i-1} + \sum_{r=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_r}(\xi_{i-1}) x_{r+1} \end{aligned} \tag{7.34}$$

for $i = 2, \dots, n$, and we let $u = \alpha_n(\xi_n)$. We specify the functions κ_i later.

Elementary calculations yield

$$\begin{cases} \dot{z}_1 = z_2 + \alpha_1(x_1) + \varphi_1(x_1)^\top \Delta_1(t, x, u) \\ \dot{z}_i = z_{i+1} + \alpha_i(\xi_i) + \varphi_i(\xi_i)^\top \Delta_i(t, x, u) \\ \qquad - \displaystyle\sum_{r=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_r}(\xi_{i-1}) \dot{x}_r, \quad i = 2, 3, \dots, n-1 \\ \dot{z}_n = \alpha_n(\xi_n) + \varphi_n(\xi_n)^\top \Delta_n(t, x, u) - \displaystyle\sum_{r=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_r}(\xi_{n-1}) \dot{x}_r \ , \end{cases} \tag{7.35}$$

or equivalently,

$$\begin{cases} \dot{z}_1 = -\big[c_1 + \kappa_1(x_1)\big] z_1 + z_2 + \varphi_1(x_1)^\top \Delta_1(t, x, u) \\ \dot{z}_i = -\big[c_i + \kappa_i(\xi_i)\big] z_i - z_{i-1} + z_{i+1} + \varphi_i(\xi_i)^\top \Delta_i(t, x, u) \\ \qquad - \displaystyle\sum_{r=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_r}(\xi_{i-1}) \varphi_r(\xi_r)^\top \Delta_r(t, x, u), \quad i = 2, 3, \dots, n-1 \\ \dot{z}_n = -\big[c_n + \kappa_n(\xi_n)\big] z_n - z_{n-1} + \varphi_n(\xi_n)^\top \Delta_n(t, x, u) \\ \qquad - \displaystyle\sum_{r=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_r}(\xi_{n-1}) \varphi_r(\xi_r)^\top \Delta_r(t, x, u) \ . \end{cases} \tag{7.36}$$

The time derivative of the function

$$V_n(z_1, ..., z_n) = \frac{1}{2}\sum_{i=1}^{n} z_i^2 \tag{7.37}$$

along the trajectories of (7.36) is

$$\begin{aligned}\dot{V}_n = &-\sum_{i=1}^{n}\left[c_i + \kappa_i(\xi_i)\right]z_i^2 + z_1\varphi_1(x_1)^\top \Delta_1(t,x,u)\\ &+\sum_{i=2}^{n} z_i\left[\varphi_i(\xi_i)^\top \Delta_i(t,x,u) - \sum_{r=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_r}(\xi_{i-1})\varphi_r(\xi_r)^\top \Delta_r(t,x,u)\right].\end{aligned}$$

From (7.31), we deduce that

$$\begin{aligned}\dot{V}_n \le &-\sum_{i=1}^{n}\left[c_i + \kappa_i(\xi_i)\right]z_i^2 + \Delta_M|z_1||\varphi_1(x_1)||x_1|\\ &+\sum_{i=2}^{n}|z_i|\left[\Delta_M|\varphi_i(\xi_i)||\xi_i| + \Delta_M\sum_{r=1}^{i-1}\left|\frac{\partial\alpha_{i-1}}{\partial x_r}(\xi_{i-1})\right||\varphi_r(\xi_r)||\xi_r|\right].\end{aligned}$$

Using the inequality $|\xi_i| \ge |\xi_r|$ for all $r \in \{1, ..., i\}$ and (7.32) gives

$$\dot{V}_n \le -\sum_{i=1}^{n}[c_i + \kappa_i(\xi_i)]z_i^2 + \Delta_M\sum_{i=1}^{n}|z_i||\xi_i|\overline{\varphi}(|\xi_i|)\Gamma_i(\xi_i)\ , \tag{7.38}$$

where $\Gamma_1(\xi_1) = 1$ and

$$\Gamma_i(\xi_i) = 1 + \sum_{r=1}^{i-1}\left|\frac{\partial\alpha_{i-1}}{\partial x_r}(\xi_{i-1})\right| \quad \text{for } i = 2, \dots, n\ . \tag{7.39}$$

If the everywhere positive functions κ_i are such that

$$\sum_{i=1}^{n}\kappa_i(\xi_i)z_i^2 \ge \Delta_M\sum_{i=1}^{n}|z_i||\xi_i|\overline{\varphi}(|\xi_i|)\Gamma_i(\xi_i)\ , \tag{7.40}$$

then we obtain the desirable inequality

$$\dot{V}_n \le -\sum_{i=1}^{n} c_i z_i^2 \tag{7.41}$$

which implies the GAS of the system because V_n is a positive definite quadratic function and the right side of (7.41) is negative definite. It remains to construct positive functions κ_i that satisfy (7.40), which we do next.

7.2.2.2 Construction of the Dominating Functions κ_i's

We now construct everywhere positive functions κ_i that satisfy (7.40), by induction.

Induction Assumption. For each $k \in \{1, ..., n\}$, there are k functions $\kappa_i : \mathbb{R}^i \to [1, \infty)$ of class C^n such that

$$\tfrac{k}{n} \sum_{i=1}^{k} \kappa_i(\xi_i) z_i^2 \geq \Delta_M \sum_{i=1}^{k} |z_i||\xi_i|\overline{\varphi}(|\xi_i|)\Gamma_i(\xi_i). \tag{7.42}$$

Step 1. The result holds for $k = 1$ because we can choose an everywhere positive function $\kappa_1 \in C^n$ such that

$$\tfrac{1}{n}\kappa_1(z_1)z_1^2 \geq \Delta_M z_1^2 \overline{\varphi}(|z_1|). \tag{7.43}$$

Step $k+1$. Assume that the induction assumption is satisfied at step k. Choose an everywhere positive function $\kappa_{k+1} \in C^n$ such that

$$\tfrac{1}{4n}\kappa_{k+1}(\xi_{k+1}) \geq \tfrac{2n\Delta_M^2}{\kappa_{k+1}(\xi_{k+1})}\overline{\varphi}^2(|\xi_{k+1}|)\Gamma_{k+1}^2(\xi_{k+1}) \, . \tag{7.44}$$

The induction assumption gives

$$\begin{aligned}
\frac{k+1}{n}\sum_{i=1}^{k+1}\kappa_i(\xi_i)z_i^2 &= \frac{k}{n}\sum_{i=1}^{k+1}\kappa_i(\xi_i)z_i^2 + \frac{1}{n}\sum_{i=1}^{k+1}\kappa_i(\xi_i)z_i^2 \\
&\geq \Delta_M \sum_{i=1}^{k}|z_i||\xi_i|\overline{\varphi}(|\xi_i|)\Gamma_i(\xi_i) + \frac{1}{n}\sum_{i=1}^{k+1}\kappa_i(\xi_i)z_i^2 \\
&= \Delta_M \sum_{i=1}^{k+1}|z_i||\xi_i|\overline{\varphi}(|\xi_i|)\Gamma_i(\xi_i) \\
&\quad + \frac{1}{n}\sum_{i=1}^{k+1}\kappa_i(\xi_i)z_i^2 \\
&\quad - \Delta_M|z_{k+1}||\xi_{k+1}|\overline{\varphi}(|\xi_{k+1}|)\Gamma_{k+1}(\xi_{k+1}).
\end{aligned} \tag{7.45}$$

Using the triangular inequality $ab \leq \frac{1}{4}a^2 + b^2$ for suitable nonnegative values a and b, we deduce that

$$\begin{aligned}
&|z_{k+1}||\xi_{k+1}|\overline{\varphi}(|\xi_{k+1}|)\Gamma_{k+1}(\xi_{k+1}) \\
&\leq \frac{\kappa_{k+1}(\xi_{k+1})z_{k+1}^2}{4n\Delta_M} + \frac{\Delta_M n}{\kappa_{k+1}(\xi_{k+1})}|\xi_{k+1}|^2\overline{\varphi}^2\big(|\xi_{k+1}|\big)\Gamma_{k+1}^2(\xi_{k+1})
\end{aligned}$$

and therefore

$$\begin{aligned}\tfrac{k+1}{n}\sum_{i=1}^{k+1}\kappa_i(\xi_i)z_i^2 \geq \Delta_M\sum_{i=1}^{k+1}|z_i||\xi_i|\overline{\varphi}\big(|\xi_i|\big)\Gamma_i(\xi_i)+\frac{1}{n}\sum_{i=1}^{k}\kappa_i(\xi_i)z_i^2 \\ +\tfrac{3}{4n}\kappa_{k+1}(\xi_{k+1})z_{k+1}^2 \\ -\tfrac{n\Delta_M^2}{\kappa_{k+1}(\xi_{k+1})}|\xi_{k+1}|^2\overline{\varphi}^2\big(|\xi_{k+1}|\big)\Gamma_{k+1}^2(\xi_{k+1})\ .\end{aligned} \tag{7.46}$$

Since $x_{k+1}=z_{k+1}+\alpha_k(\xi_k)$ for all $k\geq 1$, we get

$$|\xi_{k+1}|^2=|\xi_k|^2+(z_{k+1}+\alpha_k(\xi_k))^2 \ \leq \ 2z_{k+1}^2+|\xi_k|^2+2\alpha_k^2(\xi_k),$$

so our choice (7.44) of κ_{k+1} gives

$$\begin{aligned}\tfrac{k+1}{n}\sum_{i=1}^{k+1}\kappa_i(\xi_i)z_i^2 \geq \Delta_M\sum_{i=1}^{k+1}|z_i||\xi_i|\overline{\varphi}\big(|\xi_i|\big)\Gamma_i(\xi_i)+\frac{1}{n}\sum_{i=1}^{k}\kappa_i(\xi_i)z_i^2 \\ +\tfrac{1}{2n}\kappa_{k+1}(\xi_{k+1})z_{k+1}^2 \\ -\tfrac{2n\Delta_M^2(|\xi_k|^2+|\alpha_k(\xi_k)|^2)}{\kappa_{k+1}(\xi_{k+1})}\overline{\varphi}^2\big(|\xi_{k+1}|\big)\Gamma_{k+1}^2(\xi_{k+1})\ .\end{aligned} \tag{7.47}$$

One can easily prove that there is a function $\mho$, depending on the functions $\kappa_1,\ldots,\ \kappa_k$ but not on κ_{k+1}, such that

$$|\xi_k|^2+2\big|\alpha_k(\xi_k)\big|^2 \ \leq \ \mho(|\xi_k|)\sum_{i=1}^{k}z_i^2, \tag{7.48}$$

by induction on the components of ξ_k. Therefore,

$$\begin{aligned}&\tfrac{n\Delta_M^2(|\xi_k|^2+2|\alpha_k(\xi_k)|^2)}{\kappa_{k+1}(\xi_{k+1})}\overline{\varphi}^2\big(|\xi_{k+1}|\big)\Gamma_{k+1}^2(\xi_{k+1}) \\ \leq\ &\tfrac{n\Delta_M^2\mho(|\xi_k|)\sum_{i=1}^{k}z_i^2}{\kappa_{k+1}(\xi_{k+1})}\overline{\varphi}^2\big(|\xi_{k+1}|\big)\Gamma_{k+1}^2(\xi_{k+1})\ .\end{aligned} \tag{7.49}$$

Since $\overline{\varphi}$ and Γ_{k+1} are also independent of κ_{k+1}, we can enlarge κ_{k+1} sufficiently so that

$$\tfrac{2n\Delta_M^2(|\xi_k|^2+|\alpha_k(\xi_k)|^2)}{\kappa_{k+1}(\xi_{k+1})}\overline{\varphi}^2\big(|\xi_{k+1}|\big)\Gamma_{k+1}^2(\xi_{k+1}) \leq \tfrac{1}{n}\sum_{i=1}^{k}\kappa_i(\xi_i)z_i^2\ . \tag{7.50}$$

Combining this inequality with (7.47), we obtain

$$\frac{k+1}{n}\sum_{i=1}^{k+1}\kappa_i(\xi_i)z_i^2 \geq \Delta_M\sum_{i=1}^{k+1}|z_i||\xi_i|\overline{\varphi}\big(|\xi_i|\big)\Gamma_i(\xi_i). \tag{7.51}$$

This concludes the construction of the functions κ_i, which establishes (7.41). This proves the domination result.

7.2.3 Further Extensions

In the system (7.8), each $\dot{x}_i$ depends only on $x_1, x_2, ..., x_{i+1}$ and is affine in x_{i+1}. This assumption can be relaxed. For example, we can extend the result to systems

$$\begin{cases} \dot{x}_i = g_i(x_1, x_2, ...x_i)h_i(x_{i+1}) + f_i(x_1, x_2, ..., x_i), \quad 1 \leq i \leq n-1 \\ \dot{x}_n = g_n(x_1, x_2, ...x_n)h_n(u) + f_n(x_1, x_2, ..., x_n) \ , \end{cases} \tag{7.52}$$

where each $x_i \in \mathbb{R}$, $u \in \mathbb{R}$ is the input, each function f_i is assumed to be zero at the origin, each function g_i is everywhere positive or everywhere negative, and each real-valued function h_i is a diffeomorphism satisfying $h_i(0) = 0$. The extension proceeds by choosing new coordinates $z_1, z_2, \ldots, z_n$ (which are different from, but analogous to, the ones we chose in Sect. 7.2.1) that give the system (7.26). In the first step, we take $z_1 = x_1$ and $z_2 = g_1(x_1)h_1(x_2) + x_1 + f_1(x_1)$ to get $\dot{z}_1 = -z_1 + z_2$ and $\dot{z}_2 = \mathcal{M}_2(z_1, z_2)h_2(x_3) + \tilde{f}_2(z_1, z_2) = z_3 - z_1 - z_2$ for appropriate functions $\mathcal{M}_2$ and $\tilde{f}_2$ with $\mathcal{M}_2$ being nowhere zero, and in general, $\dot{z}_i = \mathcal{M}_i(z_1, z_2, \ldots, z_i)h_i(x_{i+1}) + \tilde{f}_i(z_1, z_2, \ldots, z_i) = z_{i+1} - z_{i-1} - z_i$ for $i = 1, 2, \ldots, n-1$ for suitable functions $\mathcal{M}_i$ and $\tilde{f}_i$. We next give another backstepping result, to help the reader understand later sections.

Consider a nonlinear time-varying system

$$\begin{cases} \dot{x} = f_x(t, x, z) \\ \dot{z} = g(t, x, z)h(u) + f_z(t, x, z) \end{cases} \tag{7.53}$$

that is periodic with a given period $T > 0$ in t, where $x \in \mathbb{R}^{n_x}$, $z \in \mathbb{R}$, $u \in \mathbb{R}$, h is a diffeomorphism satisfying $h(0) = 0$, and the function g is such that there exists an everywhere positive continuous function γ_p such that

$$\gamma_p(x, z) \leq g(t, x, z) \tag{7.54}$$

for all t, x, and z. We assume that f_x, g, h, and f_z are C^1, and that there exists a function $z_s(t, x)$ that is periodic of period T in t such that $z_s(t, 0) \equiv 0$, and such that the system

$$\dot{x} = f_x\big(t, x, z_s(t, x)\big) \tag{7.55}$$

is UGAS to 0. Finally, we assume that a strict Lyapunov function V_1 is known for the closed-loop system (7.55), with V_1 having period T in t. This gives known functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and a known positive definite function $W_1(x)$ such that

$$\alpha_1(|x|) \leq V_1(t, x) \leq \alpha_2(|x|) \tag{7.56}$$

and

$$\tfrac{\partial V_1}{\partial t}(t, x) + \tfrac{\partial V_1}{\partial x}(t, x) f(x, z_s(t, x)) \leq -W_1(x) \tag{7.57}$$

for all $(t, x) \in \mathbb{R} \times \mathbb{R}^n$.

Then

$$V_2(t, x, z) = V_1(t, x) + \tfrac{1}{2}[z - z_s(t, x)]^2 \tag{7.58}$$

admits functions $\alpha_3, \alpha_4 \in \mathcal{K}_\infty$ such that

$$\alpha_3(|(x, z)|) \leq V_2(t, x, z) \leq \alpha_4(|(x, z)|) \tag{7.59}$$

for all $t \in \mathbb{R}$ and all $(x, z) \in \mathbb{R}^n \times \mathbb{R}$. Also, its time derivative along the trajectories of (7.53) satisfies

$$\begin{aligned}
\dot{V}_2 &= \tfrac{\partial V_1}{\partial t}(t, x) + \tfrac{\partial V_1}{\partial x}(t, x) f_x(t, x, z) - \mathcal{M}(t, x, z) \\
&\quad + [z - z_s(t, x)][g(t, x, z)h(u) + f_z(t, x, z)] \\
&= \tfrac{\partial V_1}{\partial t}(t, x) + \tfrac{\partial V_1}{\partial x}(t, x) f_x(t, x, z_s(t, x)) - \mathcal{M}(t, x, z) \\
&\quad + \tfrac{\partial V_1}{\partial x}(t, x)\big[f_x(t, x, z) - f_x(t, x, z_s(t, x))\big] \\
&\quad + [z - z_s(t, x)][g(t, x, z)h(u) + f_z(t, x, z)] \\
&= -W_1(x) - \mathcal{M}(t, x, z) \\
&\quad + [z - z_s(t, x)]\big[\tfrac{\partial V_1}{\partial x}(t, x) F(t, x, z) + g(t, x, z)h(u) + f_z(t, x, z)\big]
\end{aligned} \tag{7.60}$$

where

$$F(t, x, z) \; = \; \int_0^1 \frac{\partial f_x}{\partial z}(t, x, m(z - z_s(t, x)) + z_s(t, x))\mathrm{d}m$$

and $\mathcal{M}(t, x, z) \; = \; [z - z_s(t, x)]\dot{z}_s(t, x)$.

Since g is everywhere positive, the control law

$$\begin{aligned}
&u_s(t, x, z) \\
&= h^{-1}\left(\tfrac{-[z - z_s(t,x)] - \frac{\partial V_1}{\partial x}(t,x)F(t,x,z) - f_z(t,x,z) + \dot{z}(t,x)}{g(t,x,z)}\right)
\end{aligned} \tag{7.61}$$

is well defined and yields

$$\dot{V}_2 = -W_2(t, x, z)\,, \;\; \text{where} \;\; W_2(t, x, z) = W_1(x) + [z - z_s(t, x)]^2. \tag{7.62}$$

Since W_2 is periodic in t, we can find a positive definite function $\underline{\alpha}$ such that $W_2(t, x, z) \geq \underline{\alpha}(|(x, z)|)$ everywhere, which gives the UGAS for (7.53).

7.3 Backstepping for Nonautonomous Systems

When adapting the backstepping approach to nonlinear time-varying systems, it is natural to consider the special case

$$\begin{cases} \dot{x} = \mathcal{F}(t, x, z) \\ \dot{z} = p(t)u + h(t, x, z) \end{cases} \tag{7.63}$$

of (7.53), where $x \in \mathbb{R}^{n_x}$, $z \in \mathbb{R}$, $u \in \mathbb{R}$ is the input, $p(t)$ is a bounded function, and $\mathcal{F}(t, x, z)$ and $h(t, x, z)$ satisfy

$$\mathcal{F}(t, 0, 0) = 0 \quad \text{and} \quad h(t, 0, 0) = 0$$

for all t. There are several cases where strict Lyapunov function methods lead to control laws that render (7.63) UGAS to the origin. We discuss these cases next.

7.3.1 Chained Form Systems

One motivation for studying (7.63) involves the system

$$\begin{cases} \dot{\xi}_4 = \xi_3 v_1 \\ \dot{\xi}_3 = \xi_2 v_1 \\ \dot{\xi}_2 = v_2 \\ \dot{\xi}_1 = v_1 \end{cases} \tag{7.64}$$

in chained form of order 4 with inputs v_1 and v_2. Assume that we want ξ_1 to asymptotically track the function $\sin(t)$ while ξ_2, ξ_3, and ξ_4 converge to zero. This is the problem of tracking the reference trajectory

$$(\xi_{1r}, \xi_{2r}, \xi_{3r}, \xi_{4r})(t) \;=\; \big(\sin(t), 0, 0, 0\big).$$

The time-varying change of variables

$$x_1 = \xi_1 - \xi_{1r}(t) \tag{7.65}$$

and the change of feedback

$$v_1 = \cos(t) + u_1 \tag{7.66}$$

result in

$$\begin{cases} \dot{\xi}_4 = \xi_3(\cos(t) + u_1) \\ \dot{\xi}_3 = \xi_2(\cos(t) + u_1) \\ \dot{\xi}_2 = v_2 \\ \dot{x}_1 = u_1 . \end{cases} \tag{7.67}$$

The system (7.67) can be globally uniformly asymptotically stabilized provided one knows (a) a control $v_2(t, \xi_2, \xi_3, \xi_4)$ that is periodic of period 2π in t that renders

$$\begin{cases} \dot{\xi}_4 = \xi_3 \cos(t) \\ \dot{\xi}_3 = \xi_2 \cos(t) \\ \dot{\xi}_2 = v_2 \end{cases} \tag{7.68}$$

UGAS and (b) a strict Lyapunov function ν_1 for the corresponding closed-loop system that also has period 2π in t.

Indeed, assume that the control law and strict Lyapunov function ν_1 are known. Then, there exists a positive definite function $\mathcal{W}_1(\xi_2, \xi_3, \xi_4)$ such that the time derivative of ν_1 along the trajectories of (7.68) in closed-loop with $v_{2s}(t, \xi_2, \xi_3, \xi_4)$ satisfies

$$\dot{\nu}_1 \leq -\mathcal{W}_1(\xi_2, \xi_3, \xi_4) \ . \tag{7.69}$$

Consequently, the time derivative of

$$\nu_2(t, x_1, \xi_2, \xi_3, \xi_4) \ = \ \nu_1(t, \xi_2, \xi_3, \xi_4) + \frac{1}{2} x_1^2$$

along the trajectories of (7.67) in closed-loop with $v_{2s}(t, \xi_2, \xi_3, \xi_4)$ satisfies

$$\begin{aligned} \dot{\nu}_2 \leq & -\mathcal{W}_1(\xi_2, \xi_3, \xi_4) \\ & + \left[\tfrac{\partial \nu_1}{\partial \xi_4}(t, \xi_2, \xi_3, \xi_4)\xi_3 + \tfrac{\partial \nu_1}{\partial \xi_3}(t, \xi_2, \xi_3, \xi_4)\xi_2 + x_1 \right] u_1 \ . \end{aligned} \tag{7.70}$$

The choice

$$\begin{aligned} & u_1(t, x_1, \xi_2, \xi_3, \xi_4) = \\ & -\left[\tfrac{\partial \nu_1}{\partial \xi_4}(t, \xi_2, \xi_3, \xi_4)\xi_3 + \tfrac{\partial \nu_1}{\partial \xi_3}(t, \xi_2, \xi_3, \xi_4)\xi_2 + x_1 \right] \end{aligned} \tag{7.71}$$

results in

$$\dot{\nu}_2 \leq -\mathcal{W}_2(t, x_1, \xi_2, \xi_3, \xi_4) \ , \tag{7.72}$$

where

$$\begin{aligned} & \mathcal{W}_2(t, x_1, \xi_2, \xi_3, \xi_4) = \\ & \mathcal{W}_1(\xi_2, \xi_3, \xi_4) + \left[\tfrac{\partial \nu_1}{\partial \xi_4}(t, \xi_2, \xi_3, \xi_4)\xi_3 + \tfrac{\partial \nu_1}{\partial \xi_3}(t, \xi_2, \xi_3, \xi_4)\xi_2 + x_1 \right]^2 \ . \end{aligned} \tag{7.73}$$

Using the periodicity of the relevant functions, we can easily prove that $\mathcal{W}_2$ is bounded from above and below by positive definite functions of x_1, ξ_2, ξ_3, and ξ_4. It follows that the origin of (7.67) in closed-loop with $v_{2s}(t, \xi_2, \xi_3, \xi_4)$ and $u_1(t, x_1, \xi_2, \xi_3, \xi_4)$ defined in (7.71) is UGAS.

Therefore, it suffices to stabilize the system (7.68) and build a corresponding strict Lyapunov function for the closed-loop system. To globally uniformly asymptotically stabilize (7.68), it suffices to do backstepping for systems of the form (7.63). Indeed, if we can construct a globally asymptotically stabilizing 2π periodic feedback for the special case

$$\begin{cases} \dot{x}_4 = x_3 \cos(t) \\ \dot{x}_3 = U \cos(t) \end{cases} \tag{7.74}$$

of (7.63) with input U, then the argument from Sect. 7.2.3 provides a control law that renders (7.68) UGAS to the origin.

7.3.2 Feedback Systems

A more general motivation for studying the systems (7.63) arises from systems in feedback form. Solving local tracking problems for feedback systems frequently involves designing exponentially stable controllers for linear systems of the form (7.63). To understand why, consider the simple family of systems

$$\begin{cases} \dot{\xi}_1 = \mathcal{H}_1(\xi_2) \\ \dot{\xi}_2 = \mathcal{H}_2(\xi_3) \\ \dot{\xi}_3 = u \ , \end{cases} \tag{7.75}$$

where the functions $\mathcal{H}_i$ are not necessarily differomorphisms. Dynamics of the form (7.75) are said to be in feedback form or feedback systems.

Assume that there exists a bounded periodic trajectory $(\xi_{1,r}, \xi_{2,r}, \xi_{3,r})$ such that

$$\begin{cases} \dot{\xi}_{1,r}(t) = \mathcal{H}_1(\xi_{2,r}(t)) \\ \dot{\xi}_{2,r}(t) = \mathcal{H}_2(\xi_{3,r}(t) \ . \end{cases} \tag{7.76}$$

Then the dynamics for the error variables

$$\tilde{\xi}_j = \xi_j - \xi_{j,r}(t) \ , \ j = 1, ..., 3 \tag{7.77}$$

has the form

$$\begin{cases} \dot{\tilde{\xi}}_1 = \mathcal{H}_1(\tilde{\xi}_2 + \xi_{2,r}(t)) - \mathcal{H}_1(\xi_{2,r}(t)) \\ \dot{\tilde{\xi}}_2 = \mathcal{H}_2(\tilde{\xi}_3 + \xi_{3,r}(t)) - \mathcal{H}_2(\xi_{3,r}(t)) \\ \dot{\tilde{\xi}}_3 = u - \dot{\xi}_{3,r}(t) \ . \end{cases} \tag{7.78}$$

The linear approximation of (7.78) at the origin is

$$\begin{cases} \dot{\tilde{\xi}}_1 = \mathcal{H}_1'(\xi_{2,r}(t))\tilde{\xi}_2 \\ \dot{\tilde{\xi}}_2 = \mathcal{H}_2'(\xi_{3,r}(t))\tilde{\xi}_3 \\ \dot{\tilde{\xi}}_3 = u. \end{cases} \tag{7.79}$$

This system can be stabilized if the time-varying chain of integrators

$$\begin{cases} \dot{x}_1 = \mathcal{H}_1'(\xi_{2,r}(t))x_2 \\ \dot{x}_2 = \mathcal{H}_2'(\xi_{3,r}(t))U \end{cases} \tag{7.80}$$

can be globally uniformly asymptotically stabilized, and (7.80) is also of the form (7.63). In fact, once we can stabilize (7.80), the argument from Sect. 7.2.3 provides a control law that renders the system (7.79) UGAS to the origin, as well as a strict Lyapunov function for the corresponding closed-loop dynamics, assuming (7.80) and its stabilizer have the same period.

7.3.3 Feedforward Systems

Another motivation for studying the systems (7.63) arises from feedforward systems. As in the case of feedback systems, solving tracking problems for feedforward systems often involves building exponentially stable controllers for linear systems of the form (7.63). To understand why, consider the Euler-Lagrange feedforward system

$$\begin{cases} \dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 = -\xi_1 + \epsilon \sin(\xi_3) \\ \dot{\xi}_3 = \xi_4 \\ \dot{\xi}_4 = v \end{cases} \tag{7.81}$$

with input v. For definiteness, we take $\epsilon = \frac{3}{4}$. This is the so-called translational oscillator with rotating actuator (TORA) system [56]. One can readily check that the trajectory

$$(\xi_{1,r}, \xi_{2,r}, \xi_{3,r}, \xi_{4,r})(t) = \left(\sin\left(\frac{t}{2}\right), \frac{1}{2}\cos\left(\frac{t}{2}\right), \frac{t}{2}, \frac{1}{2}\right) \tag{7.82}$$

satisfies

$$\begin{cases} \dot{\xi}_{1,r} = \xi_{2,r} \\ \dot{\xi}_{2,r} = -\xi_{1,r} + \frac{3}{4}\sin(\xi_{3,r}) \\ \dot{\xi}_{3,r} = \xi_{4,r} \\ \dot{\xi}_{4,r} = 0. \end{cases} \tag{7.83}$$

Therefore, (7.82) is an admissible trajectory of (7.81). The dynamics for the error variables $\tilde{\xi}_j = \xi_j - \xi_{j,r}(t)$ for $j = 1, ..., 4$ has the form

$$\begin{cases} \dot{\tilde{\xi}}_1 = \tilde{\xi}_2 \\ \dot{\tilde{\xi}}_2 = -\tilde{\xi}_1 + \frac{3}{4}\left[\sin(\tilde{\xi}_3 + \xi_{3,r}(t)) - \sin(\xi_{3,r}(t))\right] \\ \dot{\tilde{\xi}}_3 = \tilde{\xi}_4 \\ \dot{\tilde{\xi}}_4 = v. \end{cases} \tag{7.84}$$

To construct locally uniformly exponentially stabilizing control laws for the system (7.84), we consider its linear approximation

$$\begin{cases} \dot{\tilde{\xi}}_1 = \tilde{\xi}_2 \\ \dot{\tilde{\xi}}_2 = -\tilde{\xi}_1 + \frac{3}{4}\cos\left(\xi_{3,r}(t)\right)\tilde{\xi}_3 \\ \dot{\tilde{\xi}}_3 = \tilde{\xi}_4 \\ \dot{\tilde{\xi}}_4 = v \end{cases} \tag{7.85}$$

near the origin. Applying the backstepping approach to stabilize this system involves several steps. In the first step, we find a control law

$$\tilde{\xi}_{2s}(t, \tilde{\xi}_1)$$

such that

$$\dot{\tilde{\xi}}_1 = \tilde{\xi}_{2s}(t, \tilde{\xi}_1)$$

is UGAS. We then seek a stabilizing controller for the $(\tilde{\xi}_1, \tilde{\xi}_2)$-subsystem with $\tilde{\xi}_3$ as the fictitious input. Clearly, these two steps are equivalent to considering

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 + \frac{3}{4}\cos(\xi_{3,r}(t))u = -x_1 + \frac{3}{4}\cos\left(\frac{t}{2}\right)u\ , \end{cases} \tag{7.86}$$

which again has the structure of (7.63).

7.3.4 Other Important Cases

If the continuous function $p(t)$ in (7.63) is bounded from below by a positive constant (or bounded from above by a negative constant), then state feedbacks for (7.63) can be designed by combining the Lyapunov results of [180, 181, 182]. However, if $p(t)$ is neither everywhere positive nor everywhere negative, and therefore can take the value 0 (which is the case for the systems (7.74) and (7.86)), then constructing globally uniformly asymptotically stabilizing feedbacks for systems (7.63) is much more difficult. In this situation, neither the cancelation method nor the domination method applies, because when $p(t) = 0$, the term $p(t)u = 0$ can neither cancel nor dominate a term different from 0.

We study two cases where this obstacle can be overcome. The first case involves time-varying linear systems where $p(t)$ is periodic and takes the value 0 at discrete instants. We then study *nonlinear* systems (7.63) whose x-subsystem with z regarded as a control can be stabilized by a virtual control having the form $z_s(t,x) = p^2(t)\mu_s(t,x)$, and whose term $h(t,x,z)$ is of the form $p(t)b(t,x,z)$. Here both μ_s and b are C^1. We then show how in some cases, *bounded control laws* can be constructed through a variant of the technique.

7.4 Linear Time-Varying Systems

Consider the linear time-varying system

$$\dot{X} = A(t)X + p(t)Bu + \lambda(t) \ , \tag{7.87}$$

where $u \in \mathbb{R}$, $X \in \mathbb{R}^n$, $A : \mathbb{R} \to \mathbb{R}^{n\times n}$ is continuous and bounded, $B \in \mathbb{R}^n$ is constant, $\lambda : \mathbb{R} \to \mathbb{R}^n$ is a continuous disturbance, and $p : \mathbb{R} \to \mathbb{R}$ is a periodic function.

Later, we consider the subfamily of (7.87) consisting of systems

$$\begin{cases} \dot{x}_1 = a_{1,1}(t)x_1 + p_1(t)x_2 + \lambda_1(t) \\ \dot{x}_2 = a_{2,1}(t)x_1 + a_{2,2}(t)x_2 + p_2(t)x_3 + \lambda_2(t) \\ \quad \vdots \\ \dot{x}_n = a_{n,1}(t)x_1 + a_{n,2}(t)x_2 + ... + a_{n,n}(t)x_n + p_n(t)u + \lambda_n(t) \end{cases} \tag{7.88}$$

in feedback form, where $x_i \in \mathbb{R}$, $u \in \mathbb{R}$ is the input, and the functions $\lambda_i : \mathbb{R} \to \mathbb{R}$ are continuous. Our conditions will ensure that we can construct linear time-varying feedbacks that render (7.88) ISS with respect to the disturbances λ_i.

Assumption and Technical Lemmas

Consider a function $p : \mathbb{R} \to \mathbb{R}$ that satisfies:

Assumption 7.1 *The function p is continuous and periodic of some period $T_p > 0$. The set $H = \{t \in [0, T_p] : p(t) = 0\}$ is finite and nonempty.*

Let the elements of H be denoted by $0 \le t_1 < ... < t_k \le T_p$. We use the positive constant

$$d_m = \frac{1}{4} \min\{t_2 - t_1, ..., t_k - t_{k-1}\} \tag{7.89}$$

and the sets

$$E_d = \cup_{j=1}^{k} [t_j - d, t_j + d] \cap [0, T_p] \quad \text{and} \quad F_d = \overline{[0, T_p] \setminus E_d} \ , \tag{7.90}$$

where $d \in (0, d_m]$ is a given constant. The next lemma follows because $p^2(t)$ is continuous and positive at each point of the compact set F_d:

Lemma 7.1. *Consider a function $p : \mathbb{R} \to \mathbb{R}$ that satisfies Assumption 7.1. Let $d \in (0, d_m]$ be any constant. Then*

$$C_d = \min_{s \in F_d} p^2(s) \tag{7.91}$$

is a positive real number.

Lemma 7.2. *We have*

$$\lim_{\delta \to 0^+} \int_0^{T_p} \frac{\delta}{p^2(a) + \delta} \mathrm{d}a \;=\; 0 \tag{7.92}$$

for any function $p : \mathbb{R} \to \mathbb{R}$ that satisfies Assumption 7.1.

Proof. Fix any constants $\epsilon > 0$ and

$$d \in \left(0, \min\left\{d_m, \frac{\epsilon}{4k}\right\}\right],$$

where d_m is defined in (7.89). Then

$$\begin{aligned} \int_0^{T_p} \frac{\delta}{p^2(a) + \delta} \mathrm{d}a &= \int_{E_d} \frac{\delta}{p^2(a) + \delta} \mathrm{d}a + \int_{F_d} \frac{\delta}{p^2(a) + \delta} \mathrm{d}a \\ &\le 2kd + \int_{F_d} \frac{\delta}{p^2(a) + \delta} \mathrm{d}a \\ &\le \tfrac{\epsilon}{2} + T_p \tfrac{\delta}{C_d} \;, \end{aligned} \tag{7.93}$$

where the last inequality used the facts that

$$d \in \left(0, \frac{\epsilon}{4k}\right] \quad \text{and} \quad p^2(a) \ge C_d$$

when $a \in F_d$. Therefore,

$$\int_0^{T_p} \frac{\delta}{p^2(a) + \delta} \mathrm{d}a \;\le\; \epsilon \quad \forall \delta \in \left(0, \tfrac{\epsilon C_d}{2T_p}\right] \tag{7.94}$$

which proves the lemma. □

7.4.1 General Result for Linear Time-Varying Systems

Assumptions

Assume that the linear time-varying system (7.87) is such that Assumption 7.1 and the following are both satisfied:

Assumption 7.2 *There are known positive constants c_i and $\mathcal{L}$ and C^∞ functions $L_1 : \mathbb{R} \to \mathbb{R}^n$, $L_2 : \mathbb{R} \to \mathbb{R}^n$, and $Q : \mathbb{R} \to \mathbb{R}^{n\times n}$ such that $Q(t)$ is symmetric for all $t \in \mathbb{R}$; the function*

$$\overline{Q}(t,X) \;=\; X^\top Q(t)X \tag{7.95}$$

is such that

$$c_1|X|^2 \;\le\; \bar{Q}(t,X) \;\le\; c_2|X|^2 \;\; \forall X \in \mathbb{R}^n \tag{7.96}$$

and

$$|L_1(t)| \;\le\; \mathcal{L} \;\text{ and }\; |L_2(t)| \;\le\; c_4 \tag{7.97}$$

hold for all $t \in \mathbb{R}$; and the time derivative of $\overline{Q}(t,X)$ along the trajectories of

$$\dot{X} = A(t)X + Bv + \lambda(t) \tag{7.98}$$

in closed-loop with

$$v = L(t)\cdot X, \;\;\text{where}\;\; L(t) \;=\; L_1(t) + p(t)L_2(t) \tag{7.99}$$

satisfies

$$\dot{\overline{Q}} \;\le\; -c_3\overline{Q}(t,X) + |\lambda(t)|^2\;. \tag{7.100}$$

Remark 7.1. A simple application of the triangle inequality shows that if $\bar{Q}(t,x)$ takes the form (7.95) for some everywhere symmetric matrix $Q(t)$, and if there are positive constants c_i satisfying (7.96) for all $t \in \mathbb{R}$ and

$$\dot{\bar{Q}} \le -c_3\bar{Q}(t,X)$$

along all trajectories of $\dot{X} = A(t)X + Bv$ in closed-loop with (7.99), then the time derivative of

$$\bar{Q}_c \doteq \varepsilon\bar{Q}, \;\text{where}\; \varepsilon = \tfrac{c_1c_3}{2c_2^2}$$

along trajectories of (7.98) in closed-loop with the controller (7.99) satisfies

$$\dot{\bar{Q}}_c \;\le\; -\frac{c_3}{2}\bar{Q}_c + |\lambda|^2$$

for all disturbances λ. To see why, first notice that condition (7.96) gives $\text{spectrum}\{Q(t)\} \subseteq [c_1,c_2]$ for all $t \in \mathbb{R}$ and therefore $X^\top Q(t)Q(t)X \le c_2^2\bar{Q}(t,X)/c_1$ everywhere. Therefore, along the closed-loop trajectories of (7.98), the triangle inequality gives

$$\begin{aligned}\dot{\bar{Q}}_c &\le \varepsilon\left[-c_3\bar{Q}(t,X) + 2X^\top Q(t)\lambda(t)\right]\\ &\le \varepsilon\left[-c_3\bar{Q}(t,X) + 2\left\{\tfrac{\varepsilon}{2}X^\top Q(t)Q(t)X + \tfrac{1}{2\varepsilon}|\lambda(t)|^2\right\}\right]\\ &\le -\tfrac{c_3}{2}\bar{Q}_c(t,X) + |\lambda(t)|^2.\end{aligned} \tag{7.101}$$

Therefore, by scaling $\bar{Q}$ and c_3, we can take $\lambda \equiv 0$ in Assumption 7.2.

We also assume the following:

Assumption 7.3 *The function $p(t)$ in the system (7.87) is C^∞ and satisfies Assumption 7.1.*

Statement of Theorem

Theorem 7.1. *Assume that the system (7.87) satisfies Assumptions 7.2 and 7.3. Then there exists a constant $\delta > 0$ such that the time derivative of*

$$\begin{aligned} &\widehat{Q}(t,X) = e^{R(t)}\overline{Q}(t,X), \quad \textit{where} \\ &R(t) = -\tfrac{1}{T_p}\int_{t-T_p}^{t}\left(\int_{\ell}^{t}\frac{2\delta^2|B|^2\mathcal{L}^2}{c_1(p^2(a)+\delta)^2}\mathrm{d}a\right)\mathrm{d}\ell \end{aligned} \tag{7.102}$$

along the trajectories of (7.87) in closed-loop with

$$u(t,X) = \frac{p(t)}{p^2(t)+\delta}L_1(t)\cdot X + L_2(t)\cdot X \tag{7.103}$$

satisfies

$$\dot{\widehat{Q}} \leq -\frac{4c_3}{5}\widehat{Q}(t,X) + 2|\lambda(t)|^2. \tag{7.104}$$

Moreover,

$$c_1\exp\left(-\tfrac{2|B|^2\mathcal{L}^2}{c_1}\int_{t-T_p}^{t}\frac{\delta^2}{[p^2(a)+\delta]^2}\mathrm{d}a\right)|X|^2 \leq \widehat{Q}(t,X) \leq c_2|X|^2 \tag{7.105}$$

for all $t \in \mathbb{R}$ and $X \in \mathbb{R}^n$.

Discussion on Theorem 7.1

Remark 7.2. Assumption 7.2 is satisfied if the pair $(A(t), B)$ is stabilizable by a feedback $K(t)X$ that is C^∞ and uniformly bounded with respect to time, assuming A and K have the same period. Therefore, this assumption is not restrictive.

Remark 7.3. We will see in the proof of Theorem 7.1 that (7.104) is satisfied provided δ satisfies

$$\int_0^{T_p}\frac{\delta^2}{(p^2(a)+\delta)^2}\mathrm{d}a \leq \frac{c_1c_3T_p}{10|B|^2\mathcal{L}^2}\ . \tag{7.106}$$

The proof of Lemma 7.2 shows that (7.106) is satisfied provided

$$0 \ < \ \delta \ \leq \ \frac{c_1 c_3 C_d}{20|B|^2 \mathcal{L}^2} \ , \tag{7.107}$$

where

$$d = \min \left\{ \frac{c_3 c_1 T_p}{40k|B|^2 \mathcal{L}^2}, d_m \right\}, \tag{7.108}$$

d_m is defined in (7.89), and C_d is defined in (7.91). However, in general, much larger values for δ can be found, which is important from a practical point of view if very large controls cannot be used.

For instance, consider the case where $p(t) = \cos(t)$ and $T_p = 2\pi$. Then, Appendix A.5 gives

$$\begin{aligned} \int_0^{T_p} \frac{\delta^2}{(p^2(a) + \delta)^2} \mathrm{d}a &= \int_0^{2\pi} \frac{\delta^2}{(\cos^2(a) + \delta)^2} \mathrm{d}a \\ &= 4\delta^2 \int_0^{\frac{\pi}{2}} \frac{1}{(\cos^2(a) + \delta)^2} \mathrm{d}a \\ &\leq \frac{\pi\sqrt{\delta}(1 + 3\delta)}{(1 + \delta)^{3/2}} \ . \end{aligned} \tag{7.109}$$

Hence, (7.106) is satisfied when

$$\delta \leq \delta_A = \frac{c_1^2 c_3^2}{225|B|^4 \mathcal{L}^4} \ . \tag{7.110}$$

On the other hand, we can easily show that $d_m = \frac{\pi}{4}$ and $C_d = \sin^2(d)$. By reducing c_1, we can assume that

$$\frac{\pi c_3 c_1}{20k|B|^2 \mathcal{L}^2} \leq \frac{\pi}{4} = d_m \ . \tag{7.111}$$

Assuming (7.111), the formula (7.108) for d gives

$$C_d \ = \ \sin^2(d) \ = \ \sin^2 \left(\frac{c_3 c_1 T_p}{40k|B|^2 \mathcal{L}^2} \right),$$

and therefore (7.107) gives

$$0 < \delta \leq \delta_B = \frac{c_1 c_3}{20|B|^2 \mathcal{L}^2} \sin^2 \left(\frac{\pi c_3 c_1}{20k|B|^2 \mathcal{L}^2} \right) \ . \tag{7.112}$$

Frequently, we have

$$\frac{\sqrt{c_1 c_3}}{|B| \mathcal{L}} \leq 1,$$

in which case δ_A can be significantly larger than δ_B.

Remark 7.4. When Assumption 7.2 is satisfied, the decomposition of $L(t)$ in (7.99) as the sum of a function $L_1(t)$ and a function $p(t)L_2(t)$ is not unique. For instance, if $L(t) = L_1(t) + p(t)L_2(t)$, then we also have

$$L(t) = \tilde{L}_1(t) + p(t)\tilde{L}_2(t),$$

where $\tilde{L}_1(t) = L_1(t) + 5p(t)$ and $\tilde{L}_2(t) = L_2(t) - 5$. In particular, the trivial decomposition $L(t) = L_1(t) + p(t)L_2(t)$ with $L_2(t) = 0$ and $L_1(t) = L(t)$ is always possible. The flexibility in the choices of $L_1(t)$ and $L_2(t)$ allows different possible choices of the feedback (7.103).

Remark 7.5. If the function $p(t)$ satisfies a PE property of the type

$$\int_0^{T_p} p^2(a)\mathrm{d}a > 0 \tag{7.113}$$

but violates Assumption 7.1, then there might not exist a constant $\delta > 0$ such that (7.106) holds. Therefore, Assumption 7.3 cannot be replaced by the less restrictive assumption that $p(t)$ is a C^∞ function satisfying the PE property (7.113).

Proof of Theorem 7.1

To simplify the proof, we let $L(t) = L_1(t)$ and $L_2(t) = 0$. The case where $L_2 \neq 0$ can be easily handled by performing the preliminary change of control $u = u_1 + L_2(t)$ and replacing $A(t)$ with $A(t) + Bp(t)L_2^\top(t)$. The system (7.87) in closed-loop with (7.103) is

$$\begin{aligned} \dot{X} &= A(t)X + B\tfrac{p^2(t)}{p^2(t)+\delta}L(t)\cdot X + \lambda(t) \\ &= [A(t) + BL^\top(t)]X - B\frac{\delta}{p^2(t)+\delta}L(t)\cdot X + \lambda(t)\ . \end{aligned} \tag{7.114}$$

From (7.100) in Assumption 7.2, we immediately deduce that the time derivative of $\overline{Q}$ along the trajectories of the system (7.87) in closed-loop with (7.103) satisfies

$$\begin{aligned} \dot{\overline{Q}} &\leq -c_3\overline{Q}(t,X) + \left| -B\frac{\delta}{p^2(t)+\delta}L(t)\cdot X + \lambda(t)\right|^2 \\ &\leq -c_3\overline{Q}(t,X) + \frac{2\delta^2}{(p^2(t)+\delta)^2}|B|^2\mathcal{L}^2|X|^2 + 2|\lambda(t)|^2\ , \end{aligned} \tag{7.115}$$

where $\mathcal{L}$ is the constant from Assumption 7.2. It follows from (7.96) in Assumption 7.2 that

$$\dot{\overline{Q}} \leq -c_3\overline{Q}(t,X) + \frac{2\delta^2|B|^2\mathcal{L}^2}{(p^2(t)+\delta)^2}\frac{\overline{Q}(t,X)}{c_1} + 2|\lambda(t)|^2 \ . \tag{7.116}$$

On the other hand, the time derivative of the function $\widehat{Q}$ defined in (7.102) along the trajectories of the system (7.87), in closed-loop with (7.103), satisfies

$$\dot{\widehat{Q}}(t,X) = e^{R(t)}\left[\dot{\overline{Q}}(t,X) + \overline{Q}(t,X)\dot{R}(t)\right]. \tag{7.117}$$

Moreover,

$$\dot{R}(t) = -\frac{2\delta^2|B|^2\mathcal{L}^2}{c_1(p^2(t)+\delta)^2} + \frac{1}{T_p}\int_{t-T_p}^{t}\frac{2\delta^2|B|^2\mathcal{L}^2}{c_1(p^2(a)+\delta)^2}\mathrm{d}a. \tag{7.118}$$

Combining (7.116)-(7.118) yields

$$\begin{aligned}\dot{\widehat{Q}}(t,X) &\leq e^{R(t)}\Bigg[-c_3\overline{Q}(t,X) + 2|\lambda(t)|^2 \\ &\quad +\overline{Q}(t,X)\left(\frac{1}{T_p}\int_{t-T_p}^{t}\frac{2\delta^2|B|^2\mathcal{L}^2}{c_1(p^2(a)+\delta)^2}\mathrm{d}a\right)\Bigg] \\ &= e^{R(t)}\overline{Q}(t,X)\left[-c_3 + \frac{2|B|^2\mathcal{L}^2}{c_1T_p}\int_{t-T_p}^{t}\frac{\delta^2}{(p^2(a)+\delta)^2}\mathrm{d}a\right] \\ &\quad +2e^{R(t)}|\lambda(t)|^2 \ .\end{aligned} \tag{7.119}$$

Using the definition of $\widehat{Q}$ and the non-positivity of R, we get

$$\begin{aligned}\dot{\widehat{Q}}(t,X) \leq \widehat{Q}(t,X)&\left[-c_3 + \frac{2|B|^2\mathcal{L}^2}{c_1T_p}\int_{0}^{T_p}\frac{\delta^2}{(p^2(a)+\delta)^2}\mathrm{d}a\right] \\ &+2|\lambda(t)|^2 \ .\end{aligned} \tag{7.120}$$

Using Lemma 7.2 and the inequality

$$\int_0^{T_p}\frac{\delta^2}{(p^2(a)+\delta)^2}\mathrm{d}a \ \leq \ \int_0^{T_p}\frac{\delta}{p^2(a)+\delta}\mathrm{d}a,$$

we can choose $\delta > 0$ so that

$$\frac{2|B|^2\mathcal{L}^2}{c_1T_p}\int_0^{T_p}\frac{\delta^2}{(p^2(a)+\delta)^2}\mathrm{d}a \ \leq \ \frac{1}{5}c_3 \ . \tag{7.121}$$

This choice yields

$$\dot{\widehat{Q}}(t,X) \ \leq \ -\frac{4c_3}{5}\widehat{Q}(t,X) + 2|\lambda(t)|^2 \ . \tag{7.122}$$

Finally, one can easily prove (7.105). This proves the theorem. □

Remark 7.6. A more restrictive condition on δ than the one in (7.107) guarantees that the time derivative of the storage function

$$\widetilde{Q}(t,X) = \left[1 - \frac{1}{T_p}\int_{t-T_p}^{t}\int_{\ell}^{t} \frac{2\delta^2|B|^2\mathcal{L}^2}{c_1(p^2(a)+\delta)^2}\mathrm{d}a\,\mathrm{d}\ell\right]\overline{Q}(t,X) \tag{7.123}$$

along the trajectories of (7.87) in closed-loop with (7.103) satisfies

$$\dot{\tilde{Q}} \leq -\underline{c}\tilde{Q} + \bar{c}|\lambda|^2 \tag{7.124}$$

for suitable positive constants $\underline{c}$ and $\bar{c}$. The proof of (7.124) combines the arguments from (7.114)-(7.116) with the formula

$$\frac{d}{dt}\int_{t-T_p}^{t}\int_{\ell}^{t}\mathcal{M}(a)\,\mathrm{d}a\,\mathrm{d}\ell \;=\; T_p\mathcal{M}(t) - \int_{t-T_p}^{t}\mathcal{M}(\ell)\mathrm{d}\ell,$$

which is valid for any continuous scalar function $\mathcal{M}$. In some cases, it may be more convenient to use the Lyapunov function (7.123) instead of (7.102).

7.4.2 Linear Time-Varying Systems in Feedback Form

Notation and Assumptions

We consider the linear time-varying systems (7.88), with the following notation. Let $\Lambda_j = (\lambda_1,\ldots,\lambda_j)^\top \in \mathbb{R}^j$ and $\Lambda = \Lambda_n \in \mathbb{R}^n$. Let $\xi_j = (x_1,\ldots,x_j)^\top \in \mathbb{R}^j$ and $x = \xi_n = (x_1,\ldots,x_n)^\top \in \mathbb{R}^n$. Consider the systems

$$\begin{cases} \dot{x}_1 = a_{1,1}(t)x_1 + p_1(t)x_2 + \lambda_1(t) \\ \dot{x}_2 = a_{2,1}(t)x_1 + a_{2,2}(t)x_2 + p_2(t)x_3 + \lambda_2(t) \\ \quad\vdots \\ \dot{x}_j = a_{j,1}(t)x_1 + a_{j,2}(t)x_2 + \ldots + a_{j,j}(t)x_j + p_j(t)x_{j+1} + \lambda_j(t) \end{cases} \tag{7.125}$$

for $j = 1$ to $n-1$, which we denote in compact form by

$$\dot{\xi}_j = \mathcal{A}_j(t)\xi_{j+1} + \Lambda_j(t)\ . \tag{7.126}$$

We introduce two assumptions:

Assumption 7.4 *Each function $a_{i,j}(t)$ is C^∞ and periodic.*

Assumption 7.5 *Each function $p_i(t)$ is C^∞ and satisfies Assumption 7.1.*

We use $T_{p_i} > 0$ to denote the period of $p_i(t)$ for each i.

Statement of Main Result

Our main result for (7.88) is as follows:

Theorem 7.2. *Assume that (7.88) satisfies Assumptions 7.4-7.5. Then one can construct* n *time periodic* C^∞ *functions* $g_i(t)$*, a time periodic everywhere symmetric* C^∞ *matrix* $H(t)$*, and constants* $h_i > 0$ *such that*

$$h_1 I_n \leq H(t) \leq h_2 I_n \ \ \forall t \in \mathbb{R} \ , \tag{7.127}$$

and such that the time derivative of the function

$$\mathcal{V}(t,x) = x^\top H(t) x \tag{7.128}$$

along the trajectories of the system (7.88) in closed-loop with the feedback

$$u(t,x) \ = \ g_1(t)x_1 + ... + g_n(t)x_n \tag{7.129}$$

satisfies

$$\dot{\mathcal{V}}(t,x) \ \leq \ -\mathcal{V}(t,x) + 2|\Lambda(t)|^2 \ . \tag{7.130}$$

Remark 7.7. An immediate consequence of (7.130) is that the system (7.88) in closed-loop with the feedback (7.129) is globally ISS with respect to Λ. Moreover, the explicit formula for $\mathcal{V}$ yields the explicit ISS estimate

$$|x(t)| \ \leq \ \sqrt{\frac{h_2}{h_1}} e^{-0.5(t+t_0)} |x(t_0)| + \frac{2|\Lambda|_\infty}{\sqrt{h_1}} \tag{7.131}$$

for all $t \geq t_0 \geq 0$ along the closed-loop trajectories.

Proof of Theorem 7.2

The proof proceeds by induction. We define the *step* j *subsystems* by

$$\begin{cases} \dot{\xi}_{j-1} = \mathcal{A}_{j-1}(t)\xi_j + \Lambda_{j-1}(t) \\ \dot{x}_j = \sum_{r=1}^{j} a_{j,r}(t)x_r + p_j(t)w_j + \lambda_j(t) \end{cases} \tag{7.132}$$

if $j > 1$ and

$$\dot{x}_1 = a_{1,1}(t)x_1 + p_1(t)w_1 + \lambda_1(t) \tag{7.133}$$

if $j = 1$.

Induction Hypothesis. There are j time periodic C^∞ functions $g_{i,j}(t)$, a time periodic everywhere symmetric C^∞ matrix $H_j(t)$, and positive real numbers $h_{1,j}$ and $h_{2,j}$ such that $h_{1,j}I_j \leq H_j(t) \leq h_{2,j}I_j$ for all $t \in \mathbb{R}$ for which the following holds: The time derivative of

$$\widehat{Q}_j(t, \xi_j) = \xi_j^\top H_j(t) \xi_j \tag{7.134}$$

along the trajectories of the step j subsystem in closed-loop with the feedback

$$w_j(t, \xi_j) \doteq g_{1,j}(t) x_1 + \ldots + g_{j,j}(t) x_j \tag{7.135}$$

satisfies

$$\dot{\widehat{Q}}_j(t, \xi_j) \leq -\frac{5^{n-j}}{4^{n-j}} \widehat{Q}_j(t, \xi_j) + 2|\Lambda_j(t)|^2 \ . \tag{7.136}$$

Step 1. To show that the induction assumption is satisfied for $j = 1$, consider the one-dimensional system

$$\dot{x}_1 = a_{1,1}(t) x_1 + v + \lambda_1(t) \tag{7.137}$$

with v as the input. Let $\overline{Q}_1(t, x_1) = \frac{1}{2} x_1^2$ and

$$v(t, \xi_1) = -\left[a_{1,1}(t) + \left(\frac{5}{4} \right)^n \right] x_1 \ . \tag{7.138}$$

The system (7.137) in closed-loop with (7.138) is

$$\dot{x}_1(t) = -\left(\frac{5}{4} \right)^n x_1 + \lambda_1(t). \tag{7.139}$$

Along the trajectories of (7.139), the time derivative of $\overline{Q}_1(t, x_1)$ satisfies

$$\begin{aligned} \dot{\overline{Q}}_1(t, x_1) &= -\left(\tfrac{5}{4}\right)^n x_1^2 + \lambda_1(t) x_1 \\ &= -\left(\tfrac{5}{4}\right)^n x_1^2 + \left\{ \lambda_1(t) \left(\tfrac{4}{5}\right)^{n/2} \right\} \left\{ \left(\tfrac{5}{4}\right)^{n/2} x_1 \right\} \\ &\leq -\left(\tfrac{5}{4}\right)^n \overline{Q}_1(t, x_1) + \lambda_1^2(t) \ , \end{aligned} \tag{7.140}$$

by the triangle inequality $c_1 c_2 \leq \frac{1}{2} c_1^2 + \frac{1}{2} c_2^2$ applied to the terms in braces.

We deduce that the system (7.133) satisfies Assumption 7.2 with $c_1 = c_2 = \frac{1}{2}$, $c_3 = (5/4)^n$, $L_1(t) x = v(t, \xi_1)$ as defined in (7.138), and $L_2 \equiv 0$. Moreover, Assumption 7.5 ensures that the function $p_1(t)$ satisfies Assumption 7.3. Hence, Theorem 7.1 provides a constant $\delta_1 > 0$ such that the time derivative of

$$\widehat{Q}_1(t, \xi_1) = e^{R_1(t)} x_1^2 \tag{7.141}$$

with

$$R_1(t) \ = \ -\frac{1}{T_{p_1}} \int_{t-T_{p_1}}^{t} \left(\int_{\ell}^{t} \frac{4 \delta_1^2 \mathcal{L}_1^2}{(p_1(a)^2 + \delta_1)^2} \mathrm{d}a \right) \mathrm{d}\ell \tag{7.142}$$

and $\mathcal{L}_1 = \sup_t \{ |a_{1,1}(t) + (5/4)^n| \}$ along the trajectories of

$$\dot{x}_1 = a_{1,1}(t) x_1 + p_1(t) w_1(t, \xi_1) + \lambda_1(t) \tag{7.143}$$

with

$$w_1(t,\xi_1) \;=\; g_{1,1}(t)x_1 \;\text{ and }\; g_{1,1}(t) \;=\; -p_1(t)\frac{a_{1,1}(t)+\left(\frac{5}{4}\right)^n}{p_1^2(t)+\delta_1} \tag{7.144}$$

satisfies

$$\dot{\widehat{Q}}_1(t,\xi_1) \le -\left(\tfrac{5}{4}\right)^{n-1}\widehat{Q}_1(t,\xi_1) + 2\lambda_1^2(t)\;. \tag{7.145}$$

Therefore the induction assumption is satisfied at the first step.

Inductive Step. We assume that the induction assumption is satisfied at some step $j \in [1, n-1]$. Let us prove that it is satisfied at the step $j+1$. Consider the system

$$\begin{cases} \dot{\xi}_j = \mathcal{A}_j(t)\xi_{j+1} + \Lambda_j(t) \\ \dot{x}_{j+1} = \displaystyle\sum_{r=1}^{j+1} a_{j+1,r}(t)x_r + v + \lambda_{j+1}(t)\;, \end{cases} \tag{7.146}$$

where v is the input. We can determine a globally asymptotically stabilizing feedback for (7.146) using the following classical backstepping approach. Let $w_j(t,\xi_j)$ be the feedback provided by the induction assumption. The change of coordinates $\psi = x_{j+1} - w_j(t,\xi_j)$ gives

$$\begin{cases} \dot{x}_1 = a_{1,1}(t)x_1 + p_1(t)x_2 + \lambda_1(t) \\ \dot{x}_2 = a_{2,1}(t)x_1 + a_{2,2}(t)x_2 + p_2(t)x_3 + \lambda_2(t) \\ \quad\vdots \\ \dot{x}_j = \displaystyle\sum_{r=1}^{j} a_{j,r}(t)x_r + p_j(t)[\psi + w_j(t,\xi_j)] + \lambda_j(t) \\ \dot{\psi} = \displaystyle\sum_{r=1}^{j+1} a_{j+1,r}(t)x_r + v + \lambda_{j+1}(t) - \dot{w}_j\;. \end{cases} \tag{7.147}$$

Therefore, the ψ-subsystem becomes

$$\begin{aligned} \dot{\psi} &= \sum_{r=1}^{j+1} a_{j+1,r}(t)x_r + v + \lambda_{j+1}(t) - \sum_{\ell=1}^{j} \dot{g}_{\ell,j}(t)x_\ell \\ &\quad - \sum_{\ell=1}^{j} g_{\ell,j}(t)\left(\sum_{r=1}^{\ell} a_{\ell,r}(t)x_r + p_\ell(t)x_{\ell+1} + \lambda_\ell(t)\right) \\ &= \sum_{r=1}^{j+1} b_r(t)x_r + v + \lambda_{j+1}(t) - \sum_{\ell=1}^{j} g_{\ell,j}(t)\lambda_\ell(t) \end{aligned} \tag{7.148}$$

where

$$b_r(t) = a_{j+1,r}(t) - \dot{g}_{r,j}(t) - \sum_{\ell=r}^{j} g_{\ell,j}(t)a_{\ell,r}(t) \\ -p_{r-1}(t)g_{r-1,j}(t) \tag{7.149}$$

for $r = 2, 3, \ldots, j$ and

$$b_r(t) = \begin{cases} a_{j+1,1}(t) - \dot{g}_{1,j}(t) - \sum_{\ell=1}^{j} g_{\ell,j}(t)a_{\ell,1}(t), & r = 1 \\ a_{j+1,j+1}(t) - p_j(t)g_{j,j}(t), & r = j+1 \ . \end{cases} \tag{7.150}$$

Let $\widehat{Q}_j$ be the function provided by the induction assumption. Then the time derivative of

$$W_{j+1}(t, \xi_j, \psi) \ \doteq \ \widehat{Q}_j(t, \xi_j) + \frac{1}{2}\psi^2 \tag{7.151}$$

along the trajectories of (7.147) satisfies

$$\dot{W}_{j+1} \leq -\left(\frac{5}{4}\right)^{n-j} \widehat{Q}_j(t, \xi_j) + 2|\Lambda_j(t)|^2 + \frac{\partial \widehat{Q}_j}{\partial x_j}(t, \xi_j)p_j(t)\psi \\ +\psi \left[\sum_{r=1}^{j+1} b_r(t)x_r + v + \lambda_{j+1}(t) - \sum_{\ell=1}^{j} g_{\ell,j}(t)\lambda_\ell(t)\right] \ . \tag{7.152}$$

Choosing

$$v(t, \xi_j, \psi) = -\left[2 + \left(\frac{5}{4}\right)^{n-j}\right]\psi - \frac{\partial \widehat{Q}_j}{\partial x_j}(t, \xi_j)p_j(t) - \sum_{r=1}^{j+1} b_r(t)x_r \tag{7.153}$$

we obtain

$$\dot{W}_{j+1} \leq -\left(\frac{5}{4}\right)^{n-j} \widehat{Q}_j(t, \xi_j) + 2|\Lambda_j(t)|^2 - \left[2 + \left(\frac{5}{4}\right)^{n-j}\right]\psi^2 \\ +\psi \left(\lambda_{j+1}(t) - \sum_{\ell=1}^{j} g_{\ell,j}(t)\lambda_\ell(t)\right) \ . \tag{7.154}$$

From the triangular inequality $c_1c_2 \leq c_1^2 + \frac{1}{4}c_2^2$, we deduce that

$$\dot{W}_{j+1} \leq -\left(\tfrac{5}{4}\right)^{n-j} \widehat{Q}_j(t, \xi_j) + 2|\Lambda_j(t)|^2 - \left[1 + \left(\tfrac{5}{4}\right)^{n-j}\right]\psi^2 \\ +\tfrac{1}{4}\left(\lambda_{j+1}(t) - \sum_{m=1}^{j} g_{m,j}(t)\lambda_m(t)\right)^2 \ . \tag{7.155}$$

We easily deduce that

$$\dot{W}_{j+1} \ \leq \ -\left(\tfrac{5}{4}\right)^{n-j} W_{j+1}(t, \xi_j, \psi) + \kappa_{j+1}|\Lambda_{j+1}(t)|^2 \ , \tag{7.156}$$

where

$$\kappa_{j+1} = 2 + \sup_t \left(1 + \sum_{m=1}^{j} |g_{m,j}(t)|\right)^2 . \tag{7.157}$$

Therefore, the function

$$\overline{Q}_{j+1}(t, \xi_{j+1}) = \tfrac{W_{j+1}(t,\xi_j,\psi)}{1+\kappa_{j+1}} \tag{7.158}$$

satisfies

$$\dot{\overline{Q}}_{j+1} \leq -\left(\tfrac{5}{4}\right)^{n-j} \overline{Q}_{j+1}(t, \xi_{j+1}) + |\Lambda_{j+1}(t)|^2 \tag{7.159}$$

along the trajectories of (7.146).

Moreover, there exist positive constants γ_1 and γ_2 and a function $\Gamma : \mathbb{R} \to \mathbb{R}^{(j+1)\times(j+1)}$ such that

$$\overline{Q}_{j+1}(t, \xi_{j+1}) = \xi_{j+1}^{\top} \Gamma(t) \xi_{j+1} \ \text{ and } \ \gamma_1 |\xi_{j+1}|^2 \leq \overline{Q}_{j+1}(t, \xi_{j+1}) \leq \gamma_2 |\xi_{j+1}|^2 .$$

The existence of γ_1 follows from the periodicity of the functions $g_{i,j}(t)$. We deduce that the system

$$\begin{cases} \dot{\xi}_j = \mathcal{A}_j(t)\xi_{j+1} + \Lambda_j(t) \\ \dot{x}_{j+1} = \displaystyle\sum_{r=1}^{j+1} a_{j+1,r}(t) x_r + p_{j+1}(t) w + \lambda_{j+1}(t) \end{cases} \tag{7.160}$$

satisfies Assumption 7.2 with $L_2 = 0$, and $p_{j+1}(t)$ satisfies Assumption 7.3.

Therefore, Theorem 7.1 applies to the system (7.160). It follows that we can find a constant $\delta_{j+1} > 0$ such that if we set

$$\widehat{Q}_{j+1}(t, \xi_{j+1}) \ \doteq \ e^{R_{j+1}(t)} \overline{Q}_{j+1}(t, \xi_{j+1}), \tag{7.161}$$

where

$$R_{j+1}(t) = -\tfrac{1}{T_{p_{j+1}}} \int_{t-T_{p_{j+1}}}^{t} \left(\int_{\ell}^{t} \frac{2\mathcal{L}_{j+1}^2 \delta_{j+1}^2}{\gamma_1 (p_{j+1}(a)^2 + \delta_{j+1})^2} \mathrm{d}a \right) \mathrm{d}\ell \tag{7.162}$$

and

$$\begin{aligned} \mathcal{L}_{j+1} = 2 \max_t \Bigg\{ & \sum_{r=1}^{j+1} b_r^2(t) + \sum_{r=1}^{j} 8 p_j^2(t) (H_j)_{j,r}^2(t) \\ & 2 \left[2 + \left(\frac{5}{4}\right)^{n-j} \right]^2 \sum_{r=1}^{j} \left(g_{r,j}^2 + 1\right) \Bigg\}, \end{aligned} \tag{7.163}$$

then the time derivative of $\widehat{Q}_{j+1}(t, \xi_{j+1})$ along the trajectories of the system (7.160) in closed-loop with

$$
\begin{aligned}
w_{j+1}(t,\xi_{j+1}) &= g_{1,j+1}(t)x_1 + ... + g_{j+1,j+1}(t)x_{j+1} \\
&= -\frac{p_{j+1}(t)}{p_{j+1}^2(t)+\delta_{j+1}}\left[\left(2+\left(\frac{5}{4}\right)^{n-j}\right)\psi \right. \\
&\qquad \left. + \frac{\partial \widehat{Q}_j}{\partial x_j}(t,\xi_j)p_j(t) + \sum_{r=1}^{j+1} b_r(t)x_r \right]
\end{aligned}
\tag{7.164}
$$

satisfies

$$
\dot{\widehat{Q}}_{j+1} \leq -\left(\frac{5}{4}\right)^{n-j-1} \widehat{Q}_{j+1}(t,\xi_{j+1}) + 2|\Lambda_{j+1}(t)|^2 \ . \tag{7.165}
$$

One can easily prove that there exist a function $H_{j+1}(t)$ and positive constants $h_{1,j+1}$ and $h_{2,j+1}$ such that

$$
\begin{aligned}
\widehat{Q}_{j+1}(t,\xi_{j+1}) \ &= \ \xi_{j+1}^\top H_{j+1}(t)\xi_{j+1} \quad \text{and} \\
h_{1,j+1}I_{j+1} \leq H_{j+1}(t) &\leq h_{2,j+1}I_{j+1} \quad \forall t \in \mathbb{R} \ .
\end{aligned}
\tag{7.166}
$$

Hence, the induction assumption is satisfied at the step $j+1$. We conclude by choosing $\mathcal{V}(t,x) = \widehat{Q}_n(t,x)$.

7.4.3 Illustration: Linear System with PE Coefficients

We use Theorem 7.2 to construct a stabilizing controller and a corresponding strict Lyapunov function for

$$
\begin{cases}
\dot{x}_1 = p(t)x_2 + \lambda_1(t) \\
\dot{x}_2 = p(t)u + \frac{1}{2}x_1 + \lambda_2(t) \ ,
\end{cases}
\tag{7.167}
$$

where $p(t) = 20\cos(t)$. This system is of the form (7.88) and since $p(t)$ is C^∞ and satisfies Assumptions 7.1, it follows that Assumptions 7.4-7.5 are also satisfied. Therefore, Theorem 7.2 applies to the system (7.167). Let us now construct the feedback and strict Lyapunov function guaranteed to exist by the theorem. First consider the auxiliary system

$$
\begin{cases}
\dot{x}_1 = p(t)x_2 + \lambda_1(t) \\
\dot{x}_2 = v + \frac{1}{2}x_1 + \lambda_2(t) \ ,
\end{cases}
\tag{7.168}
$$

where v is an input, and set $x = (x_1\ x_2)^\top$. When $\lambda_1 = \lambda_2 = 0$, one can apply the classical backstepping approach to obtain exponentially stabilizing linear control laws, as follows.

Step 1. Classical Backstepping

The time-varying change of coordinates

$$X_2 = x_2 + \cos^3(t)x_1 \tag{7.169}$$

transforms (7.168) into

$$\begin{cases} \dot{x}_1 = -20\cos^4(t)x_1 + 20\cos(t)X_2 + \lambda_1(t) \\ \dot{X}_2 = v + \frac{1}{2}x_1 - 3\cos^2(t)\sin(t)x_1 + 20\cos^4(t)\big[X_2 - \cos^3(t)x_1\big] \\ \qquad\quad + \cos^3(t)\lambda_1(t) + \lambda_2(t)\ . \end{cases} \tag{7.170}$$

When $\lambda_1 \equiv 0$ and $\lambda_2 \equiv 0$, the time derivative of

$$\mathcal{G}(x_1, X_2) = \frac{1}{2}[x_1^2 + X_2^2] \tag{7.171}$$

along the trajectories of (7.170) satisfies

$$\begin{aligned} \dot{\mathcal{G}} = &-20\cos^4(t)x_1^2 \\ &+X_2\big[v + 20\cos(t)x_1 + \tfrac{1}{2}x_1 - 3\cos^2(t)\sin(t)x_1 \\ &+20\cos^4(t)(X_2 - \cos^3(t)x_1)\big]. \end{aligned} \tag{7.172}$$

Choosing

$$\begin{aligned} v(t, x_1, X_2) = &-20\cos^2(t)X_2 - 20\cos(t)x_1 - \tfrac{1}{2}x_1 \\ &+3\cos^2(t)\sin(t)x_1 - 20\cos^4(t)\big(X_2 - \cos^3(t)x_1\big) \end{aligned} \tag{7.173}$$

gives

$$\begin{aligned} \dot{\mathcal{G}} &= -20\cos^4(t)x_1^2 - 20\cos^2(t)X_2^2 \\ &\leq -20\cos^4(t)\big[x_1^2 + X_2^2\big] \\ &\leq -40\cos^4(t)\mathcal{G}(x_1, X_2). \end{aligned} \tag{7.174}$$

Let

$$\mathcal{H}(t, x_1, X_2) = \left(\int_{t-\frac{\pi}{2}}^{t} \cos^4(m)\mathrm{d}m\right)\mathcal{G}(x_1, X_2). \tag{7.175}$$

Then

$$\begin{aligned}\dot{\mathcal{H}} &= \left[\cos^4(t) - \cos^4(t - \tfrac{\pi}{2})\right] \mathcal{G}(x_1, X_2) \\ &\quad + \left(\int_{t-\frac{\pi}{2}}^{t} \cos^4(m)\mathrm{d}m\right) \dot{\mathcal{G}}(x_1, X_2) \\ &\le \left[\cos^4(t) - \sin^4(t)\right]\mathcal{G}(x_1, X_2) \\ &\quad - \left(\int_{t-\frac{\pi}{2}}^{t} \cos^4(m)\mathrm{d}m\right) 40\cos^4(t)\mathcal{G}(x_1, X_2).\end{aligned} \tag{7.176}$$

Since

$$\int_{t-\frac{\pi}{2}}^{t} \cos^4(m)\mathrm{d}m = \frac{3\pi}{16} + \frac{\sin(2t)}{2} \tag{7.177}$$

and $\sin(2t) \ge -1$ everywhere, it follows that

$$\begin{aligned}\dot{\mathcal{H}} &\le \left[\cos^4(t) - \sin^4(t) - \left\{\frac{15\pi}{2} + 20\sin(2t)\right\}\cos^4(t)\right] \mathcal{G}(x_1, X_2) \\ &= -\left[\sin^4(t) + \left\{\frac{15\pi}{2} - 1 + 20\sin(2t)\right\}\cos^4(t)\right] \mathcal{G}(x_1, X_2) \\ &\le -\left[\sin^4(t) + \left(\frac{15\pi - 42}{2}\right)\cos^4(t)\right] \mathcal{G}(x_1, X_2).\end{aligned} \tag{7.178}$$

Step 2. Nonzero Disturbances

It follows that when λ_1 and λ_2 are present,

$$\begin{aligned}\dot{\mathcal{H}} &\le -\left[\sin^4(t) + \left\{\frac{15\pi - 42}{2}\right\}\cos^4(t)\right] \mathcal{G}(x_1, X_2) \\ &\quad + \left(\int_{t-\frac{\pi}{2}}^{t} \cos^4(m)\mathrm{d}m\right) x_1\lambda_1(t) \\ &\quad + \left(\int_{t-\frac{\pi}{2}}^{t} \cos^4(m)\mathrm{d}m\right) X_2\left[\cos^3(t)\lambda_1(t) + \lambda_2(t)\right]\end{aligned} \tag{7.179}$$

along the trajectories of (7.170).

Using (7.177) and the global inequalities

$$\frac{15\pi - 42}{2} \ge 1 \quad \text{and} \quad \sin^4(t) + \cos^4(t) \ge \frac{1}{2},$$

we get

$$\begin{aligned}\dot{\mathcal{H}} &\leq -\frac{1}{2}\mathcal{G}(x_1, X_2) + \left(\frac{3\pi}{16} + \frac{\sin(2t)}{2}\right)|x_1\lambda_1(t)| \\ &\quad + \left(\frac{3\pi}{16} + \frac{\sin(2t)}{2}\right)\left|X_2[\cos^3(t)\lambda_1(t) + \lambda_2(t)]\right| \\ &\leq -\frac{1}{4}[x_1^2 + X_2^2] \\ &\quad + \left(\frac{3\pi}{16} + \frac{1}{2}\right)|x_1||\lambda_1(t)| + \left(\frac{3\pi}{16} + \frac{1}{2}\right)|X_2|\left(|\lambda_1(t)| + |\lambda_2(t)|\right).\end{aligned} \tag{7.180}$$

From the triangular inequality $c_1 c_2 \leq 2c_1^2 + \frac{1}{8}c_2^2$ for suitable non-negative values c_1 and c_2, we get

$$\dot{\mathcal{H}} \leq -\frac{1}{8}\left[x_1^2 + X_2^2\right] + 2\left(\frac{3\pi}{16} + \frac{1}{2}\right)^2 \lambda_1^2(t) + 2\left(\frac{3\pi}{16} + \frac{1}{2}\right)^2 \left(|\lambda_1(t)| + |\lambda_2(t)|\right)^2.$$

Next, observing that

$$\begin{aligned}\frac{1}{8}\left[x_1^2 + X_2^2\right] &= \frac{1}{4}\mathcal{G}(x_1, X_2) \\ &= \frac{\mathcal{H}(t, x_1, X_2)}{\frac{3\pi}{4} + 2\sin(2t)} \geq \frac{\mathcal{H}(t, x_1, X_2)}{\frac{3\pi}{4} + 2}\end{aligned} \tag{7.181}$$

gives

$$\begin{aligned}\dot{\mathcal{H}} &\leq -\frac{\mathcal{H}(t, x_1, X_2)}{\frac{3\pi}{4} + 2} + 2\left(\frac{3\pi}{16} + \frac{1}{2}\right)^2 \lambda_1^2(t) \\ &\quad +2\left(\tfrac{3\pi}{16} + \tfrac{1}{2}\right)^2\left\{|\lambda_1(t)| + |\lambda_2(t)|\right\}^2 \\ &\leq -\frac{\mathcal{H}(t, x_1, X_2)}{\frac{3\pi}{4} + 2} + 6\left(\frac{3\pi}{16} + \frac{1}{2}\right)^2\left[\lambda_1^2(t) + \lambda_2^2(t)\right].\end{aligned} \tag{7.182}$$

We now return to the original coordinates. The feedback

$$v^\sharp(t, x_1, x_2) = v\big(t, x_1, x_2 + \cos^3(t)x_1\big)$$

with v defined in (7.173) admits the decomposition

$$v^\sharp(t, x_1, x_2) \;=\; L_1(t)\cdot x + 20\cos(t)L_2(t)\cdot x\ , \tag{7.183}$$

where $L_1(t)\cdot x = -\frac{1}{2}x_1$ and

$$L_2(t)\cdot x \;=\; -\left[\cos(t) + \cos^3(t)\right]x_2 + \left[-\cos^4(t) - 1 + \frac{3}{20}\cos(t)\sin(t)\right]x_1\ .$$

Next we consider the function

$$\overline{Q}(t,x_1,x_2) = \frac{1}{12\left(\frac{3\pi}{16}+\frac{1}{2}\right)^2}\mathcal{H}\big(t,x_1,x_2+\cos^3(t)x_1\big)\ . \tag{7.184}$$

By separately considering the possibilities

$$|x_1| \geq \frac{1}{\sqrt{5}}|x_2| \quad \text{and} \quad |x_1| \leq \frac{1}{\sqrt{5}}|x_2|,$$

our choice (7.169) of X_2 gives $x_1^2 + X_2^2 \geq \frac{1}{6}|x|^2$ everywhere. Also,

$$\overline{Q}(t,x) \ \geq \ \frac{1}{24\left(\frac{3\pi}{16}+\frac{1}{2}\right)^2}\left(\frac{3\pi}{16}-\frac{1}{2}\right)\left[x_1^2+(x_2+\cos^3(t)x_1)^2\right] \tag{7.185}$$

everywhere. One can then prove that the time derivative of $\overline{Q}(t,x)$ along the trajectories of (7.168) in closed-loop with the feedback $v^\sharp(t,x)$ satisfies

$$\dot{\overline{Q}} \ \leq \ -c_3\overline{Q}(t,x) + |\lambda(t)|^2, \quad \text{and} \quad c_1|x|^2 \ \leq \ \overline{Q}(t,x) \tag{7.186}$$

where $x = (x_1, x_2)$,

$$c_1 \ = \ \frac{\frac{3\pi}{16}-\frac{1}{2}}{144\left(\frac{3\pi}{16}+\frac{1}{2}\right)^2} \quad \text{and} \quad c_3 \ = \ \frac{1}{\frac{3\pi}{4}+2}\ . \tag{7.187}$$

We deduce from Theorem 7.1 and Remark 7.3 that the feedback

$$\begin{aligned} u &= \frac{p(t)}{p^2(t)+\delta}L_1(t)x + L_2(t)x \\ &= -\frac{20\cos(t)}{400\cos^2(t)+\delta}\frac{1}{2}x_1 - \left[\cos(t)+\cos^3(t)\right]x_2 \\ &\quad + \left[-\cos^4(t) - 1 + \tfrac{3}{20}\cos(t)\sin(t)\right]x_1 \end{aligned} \tag{7.188}$$

with δ such that

$$\int_0^{2\pi} \frac{\delta^2}{(400\cos^2(t)+\delta)^2}\mathrm{d}t \ \leq \ \frac{2\pi}{5}\frac{\frac{3\pi}{16}-\frac{1}{2}}{144\left(\frac{3\pi}{8}+1\right)^2}\frac{1}{\frac{3\pi}{4}+2} \tag{7.189}$$

renders (7.167) ISS with respect to λ; see (7.106). Inequality (7.189) holds if

$$\int_0^{\frac{\pi}{2}} \frac{\left(\frac{\delta}{20}\right)^2}{(\cos^2(t)+\frac{\delta}{400})^2}\mathrm{d}t \ \leq \ \frac{\pi}{23040\left(\frac{3\pi}{8}+1\right)^3}\ . \tag{7.190}$$

Therefore, we can construct an upper bound for the admissible values of $\delta > 0$ using the proof of Lemma 7.2. We leave the construction to the reader as a simple exercise.

7.5 Nonlinear Time-Varying Systems

7.5.1 Assumptions and Notation

We consider nonlinear time-varying systems of the form (7.63). Throughout the section, we assume that all of our functions are sufficiently smooth and:

Assumption 7.6 *There is a known continuous function* $b(t,x,z)$ *such that* $h(t,x,z) = p(t)b(t,x,z)$ *holds for all* $(t,x,z) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$.

Therefore, the system we consider is

$$\begin{cases} \dot{x} = \mathcal{F}(t,x,z) \\ \dot{z} = p(t)[u + b(t,x,z)] \ . \end{cases} \tag{7.191}$$

Assumption 7.7 *The functions* $|p(t)|$ *and* $|\dot{p}(t)|$ *are uniformly bounded by a positive real number* P *and two positive numbers* T *and* γ *such that*

$$\int_t^{t+T} p^2(s)\mathrm{d}s \geq \gamma \quad \forall t \in \mathbb{R} \tag{7.192}$$

are known. Also, $p \in C^1$.

Assumption 7.8 *There are known functions* V *and* $\alpha_i \in \mathcal{K}_\infty$*, a positive definite function* W*, and a function* $\mu_s \in C^1$ *such that*

$$|\mu_s(t,x)| \leq \alpha_4(|x|) \ , \tag{7.193}$$

$$\alpha_1(|x|) \leq V(t,x) \leq \alpha_2(|x|) \ , \ \left|\frac{\partial V}{\partial x}(t,x)\right| \leq \alpha_3(|x|), \tag{7.194}$$

$$\text{and} \ \frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x)\mathcal{F}(t,x,z_s(t,x)) \leq -W(x) \tag{7.195}$$

with

$$z_s(t,x) = p^2(t)\mu_s(t,x)$$

hold for all $t \in \mathbb{R}$ *and* $x \in \mathbb{R}^n$*. Also,* z_s *has period* T *in* t.

Assumption 7.9 *There exists an everywhere positive non-decreasing function* $\mathcal{C}$ *such that*

$$\frac{\partial V}{\partial x}(t,x)\big[\mathcal{F}(t,x,a_1)-\mathcal{F}(t,x,a_2)\big] \ \leq \ \frac{1}{2}W(x)+\mathcal{C}\big([a_1-a_2]^2\big)(a_1-a_2)^2 \tag{7.196}$$

for all $t \in \mathbb{R}$*,* $x \in \mathbb{R}^n$*,* $a_1 \in \mathbb{R}$*, and* $a_2 \in \mathbb{R}$.

7.5.2 Main Result and Remarks

Our main result for this subsection is the following:

Theorem 7.3. *Assume that the system (7.191) satisfies Assumptions 7.6-7.9 for some constant T. Then for any positive constant Υ, the system is globally uniformly asymptotically stabilizable by the feedback*

$$\begin{aligned} u_s(t,x,z) &= -\Upsilon p(t)[z - z_s(t,x)] - b(t,x,z) + 2\dot{p}(t)\mu_s(t,x) \\ &+p(t)\left[\frac{\partial \mu_s}{\partial t}(t,x) + \frac{\partial \mu_s}{\partial x}(t,x)\mathcal{F}(t,x,z)\right] . \end{aligned} \tag{7.197}$$

A global strict Lyapunov function for the corresponding closed-loop system is

$$U(t,x,z) = V(t,x) + K\left(\left[\frac{T}{\Upsilon} + \int_{t-T}^{t}\int_{\ell}^{t} p^2(s)\mathrm{d}s\,\mathrm{d}\ell\right] Z^2\right) , \tag{7.198}$$

where

$$Z = z - z_s(t,x) \tag{7.199}$$

and where $K \in \mathcal{K}_\infty$ is any function such that K' is non-decreasing and

$$K'(s) \ \geq \ \frac{1}{\gamma}\mathcal{C}\left(\frac{\Upsilon}{T}s\right) + \frac{1}{2\gamma} \tag{7.200}$$

for all $s \geq 0$.

Remark 7.8. Theorem 7.3 has the following important features:

1. It does not make any linear growth assumptions on $\mathcal{F}$. The only growth restriction on $\mathcal{F}$ is Assumption 7.9.
2. The PE property in Assumption 7.7 is not very restrictive; in contrast with Assumption 7.1, the function p can be equal to zero on intervals of positive length.
3. The control law z_s, its time derivative along the trajectories, and the function h must be zero when $p(t) = 0$.

Requirement 3. has no equivalent in Theorems 7.1 and 7.2. We impose it to allow nonlinearities, and to replace Assumption 7.1 by the weaker PE property from Assumption 7.7.

Proof of Theorem 7.3

The variable defined in (7.199) gives

$$\begin{cases} \dot{x} = \mathcal{F}(t, x, Z + z_s(t, x)) \\ \dot{Z} = p(t)[u + b(t, x, z)] - \dot{z}_s . \end{cases} \tag{7.201}$$

Since

$$\begin{aligned} \dot{z}_s &= 2p(t)\dot{p}(t)\mu_s(t, x) \\ &+p^2(t)\left[\frac{\partial \mu_s}{\partial t}(t, x) + \frac{\partial \mu_s}{\partial x}(t, x)\mathcal{F}(t, x, Z + z_s(t, x))\right] , \end{aligned} \tag{7.202}$$

the choice $u = u_s$ from (7.197) gives the closed-loop system

$$\begin{cases} \dot{x} = \mathcal{F}\big(t, x, z_s(t, x)\big) + \mathcal{F}\big(t, x, Z + z_s(t, x)\big) - \mathcal{F}\big(t, x, z_s(t, x)\big) \\ \dot{Z} = -\Upsilon p^2(t) Z. \end{cases} \tag{7.203}$$

Set

$$\aleph(t, Z) = \left[\frac{T}{\Upsilon} + \int_{t-T}^{t}\int_{\ell}^{t} p^2(s)\mathrm{d}s\,\mathrm{d}\ell\right] Z^2 . \tag{7.204}$$

Then the time derivative of

$$\mathcal{U}(t, x, Z) = V(t, x) + K\big(\aleph(t, Z)\big) \tag{7.205}$$

along the trajectories of (7.203) satisfies

$$\begin{aligned} \dot{\mathcal{U}} = \dot{V} &- 2K'\big(\aleph(t, Z)\big)\left[\tfrac{T}{\Upsilon} + \int_{t-T}^{t}\left(\int_{\ell}^{t} p^2(s)\mathrm{d}s\right)\mathrm{d}\ell\right]\Upsilon p^2(t) Z^2 \\ &+K'\big(\aleph(t, Z)\big)\left[T p^2(t) - \int_{t-T}^{t} p^2(s)\mathrm{d}s\right] Z^2 \\ \leq -W(x) &+ \tfrac{\partial V}{\partial x}(t, x)[\mathcal{F}(t, x, Z + z_s(t, x)) - \mathcal{F}(t, x, z_s(t, x))] \\ &-K'\big(\aleph(t, Z)\big) Z^2 \int_{t-T}^{t} p^2(s)\mathrm{d}s , \end{aligned} \tag{7.206}$$

by Assumption 7.8. Using Assumptions 7.7 and 7.9, we obtain

$$\dot{\mathcal{U}} \leq -W(x) + \frac{1}{2}W(x) + \mathcal{C}(Z^2)Z^2 - K'\left(\aleph(t, Z)\right)\gamma Z^2 . \tag{7.207}$$

Since we assumed that K' is non-decreasing, we deduce that

$$\dot{\mathcal{U}} \leq -\frac{1}{2}W(x) + \left[\mathcal{C}(Z^2) - K'\big(\frac{T}{\Upsilon}Z^2\big)\gamma\right] Z^2 . \tag{7.208}$$

Recalling (7.200), we obtain

$$\dot{\mathcal{U}} \;\leq\; -\frac{1}{2}W(x) - \frac{1}{2}Z^2.$$

Also, there are two functions $\alpha_5, \alpha_6 \in \mathcal{K}_\infty$ such that

$$\alpha_5(|(x,z)|) \leq U(t,x,z) = \mathcal{U}\big(t,x,z - z_s(t,x)\big) \leq \alpha_6(|(x,z)|)$$

for all $t \in \mathbb{R}$ and $(x,z) \in \mathbb{R}^n \times \mathbb{R}$, as desired. □

7.6 Bounded Backstepping

7.6.1 Assumptions and Statement of Result

We next show that when the following additional conditions are imposed, we can construct *bounded* stabilizing feedbacks for our systems (7.191):

Assumption 7.10 *There is a constant $\bar{B} > 0$ such that*

$$\begin{array}{l} |b(t,x,z)| \leq \bar{B}\,, \quad |\mu_s(t,x)| \leq \bar{B}\,, \quad \text{and} \\ \left|\dfrac{\partial \mu_s}{\partial t}(t,x) + \dfrac{\partial \mu_s}{\partial x}(t,x)\mathcal{F}(t,x,z)\right| \leq \bar{B}(1+|z|)\;. \end{array} \tag{7.209}$$

hold for all $t \in \mathbb{R}$ and all $(x,z) \in \mathbb{R}^n \times \mathbb{R}$, where μ_s and b are from Assumptions 7.6 and 7.8.

Remark 7.9. If μ_s satisfies Assumption 7.10 and $p(t)$ satisfies Assumption 7.7, then the choices

$$\mathcal{M} = \max\left\{1, P^2\bar{B}\right\} \tag{7.210}$$

and $z_s = p^2(t)\mu_s$ give

$$|z_s(t,x)| \leq \mathcal{M} \tag{7.211}$$

for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$.

We use the function

$$\Omega(s) \;=\; \operatorname{sgn}(s)\int_0^{|s|}\left[1 + \max\left\{0, \frac{(a-2\mathcal{M})^3}{1+(a-2\mathcal{M})^2}\right\}\right]\mathrm{d}a, \tag{7.212}$$

where $\operatorname{sgn}(s) = 1$ (resp., -1) if $s \geq 0$ (resp., $s < 0$). The function Ω has the following key properties:

1. Ω is of class C^2;
2. $\Omega(s) = s$ when $s \in [-2\mathcal{M}, 2\mathcal{M}]$; and
3. $\Omega'(z) \geq 1$ everywhere.

We prove:

Theorem 7.4. *Assume that the system (7.191) satisfies the Assumptions 7.6-7.10, and define $\mathcal{M}$ by (7.210). Then for any constant $\Upsilon > 0$, the system is globally uniformly asymptotically stabilizable by the feedback*

$$\begin{aligned} u_s(t,x,z) = & -\Upsilon p(t) \frac{\Omega(z) - z_s(t,x)}{\Omega'(z)\sqrt{1 + (\Omega(z) - z_s(t,x))^2}} - b(t,x,z) \\ & + \frac{2\dot{p}(t)\mu_s(t,x)}{\Omega'(z)} \\ & + \frac{p(t)}{\Omega'(z)} \left[\frac{\partial \mu_s}{\partial t}(t,x) + \frac{\partial \mu_s}{\partial x}(t,x)\mathcal{F}(t,x,z) \right]. \end{aligned} \tag{7.213}$$

A global strict Lyapunov function for the corresponding closed-loop system is

$$U(t,x,z) = V(t,x) + K\big(\nu_p(t, \Omega(z) - z_s(t,x))\big) \tag{7.214}$$

where $K \in C^1$ is any $\mathcal{K}_\infty$ function with a non-decreasing first derivative such that

$$K'(s) \geq \frac{T}{2\gamma\Upsilon} \left[1 + 128\sqrt{1 + \mathcal{M}^2}\mathcal{C}\left(128\sqrt{1 + \mathcal{M}^2} \frac{s}{\sqrt{1 + 2s}} \right) \right] \tag{7.215}$$

for all $s \geq 0$,

$$\nu_p(t,Z) = \frac{1}{2}Z^2 + \frac{\Upsilon}{T}\left(\int_{t-T}^{t} \left(\int_{s}^{t} p^2(a)\mathrm{d}a \right) \mathrm{d}s \right) \frac{Z^2}{\sqrt{1 + Z^2}}\ , \tag{7.216}$$

and

$$Z = \Omega(z) - z_s(t,x).$$

Moreover, the inequality

$$|u_s(t,x,z)| \leq \Upsilon P + \bar{B} + 2P\bar{B} + P\bar{B}(4\mathcal{M} + 2) \tag{7.217}$$

holds for all $t \in \mathbb{R}$ and all $(x,z) \in \mathbb{R}^n \times \mathbb{R}$.

7.6.2 Technical Lemmas

We present two technical lemmas that form the basis for our proof of Theorem 7.4. Consider the one-dimensional system

$$\dot{\xi} = -q(t)\frac{\xi}{\sqrt{1 + \xi^2}} \tag{7.218}$$

where q is any everywhere non-negative C^1 function.

Lemma 7.3. *Assume that there exist positive constants* δ_1, δ_2, *and* T_q *such that*

$$0 \leq q(t) \leq \delta_1 \quad \text{and} \quad \int_{t-T_q}^{t} q(s)\mathrm{d}s \geq \delta_2 \quad \forall t \in \mathbb{R}. \tag{7.219}$$

Then the time derivative of

$$\nu_q(t,\xi) \doteq \frac{1}{2}\xi^2 + \frac{1}{T_q}\left(\int_{t-T_q}^{t}\int_{s}^{t} q(a)\mathrm{d}a\,\mathrm{d}s\right)\frac{\xi^2}{\sqrt{1+\xi^2}} \tag{7.220}$$

along the trajectories of (7.218) satisfies

$$\dot{\nu}_q \;\leq\; -\frac{\delta_2}{T_q}\frac{\xi^2}{\sqrt{1+\xi^2}}\;. \tag{7.221}$$

Proof. The time derivative of ν_q along the trajectories of (7.218) satisfies

$$\begin{aligned}\dot{\nu}_q &\leq -q(t)\frac{\xi^2}{\sqrt{1+\xi^2}} + \left(q(t) - \frac{1}{T_q}\int_{t-T_q}^{t} q(a)\mathrm{d}a\right)\frac{\xi^2}{\sqrt{1+\xi^2}} \\ &= -\tfrac{1}{T_q}\left(\int_{t-T_q}^{t} q(a)\mathrm{d}a\right)\frac{\xi^2}{\sqrt{1+\xi^2}}\;.\end{aligned} \tag{7.222}$$

The lemma now follows from our choice of δ_2. □

Lemma 7.4. *Let* $\mathcal{M}$ *be defined by (7.210). Then for all* $z \in \mathbb{R}$, $t \in \mathbb{R}$, *and* $x \in \mathbb{R}^n$, *we have*

$$[z - z_s(t,x)]^2 \;\leq\; 64\sqrt{1+\mathcal{M}^2}\frac{(\Omega(z) - z_s(t,x))^2}{\sqrt{1+(\Omega(z)-z_s(t,x))^2}}. \tag{7.223}$$

Proof. We consider two cases.

Case 1. $|z| \leq 2\mathcal{M}$. Then $\Omega(z) = z$, so our bound (7.211) on z_s gives

$$(z - z_s(t,x))^2 \leq \sqrt{1+9\mathcal{M}^2}\frac{(\Omega(z) - z_s(t,x))^2}{\sqrt{1+(\Omega(z)-z_s(t,x))^2}}\;. \tag{7.224}$$

Case 2. $|z| \geq 2\mathcal{M}$. By (7.211), we get

$$\big(z - z_s(t,x)\big)^2 \;\leq\; \big(|z| + |z_s(t,x)|\big)^2 \;\leq\; \big(|z| + \mathcal{M}\big)^2 \;\leq\; \tfrac{5}{2}z^2\;. \tag{7.225}$$

If $2\mathcal{M} \leq |z| \leq 4\mathcal{M}$, then

$$\big[z - z_s(t,x)\big]^2 \;\leq\; 25\mathcal{M}^2\;. \tag{7.226}$$

On the other hand, since $|\Omega(z)| \geq |z|$ for all z, we have

$$\big(\Omega(z) - z_s(t,x)\big)^2 \;\geq\; (|\Omega(z)| - \mathcal{M})^2 \;\geq\; (|z| - \mathcal{M})^2 \;\geq\; \mathcal{M}^2 \,. \tag{7.227}$$

Since the function

$$\Theta(s) \doteq \frac{s}{\sqrt{1+s}} \tag{7.228}$$

is increasing on $[0,\infty)$, we get

$$\frac{(\Omega(z) - z_s(t,x))^2}{\sqrt{1 + (\Omega(z) - z_s(t,x))^2}} \geq \frac{\mathcal{M}^2}{\sqrt{1+\mathcal{M}^2}} \,, \tag{7.229}$$

so (7.226) gives

$$\big[z - z_s(t,x)\big]^2 \;\leq\; 25\sqrt{1+\mathcal{M}^2}\,\frac{(\Omega(z) - z_s(t,x))^2}{\sqrt{1 + (\Omega(z) - z_s(t,x))^2}} \,. \tag{7.230}$$

It remains to consider the case where $|z| \geq 4\mathcal{M}$; in that case,

$$\begin{aligned} |\Omega(z)| &= |z| + \int_{2\mathcal{M}}^{|z|} \frac{(m - 2\mathcal{M})^3}{1 + (m-2\mathcal{M})^2}\mathrm{d}m \\ &= |z| + \int_0^{|z|-2\mathcal{M}} \frac{m^3}{1+m^2}\mathrm{d}m \,. \end{aligned} \tag{7.231}$$

It follows that

$$\begin{aligned} \big|\Omega(z) - z_s(t,x)\big| &\geq |z| - \mathcal{M} + \int_0^{\frac{1}{2}|z|} \frac{m^3}{1+m^2}\mathrm{d}m \\ &\geq \tfrac{1}{2}|z| + \int_0^{\frac{1}{2}|z|} \frac{m^3}{1+m^2}\mathrm{d}m \\ &\geq \int_0^{\frac{1}{2}|z|} \frac{1+m^2+m^3}{1+m^2}\mathrm{d}m \;\geq\; \int_0^{\frac{1}{2}|z|} \frac{1}{2}(1+m)\mathrm{d}m \end{aligned} \tag{7.232}$$

and therefore

$$\big|\Omega(z) - z_s(t,x)\big| \geq \tfrac{1}{4}|z| + \tfrac{1}{16}z^2 \,. \tag{7.233}$$

Recalling that (7.228) is increasing, we deduce that

$$\begin{aligned} \frac{(\Omega(z) - z_s(t,x))^2}{\sqrt{1 + (\Omega(z) - z_s(t,x))^2}} &\geq \frac{\left(\frac{1}{4}|z| + \frac{1}{16}z^2\right)^2}{\sqrt{1 + \left(\frac{1}{4}|z| + \frac{1}{16}z^2\right)^2}} \\ &= \frac{\left(\frac{1}{4} + \frac{1}{16}|z|\right)^2}{\sqrt{1 + \left(\frac{1}{4}|z| + \frac{1}{16}z^2\right)^2}}z^2 \,. \end{aligned} \tag{7.234}$$

Using the inequality

$$z^2 \geq \frac{1}{2}\big(z - z_s(t,x)\big)^2,$$

(which is valid because $|z| \geq 4\mathcal{M}$, and therefore $\frac{1}{2}z^2 \geq \frac{1}{2}\mathcal{M}^2 + \mathcal{M}|z|$), we obtain

$$\frac{(\Omega(z) - z_s(t,x))^2}{\sqrt{1 + (\Omega(z) - z_s(t,x))^2}} \geq \frac{(\frac{1}{4} + \frac{1}{16}|z|)^2}{2\sqrt{1 + (\frac{1}{4}|z| + \frac{1}{16}z^2)^2}}\big(z - z_s(t,x)\big)^2. \quad (7.235)$$

Moreover, since

$$\Theta(r^2) \; \geq \; \frac{r}{2} \;\text{ on }\; [1, \infty),$$

and since our choice (7.210) of $\mathcal{M}$ gives $\mathcal{M} \geq 1$, we get

$$\begin{aligned} \frac{\left(\frac{1}{4} + \frac{1}{16}|z|\right)^2}{\sqrt{1 + \left(\frac{1}{4}|z| + \frac{1}{16}z^2\right)^2}} \; &= \; \Theta\left(\left[\frac{1}{4}|z| + \frac{1}{16}z^2\right]^2\right)\frac{1}{z^2} \\ &\geq \; \frac{1}{2z^2}\left(\frac{|z|}{4} + \frac{1}{16}z^2\right) \; \geq \; \frac{1}{32} \end{aligned}$$

when $z \neq 0$. It follows that when $|z| \geq 4\mathcal{M}$, we have

$$(z - z_s(t,x))^2 \; \leq \; 64\frac{(\Omega(z) - z_s(t,x))^2}{\sqrt{1 + |\Omega(z) - z_s(t,x)|^2}}. \quad (7.236)$$

Finally, from (7.236), (7.230) and (7.224), we deduce that (7.223) is satisfied in all three cases. This completes the proof of Lemma 7.4. □

7.6.3 Proof of Bounded Backstepping Theorem

The inequality (7.193) in Assumption 7.8 implies that for any function K of class $\mathcal{K}_\infty$, there are two functions $\alpha_5, \alpha_6 \in \mathcal{K}_\infty$ such that

$$\alpha_5(|(x,z)|) \leq U(t,x,z) \leq \alpha_6(|(x,z)|) \quad (7.237)$$

for all $t \in \mathbb{R}$ and $(x,z) \in \mathbb{R}^n \times \mathbb{R}$. Also, the time-varying change of coordinates

$$Z = \Omega(z) - z_s(t,x) \quad (7.238)$$

transforms the system (7.191) into

$$\begin{cases} \dot{x} = \mathcal{F}\big(t, x, \Omega^{-1}(Z + z_s(t, x))\big) \\ \dot{Z} = \Omega'(z)p(t)[u + b(t, x, z)] - \dot{z}_s(t, x) \ . \end{cases} \tag{7.239}$$

This system in closed-loop with $u_s(t, x, z)$ defined in (7.213) yields

$$\begin{aligned} \dot{Z} &= -\Upsilon p^2(t) \frac{\Omega(z) - z_s(t, x)}{\sqrt{1 + [\Omega(z) - z_s(t, x)]^2}} + 2\dot{p}(t)p(t)\mu_s(t, x) \\ &\quad + p^2(t) \left[\frac{\partial \mu_s}{\partial t}(t, x) + \frac{\partial \mu_s}{\partial x}(t, x)\mathcal{F}(t, x, z) \right] - \dot{z}_s(t, x) \\ &= -\Upsilon p^2(t) \tfrac{Z}{\sqrt{1+Z^2}} \ . \end{aligned} \tag{7.240}$$

Therefore, we have the closed-loop system

$$\begin{cases} \dot{x} = \mathcal{F}\big(t, x, \Omega^{-1}(Z + z_s(t, x))\big) \\ \dot{Z} = -\Upsilon p^2(t) \tfrac{Z}{\sqrt{1+Z^2}} \ . \end{cases} \tag{7.241}$$

According to Assumption 7.8, the time derivative of V along the trajectories of (7.241) satisfies

$$\begin{aligned} \dot{V} &\le -W(x) \\ &\quad + \tfrac{\partial V}{\partial x}(t, x) \left[\mathcal{F}(t, x, \Omega^{-1}(Z + z_s(t, x))) - \mathcal{F}(t, x, z_s(t, x)) \right] . \end{aligned} \tag{7.242}$$

Using Assumption 7.9, we deduce that

$$\begin{aligned} \dot{V} &\le -\tfrac{1}{2}W(x) \\ &\quad + \mathcal{C}\left([\Omega^{-1}(Z + z_s) - z_s]^2 \right) [\Omega^{-1}(Z + z_s) - z_s]^2 \ , \end{aligned} \tag{7.243}$$

where we omit the dependence of z_s on (t, x).

Next notice that (7.223) gives

$$\left(\Omega^{-1}(Z + z_s(t, x)) - z_s(t, x) \right)^2 \ \le \ 64\sqrt{1 + \mathcal{M}^2} \frac{Z^2}{\sqrt{1 + Z^2}}. \tag{7.244}$$

Combining this inequality and (7.243), we obtain

$$\begin{aligned} \dot{V} &\le -\tfrac{1}{2}W(x) \\ &\quad + 64\sqrt{1 + \mathcal{M}^2}\, \mathcal{C} \left(64\sqrt{1 + \mathcal{M}^2} \frac{Z^2}{\sqrt{1 + Z^2}} \right) \frac{Z^2}{\sqrt{1 + Z^2}}. \end{aligned} \tag{7.245}$$

On the other hand, Lemma 7.3 with the choice $q(t) = \Upsilon p^2(t)$ implies that the time derivative of $\nu_p(t, Z)$ along the trajectories of (7.241) satisfies

$$\dot{\nu}_p(t,Z) \leq -\frac{\gamma \Upsilon}{T}\frac{Z^2}{\sqrt{1+Z^2}}\ , \tag{7.246}$$

where γ is the constant in Assumption 7.7. It follows that the time derivative of U defined in (7.214) along the trajectories of (7.241) satisfies

$$\begin{aligned} \dot{U} \leq -\tfrac{1}{2}W(x) + \Bigg[& 64\sqrt{1+\mathcal{M}^2}\,\mathcal{C}\left(64\sqrt{1+\mathcal{M}^2}\frac{Z^2}{\sqrt{1+Z^2}}\right) \\ & -K'\big(\nu_p(t,Z)\big)\frac{\gamma\Upsilon}{T}\Bigg]\frac{Z^2}{\sqrt{1+Z^2}}\ . \end{aligned} \tag{7.247}$$

Since K' is non-decreasing and $\nu_p(t,Z) \geq \frac{1}{2}Z^2$, we have

$$\begin{aligned} \dot{U} \leq -\tfrac{1}{2}W(x) + \Bigg[& 64\sqrt{1+\mathcal{M}^2}\,\mathcal{C}\left(64\sqrt{1+\mathcal{M}^2}\frac{Z^2}{\sqrt{1+Z^2}}\right) \\ & -K'\left(\frac{1}{2}Z^2\right)\frac{\gamma\Upsilon}{T}\Bigg]\frac{Z^2}{\sqrt{1+Z^2}}\ . \end{aligned} \tag{7.248}$$

From (7.215), it follows immediately that

$$\dot{U} \ \leq \ -\frac{1}{2}\left[W(x) + \frac{Z^2}{\sqrt{1+Z^2}}\right]\ . \tag{7.249}$$

Therefore, the proof of Theorem 7.4 will be complete once we establish (7.217). It is easy to prove that the first three terms in the right hand side of (7.213) are bounded by ΥP, $\bar{B}$, and $2P\bar{B}$, respectively. Also, the definition of Ω gives

$$\Omega'(|z|) \ \geq \ \frac{1+|z|}{4\mathcal{M}+2} \ \ \forall z \in \mathbb{R},$$

by separately considering the cases $|z| \geq 4\mathcal{M}+1$ and $|z| \leq 4\mathcal{M}+1$ (because if $|z| \geq 4\mathcal{M}+1$, then $\Omega'(|z|) \geq 1 + \frac{1}{2}(|z| - 2\mathcal{M}) \geq 1 + \frac{|z|}{4}$ and $\Omega'(|z|) \geq 1$ everywhere). This property combined with the last inequality of Assumption 7.10 bounds the last term of the right hand side of (7.213) by $P\bar{B}(4\mathcal{M}+2)$. This concludes the proof of Theorem 7.4.

7.7 Two-Dimensional Example

The two-dimensional system

$$\left\{ \begin{array}{l} \dot{x} = \cos(t)z \\ \dot{z} = \cos(t)u \end{array} \right. \tag{7.250}$$

satisfies Assumptions 7.6 and 7.7 with $b \equiv 0$, $p(t) = \cos(t)$, $\gamma = \pi$, $P = 1$, and $T = 2\pi$. Choosing $\mu_s(t,x) = -\cos(t)x$, $z_s(t,x) = -\cos^3(t)x$, and

$$V(t,x) = \exp\left(\int_{t-\frac{\pi}{2}}^{t} \cos^2(s)\mathrm{d}s\right)\frac{1}{2}x^2 \ , \tag{7.251}$$

one can check readily that Assumption 7.8 is satisfied. In particular,

$$\begin{aligned} \frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x)\big[-\cos^4(t)x\big] &= V(t,x)\big[\cos(2t) - 2\cos^4(t)\big] \\ &= -\frac{1}{2}V(t,x)\big[1+\cos^2(2t)\big] \\ &\leq -\frac{1}{2}V(t,x) \ \leq \ -\frac{1}{4}x^2 \end{aligned} \tag{7.252}$$

and $\left|\frac{\partial V}{\partial x}(t,x)\cos(t)\right| \leq e^{\frac{\pi}{2}}|x|$ hold for all $(t,x) \in \mathbb{R}^2$. Hence,

$$\frac{\partial V}{\partial x}(t,x)\cos(t)(a_1 - a_2) \ \leq \ \frac{1}{8}|x|^2 + 2(a_1-a_2)^2 e^{\pi} \ , \tag{7.253}$$

by the triangle inequality. We easily deduce that Assumption 7.9 is satisfied with $W(x) = \frac{1}{4}x^2$ and $\mathcal{C}(s) = 2e^{\pi}$ for all $s \in \mathbb{R}$.

Therefore, Theorem 7.3 applies. It follows that the control law

$$\begin{aligned} u_s(t,x,z) = &-\cos(t)[z+\cos^3(t)x] \\ &+2\sin(t)\cos(t)x + \cos(t)\left[\sin(t)x - \cos^2(t)z\right] \end{aligned} \tag{7.254}$$

globally uniformly asymptotically stabilizes the system (7.250). Taking

$$K(s) \ = \ \left(\frac{2}{\pi}e^{\pi} + \frac{1}{2\pi}\right)s,$$

a global strict Lyapunov function for the system (7.250) in closed-loop with (7.254) is

$$\begin{aligned} \mathcal{U}(t,x,z) = &\exp\left(\int_{t-\frac{\pi}{2}}^{t}\cos^2(s)\mathrm{d}s\right)\frac{1}{2}x^2 \\ &+\frac{4e^{\pi}+1}{2\pi}\left[2\pi + \int_{t-2\pi}^{t}\left(\int_{\ell}^{t}\cos^2(s)\mathrm{d}s\right)d\ell\right][z+\cos^3(t)x]^2 \\ = &\exp\left(\frac{\pi}{4} + \frac{1}{2}\sin(2t)\right)\frac{1}{2}x^2 \\ &+\frac{4e^{\pi}+1}{2\pi}\left[2\pi + \frac{\pi}{2}\sin(2t) + \pi^2\right][z+\cos^3(t)x]^2 \ . \end{aligned}$$

7.8 PVTOL Revisited

We now use our results to construct the necessary control laws $\tilde{u}_{1s}$ and v_{2s} to stabilize (7.6). This will complete our stabilizing feedback construction for the PVTOL model from Sect. 7.1. We prove the following:

Theorem 7.5. *Choose any positive constants ε and Υ such that*

$$0 < \varepsilon \leq \frac{1}{54^2} \quad \text{and} \quad \Upsilon \leq \frac{\tan\left(\frac{3}{2}\right)}{108} . \tag{7.255}$$

Then the feedbacks

$$\tilde{u}_{1s} = \frac{[1 - 18\cos(3t)][1 - \cos(v_{2s})] - \tilde{z}_1 - \tilde{z}_2}{\cos(v_{2s})} \quad \text{and} \tag{7.256}$$

$$\begin{aligned} v_{2s}\left(t, \tilde{\xi}_1, \tilde{\xi}_2\right) = \arctan\Bigg(& -\Upsilon p(t) \frac{\Omega(\tilde{\xi}_2) + \varepsilon p^2(t) \frac{\tilde{\xi}_1}{\sqrt{1+\tilde{\xi}_1^2}}}{\Omega'(\tilde{\xi}_2)\sqrt{1 + \left(\Omega(\tilde{\xi}_2) + \varepsilon p^2(t)\frac{\tilde{\xi}_1}{\sqrt{1+\tilde{\xi}_1^2}}\right)^2}} \\ & -\varepsilon \frac{\tilde{\xi}_1}{\sqrt{1+\tilde{\xi}_1^2}} \frac{2\dot{p}(t)}{\Omega'(\tilde{\zeta}_2)} - \varepsilon \frac{p(t)}{\Omega'(\tilde{\xi}_2)(1+\tilde{\xi}_1^2)\sqrt{1+\tilde{\xi}_1^2}} \tilde{\xi}_2 \Bigg) \end{aligned} \tag{7.257}$$

with

$$\Omega(s) = \operatorname{sgn}(s) \int_0^{|s|} \left[1 + \max\left\{ 0, \frac{(a-2)^3}{1 + (a-2)^2} \right\} \right] \mathrm{d}a \tag{7.258}$$

and $p(t) = -1 + 18\cos(3t)$ *render (7.6) UGAS to the origin.*

The rest of this section is devoted to the proof of Theorem 7.5. We will presently show that $|v_{2s}| \leq \frac{3}{2}$ everywhere. Assuming this to be true for the moment, we get $\cos(v_{2s}) \geq \cos(\frac{3}{2}) > 0$ and therefore we can select (7.256) in (7.6) to get

$$\begin{cases} \dot{\tilde{\xi}}_1 = \tilde{\xi}_2 \\ \dot{\tilde{\xi}}_2 = [-1 + 18\cos(3t) + \tilde{z}_1 + \tilde{z}_2] \tan(v_{2s}) \\ \dot{\tilde{z}}_1 = \tilde{z}_2 \\ \dot{\tilde{z}}_2 = -\tilde{z}_1 - \tilde{z}_2 . \end{cases} \tag{7.259}$$

Since the $\tilde{z}$-subsystem of (7.259) is globally exponentially stable and since $|v_{2s}| \leq \frac{3}{2}$, this leads us to consider the problem of finding a control law u bounded by $\tan\left(\frac{3}{2}\right)$ and an iISS Lyapunov function $\bar{U}$ for the system

$$\begin{cases} \dot{\tilde{\xi}}_1 = \tilde{\xi}_2 \\ \dot{\tilde{\xi}}_2 = [-1 + 18\cos(3t) + d]u \end{cases} \tag{7.260}$$

with disturbance d. Later we use the iISS Lyapunov function to prove UGAS of the full closed-loop system (7.259).

7.8.1 Analysis of Reduced System

7.8.1.1 Zero Disturbances Case

To find the iISS Lyapunov function $\bar{U}$ for (7.260), we use the simplifying notation $x = \tilde{\xi}_1$ and $z = \tilde{\xi}_2$. Moreover, for the time being, let $d = 0$. Then we obtain the two-dimensional system

$$\begin{cases} \dot{x} = z \\ \dot{z} = p(t)u. \end{cases} \tag{7.261}$$

This system is of the form (7.191). Therefore, to determine stabilizing bounded controls for (7.261), we use Theorem 7.4. Before applying Theorem 7.4 to (7.261), we show that this system satisfies Assumptions 7.6-7.10.

It satisfies Assumption 7.6 with $b \equiv 0$, and it satisfies Assumption 7.7 with $P = 54$, $T = 2\pi$ and $\gamma = 324\pi$. We choose

$$\mu_s(t,x) = -\varepsilon \frac{x}{\sqrt{1+x^2}} \quad \text{and} \quad z_s(t,x) = -\varepsilon p^2(t) \frac{x}{\sqrt{1+x^2}} \tag{7.262}$$

where $\varepsilon > 0$ is such that (7.255) holds. Let $V(t,x) = \sqrt{1+\nu(t,x)} - 1$, where

$$\begin{aligned} \nu(t,x) &= \frac{1}{2}x^2 + \frac{1}{2\pi}\left(\int_{t-2\pi}^{t}\left(\int_{s}^{t}\varepsilon p^2(a)\mathrm{d}a\right)\mathrm{d}s\right)\frac{x^2}{\sqrt{1+x^2}} \\ &= \frac{1}{2}x^2 + \varepsilon\mathcal{S}(t)\frac{x^2}{\sqrt{1+x^2}} \end{aligned} \tag{7.263}$$

and

$$\mathcal{S}(t) = 163\pi + 27\sin(6t) - 12\sin(3t)\ . \tag{7.264}$$

According to Lemma 7.3, we have

$$\frac{\partial \nu}{\partial t}(t,x) + \frac{\partial \nu}{\partial x}(t,x)z_s(t,x) \ \leq \ -162\varepsilon\frac{x^2}{\sqrt{1+x^2}}\ . \tag{7.265}$$

It follows that

$$\begin{aligned} \frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x)z_s(t,x) &\leq -81\varepsilon\frac{x^2}{\sqrt{1+x^2}\sqrt{1+\nu(t,x)}} \\ &\leq -W(x)\ , \end{aligned} \tag{7.266}$$

where

$$W(x) = 81\varepsilon \frac{x^2}{1+x^2} \,, \tag{7.267}$$

because $|\mathcal{S}(t)| \leq 691$ for all $t \in \mathbb{R}$, and ε satisfies (7.255). This allows us to prove that Assumption 7.8 is satisfied. In addition,

$$\begin{aligned} \left|\frac{\partial V}{\partial x}(t,x)\right| &= \left|\frac{x\left[1+\varepsilon\mathcal{S}(t)\frac{2+x^2}{(1+x^2)\sqrt{1+x^2}}\right]}{2\sqrt{1+\nu(t,x)}}\right| \\ &\leq \frac{[1+691\varepsilon]}{\sqrt{1+\frac{1}{2}x^2}}|x| \;\leq\; \frac{2\sqrt{2}}{\sqrt{1+x^2}}|x| \,. \end{aligned} \tag{7.268}$$

This easily gives

$$\left(\frac{\partial V}{\partial x}(t,x)\right)^2 \;\leq\; \frac{8}{1+x^2}x^2 \;=\; \frac{8}{81\varepsilon}W(x) \,. \tag{7.269}$$

Therefore, we deduce from the triangular inequality that

$$\frac{\partial V}{\partial x}(t,x)(a_1-a_2) \;\leq\; \frac{1}{2}W(x) + \frac{4}{81\varepsilon}(a_1-a_2)^2 \tag{7.270}$$

which implies that Assumption 7.9 is satisfied with $\mathcal{C} \equiv \frac{4}{81\varepsilon}$. Finally, one can easily prove that Assumption 7.10 is satisfied with $\bar{B} = \varepsilon$ and therefore Theorem 7.4 applies to the system (7.261).

From Theorem 7.4 and the fact that $\mathcal{M} = 1$ (because (7.255) is satisfied), it follows that for any $\Upsilon > 0$ satisfying (7.255), the control law

$$\begin{aligned} u_s(t,x,z) = &-\Upsilon p(t)\frac{\Omega(z)+\varepsilon p^2(t)\frac{x}{\sqrt{1+x^2}}}{\Omega'(z)\sqrt{1+\left(\Omega(z)+\varepsilon p^2(t)\frac{x}{\sqrt{1+x^2}}\right)^2}} \\ &-\varepsilon\frac{x}{\sqrt{1+x^2}}\frac{2\dot{p}(t)}{\Omega'(z)} - \varepsilon\frac{p(t)}{\Omega'(z)(1+x^2)\sqrt{1+x^2}}z \end{aligned} \tag{7.271}$$

with Ω defined by (7.258) globally uniformly asymptotically stabilizes the origin of the system (7.261). Moreover, the proof of Theorem 7.4 implies that if we take $\nu_p(t,x)$ as defined in (7.216), namely,

$$\begin{aligned} \nu_p(t,Z) &= \frac{1}{2}Z^2 + \frac{\Upsilon}{2\pi}\left(\int_{t-2\pi}^{t}\left(\int_s^t p^2(a)\mathrm{d}a\right)\mathrm{d}s\right)\frac{Z^2}{\sqrt{1+Z^2}} \\ &= \frac{1}{2}Z^2 + \Upsilon\mathcal{S}(t)\frac{Z^2}{\sqrt{1+Z^2}}, \end{aligned} \tag{7.272}$$

$\mathcal{S}(t)$ defined in (7.264), and $K \in \mathcal{K}_\infty$ such that

$$\begin{aligned} K'(s) &\geq \frac{2\pi}{2\gamma\Upsilon}\left[1+128\sqrt{1+\mathcal{M}^2}\mathcal{C}\left(128\sqrt{1+\mathcal{M}^2}\frac{s}{\sqrt{1+2s}}\right)\right] \\ &= \frac{1}{324\Upsilon}\left[1+\frac{512\sqrt{2}}{81\varepsilon}\right] \ =: \underline{K}, \end{aligned} \tag{7.273}$$

then the time derivative of

$$U(t,x,z) = \sqrt{1+\nu(t,x)} - 1 + K\left(\nu_p\left(t, \Omega(z) + \varepsilon p^2(t)\tfrac{x}{\sqrt{1+x^2}}\right)\right) \tag{7.274}$$

along the trajectories of (7.261), in closed-loop with (7.271), satisfies

$$\dot{U} \ \leq \ -\tfrac{1}{2}\left[W(x) + \tfrac{Z^2}{\sqrt{1+Z^2}}\right] , \tag{7.275}$$

where $Z = \Omega(z) - z_s(t,x)$. Let us choose $K(s) = \underline{K}s$. By (7.217), the function u_s defined in (7.271) satisfies $|u_s(t,x,z)| \leq 54\Upsilon + 433\varepsilon$. Noting that Υ satisfies (7.255) and observing that (7.255) implies that

$$\varepsilon \ \leq \ \frac{\tan\left(\frac{3}{2}\right)}{54^2},$$

we get

$$|u_s(t,x,z)| \ \leq \ \tan\left(\frac{3}{2}\right) . \tag{7.276}$$

We therefore take $v_{2s} = \arctan(u_s)$.

7.8.1.2 Nonzero Disturbances Case

Returning to the system (7.260) when d is present, we immediately deduce from the previous analysis that the time derivative of

$$U(t,\tilde{\xi}_1,\tilde{\xi}_2) = \sqrt{1+\nu(t,\tilde{\xi}_1)} - 1 + \underline{K}\nu_p\left(t, \varrho(t,\tilde{\xi}_1,\tilde{\xi}_2)\right) \tag{7.277}$$

with

$$\varrho(t,\tilde{\xi}_1,\tilde{\xi}_2) = \Omega(\tilde{\xi}_2) + \varepsilon p^2(t)\frac{\tilde{\xi}_1}{\sqrt{1+\tilde{\xi}_1^2}} \tag{7.278}$$

along the solutions of (7.260) in closed-loop with $u_s(t,\tilde{\xi}_1,\tilde{\xi}_2)$ defined in (7.271) satisfies

$$\dot{U} \ \leq \ -\frac{1}{2}W(\tilde{\xi}_1) - \frac{1}{2}\hat{\varrho}(t,\tilde{\xi}_1,\tilde{\xi}_2) + \frac{\partial U}{\partial \tilde{\xi}_2}(t,\tilde{\xi}_1,\tilde{\xi}_2) d u_s(t,\tilde{\xi}_1,\tilde{\xi}_2) , \tag{7.279}$$

where

$$\hat{\varrho} \doteq \frac{\varrho^2}{\sqrt{1+\varrho^2}}.$$

We have

$$\begin{aligned}\frac{\partial U}{\partial \tilde{\xi}_2}(t,\tilde{\xi}_1,\tilde{\xi}_2) &= \underline{K}\frac{\partial \nu_p}{\partial Z}\left(t,\varrho\left(t,\tilde{\xi}_1,\tilde{\xi}_2\right)\right)\Omega'\left(\tilde{\xi}_2\right)\\ &= \underline{K}\left[1+\Upsilon\mathcal{S}(t)\frac{2+\varrho^2(t,\tilde{\xi}_1,\tilde{\xi}_2)}{(1+\varrho^2(t,\tilde{\xi}_1,\tilde{\xi}_2))^{\frac{3}{2}}}\right]\\ &\quad\times\varrho(t,\tilde{\xi}_1,\tilde{\xi}_2)\Omega'(\tilde{\xi}_2)\ .\end{aligned} \tag{7.280}$$

Since $\mathcal{S}(t)| \leq 691$ everywhere, we deduce that

$$\left|\frac{\partial U}{\partial \tilde{\xi}_2}(t,\tilde{\xi}_1,\tilde{\xi}_2)\right| \leq \mathcal{M}_1|\varrho(t,\tilde{\xi}_1,\tilde{\xi}_2)\Omega'(\tilde{\xi}_2)| \tag{7.281}$$

where $\mathcal{M}_1 = 2\underline{K}(1+691\Upsilon)$, and therefore

$$\left|\frac{\partial U}{\partial \tilde{\xi}_2}(t,\tilde{\xi}_1,\tilde{\xi}_2)du_s(t,\tilde{\xi}_1,\tilde{\xi}_2)\right| \leq \mathcal{M}_2|\varrho(t,\tilde{\xi}_1,\tilde{\xi}_2)\Omega'(\tilde{\xi}_2)||d| \tag{7.282}$$

where $\mathcal{M}_2 = \mathcal{M}_1 \tan(3/2)$. Next, using (7.255) one can easily prove that $|\varrho(t,\tilde{\xi}_1,\tilde{\xi}_2)| \leq |\Omega(\tilde{\xi}_2)| + 19^2\varepsilon \leq |\Omega(\tilde{\xi}_2)| + 1$.

It follows that

$$\left|\frac{\partial U}{\partial \tilde{\xi}_2}(t,\tilde{\xi}_1,\tilde{\xi}_2)du_s(t,\tilde{\xi}_1,\tilde{\xi}_2)\right| \leq \mathcal{M}_2\left[|\Omega(\tilde{\xi}_2)|+1\right]\Omega'(\tilde{\xi}_2)|d|\ . \tag{7.283}$$

This inequality combined with (7.279) yields

$$\dot{U} \leq -\frac{1}{2}W(\tilde{\xi}_1) - \frac{1}{2}\hat{\varrho}(t,\tilde{\xi}_1,\tilde{\xi}_2) + \mathcal{M}_2\left[|\Omega(\tilde{\xi}_2)|+1\right]\Omega'(\tilde{\xi}_2)|d|\ . \tag{7.284}$$

Therefore, the time derivative of

$$\overline{U}(t,\tilde{\xi}_1,\tilde{\xi}_2) = \ln\left(1+U(t,\tilde{\xi}_1,\tilde{\xi}_2)\right) \tag{7.285}$$

along the closed-loop trajectories of (7.260) satisfies

$$\dot{\overline{U}} \leq \frac{-\frac{1}{2}W(\tilde{\xi}_1) - \frac{1}{2}\hat{\varrho}(t,\tilde{\xi}_1,\tilde{\xi}_2) + \mathcal{M}_2[|\Omega(\tilde{\xi}_2)|+1]\Omega'(\tilde{\xi}_2)|d|}{1+U(t,\tilde{\xi}_1,\tilde{\xi}_2)}\ . \tag{7.286}$$

Next, observe that

$$|\Omega'(\tilde{\xi}_2)| = 1+\max\left\{0, \frac{(|\tilde{\xi}_2|-2\mathcal{M})^3}{1+(|\tilde{\xi}_2|-2\mathcal{M})^2}\right\} \leq 1+|\tilde{\xi}_2| \leq 1+|\Omega(\tilde{\xi}_2)|\ .$$

It follows that

$$\dot{\overline{U}} \leq \frac{-\frac{1}{2}W(\tilde{\xi}_1) - \frac{1}{2}\hat{\varrho}(t,\tilde{\xi}_1,\tilde{\xi}_2) + \mathcal{M}_2[|\Omega(\tilde{\xi}_2)|+1]^2|d|}{1+U(t,\tilde{\xi}_1,\tilde{\xi}_2)} , \tag{7.287}$$

and therefore

$$\begin{aligned}\dot{\overline{U}} &\leq \frac{-\frac{1}{2}W(\tilde{\xi}_1)-\frac{1}{2}\hat{\varrho}(t,\tilde{\xi}_1,\tilde{\xi}_2)+\mathcal{M}_2\left[\left|\Omega(\tilde{\xi}_2)+\varepsilon p^2(t)\frac{\tilde{\xi}_1}{\sqrt{1+\tilde{\xi}_1^2}}\right|+\varepsilon p^2(t)+1\right]^2|d|}{1+U(t,\tilde{\xi}_1,\tilde{\xi}_2)} \\ &\leq \frac{-\frac{1}{2}W(\tilde{\xi}_1)-\frac{1}{2}\hat{\varrho}(t,\tilde{\xi}_1,\tilde{\xi}_2)+\mathcal{M}_2\left[\left|\varrho(t,\tilde{\xi}_1,\tilde{\xi}_2)\right|+361\varepsilon+1\right]^2|d|}{1+U(t,\tilde{\xi}_1,\tilde{\xi}_2)} \\ &\leq \frac{-\frac{1}{2}W(\tilde{\xi}_1)-\frac{1}{2}\hat{\varrho}(t,\tilde{\xi}_1,\tilde{\xi}_2)+2\mathcal{M}_2[\varrho^2(t,\tilde{\xi}_1,\tilde{\xi}_2)+4]|d|}{\sqrt{1+\nu(t,\tilde{\xi}_1)}+K\left(\frac{1}{2}\varrho^2(t,\tilde{\xi}_1,\tilde{\xi}_2)+\Upsilon\mathcal{S}(t)\frac{\varrho^2(t,\tilde{\xi}_1,\tilde{\xi}_2)}{\sqrt{1+\varrho^2(t,\tilde{\xi}_1,\tilde{\xi}_2)}}\right)} .\end{aligned} \tag{7.288}$$

It follows that one can determine a constant $\mathcal{M}_3$ such that

$$\begin{aligned}\dot{\overline{U}} \leq &\frac{-\frac{1}{2}W(\tilde{\xi}_1) - \frac{1}{2}\hat{\varrho}(t,\tilde{\xi}_1,\tilde{\xi}_2)}{\sqrt{1+\nu(t,\tilde{\xi}_1)} + K\left(\frac{1}{2}\varrho^2(t,\tilde{\xi}_1,\tilde{\xi}_2) + \Upsilon\mathcal{S}(t)\frac{\varrho^2(t,\tilde{\xi}_1,\tilde{\xi}_2)}{\sqrt{1+\varrho^2(t,\tilde{\xi}_1,\tilde{\xi}_2)}}\right)} \\ &+\mathcal{M}_3|d| .\end{aligned} \tag{7.289}$$

This implies that $\overline{U}$ is the desired iISS Lyapunov function.

7.8.2 UGAS of Full System

Standard arguments (analogous to those in [8] but generalized to time-varying periodic systems) now provide $\underline{\alpha} \in \mathcal{K}_\infty$, $\beta \in \mathcal{KL}$, and a constant $\bar{M} > 0$ such that for each $k \in \mathbb{N} \cup \{0\}$ and $t_0 \geq 0$ and each trajectory $\tilde{\xi}(t)$ of (7.260) with initial time t_0, we have the iISS estimate

$$\underline{\alpha}\left(|\tilde{\xi}(t+2k\pi)|\right) \leq \beta(|\tilde{\xi}(t_0+2k\pi)|, t-t_0) + \bar{M}\int_{t_0+2k\pi}^{t+2k\pi} |d(r)|\mathrm{d}r \tag{7.290}$$

for all $t \geq t_0$ and all exponentially decaying disturbances d. Specializing to the case where $d = \tilde{z}$ converges exponentially to zero and $k = 0$, (7.290) readily gives a $\mathcal{K}_\infty$ function $\bar{\mathcal{M}}$ such that $|\tilde{\xi}(t)| \leq \bar{\mathcal{M}}(|\xi(t_0)|)$ for all $t \geq t_0 \geq 0$ along the closed-loop trajectories. Also, for each pair (ε, b) of positive constants, we can find a positive integer $\tilde{K}$ such that

$$\left|\tilde{\xi}(t+2k\pi)\right| < \varepsilon \quad \text{when} \quad \min\{t-t_0, k\} \geq \tilde{K} \;\text{ and }\; |(\tilde{\xi}(t_0), \tilde{z}(t_0))| \leq b.$$

Therefore, we get the uniform global attractivity condition $|\tilde{\xi}(r)| < \varepsilon$ when $r \geq T + t_0$ and $|(\tilde{\xi}(t_0), \tilde{z}(t_0))| \leq b$, where $T = \tilde{K}(1+2\pi)$ depends only on ε

and b. We deduce that the origin of (7.259) in closed-loop with $v_{2s}(t, \tilde{\xi}_1, \tilde{\xi}_2)$ is UGAS. This proves the theorem. □

7.8.3 Numerical Example

To validate our feedback design, we simulated (7.6) in closed-loop with the feedbacks (7.256) and (7.257), using $\varepsilon = 1/54^2$ and the initial state $(\tilde{\zeta}_1, \tilde{\zeta}_2, \tilde{z}_1, \tilde{z}_2)(0) = (0.5, 0.5, 1, 1)$. We report the corresponding error trajectories for the positions and velocities in Figs. 7.1 and 7.2. Our simulation shows the rapid convergence of the tracking error to zero and therefore validates our findings.

7.9 Comments

Backstepping is a powerful method because it applies to general classes of nonlinear systems and simultaneously constructs Lyapunov functions and stabilizing feedbacks. Some pioneering works on backstepping include [19, 138, 179]; see [75] for other important references. Introductions to backstepping can be found in several articles and textbooks. In [70, 183], results similar to the one we presented in Sect. 7.2.3 are presented. In [148, Chap. 6], backstepping with cancelation is introduced. In [149, Chap. 6], strict feedback systems (which comprise a family of systems that is slightly more restrictive than the family (7.52)) are studied. Time varying versions of backstepping are given in [181]. A first result on bounded backstepping for time-invariant systems is in [44]. An extension to time-varying systems is given in [99]. This last extension borrows some key ideas of [66]. Our approach differs from this earlier work because of our global strict Lyapunov function constructions.

The literature on the PVTOL model is sizable. Some of this work uses the more general VTOL model

$$\begin{cases} \ddot{x} = -u_1 \sin(\theta) + \varepsilon u_2 \cos(\theta) \\ \ddot{y} = u_1 \cos(\theta) + \varepsilon u_2 \sin(\theta) - 1 \\ \ddot{\theta} = u_2 \ , \end{cases}$$

where the positive parameter ε represents the sloping of the wings of the aircraft. The model appears to have originated in [56], which developed an approximate input-output linearization method that led to asymptotic stability and bounded tracking. For a nonlinear small gain approach to the model, see [174]; and see [96] for an extension of [56] based on flatness. In [149, Chap. 6], the PVTOL model is stabilized by time-invariant feedback. See also [80], which uses an optimal control approach to design state feedbacks that give

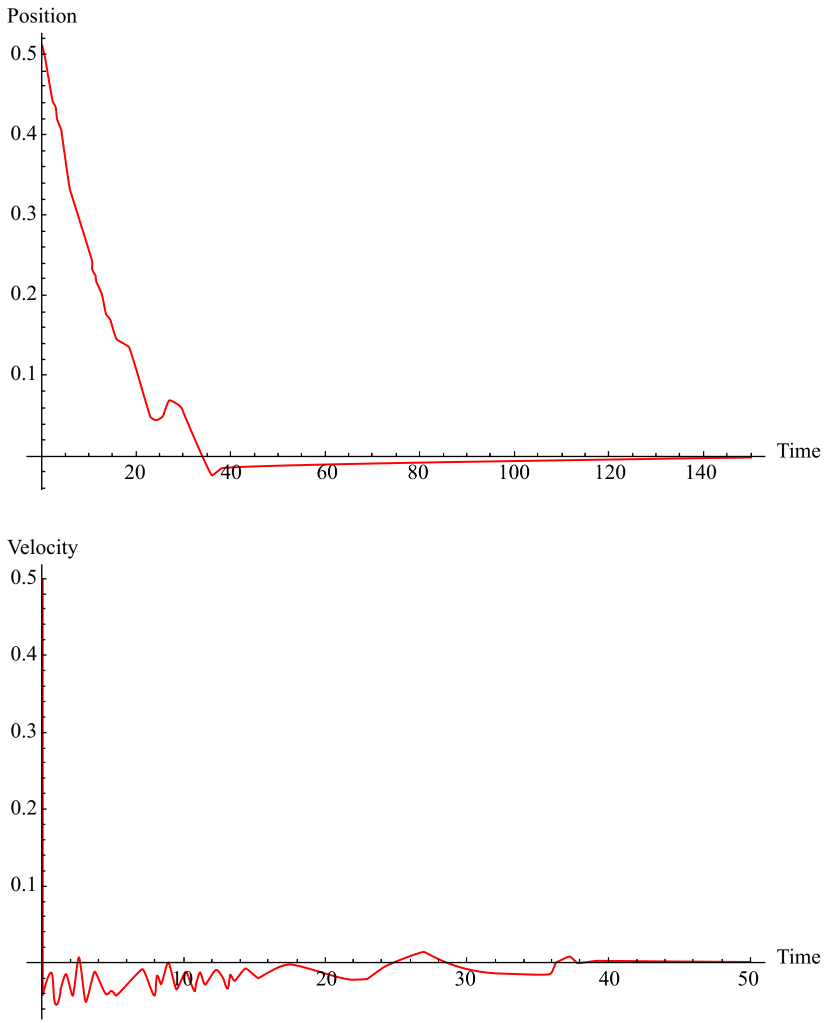

Fig. 7.1 Horizontal position and velocity components of (7.6)

robust hovering control of the PVTOL model. For internal model and output tracking approaches, see [95] and [38], respectively. Finally, see [129] for a PVTOL set up where the state is measured using a visual system that produces a delay, and [43] for state feedback designs for PVTOL models with delays in the input for cases where the velocity variables are not available for measurement. By contrast, our treatment of the PVTOL model is based on constructions of global strict Lyapunov functions.

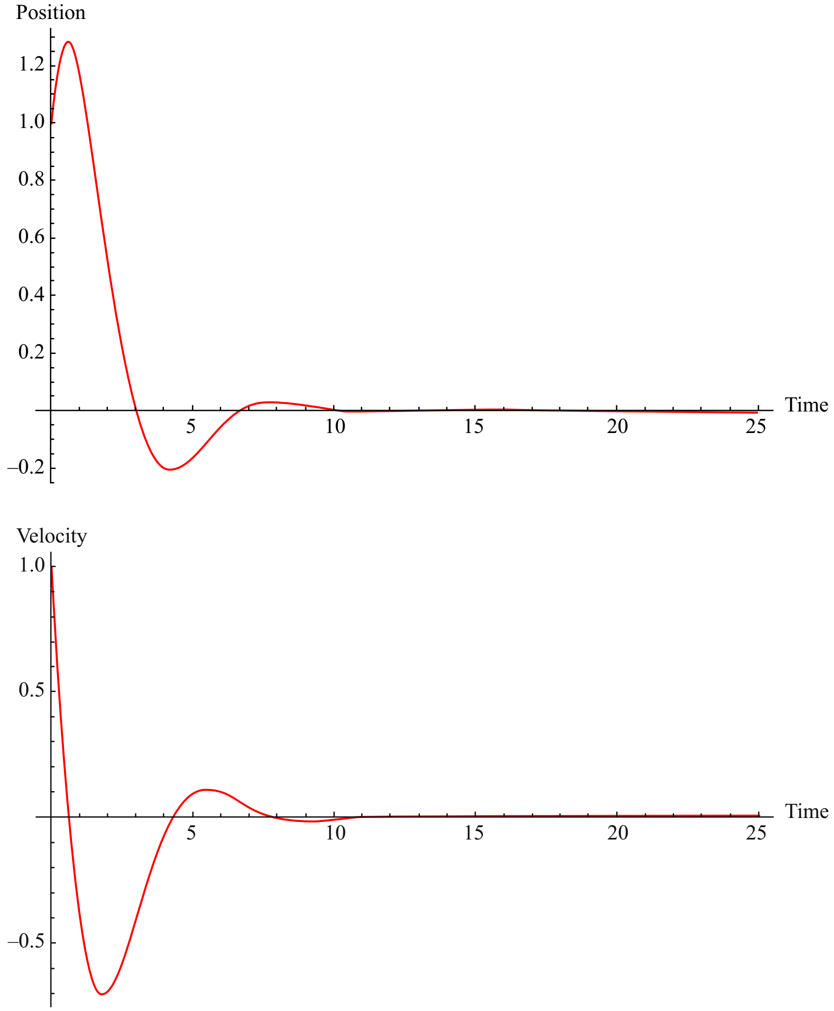

Fig. 7.2 Vertical position and velocity components of (7.6)

Chained form systems of the type (7.64) have been studied extensively. See for example [146] where they are used to control nonholonomic wheeled mobile robots and cars with multiple trailers. The TORA dynamics (7.81) has been studied by many authors; see for example [149]. The physical model consists of a platform connected to a fixed frame of reference by a spring. The platform can oscillate in the horizontal plane, and friction is assumed to be negligible. There is a rotating eccentric mass on the platform that is actuated by a DC motor. The rotating mass yields a force that can be controlled to dampen the oscillations of the platform. The control variable is the motor torque. There are several stabilizing control designs in the literature, where stability for the TORA dynamics is shown using non-strict Lyapunov functions and the LaSalle Invariant Set [149].